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On GIT stability of linear systems of hypersurfaces in projective spaces

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Abstract. In this paper, we consider the problem of classifying linear systems of hypersurfaces (of a fixed degree) in projective space up to projective equivalence. Our main result consists of a complete criterion for (semi)stability in the sense of geometric invariant theory (GIT). As an application, we inspect a few relevant geometric examples recovering, for instance, Miranda's characterization of GIT stability of pencils of plane cubics. Furthermore, we completely describe GIT stability of Halphen pencils of any index.

1. Introduction

Linear systems of hypersurfaces are ubiquitous in algebraic geometry, usually in connection with rich geometric structures, including algebraic fiber spaces. This paper aims to shed light on the problem of classifying such objects up to projective equivalence using geometric invariant theory (GIT). Our work fits in a collection of other GIT constructions of moduli spaces of linear systems of hypersurfaces, such as [7,8,11,15,17,21,26,30,33].

For a fixed positive integer n, let V denote the $(n+1)$-dimensional vector space $\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Then for each integer $r \geq 1$, the projective space $\mathbb{P}(S^r V^\vee)$ parameterizes hypersurfaces of degree r on $\mathbb{P}(V) = \mathbb{P}^n$. In particular, the space $\mathcal{X}_{k,d,n}$ of k-dimensional linear systems of hypersurfaces of degree d in $\mathbb{P}^n$ can be embedded in the projective space $\mathbb{P}(\Lambda^{k+1} S^d V^\vee)$ via the Plücker coordinates. The natural action of $\mathrm{PGL}(V)$ on V induces an action on this large projective space, hence on the invariant subvariety $\mathcal{X}_{k,d,n}$, and the problem we are interested in is the problem of parameterizing points in $\mathcal{X}_{k,d,n}$ modulo the induced $\mathrm{PGL}(V)$-action. Moreover, since the group $\mathrm{SL}(V)$ also acts naturally on $\mathcal{X}_{k,d,n}$, and it acts with the same orbits as $\mathrm{PGL}(V)$, we can (and we will) consider the $\mathrm{SL}(V)$-action on $\mathcal{X}_{k,d,n}$ instead for practical reasons.

Parameterizing the orbits of the $\mathrm{SL}(V)$-action on $\mathcal{X}_{k,d,n}$ is the same as constructing the quotient space $\mathcal{X}_{k,d,n}//\mathrm{SL}(V)$. This can be achieved algebraically via GIT by restricting to the open subset of $\mathcal{X}_{k,d,n}$ consisting of so-called (semi)stable points. In this paper, we provide a complete criterion for describing these (semi)stable points in $\mathcal{X}_{k,d,n} \subset \mathbb{P}(\Lambda^{k+1} S^d V^\vee)$ for the action of $\mathrm{SL}(V)$. Since the group $\mathrm{SL}(V)$ also acts on the

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space $\mathbb{P}(S^{d(k+1)}V^\vee)$ parameterizing hypersurfaces of degree $d(k+1)$, we achieve this by establishing a link between (semi)stability of points in $\mathcal{X}_{k,d,n}$ and (semi)stability of certain reducible hypersurfaces of degree $d(k+1)$ in $\mathbb{P}^n$ (points in $\mathbb{P}(S^{d(k+1)}V^\vee)$). Our criterion can now be stated.

Theorem 1.1 (= Theorem 3.3). *A linear system $\mathcal{L} \in \mathcal{X}_{k,d,n}$ is GIT stable (respectively, semistable) if and only if for any choice of generators $H_1, \dots, H_{k+1} \in \mathcal{L}$, the degree $d(k+1)$ hypersurface $H_1 + \cdots + H_{k+1}$, viewed as an element of $\mathbb{P}(S^{d(k+1)}V^\vee)$, is GIT stable (respectively, semistable).*

Theorem 1.1 shows that the following two questions are equivalent:

When is a linear system $\mathcal{L} \in \mathcal{X}_{k,d,n}$ GIT (semi)stable?

and

When is the union of $k+1$ (distinct) hypersurfaces generating $\mathcal{L}$ GIT (semi)stable?

This means that the problem of classifying points $\mathcal{L} \in \mathcal{X}_{k,d,n}$ can be viewed as the problem of classifying certain points in $\mathbb{P}(S^{d(k+1)}V^\vee)$. Since the dimension of $\mathbb{P}(S^{d(k+1)}V^\vee)$ is much smaller than that of $\mathbb{P}(\Lambda^{k+1}S^d V^\vee) \supset \mathcal{X}_{k,d,n}$, this equivalence makes the former problem simpler. Furthermore, we can use this equivalence to detect GIT stability of linear systems by applying several already-known criteria for stability of hypersurfaces.

To prove Theorem 1.1, we first compare the Hilbert–Mumford weights of a linear system $\mathcal{L} \in \mathcal{X}_{k,d,n}$ with the Hilbert–Mumford weights of a hypersurface lying in $\mathcal{L}$, as well as to the Hilbert–Mumford weights of the hypersurface given by the union of $k+1$ generators of $\mathcal{L}$. This is encoded in Lemmas 3.1 and 3.2. We then make use of the standard numerical criterion for GIT (semi)stability (Proposition 2.2) in both our setting and the analogous one for hypersurfaces.

Some partial criteria for (semi)stability of linear systems of hypersurfaces

In this paper, we also establish several secondary partial criteria for (semi)stability of points in $\mathcal{X}_{k,d,n}$, which are consequences of our main result and illustrate its applicability. We will now list these.

First, recall that GIT stability for hypersurfaces is closely related to an invariant of singularities called the *log canonical threshold*, as pointed out first by Hacking in [12] and by Kim–Lee in [18]. In [32], the second named author shows that a similar relationship also holds for pencils of plane curves. Here, using Theorem 1.1, we can show that this relationship holds more generally for all linear systems of hypersurfaces. The result we obtain, which we will state next, is also one of the main ingredients in the proof of Theorem 1.6.

Corollary 1.2 (= Corollary 4.1). *If $\mathcal{L} \in \mathcal{X}_{k,d,n}$ is GIT non-stable (respectively, unstable), then there exists a choice of generators $H_1, \dots, H_{k+1} \in \mathcal{L}$ such that*

$$\operatorname{lct}(\mathbb{P}^n, H_1 + \cdots + H_{k+1}) \le \frac{n+1}{d(k+1)} \quad (\textit{respectively, } <),$$

where $\operatorname{lct}(\mathbb{P}^n, H_1 + \cdots + H_{k+1})$ *denotes the log canonical threshold (Definition 2.6) of the pair* $(\mathbb{P}^n, H_1 + \cdots + H_{k+1})$.

In a different direction, we observe that if we can explicitly describe what the (semi)-stable points in $\mathcal{X}_{k-1,d,n}$ are, then we can use this description to characterize some of the (semi)stable points in $\mathcal{X}_{k,d,n}$. More precisely, we prove the following result.

Corollary 1.3 (= Corollary 4.3). *Let $k > 1$ and let $\mathcal{L} \in \mathcal{X}_{k,d,n}$ be a linear system containing at least one GIT semistable hypersurface. If $\mathcal{L}$ is GIT non-stable (respectively, unstable), then there exists a sub-linear system of dimension $k-1$ (of $\mathcal{L}$) that is GIT non-stable (respectively, unstable).*

In addition, we establish that there exist GIT stable k-dimensional linear systems of hypersurfaces of degree d in $\mathbb{P}^n$ with a base locus of dimension bigger than $n-(k+1)$. For instance, this tells us that we can construct examples of GIT stable pencils of plane curves whose members are all singular. This is summarized in the result below.

Corollary 1.4 (= Corollary 4.5). *Let $d > m$ and choose $\tilde{\mathcal{L}} \in \mathcal{X}_{k,d-m,n}$ which is GIT stable (respectively, semistable). If $\tilde{H}_1, \dots, \tilde{H}_{k+1}$ generate $\tilde{\mathcal{L}}$ and $H \subset \mathbb{P}^n$ is an arbitrary GIT semistable (respectively, stable) hypersurface of degree m, then the linear system $\mathcal{L} \in \mathcal{X}_{k,d,n}$ generated by the $k+1$ hypersurfaces $H_i := H + \tilde{H}_i$ of degree d is GIT stable.*

Finally, we further relate the non-stability of a point $\mathcal{L} \in \mathcal{X}_{k,d,n}$ to Chow stability (see, e.g., Section 2.5) of its base locus, or lack thereof. Detecting Chow stability is usually much more challenging than detecting GIT stability of hypersurfaces. In [28], Sano provides a partial criterion for establishing Chow stability of a complete intersection using an analytical argument. By using (the proof of) Theorem 1.1 and Sano's result (Theorem 1.1 in [28]), we relate Chow stability of a complete intersection to GIT stability of the corresponding linear system. Furthermore, in Appendix A, we prove that this relationship also holds in the semistable case using a purely algebro-geometric argument. More precisely, considering only linear systems $\mathcal{L} \in \mathcal{X}_{k,d,n}$ with a base locus of dimension $n-k-1$ and that we call *regular*, we prove the following.

Corollary 1.5 (= Theorem A.1, cf. Theorem 1.1 in [28] and Corollary 4.4). *Let $k \leq n-1$ and let $\mathcal{L} \in \mathcal{X}_{k,d,n}$ be a regular linear system. If $\mathcal{L}$ is generated by $H_{f_1}, \dots, H_{f_{k+1}}$ and the complete intersection $H_{f_1} \cap \dots \cap H_{f_{k+1}} \subset \mathbb{P}^n$ is Chow (semi)stable, then $\mathcal{L}$ is GIT (semi)stable.*

Some geometric applications

To further illustrate the applicability of Theorem 1.1, in Section 5.1, we revisit the work of Miranda on pencils of plane cubics [21], and in Section 6, the work of Wall on nets of conics [30]. We explicitly explain how one can recover their stability criteria by using Theorem 1.1. Furthermore, in Section 5.2, we provide a complete description of stability of certain pencils of curves of degree $3m$, called Halphen pencils of index m, by proving the result below.

Theorem 1.6 (= Theorem 5.1 + Theorem 5.11). *Let $\mathcal{P}$ be a Halphen pencil of index m and denote by Y the corresponding rational elliptic surface. If $\operatorname{lct}(Y, F) > 1/(2m)$ (respectively, $\geq$) for any fiber F, then $\mathcal{P}$ is GIT stable (respectively, semistable). Furthermore, except for GIT stable Halphen pencils of index $m = 2$ (which are given by Examples* 7.46, 7.47

and 7.55 *in* [31]) *and of index* $m = 3$ (*which are given by Example* 5.10)*, the converse statement also holds.*

In [14], the first named author introduced and studied the notions of adiabatic K-stability and log-twisted K-stability of the base for Calabi–Yau fibrations over curves. It is interesting to observe that Theorem 1.6 shows that these two notions are closely related to the notion of GIT stability for Halphen pencils of any index m. Indeed, it follows from Proposition 4.16 in [14] and Theorem 1.6 that log-twisted K-stability of the base curve of a rational elliptic surface implies GIT stability of the associated Halphen pencil.

We further observe that Theorem 1.6 above is entirely new and an application not only of Theorem 1.1 but also of Corollary 1.2, as already mentioned. More precisely, for the second part of the statement, we make use of the partial converse of Corollary 1.2 (see Lemma 2.8).

Connections to other works

Finally, we would like to point out that the criterion given by Theorem 1.1 can, a priori, be used to provide alternative descriptions of the results in [7, 8, 11, 15, 21, 30, 33]. Furthermore, since every pencil of plane curves can be seen as a holomorphic foliation of $\mathbb{P}^2$, it could also be used to provide an alternative description of the results in [2–4]. It would be interesting to explore what new insights could be gained from our results. For instance, combining the results in [11] with Theorem 1.1, it seems plausible that one should be able to describe the (semi)stable sextic threefolds in $\mathbb{P}^4$ which are given by the union of three quadrics. Here, we restrict ourselves to exploring the cases of pencils of plane cubics from [21] and that of nets of conics from [30], which we do in Sections 5.1 and 6, respectively.

We also remark that GIT stability of linear systems whose base locus is a complete intersection has been recently considered in [26]. After finishing this manuscript, we learned that Papazachariou had also generalized the work in [32] and that he had independently obtained some of the same results presented here. Even more recently, since any point $\mathcal{L} \in \mathcal{X}_{k,d,n}$ can also be viewed as a divisor of bidegree $(1, d)$ in $\mathbb{P}^k \times \mathbb{P}^n$, GIT stability of linear systems has also been considered in [17]. However, their approach is strictly computational.

