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The Calderón problem for a nonlocal diffusion equation with time-dependent coefficients

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Abstract. We investigate the Calderón problem for a nonlocal diffusion equation depending on a globally unknown isotropic coefficient $\gamma(x,t)$. The forward problem is posed on $\Omega \times (0,T)$ for a domain Ω that is bounded in one direction. We first show that the *Dirichlet-to-Neumann map* Λ_γ determines γ in the measurement set. By studying various properties of the related *nonlocal Neumann derivatives* $\mathcal{N}_\gamma$, we prove that both quantities $\langle \Lambda_\gamma f, g\rangle$ and $\langle \mathcal{N}_\gamma f, g\rangle$ carry the same information as long as $f,g\colon \mathbb{R}^n \setminus \overline{\Omega} \to \mathbb{R}$ have disjoint supports and γ is known in $\mathrm{supp}(f) \cup \mathrm{supp}(g)$. We obtain the desired global uniqueness theorem using a suitable integral identity for $\mathcal{N}_\gamma$ and the Runge approximation property. The results hold for any spatial dimension $n \geq 1$. In conclusion, the main observations of this article are twofold: (1) the information of Λ_γ is needed for exterior determination for γ, (2) the knowledge of $\mathcal{N}_\gamma$ and γ in the measurement set is enough to recover γ in the interior.

1. Introduction

In 1980, Alberto P. Calderón published his pioneering work "On an inverse boundary value problem" [5], which introduces a problem that later turned out to be of foundational importance for several imaging methods and inspired many developments in the field of inverse problems in general. He asked the question: "Can one determine the electrical conductivity of a medium by only making voltage and current measurements on the surface?" This problem is referred to as the *Calderón problem* in the literature. The mathematical setup is to consider a bounded domain $\Omega \subset \mathbb{R}^n$ with sufficiently regular boundary $\partial\Omega$, representing a conducting medium, and a positive function $\gamma(x) > 0$ in Ω which is its a priori unknown conductivity. It is known that sufficiently regular conductivities are uniquely determined by the information of current and voltage measurements on the boundary. In other words, γ can be recovered when the Cauchy data $\{u|_{\partial\Omega}, \gamma\partial_\nu u|_{\partial\Omega}\}$ is given, where u solves the conductivity equation

$$\operatorname{div}(\gamma\nabla u) = 0 \quad \text{in } \Omega. \tag{1.1}$$

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The Calderón problem was first solved in [46] in space dimension $n \geq 3$, where the authors demonstrated the fact that the conductivity can be determined uniquely by the *Dirichlet-to-Neumann map* (DN map, $u|_{\partial\Omega} \mapsto \gamma\, \partial u/\partial \nu$) of the conductivity equation (1.1). After some years, the same result has been showed in space dimension $n = 2$ in [39] and later for conductivities which are only uniformly elliptic [2].

Recently, the studies of Calderón type inverse problems have been considered for nonlocal operators as well. A prototypical example is the inverse exterior value problem for the fractional Schrödinger operator $(-\Delta)^s + q(x)$, which was first introduced and solved in [24]. The main tool in solving this Calderón problem is based on a suitable *unique continuation property* (UCP) and the closely related *Runge approximation*. By applying similar ideas, one can solve several challenging problems, some of which still stay open in the corresponding local cases. This shows that nonlocal inverse problems take advantage of the *nonlocality* of the underlying operators. For further details, we refer to [3,7–12,14,20–23,25–27,31,34–38,40–42,44,45] and the references therein. We emphasize that most of these works consider nonlocal inverse problems in which one wants to recover lower-order coefficients. On the other hand, in the articles [10,25,37,40–42], the authors study nonlocal inverse problems where one is interested in determining leading order coefficients, and hence they can be seen as full nonlocal analogies of the classical Calderón problem.

1.1. Mathematical modeling and main results

Let $\Omega \subset \mathbb{R}^n$ be an open set bounded in one direction for any $n \in \mathbb{N}$, and consider the initial exterior value problem of the *variable coefficient nonlocal diffusion equation*

$$\begin{cases} \partial_t u + \operatorname{div}_s(\Theta_\gamma \nabla^s u) = 0 & \text{in } \Omega_T, \\ u = f & \text{in } (\Omega_e)_T, \\ u(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $\Omega_e := \mathbb{R}^n \setminus \overline{\Omega}$ denotes the exterior of Ω, $0 < T < \infty$ and $0 < s < \min(1, n/2)$. Throughout this work, let us assume $\gamma \in L^\infty(\mathbb{R}^n_T)$ is a uniformly elliptic conductivity, i.e., there exists a constant $\gamma_0 > 0$ such that

$$0 < \gamma_0 \leq \gamma(x,t) \leq \gamma_0^{-1} \quad \text{for } (x,t) \in \mathbb{R}^n_T. \tag{1.3}$$

In addition, we denote the conductivity matrix by

$$\Theta_\gamma(x,y,t) := \gamma^{1/2}(x,t)\,\gamma^{1/2}(y,t)\,\mathbf{1}_{n\times n} \quad \text{for } x,y \in \mathbb{R}^n.$$

Moreover, we always use the notation

$$A_T := A \times (0,T)$$

to denote the space time cylinders, where $A \subset \mathbb{R}^n$ can be any set.

In this work, we are interested in the determination of the conductivity $\gamma(x,t)$ in $\mathbb{R}^n_T$ for the nonlocal diffusion equation (1.2), which extends recent global uniqueness results for the Calderón problem of certain nonlocal elliptic equations to their parabolic counterpart. Assuming the well-posedness of (1.2) at the moment (the proof will be given in Section 3), we can define the DN map via

$$\begin{aligned}(1.4)\qquad \langle \Lambda_\gamma f, g\rangle := \frac{C_{n,s}}{2}\int_0^T \int_{\mathbb{R}^{2n}} &\gamma^{1/2}(x,t)\gamma^{1/2}(y,t)\\ &\cdot \frac{(u_f(x,t)-u_f(y,t))(g(x,t)-g(y,t))}{|x-y|^{n+2s}}\,dx\,dy\,dt,\end{aligned}$$

for all $f, g \in C_c^\infty((\Omega_e)_T)$, where

$$C_{n,s} := \frac{4^s\Gamma(n/2+s)}{\pi^{n/2}|\Gamma(-s)|} \tag{1.5}$$

is a constant and u_f is the unique solution of (1.2). More precisely, we ask the following question:

Question 1. If we have given conductivities γ_1, γ_2 in a suitable function space such that

$$\Lambda_{\gamma_1} f|_{(W_2)_T} = \Lambda_{\gamma_2} f|_{(W_2)_T} \quad \text{for all } f \in C_c^\infty((W_1)_T),$$

where $W_1, W_2 \subset \Omega_e$ are given nonempty open sets, does $\gamma_1 = \gamma_2$ hold in $\mathbb{R}^n_T$?

In the limiting case $s = 1$, this problem and its generalizations have been studied, for example, in [6] or [19]. In these works, the authors determine the coefficients for heat equations for any spatial dimension $n \geq 2$ by using the corresponding boundary measurements, where they allow an additional uniformly elliptic coefficient ρ in front of the time derivative. On the other hand, in the works [29, 30], the inverse problem for the diffusion equation with fractional time derivative has been studied.

Next, let u_j be the solution of

$$\begin{cases} \partial_t u_j + \operatorname{div}_s(\Theta_{\gamma_j}\nabla^s u_j) = 0 & \text{in } \Omega_T,\\ u_j = f & \text{in } (\Omega_e)_T,\\ u_j(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{1.6}$$

and denote the exterior DN map of (1.6) by Λ_{γ_j}, for $j = 1, 2$. Our first theorem shows that the exterior DN maps have a unique continuation property. The proof is based on a spacetime Liouville reduction, which reduces the problem to a diffusion Schrödinger type inverse problem. By applying the Runge approximation property for certain equations, we can prove the uniqueness of the conductivity γ. The argument, however, requires the Alessandrini identity as well as the use of the UCP of the fractional Laplacian twice, once in H^s and once in $H^{2s,n/(2s)}$, the latter using a general UCP result in [28]. See Theorem 5.2 for further information on why earlier approaches that work well in the elliptic case lead to additional challenges in the studied parabolic case. In particular, the parabolic fractional Liouville reduction introduces new zeroth and first-order coefficients concerning the time derivative. This is a new and non-standard equation to be investigated further in our present work, in comparison to the elliptic case which reduces to a standard elliptic problem of the type $(-\Delta)^s + q$.

Theorem 1.1 (Global uniqueness). *Let $\Omega \subset \mathbb{R}^n$ be an open set bounded in one direction, $0 < T < \infty$, $0 < s < \min(1, n/2)$, $\gamma_0 > 0$, and let $W \subset \Omega_e$ be an open set. Assume that $\gamma_j \in \Gamma_{s,\gamma_0}(\mathbb{R}^n_T) \cap C^\infty(W_T)$ for $j = 1, 2$.*[1] *Then*

$$\Lambda_{\gamma_1} f|_{W_T} = \Lambda_{\gamma_2} f|_{W_T} \quad \text{for any } f \in C_c^\infty(W_T) \tag{1.7}$$

implies that $\gamma_1 = \gamma_2$ in $\mathbb{R}^n_T$.

In order to prove Theorem 1.1, we first need to establish an *exterior determination* result. This extends the results in [14,42] for elliptic equations and is based on a construction of special solutions to the equation (1.2) whose energies can be concentrated near a fixed point in the spacetime. Furthermore, to our best knowledge, Theorem 1.1 is the first result to recover time-dependent coefficients in the nonlocal setup.

Theorem 1.2 (Exterior determination). *Let $\Omega \subset \mathbb{R}^n$ be an open set bounded in one direction, $0 < T < \infty$, $0 < s < \min(1, n/2)$, $\gamma_0 > 0$, and let $W \subset \Omega_e$ be an open set. Assume that $\gamma_j \in \Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$ for $j = 1, 2$. Then (1.7) implies that $\gamma_1 = \gamma_2$ a.e. in W_T.*

Ideas for the proof. Let us briefly summarize the ideas for the proof of Theorem 1.1. We first prove the exterior uniqueness by using the exterior information from (1.7) such that $\gamma_1 = \gamma_2$ in W_T. Next, by considering arbitrary nonempty disjoint open subsets $W_1, W_2 \subset W \subset \Omega_e$, Theorem 1.2 implies that $\gamma_1 = \gamma_2$ in $(W_1 \cup W_2)_T$. We next introduce the *nonlocal Neumann derivatives*

$$\mathcal{N}_{\gamma_j} u(x,t) = C_{n,s} \int_\Omega \gamma_j^{1/2}(x,t)\, \gamma_j^{1/2}(y,t) \frac{u(x,t) - u(y,t)}{|x-y|^{n+2s}}\, dy, \quad (x,t) \in (\Omega_e)_T,$$

for $j = 1, 2$, where $C_{n,s}$ is the constant given by (1.5). In particular, we can prove that

$$\langle \mathcal{N}_{\gamma_1} f, g \rangle = \langle \mathcal{N}_{\gamma_2} f, g \rangle \quad \text{for any } f \in C_c^\infty((W_1)_T) \text{ and } g \in C_c^\infty((W_2)_T), \tag{1.8}$$

whenever (1.7) holds (see Lemma 6.6).

Meanwhile, we introduce the spacetime Liouville reduction, which transfers the nonlocal diffusion equation (1.2) into a Schrödinger type equation (see (5.3)). By utilizing the identity (1.8) and the Liouville reduction, we can derive a suitable integral identity (see Section 7.1). Now, applying the Runge approximation (Proposition 7.3), we can prove the interior uniqueness $\gamma_1 = \gamma_2$ in Ω_T and $(-\Delta)^s(\gamma_1^{1/2} - \gamma_2^{1/2}) = 0$ in Ω_T. Finally, using the UCP we can conclude the proof. We want to emphasize again that our theorems hold for any spatial dimension $n \in \mathbb{N}$.

1.2. Organization of the article

We first recall preliminaries related to function spaces and nonlocal operators in Section 2. In Section 3, we show the well-posedness of the forward problem (1.2) and define the exterior DN maps. We prove the exterior determination by using (1.7) in Section 4. In Section 5, we introduce the spacetime Liouville reduction, which transfers equation (1.2)

[1] The set $\Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$ is defined by (3.22) in Section 3.

into a Schrödinger type diffusion equation (5.3). We also study the well-posedness of the reduced equation. In Section 6, we introduce nonlocal Neumann derivatives for both equations (1.2) and (5.3). Finally, we prove the global uniqueness in Section 7.3 by deriving suitable integral identities and an approximation property. In Appendix A, we discuss and explain several connections between DN maps and nonlocal Neumann derivatives.

2. Preliminaries

Throughout this article, the space dimension n is a fixed positive integer and $\Omega \subset \mathbb{R}^n$ is an open set. In this section, we introduce fundamental properties of function spaces and operators which will be used in our study.

2.1. Fractional Sobolev spaces

We denote by $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ Schwartz functions and tempered distributions, respectively. We define the Fourier transform $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F} f(\xi) := \int_{\mathbb{R}^n} f(x) e^{-\mathrm{i}x\cdot\xi}\, dx,$$

which is occasionally also denoted by $\hat{f}$, where $\mathrm{i} = \sqrt{-1}$. By duality, it can be extended to the space of tempered distributions and will again be denoted by $\mathcal{F}u = \hat{u}$, where $u \in \mathcal{S}'(\mathbb{R}^n)$, and we denote the inverse Fourier transform by $\mathcal{F}^{-1}$. Next, recall that the fractional Laplacian of order $a \geq 0$ can be defined by a Fourier multiplier as follows:

$$(-\Delta)^{a/2} u = \mathcal{F}^{-1}(|\xi|^a \hat{u}(\xi)) \quad \text{for } u \in \mathcal{S}'(\mathbb{R}^n),$$

whenever the right-hand side is well defined. Given $a \geq 0$, the L^2-based fractional Sobolev space $H^a(\mathbb{R}^n) := W^{a,2}(\mathbb{R}^n)$ is given by

$$\|u\|^2_{H^a(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R}^n)} + \|(-\Delta)^{a/2} u\|^2_{L^2(\mathbb{R}^n)}.$$

In addition, the Parseval identity implies that the seminorm $\|(-\Delta)^{a/2} u\|_{L^2(\mathbb{R}^n)}$ can be expressed as

$$\|(-\Delta)^{a/2} u\|_{L^2(\mathbb{R}^n)} = \langle (-\Delta)^a u, u\rangle^{1/2}_{L^2(\mathbb{R}^n)}.$$

By duality, one extends the spaces $H^a(\mathbb{R}^n)$ to the range $a < 0$. If $\Omega \subset \mathbb{R}^n$ is an open set and $a \in \mathbb{R}$, then the fractional Sobolev spaces are defined by

$$H^a(\Omega) := \{ u|_\Omega : u \in H^a(\mathbb{R}^n)\}$$

and

$$\widetilde{H}^a(\Omega) := \text{closure of } C_c^\infty(\Omega) \text{ in } H^a(\mathbb{R}^n).$$

Meanwhile, $H^a(\Omega)$ is a Banach space with respect to the quotient norm

$$\|u\|_{H^a(\Omega)} := \inf\{ \|U\|_{H^a(\mathbb{R}^n)} : U \in H^a(\mathbb{R}^n) \text{ and } U|_\Omega = u\}.$$

2.2. Bessel potential spaces

Next, we introduce the Bessel potential spaces $H^{s,p}(\mathbb{R}^n)$ and two local variants of them, namely, $\widetilde{H}^{s,p}(\Omega)$ and $H^{s,p}(\Omega)$. The Bessel potential of order $s \in \mathbb{R}$ is the Fourier multiplier $\langle D\rangle^s : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ given by

$$\langle D\rangle^s u := \mathcal{F}^{-1}(\langle\xi\rangle^s \hat{u}),$$

where $\langle\xi\rangle = \sqrt{1+|\xi|^2}$ is the Japanese bracket. Now, for any $1 \leq p < \infty$ and $s \in \mathbb{R}$, the Bessel potential spaces $H^{s,p}(\mathbb{R}^n)$ are defined by

$$H^{s,p}(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n) : \langle D\rangle^s u \in L^p(\mathbb{R}^n)\},$$

equipped with the norm $\|u\|_{H^{s,p}(\mathbb{R}^n)} := \|\langle D\rangle^s u\|_{L^p(\mathbb{R}^n)}$. The local Bessel potential spaces $\widetilde{H}^{s,p}(\Omega)$ are now defined as the closure of $C_c^\infty(\Omega)$ in $H^{s,p}(\mathbb{R}^n)$ and endowed with the norm inherited from $H^{s,p}(\mathbb{R}^n)$. Moreover, we denote by $H^{s,p}(\Omega)$ the space of restrictions from elements in $H^{s,p}(\mathbb{R}^n)$ to Ω and endow it with the related quotient norm

$$\|u\|_{H^{s,p}(\Omega)} := \inf\{\|U\|_{H^{s,p}(\mathbb{R}^n)} : U \in H^{s,p}(\mathbb{R}^n), U|_\Omega = u\}.$$

We have $(\widetilde{H}^{s,p}(\Omega))^* = H^{-s,p'}(\Omega)$ and $\widetilde{H}^{s,p}(\Omega) = (H^{-s,p'}(\Omega))^*$ for every $1 < p < \infty$ and $s \in \mathbb{R}$. As usual, when $p = 2$, we drop the index p in the above notations and see that they are isomorphic to the spaces introduced in Section 2.1.

2.3. Some properties of nonlocal operators

The fractional Laplacian induces a bounded linear map $(-\Delta)^{s/2} : H^{t,p}(\mathbb{R}^n) \to H^{t-s,p}(\mathbb{R}^n)$ for every $1 \leq p < \infty$, $s \geq 0$ and $t \in \mathbb{R}$. Next, we introduce a special class of unbounded open sets which have a fractional Poincaré inequality.

Definition 2.1. (i) We say that an open set $\Omega_\infty \subset \mathbb{R}^n$ of the form $\Omega_\infty = \mathbb{R}^{n-k} \times \omega$, where $n \geq k > 0$ and $\omega \subset \mathbb{R}^k$ is a bounded open set, is a *cylindrical domain*.

(ii) We say that an open set $\Omega \subset \mathbb{R}^n$ is *bounded in one direction* if there exists a cylindrical domain $\Omega_\infty \subset \mathbb{R}^n$ and a rigid Euclidean motion $A(x) = Lx + x_0$, where L is a linear isometry and $x_0 \in \mathbb{R}^n$, such that $\Omega \subset A\Omega_\infty$.

Proposition 2.2 (Poincaré inequality (cf. Theorem 2.2 in [41])). *Let $\Omega \subset \mathbb{R}^n$ be an open set that is bounded in one direction. Suppose that $2 \leq p < \infty$ and $0 \leq s \leq t < \infty$, or $1 < p < 2$, $1 \leq t < \infty$ and $0 \leq s \leq t$. Then there exists $C(n,p,s,t,\Omega) > 0$ such that*

$$\|(-\Delta)^{s/2} u\|_{L^p(\mathbb{R}^n)} \leq C\|(-\Delta)^{t/2} u\|_{L^p(\mathbb{R}^n)} \quad \text{for all } u \in \widetilde{H}^{t,p}(\Omega).$$

For the rest of this article we fix $s \in (0,1)$. The fractional gradient of order s is the bounded linear operator $\nabla^s : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n}; \mathbb{R}^n)$ given by (see [10, 17, 41])

$$\nabla^s u(x,y) := \sqrt{\frac{C_{n,s}}{2}} \frac{u(x)-u(y)}{|x-y|^{n/2+s+1}}(x-y),$$

which satisfies

$$\|\nabla^s u\|_{L^2(\mathbb{R}^{2n})} = \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^n)} \leq \|u\|_{H^s(\mathbb{R}^n)} \tag{2.1}$$

for all $u \in H^s(\mathbb{R}^n)$, where $C_{n,s}$ is the constant given by (1.5). The adjoint of ∇^s is called fractional divergence of order s and denoted by div_s. More concretely, the fractional divergence of order s is the bounded linear operator

$$\mathrm{div}_s : L^2(\mathbb{R}^{2n};\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n)$$

satisfying

$$\langle \mathrm{div}_s\, u, v\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)} = \langle u, \nabla^s v\rangle_{L^2(\mathbb{R}^{2n})}$$

for all $u \in L^2(\mathbb{R}^{2n};\mathbb{R}^n)$, $v \in H^s(\mathbb{R}^n)$. One can show that (see Section 8 of [41])

$$\|\mathrm{div}_s(u)\|_{H^{-s}(\mathbb{R}^n)} \le \|u\|_{L^2(\mathbb{R}^{2n})}$$

for all $u \in L^2(\mathbb{R}^{2n};\mathbb{R}^n)$, and also

$$(-\Delta)^s u = \mathrm{div}_s(\nabla^s u)$$

weakly for all $u \in H^s(\mathbb{R}^n)$ (see Lemma 2.1 in [10]).

