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On the local geometry of the moduli space of $(2,2)$-threefolds in $\mathcal{A}_9$

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Abstract. We study the local geometry of the moduli space of intermediate Jacobians of $(2,2)$-threefolds in $\mathbb{P}^2 \times \mathbb{P}^2$. More precisely, we prove that a composition of the second fundamental form of the Siegel metric in $\mathcal{A}_9$ restricted to this moduli space, with a natural multiplication map is a nonzero holomorphic section of a vector bundle. We also describe its kernel. We use the two conic bundle structures of these threefolds, Prym theory, gaussian maps and Jacobian ideals.

1. Introduction

Following the philosophy of the work of Andreotti and Griffiths, the local geometry of subloci of $\mathcal{A}_g$ contains relevant information on the global geometry of the varieties of these loci. For example, the infinitesimal variation of Hodge structure allows recovering the curve, if the Clifford index is at least 2. The main tool to investigate the local behaviour of these subvarieties of $\mathcal{A}_g$ is the second fundamental form with respect to the Siegel metric, that is, the orbifold Kähler metric induced by the symmetric metric on the Siegel space.

This paper deals with the local geometry of the moduli space of intermediate Jacobians of $(2,2)$-threefolds in $\mathbb{P}^2 \times \mathbb{P}^2$. These threefolds have the remarkable property of having two conic bundle structures. As pointed out by Verra in [14], this translates into the fact that the intermediate Jacobian has two interpretations as the Prym variety of two allowable double covers of two plane sextics. In fact, using this, he proves that the restriction of the Prym map to the locus of double covers of plane sextics has degree two, thus giving a counterexample to the tetragonal conjecture of Donagi ([10]). He also proves a generic Torelli theorem for the period map of these threefolds in $\mathcal{A}_9$.

Our aim is to study the second fundamental form II of the locus $\mathcal{Q}$ of intermediate Jacobians of these threefolds in $\mathcal{A}_9$. One of the main difficulties is that it is non-holomorphic, hence it is very difficult to make explicit computations. Nevertheless, we show that its composition with a convenient multiplication map turns out to be holomorphic. This has

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been proven to be the case also for the Jacobian and Prym loci in $\mathcal{A}_g$ (see [6,7,9]), where this composition is given by the second gaussian map of either the canonical bundle, or the Prym-canonical one. These gaussian maps are generically of maximal rank. In contrast, an analogous composition of the second fundamental form of the moduli space of intermediate Jacobians of cubic threefolds in $\mathcal{A}_5$ with a natural multiplication map is shown to be zero in [8]. In the case of cubic threefolds, an important tool was the relation between the Jacobian ring of the cubic threefold and that of the family of plane quintics provided by the Prym construction.

On the contrary, here we prove that the composition of the second fundamental form of the moduli space of intermediate Jacobians of $(2,2)$ threefolds in $\mathcal{A}_9$ with a natural multiplication map is a nonzero holomorphic section $\mathcal{S}$ of the normal bundle $N_{\mathcal{Q}/\mathcal{A}_9}$. Again, Jacobian rings of both the threefold and the associated plane sextics, play a fundamental role.

More precisely, for a $(2,2)$ threefold T in $\mathbb{P}^2 \times \mathbb{P}^2$, there is a natural bigraded Jacobian ring $\oplus R_T^{a,b}$ such that the differential of the period map can be expressed in terms of this ring (see [11, 12]). It is the quotient of the ring of bihomogeneous polynomials $\oplus S^{a,b}$ by the bihomogeneous ideal $\oplus J_T^{a,b}$ generated by the partial derivatives of F.

As explained in Section 3, $T_{\mathcal{Q},T}$ can be identified with $R_T^{2,2}$, and moreover,

$$\mathcal{O}_{\mathcal{Q},T} \cong R_T^{4,4} \cong \mathbb{C}.$$

There are isomorphisms

$$H^{2,1}(T) \cong R_T^{1,1} \quad \text{and} \quad \Omega^1_{\mathcal{Q},T} \cong R_T^{2,2},$$

and the dual of the differential of the period map corresponds to the multiplication map

$$\nu : \operatorname{Sym}^2 R_T^{1,1} \longrightarrow R_T^{2,2}.$$

Hence, there is a natural isomorphism

$$N^*_{\mathcal{Q}/\mathcal{A}_9,JT} = \operatorname{Ker}(\operatorname{Sym}^2 R_T^{1,1} \xrightarrow{\nu} R_T^{2,2}).$$

So, the second fundamental form at JT,

$$\mathrm{II} : N^*_{\mathcal{Q}/\mathcal{A}_9,JT} \longrightarrow \operatorname{Sym}^2 \Omega^1_{\mathcal{Q},JT},$$

can be seen as a map

$$\operatorname{Ker}(\operatorname{Sym}^2 R_T^{1,1} \xrightarrow{\nu} R_T^{2,2}) \longrightarrow \operatorname{Sym}^2 R_T^{2,2}.$$

The main result of this paper is the following.

Theorem 1.1. *For a general T, the composition*

$$m \circ \mathrm{II} : N^*_{\mathcal{Q}/\mathcal{A}_9,JT} = \operatorname{Ker}(\operatorname{Sym}^2 R_T^{1,1} \xrightarrow{\nu} R_T^{2,2}) \longrightarrow R_T^{4,4} \cong \mathbb{C},$$

where $m\colon \operatorname{Sym}^2 R_T^{2,2} \longrightarrow R_T^{4,4}$ is the multiplication map, is nonzero, and via duality, it gives a canonical non-trivial holomorphic section $\mathcal{S}$ of the normal bundle.

In particular, Theorem 1.1 implies that the second fundamental form

$$\mathrm{II} : N^*_{\mathcal{Q}/\mathcal{A}_9,JT} \longrightarrow \mathrm{Sym}^2\Omega^1_{\mathcal{Q},JT}$$

is nonzero.

We remark that this is also implied by the fact that the monodromy group of the family of these threefolds is the whole symplectic group (see Proposition 4.6). Nevertheless, our result is stronger.

Moreover, for a general T, the kernel of the composition $m \circ \mathrm{II}$ is explicitly described in terms of the geometry of T (see Proposition 4.7, and Remark 4.8).

One of the main ingredients in the proof is to relate the second fundamental form II with the restriction of the second fundamental form of the Prym map $\mathcal{R}_{10} \longrightarrow \mathcal{A}_9$ to the locus of double coverings of plane sextics.

