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Examples of symplectic non-leaves

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Abstract. This paper deals with the following question: which manifolds can be realized as leaves of codimension-1 symplectic foliations (of regularity at least C^2) on closed manifolds? We first observe that leaves of symplectic foliations are necessarily strongly geometrically bounded. We show that a symplectic structure which admits an exhaustion by compacts with (convex) contact boundary can be deformed to a strongly geometrically bounded one. We then give examples of smooth manifolds which admit a strongly geometrically bounded symplectic form and can be realized as a smooth leaf, but not as a symplectic leaf for any choice of symplectic form on them. Lastly, we show that the (complex) blowup of $2n$-dimensional Euclidean space at infinitely many points admits both strongly geometrically bounded symplectic forms for which it can and cannot be realized as a symplectic leaf.

1. Introduction

The realizability problem, namely that of understanding which (open) manifolds can be leaves of foliations, was introduced by Sondow [25], and has been extensively studied since then. Unless explicitly stated otherwise, throughout this paper "foliation" will always mean codimension-1 foliation, with regularity at least C^2 (in the sense of Definition 1.1.18 in [4], i.e., both tangentially and transversely), on a closed ambient manifold. Moreover, the realization problem is intended for open manifolds.

The situation in low dimensions is very flexible: every open orientable surface is a leaf of a foliation on any given ambient manifold [6]. In higher dimensions, there are many manifolds which are not diffeomorphic to leaves of foliations. The first examples were found by [9, 14], and subsequently simply connected examples were given in [2] (see also [23]). More recently, [20, 21] have found some examples of topological manifolds with exotic smooth structures which are not diffeomorphic to smooth leaves. One can also consider foliations with additional leafwise structures; see, for instance, [2, 22, 26] for examples of Riemannian manifolds not quasi-isometric to leaves of a Riemannian foliation.

In this paper, we focus on foliations $\mathcal{F}$ on an ambient (closed) manifold M^{2n+1} which are equipped with a *leafwise symplectic structure*, i.e., a leafwise 2-form $\omega \in \Omega^2(\mathcal{F})$ which is (leafwise) closed and non-degenerate. Symplectic foliations arise naturally in many set-

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tings. For instance, they are of relevance from the perspective of Poisson geometry, see [12, 13], as they are examples of regular (corank-one) Poisson structures. Moreover, the subclass of *strong symplectic foliations*, i.e., those for which $\omega \in \Omega^2(\mathcal{F})$ extends to a closed 2-form on M, constitute a high-dimensional analogue to 3-dimensional taut foliations which behaves "rigidly" [10,16], contrary to taut high-dimensional foliations which themselves are flexible [19].

Concerning the existence problem for symplectic foliations, Bertelson, see [3] (see also [12]), found foliations which do not admit a leafwise symplectic structure, although the formal obstructions vanish. Here we instead investigate obstructions on the level of a single leaf. That is, we consider the following question:

Is there a (*symplectic*) *manifold that is not a leaf of a symplectic foliation?*

This question makes sense both with and without fixing the symplectic structure on the manifold. Furthermore, any closed symplectic manifold (W, ω) can be realized as a symplectic leaf by considering the product foliation on $W \times \mathbb{S}^1$. As such, we restrict our attention to *open* manifolds. Even though W is open, the compactness of the ambient manifold implies that the behavior of ω resembles the compact case: any leaf (W, ω) is *strongly geometrically bounded* (or SGB in short) if there exists a compatible almost complex structure J for which the associated Riemannian metric $g = \omega(\cdot, J\cdot)$ has bounded injectivity radius and scalar curvature (see Definition 3.3 and Lemma 3.4 for details).[1] We point out that non-SGB (and non-geometrically-bounded) symplectic manifolds are plentiful: for instance open symplectic manifolds with finite volume, and symplectizations of contact manifolds are not SGB and thus cannot be a leaf of a symplectic foliation.

In light of this we consider the following variations of our motivating question (in which W is always assumed to be open):

(Q1) Is there a manifold that admits a SGB symplectic structure but is not diffeomorphic to a symplectic leaf (for any choice of ω)?

(Q2) Is there a SGB symplectic manifold (W, ω) with W diffeomorphic to a symplectic leaf but (W, ω) not symplectomorphic to a symplectic leaf?