2.4. Bochner spaces

Next, we introduce some standard function spaces for time-dependent PDEs adapted to the nonlocal setting considered in this article. Let X be a Banach space and $(a,b) \subset \mathbb{R}$. Then we let $L^p(a,b;X)$ $(1 \le p < \infty)$ stand for the space of measurable functions $u\colon (a,b) \to X$ such that

$$\|u\|_{L^p(a,b;X)} := \Big(\int_a^b \|u(t)\|_X^p\,dt\Big)^{1/p} < \infty, \tag{2.2}$$

and $L^\infty(a,b;X)$ the space of measurable functions $u\colon (a,b) \to X$ such that

$$\|u\|_{L^\infty(a,b;X)} := \inf\{M : \|u(t)\|_X \le M \text{ a.e.}\} < \infty.$$

As usual, we say that $u \in L^p_{\mathrm{loc}}(a,b;X)$ if $\chi_K u \in L^p(a,b;X)$ for any compact set $K \subset (a,b)$, where χ_A denotes the characteristic function of the set A.

Moreover, if $u \in L^1_{\mathrm{loc}}(a,b;X)$ and X is a space of functions over an open set $\Omega \subset \mathbb{R}^n$, as $L^p(\Omega)$, then u is identified with a function $u(x,t)$ and $u(t)$ denotes the function $\Omega \ni x \mapsto u(x,t)$ for almost all t. This is justified from the fact that any $u \in L^q(a,b;L^p(\Omega))$ with $1 \le q, p < \infty$ can be seen as a measurable function $u\colon \Omega\times(a,b) \to \mathbb{R}$ such that the norm $\|u\|_{L^q(a,b;L^p(\Omega))}$, as defined in (2.2), is finite. Clearly, a similar statement holds for the spaces $L^q(a,b;H^{s,p}(\mathbb{R}^n))$ and their local versions. Furthermore, the distributional derivative $du/dt \in \mathcal{D}'((a,b);X)$ is identified with the derivative $\partial_t u \in \mathcal{D}'(\Omega\times(a,b))$ as long as it is well defined. Here $\mathcal{D}'((a,b);X)$ stands for all continuous linear operators from $C_c^\infty((a,b))$ to X. Given two Banach spaces X, Y such that $X \hookrightarrow Y$, we say that $u \in L^2(a,b;X)$ has a (weak) time derivative $u' := du/dt$ in $L^2(a,b;Y)$ if there exists $v \in L^2(a,b;Y)$ such that

$$\langle u', \eta\rangle := -\int_a^b u(t)\eta'(t)\,dt = \int_a^b v(t)\eta(t)\,dt$$

for $\eta \in C_c^\infty((a,b))$ (cf. [15]).

3. The forward problem of nonlocal diffusion equation

In this section, we study the well-posedness of the initial exterior problem (1.2) with possibly nonzero initial condition u_0 and the properties of the related DN maps. We start by setting up the relevant bilinear forms and then define the notion of solutions used throughout this article, which is in parallel to the theory developed for second-order parabolic equations (see, e.g., [32, 33]).

Definition 3.1 (Definition of bilinear forms and conductivity matrix). Let $\Omega \subset \mathbb{R}^n$ be an open set, $0 < s < \min(1, n/2)$ and $\gamma \in L^\infty(\mathbb{R}^n_T)$. Then we define the conductivity matrix associated with γ by

$$\Theta_\gamma : \mathbb{R}^{2n} \times (0,T) \to \mathbb{R}^{n\times n}, \quad \Theta_\gamma(x,y,t) := \gamma^{1/2}(x,t)\,\gamma^{1/2}(y,t)\,\mathbf{1}_{n\times n},$$

for $x, y \in \mathbb{R}^n$, $0 < t < T$, and the following time-dependent bilinear form for the fractional conductivity operator:

$$\begin{aligned} &B_\gamma(t;\cdot,\cdot) : H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{R}, \\ &B_\gamma(t;u,v) := \int_{\mathbb{R}^{2n}} \Theta_\gamma(t)\nabla^s u \cdot \nabla^s v \, dx\, dy. \end{aligned} \tag{3.1}$$

Moreover, if $m_\gamma \in L^\infty(0,T; H^{2s,n/(2s)}(\mathbb{R}^n))$, where

$$m_\gamma := \gamma^{1/2} - 1 \quad \text{in } \mathbb{R}^n_T \tag{3.2}$$

denotes the *background deviation*, then we let $q_\gamma(t)$: $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{R}$ be defined by

$$\langle q_\gamma(t)u, v\rangle := -\left\langle \frac{(-\Delta)^s m_\gamma}{\gamma^{1/2}}\, u, v \right\rangle_{L^2(\mathbb{R}^n)},$$

for $u, v \in H^s(\mathbb{R}^n)$. In this case, we define the time-dependent bilinear form for the related fractional Schrödinger operator with potential q_γ:

$$\begin{aligned} &B_{q_\gamma}(t;\cdot,\cdot) : H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{R}, \\ &B_{q_\gamma}(t;u,v) := \int_{\mathbb{R}^n} (-\Delta)^{s/2} u\, (-\Delta)^{s/2} v\, dx + \int_{\mathbb{R}^n} q_\gamma(t) uv\, dx \end{aligned}$$

for all $u, v \in H^s(\mathbb{R}^n)$.

Remark 3.2. If no confusion arises, we will drop the subscript γ in the definition for the conductivity matrix $\Theta_\gamma(t)$. Moreover, the boundedness and coercivity of these bilinear forms are established in Lemma 3.3.

Lemma 3.3. *Let $0 < s < \min(1, n/2)$ and suppose that $\gamma = \gamma(x,t) \in L^\infty(\mathbb{R}^n_T)$ is uniformly elliptic satisfying (1.3). If for the background deviation m_γ of γ, we have $m_\gamma \in L^\infty(0,T; H^{2s,n/(2s)}(\mathbb{R}^n))$, then there exists $C > 0$ such that*

$$|B_\gamma(t;u,v)| \le \|\gamma\|_{L^\infty(\mathbb{R}^n_T)} \|u\|_{H^s(\mathbb{R}^n)} \|v\|_{H^s(\mathbb{R}^n)} \tag{3.3}$$

and

$$|B_{q_\gamma}(t;u,v)| \le C\|u\|_{H^s(\mathbb{R}^n)} \|v\|_{H^s(\mathbb{R}^n)}.$$

Moreover, if $\Omega \subset \mathbb{R}^n$ is an open set that is bounded in one direction, then the bilinear form $B_\gamma(t;\cdot,\cdot)$ is uniformly coercive over $\widetilde{H}^s(\Omega)$, that is, there exists $c > 0$ such that

$$B_\gamma(t;u,u) \geq c\|u\|^2_{H^s(\mathbb{R}^n)} \tag{3.4}$$

for all $u \in H^s(\mathbb{R}^n)$ and a.e. $0 < t < T$.

Proof. Throughout the proof we will write m and q instead of m_γ and q_γ. Estimate (3.3) follows immediately from (2.1). Next note that, by Lemma 8.3 in [41], the uniform ellipticity of γ and the boundedness of the fractional Laplacian, we have

$$\begin{aligned} |B_q(t;u,v)| &\leq C(1 + \|q(t)\|_{L^{n/(2s)}(\mathbb{R}^n)})\,\|u\|_{H^s(\mathbb{R}^n)}\,\|v\|_{H^s(\mathbb{R}^n)} \\ &\leq C(1 + \|m(t)\|_{H^{2s,n/(2s)}(\mathbb{R}^n)})\,\|u\|_{H^s(\mathbb{R}^n)}\,\|v\|_{H^s(\mathbb{R}^n)} \\ &\leq C\gamma_0^{1/2}(1 + \|m\|_{L^\infty(0,T;H^{2s,n/(2s)}(\mathbb{R}^n))})\|u\|_{H^s(\mathbb{R}^n)}\,\|v\|_{H^s(\mathbb{R}^n)}. \end{aligned}$$

The uniform coercivity estimate (3.4) of $B_\gamma(t;\cdot,\cdot)$ follows by the uniform ellipticity of γ, (2.1) and the Poincaré inequality, cf. Proposition 2.2. ■

Definition 3.4 (Weak solutions). Let $\Omega \subset \mathbb{R}^n$ be an open set, $0 < T < \infty$, $0 < s < 1$ and assume that $\gamma \in L^\infty(\mathbb{R}^n_T)$ is uniformly elliptic. Let $u_0 \in L^2(\Omega)$, $f \in L^2(0,T;H^s(\mathbb{R}^n))$ and $F \in L^2(0,T;H^{-s}(\Omega))$.

(i) We say that $u \in L^2(0,T;H^s(\mathbb{R}^n))$ solves the nonlocal diffusion equation

$$\partial_t u + \mathrm{div}_s(\Theta_\gamma \nabla^s u) = F \quad \text{in } \Omega_T, \tag{3.5}$$

if the equation is satisfied in the sense of distributions, that is,

$$\mathbf{B}_\gamma(u,\varphi) := -\int_{\Omega_T} u\partial_t\varphi\,dx\,dt + \int_0^T B_\gamma(t;u,\varphi)\,dt = \langle F,\varphi\rangle$$

for all $\varphi \in C_c^\infty(\Omega_T)$, where $\langle\cdot,\cdot\rangle$ denotes the natural duality pairing.

(ii) We say that $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^s(\mathbb{R}^n))$ solves

$$\begin{cases} \partial_t u + \mathrm{div}_s(\Theta_\gamma \nabla^s u) = F & \text{in } \Omega_T, \\ u = f & \text{in } (\Omega_e)_T, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{3.6}$$

if the exterior value f is attained in the sense $u - f \in L^2(0,T;\widetilde{H}^s(\Omega))$ and

$$\mathbf{B}_\gamma(u,\varphi) = \langle F,\varphi\rangle + \int_\Omega u_0(x)\varphi(x,0)\,dx$$

for all $\varphi \in C_c^\infty(\Omega \times [0,T))$.

Remark 3.5. Let us briefly explain why we prescribed the initial condition in $L^2(\Omega)$ and not in $L^2(\mathbb{R}^n)$. It is known that $(\widetilde{H}^s(\Omega))^* = H^{-s}(\Omega)$ for any $s \in \mathbb{R}$ and any open set $\Omega \subset \mathbb{R}^n$. On the other hand, by density of $C_c^\infty(\Omega_T)$ in $L^2(0,T;\widetilde{H}^s(\Omega))$, it follows that equation (3.5) implies that $\partial_t u$ can be identified with an element in $L^2(0,T;H^{-s}(\Omega))$. By the trace theorem (see Theorem 1 in Section 1.2, Chapter XVIII, of [15]), this implies $u \in C([0,T];L^2(\Omega))$. Thus, $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^s(\mathbb{R}^n))$ is a solution to (3.6) if and only if $u \in L^2(0,T;H^s(\mathbb{R}^n))$ with $\partial_t u \in L^2(0,T;H^{-s}(\Omega))$ solves (3.5) in the sense of distributions, $u - f \in L^2(0,T;\widetilde{H}^s(\Omega))$ and $u(0) = u_0$ in the sense of traces. By approximation, one sees that $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^s(\mathbb{R}^n))$ is a solution of (3.5) if and only if $u \in L^2(0,T;H^s(\mathbb{R}^n))$, with $\partial_t u \in L^2(0,T;H^{-s}(\Omega))$, satisfies

$$\langle \partial_t u, \varphi\rangle_{H^{-s}(\Omega)\times\widetilde{H}^s(\Omega)} + B_\gamma(t;u,\varphi) = \langle F(t),\varphi\rangle_{H^{-s}(\Omega)\times\widetilde{H}^s(\Omega)}$$

for all $\varphi \in \widetilde{H}^s(\Omega)$ in the sense of distributions on $(0,T)$, $u - f \in L^2(0,T;\widetilde{H}^s(\Omega))$ and $u(0) = u_0$.

Theorem 3.6 (Well-posedness of the forward problem). *Let $\Omega \subset \mathbb{R}^n$ be an open set bounded in one direction, $0 < T < \infty$, $0 < s < \min(1, n/2)$ and assume that $\gamma \in L^\infty(\mathbb{R}^n_T)$ is uniformly elliptic. Assume that $F \in L^2(0,T;H^{-s}(\Omega))$, $f \in L^2(0,T;H^s(\mathbb{R}^n))$ with $\partial_t f \in L^2(0,T;H^{-s}(\mathbb{R}^n))$ and $u_0 \in L^2(\Omega)$.*

(i) *Then there exists a unique solution $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^s(\mathbb{R}^n))$, with $\partial_t u \in L^2(0,T;H^{-s}(\Omega))$, of*

$$\begin{cases} \partial_t u + \mathrm{div}_s(\Theta_\gamma \nabla^s u) = F & \text{in } \Omega_T,\\ u = f & \text{in } (\Omega_e)_T,\\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{3.7}$$

satisfying the energy estimate

$$\begin{aligned} &\|u-f\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u-f\|^2_{L^2(0,T;H^s(\mathbb{R}^n))} \\ &\quad + \|\partial_t(u-f)\|^2_{L^2(0,T;H^{-s}(\Omega))} \\ &\quad \le C\big(\|u_0\|^2_{L^2(\Omega)} + \|f(0)\|^2_{L^2(\Omega)} + \|F\|^2_{L^2(0,T;H^{-s}(\Omega))} \\ &\quad + \|\partial_t f\|^2_{L^2(0,T;H^{-s}(\Omega))} + \|\mathrm{div}_s(\Theta_\gamma \nabla^s f)\|^2_{L^2(0,T;H^{-s}(\Omega))}\big), \end{aligned} \tag{3.8}$$

for some constant $C > 0$ independent of F, f and u_0.

(ii) *If additionally the conductivity γ satisfies $m_\gamma \in L^\infty(0,T;H^{4s,n/(2s)}(\mathbb{R}^n))$ with $\partial_t \gamma \in L^\infty(\mathbb{R}^n_T)$, $F \in L^2(\Omega_T)$, $f \in H^1(0,T;L^2(\mathbb{R}^n)) \cap L^2(0,T;H^{2s}(\mathbb{R}^n))$ and $u_0 \in H^s(\mathbb{R}^n)$ such that $u_0 - f(0) \in \widetilde{H}^s(\Omega)$, then the unique solution u to (3.7) satisfies $u \in L^\infty(0,T;H^s(\mathbb{R}^n))$, $\partial_t u \in L^2(\Omega_T)$ and*

$$\begin{aligned} &\|\partial_t(u-f)\|^2_{L^2(\Omega_T)} + \|u-f\|^2_{L^\infty(0,T;H^s(\mathbb{R}^n))} \\ &\quad \le C\big(\|u_0\|^2_{H^s(\mathbb{R}^n)} + \|F\|^2_{L^2(\Omega_T)} \\ &\quad + \|f(0)\|^2_{H^s(\mathbb{R}^n)} + \|\partial_t f\|^2_{L^2(\Omega_T)} + \|f\|^2_{L^2(0,T;H^{2s}(\mathbb{R}^n))}\big), \end{aligned} \tag{3.9}$$

for some $C > 0$ independent of the data F, f and u_0.

Remark 3.7. If $F = u_0 = 0$, then we denote the unique solution of (3.7) by u_f for simplicity.

Proof of Theorem 3.6. (i) By the regularity assumptions on the exterior value f and the trace theorem (see Theorem 2 in Section 1.2, Chapter XVIII, of [15]), we see that $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^s(\mathbb{R}^n))$ is a solution of (3.7) if and only if $\widetilde{u} := u - f \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^s(\mathbb{R}^n))$ solves the homogeneous problem

$$\begin{cases} \partial_t \widetilde{u} + \operatorname{div}_s(\Theta_\gamma \nabla^s \widetilde{u}) = \widetilde{F} & \text{in } \Omega_T, \\ \widetilde{u} = 0 & \text{in } (\Omega_e)_T, \\ \widetilde{u}(0) = \widetilde{u}_0 & \text{in } \Omega, \end{cases} \tag{3.10}$$

with $\widetilde{u}_0 := u_0 - f(0) \in L^2(\Omega)$ and

$$\widetilde{F} := F - \partial_t f - \operatorname{div}_s(\Theta_\gamma \nabla^s f) \in L^2(0,T;H^{-s}(\Omega)).$$

Note that by Remark 3.5, this means that $\widetilde{u} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;\widetilde{H}^s(\Omega))$ satisfies

$$\langle \partial_t \widetilde{u}, \varphi \rangle_{H^{-s}(\Omega)\times \widetilde{H}^s(\Omega)} + B_\gamma(t;\widetilde{u},\varphi) = \langle \widetilde{F}(t), \varphi \rangle_{H^{-s}(\Omega)\times \widetilde{H}^s(\Omega)}$$

for all $\varphi \in \widetilde{H}^s(\Omega)$ in the sense of distributions on $(0,T)$, $\widetilde{u}(0) = \widetilde{u}_0$. Now one can construct the solution $\widetilde{u}$ by the classical Galerkin approximation. In fact, using Lemma 3.3, we deduce, from Theorems 1 and 2 in Sections 3.1–3.2, Chapter XVIII, of [15], that this problem has a unique solution $\widetilde{u} \in L^2(0,T;\widetilde{H}^s(\Omega))$ with $\partial_t \widetilde{u} \in L^2(0,T;H^{-s}(\Omega))$. Hence, our solution to the original problem is $u = \widetilde{u} + f$.

Next we prove that this solution is the unique solution to (3.7). Assume there are two solutions $u, v \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^s(\mathbb{R}^n))$ to (3.7). Then $w := u - v$ solves

$$\begin{cases} \partial_t w + \operatorname{div}_s(\Theta_\gamma \nabla^s w) = 0 & \text{in } \Omega_T, \\ w = 0 & \text{in } (\Omega_e)_T, \\ w(0) = 0 & \text{in } \Omega. \end{cases}$$

By approximation and integration by parts,[2]

$$\int_0^T \langle \partial_t w, \varphi \rangle \eta \, dt + \int_0^T B_\gamma(w,\varphi)\eta \, dt = 0$$

for all $\varphi \in \widetilde{H}^s(\Omega)$, $\eta \in C_c^\infty((0,T))$, and hence

$$\langle \partial_t w, \varphi \rangle + B_\gamma(w,\varphi) = 0$$

for a.e. $t \in (0,T)$. Hence, replacing φ by $w(t) \in \widetilde{H}^s(\Omega)$ and integrating the resulting equation over $(0,T)$ gives

$$\frac{\|w(T)\|^2_{L^2(\Omega)}}{2} + \int_0^T B_\gamma(w,w)\, dt = 0.$$

[2]Here and at some other instances, we simply write $\langle u, v \rangle$ when the pairing between the functions u and v is clear from the context. For example, if $u \in H^{-s}(\Omega)$ and $v \in \widetilde{H}^s(\Omega)$, then $\langle u, v \rangle$ stands for the duality pairing $\langle u, v \rangle_{H^{-s}(\Omega)\times \widetilde{H}^s(\Omega)}$.

Here we used $w(0) = 0$ in Ω and the integration by parts formula in Banach spaces. Using the uniform ellipticity of γ and Poincaré's inequality it follows that $w = 0$ and therefore $u = v$ in $\mathbb{R}^n_T$.