2. Relevant background and notations

In this section, we present the basic GIT setup for the action of the reductive group $\mathrm{SL}(n+1)$ on the projective variety $\mathcal{X}_{k,d,n}$ of k-dimensional linear systems of hypersurfaces of degree d in $\mathbb{P}^n$. We further establish the notations we will use throughout the paper. In particular, we recall what kind of tools we can use to determine when a point in $\mathcal{X}_{k,d,n}$ is (semi)stable or not, namely the numerical criterion of Hilbert–Mumford (Proposition 2.2) and the log canonical threshold (Definition 2.6). For details, see [22] and Section 8 of [19]. We work over $\mathbb{C}$ throughout the paper, and will also adopt the following conventions.

- Given any vector space W over $\mathbb{C}$ and of finite dimension, $\mathbb{P}(W)$ denotes the projective space parametrizing hyperplanes in W. The projective space of lines in W is denoted by $|W|$. Thus, $\mathbb{P}(W) = |W^\vee|$, where $W^\vee$ denotes the dual vector space of W. Moreover, we identify points in $\mathbb{P}(W)$ with non-zero linear functionals on W.
- In particular, for the vector space $V := H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ of dimension $n+1$, $\mathbb{P}(V)$ means $|V^\vee| = \mathbb{P}^n$ in this paper, and the same notations are also used for the symmetric powers of V and $V^\vee$. Hence, for each $r \geq 1$, we regard $\mathbb{P}(S^r V^\vee)$ as the complete linear system $|\mathcal{O}_{|V^\vee|}(r)| = |\mathcal{O}_{\mathbb{P}^n}(r)|$ of hypersurfaces of degree r on $|V^\vee| = \mathbb{P}(V) = \mathbb{P}^n$.

2.1. The general GIT setup

We start by recalling the notion of GIT (semi)stability in a general setting. Throughout this section, we let X be a projective variety on which the group $\mathrm{SL}(n+1)$ acts, and $\mathcal{L}$ an ample $\mathrm{SL}(n+1)$-linearized line bundle on X (see, e.g., Definition 1.6 in [23]).

Definition 2.1. A closed point $x \in X$ is *GIT semistable with respect to* $\mathcal{L}$ if there exist some $m \in \mathbb{Z}_{>0}$ and an $\mathrm{SL}(n+1)$-invariant section $s \in H^0(X, \mathcal{L}^{\otimes m})$ such that $s(x) \neq 0$. If, in addition, the $\mathrm{SL}(n+1)$-orbit of x is closed and of maximal dimension, then the point x is *GIT stable with respect to* $\mathcal{L}$.

When the projective variety X has Picard number one, as in the case we are interested in, then (semi)stability (as in Definition 2.1) is independent of the choice of $\mathcal{L}$ (see, e.g., Proposition 1.4 and Corollary 1.20 in [23]). In this case, we simply say that x is (semi)stable.

In general, the GIT machinery tells us that one can detect (semi)stability of a point $x \in X$ by a numerical invariant known as the Hilbert–Mumford weight. This can be defined in the following way. Fix any (not necessarily ample) $\mathrm{SL}(n+1)$-linearized line bundle $\mathcal{M}$ on X. Given any one-parameter subgroup $\lambda\colon \mathbb{C}^\times \to \mathrm{SL}(n+1)$, i.e., λ is a homomorphism of algebraic groups, the corresponding *Hilbert–Mumford weight* of $x \in X$ with respect to λ and $\mathcal{M}$ is the quantity

$$\mu^{\mathcal{M}}(x, \lambda) := -\text{the weight of the action of } \mathbb{C}^\times \text{ on } \mathcal{M} \otimes k(y) \text{ via } \lambda,$$

where y is the unique point in X such that $y = \lim_{t\to 0} \lambda(t) \cdot x$, and $k(y)$ is the corresponding residue field. The Hilbert–Mumford criterion (Theorem 2.1 in [23]) then tells us that a closed point $x \in X$ is GIT semistable (respectively, stable) with respect to an ample $\mathrm{SL}(n+1)$-linearized line bundle $\mathcal{L}$ if and only if $\mu^{\mathcal{L}}(x, \lambda) \geq 0$ (respectively, $\mu^{\mathcal{L}}(x, \lambda) > 0$) for any non-trivial one-parameter subgroup $\lambda\colon \mathbb{C}^\times \to \mathrm{SL}(n+1)$.

In this paper, we are interested in the projective variety $X = \mathcal{X}_{k,d,n}$. Letting $V := H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ as above, we identify $\mathcal{X}_{k,d,n}$ with the Grassmannian variety of $(k+1)$-planes in the vector space $S^d V$ of all hypersurfaces of degree d. We embed $\mathcal{X}_{k,d,n}$ into the projective space $\mathbb{P}(\Lambda^{k+1} S^d V^\vee) \simeq \mathbb{P}^N$ via the Plücker coordinates and assume that $\dim \mathbb{P}(S^d V^\vee) \geq k$. Moreover, we consider the action of $\mathrm{SL}(V) \simeq \mathrm{SL}(n+1)$ on $\mathcal{X}_{k,d,n} \subset \mathbb{P}^N$ coming from the natural linear action of $\mathrm{SL}(V)$ on V and, since $\mathrm{Pic}(\mathcal{X}_{k,d,n}) \simeq \mathbb{Z}$, we consider the GIT stability of $\mathcal{X}_{k,d,n}$ with respect to the very ample line bundle $\mathcal{L}$ corresponding to the Plücker embedding together with its unique $\mathrm{SL}(V)$-linearization.

2.2. The Plücker embedding

Next, we will describe the Plücker coordinates of a point in $\mathcal{X}_{k,d,n}$. This description will be necessary when we translate the Hilbert–Mumford criterion to our setting (Proposition 2.2).

We let $r := \dim S^d V^\vee$ and given any tuple $I = (i_0, \dots, i_{n-1})$ of non-negative integers, we write x_I to denote the monomial

$$x_0^{i_0} \cdot x_1^{i_1} \cdots x_{n-1}^{i_{n-1}} \cdot x_n^{d-i_0-\cdots-i_{n-1}}.$$

Then, if we choose $\mathcal{L} \in \mathcal{X}_{k,d,n}$ and $k+1$ hypersurfaces $H_{f_1}, \dots, H_{f_{k+1}}$ as generators, each H_{f_j} represented (in some choice of coordinates) by

$$f_j = \sum_I f_I^j x_I = 0, \tag{2.1}$$

we can represent the linear system $\mathcal{L}$ by a $(k+1) \times r$ matrix whose $(k+1) \times (k+1)$ minors are precisely the Plücker coordinates of $\mathcal{L}$. In other words, in (2.1) we assume that the collection $\{x_I\}$ is a basis for the space $S^d V^\vee$ of hypersurfaces of degree d so that each Plücker coordinate of $\mathcal{L}$ can be written as

$$M_{I_1,\dots,I_{k+1}} := \begin{vmatrix} f_{I_1}^1 & f_{I_2}^1 & \cdots & f_{I_{k+1}}^1 \\ f_{I_1}^2 & f_{I_2}^2 & \cdots & f_{I_{k+1}}^2 \\ \vdots & \vdots & \ddots & \vdots \\ f_{I_1}^{k+1} & f_{I_2}^{k+1} & \cdots & f_{I_{k+1}}^{k+1} \end{vmatrix} \tag{2.2}$$

for some choice of $k+1$ tuples $I_\ell := (i_{0,\ell}, \dots, i_{n-1,\ell})$ among the r tuples I appearing in (2.1).

2.3. The Hilbert–Mumford criterion for linear systems

We now want to state the Hilbert–Mumford criterion for a point $\mathcal{L} \in \mathcal{X}_{k,d,n}$. For this, we need to describe how one-parameter subgroups of $\mathrm{SL}(V)$ act on the Plücker coordinates. Throughout the paper, we will always assume that any one-parameter subgroup λ of $\mathrm{SL}(V)$ is normalized, meaning that given $\lambda\colon \mathbb{C}^\times \to \mathrm{SL}(V)$ we choose coordinates $(x_0 : x_1 : \dots : x_n)$ in $\mathbb{P}^n$ such that λ is expressed as

$$t \mapsto \begin{pmatrix} t^{a_0} & & & \\ & t^{a_1} & & \\ & & \ddots & \\ & & & t^{a_n} \end{pmatrix}, \tag{2.3}$$

for some $a_i \in \mathbb{Z}$ such that $a_0 \geq a_1 \geq \cdots \geq a_n$, $a_0 > 0$ and $a_0 + a_1 + \cdots + a_n = 0$. In particular, if we define

$$a_{I_1,\dots,I_{k+1}} := \sum_{l=0}^{n-1} a_l (i_{l,1} + \cdots + i_{l,k+1}) + a_n \Big(d(k+1) - \sum_{l=0}^{n-1} (i_{l,1} + \cdots + i_{l,k+1}) \Big)$$

and we pick $\mathcal{L} \in \mathcal{X}_{k,d,n}$ and generators $H_{f_1}, \dots, H_{f_{k+1}}$ as above, then the action of $\lambda(t)$ on the Plücker coordinate $M_{I_1,\dots,I_{k+1}}$ has weight $a_{I_1,\dots,I_{k+1}}$. It is given by

$$M_{I_1,\dots,I_{k+1}} \mapsto t^{a_{I_1 \dots I_k}} M_{I_1,\dots,I_{k+1}}.$$

Therefore, letting

- $\tilde{a}_{I_1,\dots,I_{k+1}} := \sum_{l=0}^{n-1}(a_l - a_n)(i_{l,1} + \cdots + i_{l,k+1})$,
- $A_\lambda := \sum_{l=0}^{n-1}(a_l - a_n) = -a_n(n+1)$, and
- $\omega(\mathcal{L}, \lambda) := \min\{\tilde{a}_{I_1,\dots,I_{k+1}} \mid M_{I_1,\dots,I_{k+1}} \neq 0\}$,

we can rephrase the standard numerical criterion of Hilbert–Mumford in our setting in the following way.

Proposition 2.2 (Hilbert–Mumford criterion). *A point $\mathcal{L} \in \mathcal{X}_{k,d,n}$ is GIT unstable (respectively, non-stable) if and only if there exists a one-parameter subgroup λ of* $\mathrm{SL}(V)$ *such that*

$$\frac{d(k+1)}{n+1} < \frac{\omega(\mathcal{L}, \lambda)}{A_\lambda} \quad (\textit{respectively}, \ \leq).$$

Proof. By Theorem 2.1 in [23], we know that a point $\mathcal{L} \in \mathcal{X}_{k,d,n}$ is unstable (respectively, non-stable) if and only if there exists a one-parameter subgroup $\lambda\colon \mathbb{C}^\times \to \mathrm{SL}(V)$ such that $\mu^{\mathcal{L}}(\mathcal{L}, \lambda) < 0$ (respectively, ≤ 0). By choosing coordinates as in (2.3), these conditions are equivalent to saying that for any non-zero Plücker coordinate $M_{I_1,\dots,I_k}$, the corresponding weight $a_{I_1,\dots,I_{k+1}}$ is positive (respectively, non-negative).

Thus, it suffices to observe that we have the following sequence of equivalences:

$$a_{I_1,\dots,I_{k+1}} > 0 \quad (\text{respectively}, \ \geq) \quad \Longleftrightarrow$$

$$\sum_{l=0}^{n-1} a_l(i_{l_1} + \cdots + i_{l_{k+1}}) + a_n\Big(d(k+1) - \sum_{l=0}^{n-1}(i_{l_1} + \cdots + i_{l_{k+1}})\Big) > 0 \quad (\text{respectively}, \ \geq)$$

$$\Longleftrightarrow \sum_{l=0}^{n-1}(a_l - a_n)(i_{l_1} + \cdots + i_{l_{k+1}}) - \frac{d(k+1)}{n+1} \cdot A_\lambda > 0 \qquad (\text{respectively}, \ \geq)$$

$$\Longleftrightarrow \frac{\tilde{a}_{I_1,\dots,I_{k+1}}}{A_\lambda} > \frac{d(k+1)}{n+1} \quad (\text{respectively}, \ \geq).$$

This completes the proof. ∎

When $k = 0$, we will also adopt the following convention. Given any point $\mathcal{L} \in \mathcal{X}_{0,d,n}$ generated by some hypersurface H_f that is given by the zero locus of some homogeneous polynomial $f = \sum_I f_I x_I$, we will write $\omega(f, \lambda)$ instead of $\omega(\mathcal{L}, \lambda)$. In other words, we set

$$\omega(f, \lambda) := \min\Big\{\sum_{l=0}^{n-1}(a_l - a_n) \cdot i_l \ \ | \ \ f_I \neq 0\Big\}.$$

Then, a crucial observation is that $\omega(-, \lambda)$ is additive in the following sense.