We show that the restriction of the composition $m \circ \mathrm{II}$ to the space of quadrics containing the Prym-canonical image of both the plane sextics is nonzero. This is done using the fact that it coincides with the composition of the second gaussian map of the Prym-canonical bundle of the plane sextics with a suitable projection.

The main technical point is the computation of this gaussian map on the quadric containing the Prym-canonical image of the sextic given by the equation of the threefold and we do this in a specific example.

The structure of the paper is as follows. In Section 2, we recall the properties of (2, 2) threefolds following [14]. In Section 3, we recall the theory of bigraded Jacobian rings, we introduce the second fundamental form II, and we recall previous results on the computation of the second fundamental form of the Prym locus. In Section 4, we give an interpretation of the second fundamental form in terms of Prym theory, and we prove the main Theorem.

2. (2, 2)-threefolds in $\mathbb{P}^2 \times \mathbb{P}^2$

In this section, we mainly recall the properties of (2, 2)-threefolds proved by Verra in [14].

We consider a threefold T in $\mathbb{P}^2 \times \mathbb{P}^2$ given by a bihomogeneous equation F of degree (2, 2). More precisely,

$$F = \sum_{0\leq i,j,k,l\leq 2} a_{ijkl}\, x_i x_j y_k y_l.$$

Let W be the image of the Segre embedding $s\colon \mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$. Then

$$T \subset \mathbb{P}^2 \times \mathbb{P}^2 \cong W \subset \mathbb{P}^8.$$

We denote $\alpha_{ik} := x_i y_k$. These are natural coordinates in $\mathbb{P}^8$, and T can be seen as a complete intersection $T = W \cap Q$, where $Q \subset \mathbb{P}^8$ is the quadric given by the equation

$$Q_0 = \sum_{0\leq i,j,k,l\leq 2} a_{ijkl}\, \alpha_{ik}\alpha_{jl}.$$

Remember that the quadratic equations $\alpha_{ik}\alpha_{jl} - \alpha_{il}\alpha_{jk}$ generate the ideal I_W of W. Then the ideal of T is $I_T = I_W + \langle Q_0\rangle$.

The projections on each factor $p_i\colon T \to \mathbb{P}^2$ provide two conic bundle structures on T. Or, in other words, if we fix a point $(x_0 : x_1 : x_2)$ in the first plane, then the equation $F(x_0, x_1, x_2, y_0, y_1, y_2)$ gives a conic whose matrix $A(x)$ has degree 2 entries in the $x_i's$, $A(x)_{kl} = \sum_{i,j} a_{ijkl} x_i x_j$. Therefore, the determinant of this matrix gives a sextic plane curve C which parametrizes the degenerate conics of the family. The lines contained in these degenerate conics define a curve $\widetilde{C}$ contained in the corresponding Grassmannian and a natural degree two map $\pi_1\colon \widetilde{C} \to C$. It was proved by Beauville in [1] that π_1 is an allowable covering of degree 2 and that its Prym variety $P(\widetilde{C}, C)$ is isomorphic (as principally polarized abelian variety) to the intermediate Jacobian JT. Similarly, using the second projection, there is another sextic plane curve D and a covering $\pi_2\colon \widetilde{D} \to D$ with the same property. In particular, $P(\widetilde{C}, C) \cong P(\widetilde{D}, D)$. This can be written as a relation of moduli spaces in the following way. Let us define the following:

$$\mathcal{Q} = \{T \subset \mathbb{P}^2 \times \mathbb{P}^2 \mid T \text{ is a } (2,2) \text{ smooth threefold}\}/\cong,$$

$$\widetilde{\mathcal{Q}} = \{(T, p_i) \mid T \in \mathcal{Q},\ p_i \text{ the two natural projections on } \mathbb{P}^2\}/\cong,$$

$$\mathcal{P}_6 = \{\text{allowable double coverings } \tilde{C} \to C, \text{ where } C \text{ is a plane sextic}\}/\cong,$$

$$\mathcal{A}_9 = \text{ the moduli space of principally polarized abelian varieties of dimension 9.}$$

Then, there is a commutative diagram,

$$\begin{tikzcd} \widetilde{\mathcal{Q}} \arrow[r, "d"] \arrow[d, "f"'] & \mathcal{P}_6 \arrow[d, "p"] \\ \mathcal{Q} \arrow[r, "j"] & \mathcal{A}_9, \end{tikzcd} \tag{2.1}$$

where d associates to the conic bundle $p_i\colon T \to \mathbb{P}^2$ the discriminant curve $C \subset \mathbb{P}^2$ and the natural allowable covering $\pi_1\colon \widetilde{C} \to C$; f is the forgetful map; $j(T) = H^{1,2}(T)/H^3(T, \mathbb{Z})$ is the intermediate Jacobian of T; and p is the Prym map restricted to $\mathcal{P}_6$. Moreover, $\widetilde{\mathcal{Q}}$, $\mathcal{Q}$ and $\mathcal{P}_6$ are irreducible of dimension 19.

The main result in [14] is the computation of the degrees of all these maps that turn out to be generically finite on their images. He proves, based on results in [3], that d and j have degree 1 and that f and p have degree 2 on their respective images. In particular, the map d is dominant, since $\widetilde{\mathcal{Q}}$ and $\mathcal{P}_6$ have the same dimension.

The main theorem in [14] states that the Prym map has degree exactly 2 when restricted to the locus of unramified double coverings of plane sextics.