Next, we show the energy estimate (3.8). By equation (3.70) in [15],

$$\begin{aligned}
&\frac{\|(u-f)(t)\|^2_{L^2(\Omega)}}{2} + \int_0^t \int_{\mathbb{R}^{2n}} \Theta_\gamma \nabla^s (u-f)(\tau) \cdot \nabla^s (u-f)(\tau)\, dx\, dy\, d\tau \\
&\quad = \frac{\|u_0 - f(0)\|^2_{L^2(\Omega)}}{2} + \int_0^t \langle \widetilde{F}(\tau), (u-f)(\tau)\rangle_{H^{-s}(\Omega)\times \widetilde{H}^s(\Omega)}\, d\tau
\end{aligned}$$

for all $t \in (0, T)$. The right-hand side can be estimated as

$$\begin{aligned}
&\frac{\|u_0 - f(0)\|^2_{L^2(\Omega)}}{2} + \int_0^t \langle \widetilde{F}(\tau), (u-f)(\tau)\rangle_{H^{-s}(\Omega)\times \widetilde{H}^s(\Omega)}\, d\tau \\
&\quad \leq C(\|u_0\|^2_{L^2(\Omega)} + \|f(0)\|^2_{L^2(\Omega)}) + \|\widetilde{F}\|_{L^2(0,T;H^{-s}(\Omega))} \|u-f\|_{L^2(0,T;\widetilde{H}^s(\Omega))} \\
&\quad \leq C(\|u_0\|^2_{L^2(\Omega)} + \|f(0)\|^2_{L^2(\Omega)}) + \big(\|F\|_{L^2(0,T;H^{-s}(\Omega))} + \|\partial_t f\|_{L^2(0,T;H^{-s}(\Omega))} \\
&\qquad + \|\mathrm{div}_s(\Theta_\gamma \nabla^s f)\|_{L^2(0,T;H^{-s}(\Omega))}\big) \|u-f\|_{L^2(0,T;H^s(\mathbb{R}^n))}.
\end{aligned} \tag{3.11}$$

On the other hand, using the uniform ellipticity of γ and the fractional Poincaré inequality, the left-hand side of (3.8) can be bounded from below by

$$\begin{aligned}
&\frac{\|(u-f)(t)\|^2_{L^2(\Omega)}}{2} + \int_0^t \int_{\mathbb{R}^{2n}} \Theta_\gamma \nabla^s (u-f)(\tau) \cdot \nabla^s (u-f)(\tau)\, dx\, dy\, d\tau \\
&\quad \geq \frac{\|(u-f)(t)\|^2_{L^2(\Omega)}}{2} + c\|\nabla^s(u-f)\|^2_{L^2(0,t;L^2(\mathbb{R}^{2n}))}, \\
&\quad \geq \frac{\|(u-f)(t)\|^2_{L^2(\Omega)}}{2} + c\|(-\Delta)^{s/2}(u-f)\|^2_{L^2(0,t;L^2(\mathbb{R}^n))} \\
&\quad \geq c\big(\|(u-f)(t)\|^2_{L^2(\Omega)} + \|u-f\|^2_{L^2(0,t;H^s(\mathbb{R}^n))}\big).
\end{aligned} \tag{3.12}$$

Hence, combining (3.11) and (3.12), we deduce

$$\begin{aligned}
&\|u-f\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u-f\|^2_{L^2(0,T;H^s(\mathbb{R}^n))} \\
&\quad \leq C(\|u_0\|^2_{L^2(\Omega)} + \|f(0)\|^2_{L^2(\Omega)}) + C\big(\|F\|_{L^2(0,T;H^{-s}(\Omega))} + \|\partial_t f\|_{L^2(0,T;H^{-s}(\Omega))} \\
&\qquad + \|\mathrm{div}_s(\Theta_\gamma \nabla^s f)\|_{L^2(0,T;H^{-s}(\Omega))}\big) \|u-f\|_{L^2(0,T;H^s(\mathbb{R}^n))}.
\end{aligned}$$

Next, recall that for all $\varepsilon > 0$ and $a, b \in \mathbb{R}$, we have the estimate $ab \leq \varepsilon a^2 + C_\varepsilon b^2$, where $C_\varepsilon > 0$. Hence, after absorbing the term $\varepsilon\|u-f\|^2_{L^2(0,T;H^s(\mathbb{R}^n))}$ on the right-hand side, we obtain

$$\begin{aligned}
&\|u-f\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u-f\|^2_{L^2(0,T;H^s(\mathbb{R}^n))} \\
&\quad \leq C\big(\|u_0\|^2_{L^2(\Omega)} + \|f(0)\|^2_{L^2(\Omega)} + \|F\|^2_{L^2(0,T;H^{-s}(\Omega))} + \|\partial_t f\|^2_{L^2(0,T;H^{-s}(\Omega))} \\
&\qquad + \|\mathrm{div}_s(\Theta_\gamma \nabla^s f)\|^2_{L^2(0,T;H^{-s}(\Omega))}\big).
\end{aligned}$$

Now, by the equation again, one knows that

$$\langle \partial_t(u-f),\varphi\rangle_{H^{-s}(\Omega)\times\widetilde{H}^s(\Omega)} + B_\gamma(t;u-f,\varphi) = \langle \widetilde{F},\varphi\rangle_{H^{-s}(\Omega)\times\widetilde{H}^s(\Omega)},$$

for any $\varphi\in\widetilde{H}^s(\Omega)$ and a.e. $t\in(0,T)$. Consequently,

$$|\langle \partial_t(u-f),\varphi\rangle_{H^{-s}(\Omega)\times\widetilde{H}^s(\Omega)}| \le |B_\gamma(t;u-f,\varphi)| + |\langle \widetilde{F},\varphi\rangle_{H^{-s}(\Omega)\times\widetilde{H}^s(\Omega)}|.$$

Integrating from 0 to T and using $(L^2(0,T;\widetilde{H}^s(\Omega)))^* = L^2(0,T;H^{-s}(\Omega))$, one can conclude that $\partial_t(u-f)\in L^2(0,T;H^{-s}(\Omega))$. Since $\partial_t f\in L^2(0,T;H^{-s}(\Omega))$, we obtain $\partial_t u\in L^2(0,T;H^{-s}(\Omega))$, as desired.

(ii) First we show that $\mathrm{div}_s(\Theta_\gamma\nabla^s f)\in L^2(\Omega_T)$. More concretely, we prove that

$$\left|\int_0^T \langle \mathrm{div}_s(\Theta_\gamma\nabla^s f),\varphi\rangle\,dt\right| \le C\|f\|_{L^2(0,T;H^{2s}(\mathbb{R}^n))}\|\varphi\|_{L^2(\Omega_T)}$$

for all $\varphi\in C_c^\infty(\Omega_T)$ and some $C>0$ independent of φ. This gives already the claim, as then, by density, $\mathrm{div}_s(\Theta_\gamma\nabla^s f)$ can be uniquely extended to an element in $L^2(\Omega_T)$ such that

$$\|\mathrm{div}_s(\Theta_\gamma\nabla^s f)\|_{L^2(\Omega_T)} \le C\|f\|_{L^2(0,T;H^{2s}(\mathbb{R}^n))}. \tag{3.13}$$

Using Remark 8.8 in [41] in every time slice, we obtain

$$\begin{aligned}\left|\int_0^T \langle \mathrm{div}_s(\Theta_\gamma\nabla^s f),\varphi\rangle\,dt\right| &= \left|\int_0^T \langle \Theta_\gamma\nabla^s f,\nabla^s\varphi\rangle\,dt\right| \\ &= \left|\int_0^T \langle(-\Delta)^{s/2}(\gamma^{1/2}f),(-\Delta)^{s/2}(\gamma^{1/2}\varphi)\rangle + \langle q_\gamma(\gamma^{1/2}f),\gamma^{1/2}\varphi\rangle\,dt\right|.\end{aligned} \tag{3.14}$$

Now note that by Corollary A.7 in [41], we have $\gamma^{1/2}\varphi\in H^s(\mathbb{R}^n)$.

On the other hand, choosing $p_1=n/(2s)$, $s_1=4s$, $p_2=2$ and $r_2=2n/(n-2s)$ as in Lemma A.6 of [41], using the Sobolev embedding and the monotonicity of Bessel potential spaces, we deduce that $m_\gamma f\in H^{2s}(\mathbb{R}^n)$, with

$$\begin{aligned}\|m_\gamma f\|_{H^{2s}(\mathbb{R}^n)} &\le C\big(\|m_\gamma\|_{L^\infty(\mathbb{R}^n)}\|f\|_{H^{2s}(\mathbb{R}^n)} \\ &\qquad + \|f\|_{L^{2n/(n-2s)}(\mathbb{R}^n)}\|m_\gamma\|^{1/2}_{H^{4s,n/(2s)}(\mathbb{R}^n)}\|m_\gamma\|^{1/2}_{L^\infty(\mathbb{R}^n)}\big) \\ &\le C\|m_\gamma\|_{L^\infty(\mathbb{R}^n)}(1+\|m_\gamma\|^{1/2}_{H^{4s,n/(2s)}(\mathbb{R}^n)})\|f\|_{H^{2s}(\mathbb{R}^n)}.\end{aligned}$$

This in turn shows $\gamma^{1/2}f\in H^{2s}(\mathbb{R}^n)$ for a.e. $t\in(0,T)$, with

$$\|\gamma^{1/2}f\|_{H^{2s}(\mathbb{R}^n)} \le (1+\|m_\gamma\|_{L^\infty(\mathbb{R}^n)})(1+\|m_\gamma\|^{1/2}_{H^{4s,n/(2s)}(\mathbb{R}^n)})\|f\|_{H^{2s}(\mathbb{R}^n)}. \tag{3.15}$$

Additionally, by the Gagliardo–Nirenberg inequality in Bessel potential spaces and the Sobolev embedding (cf. equation (18) in [13]), we have

$$\|m_\gamma\|_{H^{2s,n/s}(\mathbb{R}^n)} \le C\|m_\gamma\|^{1/2}_{H^{4s,n/(2s)}(\mathbb{R}^n)}\|m_\gamma\|^{1/2}_{L^\infty(\mathbb{R}^n)}$$

and therefore $(-\Delta)^s m_\gamma \in L^{n/s}(\mathbb{R}^n)$. Applying Hölder's inequality with $p_1 = n/s$, $p_2 = 2n/(n-2s)$, $p_3 = 2$, we can estimate

$$\begin{aligned}(3.16)\qquad &\|q_\gamma(\gamma^{1/2} f)\gamma^{1/2}\varphi\|_{L^1(\mathbb{R}^n)} \\ &\leq \|q_\gamma\|_{L^{n/s}(\mathbb{R}^n)}\|\gamma^{1/2} f\|_{L^{2n/(n-2s)}(\mathbb{R}^n)}\|\gamma^{1/2}\varphi\|_{L^2(\mathbb{R}^n)} \\ &\leq C\|m_\gamma\|^{1/2}_{H^{4s,n/(2s)}(\mathbb{R}^n)}\|m_\gamma\|_{L^\infty(\mathbb{R}^n)}\|\gamma^{1/2} f\|_{H^s(\mathbb{R}^n)}\|\varphi\|_{L^2(\mathbb{R}^n)} \\ &\leq C(1+\|m_\gamma\|^{1/2}_{H^{4s,n/(2s)}(\mathbb{R}^n)})\|m_\gamma\|^2_{L^\infty(\mathbb{R}^n)}(1+\|m_\gamma\|^{1/2}_{H^{2s,n/2s}(\mathbb{R}^n)}) \\ &\quad \cdot \|f\|_{H^s(\mathbb{R}^n)}\|\varphi\|_{L^2(\mathbb{R}^n)},\end{aligned}$$

where in the third inequality we again used Corollary A.7 in [41]. Now, using $\gamma^{1/2} f \in H^{2s}(\mathbb{R}^n)$, $\gamma^{1/2}\varphi \in H^s(\mathbb{R}^n)$ and estimates (3.15)–(3.16), we obtain, by Hölder's inequality and (3.14), the bound

$$\begin{aligned}\left|\int_0^T \langle \mathrm{div}_s(\Theta_\gamma \nabla^s f), \varphi\rangle\, dt\right| &= \left|\int_0^T \langle (-\Delta)^s(\gamma^{1/2} f), \gamma^{1/2}\varphi\rangle + \langle q_\gamma(\gamma^{1/2} f), \gamma^{1/2}\varphi\rangle\, dt\right| \\ &\leq C(1+\|\gamma\|_{L^\infty(\mathbb{R}^n_T)})(1+\|m_\gamma\|_{L^\infty(0,T;H^{4s,n/(2s)}(\mathbb{R}^n))}) \\ &\quad \cdot \|f\|_{L^2(0,T;H^{2s}(\mathbb{R}^n))}\|\varphi\|_{L^2(\Omega_T)}.\end{aligned}$$

On the other hand, by definition, $u_0 - f(0) \in \widetilde{H}^s(\Omega)$. Hence, if we set as above $\widetilde{u} = u - f$, then we see that it solves (3.10) with $\widetilde{u}_0 := u_0 - f(0) \in \widetilde{H}^s(\Omega)$ and

$$\widetilde{F} := F - \partial_t f - \mathrm{div}_s(\Theta_\gamma \nabla^s f) \in L^2(\Omega_T). \tag{3.17}$$

Now we proceed similarly to Theorem 5 in Chapter 7 of [18]. For this purpose, let us recall how the unique solution $\widetilde{u}$ in Theorems 1 and 2 from Sections 3.1–3.2, Chapter XVIII, of [15] is constructed. Since $\widetilde{H}^s(\Omega)$ is a separable Hilbert space, the finite-dimensional subspaces

$$\widetilde{H}^s_m := \mathrm{span}\{w_1, \dots, w_m\}$$

for $m \in \mathbb{N}$, where $(w_k)_{k\in\mathbb{N}} \subset \widetilde{H}^s(\Omega)$ is an orthonormal basis of $\widetilde{H}^s(\Omega)$, form a Galerkin approximation for $\widetilde{H}^s(\Omega)$ (see Section 2 in Chapter XVIII of [15] for the definition of a Galerkin approximation). Let us note that in our generality we cannot take $(w_k)_{k\in\mathbb{N}}$ to be the eigenfunctions of the fractional Laplacian $(-\Delta)^s$, as Ω is only bounded in one direction and so $\widetilde{H}^s(\Omega) \hookrightarrow L^2(\Omega)$ may fail to be compact. Observe, by density of $\widetilde{H}^s(\Omega)$ in $L^2(\Omega)$, the family $(\widetilde{H}^s_m)_{m\in\mathbb{N}}$ is also a Galerkin approximation for $L^2(\Omega)$. By Lemma 1 in Section 3.1, Chapter XVIII, of [15], there are unique solutions $\widetilde{u}_m \in C([0,T]; \widetilde{H}^s_m)$ with $\partial_t \widetilde{u}_m \in L^2(0,T; \widetilde{H}^s_m)$ and

$$\langle \partial_t \widetilde{u}_m, w_j\rangle + B_\gamma(t; \widetilde{u}_m, w_j) = \langle \widetilde{F}, w_j\rangle \tag{3.18}$$

for all $1 \leq j \leq m$ and a.e. $t \in (0,T)$, where $\widetilde{u}^m_0 \in \widetilde{H}^s_m$ are chosen in such a way that $\widetilde{u}^m_0 \to \widetilde{u}_0$ in $L^2(\Omega)$.

In fact, the solutions $\widetilde{u}_m$ can be written in the form

$$\widetilde{u}_m = \sum_{j=1}^m c^j_m w_j.$$

Here $c_m = (c_m^1, \dots, c_m^m)$ are absolutely continuous functions that solve

$$A_m \partial_t c_m + B_m(t) c_m = \widetilde{F}_m(t), \quad c_m(0) = \widetilde{u}_0^m,$$

where $A_m := (\langle w_i, w_j \rangle)_{1\le i,j\le m}$, $B_m(t) := (B_\gamma(t; w_i, w_j))_{1\le i,j\le m}$ and, finally, $\widetilde{F}_m(t) = (\langle \widetilde{F}(t), w_j \rangle)_{1\le j\le m}$. We have $\|\widetilde{u}_0^m\|_{L^2(\Omega)} \le c\|\widetilde{u}_0\|_{L^2(\Omega)}$ for some constant independent of m. Next observe that if $\widetilde{u}_0 \in \widetilde{H}^s(\Omega)$, as in our case, then we can take

$$\widetilde{u}_0^m = \sum_{j=1}^m \langle \widetilde{u}_0, w_j \rangle_{\widetilde{H}^s(\Omega)} w_j \in \widetilde{H}_m^s$$

and see that $\widetilde{u}_0^m \to \widetilde{u}_0$ in $H^s(\mathbb{R}^n)$ as $m \to \infty$. Moreover, this convergence implies

$$\|\widetilde{u}_0^m\|_{H^s(\mathbb{R}^n)} \le c\|\widetilde{u}_0\|_{H^s(\mathbb{R}^n)} \tag{3.19}$$

for some $c > 0$ independent of m. Now fix $m \in \mathbb{N}$, multiply (3.18) by $\partial_t c_m^j$ and sum j over $\{1, \dots, m\}$ to obtain

$$\langle \partial_t \widetilde{u}_m, \partial_t \widetilde{u}_m \rangle_{L^2(\Omega)} + B_\gamma(t; \widetilde{u}_m, \partial_t \widetilde{u}_m) = \langle \widetilde{F}, \partial_t \widetilde{u}_m \rangle_{L^2(\Omega)},$$

where we used (3.17) and $\partial_t \widetilde{u}_m \in \widetilde{H}_m^s \subset L^2(\Omega)$.

Observe that

$$\begin{aligned}\partial_t B_\gamma(t; \widetilde{u}_m, \widetilde{u}_m) = 2B_\gamma(t; \widetilde{u}_m, \partial_t \widetilde{u}_m) + \int_{\mathbb{R}^{2n}} \big[\big(\partial_t \gamma^{1/2}(x,t)\, \gamma^{1/2}(y,t) \\ + \gamma^{1/2}(x,t)\, \partial_t \gamma^{1/2}(y,t)\big) \nabla^s \widetilde{u}_m \cdot \nabla^s \widetilde{u}_m\big] dx\, dy.\end{aligned}$$

Hence, using the uniform ellipticity of γ, the Cauchy–Schwarz inequality and Young's inequality,

$$\begin{aligned}&\|\partial_t \widetilde{u}_m\|^2_{L^2(\Omega)} + \partial_t B_\gamma(t; \widetilde{u}_m, \widetilde{u}_m) \\ &\quad \le C\big(\|\partial_t \gamma\|_{L^\infty(\mathbb{R}^n_T)} \|\gamma\|^{1/2}_{L^\infty(\mathbb{R}^n_T)} \|\widetilde{u}_m\|^2_{H^s(\mathbb{R}^n)} + \varepsilon^{-1}\|\widetilde{F}\|^2_{L^2(\Omega)}\big) + \varepsilon\|\partial_t \widetilde{u}_m\|^2_{L^2(\Omega)}\end{aligned}$$

for some $C > 0$ only depending on the ellipticity constant γ_0 and all $\varepsilon > 0$. Taking $\varepsilon = 1/2$, we can absorb the last term on the right-hand side and after integrating over $(0,t) \subset (0,T)$, we obtain

$$\begin{aligned}&\|\partial_t \widetilde{u}_m\|^2_{L^2(\Omega_t)} + B_\gamma(t; \widetilde{u}_m, \widetilde{u}_m) \\ &\quad \le B_\gamma(0; \widetilde{u}_0^m, \widetilde{u}_0^m) + C(\|\partial_t \gamma\|_{L^\infty(\mathbb{R}^n_T)} \|\gamma\|^{1/2}_{L^\infty(\mathbb{R}^n_T)} \|\widetilde{u}_m\|^2_{L^2(0,T;H^s(\mathbb{R}^n))} + \|\widetilde{F}\|^2_{L^2(\Omega_T)}).\end{aligned}$$

Taking the supremum over $(0,T)$, using the uniform ellipticity of γ and (3.19), we get

$$\begin{aligned}&\|\partial_t \widetilde{u}_m\|^2_{L^2(\Omega_T)} + \|\widetilde{u}_m\|^2_{L^\infty(0,T;H^s(\mathbb{R}^n))} \\ &\quad \le C\big(\|\gamma\|_{L^\infty(\mathbb{R}^n_T)} \|\widetilde{u}_0\|^2_{H^s(\mathbb{R}^n)} \\ &\qquad + \|\partial_t \gamma\|_{L^\infty(\mathbb{R}^n_T)} \|\gamma\|^{1/2}_{L^\infty(\mathbb{R}^n_T)} \|\widetilde{u}_m\|^2_{L^2(0,T;H^s(\mathbb{R}^n))} + \|\widetilde{F}\|^2_{L^2(\Omega_T)}\big).\end{aligned}$$

Now the term $\|\widetilde{u}_m\|_{L^2(0,T;H^s(\mathbb{R}^n))}$ can be bounded from above using the energy estimate

$$\|\widetilde{u}_m\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\widetilde{u}_m\|^2_{L^2(0,T;H^s(\mathbb{R}^n))} \leq C(\|\widetilde{u}_0\|^2_{L^2(\Omega)} + \|\widetilde{F}\|^2_{L^2(\Omega_T)})$$

(cf. equation (3.40) in Section 3.2, Chapter XVIII, of [15]) for some $C > 0$ only depending on γ_0. This then gives

$$\|\partial_t \widetilde{u}_m\|^2_{L^2(\Omega_T)} + \|\widetilde{u}_m\|^2_{L^\infty(0,T;H^s(\mathbb{R}^n))} \leq C(\|\widetilde{u}_0\|^2_{H^s(\mathbb{R}^n)} + \|\widetilde{F}\|^2_{L^2(\Omega_T)}) \tag{3.20}$$

for some $C > 0$ only depending on γ.