Lemma 2.3. *Fix a one-parameter subgroup* λ *of* $\mathrm{SL}(V)$ *and two non-zero homogeneous polynomials* f *and* g*, not necessarily of the same degree. Then,*

$$\omega(f \cdot g, \lambda) = \omega(f, \lambda) + \omega(g, \lambda).$$

Proof. The statement follows from the fundamental theory of graded polynomial rings, but we present here a short proof for the reader's convenience.

First, write $f = \sum_I f_I x_I$ and $g = \sum_J g_J x_J$. Next, set

$$f' := \sum_{I'} f_{I'} x_{I'} \quad \text{and} \quad g' := \sum_{J'} g_{J'} x_{J'},$$

where the tuples I' (respectively, J') run over all the tuples I (respectively, J) such that we have $\omega(x_I, \lambda) = \omega(f, \lambda)$ (respectively, $\omega(x_J, \lambda) = \omega(g, \lambda)$). Then, it is routine to check that

$$\omega(f' \cdot g', \lambda) = \omega(f', \lambda) + \omega(g', \lambda) = \omega(f, \lambda) + \omega(g, \lambda).$$

Now, let $f'' := f - f'$ and $g'' := g - g'$, and observe that

$$\omega(f, \lambda) + \omega(g, \lambda) < \min\{\omega(f'' \cdot g', \lambda), \omega(f' \cdot g'', \lambda), \omega(f'' \cdot g'', \lambda)\}$$

and that

$$\omega(f \cdot g, \lambda) = \min\{\omega(f' \cdot g', \lambda), \omega(f'' \cdot g', \lambda), \omega(f' \cdot g'', \lambda), \omega(f'' \cdot g'', \lambda)\}.$$

Putting all of the above together, we obtain the assertion. ∎

We can use Lemma 2.3 to obtain a partial criterion for (semi)stability of hypersurfaces of any degree α in $\mathbb{P}^n$, which is relevant when applying our results to concrete examples, and which one can use to derive many corollaries from Theorem 3.3. We will now state and prove this partial criterion.

Proposition 2.4. *Let* $H_f = H_{f_1} + \cdots + H_{f_r}$ *be a reducible hypersurface of degree* α *in* $\mathbb{P}^n$*. If all the hypersurfaces* H_{f_i} *are GIT semistable, then* H *is GIT semistable. Furthermore, if one of the* H_{f_i} *is GIT stable, then* H *is GIT stable.*

Proof. Suppose that each H_{f_i} has degree d_i. Consider the actions of $\mathrm{SL}(V)$ on $\mathbb{P}(S^\alpha V^\vee)$ and on $\mathbb{P}(S^{d_i} V^\vee)$ for $i = 1, \dots, r$. Choose any one-parameter subgroup λ of $\mathrm{SL}(V)$. If all the H_{f_i} are semistable, then

$$\frac{\omega(f_i, \lambda)}{A_\lambda} \leq \frac{d_i}{n+1} \quad \text{for all } i = 1, \dots, r.$$

In particular,

$$\frac{\omega(f, \lambda)}{A_\lambda} = \sum_{i=1}^{r} \frac{\omega(f_i, \lambda)}{A_\lambda} \leq \sum_{i=1}^{r} \frac{d_i}{n+1} = \frac{\alpha}{n+1}, \tag{2.4}$$

where we use Lemma 2.3 in the first equality. Now, if there is some j such that H_{f_j} is stable, then we further have $\omega(f_j, \lambda)/A_\lambda < d_j/(n+1)$, and the inequality in (2.4) is strict. In both cases, the statement then follows from Proposition 2.2. ∎

Remark 2.5. Observe that in Proposition 2.4, we are neither assuming that the components H_{f_i} are of the same degree nor do we assume that they are irreducible. Moreover, Proposition 2.4 also follows from Proposition B.1 in [25]. In [25], Okawa proves that if two cycles of the same dimension in some projective space are Chow semistable, then their sum is also Chow semistable. Moreover, if one of the cycles is Chow stable, then so is their sum. We note that Chow stability of hypersurfaces is nothing but GIT stability as we consider here.

Similar to [32], Lemma 2.3 can further be used to relate GIT stability of a linear system $\mathcal{L} \in \mathcal{X}_{k,d,n}$ to GIT stability of the hypersurfaces H_{f_j} lying on $\mathcal{L}$ and, in particular, to the number $\operatorname{lct}(\mathbb{P}^n, H_{f_j})$, known as the log canonical threshold of the pair $(\mathbb{P}^n, H_{f_j})$. We will do this in Section 3. In preparation, we present next some basic notions concerning log canonical pairs and toric valuations and their connection to GIT stability. See Section 8 of [19] for more details.

2.4. The log canonical threshold

Let X be a smooth projective variety and E a prime divisor over X, that is, a prime divisor on a smooth variety Y that admits a proper birational morphism $\pi\colon Y \to X$. Then, we define the *log discrepancy* of E to be the number

$$A_X(E) = 1 + \operatorname{ord}_E(K_{Y/X}),$$

where $\operatorname{ord}_E(K_{Y/X})$ denotes the coefficient of E appearing in $K_{Y/X} := K_Y - \pi^* K_X$. In particular, we say that a pair (X, D) (where D is an effective $\mathbb{Q}$-divisor on X) is *log canonical* if for any prime divisor E over X as above one has $A_X(E) - \operatorname{ord}_E(D) \geq 0$, where $\operatorname{ord}_E(D)$ denotes the coefficient of E in $\pi^* D$. In general, given X and E as above, we can define a valuation ord_E on X that sends each rational function in $K(X)^\times = K(Y)^\times$ to its order of vanishing along E. Using these notions, we can now introduce the following algebraic invariant.

Definition 2.6. The *log canonical threshold* of a pair (X, D) is the rational number

$$\operatorname{lct}(X, D) = \inf_E \frac{A_X(E)}{\operatorname{ord}_E(D)},$$

where E runs over all prime divisors over X (i.e., all prime divisors E on some model Y as above) such that $\operatorname{ord}_E(D) \neq 0$.

It is well known that in Definition 2.6, the infimum is a minimum, and we say that a prime divisor E over X computes $\operatorname{lct}(X, D)$ if $\operatorname{lct}(X, D) = A_X(E)/\operatorname{ord}_E(D)$. Moreover, when X and Y admit a torus action, a special role is played by those prime divisors $E \subset Y$ over X which are invariant. This motivates the definition below.

Definition 2.7. When $X = \mathbb{P}^n$, a prime divisor E over $\mathbb{P}^n$ is *toric* if there exists a T-equivariant proper birational morphism $\pi\colon Y \to X$ from a normal variety with a T-action such that E is a T-invariant prime divisor on Y with respect to some maximal torus $T \subset \mathrm{SL}(n+1)$.

The usefulness, for our purposes, of this notion of toric divisors lies in the following lemma (cf. Section 8 in [19] and Proposition 4.2 in [32]).

Lemma 2.8. *Let $H_f : (f = 0) \subset \mathbb{P}^n$ be a hypersurface of degree d and let λ be a non-trivial one-parameter subgroup of* $\mathrm{SL}(n+1)$. *Then there exists a toric prime divisor E over $\mathbb{P}^n$ such that*

$$\frac{\omega(f,\lambda)}{A_\lambda} = \frac{\mathrm{ord}_E(H_f)}{A_{\mathbb{P}^n}(E)}.$$

Proof. As in (2.3), we choose coordinates $(x_0 : \ldots : x_n)$ in $\mathbb{P}^n$ that normalize λ. Then, the image of λ lies in the fixed maximal torus $T \subset \mathrm{SL}(n+1)$ of the diagonal matrices in the chosen coordinates, which is acting on $\mathbb{P}^n$, and we can consider $\mathbb{P}^n$ as a toric variety. In particular, we claim that there exists a toric prime divisor E over $\mathbb{P}^n$ and $c > 0$ such that $\omega(f,\lambda) = c \cdot \mathrm{ord}_E(H_f)$ for any non-zero homogeneous polynomial f. Indeed, we can take E to be the exceptional divisor of the weighted blow-up of $\mathbb{A}^n \cong (x_n \neq 0) \subset \mathbb{P}^n$ with respect to the weights $a_i - a_n$ for x_i/x_n. Here, we remark that $a_0 - a_n > 0$ by the assumption on λ and hence E is well defined. Then, for such E, we have that $c = A_\lambda / A_{\mathbb{P}^n}(E)$ by Proposition 5.1 in [16]. ■

This allows us to obtain the following sufficient criterion for a hypersurface of degree d in $\mathbb{P}^n$ to be GIT (semi)stable, which was first observed by Hacking (Propositions 10.2 and 10.4 in [12]), and Kim–Lee (Theorem 2.3 in [18]) in the case of plane curves.

Corollary 2.9. *If a hypersurface $H_f : (f = 0) \subset \mathbb{P}^n$ of degree d is GIT unstable (respectively, non-stable), then*

$$\frac{d}{n+1} < \frac{1}{\mathrm{lct}(\mathbb{P}^n, H_f)} \qquad (\textit{respectively}, \ \leq).$$

Moreover, the converse also holds if a toric divisor over $\mathbb{P}^n$ computes $\mathrm{lct}(\mathbb{P}^n, H_f)$.

Proof. By assumption, we can find a one-parameter subgroup λ such that

$$\frac{d}{n+1} < \frac{\omega(f,\lambda)}{A_\lambda} \qquad (\text{respectively}, \ \leq).$$

Applying Lemma 2.8 to this one-parameter subgroup, we obtain the result by observing that

$$\mathrm{lct}(\mathbb{P}^n, H_f) \leq \inf_{E:\text{toric}} \frac{A_{\mathbb{P}^n}(E)}{\mathrm{ord}_E(H_f)}, \tag{2.5}$$

which follows from Definition 2.6. ■

Remark 2.10. The inequality (2.5) can in general be strict. For example, for any Płoski curve C_d of even degree d we have $\mathrm{lct}(X, C_d) = 5/(2d)$ but

$$\inf_{E:\text{toric}} \frac{A_{\mathbb{P}^n}(E)}{\mathrm{ord}_E(C_d)} = \frac{3}{d},$$

since C_d is known to be strictly GIT semistable (cf. [9]).

Moreover, the following construction shows that toric divisors, as in Definition 2.7, appear naturally. This construction will be helpful to us in Section 5.2.

Remark 2.11. Consider $X = \mathbb{P}^n$ as a toric variety with respect to some maximal torus $T \subset \mathrm{SL}(n+1)$. Let $\pi\colon Y \to X$ be a T-equivariant proper birational morphism from a smooth variety. Let D_1 and D_2 be toric prime divisors on Y intersecting at a closed point p. Then p is T-invariant and hence the blow-up $\mu\colon Z \to Y$ at p is also T-equivariant. In particular, the exceptional divisor E of μ is toric.

2.5. Chow stability

Finally, we end this background section with a short description of the notion of Chow stability of algebraic cycles on $\mathbb{P}^n$ (cf. [22]).

Let $\mathrm{Chow}_{r,d}$ be the Chow variety parameterizing r-codimensional cycles of degree d. Consider the natural closed immersion $\iota\colon \mathrm{Chow}_{r,d} \hookrightarrow \mathbb{P}(W)$, where $W = \otimes^{r+1}(S^d(V^\vee))$, and consider $\mathcal{M} := \iota^*\mathcal{O}_{\mathbb{P}(W)}(1)$ together with its unique $\mathrm{SL}(V)$-linearization. We call an r-codimensional algebraic cycle X of $\mathbb{P}^n$ of degree d *Chow (semi)stable* if the corresponding point of $\mathrm{Chow}_{r,d}$ is GIT (semi)stable with respect to $\mathcal{M}$. For details, see Section 1.16 of [22].

3. A complete criterion for (semi)stability

Adapting the work in [32] by the second named author, we will now state and prove Lemmas 3.1 and 3.2 below, which generalize Proposition 3.5 in [32] and Corollary 3.11 in [32], and are the main ingredients in the proof of Theorem 1.1.