We will assume from now on that T, C and D are generic, in particular, all three are smooth. We are interested in the realization of C and D in $\mathbb{P}^8$. Let C' (respectively, D') be the set of double points of the conics of the first (respectively, second) conic bundle structure on T. Notice that $C', D' \subset T \subset W \subset \mathbb{P}^8$ and that $C' \cong C$ and $D' \cong D$. Moreover, C' (respectively, D') is the locus of points of T where the partial derivatives $F_{y_0}, F_{y_1}, F_{y_2}$ (respectively, $F_{x_0}, F_{x_1}, F_{x_2}$) vanish. So the corresponding ideals are

$$I_{C'} = I_T + \langle F_{y_0}, F_{y_1}, F_{y_2}\rangle \quad \text{and} \quad I_{D'} = I_T + \langle F_{x_0}, F_{x_1}, F_{x_2}\rangle.$$

In fact, C' is the Prym-canonical image of C: let $\eta \in JC$ be the 2-torsion point that defines the covering π_1. For a generic C, we have that $h^0(C, \eta(2)) = 3$, and the tensor product

$$H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, \eta(2)) \mapsto H^0(C, \omega_C \otimes \eta)$$

is an isomorphism. Then the embedding $\phi\colon C \hookrightarrow \mathbb{P}^8$ is the composition

$$C \xrightarrow{\varphi_1\times\varphi_2} \mathbb{P}^2\times\mathbb{P}^2 \xrightarrow{\cong} W \hookrightarrow \mathbb{P}^8 \cong \mathbb{P}(H^0(C,\omega_C\otimes\eta)^\vee),$$

where φ_1,φ_2 are the maps defined by the linear systems $|\mathcal{O}_C(1)|$ and $|\eta(2)|$. Thus, by definition, $C' = (\varphi_1\times\varphi_2)(C)$. The same holds for the cover π_2, and we denote by $\eta' \in JD$ the corresponding 2-torsion point.

3. Preliminaries on Jacobian rings and second fundamental forms

3.1. Bigraded Jacobian ring and second fundamental form

In this subsection, we describe the second fundamental form of $\mathcal{Q}$ in $\mathcal{A}_9$ and the multiplication maps in terms of Jacobian rings. First notice that, since all the maps in diagram (2.1) are generically finite onto their respective images, we obtain the following identifications:

$$\Omega^1_{\mathcal{Q},\bullet} \cong \Omega^1_{\tilde{\mathcal{Q}},\bullet} \cong \Omega^1_{\mathcal{P}_6,\bullet} \cong \Omega^1_{\mathcal{M}^{\mathrm{pl}}_{10},\bullet},$$

where we denote by $\mathcal{M}^{\mathrm{pl}}_{10}$ the moduli space of smooth plane sextics in $\mathcal{M}_{10}$, the moduli space of genus 10 curves.

By abuse of notation, we also denote by $\mathcal{Q}$ the closure of its image in $\mathcal{A}_9$ via the map j.

Recall that Griffiths studied the periods of hypersurfaces by means of the Jacobian ring. Later, Green extended this theory to a more general setting, covering, in particular, the case of hypersurfaces in the product of two projective spaces (see [12] and Lecture 4 in [11]). More precisely, we consider the bigraded Jacobian ring $\oplus R^{a,b}_T$ of T, which is the quotient of the ring of bihomogeneous polynomials $\oplus S^{a,b}$ by the bihomogeneous ideal $\oplus J^{a,b}_T$ generated by the partial derivatives of F.

Using the proposition on p. 45 and the theorem on p. 47 of [11], with $X = T$, $Y = \mathbb{P}^2\times\mathbb{P}^2$, $L = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(2,2)$, $n = 4$, $q = 1$, one immediately sees that $T_{\mathcal{Q},T}$ can be identified with $R^{2,2}_T$, $H^{2,1}(T) \cong R^{1,1}_T$, $\mathcal{O}_{\mathcal{Q},T} \cong R^{4,4}_T \cong \mathbb{C}$, and by duality (see part (2) of the theorem on p. 47 of [11]) we have $\Omega^1_{\mathcal{Q},T} \cong R^{2,2}_T$. Moreover, the dual of the differential of the period map ${}^t dj$ identifies with the multiplication map

$$\nu : \mathrm{Sym}^2 R^{1,1}_T \longrightarrow R^{2,2}_T.$$

Consider now the cotangent exact sequence

$$0 \to N^*_{\mathcal{Q}/\mathcal{A}_9} \to \Omega^1_{\mathcal{A}_9|\mathcal{Q}} \xrightarrow{{}^t dj} \Omega^1_{\mathcal{Q}} \to 0.$$

With the above identifications, there is a natural isomorphism

$$N^*_{\mathcal{Q}/\mathcal{A}_9,JT} = \mathrm{Ker}(\mathrm{Sym}^2 R^{1,1}_T \xrightarrow{\nu} R^{2,2}_T).$$

Denote by ∇ the Chern connection of the Siegel metric and consider the second fundamental form

$$\mathrm{II} = ({}^t dj \otimes \mathrm{Id}) \circ \nabla_{|N^*_{\mathcal{Q}/\mathcal{A}_9}} : N^*_{\mathcal{Q}/\mathcal{A}_9} \to \mathrm{Sym}^2\Omega^1_{\mathcal{Q}}.$$

Using the above identifications, at a generic point $j(T) = JT \in \mathcal{Q}$, the second fundamental form can be seen as a map

$$\mathrm{II} : \mathrm{Ker}(\mathrm{Sym}^2 R_T^{1,1} \xrightarrow{\nu} R_T^{2,2}) \to \mathrm{Sym}^2 R_T^{2,2}.$$

We can compose II with the multiplication map

$$m : \mathrm{Sym}^2 R_T^{2,2} \longrightarrow R_T^{4,4} \cong \mathbb{C},$$

and we get a map $m \circ \mathrm{II} : N^*_{\mathcal{Q}/\mathcal{A}_9} \longrightarrow \mathcal{O}_{\mathcal{Q}}$.

We will prove that $m \circ \mathrm{II}$ is holomorphic and nonzero, hence, by duality, it gives a nonzero holomorphic section $\mathcal{S}$ of the normal bundle $N_{\mathcal{Q}/\mathcal{A}_9}$.

3.2. Second fundamental form of the Prym locus

In this subsection, we recall some results of [6] on the second fundamental form of the Prym map

$$p_g : \mathcal{R}_g \to \mathcal{A}_{g-1}.$$

Recall that $\mathcal{R}_g$ denotes the moduli space of pairs (C, η) of smooth curves C of genus g and a non-trivial two-torsion point $\eta \in JC[2]$, and p_g maps (C, η) to the Prym variety associated with the attached double covering. The codifferential of the Prym map at the point (C, η) is naturally identified with the multiplication map

$$\beta : \mathrm{Sym}^2 H^0(C, \omega_C \otimes \eta) \longrightarrow H^0(C, \omega_C^2).$$

Hence, we can identify the conormal bundle $N^*_{\mathcal{R}_g/\mathcal{A}_{g-1},(C,\eta)}$ with the kernel $I_2(\omega_C \otimes \eta)$ of this multiplication map. Denote by

$$\rho : I_2(\omega_C \otimes \eta) \longrightarrow \mathrm{Sym}^2 H^0(C, \omega_C^2)$$

the second fundamental form of the Prym map at (C, η). We recall the following.