By Lemma 3 in Section 3.3, Chapter XVIII, of [15], we know that, up to subsequences,

(i) $\widetilde{u}_m \rightharpoonup \widetilde{u}$ in $L^2(0,T;H^s(\mathbb{R}^n))$ and

(ii) $\widetilde{u}_m \overset{*}{\rightharpoonup} \widetilde{u}$ in $L^\infty(0,T;L^2(\Omega))$

as $m \to \infty$. Using (i), (ii), (3.20) and the usual compactness arguments, we deduce that $\partial_t \widetilde{u} \in L^2(\Omega_T)$, $\widetilde{u} \in L^\infty(0,T;H^s(\mathbb{R}^n))$ and that

$$\begin{aligned}\|\partial_t \widetilde{u}\|^2_{L^2(\Omega_T)} &+ \|\widetilde{u}\|^2_{L^\infty(0,T;H^s(\mathbb{R}^n))} \\ &\leq C(\|\widetilde{u}_0\|^2_{H^s(\mathbb{R}^n)} + \|\widetilde{F}\|^2_{L^2(\Omega_T)}) \\ &\leq C(\|u_0\|^2_{H^s(\mathbb{R}^n)} + \|f(0)\|^2_{H^s(\mathbb{R}^n)} + \|F\|^2_{L^2(\Omega_T)} \\ &\quad + \|\partial_t f\|^2_{L^2(\Omega_T)} + \|f\|^2_{L^2(0,T;H^{2s}(\mathbb{R}^n))}),\end{aligned} \tag{3.21}$$

where $C > 0$ only depends on γ. More concretely, we have used here that if $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ is a Hilbert space, $(U_k)_{k\in\mathbb{N}} \subset L^2(0,T;\mathcal{H})$ satisfies $\|U_k\|_{L^\infty(0,T;\mathcal{H})} \leq C$ uniformly in $k \in \mathbb{N}$ and $U_k \rightharpoonup U$ in $L^2(0,T;\mathcal{H})$ as $k \to \infty$ for some $U \in L^2(0,T;\mathcal{H})$, then one has $U \in L^\infty(0,T;\mathcal{H})$ with $\|U\|_{L^\infty(0,T;\mathcal{H})} \leq C$. To see this, let us fix some $V \in \mathcal{H}$ and note that Lebesgue's differentiation theorem together with our assumptions ensure the estimate

$$\begin{aligned}|\langle U(t), V\rangle_{\mathcal{H}}| &= \lim_{h\to 0}\left| \fint_t^{t+h} \langle U(\tau), V\rangle_{\mathcal{H}}\, d\tau\right| \\ &= \lim_{h\to 0}\lim_{k\to\infty}\left| \fint_t^{t+h} \langle U_k(\tau), V\rangle_{\mathcal{H}}\, d\tau\right| \leq C\|V\|_{\mathcal{H}}\end{aligned}$$

for a.e. $0 < t < T$. Taking $V = U(t) \in \mathcal{H}$, this guarantees that $U \in L^\infty(0,T;\mathcal{H})$ with $\|U\|_{L^\infty(0,T;\mathcal{H})} \leq C$, as we wanted to see. Additionally, in the second inequality of the estimate (3.21), we used the definition of $\widetilde{u}_0$, $\widetilde{F}$ and (3.13). This establishes estimate (3.9) and we can conclude the proof. ■

Because of this well-posedness result, we make the following definition.

Definition 3.8. Let $\Omega \subset \mathbb{R}^n$ be an open set, $0 < T < \infty$, $0 < s < \min(1, n/2)$ and $\gamma_0 > 0$. Then we define the data spaces $X_s(\Omega_T)$, $\widetilde{X}_s(\Omega_T)$ and the class of admissible conductivities $\Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$ by

$$\begin{aligned}X_s(\Omega_T) &:= \big\{(f,u_0) \in L^2(0,T;H^{2s}(\mathbb{R}^n)) \times H^s(\mathbb{R}^n) : \\ &\qquad \partial_t f \in L^2(0,T;L^2(\mathbb{R}^n)),\ u_0 - f(0) \in \widetilde{H}^s(\Omega)\big\}, \\ \widetilde{X}_s(\Omega_T) &:= \{f \in L^2(0,T;H^{2s}(\mathbb{R}^n)) : \partial_t f \in L^2(0,T;L^2(\mathbb{R}^n)),\ f(0) = 0\},\end{aligned}$$

and

$$\Gamma_{s,\gamma_0}(\mathbb{R}^n_T) := \{\gamma \in C^1_t C_x(\mathbb{R}^n_T) : \gamma \text{ satisfies (1.3)}, \partial_t\gamma \in L^\infty(\mathbb{R}^n_T) \text{ and } m_\gamma \in C([0,T]; H^{4s+\varepsilon, n/(2s)}(\mathbb{R}^n)) \text{ for some } \varepsilon > 0\}. \tag{3.22}$$

Here, the space $C^k_t C^\ell_x(\mathbb{R}^n_T)$, $k, \ell \in \mathbb{N}_0$, consists of all functions which are k-times continuously differentiable in the time variable t and ℓ-times in the space variable x.

With this notation at hand, the above theorem can be rewritten as follows.

Corollary 3.9. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $0 < T < \infty$, $0 < s < \min(1, n/2)$ and $\gamma_0 > 0$. Then for all $(f, u_0) \in X_s(\Omega_T)$ and $\gamma \in \Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$, there is a unique solution $u_{f,u_0} \in L^\infty(0,T; H^s(\mathbb{R}^n))$, with $\partial_t u_{f,u_0} \in L^2(\Omega_T)$ satisfying*

$$\begin{aligned} &\|\partial_t(u_{f,u_0} - f)\|^2_{L^2(\Omega_T)} + \|u_{f,u_0} - f\|^2_{L^\infty(0,T;H^s(\mathbb{R}^n))} \\ &\quad \leq C(\|u_0\|^2_{H^s(\mathbb{R}^n)} + \|f(0)\|^2_{H^s(\mathbb{R}^n)} + \|\partial_t f\|^2_{L^2(\Omega_T)} + \|f\|^2_{L^2(0,T;H^{2s}(\mathbb{R}^n))}), \end{aligned}$$

for some constant $C > 0$ independent of f, u_0 and u_{f,u_0}.

Proof. This is an immediate consequence of Theorem 3.6 by taking $F = 0$. ∎

With the well-posedness at hand, we can define the DN map (1.4), which was introduced in Section 1, rigorously. Similarly, as in the nonlocal elliptic case (see [16] or Appendix A), we have the following definition.

Definition 3.10 (The DN map). Let $\Omega \subset \mathbb{R}^n$ be an open set bounded in one direction, $0 < T < \infty$, $0 < s < \min(1, n/2)$, $\gamma_0 > 0$ and $\gamma \in \Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$. Then we define the DN map Λ_γ by

$$\begin{aligned} \langle \Lambda_\gamma f, g\rangle &:= \int_0^T B_\gamma(u_f, g)\,dt \\ &= \frac{C_{n,s}}{2}\int_0^T \int_{\mathbb{R}^{2n}} \gamma^{1/2}(x,t)\gamma^{1/2}(y,t) \\ &\qquad \cdot \frac{(u_f(x,t) - u_f(y,t))(g(x,t) - g(y,t))}{|x-y|^{n+2s}}\,dx\,dy\,dt \end{aligned} \tag{3.23}$$

for all $f, g \in C^\infty_c((\Omega_e)_T)$, where u_f is the unique solution of (1.2).

4. Exterior determination

The main goal of this section is to prove Theorem 1.2. We first establish an energy estimate that allows us to deduce that the Dirichlet energies of suitable special solutions concentrate in the exterior.

Lemma 4.1. *Suppose that $W \subset \Omega_e$ is an open nonempty set with finite measure and $\operatorname{dist}(W, \Omega) > 0$. Let u_f be the unique solution to (3.7) with $f \in C^\infty_c(W_T)$, $F \equiv 0$ and $u_0 \equiv 0$. Then*

$$\|u_f - f\|_{L^\infty(0,T;L^2(\Omega))} + \|u_f - f\|_{L^2(0,T;H^s(\mathbb{R}^n))} \leq C\|f\|_{L^2(W_T)},$$

where the constant $C > 0$ does not depend on $f \in C^\infty_c(W_T)$.

Proof. By applying the energy estimate (3.8) in Theorem 3.6, we obtain

$$\begin{aligned}&\|u_f - f\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_f - f\|^2_{L^2(0,T;H^s(\mathbb{R}^n))}\\ &\quad\leq C(\|\partial_t f\|^2_{L^2(0,T;H^{-s}(\Omega))} + \|\mathrm{div}_s(\Theta_\gamma \nabla^s f)\|^2_{L^2(0,T;H^{-s}(\Omega))}).\end{aligned}$$

Since, f is compactly supported in $W_T \subset (\Omega_e)_T$, the first contribution in the above estimate is zero. By the proof of Lemma 3.1 in [42],

$$\|\mathrm{div}_s(\Theta_\gamma \nabla^s f)(t)\|_{H^{-s}(\Omega)} \leq C\|f(t)\|_{L^2(W)}$$

for a.e. $t \in (0,T)$ and some $C > 0$ only depending on n, s, W and $\|\gamma\|_{L^\infty(\mathbb{R}^n_T)}$. Hence,

$$\|u_f - f\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_f - f\|^2_{L^2(0,T;H^s(\mathbb{R}^n))} \leq C\|f\|^2_{L^2(W_T)},$$

and we can conclude the proof. ■

Proof of Theorem 1.2. First, let γ denote either of the two diffusion coefficients γ_1 or γ_2. Using the Sobolev embedding, we may assume that $\gamma \in C_b(W_T)$. Now, by Lemma 5.5 in [14], for any $x_0 \in W$, there exists $(\phi_N)_{N\in\mathbb{N}} \subset C_c^\infty(W)$ such that $\|\phi_N\|_{H^s(\mathbb{R}^n)} = 1$, $\|\phi_N\|_{L^2(\mathbb{R}^n)} \to 0$ as $N \to \infty$ and $\mathrm{supp}(\phi_N) \to \{x_0\}$. Moreover, Proposition 1.5 in [14] implies that $B_{\gamma(\cdot,t_0)}(\phi_N, \phi_N) \to \gamma(x_0,t_0)$ as $N \to \infty$ for any $t_0 \in (0,T)$. Next let $\eta \in C_c^\infty((0,T))$ and define $\Phi_N := \eta\phi_N \in C_c^\infty(W_T)$. It follows that

$$\int_0^T B_\gamma(\Phi_N,\Phi_N)\,dt = \int_0^T \eta^2(t) B_\gamma(\phi_N,\phi_N)\,dt.$$

By the dominated convergence theorem, we obtain

$$\lim_{N\to\infty}\int_0^T B_\gamma(\Phi_N,\Phi_N)\,dt = \int_0^T \eta^2(t)\gamma(x_0,t)\,dt. \tag{4.1}$$

Let us now consider the solutions u_N to the equation

$$\begin{cases}\partial_t u + \mathrm{div}_s(\Theta_\gamma \nabla^s u) = 0 & \text{in } \Omega_T,\\ u = \Phi_N & \text{in } (\Omega_e)_T,\\ u(x,0) = 0 & \text{in } \Omega,\end{cases}$$

for $N \in \mathbb{N}$. By the definition of the DN map (3.23), we have

$$\begin{aligned}\langle \Lambda_\gamma \Phi_N, \Phi_N\rangle &= \int_0^T B_\gamma(u_N,\Phi_N)\,dt\\ &= \int_0^T B_\gamma(u_N - \Phi_N,\Phi_N)\,dt + \int_0^T B_\gamma(\Phi_N,\Phi_N)\,dt.\end{aligned} \tag{4.2}$$

Next, note that Lemma 4.1 implies

$$\begin{aligned}\left|\int_0^T B_\gamma(u_N - \Phi_N,\Phi_N)\,dt\right| &\leq C\int_0^T \|(u_N - \Phi_N)(\cdot,t)\|_{H^s(\mathbb{R}^n)}\|\Phi_N(\cdot,t)\|_{H^s(\mathbb{R}^n)}\,dt\\ &\leq C\left(\int_0^T \|(u_N - \Phi_N)(\cdot,t)\|^2_{H^s(\mathbb{R}^n)}\,dt\right)^{1/2}\\ &\leq C\|\Phi_N\|_{L^2(0,T;L^2(\mathbb{R}^n))} = C\|\eta\|_{L^2((0,T))}\|\phi_N\|_{L^2(W)},\end{aligned}$$

and hence

$$\lim_{N\to\infty}\int_0^T B_\gamma(u_N-\Phi_N,\Phi_N)\,dt=0. \tag{4.3}$$

We obtain from (4.1)–(4.3) that

$$\lim_{N\to\infty}\langle\Lambda_\gamma\Phi_N,\Phi_N\rangle=\int_0^T\eta^2(t)\gamma(x_0,t)\,dt. \tag{4.4}$$

Hence, applying the identity (4.4) to $\gamma=\gamma_1$ and $\gamma=\gamma_2$, and subtracting them, with (1.7) at hand, we deduce

$$\int_0^T(\gamma_1(x_0,t)-\gamma_2(x_0,t))\eta\,dt=0$$

for all $\eta\in C_c^\infty((0,T))$ with $\eta\geq 0$. This implies $\gamma_1(x_0,t)\geq\gamma_2(x_0,t)$ a.e. Interchanging the role of γ_1 and γ_2, we also obtain the reversed inequality and deduce by continuity that $\gamma_1(x_0,t)=\gamma_2(x_0,t)$ for all $t\in(0,T)$. Since this construction can be done for any $x_0\in W$, we have $\gamma_1=\gamma_2$ in W_T. ∎

Remark 4.2. Note that we also obtain a Lipschitz stability estimate for the exterior determination problem with partial data as in the elliptic case [14]. Moreover, one can easily observe that in contrast to Λ_γ the new DN map $\mathcal{N}_\gamma$, which is defined in Section 6, satisfies $\langle\mathcal{N}_\gamma\Phi_N,\Phi_N\rangle\to 0$ as $N\to\infty$.

5. The spacetime Liouville reduction

In this section, we derive the spacetime Liouville reduction.

Lemma 5.1 (Auxiliary lemma). *Let $\Omega\subset\mathbb{R}^n$ be an open set bounded in one direction, $0<T<\infty$, $0<s<\min(1,n/2)$, and $V\subset\Omega_e$ a nonempty open set. Assume that $\gamma\in L^\infty(\mathbb{R}^n_T)$ with background deviation $m_\gamma\in L^\infty(0,T;H^{2s,n/(2s)}(\mathbb{R}^n))$ satisfies $\gamma\geq\gamma_0>0$ for some positive constant γ_0. Then the following assertions hold*:

(i) *For any $\psi\in L^2(0,T;\widetilde{H}^s(V))$, we have $\gamma^{1/2}\psi,\gamma^{-1/2}\psi\in L^2(0,T;\widetilde{H}^s(V))$ and*

$$\|\gamma^{1/2}\psi\|_{L^2(0,T;H^s(\mathbb{R}^n))}\lesssim(1+\|m_\gamma\|_{L^\infty(\mathbb{R}^n_T)}+\|m_\gamma\|_{L^\infty(0,T;H^{2s,n/(2s)}(\mathbb{R}^n))})\cdot\|\psi\|_{L^2(0,T;H^s(\mathbb{R}^n))}$$

and

$$\|\gamma^{-1/2}\psi\|_{L^2(0,T;H^s(\mathbb{R}^n))}\lesssim\big(1+\|m_\gamma\|_{L^\infty(\mathbb{R}^n_T)}+\|m_\gamma\|_{L^\infty(0,T;H^{2s,n/(2s)}(\mathbb{R}^n))}+\|m_\gamma\|^{2s}_{L^\infty(0,T;H^{2s,n/(2s)}(\mathbb{R}^n))}\big)\|\psi\|_{L^2(0,T;H^s(\mathbb{R}^n))}.$$

(ii) *Let $u,\varphi\in L^2(0,T;H^s(\mathbb{R}^n))$. Then*

$$\int_{t_1}^{t_2}\langle\Theta_\gamma\nabla^s u,\nabla^s\varphi\rangle_{L^2(\mathbb{R}^{2n})}\,dt=\int_{t_1}^{t_2}\langle(-\Delta)^{s/2}(\gamma^{1/2}u),(-\Delta)^{s/2}(\gamma^{1/2}\varphi)\rangle_{L^2(\mathbb{R}^n)}\,dt+\int_{t_1}^{t_2}\langle q_\gamma\gamma^{1/2}u,\gamma^{1/2}\varphi\rangle_{L^2(\mathbb{R}^n)}\,dt$$

for all $0\leq t_1<t_2\leq T$.

Proof. (i) First we show that $\gamma^{1/2}\psi \in L^2(0,T;\widetilde{H}^s(V))$ for any $\psi \in L^2(0,T;\widetilde{H}^s(V))$. Decomposing $\gamma^{1/2}\psi$ as $m_\gamma\psi + \psi$, we deduce from Corollary A.7 in [41] that

$$\begin{aligned}(5.1)\quad &\|\gamma^{1/2}\psi\|^2_{L^2(0,T;H^s(\mathbb{R}^n))}\\ &\leq C(\|m_\gamma\psi\|^2_{L^2(0,T;H^s(\mathbb{R}^n))} + \|\psi\|^2_{L^2(0,T;H^s(\mathbb{R}^n))})\\ &\leq C\int_0^T (\|m_\gamma\|^2_{L^\infty(\mathbb{R}^n)} + \|m_\gamma\|_{H^{2s,n/2s}(\mathbb{R}^n)}\|m_\gamma\|_{L^\infty(\mathbb{R}^n)})\|\psi\|^2_{H^s(\mathbb{R}^n)}\,dt\\ &\quad + C\|\psi\|^2_{L^2(0,T;H^s(\mathbb{R}^n))}\\ &\leq C(1 + \|m_\gamma\|^2_{L^\infty(\mathbb{R}^n_T)} + \|m_\gamma\|_{L^\infty(0,T;H^{2s,n/(2s)}(\mathbb{R}^n))})\|\psi\|^2_{L^2(0,T;H^s(\mathbb{R}^n))}.\end{aligned}$$

Hence, we have $\gamma^{1/2}\psi \in L^2(0,T;H^s(\mathbb{R}^n))$.