Lemma 3.1. *Let $\mathcal{L} \in \mathcal{X}_{k,d,n}$ and let $H_{f_1}, \dots, H_{f_l} \in \mathcal{L}$ be $l \le k+1$ linearly independent hypersurfaces. Then, for any one-parameter subgroup $\lambda\colon \mathbb{C}^\times \to \mathrm{SL}(V)$,*

$$\sum_{j=1}^{l} \omega(f_j, \lambda) \le \omega(\mathcal{L}, \lambda).$$

Proof. Choose $k-l+1$ linearly independent hypersurfaces in $\mathcal{L}$, say $H_{f_{l+1}}, \dots, H_{f_{k+1}}$, so that $H_{f_1}, \dots, H_{f_l}, H_{f_{l+1}}, \dots, H_{f_{k+1}}$ generate $\mathcal{L}$. Choose λ and fix $I_1, \dots, I_{k+1}$ such that $\omega(\mathcal{L}, \lambda) = \tilde{a}_{I_1,\dots,I_{k+1}}$. Then $M_{I_1,\dots,I_{k+1}} \ne 0$ and the corresponding determinant as in (2.2) is non-zero by definition. In particular, each row i and each column j of (2.2) must contain a non-zero element. Thus, there exists a permutation $\sigma \in S_{k+1}$ such that $f_{I_j}^{\sigma(j)} \ne 0$ for all $j = 1, \dots, k+1$, and it follows that

$$\begin{aligned}\omega(\mathcal{L}, \lambda) = \tilde{a}_{I_1,\dots,I_{k+1}} &= \sum_{l=0}^{n-1} (a_l - a_n)(i_{l,1} + \dots + i_{l,k+1}) \\ &= \sum_{j=1}^{k+1} \underbrace{\Big(\sum_{l=0}^{n-1} (a_l - a_n) \cdot i_{l,j} \Big)}_{\ge \omega(f_{\sigma(j)}, \lambda)} \ge \sum_{j=1}^{k+1} \omega(f_j, \lambda) \ge \sum_{j=1}^{l} \omega(f_j, \lambda).\end{aligned}$$

This completes the proof. ∎

Lemma 3.2. *Let $\mathcal{L} \in \mathcal{X}_{k,d,n}$, let $\lambda\colon \mathbb{C}^\times \to \mathrm{SL}(V)$ denote a one-parameter subgroup and $H_{f_1} \in \mathcal{L}$ a hypersurface. Then, there exist $H_{f_2}, \dots, H_{f_{k+1}} \in \mathcal{L}$ such that the $k+1$ hypersurfaces $H_{f_1}, H_{f_2}, \dots, H_{f_{k+1}}$ generate $\mathcal{L}$ and*

$$\omega(\mathcal{L}, \lambda) = \sum_{j=1}^{k+1} \omega(f_j, \lambda).$$

Proof. Fix $\mathcal{L} \in \mathcal{X}_{k,d,n}$, $\lambda\colon \mathbb{C}^\times \to \mathrm{SL}(V)$ and $H_{f_1} \in \mathcal{L}$ as above. We claim that we can find hypersurfaces $H_{f_2}, \dots, H_{f_{k+1}} \in \mathcal{L}$ all distinct and tuples $I_1, \dots, I_{k+1}$ such that

(i) the hypersurfaces $H_{f_1}, H_{f_2}, \dots, H_{f_{k+1}}$ generate $\mathcal{L}$,

(ii) the Plücker coordinate $M_{I_1,\dots,I_{k+1}}$ is non-zero and is given by

$$M_{I_1,\dots,I_{k+1}} = \begin{vmatrix} f^1_{I_1} & f^1_{I_2} & f^1_{I_3} & \dots & f^1_{I_{k+1}} \\ 0 & f^2_{I_2} & f^2_{I_3} & \dots & f^2_{I_{k+1}} \\ 0 & 0 & f^3_{I_3} & \dots & f^3_{I_{k+1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & f^{k+1}_{I_{k+1}} \end{vmatrix} \neq 0,$$

and

(iii) $\omega(f_j, \lambda) = \sum_{l=0}^{n-1} (a_l - a_n) \cdot i_{l,j}$ for all $j = 1, \dots, k+1$.

If our claim holds, then we have

$$\begin{aligned} \omega(\mathcal{L}, \lambda) \le \tilde{a}_{I_1,\dots,I_{k+1}} &= \sum_{l=0}^{n-1} (a_l - a_n)(i_{l,1} + \dots + i_{l,k+1}) \\ &= \sum_{j=1}^{k+1} \Big(\sum_{l=0}^{n-1} (a_l - a_n) \cdot i_{l,j} \Big) = \sum_{j=1}^{k+1} \omega(f_j, \lambda). \end{aligned}$$

Hence, $\omega(\mathcal{L}, \lambda) = \sum_{j=1}^{k+1} \omega(f_j, \lambda)$ holds by Lemma 3.1. Thus, it suffices to prove our claim, which we do next.

First, we choose any k hypersurfaces $H_{f_2}, \dots, H_{f_{k+1}} \in \mathcal{L}$ such that $H_{f_1}, H_{f_2}, \dots, H_{f_{k+1}}$ generate $\mathcal{L}$. We will show that up to replacing each H_{f_j} ($j-1$ times), for $j = 2, \dots, k+1$, we can find $k+1$ tuples $I_1, \dots, I_{k+1}$ as above such that (i), (ii) and (iii) hold.

Indeed, fix I_1 such that $f^1_{I_1} \neq 0$ and $\omega(f_1, \lambda) = \sum_{l=0}^{n-1} (a_l - a_n) \cdot i_{l,1}$. Then by replacing f_j with

$$f'_j = f_j - \frac{f^j_{I_1}}{f^1_{I_1}} f_1$$

for all $j = 2, \dots, k+1$, we have that $f^j_{I_1} = 0$ for all $j = 2, \dots, k+1$. In particular, if we choose $I_2, \dots, I_{k+1}$ and we let $M(I_2, \dots, I_{k+1})$ denote the sub-determinant of (2.2) computed by removing the first row and the first column, then the corresponding Plücker coordinate $M_{I_1 I_2 \dots I_{k+1}} = f^1_{I_1} M(I_2, \dots, I_{k+1})$ will be non-zero whenever $M(I_2, \dots, I_{k+1}) \neq 0$.

Therefore, if we now fix I_2 such that $f^2_{I_2} \neq 0$ and $\omega(f_2, \lambda) = \sum_{l=0}^{n-1}(a_l - a_n) \cdot i_{l,2}$, then by replacing f_j with

$$f'_j = f_j - \frac{f^j_{I_2}}{f^2_{I_2}} f_2$$

for all $j = 3, \dots, k+1$, we have that $f^j_{I_2} = 0$ for all $j = 3, \dots, k+1$, and thus that $M_{I_1 I_2 I_3 \dots I_{k+1}} \neq 0$ for all choices of tuples $I_3, \dots, I_{k+1}$ such that

$$\begin{vmatrix} f^3_{I_3} & f^3_{I_4} & \cdots & f^3_{I_{k+1}} \\ f^4_{I_3} & f^4_{I_4} & \cdots & f^4_{I_{k+1}} \\ \vdots & \vdots & \ddots & \vdots \\ f^{k+1}_{I_3} & f^{k+1}_{I_4} & \cdots & f^{k+1}_{I_{k+1}} \end{vmatrix} \neq 0.$$

By iterating this argument $k+1$ times, we find the desired tuples $I_1, I_2, \dots, I_{k+1}$ and the desired generators for $\mathcal{L}$, which satisfy (i), (ii) and (iii) by construction. ■

As a consequence, using both Lemmas 3.1 and 3.2, we can finally state and prove the following slightly stronger version of Theorem 1.1.

Theorem 3.3. *Let $\mathcal{L} \in \mathcal{X}_{k,d,n}$ and fix a hypersurface $H_{f_1} \in \mathcal{L}$. Then, $\mathcal{L}$ is GIT stable (respectively, semistable) if and only if, for any choice of k hypersurfaces $H_{f_2}, \dots, H_{f_{k+1}}$ in $\mathcal{L}$ such that $H_{f_1}, H_{f_2}, \dots, H_{f_{k+1}}$ generate $\mathcal{L}$, the degree $d(k+1)$ hypersurface $H_{f_1} + \cdots + H_{f_{k+1}}$ is GIT stable (respectively, semistable).*

Proof. Suppose first that the linear system $\mathcal{L}$ is GIT stable (respectively, semistable) and choose any one-parameter subgroup λ of $\mathrm{SL}(V)$. Given $H_{f_2}, \dots, H_{f_{k+1}} \in \mathcal{L}$ such that $H_{f_1}, H_{f_2}, \dots, H_{f_{k+1}}$ generate $\mathcal{L}$, by Lemmas 2.3, 3.1 and Proposition 2.2, we have

$$\frac{d(k+1)}{n+1} \overset{(\text{respectively}, \geq)}{>} \frac{\omega(\mathcal{L}, \lambda)}{A_\lambda} \geq \sum_{j=1}^{k+1} \frac{\omega(f_j, \lambda)}{A_\lambda} = \frac{\omega(f_1 \cdots f_{k+1}, \lambda)}{A_\lambda},$$

and we conclude that $H_{f_1} + \cdots + H_{f_{k+1}}$ is GIT stable (respectively, semistable). Here, we apply Proposition 2.2 with $k = 0$ and with d replaced by $d(k+1)$.

Choose now any one-parameter subgroup λ of $\mathrm{SL}(V)$ for the converse. Given the fixed hypersurface $H_{f_1} \in \mathcal{L}$, by Lemma 3.2 we can find k hypersurfaces $H_{f_2}, \dots, H_{f_{k+1}} \in \mathcal{L}$ such that $H_{f_1}, H_{f_2}, \dots, H_{f_{k+1}}$ generate $\mathcal{L}$ and

$$\frac{\omega(\mathcal{L}, \lambda)}{A_\lambda} = \sum_{j=1}^{k+1} \frac{\omega(f_j, \lambda)}{A_\lambda} = \frac{\omega(f_1 \cdots f_{k+1}, \lambda)}{A_\lambda}.$$

Because the hypersurface $H_{f_1} + \cdots + H_{f_{k+1}}$ is GIT stable (respectively, semistable), we have (again by applying Proposition 2.2 with $k = 0$ and with d replaced by $d(k+1)$)

$$\frac{\omega(f_1 \cdots f_{k+1}, \lambda)}{A_\lambda} < \frac{d(k+1)}{n+1} \quad (\text{respectively}, \ \leq)$$

and it follows from Proposition 2.2 that $\mathcal{L}$ is GIT stable (respectively, semistable). ■

4. Some partial criteria for (semi)stability

We now collect several partial criteria for (semi)stability of points in $\mathcal{X}_{k,d,n}$, which emanate from our main result except for Corollary 4.4. First, we state and prove a slightly more precise version of Corollary 1.2.

Corollary 4.1. *Let $\mathcal{L} \in \mathcal{X}_{k,d,n}$ and let $H_1 \in \mathcal{L}$. If $\mathcal{L}$ is GIT non-stable (respectively, unstable), then there exists a choice of k hypersurfaces $H_2, \dots, H_{k+1} \in \mathcal{L}$ such that $H_1, H_2, \dots, H_{k+1}$ generate $\mathcal{L}$ and*

$$\operatorname{lct}(\mathbb{P}^n, H_1 + \cdots + H_{k+1}) \leq \frac{n+1}{d(k+1)} \quad (\textit{respectively}, \ <).$$

If a toric prime divisor E over $\mathbb{P}^n$ computes $\operatorname{lct}(\mathbb{P}^n, H_1 + \cdots + H_{k+1})$, then the converse also holds.

Proof. By Theorem 3.3, if $\mathcal{L} \in X$ is as in the statement, then there is a choice of generators of $\mathcal{L}$ such that their summation (union) is a non-stable (respectively, unstable) hypersurface of degree $d(k+1)$. This hypersurface satisfies the above inequality involving the log canonical threshold by Lemma 2.8. ■

Next, we obtain the following criterion for pencils of hypersurfaces.

Corollary 4.2. *Let $\mathcal{P} \in \mathcal{X}_{1,d,n}$ be a pencil and choose $H_f \in \mathcal{P}$. Then $\mathcal{P}$ is GIT non-stable (respectively, unstable) if and only if there exists $H_g \in \mathcal{P}$ other than H_f such that $H_f + H_g$ is GIT non-stable (respectively, unstable).*

Proof. Here, we are simply restating Theorem 3.3 in the case $k = 1$. Note that any two distinct members give a generating set for a pencil. ■

We also have the result below, which states that GIT stability of linear systems can be described inductively in some cases.

Corollary 4.3. *Let $k > 1$ and $\mathcal{L} \in \mathcal{X}_{k,d,n}$ be a linear system containing at least one GIT semistable hypersurface (e.g., a smooth one). If $\mathcal{L}$ is GIT non-stable (respectively, unstable), then there exists a sub-linear system of dimension $k-1$ that is GIT non-stable (respectively, unstable).*

Proof. Let $\mathcal{L} \in \mathcal{X}_{k,d,n}$ be as in the statement and pick $H_{f_1} \in \mathcal{L}$ semistable. If the linear system $\mathcal{L}$ is non-stable (respectively, unstable), then by combining Proposition 2.2 with Lemma 3.2, we can find $H_{f_2}, \dots, H_{f_{k+1}} \in \mathcal{L}$ such that the $k+1$ hypersurfaces $H_{f_1}, H_{f_2}, \dots, H_{f_{k+1}}$ generate $\mathcal{L}$ and the hypersurface $H_{f_2} + \cdots + H_{f_{k+1}}$ of degree dk is non-stable (respectively, unstable). Therefore, Theorem 3.3 shows that the sub-linear system of $\mathcal{L}$ generated by $H_{f_2}, \dots, H_{f_{k+1}}$ of dimension $k-1$ is GIT non-stable (respectively, unstable). ■

Moreover, we obtain the following criterion for *regular* linear systems $\mathcal{L} \in \mathcal{X}_{k,d,n}$, meaning linear systems with a base locus of dimension $n-k-1$.

