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Irregular fibrations of derived equivalent varieties

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Abstract. We study the behavior of irregular fibrations of a variety under derived equivalence of its bounded derived category. In particular, we prove the derived invariance of the existence of an irregular fibration over a variety of general type, extending the case of irrational pencils onto curves of genus $g \geq 2$. We also prove that a derived equivalence of such fibrations induces a derived equivalence between their general fibers.

1. Introduction

In this paper, we investigate the invariance of irregular fibrations under derived equivalence. An *irregular fibration* is a surjective morphism with connected fibers from a smooth projective variety onto a normal projective variety of positive dimension admitting a desingularization of maximal Albanese dimension. (This means that the Albanese map of this smooth model is generically finite onto its image. This property does not depend on the chosen desingularization.) Two smooth projective complex varieties X and Y are *derived equivalent* if there exists an equivalence of triangulated categories $\Phi\colon \mathbf{D}(X) \xrightarrow{\sim} \mathbf{D}(Y)$ between their bounded derived categories of coherent sheaves. The theorem below is the main result of the paper. It concerns the derived invariance of irregular fibrations $f\colon X \to V$ onto varieties of general type, i.e., such that one (and hence any) resolution of singularities of V is of general type. These fibrations can be regarded as a higher-dimensional analogue of the notion of irrational pencils over smooth curves of genus $g \geq 2$. It turns out that the mere existence of an irregular fibration imposes quite strong restrictions on the geometry of Fourier–Mukai partners.

Theorem 1.1. *Suppose that $\mathbf{D}(X) \simeq \mathbf{D}(Y)$ and that X carries an irregular fibration $f\colon X \to V$ such that V is of general type. Then:*

- (i) *Y admits an irregular fibration $h\colon Y \to W$ such that W is birational to V;*
- (ii) *The general fibers of f and h are derived equivalent;*
- (iii) *If the (anti)canonical line bundle of the general fiber of f is big, then X and Y are K-equivalent.*

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Recall that two varieties X and Y are *K-equivalent* if there exists a third smooth projective variety Z and birational morphisms

$$\begin{tikzcd} & Z \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\ X & & Y \end{tikzcd}$$

such that $p_X^* \omega_X \simeq p_Y^* \omega_Y$. An important aspect of the K-equivalence relation is that it preserves the Hodge numbers, as proved by Kontsevich's motivic integration theory.

Theorem 1.1 provides generalizations of several previously known results. First of all, item (iii) above should be seen as a relative version of Kawamata's birational reconstruction theorem [13].[1] Moreover, it generalizes Theorem 6 in [20], where the case of irregular fibrations onto curves of genus $g \geq 2$ was considered.

If one restricts to somewhat more specific irregular fibrations, one obtains a stronger result. Namely, assume that, beyond being of general type, V admits a morphism $c_V \colon V \to \operatorname{Alb} \widetilde{V}$ which is finite onto its image and such that the composition $\widetilde{V} \to V \xrightarrow{c_V} \operatorname{Alb} \widetilde{V}_X$ equals the Albanese map of a desingularization $\widetilde{V}$ (cf. Section 3.2 of [19]). Note that this is precisely what happens when $\dim V = 1$.

Proposition 1.2. *Let $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. Under the above assumption, if ω_X or ω_X^{-1} is f-ample, then X is isomorphic to Y.*

We now present the second main result of the paper, which generalizes Theorem 1 in [19]. We say that two irregular fibrations $f_1 \colon X \to V_1$ and $f_2 \colon X \to V_2$ of a variety X are *equivalent* if there exists a birational map $\sigma \colon V_1 \dashrightarrow V_2$ such that $f_2 = \sigma \circ f_1$.

Theorem 1.3. *Suppose $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. There exists a base-preserving bijection between the sets of equivalence classes of irregular fibrations of X and Y onto normal projective varieties of general type.*

We refer the reader to Theorem 4.6 for the proof of Theorem 1.3 and a more precise statement. See also Remark 4.7.

To *any* fibration $f \colon X \to V$ onto a normal projective variety V there is attached an abelian subvariety $\widehat{B_V}$ of $\operatorname{Pic}^0 X$ as follows. Let $\tilde{f} \colon \widetilde{X} \to \widetilde{V}$ be a *non-singular representative* of the fibration f, namely a commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\rho"] \arrow[d, "\tilde{f}"] & X \arrow[d, "f"] \\ \widetilde{V} \arrow[r, "\theta"] & V, \end{tikzcd} \tag{1.1}$$

where ρ and θ are birational morphisms from smooth projective varieties and $\tilde{f}$ is a fibration.

[1] Actually, by the first works of Kawamata, Kollár and Viehweg on the Iitaka conjecture, if in part (iii) of Theorem 1.1 the general fiber of f is of general type, then X itself is of general type because V_X is so (see, e.g., Theorem 1.2.9 and Problem 1.1.2 in [7]). However, we do not use this result here. Our approach is self-contained and moreover it also works *verbatim* when the anticanonical line bundle of the general fiber of f is big.

By noting that the push-forward map $\rho_*\colon \operatorname{Pic}^0 \widetilde{X} \to \operatorname{Pic}^0 X$ is an isomorphism, we define the abelian subvariety

$$\widehat{B_V} \stackrel{\text{def}}{=} \rho_* \tilde{f}^* \operatorname{Pic}^0 \widetilde{V} \subset \operatorname{Pic}^0 X.$$

It is easy to check that $\widehat{B_V}$ is well defined, i.e., it does not depend on the choice of the non-singular representative. What happens is that, if $\widetilde{V}$ is of general type and of maximal Albanese dimension as in Theorem 1.1, then $\widehat{B_V}$ is a *Rouquier-stable subvariety* with respect to any exact equivalence $\mathbf{D}(X) \simeq \mathbf{D}(Y)$ (cf. Lemma 4.3). The notion of Rouquier-stable subvarieties was introduced in [4] in order to study the derived invariance of certain relative canonical rings. Briefly, it refers to abelian subvarieties $\widehat{B_X} \subset \operatorname{Pic}^0 X$ that are mapped isomorphically via the Rouquier isomorphism (2.1) to abelian subvarieties $\widehat{B_Y}$ of $\operatorname{Pic}^0 Y$. We refer the reader to Sections 2 and 3 for the definition and main properties of Rouquier-stable subvarieties. The above fact allows us to apply the general results of Section 3 to the setting of Theorems 1.1 and 1.3. The proof of (ii) also builds on the latest relativization technique for the kernel [17].

Finally, we note that in fact we prove slightly more general results than those of Theorem 1.1, although in a little less geometric settings (see Theorems 4.1 and 4.4). For instance, we record the following particular case of Theorem 4.4.

Corollary 1.4. *Let X and Y be derived equivalent varieties. Assume that $\chi(X, \mathcal{O}_X) \neq 0$. If the (anti)canonical line bundle of the general fiber of the Albanese map of X is big, then X and Y are K-equivalent.*

In another direction, even if the base V of an irregular fibration as in (1.1) is not of general type, then in any case $\widehat{B_V}$ contains a certain Rouquier-stable subvariety (namely, Pic^0 of the base of the Iitaka fibration of V), leading to the following result extending Theorem 1.1 (i).

Theorem 1.5. *Suppose $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. If X admits an irregular fibration $f\colon X \to V$, then there exists a fibration $h\colon Y \to W$ of Y onto a normal projective variety W which is birational to the base of the Iitaka fibration of V. In particular, we have $\dim W = \operatorname{kod}(V)$ and any smooth model of W is of maximal Albanese dimension.*

This quite satisfactorily answers the problem of understanding in which manner an arbitrary irregular fibration of a given variety varies under derived equivalence of its bounded derived category. Moreover, in [25] Popa conjectured the derived invariance of non-vanishing canonical loci (see also Section 2.1.5 below) figuring out that the geometric meaning of his conjecture is that derived equivalent varieties should have the same type of fibrations over lower-dimensional irregular varieties, this allowing for more geometric tools in the study of Fourier–Mukai partners. Popa proved a version of this principle assuming his conjecture (see Corollary 3.4 in [25]). Here we notice that, although Popa's conjecture is still open, our techniques allow to get an unconditional proof of his result:

Theorem 1.6. *Suppose $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. If X admits a fibration $f\colon X \to Z$ onto a normal projective m-dimensional variety Z whose Albanese map[2] is not surjective, then Y admits an irregular fibration $h\colon Y \to W$ onto a variety of general type W, with $0 < \dim W \leq m$.*

[2]Actually the Albanese map of a desingularization of Z.