Theorem 3.1 (Theorem 2.1 in [6]). *The diagram*

$$\begin{tikzcd} I_2(\omega_C \otimes \eta) \arrow[r, "\rho"] \arrow[d, "\mu_2"'] & \mathrm{Sym}^2 H^0(\omega_C^{\otimes 2}) \arrow[dl, "\beta"] \\ H^0(\omega_C^{\otimes 4}) & \end{tikzcd}$$

is commutative up to scalar, where μ_2 is the second gaussian (or Wahl) map.

For the reader's convenience, we recall the definition of the second gaussian map in local coordinates. For more details on gaussian maps, see [15].

Take a basis $\{\omega_1, \dots, \omega_{g-1}\}$ of $H^0(C, \omega_C \otimes \eta)$, choose a local coordinate z, and a local frame l for the line bundle η, so that locally we write $\omega_i = h_i(z) dz \otimes l$. A quadric $Q \in I_2(\omega_C \otimes \eta)$ has the following expression:

$$Q = \sum_{i,j=1}^{g-1} a_{ij}\, \omega_i \odot \omega_j,$$

with
$$\sum_{i,j} a_{ij}\, h_i(z) h_j(z) = 0.$$
Then
$$\mu_2(Q) := \sum_{i,j} a_{ij}\, h_i'(z) h_j'(z) (dz)^4 = -\sum_{i,j} a_{ij}\, h_i''(z) h_j(z) (dz)^4. \tag{3.1}$$

4. Prym theory and proof of the main theorem

The proof of Theorem 1.1 relies on the fact that $\mathcal{Q}$ can be seen as the image of the Prym map restricted to $\mathcal{P}_6$. Let C be a smooth sextic plane curve, and let $R_C^\bullet$ be the Jacobian ring of C. So the identification
$$\Omega^1_{\mathcal{M}_{10}^{\mathrm{pl}},C} \cong R_C^6$$
holds. Then, we use Prym theory to express the normal sheaf of $\mathcal{Q}$ in $\mathcal{A}_9$ in another way. Consider the restriction p of the Prym map p_{10} to $\mathcal{P}_6$:
$$\mathcal{P}_6 \hookrightarrow \mathcal{R}_{10} \xrightarrow{p_{10}} \mathcal{A}_9.$$
Recall that the map p is generically finite of degree 2. The transpose of dp_{10} at a point $(C,\eta) \in \mathcal{R}_{10}$ is the multiplication map $\mathrm{Sym}^2 H^0(C, \omega_C \otimes \eta) \longrightarrow H^0(C, \omega_C^2)$. In a generic point $(C,\eta) \in \mathcal{P}_6$ it is surjective by (3.6) in [14]. Thus, we have
$$N^*_{\mathcal{R}_{10}/\mathcal{A}_9,(C,\eta)} \cong I_2(\omega_C \otimes \eta),$$
which is the vector space of equations of quadrics through C', the Prym-canonical image of C. We have a short exact sequence of conormal bundles:
$$0 \longrightarrow I_2(\omega_C \otimes \eta) \longrightarrow N^*_{\mathcal{Q}/\mathcal{A}_9,JT} \longrightarrow N^*_{\mathcal{P}_6/\mathcal{R}_{10},(C,\eta)} = J_C^6/\langle C\rangle \longrightarrow 0.$$
Here, $J_C^\bullet$ means the Jacobian ideal of C, and $\langle C\rangle$ the subideal generated by the equation of the sextic C.

We have the following diagram:
$$\begin{tikzcd}
N^*_{\mathcal{R}_{10}/\mathcal{A}_9,(C,\eta)} \cong I_2(\omega_C \otimes \eta) \arrow[r, "\rho"] \arrow[d, hook] & \mathrm{Sym}^2 \Omega^1_{\mathcal{R}_{10},(C,\eta)} \cong \mathrm{Sym}^2 H^0(C, \omega_C^2) \arrow[d] \\
N^*_{\mathcal{Q}/\mathcal{A}_9,JT} \arrow[r] & \mathrm{Sym}^2 \Omega^1_{\mathcal{P}_6,(C,\eta)} \cong \mathrm{Sym}^2 R_C^6,
\end{tikzcd}$$
where ρ is the second fundamental form of the Prym map p_{10} and the second horizontal row is another way of representing the map II using Prym theory. Notice that the first vertical arrow is the inclusion of the conormal bundles, while the second vertical arrow is induced by projection of cotangent bundles.

Recall that, by Theorem 3.1, the composition of ρ with the multiplication map

$$\mathrm{Sym}^2 H^0(C,\omega_C^2) \longrightarrow H^0(C,\omega_C^4)$$

is the second gaussian map μ_2: $I_2(\omega_C \otimes \eta) \longrightarrow H^0(C,\omega_C^4)$.

Observe that

$$H^0(C,\omega_C^2) \cong H^0(C,\mathcal{O}_C(6))$$

and there is a map $H^0(C,\mathcal{O}_C(6)) \to R_C^6$ induced by the projection $H^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(6)) \to R_C^6$, since the equation of C belongs to the Jacobian ideal of C. Analogously, we have a map

$$\tau : H^0(C,\mathcal{O}_C(12)) \longrightarrow R_C^{12}.$$

So we have the following commutative diagram:
(4.1)

$$\begin{tikzcd}
I_2(\omega_C \otimes \eta) \arrow[r, "\rho"] \arrow[d] \arrow[rrr, bend left, "\mu_2"] & Sym^2 H^0(C,\omega_C^2) = Sym^2 H^0(C,\mathcal{O}_C(6)) \arrow[rr, "m"] \arrow[d] & & H^0(C,\mathcal{O}_C(12)) \arrow[d, "\tau"] \\
N^*_{\mathcal{Q}/\mathcal{A}_9,JT} \arrow[r, "\mathrm{II}"] & \mathrm{Sym}^2 R_C^6 \arrow[rr, "m"] & & R_C^{12} = \mathbb{C}.
\end{tikzcd}$$

We will prove that the composition $\tau \circ \mu_2$ is nonzero, showing on an explicit example that $\tau \circ \mu_2(F) \neq 0$, where

$$F = \sum_{0 \le i,j,k,l \le 2} a_{ijkl}\, x_i x_j y_k y_l = \sum_{0 \le i,j,k,l \le 2} a_{ijkl}\, \alpha_{ik} \alpha_{jl} \in I_2(\omega_C \otimes \eta)$$

denotes as usual the equation of the threefold T.