Corollary 4.4. *Let $k \leq n-1$ and let $\mathcal{L} \in \mathcal{X}_{k,d,n}$ be a regular linear system generated by $H_{f_1}, \dots, H_{f_{k+1}}$. If the complete intersection $H_{f_1} \cap \cdots \cap H_{f_{k+1}} \subset \mathbb{P}^n$ is Chow stable, then $\mathcal{L}$ is GIT stable.*

Proof. Choose arbitrary generators $H_{f_1}, \dots, H_{f_{k+1}}$ of $\mathcal{L}$ as above. It is shown in Theorem 1.1 of [28] that if the base locus of $\mathcal{L}$ is Chow stable, then for any one-parameter subgroup $\lambda \colon \mathbb{C}^\times \to \mathrm{SL}(n+1)$, we have that

$$\sum_{i=1}^{k+1} \mu^{\mathcal{L}}(H_{f_i}, \lambda) < 0,$$

where we consider the usual Hilbert–Mumford weights for hypersurfaces of degree d, that is, we consider the $\mathrm{SL}(n+1)$-action on $\mathbb{P}(S^d V^\vee)$ and we choose the ample line bundle $\mathcal{L} = \mathcal{O}(1)$, together with its unique $\mathrm{SL}(n+1)$-linearization. Thus, the assertion follows from Lemmas 3.1 and 3.2 and Proposition 2.2. An alternative proof is also provided in Appendix A. ∎

Although this last result is not a direct consequence of Theorem 3.3, the idea of relating Sano's result (Theorem 1.1 in [28]) on Chow stability of complete intersections to GIT stability of linear systems is new. It can be seen as an application of the fundamental lemmas from Section 3 and is included here for this reason.

Finally, we remark that many concrete examples of GIT stable linear systems of hypersurfaces that are not complete intersections can be constructed. The following is also a consequence of our main theorem.

Corollary 4.5. *Let $d > m$ and let $\tilde{\mathcal{L}}$ be a k-dimensional linear system of hypersurfaces of degree $d-m$ in $\mathbb{P}^n$ that is GIT stable (respectively, semistable). If $\tilde{H}_1, \dots, \tilde{H}_{k+1}$ generate $\tilde{\mathcal{L}}$ and $H \subset \mathbb{P}^n$ is an arbitrary GIT semistable (respectively, stable) hypersurface of degree m, then the linear system $\mathcal{L}$ generated by the $k+1$ hypersurfaces $H_i := H + \tilde{H}_i$ of degree d is GIT stable.*

Proof. Choose generators $H_{f_1}, \dots, H_{f_{k+1}}$ for $\mathcal{L}$. Then

$$H_{f_1} + \cdots + H_{f_{k+1}} = (k+1)H + \tilde{H}_{g_1} + \cdots + \tilde{H}_{g_{k+1}},$$

where the $\tilde{H}_{g_i}$ generate $\tilde{\mathcal{L}}$. Now, since $\tilde{\mathcal{L}}$ is stable (respectively, semistable), it follows from Theorem 3.3 that the hypersurface $\tilde{H}_{g_1} + \cdots + \tilde{H}_{g_{k+1}}$ is stable (respectively, semistable). In particular, since H is semistable (respectively, stable), the hypersurface $H_{f_1} + \cdots + H_{f_{k+1}}$ is stable by Proposition 2.4. Thus, the linear system $\mathcal{L}$ is stable by Theorem 3.3. ∎

5. Stability of pencils of plane curves

This section will show how our stability criterion can be used to recover the results in [21]. In addition, we will also provide a complete description of the stability of so-called Halphen pencils of index m, which are pencils of plane curves of degree $3m$ that have exactly nine base points, each of multiplicity m.

5.1. Pencils of plane cubics revisited

The work of Miranda in [21] describes GIT stability of a pencil of plane cubics (with general member smooth) in terms of the types of singular fibers appearing in the corresponding rational elliptic surface, which is obtained by blowing up $\mathbb{P}^2$ at the nine base points of the pencil. Using Kodaira's notation for the singular fibers, he proves the following.

Theorem ([21]). *A pencil of plane cubics $\mathcal{P}$ is stable if and only if $\mathcal{P}$ contains a smooth member and every fiber of the corresponding rational elliptic surface $X_{\mathcal{P}}$ is reduced. Moreover, if $\mathcal{P}$ contains a smooth member, then $\mathcal{P}$ is semistable if and only if $X_{\mathcal{P}}$ does not contain a fiber of type* II^*, III^* *or* IV^*.

We will now describe how we can use Corollary 4.2 to recover Miranda's result. First, observe that by Theorem 2.3 in [29] and Corollary 4.2, a pencil of plane cubics $\mathcal{P}$ is stable if and only if for any two (distinct) cubics C_f, C_g in $\mathcal{P}$ the sextic curve $C_f + C_g$ satisfies the following conditions:

(i) it does not contain a multiple line as a component,

(ii) it does not have consecutive triple points, and

(iii) it does not have a singular point of multiplicity at least four.

Therefore, a pencil of plane cubics $\mathcal{P}$ is stable if and only if the following conditions hold:

(i') $\mathcal{P}$ contains a smooth member,

(ii') any curve in $\mathcal{P}$ is reduced, and

(iii') any curve in $\mathcal{P}$ is either smooth or has, at worst, one node as a singularity at a base point of $\mathcal{P}$.

It is routine to check that conditions (ii') and (iii') hold if and only if every fiber of $X_{\mathcal{P}}$ is reduced.

Finally, by [29] and Corollary 4.2, $\mathcal{P}$ is unstable if and only if we can find two cubics C_f and C_g in $\mathcal{P}$ such that the sextic curve $C_f + C_g$ satisfies one of the following conditions:

(a) it has a line as a component, and it has a triple point on that line, which remains a triple point with a threefold tangent under a blow-up;

(b) it has a quadruple point, which has a threefold or a fourfold tangent;

(c) it has a singular point of multiplicity at least five.

Assuming that $\mathcal{P}$ has a smooth member, we see that one of the conditions (a), (b), or (c) holds if and only if, up to relabeling, one of the following situations occurs:

(a') C_f is a triple line and C_g is arbitrary;

(b') C_f is the union of a double line and another line, and C_g is tangent to the double line at the intersection point (of the two lines);

(c') C_f is the union of a double line and another line, and C_g intersects the double line at a single point.

Now, if that is the case, then the fiber of $X_{\mathcal{P}}$ corresponding to C_f will have a component of multiplicity three and hence it must be of type II^*, III^* or IV^*. Conversely, if $X_{\mathcal{P}}$

contains a fiber of type II^*, III^* or IV^*, then by Proposition 4.2 in [31], we know that $\mathcal{P}$ is the pencil $\lambda C_f + \mu C_g = 0$, where C_f consists of either a triple line or a double line and another line. This is the content of Lemma 6.4 in [21], which is the hardest step in the proof of Theorem 6.1 in [21]. Moreover, if C_f consists of a double line and another line, then C_f and C_g must intersect as in (b') or (c'). Otherwise, blowing up the nine base points of $\mathcal{P}$ would not yield a fiber with a component of multiplicity three.

5.2. Halphen pencils

We now provide a complete description of the stability of certain pencils of plane curves of degree $3m$, called Halphen pencils of index m. First studied by the French mathematician Georges Henri Halphen in [13], these pencils are characterized by the property that they have precisely nine base points (possibly infinitely near), each of multiplicity m, and their general members are integral curves. For details, see [10]. The case of index $m = 1$ consists of pencils of plane cubics with a smooth member. The case $m = 2$ is also well understood, and it has been considered by the second named author in [31] and [33]. Here, we address all other cases.

Given a Halphen pencil $\mathcal{P}$ of index m, taking the minimal resolution of its base points, one obtains a so-called rational elliptic surface of index m, that is, a (smooth and projective) rational surface Y that comes equipped with a relatively minimal genus-one fibration which is given by the linear system $|-mK_Y|$ and has precisely one multiple fiber of multiplicity m. As in [21] and [33], our strategy is to explore the geometry of Y to describe the stability of $\mathcal{P}$. We will first prove the following.

Theorem 5.1. *Let $\mathcal{P}$ be a Halphen pencil of index $m \geq 1$ and let Y denote the corresponding rational elliptic surface. If $\mathrm{lct}(Y, F) > 1/(2m)$ (respectively, $\geq$) for any fiber F, then $\mathcal{P}$ is GIT stable (respectively, semistable).*

Remark 5.2. The table below gives the number $\mathrm{lct}(Y, F)$, depending on the type of the fiber F according to Kodaira's notation.

$\mathrm{lct}(Y, F)$	Type of F	$\mathrm{lct}(Y, F)$	Type of F
$1/m$	${}_m\mathrm{I}_n$	$1/2$	I_n^*
$5/6$	II	$1/6$	II^*
$3/4$	III	$1/4$	III^*
$2/3$	IV	$1/3$	IV^*

Remark 5.3. The condition $\mathrm{lct}(Y, F) > 1/(2m)$ is known to be equivalent to the notion of uniform adiabatic K-stability of rational elliptic surfaces of index m introduced in [14]. In particular, Theorem 5.1 implies that GIT stability of Halphen pencils is closely related to the existence of Kähler metrics with constant scalar curvature on the corresponding rational elliptic surfaces.

We need the following lemma to prove Theorem 5.1.

Lemma 5.4. *Let $\mathcal{P}$ be a pencil of plane curves of degree $m \cdot s$ with s^2 base points (possibly infinitely near), each of multiplicity m. Suppose that $\mathcal{P}$ contains at least one integral curve. Let $\pi\colon Y \to \mathbb{P}^2$ denote the s^2-fold blow-up which resolves $\mathcal{P}$, and let $q\colon Y \to \mathbb{P}^1$*

be the morphism induced by $\mathcal{P}$. *Then there exists a curve* $C_f \in \mathcal{P}$ *such that for any other curve* $C_g \in \mathcal{P}$, *we have that*

$$K_Y + \frac{1}{2m}(F_f + F_g) = \pi^*\left(K_{\mathbb{P}^2} + \frac{1}{2m}(C_f + C_g)\right),$$

where F_f *and* F_g *are the corresponding fibers of* q.

Proof. We choose C_f as a curve in $\mathcal{P}$ corresponding to a smooth fiber of q. Given any other curve C_g in $\mathcal{P}$, let F_g be the fiber of q corresponding to C_g, let $\pi^*\mathcal{P}$ be the pencil on Y generated by π^*C_f and π^*C_g, and let D be the fixed part of $\pi^*\mathcal{P}$. Then, it is routine to check that $m \cdot K_{Y/\mathbb{P}^2} = \mathrm{D}$. In particular,

$$\pi^*\left(K_{\mathbb{P}^2} + \frac{1}{2m}(C_f + C_g)\right) = K_Y - K_{Y/\mathbb{P}^2} + \frac{1}{2m}(F_f + F_g + 2\mathrm{D}) = K_Y + \frac{1}{2m}(F_f + F_g).$$

Thus, we obtain the assertion. ∎

Then, the argument is the one that follows.

Proof of Theorem 5.1. We first observe that Halphen pencils of any index m satisfy the assumptions of Lemma 5.4. We can choose C_f as a curve corresponding to any smooth fiber of the corresponding genus-one fibration. Now, choose any other curve $C_g \in \mathcal{P}$. Lemma 5.4 implies that

$$\mathrm{lct}(Y, F_f + F_g) > \frac{1}{2m} \text{ (respectively, } \geq) \iff \mathrm{lct}(\mathbb{P}^2, C_f + C_g) > \frac{1}{2m} \text{ (respectively, } \geq).$$

Note that $\mathrm{lct}(Y, F_f) = 1$. Thus, $\mathrm{lct}(Y, F_f + F_g) > 1/(2m)$ (respectively, $\geq$) by assumption. Now, because C_g was arbitrary, it follows from Corollary 4.1 that $\mathcal{P}$ is GIT stable (respectively, semistable). ∎

In particular, the following two statements hold (see Remark 5.2). These are direct consequences of Theorem 5.1.