Next recall that if $T > 0$ and if X is a Banach space with dense subset X_0, then $C_c^\infty((0,T)) \otimes X_0$ is dense in $L^2(0,T;X)$. Let $(\rho_\varepsilon)_{\varepsilon>0} \subset C_c^\infty(\mathbb{R}^n)$ be a standard mollifier and choose a sequence $(\psi_k)_{k\in\mathbb{N}} \subset C_c^\infty((0,T)) \otimes C_c^\infty(V)$ such that $\psi_k \to \psi$ in $L^2(0,T;\widetilde{H}^s(V))$. The sequence $(\gamma^{1/2} * \rho_{\varepsilon_k})\psi_k$, $k \in \mathbb{N}$, belongs to $L^2(0,T;\widetilde{H}^s(V))$, where $\varepsilon_k \to 0$ as $k \to \infty$. Hence, if we can show that $(\gamma_i^{1/2} * \rho_{\varepsilon_k})\psi_k \to \gamma_i^{1/2}\psi$ in $L^2(0,T;H^s(\mathbb{R}^n))$ as $k \to \infty$, then it follows that $\gamma^{1/2}\psi \in L^2(0,T;\widetilde{H}^s(V))$. We can estimate

$$\begin{aligned}(5.2)\qquad &\|\gamma^{1/2}\psi - (\gamma^{1/2} * \rho_{\varepsilon_k})\psi_k\|_{L^2(0,T;H^s(\mathbb{R}^n))}\\ &\leq \|m_\gamma\psi - m_\gamma^k\psi_k\|_{L^2(0,T;H^s(\mathbb{R}^n))} + \|\psi - \psi_k\|_{L^2(0,T;H^s(\mathbb{R}^n))}\\ &\leq \|(m_\gamma - m_\gamma^k)\psi\|_{L^2(0,T;H^s(\mathbb{R}^n))} + \|m_\gamma^k(\psi - \psi_k)\|_{L^2(0,T;H^s(\mathbb{R}^n))}\\ &\quad + \|\psi - \psi_k\|_{L^2(0,T;H^s(\mathbb{R}^n))},\end{aligned}$$

where we have set $m_\gamma^k = m_{\gamma_i} * \rho_{\varepsilon_k}$.

Now, for the second term in the right-hand side of (5.2), we can apply estimate (5.1), but all the terms involving m_γ^k are uniformly bounded for $k \in \mathbb{N}$ by using the Young's inequality and the fact that Bessel potentials commute with convolution. Hence, the second and third terms go to zero as $k \to \infty$. For the first term in the right-hand side of (5.2), we observe that, by Corollary A.7 in [41], $(m_\gamma - m_\gamma^k)\psi \to 0$ in $H^s(\mathbb{R}^n)$ as $k \to \infty$ for a.e. $t \in (0,T)$ and

$$\|(m_\gamma - m_\gamma^k)\psi\|_{H^s(\mathbb{R}^n)} \leq C(\|m_\gamma\|_{L^\infty(\mathbb{R}^n)} + \|m_\gamma\|^{1/2}_{H^{2s,n/(2s)}(\mathbb{R}^n)}\|m_\gamma\|^{1/2}_{L^\infty(\mathbb{R}^n)})\|\psi\|_{H^s(\mathbb{R}^n)},$$

for a.e. $t \in (0,T)$.

With the above estimate at hand, let us use Young's inequality again and the fact that the Bessel potentials commute with convolution. Since $m_\gamma \in L^\infty(0,T;H^{2s,n/(2s)}(\mathbb{R}^n))$, the term in brackets is uniformly bounded in t, and thus Lebesgue's dominated convergence theorem implies that $(m_\gamma - m_\gamma^k)\psi \to 0$ in $L^2(0,T;H^s(\mathbb{R}^n))$ as $k \to \infty$. Therefore, the assertion follows.

Similarly, one can prove $\gamma^{-1/2}\psi \in L^2(0,T;\widetilde{H}^s(V))$ for any $\psi \in L^2(0,T;\widetilde{H}^s(V))$. Indeed, it essentially follows from the decomposition $\gamma^{-1/2} = 1 - m_\gamma/(m_\gamma + 1)$ and the fact that the second term has the same regularity properties as m_γ. More concretely, from the proof of Theorem 8.6 in [41] and [1], p. 156, it follows that

$$\Big\|\frac{m_\gamma}{m_\gamma+1}\Big\|_{L^\infty(0,T;H^{2s,n/(2s)}(\mathbb{R}^n))} \leq C\big(\|m_\gamma\|_{L^\infty(0,T;H^{2s,n/(2s)}(\mathbb{R}^n))} + \|m_\gamma\|^{2s}_{L^\infty(0,T;H^{2s,n/(2s)}(\mathbb{R}^n))}\big),$$

and hence we can repeat the above argument by using the smooth approximation of the function $m_\gamma^k/(m_\gamma^k+1)$ this time. Thus, we conclude that $\gamma^{-1/2}\psi \in L^2(0,T;\widetilde{H}^s(V))$ for all $\psi \in L^2(0,T;\widetilde{H}^s(V))$.

(ii) Note that due to our regularity assumptions, we can apply Lemma 4.1 in [14] or Remark 8.8 in [41] in every time slice to obtain

$$\begin{aligned}&\langle \Theta_\gamma \nabla^s u, \nabla^s\varphi\rangle_{L^2(\mathbb{R}^{2n})}\\ &= \langle (-\Delta)^{s/2}(\gamma^{1/2}u), (-\Delta)^{s/2}(\gamma^{1/2}\varphi)\rangle_{L^2(\mathbb{R}^n)} - \langle (-\Delta)^{s/2}m_\gamma, (-\Delta)^{s/2}(\gamma^{1/2}u\varphi)\rangle_{L^2(\mathbb{R}^n)}\\ &= \langle (-\Delta)^{s/2}(\gamma^{1/2}u), (-\Delta)^{s/2}(\gamma^{1/2}\varphi)\rangle_{L^2(\mathbb{R}^n)} + \langle q_\gamma\gamma^{1/2}u, \gamma^{1/2}\varphi\rangle_{L^2(\mathbb{R}^n)},\end{aligned}$$

for a.e. $t \in (0,T)$ and all $u,\varphi \in L^2(0,T;H^s(\mathbb{R}^n))$, where m_γ and q_γ are the functions defined by (3.2) and (5.4), respectively. Finally, note that, by the properties of the fractional Laplacian, the fact that $(u,v) \mapsto q_\gamma uv$ is bilinear and bounded as a map from $L^2(0,T;H^s(\mathbb{R}^n)) \times L^2(0,T;H^s(\mathbb{R}^n))$ to $L^1(0,T;L^1(\mathbb{R}^n))$ (cf. Corollary A.11 in [41]) and the assertion (i), all terms appearing in the above identity are in $L^1((0,T))$. ■

Now, we are ready to introduce the Liouville reduction.

Theorem 5.2 (Fractional spacetime Liouville reduction). *Let $\Omega \subset \mathbb{R}^n$ be an open set bounded in one direction, $0 < T < \infty$, $0 < s < \min(1, n/2)$, $\gamma_0 > 0$ and $\gamma \in \Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$.*

(i) *If $F \in L^2(\Omega_T)$, $(f,u_0) \in X_s(\Omega_T)$ and u is the unique solution to (3.7), then $v := \gamma^{1/2}u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^s(\mathbb{R}^n))$ solves*

$$\begin{cases}\partial_t(\gamma^{-1}v) + ((-\Delta)^s + Q_\gamma)v = G & \text{in } \Omega_T,\\ v = g & \text{in } (\Omega_e)_T,\\ v(0) = v_0 & \text{in } \Omega,\end{cases} \tag{5.3}$$

with $G = \gamma^{-1/2}F \in L^2(\Omega_T)$, $(g,v_0) = (\gamma^{1/2}f, \gamma^{1/2}u_0) \in X_s(\Omega_T)$ and

$$Q_\gamma = q_\gamma + \frac{\partial_t\gamma}{2\gamma^2}, \quad \text{with } q_\gamma = -\frac{(-\Delta)^s m_\gamma}{\gamma^{1/2}}. \tag{5.4}$$

(ii) *For all $G \in L^2(\Omega_T)$ and $(g,v_0) \in X_s(\Omega_T)$, there exists a unique solution $v \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^s(\mathbb{R}^n))$ to (5.3). Moreover, it is given by $v = \gamma^{1/2}u$, where u is the solution to*

$$\begin{cases}\partial_t u + \operatorname{div}_s(\Theta_\gamma\nabla^s u) = F & \text{in } \Omega_T,\\ u = f & \text{in } (\Omega_e)_T,\\ u(0) = u_0 & \text{in } \Omega,\end{cases} \tag{5.5}$$

with $F = \gamma^{1/2}G$, $f = \gamma^{-1/2}g$ and $u_0 = \gamma^{-1/2}v_0$.

Proof. (i) Note that $G \in L^2(\Omega_T)$ and by Lemma 5.1, we have $g \in L^2(0,T;H^s(\mathbb{R}^n))$, $v_0 \in H^s(\mathbb{R}^n)$ and $v - g \in L^2(0,T;\widetilde{H}^s(\Omega))$. By the assumptions on f,γ,u, we have $\partial_t(\gamma^{1/2}f) \in L^2(\mathbb{R}^n_T)$ and $\partial_t v \in L^2(\Omega_T)$. This implies that

$$v_0 - g(0) = \gamma^{1/2}(0)(u_0 - f(0)) \in \widetilde{H}^s(\Omega).$$

Therefore, we have shown that $(g, v_0) \in X_s(\Omega_T)$.

Arguing as above, we have that v has the same regularity properties as u. Thus, it remains to prove that v solves (5.3). By definition, u satisfies

$$-\int_{\Omega_T} u\partial_t\varphi\,dx\,dt + \int_0^T B_\gamma(t;u,\varphi)\,dt = \int_{\Omega_T} F\varphi\,dx\,dt + \int_\Omega u_0(x)\varphi(x,0)\,dx \tag{5.6}$$

for all $\varphi \in C_c^\infty(\Omega \times [0,T))$. Observe, as in the case $s = 1$ (cf. Lemma 4.12 in [33]), the space $C_c^\infty(\Omega \times [0,T))$ is dense in

$$W_s(\Omega_T) := \{\varphi \in L^2(0,T;\widetilde{H}^s(\Omega)) : \partial_t\varphi \in L^2(\Omega_T) \text{ and } \varphi(T) = 0\},$$

which is endowed with the natural norm

$$\|u\|^2_{W_s(\Omega_T)} := \|\partial_t u\|^2_{L^2(\Omega_T)} + \|u\|^2_{L^2(0,T;H^s(\mathbb{R}^n))}.$$

This can easily be seen by using a cutoff function in time as in Exercise 8.8 of [4], and using the density of $C_c^\infty((0,T)) \otimes C_c^\infty(\Omega)$ in $L^2(0,T;\widetilde{H}^s(\Omega))$.

Now, as we proved above, the space $W_s(\Omega_T)$ is invariant under multiplication with either $\gamma^{1/2}$ or $\gamma^{-1/2}$. Moreover, we have

$$\begin{aligned} u\partial_t\varphi &= \frac{1}{\gamma^{1/2}}(\gamma^{1/2}u)\,\partial_t\frac{\gamma^{1/2}\varphi}{\gamma^{1/2}} \\ &= \frac{1}{\gamma}(\gamma^{1/2}u)\,\partial_t(\gamma^{1/2}\varphi) - \frac{1}{2\gamma^2}(\gamma^{1/2}u)\,(\gamma^{1/2}\varphi)\,\partial_t\gamma, \end{aligned} \tag{5.7}$$

for all $\varphi \in W_s(\Omega_T)$. Hence, using the (space) Liouville reduction (see Remark 8.8 in [41]) in every time slice for all $\varphi \in W_s(\Omega_T)$, the identity (5.6) implies

$$\begin{aligned} &-\int_{\Omega_T} u\partial_t\varphi\,dx\,dt + \int_0^T \langle(-\Delta)^{s/2}(\gamma^{1/2}u), (-\Delta)^{s/2}(\gamma^{1/2}\varphi)\rangle\,dt \\ &\quad + \int_{\Omega_T} q_\gamma(\gamma^{1/2}u)(\gamma^{1/2}\varphi)\,dx\,dt \\ &= \int_{\Omega_T} F\varphi\,dx\,dt + \int_\Omega \frac{\gamma^{1/2}(x,0)\,u_0(x)\,\gamma^{1/2}(x,0)\,\varphi(x,0)}{\gamma(x,0)}\,dx. \end{aligned}$$

Inserting (5.7) yields

$$\begin{aligned} &-\int_{\Omega_T} \frac{(\gamma^{1/2}u)\partial_t(\gamma^{1/2}\varphi)}{\gamma}\,dx\,dt + \int_0^T \langle(-\Delta)^{s/2}(\gamma^{1/2}u), (-\Delta)^{s/2}(\gamma^{1/2}\varphi)\rangle\,dt \\ &\quad + \int_{\Omega_T} Q_\gamma(\gamma^{1/2}u)(\gamma^{1/2}\varphi)\,dx\,dt \\ &= \int_{\Omega_T} (\gamma^{-1/2}F)(\gamma^{1/2}\varphi)\,dx\,dt + \int_\Omega \frac{\gamma^{1/2}(x,0)\,u_0(x)\,\gamma^{1/2}(x,0)\,\varphi(x,0)}{\gamma(x,0)}\,dx. \end{aligned} \tag{5.8}$$

Hence, choosing $\varphi = \gamma^{-1/2}\psi$ with $\psi \in W_s(\Omega_T)$, we see that v satisfies

$$-\int_{\Omega_T} \gamma^{-1} v \partial_t \psi \, dx\, dt + \int_0^T \langle (-\Delta)^{s/2} v, (-\Delta)^{s/2}\psi \rangle \, dt + \int_{\Omega_T} Q_\gamma v \psi \, dx\, dt = \int_{\Omega_T} G\psi \, dx\, dt + \int_\Omega \frac{v_0(x)\psi(x,0)}{\gamma(x,0)} \, dx, \tag{5.9}$$

for all $\psi \in W_s(\Omega_T)$. Therefore v is a solution of (5.3) as claimed.

(ii) Existence and uniqueness of solutions in $H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^s(\mathbb{R}^n))$ easily follows from (i) by choosing the data F, f, u_0 appropriately and by observing that $\gamma^{-1/2}$ has precisely the same regularity properties as $\gamma^{1/2}$. In fact, assuming that $v \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^s(\mathbb{R}^n))$ solves (5.3), then, as in (i), we deduce $u := \gamma^{-1/2} v \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^s(\mathbb{R}^n))$, $F := \gamma^{1/2} G \in L^2(\Omega)$ and $(f, u_0) := (\gamma^{1/2} g, \gamma^{1/2} v_0) \in X_s(\Omega_T)$.

By the definition, v solves (5.9) for any $\psi \in W_s(\Omega_T)$. Replacing ψ by $\gamma^{1/2}\varphi$ and inserting these definitions of F, u_0 and f, we get (5.8). Plugging the identity (5.7) and using the slicewise Liouville reduction, we see that u solves (5.5). Now, Theorem 3.6 gives the existence of such a solution u and by (i) the function v solves (5.3). On the other hand, if v_1, v_2 are two solutions of (5.3), then arguing as above we see that $u_i := \gamma^{-1/2} v_i$ for $i = 1, 2$ solve (5.5). Since solutions to the nonlocal diffusion equation (5.5) are unique, we deduce that $u_1 = u_2$ and thus $v_1 = v_2$. ∎

Next, we will prove well-posedness for the diffusion equation derived by the Liouville reduction and its adjoint equation under the milder assumption $G \in L^2(0,T;H^{-s}(\Omega))$ but $g \in C_c^\infty((\Omega_e)_T)$. Moreover, we will see that u is the solution to (3.7) if and only if v is the unique solution to (5.3) of the form $v = \gamma^{-1/2} u$.

Definition 5.3. If $u \in L^1_{\mathrm{loc}}(V_T)$ for some open set $V \subset \mathbb{R}^n$, then we set

$$u^*(x,t) := u(x, T-t) \quad \text{for all } (x,t) \in V_T.$$

Proposition 5.4 (Well-posedness). *Let $\Omega \subset \mathbb{R}^n$ be an open set bounded in one direction, $0 < T < \infty$, $0 < s < \min(1, n/2)$, $\gamma_0 > 0$ and $\gamma \in \Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$. If $g \in C_c^\infty((\Omega_e)_T)$ and $G \in L^2(0,T;H^{-s}(\Omega))$, then there exist unique solutions v, $v^* \in L^2(0,T;H^s(\mathbb{R}^n))$ with $\partial_t v$ and $\partial_t v^* \in L^2(0,T,;H^{-s}(\Omega))$ of*

$$\begin{cases} \partial_t(\gamma^{-1} v) + ((-\Delta)^s + Q_\gamma) v = G & \text{in } \Omega_T, \\ v = g & \text{in } (\Omega_e)_T, \\ v(0) = 0 & \text{in } \Omega, \end{cases} \tag{5.10}$$

and

$$\begin{cases} -\gamma^{-1}\partial_t v^* + ((-\Delta)^s + Q_\gamma) v^* = G & \text{in } \Omega_T, \\ v^* = g & \text{in } (\Omega_e)_T, \\ v^*(T) = 0 & \text{in } \Omega, \end{cases} \tag{5.11}$$

respectively. Here Q_γ is the function (5.4) given by the Liouville reduction.

Proof. Let us prove the uniqueness of solutions to (5.10). Suppose that v_1, v_2 are solutions of (5.10), and consider $\tilde{v} := v_1 - v_2$. Then $\tilde{v}$ is the solution to

$$\begin{cases} \partial_t(\gamma^{-1}\tilde{v}) + ((-\Delta)^s + Q_\gamma)\tilde{v} = 0 & \text{in } \Omega_T, \\ \tilde{v} = 0 & \text{in } (\Omega_e)_T, \\ \tilde{v}(0) = 0 & \text{in } \Omega. \end{cases} \tag{5.12}$$

Multiplying (5.12) by $\tilde{v}$, it is not hard to see

$$\int_\Omega \partial_t(\gamma^{-1}\tilde{v})\,\tilde{v}\,dx + \int_{\mathbb{R}^n} |(-\Delta)^{s/2}\,\tilde{v}|^2\,dx + \int_\Omega Q_\gamma|\tilde{v}|^2\,dx = 0, \tag{5.13}$$

where the first integral has to be understood as the duality pairing between $\widetilde{H}^s(\Omega)$ and $H^{-s}(\Omega)$. Meanwhile, notice that the first term of the above identity can be expressed as

$$\int_\Omega \partial_t(\gamma^{-1}\tilde{v})\,\tilde{v}\,dx = \frac{\partial_t}{2}\int_\Omega \gamma^{-1}|\tilde{v}|^2\,dx + \int_\Omega |\tilde{v}|^2\gamma^{-1/2}\,\partial_t(\gamma^{-1/2})\,dx. \tag{5.14}$$

We next plug (5.14) into (5.13), which give rises to

$$\begin{aligned} \frac{\partial_t}{2}\int_\Omega \gamma^{-1}|\tilde{v}|^2\,dx + \int_{\mathbb{R}^n} |(-\Delta)^{s/2}\tilde{v}|^2\,dx &= -\int_\Omega Q_\gamma|\tilde{v}|^2\,dx - \int_\Omega \gamma^{-1/2}\partial_t(\gamma^{-1/2})|\tilde{v}|^2\,dx \\ &\le C\int_\Omega \gamma^{-1}|\tilde{v}|^2\,dx, \end{aligned}$$

for a constant $C > 0$ independent of $\tilde{v}$, where we used that $\gamma \in C_t^1 C_x(\mathbb{R}^n_T)$ is uniformly elliptic with $\partial_t\gamma \in L^\infty(\mathbb{R}^n_T)$ and (3.16).

Thus, we obtain

$$\partial_t\|\gamma^{-1/2}\,\tilde{v}\|^2_{L^2(\Omega)} \le C\Big(\partial_t\int_\Omega \gamma^{-1}|\tilde{v}|^2\,dx + \int_{\mathbb{R}^n} |(-\Delta)^{s/2}\,\tilde{v}|^2\,dx\Big) \le C\|\gamma^{-1/2}\,\tilde{v}\|^2_{L^2(\Omega)},$$

and the Gronwall's inequality implies that

$$\|\gamma^{-1/2}(\cdot,t)\tilde{v}(\cdot,t)\|^2_{L^2(\Omega)} \le e^{Ct}\|\gamma^{-1/2}(\cdot,0)\tilde{v}(\cdot,0)\|^2_{L^2(\Omega)} = 0 \quad \text{for } t \in (0,T),$$

where we used the initial condition is 0. This shows $\tilde{v} = 0$ in Ω_T, as desired.

When $\gamma \in C_t^1 C_x(\mathbb{R}^n_T)$ is uniformly elliptic with $\partial_t\gamma \in L^\infty(\mathbb{R}^n_T)$, the proof of well-posedness of either (5.10) or (5.11) are similar. More precisely, one can use the relation

$$\partial_t(\gamma^{-1}v) = \gamma^{-1}\partial_t v + \partial_t(\gamma^{-1})v$$

to rewrite equation (5.10) as

$$\begin{cases} \gamma^{-1}\partial_t v + ((-\Delta)^s + \widetilde{Q}_\gamma)v = G & \text{in } \Omega_T, \\ v = g & \text{in } (\Omega_e)_T, \\ v(0) = 0 & \text{in } \Omega, \end{cases} \tag{5.15}$$

where $\widetilde{Q}_\gamma := Q_\gamma + \partial_t(\gamma^{-1})$ in Ω_T. Now, it is not hard to see the well-posedness of (5.15) and (5.11) are the same by reversing the time variable $t \to T - t$ as in Definition 5.3.