Remark 4.1. Observe that, for a general T, by symmetry, similar diagrams and maps exist for the curve D. Notice that, by the proof of Proposition 5.3 in [14], the intersection of $I_2(\omega_C \otimes \eta)$ and $I_2(\omega_D \otimes \eta')$ is the vector space of the equations of the quadrics containing the threefold T, which is 10-dimensional. Therefore, the sum $I_2(\omega_C \otimes \eta) + I_2(\omega_D \otimes \eta')$ has dimension 26 and hence it equals $N^*_{\mathcal{Q}/\mathcal{A}_9,JT}$.

To show that the map $m \circ \mathrm{II}$ is holomorphic, first notice that $I_2(\omega_C \otimes \eta)$ is the fibre of a holomorphic vector bundle $\mathcal{I}_2$ over $\mathcal{P}_6$ at the point (C,η), and analogously, $I_2(\omega_D \otimes \eta')$ is the fibre a holomorphic vector bundle $\mathcal{I}'_2$ over (D,η'). We have shown that $N^*_{\mathcal{Q}/\mathcal{A}_9} = \mathcal{I}_2 + \mathcal{I}'_2$. The second gaussian map μ_2 is a holomorphic map between vector bundles over $\mathcal{P}_6$ (see, e.g., [6]), and also τ varies holomorphically. Since the restriction of $m \circ \mathrm{II}$ to $\mathcal{I}_2$ is the composition $\tau \circ \mu_2$ (and analogously for the restriction to $\mathcal{I}'_2$), we have shown that $m \circ \mathrm{II}$ gives a holomorphic section of the normal bundle $N^*_{\mathcal{Q}/\mathcal{A}_9}$.

Denote by $A(x)$ the symmetric matrix given by

$$A(x)_{kl} = \sum_{i,j} a_{ijkl}\, x_i x_j,$$

and by $\hat{A}_{ij}(x)$ the cofactor, that is, the product of the determinant of the matrix obtained from $A(x)$ by removing the i-th row and the j-th column with $(-1)^{i+j}$.

Lemma 4.2. *We have the following commutative diagram*:

$$\begin{tikzcd} C \arrow[r, "\varphi_2"] \arrow[d, "\varphi_1"] & \mathbb{P}^2 \arrow[d, "v_2"] \\ \mathbb{P}^2 \arrow[r, "h"] & \mathbb{P}^5, \end{tikzcd} \tag{4.2}$$

where $h(x) = (\hat{A}_{ij}(x))_{i,j}$, v_2 is the Veronese map, and φ_1 and φ_2 are the maps defined at the end of Section 2.

Proof. Since for any $x \in C$, $\phi(x) = (\varphi_1 \times \varphi_2)(x)$ is the intersection point of the two lines corresponding to the singular conic $A(x)$, $\varphi_2(x) = (y_0 : y_1 : y_2)$ is the point in $\mathbb{P}^2$ given by $\mathrm{Ker}(A(x))$. So there exist nonzero constants λ_i such that

$$\lambda_i(y_0, y_1, y_2) = ((\hat{A}_{i0}(x)), (\hat{A}_{i1}(x)), (\hat{A}_{i2}(x))), \quad \text{for all } i = 1, 2, 3.$$

By the symmetry of $A(x)$, we can identify $\lambda_i = y_i$ for all $i = 0, 1, 2$. So the diagram is commutative and, if we restrict to C, we have

$$y_i y_j = \hat{A}_{ij}(x). \qquad \blacksquare$$

Now choose a local coordinate z on C, a local frame l for $\mathcal{O}_C(1)$ and a local frame σ for η. Then locally we write $x_i = f_i(z)l$, $y_i = g_i(z)l^2\sigma$ and $\alpha_{ij} = f_i(z)g_j(z)l^3\sigma$. So, by (3.1), the local expression for $\mu_2(F)$ is given by

$$\mu_2(F) = -\sum_{i,j,k,l} a_{ijkl}(f_i g_k)''(f_j g_l)(dz)^2 l^6 = \sum_{i,j,k,l} a_{ijkl}(f_i g_k)'(f_j g_l)'(dz)^2 l^6.$$

First, notice that since $F \in I_2(\omega_C \otimes \eta)$, we have

$$\sum_{i,j,k,l} a_{ijkl}(f_i g_k)(z)(f_j g_l)(z) = 0,$$

hence, taking the derivative with respect to z, we get

$$\begin{aligned} 0 &= \sum_{i,j,k,l} a_{ijkl}(f_i g_k)'(f_j g_l) = \sum_{i,j,k,l} a_{ijkl} f_i' f_j g_k g_l + \sum_k g_k' \Big(\sum_{i,j,l} a_{ijkl} f_i f_j g_l \Big) \\ &= \sum_{i,j,k,l} a_{ijkl} f_i' f_j g_k g_l + \sum_k g_k' \Big(\sum_l A_{kl}(z) g_l(z) \Big) = \sum_{i,j,k,l} a_{ijkl} f_i' f_j g_k g_l, \end{aligned}$$

where $A_{kl}(z) = \sum_{i,j} a_{ijkl} f_i(z) f_j(z)$, and where the last equality holds since the point $(y_0, y_1, y_2)^t$ is in the kernel of the matrix $A(x)$, hence $\sum_l A_{kl}(z) g_l(z) = 0$.