Proof. If the Albanese map of (a desingularization of) Z is not surjective, then its image admits a fiber bundle structure g onto a positive dimensional variety of maximal Albanese dimension and of general type (see Theorem 10.9 and Corollary 10.5 in [29]). By taking the Stein factorization of the composition $g \circ a_Z$, we see that Z admits a fibration g' onto a normal projective variety of general type, maximal Albanese dimension and positive dimension $\leq m$. So Theorem 1.1 (i) (and its proof) may apply to the composition $g' \circ f$. ∎

Notation. Our ground field is the field of complex numbers $\mathbb{C}$. A variety means an irreducible smooth projective variety, unless otherwise stated. A fibration is a surjective morphism of normal projective varieties with connected fibers. If X is a variety, we denote by $\mathbf{D}(X) := \mathbf{D}^b\big(\mathscr{C}oh(X)\big)$ the bounded derived category of coherent sheaves on X. The Albanese map of X is denoted by $a_X \colon X \to \operatorname{Alb} X$. The irregularity of X is $q(X) := h^{1,0}(X) = \dim \operatorname{Alb} X$. If we fix a Poincaré line bundle $\mathscr{P}$ on $X \times \operatorname{Pic}^0 X$, we denote by $P_\alpha := \mathscr{P}|_{X \times \{\alpha\}}$ the line bundle parameterized by the point $\alpha \in \operatorname{Pic}^0 X$. Given a morphism of abelian varieties $\varphi \colon A \to B$, we denote by $\widehat{\varphi} \colon \widehat{B} \to \widehat{A}$ the dual morphism of φ.

2. Rouquier-stable subvarieties

Let X and Y be smooth projective complex varieties and let $\Phi \colon \mathbf{D}(X) \xrightarrow{\sim} \mathbf{D}(Y)$ be an exact equivalence between the derived categories of X and Y. The equivalence Φ induces a functorial isomorphism of algebraic groups

$$\varphi : \operatorname{Aut}^0 X \times \operatorname{Pic}^0 X \xrightarrow{\sim} \operatorname{Aut}^0 Y \times \operatorname{Pic}^0 Y \tag{2.1}$$

known as the *Rouquier isomorphism* [27]. This isomorphism has been employed by Orlov in [24] in order to classify derived equivalences of abelian varieties. Moreover, it plays a crucial role in Popa–Schnell's proof of the derived invariance of the irregularity [26]. Other applications are contained in [18], [20], [5] and [17].

The main difficulty in dealing with the Rouquier isomorphism is that, in general, it does not respect the factors. Namely, there exist equivalences such that

$$\varphi\big(\{\mathrm{id}_X\} \times \operatorname{Pic}^0 X\big) \neq \{\mathrm{id}_Y\} \times \operatorname{Pic}^0 Y.$$

For instance, this is the case of the Fourier–Mukai–Poincaré transform between an abelian variety and its dual. Quite naturally, one is led to consider *Rouquier-stable subvarieties* as introduced in [4].

Definition 2.1. An abelian subvariety $\widehat{B_X} \subset \operatorname{Pic}^0 X$ is *Rouquier-stable* (with respect to the equivalence Φ), if the induced Rouquier isomorphism (2.1) satisfies

$$\varphi\big(\{\mathrm{id}_X\} \times \widehat{B_X}\big) \subseteq \{\mathrm{id}_Y\} \times \operatorname{Pic}^0 Y.$$

We denote by $\widehat{B_Y}$ the abelian subvariety

$$\widehat{B_Y} := p_{\operatorname{Pic}^0 Y}\big(\varphi\big(\{\mathrm{id}_X\} \times \widehat{B_X}\big)\big)$$

of $\operatorname{Pic}^0 Y$, where $p_{\operatorname{Pic}^0 Y} \colon \operatorname{Aut}^0 Y \times \operatorname{Pic}^0 Y \to \operatorname{Pic}^0 Y$ is the projection onto the second factor. By a slight abuse of notation, we simply write $\widehat{B_Y} = \varphi(\widehat{B_X})$.

We refer the reader to Section 2.1 for several examples of Rouquier-stable subvarieties.

A Rouquier-stable subvariety $\widehat{B_X} \subseteq \operatorname{Pic}^0 X$ induces two morphisms. The first one, $b_X\colon X \to B_X$, is given as the composition

$$\begin{tikzcd} X \arrow[r, "a_X"] \arrow[rr, bend left, "b_X"] & \operatorname{Alb} X \arrow[r, "\pi_{B_X}"] & B_X \end{tikzcd}$$

of the Albanese map $a_X\colon X \to \operatorname{Alb} X$ with the dual morphism π_{B_X} of the inclusion $\widehat{B_X} \subseteq \operatorname{Pic}^0 X$. The second, $b_Y\colon Y \to B_Y$, is defined similarly as the composition

$$\begin{tikzcd} Y \arrow[r, "a_Y"] \arrow[rr, bend left, "b_Y"] & \operatorname{Alb} Y \arrow[r, "\pi_{B_Y}"] & B_Y. \end{tikzcd}$$

We refer to the morphisms $b_X\colon X \to B_X$ and $b_Y\colon Y \to B_Y$ as a (*pair of*) *Rouquier-stable morphisms*. By taking the Stein factorization, we have commutative diagrams

$$\begin{tikzcd} X \arrow[r, "a_X"] \arrow[d, "s_X"'] \arrow[dr, "b_X"] & \operatorname{Alb} X \arrow[d, "\pi_{B_X}"] \\ X' \arrow[r, "b'_X"] & B_X, \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "a_Y"] \arrow[d, "s_Y"'] \arrow[dr, "b_Y"] & \operatorname{Alb} Y \arrow[d, "\pi_{B_Y}"] \\ Y' \arrow[r, "b'_Y"] & B_Y, \end{tikzcd}$$

where $s_X\colon X \to X'$ and $s_Y\colon Y \to Y'$ are fibrations onto normal projective varieties and $b'_X\colon X' \to B_X$ and $b'_Y\colon Y' \to B_Y$ are finite morphism onto their images. A result of [4] shows that the finite components of these Stein factorizations are isomorphic. More precisely, there exists an isomorphism $\psi\colon Y' \xrightarrow{\sim} X'$ such that the diagram

$$\begin{tikzcd} X' \arrow[d, "b'_X"] & Y' \arrow[l, "\psi"'] \arrow[d, "b'_Y"] \\ B_X & B_Y \arrow[l, "\widehat{\varphi}"'] \end{tikzcd}$$

is commutative, where $\widehat{\varphi}$ is the dual isomorphism. In Section 3 we will recall this fact and, building on [17], we will show that the general fibers of s_X and s_Y are derived equivalent.

2.1. Examples

In this subsection, we present a few examples of Rouquier-stable subvarieties. Let us fix an equivalence $\Phi\colon \mathbf{D}(X) \to \mathbf{D}(Y)$ of triangulated categories and let $\varphi\colon \operatorname{Aut}^0 X \times \operatorname{Pic}^0 X \to \operatorname{Aut}^0 Y \times \operatorname{Pic}^0 Y$ be the induced Rouquier isomorphism. We point out that, aside from Section 2.1.4, all the examples presented below are intrinsically Rouquier-stable, i.e., they are stable with respect to any equivalence.

2.1.1. The trivial example. The subset $\{\hat{0}\} \subset \operatorname{Pic}^0 X$ is Rouquier-stable. The induced pair of Rouquier-stable morphisms are the constant maps $X \to \{0\}$ and $Y \to \{0\}$.

2.1.2. A numerical condition. Let $a(X) := \dim \mathrm{Alb}(\mathrm{Aut}^0 X)$. If $q(X) > a(X)$, then there exists a Rouquier-stable subvariety of positive dimension of $\mathrm{Pic}^0 X$. For a proof of this fact we adopt the terminology of [26], pp. 532–533. Set $G_X = \mathrm{Aut}^0 X$. Then

$$\dim(\ker(\pi)_0) = \dim\big(\ker\big(\mathrm{Pic}^0 X \to \widehat{A}\big)_0\big) = q(X) - \dim \widehat{A} \geq q(X) - a(X) > 0,$$

so that $\ker(\pi)_0$ is an abelian variety of positive dimension, which must be Rouquier-stable.

2.1.3. Affine automorphism group. This is a special case of the above (2.1.2). Following [26], p. 534, $\mathrm{Aut}^0 X$ is an affine algebraic group if and only if $\mathrm{Aut}^0 Y$ is so. In this case, since $\mathrm{Pic}^0 X$ is projective and the irregularity is a derived invariant (cf. Corollary B in [26]), one has

$$\varphi\big(\{\mathrm{id}_X\} \times \mathrm{Pic}^0 X\big) = \{\mathrm{id}_Y\} \times \mathrm{Pic}^0 Y, \tag{2.2}$$

so that $\mathrm{Pic}^0 X$ is Rouquier-stable.[3] The induced pair of Rouquier-stable morphisms are the Albanese maps a_X and a_Y themselves. Instances of varieties with affine automorphism group $\mathrm{Aut}^0(-)$ are varieties with non-vanishing Euler characteristic $\chi(X, \mathcal{O}_X) \neq 0$ (see Corollary 2.6 in [26]), and varieties with big (anti)canonical line bundle (see, e.g., Proposition 2.26 in [3] for a detailed proof of this folklore result. Besides, the automorphism group $\mathrm{Aut}(-)$ of a variety of general type is finite by a classical result of Matsumura, see Corollary 2 in [21]).