Now, by slight modification of the proof of Theorem 5.2, one knows that v is the unique solution to (5.10) if and only if u is the solution to (3.7) with $G = \gamma^{-1/2}F \in L^2(0,T;H^{-s}(\Omega))$ and $g = \gamma^{1/2}f \in L^2(0,T;H^s(\mathbb{R}^n))$ with $\partial_t g \in L^2(\mathbb{R}^n_T)$. Hence, applying Theorem 3.6 for the solution u of (3.7), one has $u \in L^2(0,T;H^s(\mathbb{R}^n))$ with $\partial_t u \in L^2(0,T;H^{-s}(\Omega))$, and the same holds true for v. This proves the assertion. ∎

Remark 5.5. Combining similar arguments as in the proofs of Proposition 5.4 and Theorem 3.6, one may derive the well-posedness of the initial-exterior value problem of

$$\begin{cases} \partial_t(\gamma^{-1}v) + ((-\Delta)^s + Q)v = G & \text{in } \Omega_T, \\ v = g & \text{in } (\Omega_e)_T, \\ v(0) = 0 & \text{in } \Omega, \end{cases}$$

under suitable regularity assumptions of Q, g and G. Here the potential Q may not be of the same form as the function Q_γ given by the spacetime Liouville reduction (5.4).

6. Nonlocal Neumann derivative and new DN maps

Motivated by Lemma 3.3 in [16], for a given function u, we define the analogous *nonlocal normal derivative* in the exterior domain by

$$\mathcal{N}_\gamma u(x,t) = C_{n,s}\int_\Omega \gamma^{1/2}(x,t)\,\gamma^{1/2}(y,t)\,\frac{u(x,t)-u(y,t)}{|x-y|^{n+2s}}\,dy, \quad (x,t)\in(\Omega_e)_T,$$

where $C_{n,s}$ is the constant given by (1.5). We note that in the elliptic case similar data was considered in [24] for the fractional Schrödinger equation. However, as in our problem the coefficients are assumed to be unknown in the exterior, the Neumann and DN data in fact contain different information. We now proceed with the Neumann data which contains less information than the DN data. In particular, our results in this section are more general than the corresponding results which would rely on the DN data. This matter is discussed in detail in Appendix A.

6.1. Alternative definition of the DN map

Let us make a new definition of the DN map.

Definition 6.1 (New DN map for nonlocal diffusion equation). Let $\Omega \subset \mathbb{R}^n$ be an open set bounded in one direction, $0 < T < \infty$, $0 < s < \min(1, n/2)$, $\gamma_0 > 0$ and $\gamma \in \Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$. Then we define the exterior DN map $\mathcal{N}_\gamma$ by

$$\langle \mathcal{N}_\gamma f, g\rangle = \frac{C_{n,s}}{2}\int_0^T \int_{\mathbb{R}^{2n}\setminus(\Omega_e\times\Omega_e)} \gamma^{1/2}(x,t)\,\gamma^{1/2}(y,t) \cdot \frac{(u_f(x,t)-u_f(y,t))(g(x,t)-g(y,t))}{|x-y|^{n+2s}}\,dx\,dy\,dt,$$

for all $f, g \in C_c^\infty((\Omega_e)_T)$, where u_f is the unique solution (see Corollary 3.9 for the well-posedness) of

$$\begin{cases} \partial_t u + \mathrm{div}_s(\Theta_\gamma \nabla^s u) = 0 & \text{in } \Omega_T, \\ u = f & \text{in } (\Omega_e)_T, \\ u(0) = 0 & \text{in } \Omega. \end{cases}$$

Proposition 6.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set bounded in one direction, $0 < T < \infty$, $0 < s < \min(1, n/2)$, $\gamma_0 > 0$ and $\gamma \in \Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$. Let $f, g \in C_c^\infty((\Omega_e)_T)$ and denote by $u_f \in L^2(0,T; H^s(\mathbb{R}^n))$ the unique solutions to*

$$\begin{cases} \partial_t u + \mathrm{div}_s(\Theta_\gamma \nabla^s u) = 0 & \textit{in } \Omega_T, \\ u = f & \textit{in } (\Omega_e)_T, \\ u(0) = 0 & \textit{in } \Omega. \end{cases}$$

Let also $v_g \in L^2(0,T; H^s(\mathbb{R}^n))$ be any function satisfying $\partial_t v_g \in L^2(0,T; H^{-s}(\Omega))$ and $v_g - g \in L^2(0,T; \widetilde{H}^s(\Omega))$. Then

$$\begin{aligned} \langle \mathcal{N}_\gamma f, g \rangle = &\int_{\Omega_T} \partial_t u_f v_g \, dx\, dt + \int_0^T B_\gamma(u_f, v_g)\, dt \\ &- \frac{C_{n,s}}{2} \int_0^T \int_{\Omega_e \times \Omega_e} \gamma^{1/2}(x,t)\gamma^{1/2}(y,t) \\ &\cdot \frac{(f(x,t) - f(y,t))(g(x,t) - g(y,t))}{|x-y|^{n+2s}}\, dx\, dy\, dt. \end{aligned} \tag{6.1}$$

Proof. By the definition (3.1) of the bilinear form B_γ,

$$\begin{aligned} \langle \mathcal{N}_\gamma u, g \rangle &= \frac{C_{n,s}}{2} \int_0^T \int_{\mathbb{R}^{2n} \setminus (\Omega_e \times \Omega_e)} \gamma^{1/2}(x,t)\gamma^{1/2}(y,t) \\ &\quad \cdot \frac{(u_f(x,t) - u_f(y,t))(g(x,t) - g(y,t))}{|x-y|^{n+2s}}\, dx\, dy\, dt \\ &= \int_0^T B_\gamma(u_f, g)\, dt - \frac{C_{n,s}}{2} \int_0^T \int_{\Omega_e \times \Omega_e} \gamma^{1/2}(x,t)\gamma^{1/2}(y,t) \\ &\quad \cdot \frac{(u_f(x,t) - u_f(y,t))(g(x,t) - g(y,t))}{|x-y|^{n+2s}}\, dx\, dy\, dt. \end{aligned}$$

First, note that by writing

$$u_f = (u_f - f) + f$$

and using $(u_f - f)(\cdot, t) \in \widetilde{H}^s(\Omega)$ for a.e. $t \in (0,T)$, the last term is equal to

$$-\frac{C_{n,s}}{2} \int_0^T \int_{\Omega_e \times \Omega_e} \gamma^{1/2}(x,t)\gamma^{1/2}(y,t) \frac{(f(x,t) - f(y,t))(g(x,t) - g(y,t))}{|x-y|^{n+2s}}\, dx\, dy\, dt.$$

On the other hand, by using cutoff functions η_m in time vanishing near $t = 0$ and $t = T$ but equal to one on the support of g, one obtains

$$\begin{aligned}
\int_0^T B_\gamma(u_f, g)\, dt &= \lim_{m\to\infty} \int_0^T B_\gamma(u_f, \eta_m g)\, dt \\
&= \lim_{m\to\infty} \Big(-\int_0^T B_\gamma(u_f, \eta_m(v_g - g))\, dt + \int_0^T B_\gamma(u_f, \eta_m v_g)\, dt \Big) \\
&= -\lim_{m\to\infty} \int_{\Omega_T} u_f \partial_t(\eta_m(v_g - g))\, dx\, dt + \int_0^T B_\gamma(u_f, v_g)\, dt \\
&= \lim_{m\to\infty} \int_{\Omega_T} \partial_t u_f \eta_m v_g\, dx\, dt + \int_0^T B_\gamma(u_f, v_g)\, dt \\
&= \int_{\Omega_T} \partial_t u_f v_g\, dx\, dt + \int_0^T B_\gamma(u_f, v_g)\, dt.
\end{aligned}$$

This concludes the proof. ■

We next define the DN map for the spacetime Liouville reduction equation by the corresponding nonlocal Neumann derivative.

Definition 6.3. Let $\Omega \subset \mathbb{R}^n$ be an open set bounded in one direction, and let $0 < T < \infty$, $0 < s < \min(1, n/2)$, $\gamma_0 > 0$ and $\gamma \in \Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$. Then we define the exterior DN map $\mathcal{N}_{Q_\gamma}$ by

(6.2)
$$\langle \mathcal{N}_{Q_\gamma} f, g\rangle = \frac{C_{n,s}}{2} \int_0^T \int_{\mathbb{R}^{2n}\setminus(\Omega_e\times\Omega_e)} \frac{(v_f(x,t) - v_f(y,t))(g(x,t) - g(y,t))}{|x-y|^{n+2s}}\, dx\, dy\, dt,$$

for all $f, g \in C_c^\infty((\Omega_e)_T)$, where v_f is the unique solution (see Theorem 5.2 and Proposition 5.4) of

$$\begin{cases} \partial_t(\gamma^{-1} v) + ((-\Delta)^s + Q_\gamma)v = 0 & \text{in } \Omega_T, \\ v = g & \text{in } (\Omega_e)_T, \\ v(0) = 0 & \text{in } \Omega, \end{cases}$$

and $C_{n,s}$ is the constant given by (1.5).

To prove Theorem 1.1, let us derive a useful representation formula of (6.2).

Proposition 6.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set bounded in one direction, $0 < T < \infty$, $0 < s < \min(1, n/2)$, $\gamma_0 > 0$ and $\gamma \in \Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$. Let $f, g \in C_c^\infty((\Omega_e)_T)$ and denote by u_f the unique solutions to*

$$\begin{cases} \partial_t(\gamma^{-1} u) + ((-\Delta)^s + Q_\gamma)u = 0 & \text{in } \Omega_T, \\ u = f & \text{in } (\Omega_e)_T, \\ u(0) = 0 & \text{in } \Omega. \end{cases} \tag{6.3}$$

Let also $v_g \in L^2(0,T;H^s(\mathbb{R}^n))$ be any function satisfying $\partial_t v_g \in L^2(0,T,H^{-s}(\Omega))$ and $v_g - g \in L^2(0,T;\widetilde{H}^s(\Omega))$. Then

$$\begin{aligned}&\langle \mathcal{N}_{Q_\gamma} f, g\rangle \\ &= \int_{\Omega_T} \partial_t(\gamma^{-1}u_f)v_g\,dx\,dt + \int_{\mathbb{R}^n_T} (-\Delta)^{s/2}u_f(-\Delta)^{s/2}v_g\,dx\,dt + \int_{\Omega_T} Q_\gamma u_f v_g\,dx\,dt \\ &\quad - \frac{C_{n,s}}{2}\int_0^T\int_{\Omega_e\times\Omega_e} \frac{(f(x,t)-f(y,t))(g(x,t)-g(y,t))}{|x-y|^{n+2s}}\,dx\,dy\,dt.\end{aligned}$$

Remark 6.5. Observe that the last term in (6.1) is independent of Q_γ. Therefore, in this case the corresponding DN map Λ_{Q_γ} can be defined by

$$\begin{aligned}\langle \Lambda_{Q_\gamma} f, g\rangle &= \int_{\Omega_T} \partial_t(\gamma^{-1}u_f)v_g\,dx\,dt + \int_{\mathbb{R}^n_T} (-\Delta)^{s/2}u_f(-\Delta)^{s/2}v_g\,dx\,dt \\ &\quad + \int_{\Omega_T} Q_\gamma u_f v_g\,dx\,dt\end{aligned}$$

for $f, g \in C_c^\infty((\Omega_e)_T)$, and it contains the same information as $\mathcal{N}_{Q_\gamma}$.

Proof of Proposition 6.4. As in the proof of Proposition 6.2, we have

$$\begin{aligned}\langle \mathcal{N}_{Q_\gamma} f, g\rangle &= \int_{\mathbb{R}^n_T} (-\Delta)^{s/2}u_f(-\Delta)^{s/2}g\,dx\,dt \\ &\quad - \frac{C_{n,s}}{2}\int_0^T\int_{\Omega_e\times\Omega_e} \frac{(f(x,t)-f(y,t))(g(x,t)-g(y,t))}{|x-y|^{n+2s}}\,dx\,dy\,dt.\end{aligned}$$

Next, as in the proof of Proposition 6.2, we use a sequence of cutoff functions $(\eta_m)_{m\in\mathbb{N}} \subset C_c^\infty((0,T))$ to deduce the identity

$$\begin{aligned}&\int_{\mathbb{R}^n_T} (-\Delta)^{s/2}u_f(-\Delta)^{s/2}g\,dx\,dt \\ &= \lim_{m\to\infty} -\int_{\mathbb{R}^n_T} (-\Delta)^{s/2}u_f(-\Delta)^{s/2}(\eta_m(v_g-g))\,dx\,dt \\ &\quad + \lim_{m\to\infty}\int_{\mathbb{R}^n_T} (-\Delta)^{s/2}u_f(-\Delta)^{s/2}(\eta_m v_g)\,dx\,dt \\ &= \lim_{m\to\infty}\Big(-\int_{\Omega_T} \gamma^{-1}u_f\partial_t(\eta_m(v_g-g))\,dx\,dt + \int_{\Omega_T} Q_\gamma u_f(\eta_m(v_g-g))\,dx\,dt\Big) \\ &\quad + \int_{\mathbb{R}^n_T} (-\Delta)^{s/2}u_f(-\Delta)^{s/2}v_g\,dx\,dt \\ &= \int_{\Omega_T} \partial_t(\gamma^{-1}u_f)v_g\,dx\,dt + \int_{\mathbb{R}^n_T} (-\Delta)^{s/2}u_f(-\Delta)^{s/2}v_g\,dx\,dt \\ &\quad + \int_{\Omega_T} Q_\gamma u_f v_g\,dx\,dt.\end{aligned}$$

This completes the proof. ∎

6.2. Relation between DN map and nonlocal Neumann derivative

Let us consider two arbitrary nonempty open subsets $W_1, W_2 \subset W$, with $W_1 \cap W_2 = \emptyset$, where $W \subset \Omega_e$ denotes the open set in the statements of either Theorem 1.1 or Theorem 1.2. Meanwhile, with the exterior determination result (Theorem 1.2) at hand, one already knows that $\gamma_1 = \gamma_2$ in W_T, provided that $\Lambda_{\gamma_1} f|_{W_T} = \Lambda_{\gamma_2} f|_{W_T}$ for any $f \in C_c^\infty(W_T)$. Adopting these notations, one immediately has $\gamma_1 = \gamma_2$ in $(W_1 \cup W_2)_T$. Then we can derive the following relation.

Lemma 6.6. *Let $\Omega \subset \mathbb{R}^n$ be an open set bounded in one direction, $0 < T < \infty$, $0 < s < \min(1, n/2)$, $\gamma_0 > 0$ and $\gamma_j \in \Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$ for $j = 1, 2$. Assume that $W_1, W_2 \subset \Omega_e$ are two nonempty open disjoint sets and $\Gamma \in \Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$ are such that $\gamma_1(x,t) = \gamma_2(x,t) = \Gamma(x,t)$ for all $(x,t) \in (W_1 \cup W_2)_T$. Then we have*

$$\Lambda_{\gamma_1} f|_{(W_2)_T} = \Lambda_{\gamma_2} f|_{(W_2)_T} \quad \text{for any } f \in C_c^\infty((W_1)_T)$$

if and only if

$$\langle \mathcal{N}_{\gamma_1} f, g\rangle = \langle \mathcal{N}_{\gamma_2} f, g\rangle \quad \text{for any } f \in C_c^\infty((W_1)_T) \text{ and } g \in C_c^\infty((W_2)_T).$$

Proof. We have, for any $f \in C_c^\infty((W_1)_T)$ and $g \in C_c^\infty((W_2)_T)$,

$$\langle \Lambda_\gamma f, g\rangle = \frac{C_{n,s}}{2}\int_0^T \int_{\mathbb{R}^{2n}} \gamma^{1/2}(x,t)\,\gamma^{1/2}(y,t)\frac{(u_f(x,t)-u_f(y,t))(g(x,t)-g(y,t))}{|x-y|^{n+2s}}\,dx\,dy\,dt,$$

by using Definition 3.10. Thus, combining with Definition 6.1, one has that

$$\begin{aligned}(6.4)\qquad \langle \Lambda_{\gamma_1} f, g\rangle &= \langle \mathcal{N}_{\gamma_1} f, g\rangle + \frac{C_{n,s}}{2}\int_0^T \int_{\Omega_e\times\Omega_e} \gamma_1^{1/2}(x,t)\gamma_1^{1/2}(y,t) \\ &\quad \cdot \frac{(f(x,t)-f(y,t))(g(x,t)-g(y,t))}{|x-y|^{n+2s}}\,dx\,dy\,dt \\ &= \langle \mathcal{N}_{\gamma_2} f, g\rangle + \frac{C_{n,s}}{2}\int_0^T \int_{\Omega_e\times\Omega_e} \gamma_2^{1/2}(x,t)\gamma_2^{1/2}(y,t) \\ &\quad \cdot \frac{(f(x,t)-f(y,t))(g(x,t)-g(y,t))}{|x-y|^{n+2s}}\,dx\,dy\,dt = \langle \Lambda_{\gamma_2} f, g\rangle,\end{aligned}$$

where we used that $u_f = f$ in $(\Omega_e)_T$.