So, differentiating with respect to z the polynomial equation

$$\sum_{i,j,k,l} a_{ijkl} f_i' f_j g_k g_l = 0,$$

we obtain

$$\sum_{i,j,k,l} a_{ijkl}\, f_i''\, f_j\, g_k\, g_l + \sum_{i,j,k,l} a_{ijkl}\, f_i'\, f_j'\, g_k\, g_l + 2\sum_{i,j,k,l} a_{ijkl}\, f_i'\, f_j\, g_k'\, g_l = 0. \tag{4.3}$$

Now

$$\begin{aligned}\mu_2(F) &= -\sum_{i,j,k,l} a_{ijkl}(f_i g_k)''(f_j g_l)\,(dz)^2\, l^6\\ &= -\Big(\sum_{i,j,k,l} a_{ijkl}\, f_i''\, f_j\, g_k\, g_l + 2\sum_{i,j,k,l} a_{ijkl}\, f_i'\, f_j\, g_k'\, g_l\Big)(dz)^2\, l^6\\ &= \sum_{i,j,k,l} a_{ijkl}\, f_i'\, f_j'\, g_k\, g_l\,(dz)^2\, l^6,\end{aligned}$$

where the last equality follows from (4.3) and we used that

$$\sum_{i,j,k,l} a_{ijkl}\, f_i\, f_j\, g_k''\, g_l = \sum_k g_k'' \sum_l A_{kl}(z)\, g_l(z) = 0.$$

So we have found the following local expression for $\mu_2(F)$:

$$\mu_2(F) = \sum_{i,j,k,l} a_{ijkl}\, f_i'\, f_j'\, g_k\, g_l\,(dz)^2\, l^6. \tag{4.4}$$

Proposition 4.3. *The map $\tau \circ \mu_2$ is nonzero.*

Proof. We show this by exhibiting an explicit example where the map is nonzero.

Consider the smooth $(2,2)$-threefold $X = \{F = 0\}$, where

$$\begin{aligned}F &= 6x_0^2 y_0^2 + 2\lambda x_2^2 y_0 y_1 + 2\lambda x_1^2 y_0 y_2 + 6x_1^2 y_1^2 + 2\lambda x_0^2 y_1 y_2 + 6x_2^2 y_2^2\\ &= 6\alpha_{00}^2 + 2\lambda\alpha_{20}\alpha_{21} + 2\lambda\alpha_{10}\alpha_{12} + 6\alpha_{11}^2 + 2\lambda\alpha_{01}\alpha_{02} + 6\alpha_{22}^2,\end{aligned} \tag{4.5}$$

with $\lambda^3 = -108$. Thus, the equation of the plane sextic attached to the first projection p_1 is the determinant of the following symmetric matrix:

$$A = \begin{pmatrix} 6x_0^2 & \lambda x_2^2 & \lambda x_1^2 \\ \lambda x_2^2 & 6x_1^2 & \lambda x_0^2 \\ \lambda x_1^2 & \lambda x_0^2 & 6x_2^2 \end{pmatrix}, \quad \det(A) = -6\lambda^2(x_0^6 + x_1^6 + x_2^6).$$

So the plane curve C is the Fermat sextic with equation $G := x_0^6 + x_1^6 + x_2^6 = 0$. We will show that $\tau \circ \mu_2(F) \neq 0$. Consider the affine chart $x_0 \neq 0$, set $w_i = x_i/x_0$, $i = 1, 2$, and let $h(w_1, w_2) := G(1, w_1, w_2) = 1 + w_1^6 + w_2^6$. Assume that $\partial h/\partial w_2 = 6w_2^5 \neq 0$, so $z := w_1$ is a local coordinate, hence $w_2 = f_2(z)$. In this local setting, we have $z = w_1$, $l = x_0$, $x_1 = w_1 x_0 = zl$ and $x_2 = w_2 x_0 = f_2(z) l$. So in the above notation, we have $f_0(z) = 1$, $f_1(z) = z$ and $f_2(z) = w_2(z)$, and since $h(z, f_2(z)) = 0$, we have

$$\partial h/\partial w_1 + f_2'(z)\, \partial h/\partial w_2 = 0,$$

so $f_2'(z) = -h_{w_1}/h_{w_2}$ (where $h_{w_i} = \partial h/\partial w_i$), while clearly $f_0' = 0$, $f_1' = 1$.

Then, by (4.4), we have

$$
\begin{aligned}
\mu_2(F) &= \sum_{i,j,k,l} a_{ijkl}\, f_i' f_j' g_k g_l\,(dz)^2 l^6 \\
&= \sum_{k,l} (g_k g_l)\, l^4 \left(a_{11kl}(f_1')^2 + 2a_{12kl}\, f_1' f_2' + a_{22kl}(f_2')^2\right)(dz)^2 l^2 \\
&= \sum_{k,l} (g_k g_l)\, l^4 \left(a_{11kl} - 2a_{12kl}(h_{w_1}/h_{w_2}) + a_{22kl}(h_{w_1}/h_{w_2})^2\right)(dz)^2 l^2.
\end{aligned}
$$

Recall that by (4.2) we have $\hat{A}_{kl}(z) = (g_k g_l)\, l^4$, thus

$$
\begin{aligned}
\mu_2(F) &= \sum_{k,l} \hat{A}_{kl}(z)\,(a_{11kl} - 2a_{12kl}(h_{w_1}/h_{w_2}) + a_{22kl}(h_{w_1}/h_{w_2})^2)(dz)^2 l^2 \\
&= \sum_{k,l} \hat{A}_{kl}(z)\,(a_{11kl} - 2a_{12kl}(h_{w_1}/h_{w_2}) + a_{22kl}(h_{w_1}/h_{w_2})^2)(h_{w_2})^2 l^8,
\end{aligned}
$$

since, by adjunction, we can write $dz = h_{w_2} l^3$. So we have

$$
\begin{aligned}
\mu_2(F) &= \sum_{k,l} \hat{A}_{kl}(a_{11kl} - 2a_{12kl}(G_{x_1}/G_{x_2}) + a_{22kl}(G_{x_1}/G_{x_2})^2)\,\frac{(G_{x_2})^2}{x_0^{10}}\, x_0^8 \\
&= \sum_{k,l} \hat{A}_{kl}(a_{11kl}(G_{x_2})^2 - 2a_{12kl}\, G_{x_1} G_{x_2} + a_{22kl}(G_{x_1})^2)\,\frac{1}{x_0^2}.
\end{aligned}
\tag{4.6}
$$

Now, using our equation and computing the minors of the matrix A, we obtain

$$
\begin{aligned}
\mu_2(F) &= \frac{36}{x_0^2}\,(x_1^{10}(2\lambda\, \hat{A}_{01} + 6\hat{A}_{22}) + x_2^{10}(2\lambda\, \hat{A}_{02} + 6\hat{A}_{11})) \\
&= -18\lambda^2\, \frac{36}{x_0^2}\,(x_1^{10} x_2^4 + x_2^{10} x_1^4) = -18\lambda^2\, \frac{36}{x_0^2}\, x_1^4 x_2^4 (x_1^6 + x_2^6).
\end{aligned}
$$