Remark 1.1. There are several variations of the above questions that we do not consider here, but are nevertheless interesting. Indeed, recall that we assume all foliations to have codimension 1 and C^2-regularity, both transversely and tangentially. However, one could consider the questions above also for foliations of higher codimension, and for foliations of lower (transverse) regularity. This seems to be a delicate question already for topological manifolds and for foliations not equipped with an additional geometric structure. Another related question is whether any manifold admitting a symplectic structure also admits a SGB symplectic structure (see Theorem 1.2 below for a partial answer).

In this paper, we answer positively to both questions. We start by proving a symplectic analogue of Greene's result [11] on the existence of Riemannian metrics of bounded geometry on open manifolds. In order to state our result, we define *exhaustion of contact type* as any exhaustion by compacts $\mathcal{K} = \{K_n\}_{n \in \mathbb{N}}$ on a symplectic manifold (W, ω) such that ∂K_n is a hypersurface of (convex) contact type. We then prove the following.

[1] As the name suggests, SGB is stronger, and implies, the usual notion of geometrically boundedness (see, e.g., Definition 4.1.1 in [24]).

Theorem 1.2. *Let (W, ω) be a symplectic manifold admitting an exhaustion $\mathcal{K}$ of contact type. Then ω is homotopic through symplectic forms to a SGB symplectic form ω'.*

We point out that, on each of the compact sets $K_n \setminus \mathcal{O}p(K_{n-1})$, where $\mathcal{O}p(K_{n-1})$ denotes an arbitrarily small open neighborhood of K_{n-1}, the symplectic form ω' (and the whole homotopy) is just a rescaling of ω by a constant depending on n; in particular, the less obvious part of the construction happens near the boundaries of the K_n's, where ω simply gets "stretched" along a collar (see the proof in Section 3 for more details). Notice also that the above result is well known for Liouville manifolds of finite-type, see Proposition 2.2 and Remark 2.3 in [8]; more precisely, in the finite-type case the homotopy is not necessary, and ω is directly SGB.

In order to answer Question (Q1), we prove the following.

Theorem 1.3. *Let W^{2n} be an open manifold with a finite number k of ends. Suppose that, for $i = 1, \dots, k$, the i-th end has a neighborhood diffeomorphic to $N_i \times [0, \infty)$, where N_i has trivial π_1 and H^2. If there is only one end,* additionally *assume that the compact $W \setminus N_1 \times (0, \infty)$ has non-trivial π_1 or non-trivial H^2. Then W is not diffeomorphic to a leaf of a symplectic foliation.*

Notice that these manifolds have cylindrical ends and, as such, they can be realized as leaves of a smooth foliation. For example, consider the truncation W_{tr} of W, which is the compact manifold with boundary obtained by removing the cylindrical end. Using a turbulization procedure (see, for instance, Example 3.3.11 in [4]), we obtain a foliation on $\mathbb{S}^1 \times W_{tr}$ (tangent to the boundary) whose non-compact leaves are all diffeomorphic to W. Hence, gluing two such Reeb components gives a foliation on a closed manifold containing W as a leaf.

Moreover, it is not difficult to find manifolds satisfying the hypotheses of Theorem 1.3 and admitting a SGB symplectic structure. For example, (connected) Weinstein manifolds (W, ω) of finite type are open symplectic manifolds with only one end, which is cylindrical over a (connected) contact manifold. Moreover, if W admits a handle decomposition without 1- and 2-handles, then the ends have trivial π_1 and H^2. On the other hand, via a symplectic blowup, one may easily arrange non-trivial H^2 on a compact subset of W, while preserving the SGB property. Hence, the following corollary is a direct consequence of Theorem 1.3, and gives a positive answer to Question (Q1).

Corollary 1.4. *Let (W, ω) be a Weinstein manifold of finite type without* 1*-and* 2*-handles, and denote by (W', ω') the symplectic blowup at any point of W. Then ω' is SGB, but W' is not diffeomorphic to a leaf of a symplectic foliation.*

In particular, for instance $\mathbb{R}^{2n}$ blown-up at a point admits a SGB symplectic structure but is not diffeomorphic to a symplectic leaf. In fact, although $(\mathbb{R}^{2n}, \omega_{\text{std}})$ itself does not satisfy the conditions of Theorem 1.3, part of the proof of the latter (and an explicit construction) gives the following.

Corollary 1.5. *If $n \geq 2$, then $\mathbb{R}^{2n}$ is not diffeomorphic to a proper symplectic leaf. On the other hand, $(\mathbb{R}^{2n}, \omega_{\text{std}})$ can be realized as a non-proper leaf of a symplectic foliation.*

Recall that a leaf L is *proper* if it does not accumulate onto itself.