On the other hand, one can see that

$$\begin{aligned}(6.5)\qquad &\int_0^T \int_{\Omega_e\times\Omega_e} \gamma_1^{1/2}(x,t)\gamma_1^{1/2}(y,t)\,\frac{(f(x,t)-f(y,t))(g(x,t)-g(y,t))}{|x-y|^{n+2s}}\,dx\,dy\,dt \\ &= \int_0^T \int_{\Omega_e\times\Omega_e} \gamma_2^{1/2}(x,t)\gamma_2^{1/2}(y,t)\,\frac{(f(x,t)-f(y,t))(g(x,t)-g(y,t))}{|x-y|^{n+2s}}\,dx\,dy\,dt,\end{aligned}$$

where we used that $f \in C_c^\infty((W_1)_T)$, $g \in C_c^\infty((W_2)_T)$ with $\gamma_1 = \gamma_2$ in $(W_1 \cup W_2)_T$ and $W_1 \cap W_2 = \emptyset$. Finally, insert (6.5) into (6.4), and we can see the assertion is true. This completes the proof. ∎

Theorem 6.7 (Relation of DN maps). *Let $\Omega \subset \mathbb{R}^n$ be an open set bounded in one direction, $0 < T < \infty$, $0 < s < \min(1, n/2)$, $\gamma_0 > 0$ and $\gamma_j \in \Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$ for $j = 1, 2$. Assume that $W_1, W_2 \subset \Omega_e$ are two nonempty open disjoint sets and $\Gamma \in \Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$ are such that $\gamma_1(x,t) = \gamma_2(x,t) = \Gamma(x,t)$ for all $(x,t) \in (W_1 \cup W_2)_T$ and $\Gamma \in C^\infty((W_1 \cup W_2)_T)$. Then for $f \in C_c^\infty((W_1)_T)$, $g \in C_c^\infty((W_2)_T)$, we have*

$$\langle \mathcal{N}_{\gamma_1} f, g\rangle = \langle \mathcal{N}_{\gamma_2} f, g\rangle$$

if and only if

$$\langle \mathcal{N}_{Q_{\gamma_1}}(\Gamma^{1/2} f), (\Gamma^{1/2} g)\rangle = \langle \mathcal{N}_{Q_{\gamma_2}}(\Gamma^{1/2} f), (\Gamma^{1/2} g)\rangle.$$

Proof. By the Liouville reduction (cf. Theorem 5.2),

$$\begin{aligned}
&\int_{\Omega_T} \partial_t u_f v_g \,dx\,dt + \int_0^T B_\gamma(u_f, v_g)\,dt \\
&= \int_{\Omega_T} \partial_t(\gamma^{-1}(\gamma^{1/2}u_f))(\gamma^{1/2}v_g)\,dx\,dt + \int_{\mathbb{R}^n_T} (-\Delta)^{s/2}(\gamma^{1/2}u_f)(-\Delta)^{s/2}(\gamma^{1/2}v_g)\,dx\,dt \\
&\quad + \int_{\mathbb{R}^n_T} q_\gamma(\gamma^{1/2}u_f)(\gamma^{1/2}v_g)\,dx\,dt + \int_{\Omega_T} \frac{\partial_t\gamma}{2\gamma^2}(\gamma^{1/2}u_f)(\gamma^{1/2}v_g)\,dx\,dt \\
&= \int_{\Omega_T} \partial_t(\gamma^{-1}(\gamma^{1/2}u_f))(\gamma^{1/2}v_g)\,dx\,dt + \int_{\mathbb{R}^n_T} (-\Delta)^{s/2}(\gamma^{1/2}u_f)(-\Delta)^{s/2}(\gamma^{1/2}v_g)\,dx\,dt \\
&\quad + \int_{\Omega_T} Q_\gamma(\gamma^{1/2}u_f)(\gamma^{1/2}v_g)\,dx\,dt + \int_{(\Omega_e)_T} q_\gamma(\gamma^{1/2}u_f)(\gamma^{1/2}v_g)\,dx\,dt \\
&= \langle \mathcal{N}_{Q_\gamma}(\Gamma^{1/2} f), (\Gamma^{1/2} g)\rangle + \int_{(\Omega_e)_T} q_\gamma(\gamma^{1/2} f)(\gamma^{1/2} g)\,dx\,dt \\
&\quad + \frac{C_{n,s}}{2}\int_0^T \int_{\Omega_e\times\Omega_e} \frac{(f(x,t)-f(y,t))(g(x,t)-g(y,t))}{|x-y|^{n+2s}}\,dx\,dy\,dt,
\end{aligned}$$

since $w_f := \gamma^{1/2}u_f$ solves (6.3) with exterior condition $\Gamma^{1/2} f$ and $v_{\Gamma^{1/2}g} := \gamma^{1/2}v_g$ is an extension of $\gamma^{1/2}g$ with the same regularity properties as v_g. Therefore,

$$\begin{aligned}
&\langle \mathcal{N}_\gamma f, g\rangle - \langle \mathcal{N}_{Q_\gamma}(\Gamma^{1/2} f), (\Gamma^{1/2} g)\rangle \\
&\qquad = \int_{(\Omega_e)_T} q_\gamma(\gamma^{1/2} f)(\gamma^{1/2} g)\,dx\,dt \\
&\qquad\quad - \frac{C_{n,s}}{2}\int_0^T \int_{\Omega_e\times\Omega_e} \frac{(f(x,t)-f(y,t))(g(x,t)-g(y,t))}{|x-y|^{n+2s}}\,dx\,dy\,dt \\
&\qquad\quad + \frac{C_{n,s}}{2}\int_0^T \int_{\Omega_e\times\Omega_e} \gamma^{1/2}(x,t)\,\gamma^{1/2}(y,t) \\
&\qquad\quad \cdot \frac{(f(x,t)-f(y,t))(g(x,t)-g(y,t))}{|x-y|^{n+2s}}\,dx\,dy\,dt.
\end{aligned}$$

Since $W_1 \cap W_2 = \emptyset$, it follows that

$$\begin{aligned}\langle \mathcal{N}_\gamma f, g\rangle - \langle \mathcal{N}_{Q_\gamma}(\Gamma^{1/2} f), (\Gamma^{1/2} g)\rangle \\ = -\frac{C_{n,s}}{2}\int_0^T \int_{\Omega_e\times\Omega_e} (1-\Gamma^{1/2}(x,t)\,\Gamma^{1/2}(y,t)) \\ \cdot \frac{(f(x,t)g(y,t)-f(y,t)g(x,t))}{|x-y|^{n+2s}}\,dx\,dy\,dt.\end{aligned}$$

Now this expression on the right-hand side does not depend on the conductivities and so we see that

$$\langle \mathcal{N}_{\gamma_1} f, g\rangle = \langle \mathcal{N}_{\gamma_2} f, g\rangle \iff \langle \mathcal{N}_{Q_{\gamma_1}} f, g\rangle = \langle \mathcal{N}_{Q_{\gamma_2}} f, g\rangle,$$

for any $f \in C_c^\infty((W_1)_T)$ and $g \in C_c^\infty((W_2)_T)$. This proves the assertion. ■

6.3. Adjoint DN map

Let us introduce the adjoint DN map which then will be used to prove a suitable integral identity.

Definition 6.8 (Adjoint DN map). Let $\Omega \subset \mathbb{R}^n$ be an open set bounded in one direction, $0 < T < \infty$, $0 < s < \min(1, n/2)$, $\gamma_0 > 0$ and $\gamma \in \Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$. Then we define the adjoint exterior DN map $\mathcal{N}^*_{Q_\gamma}$ by

$$\langle \mathcal{N}^*_{Q_\gamma} f, g\rangle = \frac{C_{n,s}}{2}\int_0^T \int_{\mathbb{R}^{2n}\setminus(\Omega_e\times\Omega_e)} \frac{(u_f(x,t)-u_f(y,t))(g(x,t)-g(y,t))}{|x-y|^{n+2s}}\,dx\,dy\,dt$$

for all $f, g \in C_c^\infty((\Omega_e)_T)$, where u_f is the unique solution to

$$\begin{cases} -\gamma^{-1}\partial_t v + ((-\Delta)^s + Q_\gamma)v = 0 & \text{in } \Omega_T, \\ v = f & \text{in } (\Omega_e)_T, \\ v(T) = 0 & \text{in } \Omega, \end{cases}$$

and $C_{n,s}$ is the constant given by (1.5).

We make the following simple observations.

Lemma 6.9 (Properties of adjoint DN map). *Let $\Omega \subset \mathbb{R}^n$ be an open set bounded in one direction, $0 < T < \infty$, $0 < s < \min(1, n/2)$, $\gamma_0 > 0$ and $\gamma \in \Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$. If $f, g \in C_c^\infty((\Omega_e)_T)$ and u_f is the unique solution to*

$$\begin{cases} -\gamma^{-1}\partial_t u + ((-\Delta)^s + Q_\gamma)u = 0 & \text{in } \Omega_T, \\ u = f & \text{in } (\Omega_e)_T, \\ u(T) = 0 & \text{in } \Omega, \end{cases}$$

then the following hold:

(i) *For any extension v_g of g with $v_g \in L^2(0,T;H^s(\mathbb{R}^n))$ and $\partial_t v_g \in L^2(0,T,H^{-s}(\Omega))$,*

$$\begin{aligned}\langle \mathcal{N}^*_{Q_\gamma} f, g\rangle = &-\int_{\Omega_T} \gamma^{-1}\partial_t u_f v_g \,dx\,dt + \int_{\mathbb{R}^n_T} (-\Delta)^{s/2} u_f (-\Delta)^{s/2} v_g \,dx\,dt \\ &+ \int_{\Omega_T} Q_\gamma u_f v_g \,dx\,dt \\ &- \frac{C_{n,s}}{2}\int_0^T \int_{\Omega_e\times\Omega_e} \frac{(f(x,t)-f(y,t))(g(x,t)-g(y,t))}{|x-y|^{n+2s}}\,dx\,dy\,dt.\end{aligned}$$

(ii) *We have $\langle \mathcal{N}^*_{Q_\gamma} f, g\rangle = \langle \mathcal{N}_{Q_\gamma} g, f\rangle$.*

Proof. (i) This follows from a similar calculation as in Proposition 6.4.

(ii) Let u_g be the solution to (6.3) as the exterior data f is replaced by g. Since $u_f(T) = 0$, $u_g(0) = 0$, we have

$$\int_{\Omega_T} (\partial_t(\gamma^{-1}u_g)u_f + u_g\gamma^{-1}\partial_t u_f))\,dx\,dt = 0.$$

This immediately shows the claim. ■

7. The global uniqueness

We split this final section into several parts. We first establish the integral identity and the Runge approximations in Sections 7.1 and 7.2, respectively. Combined with these two statements, one can prove the interior uniqueness in Section 7.3. Finally, we show the UCP of exterior DN maps, which together with the work of Section 4 imply the global uniqueness result of Theorem 1.1.

7.1. Integral identity

One of the key material to prove the interior uniqueness is to derive a suitable integral identity.

Proposition 7.1 (Integral identity). *Let $\Omega \subset \mathbb{R}^n$ be an open set bounded in one direction, $0 < T < \infty$, $0 < s < \min(1, n/2)$, $\gamma_0 > 0$ and $\gamma_j \in \Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$ for $j = 1,2$. Assume that $W_1, W_2 \subset \Omega_e$ are two nonempty open sets and $\Gamma \in \Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$ are such that $\gamma_1(x,t) = \gamma_2(x,t) = \Gamma(x,t)$ for all $(x,t) \in (W_1 \cup W_2)_T$ and $\Gamma \in C^\infty((W_1 \cap W_2)_T)$. Then for $f \in C_c^\infty((W_1)_T)$, $g \in C_c^\infty((W_2)_T)$, we have*

$$\langle (\mathcal{N}_{Q_{\gamma_1}} - \mathcal{N}_{Q_{\gamma_2}}) f, g\rangle = \int_{\Omega_T} (\gamma_2^{-1} - \gamma_1^{-1}) v_f \partial_t v_g \,dx\,dt + \int_{\Omega_T} (Q_{\gamma_1} - Q_{\gamma_2}) v_g v_f \,dx\,dt,$$

where v_f is the unique solution to (6.3) with $\gamma = \gamma_1$ and v_g is the unique solution to the adjoint equation

$$\begin{cases} -\gamma_2^{-1}\partial_t w + ((-\Delta)^s + Q_{\gamma_2})w = 0 & \text{in } \Omega_T, \\ w = g & \text{in } (\Omega_e)_T, \\ w(T) = 0 & \text{in } \Omega. \end{cases}$$

Proof. By Lemma 6.9 and Proposition 6.4, we have

$$
\begin{aligned}
\langle(\mathcal{N}_{Q_{\gamma_1}} - \mathcal{N}_{Q_{\gamma_2}})f, g\rangle &= \langle\mathcal{N}_{Q_{\gamma_1}} f, g\rangle - \langle\mathcal{N}_{Q_{\gamma_2}} f, g\rangle = \langle\mathcal{N}_{Q_{\gamma_1}} f, g\rangle - \langle\mathcal{N}^*_{Q_{\gamma_2}} g, f\rangle \\
&= \int_{\Omega_T} \partial_t(\gamma_1^{-1} v_f) v_g \, dx\, dt + \int_{\mathbb{R}^n_T} (-\Delta)^{s/2} v_f (-\Delta)^{s/2} v_g \, dx\, dt + \int_{\Omega_T} Q_{\gamma_1} v_f v_g \, dx\, dt \\
&\quad - \frac{C_{n,s}}{2} \int_0^T \int_{\Omega_e \times \Omega_e} \frac{(f(x,t) - f(y,t))(g(x,t) - g(y,t))}{|x-y|^{n+2s}} \, dx\, dy\, dt \\
&\quad + \int_{\Omega_T} \gamma_2^{-1} \partial_t v_g v_f \, dx\, dt - \int_{\mathbb{R}^n_T} (-\Delta)^{s/2} v_g (-\Delta)^{s/2} v_f \, dx\, dt - \int_{\Omega_T} Q_{\gamma_2} v_g v_f \, dx\, dt \\
&\quad + \frac{C_{n,s}}{2} \int_0^T \int_{\Omega_e \times \Omega_e} \frac{(f(x,t) - f(y,t))(g(x,t) - g(y,t))}{|x-y|^{n+2s}} \, dx\, dy\, dt \\
&= \int_{\Omega_T} \partial_t(\gamma_1^{-1} v_f) v_g \, dx\, dt + \int_{\Omega_T} \gamma_2^{-1} v_f \partial_t v_g \, dx\, dt + \int_{\Omega_T} (Q_{\gamma_1} - Q_{\gamma_2}) v_g v_f \, dx\, dt \\
&= \int_{\Omega_T} (\gamma_2^{-1} - \gamma_1^{-1}) v_f \partial_t v_g \, dx\, dt + \int_{\Omega_T} (Q_{\gamma_1} - Q_{\gamma_2}) v_g v_f \, dx\, dt,
\end{aligned}
$$

where we used for the integration by parts that $v_f(0) = 0$ and $v_g(T) = 0$. ∎

7.2. Approximation property

To prove the interior uniqueness result of γ, we derive an approximation property of solutions to the Schrödinger type equations. First, we introduce the source to solution map, which is usually called Poisson operator. Assume that $\Omega \subset \mathbb{R}^n$ is an open set bounded in one direction, $0 < T < \infty$, $0 < s < \min(1, n/2)$, $\gamma_0 > 0$, $\gamma \in \Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$ and $W \subset \Omega_e$ is a nonempty open set. With the well-posedness of (5.3), we can define the Poisson operator P_γ as follows:

$$
P_\gamma : C_c^\infty(W_T) \to H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^s(\mathbb{R}^n)), \quad f \mapsto v_f, \tag{7.1}
$$

where $v_f \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^s(\mathbb{R}^n))$ is the unique solution of

$$
\begin{cases}
\partial_t(\gamma^{-1} v) + ((-\Delta)^s + Q_\gamma) v = 0 & \text{in } \Omega_T, \\
v = f & \text{in } (\Omega_e)_T, \\
v(0) = 0 & \text{in } \Omega,
\end{cases} \tag{7.2}
$$

with $v_f - f \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; \widetilde{H}^s(\Omega))$.

Next, before studying the Runge approximation for equation (7.2), let us recall the UCP for the fractional Laplacian (see, e.g., Theorem 1.2 in [24] for functions in H^r, or Theorem 2.2 in [28] for functions in $H^{r,p}$).

Proposition 7.2 (Unique continuation for the fractional Laplacian). *For $n \in \mathbb{N}$ and $s \in (0,1)$, let $w \in H^{-r}(\mathbb{R}^n)$ for some $r \in \mathbb{R}$. Given a nonempty open subset $W \subset \mathbb{R}^n$, then $w = (-\Delta)^s w = 0$ in W implies that $w \equiv 0$ in $\mathbb{R}^n$.*

Proposition 7.3 (Runge approximation). *Let $\Omega \subset \mathbb{R}^n$ be an open set bounded in one direction, $0 < T < \infty$, $0 < s < \min(1, n/2)$, $\gamma_0 > 0$ and $\gamma \in \Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$, and let $W \subset \Omega_e$ be a nonempty open set. Let P_γ be the Poisson operator given by (7.1), and define*

$$\mathcal{R} := \{v_f - f : f \in C_c^\infty(W_T)\}.$$

Then the set $\mathcal{R}$ is dense in $L^2(0,T;\widetilde{H}^s(\Omega))$.

Proof. By using Theorem 5.2, one has $\mathcal{R} \subset L^2(0,T;\widetilde{H}^s(\Omega))$. In order to show the density, by the Hahn–Banach theorem, we need to show that if $F \in (L^2(0,T;\widetilde{H}^s(\Omega)))^* = L^2(0,T;H^{-s}(\Omega))$ is such that[3] $\langle F, w\rangle = 0$ for any $w \in \mathcal{R}$, then F must be zero. Via $\langle F, w\rangle = 0$ for any $w \in \mathcal{R}$, we have

$$\langle F, P_\gamma f - f\rangle = 0 \quad \text{for any } f \in C_c^\infty(W_T).$$

We next claim that

$$\langle F, P_\gamma f - f\rangle = -\int_{\mathbb{R}^n_T} (-\Delta)^{s/2} f (-\Delta)^{s/2}\varphi \, dx\, dt \quad \text{for any } f \in C_c^\infty(W_T), \tag{7.3}$$

where $\varphi \in L^2(0,T;H^s(\mathbb{R}^n))$ with $\partial_t\varphi \in L^2(0,T;H^{-s}(\Omega))$ (see Proposition 5.4) is the unique solution of the adjoint equation

$$\begin{cases} -\gamma^{-1}\partial_t\varphi + ((-\Delta)^s + Q_\gamma)\varphi = F & \text{in } \Omega_T, \\ \varphi = 0 & \text{in } (\Omega_e)_T, \\ \varphi(T) = 0 & \text{in } \Omega. \end{cases}$$

In fact, by direct computations, one has that $v_f = P_\gamma f$ and

$$\begin{aligned} &\langle F, P_\gamma f - f\rangle \\ &= \int_{\Omega_T} (-\gamma^{-1}\partial_t\varphi + Q_\gamma)(v_f - f)\, dx\, dt + \int_{\mathbb{R}^n_T} (-\Delta)^{s/2}\varphi(-\Delta)^{s/2}(v_f - f)\, dx\, dt \\ &= \int_{\Omega_T} (-\gamma^{-1}\partial_t\varphi + Q_\gamma\varphi)v_f\, dx\, dt + \int_{\mathbb{R}^n_T} (-\Delta)^{s/2}\varphi(-\Delta)^{s/2} v_f\, dx\, dt \\ &\quad - \int_{\mathbb{R}^n_T} (-\Delta)^{s/2}\varphi(-\Delta)^{s/2} f\, dx\, dt \\ &= \underbrace{\int_{\Omega_T} (\partial_t(\gamma^{-1}v_f) + Q_\gamma v_f)\varphi\, dx\, dt + \int_{\mathbb{R}^n_T} (-\Delta)^{s/2}v_f(-\Delta)^{s/2}\varphi\, dx\, dt}_{=0, \text{ since } v_f = P_\gamma f} \\ &\quad - \int_{\mathbb{R}^n_T} (-\Delta)^{s/2}\varphi(-\Delta)^{s/2} f\, dx\, dt \\ &= -\int_{\mathbb{R}^n_T} (-\Delta)^{s/2}\varphi(-\Delta)^{s/2} f\, dx\, dt, \end{aligned}$$

[3] Note that here $\langle F, w\rangle = \int_{\Omega_T} F w \, dx\, dt$ denotes the duality pairing, for $F \in L^2(0,T;H^{-s}(\Omega))$ and $w \in L^2(0,T;\widetilde{H}^s(\Omega))$.

where we used that $v_f(0) = \varphi(T) = 0$ for the integration by parts in the third equality sign and the integral involving the time derivative has to be understood in a weak sense. This shows identity (7.3). Finally, identity (7.3) is equivalent to

$$(-\Delta)^s \varphi = 0 \quad \text{in } W_T.$$

Thus, the function φ satisfies

$$\varphi = (-\Delta)^s \varphi = 0 \quad \text{in } W_T,$$

and then, by Proposition 7.2, we have

$$\varphi \equiv 0 \quad \text{in } \mathbb{R}^n_T,$$

so that $F \equiv 0$ in $\mathbb{R}^n_T$. In summary, we showed that the set $\mathcal{R}$ is dense in $L^2(0,T;\widetilde{H}^s(\Omega))$. This proves the assertion. ■

Remark 7.4. (i) By Proposition 7.3, we know that given any $\phi \in L^2(0,T;\widetilde{H}^s(\Omega))$, there exists a sequence of solutions $\{v_{f_k}\}_{k\in\mathbb{N}} \in L^2(0,T;H^s(\mathbb{R}^n))$ to (7.2), with $f = f_k$, such that

$$v_{f_k} - f_k \to \phi \quad \text{in } L^2(0,T;\widetilde{H}^s(\Omega)) \text{ as } k \to \infty.$$

Since v_{f_k} is a solution, by applying Proposition 5.4, we see that $\partial_t v_{f_k} \in L^2(0,T;H^{-s}(\Omega))$. Now assume that the time derivative of ϕ belongs to $L^2(0,T;H^{-s}(\Omega))$. Then we have

$$\begin{aligned}\lim_{k\to\infty}\int_{\Omega_T} \partial_t(v_{f_k} - f_k)\varphi\,dx\,dt &= -\lim_{k\to\infty}\int_{\Omega_T}(v_{f_k} - f_k)\partial_t\varphi\,dx\,dt\\ &= -\int_{\Omega_T}\phi\,\partial_t\varphi\,dx\,dt\end{aligned}$$

for any $\varphi \in L^2(0,T;\widetilde{H}^s(\Omega))$ with $\partial_t\varphi \in L^2(0,T;H^{-s}(\Omega))$ and $\varphi(T) = 0$. If, additionally, $\phi(0) = 0$ or $\varphi(0) = 0$, then

$$\lim_{k\to\infty}\int_{\Omega_T}\partial_t(v_{f_k} - f_k)\varphi\,dx\,dt = \int_{\Omega_T}(\partial_t\phi)\varphi\,dx\,dt.$$

(ii) By using similar arguments as in the proof of Proposition 7.3, one can show that the Runge approximation holds for the adjoint diffusion equation

$$\begin{cases} -\gamma^{-1}\partial_t v^* + ((-\Delta)^s + Q_\gamma)v^* = 0 & \text{in } \Omega_T,\\ v^* = g & \text{in } (\Omega_e)_T,\\ v^*(T) = 0 & \text{in } \Omega.\end{cases}$$

In other words, given a nonempty open set $W \subset \Omega_e$, the set

$$\mathcal{R}^* := \{v_g^* - g;\, g \in C_c^\infty(W_T)\}$$

is dense in $L^2(0,T;\widetilde{H}^s(\Omega))$.