2.1.4. Strongly filtered equivalences. In the paper [16], the authors introduce a notion of equivalence called *strongly filtered*.[4] For this type of equivalence, the formula (2.2) continues to hold. As suggested in [26] and [16], the level of mixedness of the Rouquier isomorphism could be interpreted as a measure of the complexity of a derived equivalence from the point of view of birational geometry. For instance, in [16] it is proved that a strongly filtered equivalence of smooth projective threefolds with positive irregularity induces a birational isomorphism.

2.1.5. Cohomological support loci. Given a coherent sheaf $\mathcal{F}$ on a variety X, the *cohomological support loci attached to $\mathcal{F}$* are the algebraic closed subsets

$$V^i(X, \mathcal{F}) = \{\alpha \in \mathrm{Pic}^0 X \mid H^i(X, \mathcal{F} \otimes P_\alpha) \neq 0\}.$$

Let us denote by $V^i(X, \mathcal{F})_0$ the union of the irreducible components of $V^i(X, \mathcal{F})$ passing through the origin. By Claim 3.3 in [18], one has

$$\varphi\big(\{\mathrm{id}_X\} \times V^i(X, \Omega_X^j \otimes \omega_X^{\otimes m})_0\big) \subseteq \{\mathrm{id}_Y\} \times \mathrm{Pic}^0 Y \tag{2.3}$$

for all $i, j \geq 0$ and all $m \in \mathbb{Z}$. In particular, any abelian subvariety of $\mathrm{Pic}^0 X$ that is contained in some $V^i(X, \Omega_X^j \otimes \omega_X^{\otimes m})_0$ is Rouquier-stable. Moreover, since φ is an isomorphism of algebraic groups, the abelian subvarieties of $\mathrm{Pic}^0 X$ generated by the

[3] Proof: Since $\mathrm{Aut}^0 Y$ is affine, the composition $\{\mathrm{id}_X\} \times \mathrm{Pic}^0 X \xrightarrow{\varphi} \mathrm{Aut}^0 Y \times \mathrm{Pic}^0 Y \to \mathrm{Aut}^0 Y$ is constant.

[4] An equivalence is *strongly filtered* if it preserves the codimension filtration on the numerical Chow ring, together with the Hochschild–Kostant–Rosenberg filtrations on the Hochschild homology and cohomology.

loci $V^i(X, \Omega_X^j \otimes \omega_X^{\otimes m})_0$ (or by any of their irreducible components) are Rouquier-stable. It is also known by Proposition 3.1 in [18] that the Rouquier isomorphism preserves the full loci $V^0(X, \omega_X^{\otimes m})$, namely

$$\varphi\big(\{\mathrm{id}_X\} \times V^0(X, \omega_X^{\otimes m})\big) = \{\mathrm{id}_Y\} \times V^0(Y, \omega_Y^{\otimes m}), \quad \forall m \in \mathbb{Z}\,. \tag{2.4}$$

Note that the same behavior is expected to hold for the loci $V^i(X, \omega_X)_0$ for any i, see Conjecture 11 in [20].

2.1.6. The Albanese–Iitaka morphism. Suppose that the Kodaira dimension of X is non-negative. We can choose a smooth birational modification $\widetilde{X} \to X$ such that the Iitaka fibration is represented by a morphism $f_{\widetilde{X}}\colon \widetilde{X} \to Z_X$, with Z_X smooth. More concretely, we have a commutative diagram

$$\begin{tikzcd}
\widetilde{X} \arrow[r] \arrow[rr, bend left, "a_{\widetilde{X}}"] \arrow[dr, "f_{\widetilde{X}}"'] & X \arrow[r, "a_X"] \arrow[d, dashed] \arrow[dr, "c_X"] & \mathrm{Alb}\, X \arrow[d, "\pi_X"] \\
 & Z_X \arrow[r, "a_{Z_X}"] & \mathrm{Alb}\, Z_X,
\end{tikzcd}$$

where a_X, $a_{\widetilde{X}}$ and a_{Z_X} are Albanese maps, and π_X is a fibration of abelian varieties induced by $f_{\widetilde{X}}$ (cf. Lemma 11.1 (a) in [9]). Note that $\mathrm{Alb}\, Z_X$, π_X and $c_X = \pi_X \circ a_X$ only depend on X, and not on the modification $\widetilde{X}$ we fixed. In [4], the morphism c_X is called the *Albanese–Iitaka morphism* of X. It follows from (2.4) that the Rouquier isomorphism acts as

$$\varphi\big(\{\mathrm{id}_X\} \times \widehat{\pi}_X(\mathrm{Pic}^0 Z_X)\big) = \{\mathrm{id}_Y\} \times \widehat{\pi}_Y(\mathrm{Pic}^0 Z_Y)$$

(see the proof of Lemma 3.4 in [5]). In particular, $\widehat{\pi}_X(\mathrm{Pic}^0 Z_X)$ is a Rouquier-stable subvariety and the Albanese–Iitaka morphisms c_X and c_Y are Rouquier-stable. This fact was already noted (and used) in [5], and moreover, it is particularly useful in [4].

2.1.7. Fibrations over varieties of general type. Let $f\colon X \to V$ be an irregular fibration onto a normal projective variety of general type. By keeping notation as in (1.1), we will show in Lemma 4.3 that the abelian subvariety $\rho_* \widetilde{f}^* \mathrm{Pic}^0 \widetilde{V} \subset \mathrm{Pic}^0 X$ is Rouquier-stable. In particular, the abelian subvarieties attached to χ-positive fibrations considered in [19] are Rouquier-stable (cf. Remark 14 in [19]).

3. The Stein factorization of a Rouquier-stable morphism

In this section, we study the effects of the existence of a non-trivial Rouquier-stable subvariety. Informally speaking, one such subvariety turns a derived equivalence into a relative equivalence, at least generically.

Let $\Phi\colon \mathbf{D}(X) \to \mathbf{D}(Y)$ be an equivalence of triangulated categories. Let $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ be the natural projections onto the first and second factor, respectively.

By Orlov's representability theorem (Theorem 2.2 in [23]), there exists an object $\mathcal{E}$ in $\mathbf{D}(X \times Y)$ that is unique up to isomorphism and such that

$$\Phi(-) \simeq \Phi_{\mathcal{E}}(-) := \mathbf{R}q_*\big(p^*(-) \overset{\mathbf{L}}{\otimes} \mathcal{E}\big).$$

We denote by $\varphi_{\mathcal{E}}$ the Rouquier isomorphism induced by $\Phi_{\mathcal{E}}$.

Theorem 3.1. *Let $\Phi_{\mathcal{E}}\colon \mathbf{D}(X) \to \mathbf{D}(Y)$ be an equivalence and let $\widehat{B_X} \subset \operatorname{Pic}^0 X$ be a Rouquier-stable subvariety. Moreover, let $b_X\colon X \to B_X$ and $b_Y\colon Y \to B_Y$ be the induced pair of Rouquier-stable morphisms. By considering the Stein factorizations of b_X and b_Y, we have the following commutative diagrams*:

$$\begin{tikzcd} X \arrow[r, "a_X"] \arrow[d, "s_X"'] \arrow[dr, "b_X"] & \operatorname{Alb} X \arrow[d, "\pi_{B_X}"] \\ X' \arrow[r, "b'_X"] & B_X, \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "a_Y"] \arrow[d, "s_Y"'] \arrow[dr, "b_Y"] & \operatorname{Alb} Y \arrow[d, "\pi_{B_Y}"] \\ Y' \arrow[r, "b'_Y"] & B_Y, \end{tikzcd} \tag{3.1}$$

where s_X and s_Y are surjective morphisms with connected fibers, and b'_X and b'_Y are finite morphisms onto their images. Then:

(i) *There exists an isomorphism $\psi\colon Y' \xrightarrow{\sim} X'$ of normal projective varieties such that the kernel $\mathcal{E}$ is set-theoretically supported on the fiber product $X \times_{Y'} Y$ defined as follows*:

$$\begin{tikzcd} X \times_{Y'} Y \arrow[r] \arrow[d] & X \arrow[d, "\psi^{-1}\circ s_X"] \\ Y \arrow[r, "s_Y"] & Y'. \end{tikzcd}$$

(ii) *The finite parts of the Stein factorizations are isomorphic, i.e., the following diagram commutes*:

$$\begin{tikzcd} X' \arrow[d, "b'_X"] & Y' \arrow[l, "\sim"', "\psi"] \arrow[d, "b'_Y"] \\ B_X & B_Y. \arrow[l, "\sim"', "\widehat{\varphi_{\mathcal{E}}}"] \end{tikzcd}$$