Corollary 5.5. *All Halphen pencils of index* $m > 3$ *are GIT stable. Moreover, a Halphen pencil of index* $m = 3$ *is always GIT semistable, and it is stable whenever the corresponding rational elliptic surface does not contain a fiber of type* II^*.

Corollary 5.6. *Let* $\mathcal{P}$ *be a Halphen pencil of index* $m = 2$. *If the rational elliptic surface* Y *corresponding to* $\mathcal{P}$ *does not contain a fiber of type* II^* *or* III^*, $\mathcal{P}$ *is GIT stable. Moreover, if* Y *does have a fiber of type* III^*, *then* $\mathcal{P}$ *is always semistable* (*cf.* [33]).

We will now consider the converse statement to that of Theorem 5.1. For this, we first observe that the multiple cubic mC in a Halphen pencil $\mathcal{P}$ of index $m > 1$ satisfies $\mathrm{lct}(\mathbb{P}^2, mC) = 1/m$ by Proposition 4.9 in [31], and that the following two lemmas hold.

Lemma 5.7 (Theorem 1.1 in [31]). *If* $m > 1$ *and* F_f *is a reduced fiber of* Y, *then the corresponding curve* C_f *is reduced and we have that*

$$\frac{1}{m} < \mathrm{lct}(\mathbb{P}^2, C_f) \leq \mathrm{lct}(Y, F_f).$$

Lemma 5.8. *If $m > 1$ and Y contains a fiber F such that* $\mathrm{lct}(Y, F) \leq 1/(2m)$ *but the corresponding plane curve $C_g \in \mathcal{P}$ is GIT semistable, then $\mathcal{P}$ is GIT stable.*

Proof. First, note that F is a fiber of type II^* or III^* by Remark 5.2. Moreover, since the Picard number of Y is ten, any other non-multiple fiber of Y is reduced. Thus, we can show that any member $C \in \mathcal{P}$ is GIT semistable. Indeed, when C is the multiple cubic, it follows from Proposition 4.9 in [31] and Corollary 2.9 that C is GIT semistable. On the other hand, if C corresponds to a reduced fiber of Y, then C is GIT stable by Lemma 5.7 and Corollary 2.9.

Take now $C_f \in \mathcal{P}$ corresponding to a smooth fiber F_f. This curve C_f is GIT stable, as shown in the first paragraph. In particular, given any other curve $C_g \in \mathcal{P}$, the curve $C_f + C_g$ of degree $6m$ is GIT stable by Proposition 2.4 since C_g is GIT semistable. Therefore, it follows from Corollary 4.2 that the pencil $\mathcal{P}$ is stable as asserted. ∎

In general, the converse of Theorem 5.1 does not hold. The following two explicit examples show that there are both stable and non-stable Halphen pencils of index $m = 3$ yielding a fiber of type II^*. Nonetheless, we can show that Examples 7.46, 7.47 and 7.55 in [31] and Example 5.10 are precisely the only GIT stable counterexamples to the converse of Theorem 5.1. This will be done in the proof of Theorem 5.11 below.

Example 5.9. Consider the cubic C given by $z^2y + x(y^2 + xz) = 0$. Let Q be the conic $y^2 + xz = 0$ and let L be the line $y = 0$. Then the pencil generated by $C_f = 2Q + 5L$ and $C_g = 3C$ is a Halphen pencil of index three, say $\mathcal{P}$, such that the corresponding rational elliptic surface contains a fiber of type II^* and such pencil is GIT non-stable.

To see why, let λ be the one-parameter subgroup determined (in these coordinates) by $a_0 = 1, a_1 = 0$ and $a_2 = -1$. Then, arguing as in Lemma 3.3 of [9], we compute $\omega(fg, \lambda) = 18$ and it follows from Lemma 3.1 that

$$6 = \frac{\omega(fg,\lambda)}{3} = \frac{\omega(f,\lambda) + \omega(g,\lambda)}{3} \leq \frac{\omega(\mathcal{P},\lambda)}{3} = \frac{\omega(\mathcal{P},\lambda)}{(a_0 - a_2) - (a_1 - a_2)}.$$

Thus, $\mathcal{P}$ is non-stable by Proposition 2.2.

Example 5.10. Consider the cubic C given by $(y^2 + xz)(\alpha y + z) + \beta y x^2 = 0$, for some $\alpha, \beta \neq 0$. Let Q be the conic $y^2 + xz = 0$ and L be the line $y = 0$. Then the pencil $\mathcal{P}$ generated by $C_f = 4Q + L$ and $C_g = 3C$ is a Halphen pencil of index three such that the corresponding rational elliptic surface contains a fiber F of type II^*. Note that $\mathrm{lct}(\mathbb{P}^2, Q + L) = 1$ and $3Q$ is semistable. By Corollary 2.9 and Proposition 2.4, C_f is also semistable. Thus, $\mathcal{P}$ is GIT stable by Lemma 5.8 since F corresponds to the semistable curve C_f.

It is also essential to observe that Example 5.10 is unique up to a choice of projective coordinates, as we now explain. Arguing as in Section 5 of [31], we can show that if a Halphen pencil of index $m = 3$ yields a fiber F of type II^* and the curve corresponding to F consists in the union of a conic taken with multiplicity four and a line (as in Example 5.10), then the conic and the line must intersect at two distinct points, say P_1 and P_2. Moreover, up to relabeling, the unique triple cubic in the pencil, say $3C$, is such that C intersects Q (respectively, L) with multiplicity five (respectively, one) at P_1 and it intersects Q (respectively, L) with multiplicity one (respectively, two) at P_2. In particular, to obtain the corresponding rational elliptic surface Y, we must blow up the plane seven times at P_1 and two times at P_2.

Even more is true. If Q, C and L are as in the previous paragraph, then up to a change of coordinates, we may assume that $P_1 = (0:0:1)$, $P_2 = (1:0:0)$, Q is the conic given by $y^2 + xz = 0$ and L is the line given by $y = 0$. This further implies that C can be taken to be a cubic as in Example 5.10. Indeed, up to scaling, C is given by

$$xz^2 + \alpha y^3 + \beta x^2 y + axy^2 + bxyz + cy^2 z + dyz^2 = 0, \quad \text{with } \alpha \neq 0,$$

and we further know that the line $x = 0$ is tangent to C at P_1 (with multiplicity two since P_1 cannot be a flex). Thus, we may further assume that $d = 0$ and $c = 1$. We can then rewrite the defining polynomial of C as

$$(y^2 + xz)(\alpha y + z) + xy(\beta x + ay + (b - \alpha)z) = 0.$$

Hence, if $I_{P_1}(C, Q)$ denotes the intersection multiplicity of C and Q at P_1, then

$$I_{P_1}(C, Q) = I_{P_1}(Q, L) + I_{P_1}(Q, L') + I_{P_1}(Q, L'') = 3 + I_{P_1}(Q, L''),$$

where $L' : x = 0$ and $L'' : \beta x + ay + (b - \alpha)z = 0$. Furthermore, since the intersection multiplicity $I_{P_1}(C, Q)$ is equal to five, we must have $I_{P_1}(Q, L'') = 2$, which tells us that $L'' = L'$. Hence, $a = 0$ and $b = \alpha$. In other words, the cubic C is precisely as in Example 5.10, as claimed.

We can now prove the following theorem.

Theorem 5.11. *Let $\mathcal{P}$ be a Halphen pencil of index m and let Y be the corresponding rational elliptic surface. Assume that $\mathcal{P}$ is not given by Examples* 7.46*,* 7.47*,* 7.55 *in* [31] *or Example* 5.10 (*up to a projective change of coordinates*)*. If* $\operatorname{lct}(Y, F) < 1/(2m)$ (*respectively,* $\leq 1/(2m)$) *for some fiber F, then $\mathcal{P}$ is GIT unstable* (*respectively, GIT non-stable*)*.*

Proof. The case when $m = 1$ follows from [21] and Remark 5.2. Therefore, by Corollaries 5.5 and 5.6, it suffices to deal with the cases $m = 2$ and $m = 3$.

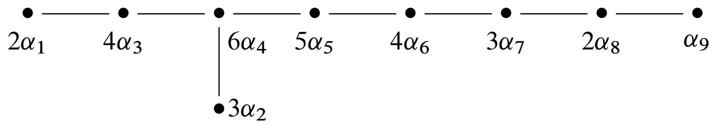
Observe now that, in view of Lemma 5.4 and Corollary 4.1, if there exists a toric divisor $E \subset Y$ over $\mathbb{P}^2$ (see Definition 2.7) and a fiber F of Y such that

$$\frac{A_{\mathbb{P}^2}(E)}{\operatorname{ord}_E(F)} \leq \frac{1}{2m}$$

(respectively, $<$), then the Halphen pencil $\mathcal{P}$ is GIT non-stable (respectively, unstable). We call such toric divisors *destabilizing*, and we say that E makes $\mathcal{P}$ non-stable (respectively, unstable). Thus, in what follows, we will argue that there exists a destabilizing toric divisor for all Halphen pencils of index $m = 2$ and $m = 3$ except for Examples 7.46, 7.47 and 7.55 in [31] and Example 5.10.

When $m = 2$, all Halphen pencils have been completely classified in [31]. Here we simply observe that except for Examples 7.46, 7.47 and 7.55 in [31], we can always find destabilizing toric divisors for all Halphen pencils of index two such that Y contains a fiber F of type II* or III* (see [21] and Section 7 of [31]). Therefore, we focus on the case when $\mathcal{P}$ is a Halphen pencil of index $m = 3$, which is not as in Example 5.10 and Y has a fiber F of type II*. We claim that there exists a toric divisor computing $\operatorname{lct}(Y, F)$ (hence a destabilizing toric divisor) for such pencils.

To prove the claim, the key ingredients we will use are Lemma 4.1 in [31], the fact that (-1)-curves are trisections of Y and Remark 2.11. Recall first that a fiber F of type II* has the dual graph below.



Thus, to prove the claim, we aim to show that the rational curve α_4 is always toric (Definition 2.7).

Fix a nine-fold blow-up $\pi : Y \to \mathbb{P}^2$ and let m_j denote the multiplicity of α_j in F for $j = 1, \dots, 9$. Since $\mathcal{P}$ is a pencil of plane curves of degree nine, there are four cases to consider: (1) α_2 is the strict transform of a cubic, (2) α_2 is the strict transform of a conic, (3) α_2 is the strict transform of a line L, and (4) α_2 is exceptional. We will consider each of these cases separately.

Case (1). This case cannot happen. There exists a unique triple cubic in $\mathcal{P}$, and that corresponds to the unique multiple fiber (see, e.g., Corollary 5.6.3 in [10]).

Case (2). Again, this case cannot happen. By contradiction, assume that $\pi_*(\alpha_2)$ is a conic. Note that $\deg \pi_*(\sum_{j \neq 2} m_j \alpha_j) = 3$, where $\deg C$ denotes the degree of a curve C in $\mathbb{P}^2$. If α_1 is non-exceptional, then $\pi_*(\alpha_1)$ and $\pi_*(\alpha_9)$ must be lines in $\mathbb{P}^2$. Note that α_j for $3 \leq j \leq 8$ are all exceptional. By Lemma 4.1 in [31] and the fact that a (-1)-curve of Y is a trisection, there exists a π-exceptional (-1)-curve E such that $\alpha_8 \cdot E = 1$ but $\alpha_j \cdot E = 0$ for $3 \leq j \leq 7$. If we blow down α_j for $3 \leq j \leq 8$ and E, then we have that $(\pi_*(\alpha_9))^2 \geq 4$, which is impossible. Now, if α_1 is exceptional, then there are three possibilities:

(a) $\pi_*(\alpha_7)$ is a line,

(b) $\pi_*(\alpha_8)$ and $\pi_*(\alpha_9)$ are lines, or

(c) all α_j are exceptional except for α_2 and α_9.

We claim that none of these cases can occur either.

First, in the case (a), all α_j are exceptional except for α_2 and α_7. By Lemma 4.1 in [31] and the fact that a (-1)-curve is a trisection of Y, there exists a π-exceptional (-1)-curve E such that $\alpha_1 \cdot E = 1$ but $\alpha_j \cdot E = 0$ for $3 \leq j \leq 6$. By contracting E, α_1, $\alpha_3, \alpha_4, \alpha_5$ and α_6, we have that $\pi_*(\alpha_2) \cdot \pi_*(\alpha_7) \geq 3$. However, $\pi_*(\alpha_2) \cdot \pi_*(\alpha_7) = 2$ since $\pi_*(\alpha_7)$ is a line and $\pi_*(\alpha_2)$ is a conic. This is a contradiction.