7.3. Interior determination and proof of Theorem 1.1

Let us state the interior uniqueness result.

Theorem 7.5 (Interior uniqueness). *Let $\Omega \subset \mathbb{R}^n$ be an open set bounded in one direction, $0 < T < \infty$, $0 < s < \min(1, n/2)$, $\gamma_0 > 0$ and $\gamma_j \in \Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$ for $j = 1, 2$. Assume that $W_1, W_2 \subset \Omega_e$ are two disjoint nonempty open sets and $\Gamma \in \Gamma_{s,\gamma_0}(\mathbb{R}^n_T)$ are such that $\gamma_1(x,t) = \gamma_2(x,t) = \Gamma(x,t)$ for all $(x,t) \in (W_1 \cup W_2)_T$ and $\Gamma \in C^\infty((W_1 \cup W_2)_T)$. Then*

$$\langle \mathcal{N}_{\gamma_1} f, g\rangle = \langle \mathcal{N}_{\gamma_2} f, g\rangle \tag{7.4}$$

if and only if

$$\gamma_1 = \gamma_2 \quad \text{and} \quad Q_{\gamma_1} = Q_{\gamma_2} \quad \text{in } \Omega_T.$$

Proof. Via Theorem 6.7, for $g \in C_c^\infty((W_2)_T)$, one has

$$\langle \mathcal{N}_{\gamma_1} f, g\rangle = \langle \mathcal{N}_{\gamma_2} f, g\rangle$$

if and only if

$$\langle \mathcal{N}_{Q_{\gamma_1}}(\Gamma^{1/2} f), (\Gamma^{1/2} g)\rangle = \langle \mathcal{N}_{Q_{\gamma_2}}(\Gamma^{1/2} f), (\Gamma^{1/2} g)\rangle. \tag{7.5}$$

Since Γ is uniformly elliptic and smooth on $(W_1 \cup W_2)_T$, condition (7.5) implies

$$\langle \mathcal{N}_{Q_{\gamma_1}} f, g\rangle = \langle \mathcal{N}_{Q_{\gamma_2}} f, g\rangle$$

for all $f \in C_c^\infty((W_1)_T)$ and $g \in C_c^\infty((W_2)_T)$. Moreover, by Proposition 7.1, one has

$$\int_{\Omega_T} (\gamma_2^{-1} - \gamma_1^{-1}) v_f\, \partial_t v_g^*\, dx\, dt + \int_{\Omega_T} (Q_{\gamma_1} - Q_{\gamma_2}) v_g^* v_f\, dx\, dt = 0, \tag{7.6}$$

where $v_f \in \mathcal{H} := H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^s(\mathbb{R}^n))$ and $v_g^* \in \mathcal{H}$ are, respectively, the solutions to

$$\begin{cases} \partial_t(\gamma_1^{-1} v_f) + ((-\Delta)^s + Q_{\gamma_1}) v_f = 0 & \text{in } \Omega_T, \\ v_f = f & \text{in } (\Omega_e)_T, \\ v_f(0) = 0 & \text{in } \Omega, \end{cases} \tag{7.7}$$

and

$$\begin{cases} -\gamma_2^{-1} \partial_t v_g^* + ((-\Delta)^s + Q_{\gamma_2}) v_g^* = 0 & \text{in } \Omega_T, \\ v_g^* = g & \text{in } (\Omega_e)_T, \\ v_g^*(T) = 0 & \text{in } \Omega, \end{cases} \tag{7.8}$$

Step 1. $Q_{\gamma_1} = Q_{\gamma_2}$ *in* Ω_T.

Take $\Omega' \Subset \Omega$ and $\psi \in C_c^\infty(\Omega)$ with $\psi|_{\overline{\Omega'}} = 1$. By extending ψ for all times trivially, we have $\psi \in \mathcal{H}$. Then take $\phi \in C_c^\infty(\Omega'_T)$ and apply the Runge approximation (Proposition 7.3

and Remark 7.4) to find sequences $\{f_\ell\}_{\ell=1}^\infty \subset C_c^\infty((W_1)_T)$ and $\{g_k\}_{k=1}^\infty \subset C_c^\infty((W_2)_T)$ such that

$$v_{f_\ell} - f_\ell \to \phi \quad \text{and} \quad v_{g_k}^* - g_k \to \psi \quad \text{as } \ell, k \to \infty.$$

Here $v_{f_\ell} \in \mathcal{H}$ and $v_{g_k}^* \in \mathcal{H}$ are the solutions to (7.7) and (7.8) with $f = f_\ell$ and $g = g_k$, respectively. Hence,

$$\lim_{\ell,k\to\infty} \int_{\Omega_T} (\gamma_2^{-1} - \gamma_1^{-1}) v_{f_\ell} \partial_t v_{g_k}^* \, dx\, dt = 0$$

and

$$\lim_{\ell,k\to\infty} \int_{\Omega_T} (Q_{\gamma_1} - Q_{\gamma_2}) v_{g_k}^* v_{f_\ell} \, dx\, dt = \int_{\Omega_T} (Q_{\gamma_1} - Q_{\gamma_2}) \psi\phi \, dx\, dt.$$

Using that $\phi\psi = \phi$, we deduce

$$\int_{\Omega_T} (Q_{\gamma_1} - Q_{\gamma_2}) \phi \, dx\, dt = 0,$$

for any possible $\phi \in C_c^\infty(\Omega_T)$. Thus, one can conclude that $Q_{\gamma_1} = Q_{\gamma_2}$ in Ω_T.

Step 2. $\gamma_1 = \gamma_2$ *in* Ω_T.

Plugging $Q_{\gamma_1} = Q_{\gamma_2}$ in Ω_T into (7.6), we have

$$\int_{\Omega_T} (\gamma_2^{-1} - \gamma_1^{-1}) v_f \partial_t v_g^* \, dx\, dt = 0,$$

for any $f \in C_c^\infty((W_1)_T)$ and $g \in C_c^\infty((W_2)_T)$. Take $\Omega' \Subset \Omega$, $\eta \in C_c^\infty(\Omega)$ with $\eta|_{\overline{\Omega'}} = 1$ and set $\psi(\cdot, t) = t\eta$. By repeating the arguments in Step 1, with the Runge approximation at hand, one can also conclude that

$$\int_{\Omega_T} (\gamma_2^{-1} - \gamma_1^{-1}) \phi \, dx\, dt = 0,$$

for any possible $\phi \in C_c^\infty(\Omega_T)$. This ensures $\gamma_1 = \gamma_2$ in Ω_T. ∎

Proof of Theorem 1.1. First, we apply Theorem 1.2 to deduce that $\gamma_1 = \gamma_2$ in W_T. Then we choose two nonempty disjoint open sets $W_1, W_2 \subset W$. By Lemma 6.6, condition (1.7) implies that the identity (7.4) holds for W_1, W_2 as chosen initially. Now, by using Theorem 7.5, we can conclude that

$$\gamma_1 = \gamma_2 \quad \text{and} \quad Q_{\gamma_1} = Q_{\gamma_2} \quad \text{in } \Omega_T.$$

This in turn implies

$$0 = Q_{\gamma_1} - Q_{\gamma_2} = (-\Delta)^s (m_{\gamma_2} - m_{\gamma_1}) \quad \text{in } \Omega,$$

for a.e. $t \in (0, T)$. Hence, by the UCP (see Theorem 2.2 in [28]), it follows that $\gamma_1 = \gamma_2$ in $\mathbb{R}^n_T$. ∎

A. Discussion of nonlocal normal derivatives and DN maps

In this section, we provide the motivation behind the definition of the nonlocal Neumann derivatives $\mathcal{N}_\gamma$ and $\mathcal{N}_{Q_\gamma}$, which underly Definitions 6.1 and 6.3. We restrict here our attention to time-independent functions for simplicity. First recall that from Lemma 3.3 in [16], one has the nonlocal integration by parts formula

$$\begin{aligned}\int_{\Omega_e} (\mathcal{N}_s u)v\,dx &+ \int_\Omega ((-\Delta)^s u)v\,dx\\ &= \frac{C_{n,s}}{2}\int_{\mathbb{R}^{2n}\setminus(\Omega_e\times\Omega_e)} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy\end{aligned}$$

for all $u, v \in C^2(\mathbb{R}^n)$, where

$$\mathcal{N}_s u(x) := C_{n,s}\int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy \tag{A.1}$$

denotes the nonlocal normal derivative for sufficiently regular functions $u\colon \mathbb{R}^n \to \mathbb{R}$ and $x \in \Omega_e$. Here $C_{n,s}$ is the same constant given by (1.5).

Next, we want to show that a similar formula holds for the fractional conductivity operator studied in this work. For simplicity, assume $u, \phi \in C_c^\infty(\mathbb{R}^n)$ and denote the duality pairing between $H^s(\mathbb{R}^n)$ and $H^{-s}(\mathbb{R}^n)$ by $\langle\cdot,\cdot\rangle$. Then we have

$$\begin{aligned}\langle \mathrm{div}_s(\Theta_\gamma \nabla^s u), \phi\rangle &= \frac{C_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\gamma^{1/2}(x)\gamma^{1/2}(y)}{|x-y|^{n+2s}}(u(x)-u(y))(\phi(x)-\phi(y))\,dx\,dy\\ &= \frac{C_{n,s}}{2}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \frac{\gamma^{1/2}(y)}{|x-y|^{n+2s}}(u(x)-u(y))\,dy\right)\gamma^{1/2}(x)\phi(x)\,dx\\ &\quad - \frac{C_{n,s}}{2}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \frac{\gamma^{1/2}(x)}{|x-y|^{n+2s}}(u(x)-u(y))\,dx\right)\gamma^{1/2}(y)\phi(y)\,dy\\ &= C_{n,s}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \frac{\gamma^{1/2}(y)}{|x-y|^{n+2s}}(u(x)-u(y))\,dy\right)\gamma^{1/2}(x)\phi(x)\,dx\\ &= \int_{\mathbb{R}^n} L_\gamma^s u(x)\phi(x)\,dx,\end{aligned}$$

where we set

$$L_\gamma^s u(x) := C_{n,s}\,\gamma^{1/2}(x)\int_{\mathbb{R}^n} \frac{\gamma^{1/2}(y)}{|x-y|^{n+2s}}\,(u(x)-u(y))\,dy. \tag{A.2}$$

Now, let us define the (general) nonlocal Neumann derivative by

$$\mathcal{N}_s^\gamma u(x) := C_{n,s}\,\gamma^{1/2}(x)\int_\Omega \gamma^{1/2}(y)\,\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy \tag{A.3}$$

for $x \in \Omega_e$ and sufficiently regular functions $u\colon \mathbb{R}^n \to \mathbb{R}$ (in this section we write the superscript γ to distinguish it from the normal derivative in (A.1)). With this definition, we have

$$\frac{C_{n,s}}{2}\int_{\mathbb{R}^{2n}\setminus(\Omega_e\times\Omega_e)}\gamma^{1/2}(x)\,\gamma^{1/2}(y)\,\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy = \int_{\Omega} L^s_\gamma u(x)\,v(x)\,dx + \int_{\Omega_e} v(x)\,\mathcal{N}^\gamma_s u(x)\,dx. \tag{A.4}$$

To see this, observe that $\mathbb{R}^{2n}\setminus(\Omega_e\times\Omega_e) = (\Omega\times\mathbb{R}^n)\cup(\Omega_e\times\Omega)$ and hence

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}\setminus(\Omega_e\times\Omega_e)}\gamma^{1/2}(x)\,\gamma^{1/2}(y)\,\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy\\
&\quad=\int_{\mathbb{R}^{2n}\setminus(\Omega_e\times\Omega_e)}\gamma^{1/2}(x)\,\gamma^{1/2}(y)\,\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,v(x)\,dx\,dy\\
&\qquad-\int_{\mathbb{R}^{2n}\setminus(\Omega_e\times\Omega_e)}\gamma^{1/2}(x)\,\gamma^{1/2}(y)\,\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,v(y)\,dx\,dy\\
&\quad=2\int_{\mathbb{R}^{2n}\setminus(\Omega_e\times\Omega_e)}\gamma^{1/2}(x)\,\gamma^{1/2}(y)\,\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,v(x)\,dx\,dy\\
&\quad=2\int_{\Omega}v(x)\Big(\int_{\mathbb{R}^n}\gamma^{1/2}(x)\,\gamma^{1/2}(y)\,\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy\Big)\,dx\\
&\qquad+2\int_{\Omega_e}v(x)\Big(\int_{\Omega}\gamma^{1/2}(x)\,\gamma^{1/2}(y)\,\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy\Big)\,dx.
\end{aligned}$$

By (A.2) and (A.3), this implies the identity (A.4). From (A.4), we make the following observations.

(i) There holds

$$\frac{C_{n,s}}{2}\int_{\mathbb{R}^{2n}\setminus(\Omega_e\times\Omega_e)}\gamma^{1/2}(x)\,\gamma^{1/2}(y)\,\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy = \int_{\Omega_e} v(x)\,\mathcal{N}^\gamma_s u(x)\,dx$$

for all $v\in C^\infty_c(\Omega_e)$, and so coincides with our weak formulation.

(ii) We have

$$\begin{aligned}
&\frac{C_{n,s}}{2}\int_{\mathbb{R}^{2n}\setminus(\Omega_e\times\Omega_e)}\gamma^{1/2}(x)\,\gamma^{1/2}(y)\,\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy\\
&\quad=\int_{\Omega}L^s_\gamma u(x)\,v(x)\,dx\\
&\quad=\frac{C_{n,s}}{2}\int_{\mathbb{R}^{2n}}\frac{\gamma^{1/2}(x)\gamma^{1/2}(y)}{|x-y|^{n+2s}}(u(x)-u(y))(v(x)-v(y))\,dx\,dy
\end{aligned}$$

for all $v\in C^\infty_c(\Omega)$.

Note that all observations above hold for a general symmetric kernel $K(x,y)$. Combining the assertions (i) and (ii), we see that:

(a) The notion of solutions in the survey article for elliptic nonlocal equations in [43] and the definitions adapted in this article are the same. The former one have the advantage that one can study solutions to nonlocal Dirichlet problems, where the exterior conditions f are less regular.

(b) We have
$$\langle \mathcal{N}_s^\gamma f, g\rangle = \langle \mathcal{N}_s^\gamma f, g'\rangle,$$
whenever $g, g' \in H^s(\mathbb{R}^n)$ satisfy $g - g' \in \widetilde{H}^s(\Omega)$, where $\mathcal{N}_s^\gamma f$ is the nonlocal normal derivative of the unique solution u_f to the homogeneous fractional conductivity equation with exterior value f. Hence, it is again well defined on the trace space $X = H^s(\mathbb{R}^n)/\widetilde{H}^s(\Omega)$.

Moreover, let us point out that in [41,42] we used the following definition of DN map:

$$\langle \Lambda_\gamma f, g\rangle = \frac{C_{n,s}}{2}\int_{\mathbb{R}^{2n}} \gamma^{1/2}(x)\gamma^{1/2}(y)\frac{(u_f(x)-u_f(y))(g(x)-g(y))}{|x-y|^{n+2s}}\,dx\,dy$$

for all $f, g \in C_c^\infty(\Omega_e)$. These two are related as follows:

$$\begin{aligned}
&\langle \Lambda_\gamma f, g\rangle \\
&\quad = \langle \mathcal{N}_s^\gamma f, g\rangle + \frac{C_{n,s}}{2}\int_{\Omega_e\times\Omega_e} \gamma^{1/2}(x)\gamma^{1/2}(y)\frac{(u_f(x)-u_f(y))(g(x)-g(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad = \langle \mathcal{N}_s^\gamma f, g\rangle + \frac{C_{n,s}}{2}\int_{\Omega_e\times\Omega_e} \gamma^{1/2}(x)\gamma^{1/2}(y)\frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{n+2s}}\,dx\,dy.
\end{aligned}$$

Remark A.1. We may now observe that the additional information on the set $\Omega_e \times \Omega_e$ precisely allows to carry out the exterior determination. This shows that Λ_γ carries more information.

As a matter of fact, the definition $\mathcal{N}_s^\gamma$ is natural since it has a clear PDE interpretation, although we cannot prove with it our exterior determination result.

Finally, we discuss the situation for constant coefficient operators like the fractional Schrödinger equation

$$\begin{cases} ((-\Delta)^s + q)u = 0 & \text{in } \Omega, \\ u = f & \text{in } \Omega_e. \end{cases} \tag{A.5}$$

In [24], the authors defined the DN map Λ_q related to this exterior value problem by

$$\langle \Lambda_q f, g\rangle = \int_{\mathbb{R}^n} (-\Delta)^{s/2}u_f(-\Delta)^{s/2}v_g\,dx + \int_\Omega q u_f v_g\,dx$$

for all $f, g \in H^s(\mathbb{R}^n)/\widetilde{H}^s(\Omega)$, where $u_f \in H^s(\mathbb{R}^n)$ is the weak solution to (A.5) and $v_g \in H^s(\mathbb{R}^n)$ an extension of g. In the special case $g \in C_c^\infty(\Omega_e)$, one has

$$\langle \Lambda_q f, g\rangle = \int_{\mathbb{R}^n} (-\Delta)^{s/2}u_f(-\Delta)^{s/2}g\,dx,$$

since we are only integrating over Ω in the potential, and so q is only implicitly contained in the definition of Λ_q. Then they showed in Lemma 3.1 of [24] that if $\Omega \Subset \mathbb{R}^n$ is smooth and $q \in C_c^\infty(\Omega)$, this DN map is simply the restriction $(-\Delta)^s u_f|_{\Omega_e}$ (as long as the data f, g are sufficiently regular), and in the case $f \in C_c^\infty(\Omega_e)$,

$$\Lambda_q f = \mathcal{N}_s f - mf + (-\Delta)^s f|_{\Omega_e},$$

where

$$m(x) = C_{n,s} \int_\Omega \frac{dy}{|x-y|^{n+2s}}$$

(cf. Lemma A.2 in [24]). But this implies in this case that

$$\Lambda_{q_1} = \Lambda_{q_2} \iff \mathcal{N}_s^1 = \mathcal{N}_s^2.$$

If q is possibly nontrivial in the exterior, then the notion of solutions to (A.5) is not affected if one introduces the related bilinear form by

$$B_q(u,v) := \int_{\mathbb{R}^n} (-\Delta)^{s/2} u \, (-\Delta)^{s/2} v \, dx + \int_{\mathbb{R}^n} quv \, dx$$

for $u, v \in H^s(\mathbb{R}^n)$. This approach was, for example, carried out in [41] and [45]. But then the natural DN map becomes

$$\langle \widetilde{\Lambda}_q f, g\rangle := \int_{\mathbb{R}^n} (-\Delta)^{s/2} u_f \, (-\Delta)^{s/2} v_g \, dx + \int_{\mathbb{R}^n} q u_f v_g \, dx$$

for all $f, g \in H^s(\mathbb{R}^n)/\widetilde{H}^s(\Omega)$, where v_g is any representative of g. These two definitions of DN maps are related as follows:

$$\langle \widetilde{\Lambda}_q f, g\rangle = \langle \Lambda_q f, g\rangle + \int_{\Omega_e} q u_f v_g \, dx = \langle \Lambda_q f, g\rangle + \int_{\Omega_e} qfg \, dx. \tag{A.6}$$

Hence, in general, if q is not zero in the exterior, these two definitions of DN maps are not equivalent and the latter helps to acquire information in the exterior. Therefore, if f and g in (A.6) have disjoint support, then they are equivalent and precisely this lack of knowledge leads to counterexamples to uniqueness (cf. [40]).

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