(iii) *The fibers $s_X^{-1}(\psi(y'))$ and $s_Y^{-1}(y')$ are derived equivalent for y' general in Y'.*

Proof. The proofs of (i) and (ii) are given in [4] (see Section 8.1 of *loc. cit.* and, especially, Theorem 8.1.1) and follow the general strategy of Theorem 1 in [19]. We just recall the main points of the proof of (i) for the reader's convenience. Denote by

$$p'\colon X' \times Y' \to X' \quad \text{and} \quad q'\colon X' \times Y' \to Y'$$

the natural projections. Let

$$\operatorname{Supp}(\mathcal{E}) := \bigcup_j \operatorname{Supp}(\mathcal{H}^j(\mathcal{E})) \subseteq X \times Y$$

be the support of $\mathcal{E}$, equipped with the reduced scheme structure. Then the projections $p'\colon (s_X \times s_Y)(\mathrm{Supp}(\mathcal{E})) \to X'$ and $q'\colon (s_X \times s_Y)(\mathrm{Supp}(\mathcal{E})) \to Y'$ have finite fibers. Moreover, they are surjective with connected fibers. In other words, $(s_X \times s_Y)(\mathrm{Supp}(\mathcal{E}))$ dominates isomorphically both X' and Y'. Hence the map

$$\psi := (p' \circ q'^{-1}) : Y' \to X'$$

is an isomorphism and $(s_X \times s_Y)(\mathrm{Supp}(\mathcal{E})) = \mathrm{Graph}(\psi)$. In particular, we have

$$\mathrm{Supp}(\mathcal{E}) \subset (s_X \times s_Y)^{-1}(\mathrm{Graph}(\psi)) = X \times_{Y'} Y.$$

In order to prove (iii), we denote by $\tau\colon U \hookrightarrow Y'$ a smooth open subvariety over which both s_Y and $\psi^{-1} \circ s_X$ are smooth morphisms, and define the preimages

$$X_U := (\psi^{-1} \circ s_X)^{-1}(U) \quad \text{and} \quad Y_U := s_Y^{-1}(U).$$

By a slight abuse of notation, we continue to denote by $\psi^{-1} \circ s_X$ and s_Y the two restrictions $(\psi^{-1} \circ s_X)|_{X_U}\colon X_U \to U$ and $s_Y|_{Y_U}\colon Y_U \to U$, respectively. Moreover, we consider the closed subscheme

$$Z := X_U \times_U Y_U \overset{\ell}{\hookrightarrow} X_U \times Y_U. \tag{3.2}$$

As $\mathcal{E}$ is set-theoretically supported on $X \times_{Y'} Y$, the derived restriction $k^* \mathcal{E}$ is set-theoretically supported on Z, where $k\colon X_U \times Y_U \hookrightarrow X \times Y$ is the inclusion map. Denote by

$$i\colon X_U \times Y \hookrightarrow X \times Y \quad \text{and} \quad j\colon X_U \times Y_U \hookrightarrow X_U \times Y$$

the open immersions so that $k = i \circ j$.

Claim 3.2. *The kernel $\mathcal{E}_U := k^* \mathcal{E} \in \mathbf{D}(X_U \times Y_U)$ defines an equivalence of bounded derived categories*

$$\Phi_{\mathcal{E}_U} : \mathbf{D}(X_U) \to \mathbf{D}(Y_U)$$

(*the functor is well defined as the support of $\mathcal{E}_U$ is proper over both X_U and Y_U*).

Proof. Let $n = \dim X$. The claim is proved in Section 3.18 of [17]. For the reader's ease, we reproduce here the argument. Denote by $\mathrm{ad}(\mathcal{E})$ the adjoint kernel:

$$\mathrm{ad}(\mathcal{E}) := \mathcal{E}^{\overset{\mathbf{R}}{\vee}} \otimes p^*\omega_X[n] \simeq \mathcal{E}^{\overset{\mathbf{R}}{\vee}} \otimes q^*\omega_Y[n]$$

in $\mathbf{D}(X \times Y)$ (the superscript $\overset{\mathbf{R}}{\vee}$ denotes the derived dual). By considering the Fourier–Mukai transform in the other direction,

$$\Psi_{\mathrm{ad}(\mathcal{E})}(-) := \mathbf{R}\, p_*(q^*(-) \overset{\mathbf{L}}{\otimes} \mathrm{ad}(\mathcal{E}))\colon \mathbf{D}(Y) \to \mathbf{D}(X),$$

one can check that $\Psi_{\mathrm{ad}(\mathcal{E})}$ is a quasi-inverse of $\Phi_{\mathcal{E}}$. By denoting by p_{ij} the projections from $X \times Y \times X$ onto the i-th and j-th factors, it follows

$$\mathbf{R}\, p_{13*}(p_{12}^* \mathcal{E} \overset{\mathbf{L}}{\otimes} p_{32}^*\mathrm{ad}(\mathcal{E})) \simeq \delta_{X*}\, \mathcal{O}_X,$$

where $\delta_X \colon X \hookrightarrow X \times X$ is the diagonal embedding. Let a_{ij} be the projection from $X_U \times Y_U \times X_U$ onto the i-th and j-th factors and set $\mathrm{ad}(\mathcal{E})_U := k^*\mathrm{ad}(\mathcal{E})$. Moreover, let $\delta_{X_U} \colon X_U \hookrightarrow X_U \times X_U$ be the diagonal embedding of X_U. By pulling-back the above isomorphism under the open immersion $r \colon X_U \times X_U \to X \times X$, and by noting that $i^* \mathcal{E}$ is supported on $X_U \times Y_U$ so that $i^* \mathcal{E} \simeq \mathbf{R} j_* \mathcal{E}_U$ and $i^*\mathrm{ad}(\mathcal{E}) \simeq \mathbf{R} j_* \mathrm{ad}(\mathcal{E})_U$ (see (1.4.3.4) on p. 45 of [1] or the proof of Lemma 36.6.2 in [28]), we have the isomorphism

$$\mathbf{R} a_{13*}(a_{12}^* \mathcal{E}_U \overset{\mathbf{L}}{\otimes} a_{32}^* \mathrm{ad}(\mathcal{E})_U) \simeq \delta_{X_U *} \mathcal{O}_{X_U} .$$

Similarly, we can prove

$$\mathbf{R} a_{13*}(a_{12}^* \mathrm{ad}(\mathcal{E})_U \overset{\mathbf{L}}{\otimes} a_{32}^* \mathcal{E}_U) \simeq \delta_{X_U *} \mathcal{O}_{X_U}$$

and that $\Phi_{\mathcal{E}_U}$ is an equivalence. ∎

Claim 3.3. *The restricted kernel $\mathcal{E}_U$ is isomorphic to a pushforward $\ell_* \mathcal{C}$ for some object $\mathcal{C}$ in $\mathbf{D}(Z)$, where ℓ is defined in* (3.2)*.*

Assuming the above Claim 3.3 for a moment, we conclude the proof as follows. From the isomorphism $\mathcal{E}_U \simeq \ell_* \mathcal{C}$ we have that

$$\Phi_{\mathcal{E}_U} = \Phi_{\mathcal{C}} : \mathbf{D}(X_U) \to \mathbf{D}(Y_U)$$

is a *relative* integral functor. As showed in Propositions 2.15 and 2.10 of [11], the derived restriction

$$\mathcal{C}_u := \mathcal{C}|_{(\psi^{-1} \circ s_X)^{-1}(u) \times s_Y^{-1}(u)}$$

induces a derived equivalence

$$\Phi_{\mathcal{C}_u} : \mathbf{D}(s_X^{-1}(\psi(u))) \to \mathbf{D}(s_Y^{-1}(u))$$

for any closed point $u \in U$ if $\Phi_{\mathcal{E}_U}$ is an equivalence. Therefore, we get (iii). ∎

Remark 3.4. The equivalence $\Phi_{\mathcal{E}_U} \colon \mathbf{D}(X_U) \to \mathbf{D}(Y_U)$ is U*-linear* in the sense that for all $\mathcal{F}$ in $\mathbf{D}(X_U)$ and $\mathcal{G}$ in $\mathbf{D}(U)$, there are bifunctorial isomorphisms

$$\Phi_{\mathcal{E}_U}\big(\mathcal{F} \overset{\mathbf{L}}{\otimes} (\psi^{-1} \circ s_X)^* \mathcal{G}\big) \simeq \Phi_{\mathcal{E}_U}(\mathcal{F}) \overset{\mathbf{L}}{\otimes} s_Y^* \mathcal{G}$$

(cf. Lemma 2.33 in [15]).