Second, in the case (b), we see that by Lemma 4.1 in [31] there is a π-exceptional (-1)-curve E intersecting either α_7, or both α_1 and α_9 since E is a trisection of Y. If the former holds, then we can show that $(\pi_*(\alpha_8))^2 \geq 5$ by contracting E and all π-exceptional α_j's. Otherwise, we have that $(\pi_*(\alpha_9))^2 \geq 5$ in a similar way. In any case, we reach a contradiction.

Finally, we deal with the case (c). Note that $\pi_*(\alpha_9)$ is a cubic. By Lemma 4.1 in [31], there exists a π-exceptional (-1)-curve E_1 intersecting α_1 or α_8. Let E_2 be the other π-exceptional (-1)-curve. Since E_1 and E_2 are trisections of Y, we have $E_1 \cdot \alpha_2 \leq 1$ and $E_2 \cdot \alpha_2 \leq 1$. If E_1 intersects α_8, then $(\pi_*(\alpha_2))^2 \leq 2$ when we contract E_1, E_2 and the exceptional α_j's. This contradicts the assumption that $\pi_*(\alpha_2)$ is a conic. If E_1 intersects α_1, then E_1 also intersects α_9 since E_1 is a trisection and we can show that $(\pi_*(\alpha_9))^2 \geq 7$ in $\mathbb{P}^2$, which is again a contradiction.

Case (3). Assume that $\pi_*(\alpha_2) = L$. In this case, α_4 is always an exceptional divisor of a blow-up at a point p' infinitely near to a point $p \in L$ as in Remark 2.11. In particular, α_4 is toric. Note that Example 5.9 is included in this case.

Case (4). First, we claim that we may assume that α_4 is an exceptional curve. Indeed, if α_4 is non-exceptional, then $\pi_*(\alpha_4)$ is a line. In particular, α_4 is automatically toric. Thus, from now on, we assume that α_4 is exceptional. Then, by Lemma 4.1 in [31], either α_3 or α_5 must be non-exceptional. If either comes from a line, then α_4 is toric by Remark 2.11. Otherwise, α_3 comes from a conic and α_5 is exceptional since $\deg \pi_*(\sum m_j \alpha_j) = 9$. Then α_9 must come from a line, and we are in the situation of Example 5.10 since, as already explained, such an example is unique up to a choice of projective coordinates.

The above four cases show that α_4 is always toric, except for a Halphen pencil as in Example 5.10, as asserted. Moreover, since α_4 computes $\mathrm{lct}(Y, F)$, α_4 makes $\mathcal{P}$ non-stable, which completes the proof. ∎

6. Stability of nets of conics

In this section, we will describe how one can apply the stability criterion given in Theorem 3.3 to obtain an alternative geometric description of stability of nets of conics. GIT stability of linear systems of quadrics in general has been explored in [6, 7, 11, 30]. Here, we provide an explicit description of stability of nets of conics using Theorem 1.1, and we also explain how our description agrees with the description in [30] by C.T.C. Wall. We further point out that the connection between the two approaches can also be somewhat easily read off from Table 1 in [1].

First, we recall that Theorem 1.1 together with the work of Shah in [29] tell us that a net of conics $\mathcal{N}$ is stable if and only if for any choice of generators C_f, C_g and C_h for $\mathcal{N}$, the sextic curve $C_f + C_g + C_h$ satisfies conditions (i), (ii) and (iii) from Section 5.1. Therefore, we obtain the following criterion.

Proposition 6.1. *A net of conics $\mathcal{N}$ is GIT stable if and only if the following conditions hold*:

- (i") *$\mathcal{N}$ does not contain a double line*;
- (ii") *at a base point of $\mathcal{N}$ any conic in $\mathcal{N}$ is smooth* (*in particular, the base locus is zero-dimensional and $\mathcal{N}$ contains a smooth member*);
- (iii") *at a base point of $\mathcal{N}$, no three conics in $\mathcal{N}$ are mutually tangent*; *and*
- (iv") *every pencil contained in $\mathcal{N}$ has a smooth member.*

Proof. If $\mathcal{N}$ is non-stable, then by Theorem 1.1 there exists a choice of generators C_f, C_g and C_h for $\mathcal{N}$ such that the sextic $C_f + C_g + C_h$ is non-stable. Then, up to relabeling, by Theorem 2.3 in [29], one of the following holds:

Case I. C_h is a double line,

Case II. C_h is the union of two lines that intersect at a base point of the net,

Case III. C_g and C_h form a pencil such that any member is singular,

Case IV. the three conics are mutually tangent at a base point of the net.

In other words, at least one among the conditions (i") through (iv") does not hold.

Conversely, if one of these conditions does not hold, we can find generators C_f, C_g and C_h for $\mathcal{N}$ as above. In particular, the sextic curve $C_f + C_g + C_h$ either contains a multiple line, a point of multiplicity at least four, or a consecutive triple point. In any case, the sextic $C_f + C_g + C_h$ is not stable by Theorem 2.3 in [29]. Hence, $\mathcal{N}$ is not stable by Theorem 1.1. ∎

We can also prove that the following two lemmas hold.

Lemma 6.2. *If a net of conics $\mathcal{N}$ has a base point p, then there exists a conic in $\mathcal{N}$ that is singular at p.*

Proof. Choose generators C_f, C_g and C_h for $\mathcal{N}$ and choose coordinates $(x_0 : x_1 : x_2)$ in $\mathbb{P}^2$. If $p = (a : b : c)$ is a base point of $\mathcal{N}$, then (a, b, c) is a non-trivial element in the kernel of the matrix

$$A := \begin{pmatrix} \partial C_f/\partial x_0 & \partial C_f/\partial x_1 & \partial C_f/\partial x_2 \\ \partial C_g/\partial x_0 & \partial C_g/\partial x_1 & \partial C_g/\partial x_2 \\ \partial C_h/\partial x_0 & \partial C_h/\partial x_1 & \partial C_h/\partial x_2 \end{pmatrix}.$$

In particular, $\det A = \det A^{\mathsf{T}} = 0$, which implies that we can find $(\lambda : \mu : \nu) \in \mathbb{P}^2$ such that

$$\begin{aligned} \lambda \frac{\partial C_f}{\partial x_0} + \mu \frac{\partial C_g}{\partial x_0} + \nu \frac{\partial C_h}{\partial x_0} &= 0, \\ \lambda \frac{\partial C_f}{\partial x_1} + \mu \frac{\partial C_g}{\partial x_1} + \nu \frac{\partial C_h}{\partial x_1} &= 0, \\ \lambda \frac{\partial C_f}{\partial x_2} + \mu \frac{\partial C_g}{\partial x_2} + \nu \frac{\partial C_h}{\partial x_2} &= 0. \end{aligned}$$

The above equations tell us that the curve $\lambda C_f + \mu C_g + \nu C_h$ is singular at p. ∎

Lemma 6.3. *If a net of conics $\mathcal{N}$ does not contain a double line and it does not have a base point, then every pencil contained in $\mathcal{N}$ has a smooth member.*

Proof. We first observe that if a pencil of conics $\mathcal{P}$ does not contain a smooth member, then one of the following conditions holds (up to change of coordinates in $\mathbb{P}^2$):

(1) the base locus of $\mathcal{P}$ is one-dimensional,

(2) $\mathcal{P}$ is generated by $C_f : x_0^2 = 0$ and $C_g : x_1^2 = 0$,

(3) $\mathcal{P}$ is generated by $C_f : x_0 x_1 = 0$ and $C_g : (a x_0 + b x_1)(c x_0 + d x_1) = 0$ for some $a, b, c, d \in \mathbb{C}$ that are not all zero.

Now, since $\mathcal{N}$ does not contain a double line and it does not have a base point, it is clear that $\mathcal{N}$ cannot contain a pencil as in (1) or (2). Thus, it suffices to check that $\mathcal{N}$ cannot contain a pencil as in (3). In fact, a pencil $\mathcal{P}$ as in (3) always contains a double line, namely the curve $\lambda C_f + C_g$, where $\lambda = 2\sqrt{abcd} - (ac + bd)$. ■

Therefore, Proposition 6.1 can be sharpened, yielding the following result.

Proposition 6.4. *A net of conics $\mathcal{N}$ is GIT stable if and only if $\mathcal{N}$ does not contain a double line and does not have a base point.*

Now, by [29] and Corollary 4.2, we further know that $\mathcal{N}$ is unstable if and only if we can find three conics C_f, C_g and C_h in $\mathcal{N}$ such that the sextic curve $C_f + C_g + C_h$ satisfies one of the conditions (a), (b) or (c) from Section 5.1. In particular, we obtain the result below.

Proposition 6.5. *A net of conics $\mathcal{N}$ is unstable if and only if we can find three conics C_f, C_g and C_h in $\mathcal{N}$ such that one of the following (non-mutually-exclusive) conditions holds:*

(a”) *the base locus $C_f \cap C_g \cap C_h$ of $\mathcal{N}$ is one-dimensional;*

(b”) *$C_f = 2L$, C_g is tangent to L at a base point of $\mathcal{N}$, and C_h is arbitrary;*

(c”) *$C_f = L + L'$ and either*
- *$C_g = L + L''$ and C_h is smooth and tangent to C_f at the intersection point $L \cap L'$; or*
- *C_g and C_h are smooth and C_f, C_g and C_h are mutually tangent at $L \cap L'$;*

(d”) *C_f and C_g intersect at a base point of $\mathcal{N}$ which is a singular point of both C_f and C_g.*

Corollary 6.6. *A net of conics with no base points is GIT semistable.*

Remark 6.7. Note that if a net of conics $\mathcal{N}$ contains a smooth member and is unstable, then by Corollary 4.3 and 1.12 in [22], we can find two conics $C_f, C_g \in \mathcal{N}$ such that $C_f + C_g$ has a triple point. This means that, up to relabeling, we can find two (distinct) conics $C_f, C_g \in \mathcal{N}$ such that C_f is singular at a point in the intersection $C_f \cap C_g$.

Finally, we explain how our criterion agrees with the criterion in [30], which we state next.

Theorem ([30]). *A net of conics is GIT stable (respectively, semistable) if and only if the corresponding discriminant cubic curve is smooth (respectively, has at worst nodes).*

We first note that any conic $C_f \in \mathbb{P}^2$ can be described by an equation of the form

$$x^{\mathsf{T}} A_f x = \sum_{i=0}^{2} \sum_{j=0}^{2} f_{ij} x_i x_j = 0,$$

where $A_f = (f_{ij})$ is a symmetric 3×3 matrix. In particular, given a net of conics, say $\lambda C_f + \mu C_g + \nu C_h = 0$, its discriminant is the ternary cubic form

$$\Delta = \Delta(\lambda, \mu, \nu) := \det(\lambda A_f + \mu A_g + \nu A_h).$$

We also note that the singularities of $\Delta = 0$ do not depend on the choice of generators C_f, C_g, and C_h. We can thus prove the following.

Proposition 6.8. *The plane cubic $\Delta = 0$ is singular if and only if there exists a choice of generators $C_{f'}$, $C_{g'}$ and $C_{h'}$ such that the sextic curve $C_{f'} + C_{g'} + C_{h'}$ is GIT non-stable.*

Proof. Write

$$\Delta = \sum_{i,j} \delta_{ij}\, \lambda^i \mu^j \nu^{3-i-j}$$

and assume that the cubic $C : \Delta = 0$ is singular. Then there exists a choice of generators $C_{f'}$, $C_{g'}$ and $C_{h'}$ (equivalently, a choice of coordinates $(\lambda : \mu : \nu)$) such that $\delta_{00} = \delta_{01} = \delta_{10} = 0$. In other words, we may assume that C is singular at $(0 : 0 : 1)$.

In particular, $\det(A_{h'}) = \delta_{00} = 0$ if and only if $C_{h'}$ is singular. Now, if $C_{h'}$ is a double line, then $C_{f'} + C_{g'} + C_{h'}$ is non-stable by [29]. Similarly, if $C_{h'}$ is the union of two lines, then we claim that the intersection point of the two lines is a base point of the net, which implies that $C_{f'} + C_{g'} + C_{h'}$ contains a point of multiplicity at least four. Hence, it is non-stable by [29]. In fact, we can choose coordinates $(x_0 : x_1 : x_2)$ such that $C_{h'}$ is given by $x_0 x_1 = 0$. In particular,

$$\delta_{01} = -g'_{22} \quad \text{and} \quad \delta_{10} = -f'_{22},$$

which implies $(0 : 0 : 1) \in C_{f'} \cap C_{g'}$.

Conversely, assume that there exists a choice of generators $C_{f'}$, $C_{g'}$ and $C_{h'}$ such that $C_{f'} + C_{g'} + C_{h'}$ is non-stable. Then, as in the proof of Proposition 6.1 and up to relabeling, one of the following conditions holds:

- Case I. $C_{h'}$ is a double line.
- Case II. $C_{h'}$ is the union of two lines that intersect at a base point of the net.
- Case III. $C_{g'}$ and $C_{h'}$ form a pencil such that any member is singular.
- Case IV. The three conics are mutually tangent at a base point of the net.