Since we are on the curve C, we have $x_1^6 + x_2^6 = -x_0^6$, so finally we get

$$
\mu_2(F) = 18\lambda^2\, \frac{36}{x_0^2}\, x_1^4 x_2^4 x_0^6 = (18 \cdot 36)\,\lambda^2 x_0^4 x_1^4 x_2^4,
$$

which is not in the Jacobian ideal of C that is generated by x_0^5, x_1^5, x_2^5. Notice that if $\partial h/\partial w_1 \neq 0$, then w_2 is a local coordinate, so in this case we have $z = w_2$, $w_1 = f_1(z)$, $f_2(z) = z$, $f_0(z) = 1$, $dw_2 = -(\partial h/\partial w_1)x_0^3$ and $f_1'(z) = -h_{w_2}/h_{w_1}$, and the computation is the same. By symmetry, an analogous computation holds in the other affine charts. So $\mu_2(F)$ is the restriction of the polynomial $(18 \cdot 36)\,\lambda^2 x_0^4 x_1^4 x_2^4 \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(12))$ to C, and hence $\tau \circ \mu_2(F) \neq 0$. ∎

Remark 4.4. Notice that formula (4.6) is valid for any equation F of a (2, 2)-threefold, where G is the equation of one of the sextics. Nevertheless, the computation can be quite difficult if the equations F and G are not as simple as in the above example. Moreover, showing that $\mu_2(F)$ does not belong to the Jacobian ideal can also be quite complicated in general.

Recall that by (2.1), we have an identification

$$R_T^{2,2} \cong \Omega^1_{\mathcal{Q},JT} \cong \Omega^1_{\mathcal{P}_6,(C,\eta)} \cong R_C^6.$$

These identifications can also be seen as induced by the pullback via the map $\phi = \varphi_1 \times \varphi_2$:

$$\phi^* : H^0(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(2,2)) \longrightarrow H^0(C, \mathcal{O}_C(6)).$$

In fact, we show the following.

Lemma 4.5. *The space $J_T^{2,2}$ maps to $J_C^6/\langle C\rangle$ via φ^*, and therefore there is an isomorphism $R_T^{2,2} \cong R_C^6$ and a commutative diagram*

$$\begin{tikzcd} \operatorname{Sym}^2 R_T^{2,2} \arrow[r, "m"] \arrow[d, "\cong"] & R_T^{4,4} \cong \mathbb{C} \arrow[d, "\cong"] \\ \operatorname{Sym}^2 R_C^6 \arrow[r, "m"] & R_C^{12} \cong \mathbb{C}. \end{tikzcd} \tag{4.7}$$

Proof. Clearly, all the elements of the form $y_j F_{y_i}$ map to zero, since they are in the ideal of $C' = \phi(C)$. So we have to show that all the elements $x_j F_{x_i}$ map to $J_C^6/\langle C\rangle$. In fact, we show that

$$\phi^*(F_{x_i}) = \partial_{x_i}(\det A(x)) \in J_C^6/\langle C\rangle,$$

where the coefficients of the symmetric matrix $(A(x))_{kl} = \sum_{i,j} a_{ijkl} x_i x_j$ are quadrics in the coordinates x_i.

Using (4.2), we have

$$\phi^*(F_{x_i}) = \phi^*\Big(\partial_{x_i}\Big(\sum_{ij} A(x)_{ij}\, y_i\, y_j\Big)\Big) = \partial_{x_i}\Big(\sum_{ij} A(x)_{ij}\, \hat{A}_{ij}(x)\Big) = \partial_{x_i}(\det A(x)).$$

In the same way, one immediately sees that ϕ^* induces an isomorphism $R_T^{4,4} \cong R_C^{12}$ and we have a commutative diagram as (4.7). ∎

Proof of the main Theorem 1.1. In Proposition 4.3, we have proven that $\tau \circ \mu_2$ is nonzero. Diagram (4.1) together with Lemma 4.5 show that the restriction of $m \circ \mathrm{II}$ to $I_2(\omega_C \otimes \eta)$ is identified with $\tau \circ \mu_2$, hence it is nonzero for a generic T. So by Remark 4.1, $m \circ \mathrm{II}$ gives a non-trivial holomorphic section $\mathcal{S}$ of the normal bundle. ∎

We observe that the fact that the second fundamental form is nonzero is also implied by the following proposition, by a result of Moonen ([13]).

Proposition 4.6. *The monodromy group of the family $\mathcal{Q}$ is the symplectic group. Hence, the special Mumford–Tate group of a generic $(2,2)$-threefold is the whole symplectic group.*

Proof. We prove that the monodromy group of $\mathcal{P}_6$ is the full symplectic group $\mathrm{Sp}(18,\mathbb{Z})$.

Recall that the monodromy map of plane sextics,

$$\pi_1(\mathcal{M}_{10}^{\mathrm{pl}}, C) \longrightarrow \mathrm{Sp}(H^1(C,\mathbb{Z})),$$

is surjective (see Théorème 4 in [2]). In particular, reducing coefficients modulo 2, this implies that $\mathcal{P}_6$ is irreducible.

For any symplectic basis $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$ of $H^1(C, \mathbb{Z})$, where $\eta = \alpha_g \pmod 2$, there is a natural symplectic basis in $H^1(P(C, \eta), \mathbb{Z})$ (see, e.g., Proposition 12.4.2 in [4]).

Consider a Lefschetz pencil around a Prym semi-abelian variety attached to a covering $\tilde{C}_0 \to C_0$ such that C_0 has only one node and the two preimages of the node are two nodes of $\tilde{C}_0$. Then, writing explicitly the Picard Lefschetz transformation in terms of the basis defined above, one easily checks that the associated monodromy generates the whole symplectic group. ∎

We will now give a complete description of the kernel of $m \circ \mathrm{II}$, that can be interesting for a deeper understanding of the second fundamental form. First, we describe the kernel of

$$\tau \circ \mu_2 : I_2(\omega_C \otimes \eta) \to R_C^{12}.$$

Denote by

$$L := \mathcal{O}_C(1) \quad \text{and} \quad M := \mathcal{O}_C(2) \otimes \eta,$$

so that $L \otimes M \cong \omega_C \otimes \eta$. Take a point $p \in C$ and consider the two line bundles

$$L(-p) = \mathcal{O}_C(1) \otimes \mathcal{O}_C(-p) \quad \text{and} \quad M(p) = \mathcal{O}_C(2) \otimes \eta \otimes \mathcal{O}_C(p).$$

Then clearly

$$L(-p) \otimes M(p) \cong \omega_C \otimes \eta.$$

Since L is base point free, we have $h^0(C, L(-p)) = 2$, and we fix a basis $\{s_1, s_2\}$ of $H^0(C, L(-p))$. We have a map

$$\begin{aligned} &\bigwedge^2 H^0(M(p)) \to I_2(\omega_C \otimes \eta), \\ &(t_1 \wedge t_2) \mapsto Q(t_1, t_2) := s_1 t_1 \odot s_2 t_2 - s_1 t_2 \odot s_2 t_1, \end{aligned} \tag{4.8}$$

where $\alpha \odot \beta := \alpha \otimes \beta + \beta \otimes \alpha$.