Proof of Claim 3.3. This is an application of the criterion Theorem 1.1 in [17]. More precisely, we need to verify the conditions (3.10) below, in order to apply Theorem 1.1 in [17] and hence to get our result. In what follows, we argue similarly to the proof of Lemma 4.11 in [17]. Recall the commutative diagram

$$\begin{tikzcd}
& X \arrow[dl, "b_X"'] \arrow[d, "s_X"] & Y \arrow[d, "s_Y"'] \arrow[dr, "b_Y"] & \\
B_X & X' \arrow[l, "b'_X"'] & Y' \arrow[l, "\psi"'] \arrow[r, "b'_Y"] & B_Y . \arrow[lll, bend left, "\widehat{\varphi}_{\mathcal{E}}"]
\end{tikzcd}$$

Let us define the morphisms

$$u_X\colon Y' \to B_X \times Y', \quad p \mapsto (b'_X(\psi(p)), p)$$

and

$$u_Y\colon Y' \to B_Y \times Y', \quad p \mapsto (b'_Y(p), p) = ((\widehat{\varphi_{\mathcal{E}}})^{-1}(b'_X(\psi(p))), p).$$

Lemma 3.5. *One has*

$$p_{12}^*(b_X \times \mathrm{id}_{Y'})^*(u_{X*}\mathcal{O}_{Y'}) \otimes p_{13}^*\mathcal{E} \simeq p_{32}^*(b_Y \times \mathrm{id}_{Y'})^*(u_{Y*}\mathcal{O}_{Y'}) \otimes p_{13}^*\mathcal{E} \tag{3.3}$$

in $\mathbf{D}(X \times Y' \times Y)$*, where we dropped the derived notation* $\mathbf{R}$ *and* $\mathbf{L}$ *for simplicity.*

Proof. The isomorphism $(\widehat{\varphi_{\mathcal{E}}})^{-1} \times \varphi_{\mathcal{E}}\colon B_X \times \widehat{B_X} \to B_Y \times \widehat{B_Y}$ preserves Poincaré line bundles, that is, $((\widehat{\varphi_{\mathcal{E}}})^{-1} \times \varphi_{\mathcal{E}})^*\mathcal{P} \simeq \mathcal{Q}$ where $\mathcal{P}$ and $\mathcal{Q}$ are normalized Poincaré line bundles on $B_Y \times \widehat{B_Y}$ and $B_X \times \widehat{B_X}$, respectively. By Theorem 1.1 in [22], we have an equivalence of derived categories

$$\mathbf{D}(\widehat{B_X} \times Y') \xrightarrow{\simeq} \mathbf{D}(B_X \times Y'), \quad \mathcal{G} \mapsto p_{2*}(p_1^*\mathcal{G} \otimes \overline{p}_{12}^*\mathcal{Q}),$$

where

$$\overline{p}_{12} : B_X \times \widehat{B_X} \times Y' \simeq (\widehat{B_X} \times Y') \times_{Y'} (B_X \times Y') \to B_X \times \widehat{B_X}.$$

Moreover, the following diagram is commutative:

$$\begin{tikzcd}
\mathbf{D}(\widehat{B_X} \times Y') \arrow[r, "\simeq"] \arrow[d, "(\varphi_{\mathcal{E}} \times \mathrm{id})_*"] & \mathbf{D}(B_X \times Y') \arrow[d, "((\widehat{\varphi_{\mathcal{E}}})^{-1} \times \mathrm{id})_*"] \\
\mathbf{D}(\widehat{B_Y} \times Y') \arrow[r, "\simeq"] & \mathbf{D}(B_Y \times Y'),
\end{tikzcd} \tag{3.4}$$

where the bottom equivalence is similarly defined, using $\mathcal{P}$ instead of $\mathcal{Q}$. In particular, there exists a unique object $\mathcal{G} \in \mathbf{D}(\widehat{B_X} \times Y')$ such that

$$u_{X*}\mathcal{O}_{Y'} \simeq p_{2*}(p_1^*\mathcal{G} \otimes \overline{p}_{12}^*\mathcal{Q}). \tag{3.5}$$

Let us consider the commutative diagram:

$$\begin{tikzcd}
X \times \widehat{B_X} \times Y' \times Y \arrow[r, "p_{134}"] \arrow[d, "p_{123}"] \arrow[rr, bend left, "p_{14}"] & X \times Y' \times Y \arrow[r] \arrow[d] & X \times Y \\
X \times \widehat{B_X} \times Y' \arrow[r] \arrow[d, "b_X \times \mathrm{id}_{\widehat{B_X} \times Y'}"] & X \times Y' \arrow[d, "b_X \times \mathrm{id}_{Y'}"] & \\
B_X \times \widehat{B_X} \times Y' = (\widehat{B_X} \times Y') \times_{Y'} (B_X \times Y') \arrow[r, "p_2"] \arrow[d, "p_1"] & B_X \times Y' & \\
\widehat{B_X} \times Y' & &
\end{tikzcd}$$

When needed, we identify $X \times \widehat{B_X} \times Y' \times Y$ with $X \times \widehat{B_Y} \times Y' \times Y$ via the isomorphism $\mathrm{id}_X \times \varphi_{\mathcal{E}} \times \mathrm{id}_{Y'} \times \mathrm{id}_Y$ in order to lighten notation. From this and (3.5), by using flat base change and the projection formula, one obtains

$$\begin{aligned} p_{12}^*(b_X \times \mathrm{id}_{Y'})^*(u_{X*}\mathcal{O}_{Y'}) \otimes p_{13}^*\mathcal{E} &\simeq p_{12}^*(b_X \times \mathrm{id}_{Y'})^*(p_{2*}(p_1^*\mathcal{G} \otimes \overline{p}_{12}^*\mathcal{Q})) \otimes p_{13}^*\mathcal{E} \\ &\simeq p_{134*}(p_{123}^*(b_X \times \mathrm{id}_{\widehat{B_X}\times Y'})^*(p_1^*\mathcal{G} \otimes \overline{p}_{12}^*\mathcal{Q})) \otimes p_{13}^*\mathcal{E} \\ &\simeq p_{134*}(p_{23}^*\mathcal{G} \otimes p_{12}^*\mathcal{Q}_X \otimes p_{14}^*\mathcal{E}), \end{aligned}$$

where

$$\mathcal{Q}_X := (b_X \times \mathrm{id}_{\widehat{B_X}})^*\mathcal{Q}$$

and in the last equality we also used the commutative diagram

$$\begin{tikzcd} X \times \widehat{B_X} \times Y' \times Y \arrow[r, "p_{123}"] \arrow[dr, "p_{12}"] & X \times \widehat{B_X} \times Y' \arrow[r, "b_X \times \mathrm{id}_{\widehat{B_X}\times Y'}"] \arrow[d] & B_X \times \widehat{B_X} \times Y' \arrow[d, "\overline{p}_{12}"] \\ & X \times \widehat{B_X} \arrow[r, "b_X \times \mathrm{id}_{\widehat{B_X}}"] & B_X \times \widehat{B_X}. \end{tikzcd}$$

By Lemma 4.10 in [17],[5] one has that

$$p_{12}^*\mathcal{Q}_X \otimes p_{14}^*\mathcal{E} \simeq (\mathrm{id}_X \times \varphi_{\mathcal{E}} \times \mathrm{id}_{Y'\times Y})^*(p_{42}^*\mathcal{P}_Y \otimes p_{14}^*\mathcal{E}), \tag{3.6}$$

where $\mathcal{P}_Y := (b_Y \times \mathrm{id}_{\widehat{B_Y}})^*\mathcal{P}$. Therefore, we get

$$\begin{aligned} &p_{134*}(p_{23}^*\mathcal{G} \otimes p_{12}^*\mathcal{Q}_X \otimes p_{14}^*\mathcal{E}) \\ &\quad\simeq p_{134*}((\mathrm{id}_X \times \varphi_{\mathcal{E}} \times \mathrm{id}_{Y'\times Y})^*(p_{23}^*(\varphi_{\mathcal{E}} \times \mathrm{id}_{Y'})_*\mathcal{G} \otimes p_{42}^*\mathcal{P}_Y \otimes p_{14}^*\mathcal{E})) \end{aligned} \tag{3.7}$$

and, by arguing as before (in the reverse order) and using the commutativity of (3.4), we see that the right-hand side in (3.7) is isomorphic to $p_{32}^*(b_Y \times \mathrm{id}_{Y'})^*(u_{Y*}\mathcal{O}_{Y'}) \otimes p_{13}^*\mathcal{E}$, as desired. ∎

Let

$$\delta_0, \delta_1 : X_U \times Y_U \to X_U \times U \times Y_U$$

be defined as $\delta_0((x,y)) = (x, \psi^{-1}(s_X(x)), y)$ and $\delta_1((x,y)) = (x, s_Y(y), y)$. Note that we may (and do) assume that $U = \psi^{-1}(b'_X)^{-1}(V)$, where $V \subseteq B_X$ is an open subscheme such that b_X is flat over it. Consider the commutative diagram

$$\begin{tikzcd} & X \times Y' \times Y \arrow[r, "p_{12}"] & X \times Y' \arrow[r, "b_X \times \mathrm{id}_{Y'}"] & B_X \times Y' \\ X_U \times U \times Y_U \arrow[r, hook] \arrow[rrr, bend left, "\alpha_0"] & X_U \times U \times Y \arrow[u, hook] \arrow[rr] & & V \times U. \arrow[u, hook] \end{tikzcd}$$

[5] In loc. cit., the authors assume $\widehat{B_X} = \mathrm{Pic}^0 X$. However, their proof works in our more general situation as well (see also Lemma 3.1 in [26]).