We will show that the cubic C is singular in all four cases.

If either one of Cases I or II holds, then we can find coordinates (x_0, x_1, x_2) in $\mathbb{P}^2$ such that $\delta_{00} = \det(A_{h'}) = 0$ and

$$\delta_{01} = -g'_{22}\,(h'_{01})^2 \quad \text{and} \quad \delta_{10} = -f'_{22}\,(h'_{01})^2.$$

In Case I, we can assume that C'_h is given by $x_0^2 = 0$. Hence, $h'_{01} = 0$. Furthermore, we can assume in Case II that C'_h is given by $x_0 x_1 = 0$, which is singular at $(0 : 0 : 1)$, and since such point is a base point of the net, we further have $f'_{22} = g'_{22} = 0$. In any case, we conclude that $\delta_{00} = \delta_{01} = \delta_{10} = 0$ and the cubic C is singular at $(0 : 0 : 1)$.

Next, if Case III holds, then the cubic C contains the line $\lambda = 0$. Therefore, it is also singular.

Finally, assume that Case IV holds. Since we can always find a singular member in the net and we have already considered Cases I and II, we may choose coordinates (x_0, x_1, x_2) in $\mathbb{P}^2$ such that $C_{h'}$ is given by $x_0 x_1 = 0$ and the base point in question is $(0 : 1 : 0)$. The case when $C_{f'}$ or $C_{g'}$ is singular is included in Case III. Thus, we may assume that the curves $C_{f'}$ and $C_{g'}$ are such that $g'_{12} = g'_{11} = f'_{12} = f'_{11} = 0$ but $f'_{22} \cdot g'_{22} \neq 0$. Then C is the singular cubic given by

$$(f'_{22}\lambda + g'_{22}\mu)(f'_{01}\lambda + g'_{01}\mu + \nu)^2 = 0.$$

This completes the proof. ∎

Similar reasoning by making convenient choices of coordinates $(x_0 : x_1 : x_2)$ in $\mathbb{P}^2$ also proves the following.

Proposition 6.9. *The plane cubic $C : \Delta = 0$ has a singularity at worst a node if and only if for any choice of generators $C_{f'}$, $C_{g'}$ and $C_{h'}$, the sextic curve $C_{f'} + C_{g'} + C_{h'}$ is GIT semistable.*

Proof. The classification of nets of conics up to projective equivalence in Table 1 of [1] tells us that we have the correspondence given in the table below.

The cubic C	The net generated by $C_{f'}$, $C_{g'}$ and $C_{h'}$	$C_{f'} + C_{g'} + C_{h'}$
$2\lambda^2\mu + \nu^3 = 0$	$\lambda x_0 x_1 + \mu x_2^2 + \nu(x_0^2 + x_1 x_2) = 0$	$x_0 x_1 x_2^2(x_0^2 + x_1 x_2) = 0$
$\nu(\nu\lambda + \mu^2) = 0$	$\lambda x_0^2 + \mu x_0 x_1 + \nu(x_1 + x_2)x_2 = 0$	$x_0^3 x_1 x_2(x_1 + x_2) = 0$
$\nu\mu^2 = 0$	$\lambda x_0 x_2 + \mu x_0 x_1 + \nu x_2^2 = 0$	$x_0^2 x_1 x_2^3 = 0$
$\nu^2\mu = 0$	$\lambda x_1^2 + \mu x_2^2 + \nu x_0 x_1 = 0$	$x_0 x_1^3 x_2^2 = 0$
$\nu\mu^2 = 0$	$\lambda x_0 x_2 + \mu x_1 x_2 + \nu(x_0^2 + x_2^2) = 0$	$x_0 x_1 x_2^2(x_0^2 + x_2^2) = 0$
$\nu^3 = 0$	$\lambda x_1^2 + \mu x_0 x_1 + \nu(x_0^2 + x_1 x_2) = 0$	$x_0 x_1^3(x_0^2 + x_1 x_2) = 0$
$\Delta \equiv 0$	$\lambda x_0^2 + \mu x_1^2 + \nu(x_0 + x_1)^2 = 0$	$x_0^2 x_1^2(x_0 + x_1)^2 = 0$
$\Delta \equiv 0$	$\lambda x_0^2 + \mu x_0 x_1 + \nu x_0 x_2 = 0$	$x_0^4 x_1 x_2 = 0$

The assertion immediately follows from this correspondence. ∎

Remark 6.10. The above correspondence also follows from the classification of symmetric determinantal representations of plane cubics, which can be found in Table 7 in the Appendix of [27].

A final remark. Note that any net of conics $\mathcal{N}$ can be regarded as a divisor of bidegree $(1, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. It defines a Fano threefold X in the family № 2.24. In Section 2 of [30], it is shown that stability of X under the action of PGL(3) × PGL(3) agrees with stability of $\mathcal{N}$ under the action of PGL(3). Here, we further observe that the Jacobian criterion gives us the following. If X is smooth, then either conditions (ii”) and (iii”) in Proposition 6.1 hold, or $\mathcal{N}$ does not have a base point. In particular, we obtain the following result by combining this last observation with Lemma 6.2 and Propositions 6.1, 6.4, and 6.5.

Proposition 6.11. *Let X be a smooth Fano threefold in the family № 2.24. Then X is always GIT semistable, and X is GIT strictly semistable whenever $\mathcal{N}$ contains a double line.*

As a consequence, we can recover Corollary 4.7.8 in [5] by Lemma 4.7.7 in [5] and by the same argument of Theorem 3.4 in [24] (see also the arguments in Appendix A). Therefore, X is strictly K-semistable whenever $\mathcal{N}$ contains a double line and has no base points.

A. Chow stability of complete intersections

In this appendix, we extend Corollary 4.4 to the semistable case using a purely algebro-geometric argument. The precise statement and its proof are given below.

Theorem A.1. *Let $X \subset \mathbb{P}^n$ be a complete intersection defined by degree d hypersurfaces $H_1, \ldots, H_k$, where $k \leq n$. If X is Chow (semi)stable, then the linear system $\mathcal{L}$ generated by $H_1, \ldots, H_k$ is GIT (semi)stable.*

Proof. Suppose the H_i are represented by homogeneous polynomials h_i, each of degree d, and let $(h_1, \ldots, h_k)$ denote $\mathcal{L}$. As before, we consider $(h_1, \ldots, h_k)$ as a point in the Grassmannian variety G parametrizing k-dimensional linear subspaces of $H^0(\mathbb{P}^n, \mathcal{O}(d))$.

Let $\mathcal{W} \subset G \times H^0(\mathbb{P}^n, \mathcal{O}(d))$ be the universal subspace associated with G and let $\mathcal{X} \subset G \times \mathbb{P}^n$ be the closed subscheme defined by the ideal generated by the image of $\mathcal{W} \otimes \mathcal{O}(-d) \to \mathcal{O}_{G \times \mathbb{P}^n}$. Furthermore, let $U \subset G$ be the locus of points p such that the fiber $\mathcal{X}_p$ has dimension $n-k$. Then U is Zariski open, $\mathcal{X}|_U \to U$ is flat and there exists a morphism $q\colon U \to \mathrm{Chow}_{k,dk}$. This morphism is the composition of the natural map from U to the Hilbert scheme induced by the flat family $\mathcal{X}|_U \to U$ and the Hilbert–Chow morphism (cf. Section 5.4 of [23]). We fix an ample line bundle M on $\mathrm{Chow}_{k,dk}$ as in Section 2.5. Now, recall that Chow stability of a cycle is defined as GIT stability of the corresponding point in $\mathrm{Chow}_{k,dk}$ with respect to M.

Let L be an ample generator of $\mathrm{Pic}(G) \cong \mathbb{Z}$ and let Y be the normalized graph of the rational map $G \dashrightarrow \mathrm{Chow}_{k,dk}$ induced by q. Note that Y admits an $\mathrm{SL}(n+1)$-left action compatible with q. Let $p_1\colon Y \to G$ and $p_2\colon Y \to \mathrm{Chow}_{k,dk}$ be the projections.

Claim. *There is an* $\mathrm{SL}(n+1)$*-equivariant linear equivalence*

$$a p_1^* L - bE \sim p_2^* M \tag{A.1}$$

for some $a \in \mathbb{Z}_{>0}$ and $b \in \mathbb{Z}_{\geq 0}$, where E is a p_1-exceptional Cartier (effective) divisor.

Proof of Claim. We first show that there exist a one-parameter subgroup λ_0 of $\mathrm{SL}(n+1)$ and a point $p_0 \in U$ such that the sign of $\mu^L(p_0, \lambda_0)$ coincides with that of $\mu^M(q(p_0), \lambda_0)$. In fact, we can exhibit p_0 and λ_0 such that λ_0 destabilizes both of p_0 and $q(p_0)$ as follows. Recall that $\mathrm{SL}(n+1)$ acts on G. Given $g \in \mathrm{SL}(n+1)$, g maps $(h_1, \ldots, h_k)$ to $(g(h_1), \ldots, g(h_k))$, where

$$g(h_i)(gx) = h_i(x) \quad \text{for any } x \in \mathbb{P}^n.$$

Moreover, $q\colon U \to \mathrm{Chow}_{k,dk}$ is $\mathrm{SL}(n+1)$-equivariant for this action. Consider $p_0 \in U$ to be the point corresponding to the linear system $(x_0^d, x_1^d, \ldots, x_{k-1}^d)$ and let λ_0 be a one-parameter subgroup of $\mathrm{SL}(n+1)$ acting on the subspace in $\mathbb{P}^n$ spanned by $x_0, \ldots, x_{k-1}$ with weight $-(n-k+1)$ and on the subspace spanned by $x_k, \ldots, x_n$ with weight k. Then λ_0 destabilizes both of p_0 and $q(p_0)$.

Now, because $\rho(G) = 1$, we know that there exist some integers $a, b \in \mathbb{Z}$ such that $p_2^* M \sim a p_1^* L - bE$, where E is a p_1-exceptional effective Cartier divisor. Indeed, we consider $p_{1*} p_2^* M$ as a divisor on G and then $p_{1*} p_2^* M \sim aL$ for some a. Thus, we have $p_2^* M = a p_1^* L - bE$ as a divisor for some b. We see that $b \geq 0$ by Lemma 3.39 in [20] since $p_2^*M - ap_1^*L = -bE$ is p_1-nef. Moreover, by Proposition 1.4 in [23], we may further assume that the above linear equivalence preserves the $\mathrm{SL}(n+1)$-linearizations.

Finally, observe that because E is fixed by $\mathrm{SL}(n+1)$, we can consider $\mathcal{O}_Y(-E)$ as an $\mathrm{SL}(n+1)$-stable subsheaf of $\mathcal{O}_Y$. Moreover, around $p_0 \in U$ we have that G and Y are isomorphic via p_1. Thus, letting $\tilde{p_0} \in Y$ be the point corresponding to p_0, we can compare $\mu^{p_2^*M}(\tilde{p_0}, \lambda_0)$ with $\mu^{ap_1^*L-bE}(\tilde{p_0}, \lambda_0)$ to conclude $a > 0$. Note that λ_0 acts on the fiber $\mathcal{O}_Y(-E) \otimes k(\tilde{p_0}) = \mathcal{O}_Y \otimes k(\tilde{p_0})$ trivially since $\tilde{p_0} \notin E$. Therefore, (A.1) holds. ■

Finally, we show that Theorem A.1 indeed follows from our claim. Given any $p \in U$ corresponding to $(h_1, \dots, h_k)$ and any arbitrary non-trivial one-parameter subgroup λ of $\mathrm{SL}(n+1)$, if we let $\tilde{p} \in Y$ be the point corresponding to p, then

$$\begin{aligned} a\mu^L(p,\lambda) = a\mu^{p_1^*L}(\tilde{p},\lambda) &= \mu^{p_2^*M}(\tilde{p},\lambda) - b\mu^{\mathcal{O}_Y(-E)}(\tilde{p},\lambda) \\ &\geq \mu^{p_2^*M}(\tilde{p},\lambda) > 0, \quad \text{(respectively, } \geq 0) \end{aligned}$$

whenever X is Chow (semi)stable by Theorem 2.1 in [23]. Here, we remark that for any $\mathrm{SL}(n+1)$-stable effective Cartier divisor E on Y, $\mu^{\mathcal{O}_Y(-E)}(y,\lambda) \leq 0$ for any closed point $y \in Y$ and one-parameter subgroup $\lambda\colon \mathbb{C}^\times \to \mathrm{SL}(n+1)$. Thus, we obtain the assertion. ■

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