Let us now go back to (Q2), to which we give a positive answer using again the symplectic blowup procedure, but this time at infinitely many points.

Theorem 1.6. *Let $n \geq 2$, and let (W^{2n}, ω) be a symplectic manifold admitting an exhaustion of contact type. Assume moreover that W has finitely many ends, and that there is a compact $K \subset W$ such that $H_{2n-2}(W \setminus K; \mathbb{Z}) = 0$. Then, there is a SGB symplectic manifold (W', ω'), with W' obtained, as a smooth manifold, from W by (complex) blowup at infinitely many points, which is not symplectomorphic to a leaf of a symplectic foliation.*

More precisely, if $\{K_n\}_{n\in\mathbb{N}}$ is the given exhaustion of contact type, then we symplectically blowup at a countable sequence of points $\{p_n\}_{n\in\mathbb{N}}$, where $p_n \in K_n \setminus K_{n-1}$. At each of these points, the blowup is performed so that the resulting copies of $\mathbb{CP}^{n-1}$ have different ω'-volumes. The difference in volume is essential, as the result of blowing up with the same size can sometimes be realized as a leaf. Indeed, we have the following positive answer to Question (Q2).

Corollary 1.7. *Let W be the result of (complex) blowing up $\mathbb{R}^{2n}$ at a sequence of points going to infinity. Then W is diffeomorphic to a leaf of a symplectic foliation, but also admits a SGB symplectic form ω for which (W, ω) is not symplectomorphic to a leaf of a symplectic foliation.*

The fact that W is diffeomorphic to a leaf of a symplectic foliation follows from an explicit construction. In this case, the induced volumes of the $\mathbb{CP}^{n-1}$ resulting from blowing up are all the same. On the other hand, in the second part of the conclusion, ω is constructed so that the induced volumes on the $\mathbb{CP}^{n-1}$ are unbounded, so that the conclusion follows from Theorem 1.6.

Outline. In Section 2, we recall some notions and results from smooth foliation theory, describe the symplectic (almost-)periodicity notion for ends of leaves of symplectic foliations, and prove that proper leaves of symplectic foliations are "symplectically almost periodic" (see Theorem 2.7). In Section 3, we define strongly geometrically bounded symplectic manifolds, and prove Theorem 1.2. In Section 4, we describe topological obstructions for a manifold to be diffeomorphic to a proper or non-proper symplectic leaf, proving in particular Theorem 1.3 and Theorem 1.5 above. Lastly, in Section 5, we study examples coming from the symplectic blowup construction, thus proving Theorem 1.6 and Theorem 1.7.

2. Ends of manifolds and accumulation of leaves

Intuitively, the ends of a manifold represent the (topologically) different ways to go to infinity. In order to make this precise, consider an exhaustion by compacts $\mathcal{K} = \{K_i\}_{i \in \mathbb{N}}$ of a manifold $W = \bigcup_{i \in \mathbb{N}} K_i$. The *endset* $\mathcal{E}_{\mathcal{K}}(W)$ is the set of sequences

$$U_1 \supset U_2 \supset U_3 \supset \cdots,$$

where U_i is a connected component of $W \setminus K_i$; each element e of $\mathcal{E}_{\mathcal{K}}(W)$ is called an *end* of W. One can see that the endsets $\mathcal{E}(W)$ associated to two different exhaustion by compacts are in natural bijection; thus, we will just denote it $\mathcal{E}(W)$ from now on. We lastly call *neighborhood* of an end $e = \{U_i\}_{i\in\mathbb{N}}$ any open set V such that $U_n \subset V$ for some n.

A leaf L of a foliation on M, or more precisely an end e of L, is said to *accumulate* onto a leaf L' if

$$\overline{U} \cap L' \neq \emptyset,$$

(where $\overline{U}$ denotes the closure of U in M) for some (and hence any) neighborhood U of e. Since on a closed manifold any non-compact leaf has non-empty limit set, we obtain the following invariant. A leaf L is said to be at *depth* 0 if it is compact, and at *depth* k if $\overline{L} \setminus L$ is a union of leaves at depth $< k$; lastly, L is at *infinite depth* if it is not at any finite depth.

The way leaves at finite depth can accumulate closely resembles the way the interior leaves of a Reeb component spiral around the boundary. That is, each end has a neighborhood which spirals (Definition 2.2) onto a leaf at lower level.