Denote by q_{ij} the projections from $X_U \times U \times Y_U$ onto the i-th and j-th factors, and by p_1 and p_2 the two projections from $X_U \times Y_U$. Then, the restriction of the left-hand side in (3.3) to $X_U \times U \times Y_U$ is isomorphic to

$$\alpha_0^*((\mathbf{R}\, u_{X*}\, \mathcal{O}_{Y'})|_{V\times U}) \overset{\mathbf{L}}{\otimes} q_{13}^*\, \mathcal{E}_U \simeq \alpha_0^*\, \mathbf{R}\, u_{X,U*}\, \mathcal{O}_U \overset{\mathbf{L}}{\otimes} q_{13}^*\, \mathcal{E}_U,$$

where $u_{X,U}\colon U \to V \times U$ is the restriction of u_X. Consider the cartesian diagram

$$\begin{tikzcd} X_U \times Y_U \arrow[r, "(\psi^{-1}\circ s_X)\circ p_1"] \arrow[d, "\delta_0"] & U \arrow[d, "u_{X,U}"] \\ X_U \times U \times Y_U \arrow[r, "\alpha_0"] & V \times U. \end{tikzcd}$$

By flat base change and the projection formula, one has

$$\begin{aligned} \alpha_0^*\, \mathbf{R}\, u_{X,U*}\, \mathcal{O}_U \overset{\mathbf{L}}{\otimes} q_{13}^*\, \mathcal{E}_U &\simeq \mathbf{R}\, \delta_{0*}\, \mathcal{O}_{X_U\times Y_U} \overset{\mathbf{L}}{\otimes} q_{13}^*\, \mathcal{E}_U \\ &\simeq \mathbf{R}\, \delta_{0*}\delta_0^* q_{13}^*\, \mathcal{E}_U \simeq \mathbf{R}\, \delta_{0*}\, \mathcal{E}_U\,. \end{aligned}$$

Similarly, the restriction of the right-hand side in (3.3) to $X_U \times U \times Y_U$ is isomorphic to $\mathbf{R}\, \delta_{1*}\, \mathcal{E}_U$. In this way, we get an isomorphism

$$\mathbf{R}\, \delta_{0*}\, \mathcal{E}_U \xrightarrow{\simeq} \mathbf{R}\, \delta_{1*}\, \mathcal{E}_U\,. \tag{3.8}$$

Let us also note that the pushforward of (3.8) through the projection q_{13}.

$$\mathcal{E}_U \simeq \mathbf{R} q_{13*}\, \mathbf{R}\, \delta_{0*}\, \mathcal{E}_U \to \mathbf{R} q_{13*}\, \mathbf{R}\, \delta_{1*}\, \mathcal{E}_U \simeq \mathcal{E}_U\,. \tag{3.9}$$

is the identity morphism of $\mathcal{E}_U$ (cf. Section 4.15 of [17]).

Moreover, as by Claim 3.2 the functor $\Phi_{\mathcal{E}_U}$ is in particular fully faithful, it holds true that

$$\operatorname{Ext}^i\big(\mathbf{L}\, i_x^*(\mathcal{E}_U), \mathbf{L}\, i_x^*(\mathcal{E}_U)\big) = \operatorname{Ext}^i\big(\Phi_{\mathcal{E}_U}(\mathbb{C}(x)), \Phi_{\mathcal{E}_U}(\mathbb{C}(x))\big) = 0$$

for all $i < 0$ and for any closed point $x \in X_U$, where $i_x\colon \{x\} \times Y_U \hookrightarrow X_U \times Y_U$. Therefore, by cohomology and base change for complexes (Section 7.7 of [8]), one has that $\mathbf{R}\, p_{1*}\, \mathbf{R}\, \mathcal{H}om(\mathcal{E}_U, \mathcal{E}_U)$ lies in $\mathbf{D}^{\geq 0}(X_U)$.

Hence,

$$\mathbf{R}\delta_{0*}\mathcal{E}_U \simeq \mathbf{R}\delta_{1*}\mathcal{E}_U, \quad (3.9)\text{ holds}, \quad \text{and} \quad \mathbf{R}p_{1*}\mathbf{R}\mathcal{H}om(\mathcal{E}_U, \mathcal{E}_U) \in \mathbf{D}^{\geq 0}(X_U). \tag{3.10}$$

At this point, the statement follows from Theorem 1.1 in [17]. ∎

4. Irregular fibrations under derived equivalence

We begin by recalling some definitions. We continue to denote by X a smooth projective complex variety of dimension n. The irregularity of a normal projective variety is defined as the irregularity of any of its resolution of singularities.

A *non-singular representative of* a fibration $f\colon X \to V$ onto a normal projective variety V is a commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\rho"] \arrow[d, "\tilde{f}"] & X \arrow[d, "f"] \\ \widetilde{V} \arrow[r, "\theta"] & V, \end{tikzcd}$$

where ρ and θ are birational morphisms from smooth projective varieties and $\tilde{f}$ is a fibration. We define the abelian subvariety

$$\widehat{B_V} \stackrel{\mathrm{def}}{=} \rho_* \tilde{f}^* \operatorname{Pic}^0 \widetilde{V} \subset \operatorname{Pic}^0 X.$$

Theorem 4.1. *Let X and Y be smooth projective varieties and let $\Phi\colon \mathbf{D}(X) \to \mathbf{D}(Y)$ be an equivalence. If $f\colon X \to V$ is an irregular fibration such that $\widehat{B_V}$ is Rouquier-stable, then Y admits a fibration $h\colon Y \to W$ onto a variety W that is birational to V. Moreover, the general fibers of f and h are derived equivalent.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "a_{\widetilde{X}}"] \arrow[d, "\tilde{f}"'] & \operatorname{Alb} \widetilde{X} \arrow[d, "\pi"] \\ \widetilde{V} \arrow[r, "a_{\widetilde{V}}"] & \operatorname{Alb} \widetilde{V}, \end{tikzcd}$$

where π is the fibration induced by $\tilde{f}$ at the level of Albanese varieties. By definition, there is an isomorphism $B_V \cong \operatorname{Alb} \widetilde{V}$ and $\pi \circ a_X = b_X$, where $b_X\colon X \to B_V$ is the Rouquier-stable morphism induced by $\widehat{B_V} \subset \operatorname{Pic}^0 X$ (see Section 2). Since

$$a_{\widetilde{X}} = (a_X \circ \rho)\colon \widetilde{X} \to X \to \operatorname{Alb} X \simeq \operatorname{Alb} \widetilde{X} \quad \text{and} \quad \rho_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_X,$$

we have that X' (notations as in (3.1)) is isomorphic to the base of the Stein factorization of $\pi \circ a_{\widetilde{X}} = a_{\widetilde{V}} \circ \tilde{f}$. But $\tilde{f}_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{V}}$, hence X' is also isomorphic to the base of the Stein factorization of $a_{\widetilde{V}}$, which is birational to $\widetilde{V}$ as $a_{\widetilde{V}}$ is generically finite onto its image. Namely, $\tilde{f}$ is a non-singular representative of the fibration $s_X\colon X \to X'$ too, and we have the following commutative diagram:

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\rho"] \arrow[d, "\tilde{f}"] \arrow[rr, bend left, "a_{\widetilde{X}}"] & X \arrow[r, "a_X"] \arrow[d, "s_X"] & \operatorname{Alb} \widetilde{X} \arrow[d, "\pi"] \\ \widetilde{V} \arrow[r] \arrow[rr, bend right, "a_{\widetilde{V}}"'] & X' \arrow[r, "b'_X"] & \operatorname{Alb} \widetilde{V}. \end{tikzcd} \tag{4.1}$$

In particular, X' is birational to V.

Let $\widehat{B_Y} := \varphi(\widehat{B_V})$ and let $b_Y : Y \to B_Y$ be the corresponding Rouquier-stable morphism. Moreover, consider the Stein factorization of b_Y:

$$\begin{tikzcd} Y \arrow[r, "s_Y"] \arrow[rr, bend left, "b_Y"] & Y' \arrow[r] & B_Y, \end{tikzcd}$$

where the first morphism is a fibration and the second is finite onto its image. By Theorem 3.1, there exists an isomorphism $X' \simeq Y'$. In particular, by taking $h := s_Y : Y \to W := Y'$, we have that V and W are birational. The second statement follows from the above construction and Theorem 3.1 (iii). ■

Recall from the Introduction that two irregular fibrations $f_1 : X \to V_1$ and $f_2 : X \to V_2$ are *equivalent* if there exists a birational map $\sigma : V_1 \dashrightarrow V_2$ such that $f_2 = \sigma \circ f_1$. We record for later use the following consequence of the proof of the previous theorem.