Proposition 4.7. *For a general T, the quadrics defined in (4.8) generate the kernel of the map $\tau \circ \mu_2$.*

Proof. A direct computation using formula (3.1) (see, e.g., Lemma 2.2 in [5]) shows that we have

$$\begin{aligned} &\mu_2 : I_2(\omega_C \otimes \eta) \to H^0(C, \omega_C^{\otimes 4}) \cong H^0(C, \mathcal{O}_C(12)), \\ &\mu_2(Q(t_1, t_2)) = \mu_{1,L(-p)}(s_1 \wedge s_2) \cdot \mu_{1,M(p)}(t_1 \wedge t_2), \end{aligned}$$

where

$$\begin{aligned} \mu_{1,L(-p)} : \bigwedge^2 H^0(C, L(-p)) \cong \langle s_1 \wedge s_2 \rangle \to H^0(C, L^{\otimes 2}(-2p) \otimes \omega_C) \\ \cong H^0(C, \mathcal{O}_C(5)(-2p)), \end{aligned}$$

and

$$\mu_{1,M(p)} : \bigwedge^2 H^0(C, M(p)) \to H^0(C, M^{\otimes 2}(2p) \otimes \omega_C) \cong H^0(C, \mathcal{O}_C(7)(2p))$$

denote the first gaussian maps of the line bundles $L(-p)$ and $M(p)$.

In local coordinates, if $s_i = f_i(z)l$, where l is a local frame for $L(-p)$, we have

$$\mu_{1,L(-p)}(s_1 \wedge s_2) = (f_1' f_2 - f_1 f_2')l^2\,dz.$$

Then, clearly we obtain

$$\operatorname{div}(\mu_{1,L(-p)}(s_1 \wedge s_2)) = \operatorname{div}(\mathrm{Pol}_p(C)_{|C}) - 2p,$$

where $\mathrm{Pol}_p(C)$ denotes the polar of the plane sextic C with respect to p.

Choose now $t_1 \in H^0(C, M(-2p)) \subset H^0(C, M(p))$, and assume that in local coordinates we have $t_i = g_i(z)\sigma$, where σ is a local frame for $M(p)$. Then we have

$$\mu_{1,M(p)}(t_1 \wedge t_2) = (g_1' g_2 - g_1 g_2')\sigma^2\,dz.$$

Since $\operatorname{ord}_p g_1 = 3$, clearly

$$\operatorname{div}(\mu_{1,M(p)}(t_1 \wedge t_2)) = 2p + E,$$

where $\mathcal{O}_C(E) \cong \mathcal{O}_C(7)$. Since the restriction to C gives a surjective map

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(7)) \to H^0(C, \mathcal{O}_C(7)),$$

there exists a polynomial $G \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(7))$, such that $\operatorname{div}(G_{|C}) = E$. Then

$$\operatorname{div}(\mu_2(Q(t_1, t_2))) = \operatorname{div}(\mathrm{Pol}_p(C)_{|C}) - 2p + 2p + E = \operatorname{div}((\mathrm{Pol}_p(C) \cdot G)_{|C}),$$

hence $\tau \circ \mu_2(Q(t_1, t_2)) = 0$, since $\mathrm{Pol}_p(C) \cdot G \in J_C^{12}$.

It remains to show that varying the point $p \in C$, and choosing $t_1 \in H^0(C, M(-2p)) \subset H^0(C, M(p))$ (which is 1-dimensional for p general in C), and varying $t_2 \in H^0(C, M(p))$, the quadrics $\{Q(t_1, t_2)\}$ generate a 17-dimensional subspace of $I_2(\omega_C \otimes \eta)$.

We will show it in the example (4.5), and this is enough, since it is an open property. Let us now explain how we compute this dimension: first observe that the choice of s_1 corresponds to the choice of a line l_1 through p, hence we have

$$\operatorname{div}(s_1) = p + D_5,$$

where D_5 is an effective divisor of degree 5. The same holds for s_2, so

$$\operatorname{div}(s_2) = p + D_5'.$$

Thus, by our choice of t_1, $s_1 t_1$ is a generator of the one-dimensional vector space $H^0(C, \omega_C \otimes \eta(-D_5 - 3p))$, and $s_2 t_1$ is a generator of $H^0(C, \omega_C \otimes \eta(-D_5' - 3p))$. To choose the forms $s_1 t_2$ and $s_2 t_2$, namely to choose $t_2 \in H^0(C, M(p))$ general, it is equivalent to choose three general points $q_1, q_2, q_3 \in C$ so that

$$\langle s_1 t_2 \rangle = H^0(C, \omega_C \otimes \eta(-D_5 - q_1 - q_2 - q_3))$$

and

$$\langle s_2 t_2 \rangle = H^0(C, \omega_C \otimes \eta(-D_5' - q_1 - q_2 - q_3)).$$

Now we have to check that varying a point p on C, choosing two lines through p, and three points q_1, q_2 and q_3 on C, one obtains a linear space of quadrics constructed as above of dimension 17. To do this computation of the dimension of this vector space of quadrics on the example (4.5), we used a MAGMA script.

This shows that, for a generic (C, η), the quadrics we considered generate the kernel of $\tau \circ \mu_2$. ∎

Remark 4.8. By Remark 4.1, we know that

$$I_2(\omega_C \otimes \eta) + I_2(\omega_D \otimes \eta') = N^*_{\mathcal{Q}/\mathcal{A}_9,T}.$$

Hence the kernel of $m \circ \mathrm{II}$ is generated by the quadrics of $I_2(\omega_C \otimes \eta)$ and $I_2(\omega_D \otimes \eta)$ that we have just described.

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