Theorem 2.1 (Theorem 8.4.6 in [4]). *Let $\mathcal{F}$ be a codimension-one C^2-foliation. If L is a leaf of $\mathcal{F}$ at depth k, then*

$$L = A \cup B^1 \cup \cdots \cup B^q,$$

where A is a compact, connected, $(n-1)$-dimensional manifold with boundary components $N^1, \ldots, N^q$, and

(1) $A \cap B^j = N^j$, $1 \leq j \leq 1$;

(2) $B^i \cap B^j = \emptyset$, $i \neq j$;

(3) B^j *spirals on a leaf* L^j *at depth*[2] *at most* $k-1$, $1 \leq j \leq q$;

(4) *for at least one value of* j, L^j *is at depth* $k-1$.

Before stating the precise definition of spiraling, let us point out two properties of the spiraling in a Reeb component. Firstly, on a neighborhood of the boundary there is a projection (e.g. along the leaves of an auxiliary transverse 1-dimensional foliation) onto the boundary leaf. Secondly, the end of each interior leaf can be written as an infinite union of diffeomorphic pieces and the projection is injective on each piece.

Definition 2.2. A neighborhood of an end $B \subset L$ is said to *spiral* onto a leaf $\widetilde{L}$ if there exists a projection $\pi\colon B \to \widetilde{L}$ (obtained by projecting along the leaves of an auxiliary transverse 1-dimensional foliation), and a (closed, connected) codimension-1 submanifold $N \subset \widetilde{L}$, called the *juncture*, such that

(i) there exists a decomposition $B = \bigcup_{j=0}^{\infty} B_j$, with $\partial B_j = N_j \sqcup N_{j+1}$ and $\mathrm{int}(B_i) \cap \mathrm{int}(B_j) = \emptyset$ if $i \neq j$;

(ii) the projection π maps each N_i diffeomorphically onto N for each $i \in \mathbb{N}$;

(iii) the restriction of π to $B_i \setminus N_{i+1}$ is injective for each $i \in \mathbb{N}$;

(iv) for each $p \in \widetilde{L}$, the sequence $\pi^{-1}p = \{q_i\}_{i \in \mathbb{N}}$, where $q_i \in B_i$, converges monotonically (as a sequence in a small interval transverse to $\mathcal{F}$) to p.

Evidently, the interior of each B_i is diffeomorphic to $\widetilde{L} \setminus N$, and B is an *infinite repetition* of L:

$$B = B_0 \cup_{N_1} B_1 \cup_{N_2} B_2 \cup_{N_3} \cdots$$

[2]Note that the statement in [4] talks about levels instead of depth. However, in light of Corollary 8.3.16 in [4], we can rephrase it in terms of depth.

In this case, we also say that the end (of which B is a neighborhood) is *periodic with period* $\widetilde{L} \setminus N$.

Theorem 2.1 has a nice consequence on the asymptotic behaviour of proper leaves with finitely many ends, which appeared as part of Theorem 2.5 in [21]. For the reader's convenience, we restate it here explicitly and give a detailed proof.

Proposition 2.3. *Let* L *be a proper leaf with finitely many ends. Then,* L *is totally proper and* $\operatorname{depth}(L) = 1$.

Proof. We start by proving that L is a *totally proper* leaf, meaning that each leaf in the closure $\overline{L}$ is proper. For this, we first recall that a subset $N \subset M$ is a *local minimal set* if there is an open set $U \subset M$ which is $\mathcal{F}$-saturated (i.e., union of leaves) such that N is a minimal set of the restriction $\mathcal{F}|_U$. We also recall that a local minimal set can be of three types:

(1) an open saturated set of $\mathcal{F}$;

(2) a single proper leaf (cf. Proposition 8.1.19 in [4] again);

(3) an exceptional local minimal set, by which we mean that its closure is transversely a Cantor set.

Going back to the proof, since L is a proper leaf, it is a local minimal set, see Proposition 8.1.19 in [4]. Hence its closure $\overline{L}$ is a finite union of local minimal sets, see Corollary 8.3.12 in [4].

We claim that all the minimal sets of $\overline{L}$ are of the second type. First, observe that a local minimal set contained in $\overline{L}$ cannot be of the first type, this would violate the properness of L. Moreover, because L has finitely many ends by assumption, Duminy's theorem (Theorem 4.3.12 in [7]) tells us that $\overline{L}$ does not contain any exceptional minimal sets. We conclude that every leaf in $\overline{L}$ is proper, that is, L is a totally proper leaf.