Lemma 4.2. *The irregular fibration $f : X \to V$ we started with is equivalent to s_X. In particular, the general fibers of f and s_X are isomorphic.*

It turns out that irregular fibrations onto varieties of general type provide a natural geometric framework where the Rouquier-stableness assumption of Theorem 4.1 is automatically satisfied.

Lemma 4.3. *If $f : X \to V$ is an irregular fibration and V is of general type, then the associated abelian variety $\widehat{B_V}$ is Rouquier-stable with respect to any equivalence.*

Proof. We aim to prove that $\widehat{B_V}$ is contained in a Rouquier-stable subvariety. Take notations as in Subsection 2.1.5. By Kollár's decomposition theorem [14] one has that

$$\tilde{f}^* V^0(\tilde{V}, \omega_{\tilde{V}}) \subseteq V^k(\tilde{X}, \omega_{\tilde{X}}),$$

where k is the dimension of the generic fiber of $\tilde{f}$ (see Lemma 6.3 in [18]). Therefore,

$$\rho_* \tilde{f}^* V^0(\tilde{V}, \omega_{\tilde{V}}) \subseteq V^k(X, \omega_X).$$

The Rouquier isomorphism φ induces a map

$$\operatorname{Pic}^0 Y \to \operatorname{Aut}^0 X, \quad \beta \mapsto p_{\operatorname{Aut}^0 X}(\varphi^{-1}(\mathrm{id}_Y, \beta)),$$

whose image is an abelian variety denoted by $A \subseteq \operatorname{Aut}^0 X$. If A is trivial, then $\operatorname{Pic}^0 X$ is Rouquier-stable by definition. So we may assume that $\dim A > 0$. Now take a general point $x_0 \in X$ and consider the orbit map

$$g : A \to X, \quad \xi \mapsto \xi(x_0).$$

Using Brion's results on the action of a non-affine algebraic group on smooth projective varieties ([2], see also Section 2 of [26]), it can be proved that $V^k(X, \omega_X)$ is contained in the subgroup $\ker(g^* : \operatorname{Pic}^0 X \to \widehat{A})$ of $\operatorname{Pic}^0 X$ (see formula (8) on p. 524 of [18]). In particular, this yields the inclusion

$$\rho_* \tilde{f}^* V^0(\tilde{V}, \omega_{\tilde{V}}) \subseteq \ker(g^*).$$

By assumption, $\widetilde{V}$ is of maximal Albanese dimension and of general type, hence Theorem 1 in [6] says that $V^0(\widetilde{V}, \omega_{\widetilde{V}})$ generates $\operatorname{Pic}^0 \widetilde{V}$ as a group. Therefore, from the above discussion we get

$$\widehat{B_V} = \rho_* \tilde{f}^* \operatorname{Pic}^0 \widetilde{V} \subseteq \ker(g^*).$$

Moreover, since $\widehat{B_V}$ is an abelian subvariety, it is actually contained in the connected component $(\ker(g^*))_0$ of $\ker(g^*)$ through the origin. Now we employ the fact that $(\ker(g^*))_0$ is Rouquier-stable as in p. 533 of [26]. ∎

4.1. Proof of Theorem 1.1 (i) and (ii)

The proof of Theorem 1.1 (i) and (ii) follows from Theorem 4.1 and Lemma 4.3.

4.2. Proof of Theorem 1.1 (iii)

Let $s_X\colon X \to X'$ (respectively, $s_Y\colon Y \to Y'$) be the fibration induced by $\widehat{B_V}$ (respectively, by $\varphi(\widehat{B_V})$). We know that

$$W := Y' \simeq X'$$

thanks to Theorem 3.1 and, moreover,

$$\operatorname{Supp}(\mathcal{E}) \subseteq X \times_W Y, \tag{4.2}$$

where $\mathcal{E} \in \mathbf{D}(X \times Y)$ is the kernel of the equivalence. At this point, the proof is a relative version of Kawamata's technique [13]. Let $Z \subseteq \operatorname{Supp}(\mathcal{E})$ be an irreducible component such that the first projection $\pi_X\colon Z \to X$ is surjective (see Corollary 6.5 in [12]). In particular, the inequality $\dim X \leq \dim Z$ holds. Denote by $\pi_Y\colon Z \to Y$ the second projection. From (4.2) we get a *commutative* diagram

$$\begin{tikzcd} & Z \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X \arrow[dr, "s_X"'] & & Y \arrow[dl, "s_Y"] \\ & W. & \end{tikzcd} \tag{4.3}$$

Note that, for any point $w \in W$, one has

$$\begin{aligned} \pi_X^{-1}(s_X^{-1}(w)) &= Z \cap (s_X^{-1}(w) \times s_Y^{-1}(w)) \subseteq \operatorname{Supp}(\mathcal{E}) \cap (s_X^{-1}(w) \times s_Y^{-1}(w)) \\ &= \operatorname{Supp}(\mathbf{L}\iota^* \mathcal{E}) \end{aligned}$$

where

$$\iota\colon s_X^{-1}(w) \times s_Y^{-1}(w) \hookrightarrow X \times Y$$

is the inclusion map (the last equality is Lemma 3.29 in [12]). Thanks to Lemma 4.2, for a general $w \in W$ the (anti)canonical bundle of $s_X^{-1}(w)$ is big, which implies, by an argument of Kawamata in [13], that the morphism $\pi_Y^{-1}(s_Y^{-1}(w)) \xrightarrow{\pi_Y} s_Y^{-1}(w)$ is generically finite. We briefly sketch this for the reader's convenience. Set

$$X_w := s_X^{-1}(w), \quad Y_w := s_Y^{-1}(w), \quad \text{and} \quad Z_w := Z \cap (s_X^{-1}(w) \times s_Y^{-1}(w))$$

for a general point $w \in W$. Moreover, let $\nu_w : \widetilde{Z}_w \to Z_w$ be the normalization and assume that ω_{X_w} is big (the other case is completely analogue). By Kodaira's lemma, for $m \gg 0$ one has that

$$\omega_{X_w}^{\otimes m} \simeq \mathcal{O}_{X_w}(H) \otimes \mathcal{O}_{X_w}(D),$$

where H is an ample divisor and D is an effective divisor on X_w. We now prove that the morphism

$$(\pi_Y \circ \nu_w) : \widetilde{Z}_w \setminus \nu_w^{-1}\pi_X^{-1}(D) \to Y_w$$

is finite. Suppose by contradiction that there exists an irreducible curve $C \subseteq \widetilde{Z}_w$ contracted by $\pi_Y \circ \nu_w$ and such that $C \not\subseteq \nu_w^{-1}\pi_X^{-1}(D)$. Then

$$0 = \deg \left(\nu_w^* \pi_Y^* (\omega_{Y_w})\right)\big|_C = \deg \left(\nu_w^* \pi_X^* (\omega_{X_w})\right)\big|_C \geq \frac{1}{m} \deg \left(\nu_w^* \pi_X^* \mathcal{O}(H)\right)\big|_C > 0,$$

where the second equality is an application of Lemma 6.6 in [12].

In particular, $\pi_Y : Z \to Y$ is generically finite and $\dim Z \leq \dim Y$. But we already know that

$$\dim Y = \dim X \leq \dim Z.$$

Therefore, $\dim X = \dim Z$. At this point, another well-established argument due to Kawamata [13] says that X and Y are K-equivalent (see also p. 149 in [12], or Lemma 15 in [20]). This concludes the proof of (iii).

The argument we just employed also provides a further generalization of Kawamata's birational reconstruction theorem (see the Introduction).

Theorem 4.4. *Let $\Phi : \mathbf{D}(X) \to \mathbf{D}(Y)$ be an equivalence and let $\widehat{B_X} \subset \operatorname{Pic}^0 X$ be a Rouquier-stable subvariety. If the (anti)canonical line bundle of the general fiber of the Rouquier-stable morphism b_X is big, then X and Y are K-equivalent.*

Remark 4.5. If ω_X (respectively, ω_X^{-1}) is big as in Kawamata's theorem, then $\operatorname{Aut}^0 X$ is an affine algebraic group (see Subsection 2.1.3). Hence the whole $\operatorname{Pic}^0 X$ is Rouquier-stable and ω_X (respectively, ω_X^{-1}) is obviously b_X-big (note that $b_X = a_X$ if $\widehat{B_X} = \operatorname{Pic}^0 X$). Moreover, as recalled in Subsection 2.1.3, varieties with non-zero Euler characteristic have affine automorphism group $\operatorname{Aut}^0(-)$. Hence Corollary 1.4 of the Introduction is a particular case of the above Theorem 4.4.

4.3. Proof of Theorem 1.3

An *irregular k-fibration* is an irregular fibration onto a variety of dimension k. For any variety X and integer $0 < k < n := \dim X$, we define the following set:

$$G_X \stackrel{\text{def}}{=} \{\text{equivalence classes of irregular } k\text{-fibrations } f : X \to V \\ \text{such that } V \text{ is of general type and } 0 < k < \dim X\}.$$

We aim to prove Theorem 1.3. Indeed, we prove the more precise Theorem 4.6 below.

Theorem 4.6. *Let $\Phi\colon \mathbf{D}(X) \to \mathbf{D}(Y)$ be a derived equivalence. There exists a bijective correspondence $\mu_\Phi\colon G_X \to G_Y$ such that if $\mu_\Phi(f\colon X \to V) = (h\colon Y \to W)$, then the varieties V and W are birational. Moreover, the generic fibers of f and h are derived equivalent.*

In the rest of this section, we prove the above theorem. The function μ_Φ is defined by Theorem 4.1: we take the Stein factorization of the Rouquier-stable morphism b_Y,

$$b_Y\colon Y \xrightarrow{h:=s_Y} W := Y' \xrightarrow{b'_Y} B_Y,$$

where $\widehat{B_Y} := \varphi(\widehat{B_V})$. In particular, we already know that V and W are birational and that the generic fibers of f and h are derived equivalent.

Now we turn to prove that μ_Φ is a bijection. Take notations as in the proof of Theorem 4.1. By Theorem 3.1 (ii) and (4.1), there exists an isomorphism of varieties $\psi\colon W \xrightarrow{\sim} X'$ such that the diagram

$$\begin{tikzcd}
\widetilde{V} \arrow[r] \arrow[rr, bend left, "a_{\widetilde{V}}"] & X' \arrow[r, "b'_X"] & B_V \simeq \operatorname{Alb}\widetilde{V} \\
\widetilde{W} \arrow[u, dashed] \arrow[r] & W \arrow[u, "\psi"] \arrow[r, "b'_Y"] & B_Y \arrow[u, "\widehat{\varphi}"']
\end{tikzcd}$$

is commutative. Hence we get that $B_Y \simeq \operatorname{Alb}\widetilde{W}$, and moreover that the bottom composition is isomorphic to the Albanese map $a_{\widetilde{W}}$ of a resolution $\widetilde{W}$ of W.

Now let

$$\begin{tikzcd}
\widetilde{Y} \arrow[r, "\sigma"] \arrow[d, "\widetilde{h}"] & Y \arrow[r] \arrow[d, "h"] & \operatorname{Alb} Y \arrow[d] \\
\widetilde{W} \arrow[r, "\xi"] & W \arrow[r] & \operatorname{Alb}\widetilde{W} \simeq B_Y,
\end{tikzcd}$$

where the left vertical morphism is a non-singular representative of the irregular fibration h, and the right-hand one is the fibration induced by the universal property of the Albanese variety. Then we have

$$\widehat{B_Y} \simeq \sigma_* \widetilde{h}^* \operatorname{Pic}^0 \widetilde{W} =: \widehat{B_W} \subseteq \operatorname{Pic}^0 Y.$$

Hence

$$\varphi(\widehat{B_V}) = \widehat{B_Y} \simeq \widehat{B_W}.$$

At this point, if we apply $\mu_{\Phi^{-1}}$ to h, where Φ^{-1} is a quasi-inverse of Φ, we get

$$\mu_{\Phi^{-1}}(h) = \mu_{\Phi^{-1}}(\mu_\Phi(f)) = s_X$$

thanks to the functoriality of the Rouquier isomorphism. By Lemma 4.2 f and s_X are equivalent fibrations of X, so $\mu_{\Phi^{-1}} \circ \mu_\Phi = \mathrm{id}_{G_X}$ and, since the role of X and Y can be symmetrically exchanged, we also get $\mu_\Phi \circ \mu_{\Phi^{-1}} = \mathrm{id}_{G_Y}$ by the same reasoning.

Remark 4.7. Let us restrict ourselves to irregular fibrations $f: X \to V$ onto varieties V admitting a morphism $c_V: V \to \operatorname{Alb}\widetilde{V}$ finite onto its image and such that the composition $\widetilde{V} \xrightarrow{\rho} V \xrightarrow{c_V} \operatorname{Alb}\widetilde{V}$ equals the Albanese map of a desingularization $\widetilde{V}$.[6] In this case, two fibrations $f: X \to V$ and $f': X \to V'$ are equivalent if there exists an isomorphism $\sigma: V \to V'$ such that $f' = \sigma \circ f$. Then the bijection of Theorem 4.6 is base-preserving in a stronger sense: namely, V is isomorphic to W. In fact, there exists an isomorphism $\sigma: V \xrightarrow{\sim} X'$ such that $s_X = \sigma \circ f$ (see Lemma 19 in [19]).

4.4. Proof of Proposition 1.2

Once the K-equivalence among X and Y has been proved by Theorem 1.1 (iii), Proposition 1.2 follows at once by standard arguments (see [20], p. 304). Namely, if the rational map $\psi: Y \dashrightarrow X$ induced by the K-equivalence is not a morphism, there exists a curve $C \subseteq Z$ that is contracted by π_Y but not by π_X (see (4.3)). So

$$(\pi_X^*\omega_X \cdot C) = (\pi_Y^*\omega_Y \cdot C) = 0.$$

On the other hand, by Lemma 4.2 and the above Remark 4.7, we see that ω_X is f-(anti)ample if and only if it is s_X-(anti)ample, and, since $\pi_X(C)$ is contained in a fiber of s_X, one gets $(\omega_X \cdot \pi_X(C)) \neq 0$. This gives a contradiction. Hence ψ is a crepant birational morphism between smooth projective varieties, hence an isomorphism.

4.5. Proof of Theorem 1.5

For the proof of Theorem 1.5, we apply the main result of [6]. Let $f: X \to V$ be an irregular fibration of X. Since by definition V is of maximal Albanese dimension, one has that $\operatorname{kod}(V) \geq 0$. Then it makes sense to consider the Iitaka fibration of V, which by definition is the Iitaka fibration of a non-singular model of V. So we get the following commutative diagram:

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\rho"] \arrow[d, "\tilde{f}"] & X \arrow[r, "a_X"] & \operatorname{Alb} X \arrow[d, "\pi"] \\
\widetilde{V} \arrow[rr, "a_{\widetilde{V}}"] \arrow[d, "g"] & & B_V = \operatorname{Alb}\widetilde{V} \arrow[d, "\kappa"] \\
Z_V \arrow[rr, "a_{Z_V}"] & & \operatorname{Alb} Z_V,
\end{tikzcd} \tag{4.4}$$

where Z_V is a smooth projective variety of dimension $\dim Z_V = \operatorname{kod}(V)$, and κ is the fibration between Albanese varieties induced by the Iitaka fibration g of V.

By Theorem 2.3 in [6], the abelian variety $\widehat{\kappa}(\operatorname{Pic}^0 Z_V)$ is contained in the abelian subvariety of $\operatorname{Pic}^0\widetilde{V}$ generated by $V^0(\widetilde{V}, \omega_{\widetilde{V}})$.[7] Since in the proof of Lemma 4.3 we

[6] This is precisely what happens if $\dim V = 1$.

[7] For the sake of clarity, let us say that we are applying Theorem 2.3 in [6] to the generically finite morphism $a_{\widetilde{V}}$, and the variety $\operatorname{Pic}^0 S$ in *loc. cit.* coincides with our $\widehat{\kappa}(\operatorname{Pic}^0 Z_V)$.

observed that $\rho_* \tilde{f}^* V^0(\widetilde{V}, \omega_{\widetilde{V}})$ is contained in a subgroup of $\operatorname{Pic}^0 X$ whose connected component through the origin is a Rouquier-stable subvariety, it follows from the commutativity of (4.4) that $\widehat{\pi}(\widehat{\kappa}(\operatorname{Pic}^0 Z_V))$ is a Rouquier-stable subvariety of $\operatorname{Pic}^0 X$. Hence, by taking the Stein factorization of the morphism induced by $\varphi(\widehat{\pi}(\widehat{\kappa}(\operatorname{Pic}^0 Z_V))) \subseteq \operatorname{Pic}^0 Y$, we obtain a fibration $h: Y \to W$. Since the base of the Stein factorization of the composition $\kappa \circ \pi \circ a_X$ is equal to the base of the Stein factorization of a_{Z_V}, which is generically finite onto its image (Proposition 2.1 (a) in [10]), we see that W is birational to Z_V by Theorem 3.1 (i).

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