To see that L is at depth 1, observe that Corollaries 8.3.10 and 8.3.16 in [4] imply that any totally proper leaf is at finite depth. Since L has finitely many ends, one can then conclude from Corollary 8.4.7 in [4] that $\operatorname{depth}(L) = 1$. ∎

For symplectic foliations, the above discussion implies that the ends of totally proper leaves are not only smoothly periodic, but also symplectically. We consider two notions of periodicity for symplectic leaves.

Definition 2.4. An end e of a symplectic manifold (W, ω) is called:

- *symplectically periodic* if it can be represented by a sequence $\{h^n(U)\}_{n\in\mathbb{N}}$, where $h\colon U \to h(U) \subset U$ is a symplectomorphism and U is a neighborhood of e;
- *symplectically almost periodic* if it can be represented by a sequence $\{h^n(U)\}_{n\in\mathbb{N}}$, where $h\colon U \to h(U) \subset U$ is a diffeomorphism and U is a neighborhood of e such that there exists a symplectic form $\omega_\infty \in \Omega^2(\overline{U} \setminus h(U))$ satisfying

$$(h^n)^*\omega \xrightarrow{n\to\infty} \omega_\infty,$$

where $\overline{U}$ is the closure of U inside the leaf, and the convergence is with respect to the C^0-Whitney topology on the compact set $\overline{U} \setminus h(U)$.

As the nomenclature suggests, a symplectically periodic end is, in particular, symplectically almost periodic. The smoothness of the leafwise symplectic form together with Theorem 2.1 then immediately implies the following.

Corollary 2.5. *Let L be a leaf of a symplectic foliation $(\mathcal{F}, \omega)$ on a compact manifold. If L is at finite depth, then it has symplectically almost periodic ends.*

Although we will not need it explicitly in the following, we point out that, in the setting of *strong* symplectic foliations, one arrange that the fibers of the projection π in Definition 2.2 are tangent to the kernel of any closed extension of the leafwise symplectic form to the ambient manifold. As the flow of any vector field in such kernel foliation preserves the leafwise symplectic structure, one gets the following stronger variant of Theorem 2.5 in the strong symplectic foliated setting.

Corollary 2.6. *Let L be a leaf of a strong symplectic foliation $(\mathcal{F}, \omega)$ on a compact M. If L is totally proper and at depth 1, then $(L, \omega|L)$ has symplectically periodic ends.*

By combining Proposition 2.3 and Corollaries 2.5 and 2.6 above, we conclude the following.

Theorem 2.7. *Let (W, ω) be an open symplectic manifold with a finite number of ends. If it is symplectomorphic to a proper leaf of a symplectic foliation, then ω is symplectically almost periodic. Moreover, if the symplectic foliation is strong, then ω is symplectically periodic.*

3. Geometrically bounded symplectic manifolds

Although the leaves we consider are non-compact, the compactness of the ambient manifold implies that the leafwise symplectic structure behaves as on compact manifolds.

Definition 3.1. A symplectic form ω on W^{2n} is said to be *geometrically bounded* (GB) if there are an almost complex structure J and a complete Riemannian metric g satisfying the following conditions:

(GB1) There are strictly positive constants A and B such that, for each $u, v \in TW$,

$$\omega(u, Ju) \geq A\|u\|_g^2 \quad \text{and} \quad |\omega(u, v)| \leq B\|u\|_g \|v\|_g.$$

(GB2) The metric g has sectional curvature $\sec(g, W)$ bounded from above and injectivity radius $\mathrm{inj}(g, W)$ bounded from below by a positive constant.

In this case, we also call (ω, J, g) a *geometrically bounded* (GB) *triple*.

Remark 3.2. Asking that the Riemannian metric g is complete in above definition is actually redundant. Indeed, any Riemannian metric with injectivity radius bounded below by a strictly positive constant is complete; cf. Lemma 2.1 in [15].

Each almost complex structure J which is compatible with ω defines a Riemannian metric given by $g_{\omega,J} = \omega(\cdot, J\cdot)$. However, in general the triple $(\omega, J, g_{\omega,J})$ might not be GB if W is open; although condition (GB1) is trivially satisfied (with constants $A = B = 1$), condition (GB2) might not be.

For example, the symplectization $(\mathbb{R} \times N, \omega := \mathrm{d}(e^t \alpha))$ of a contact manifold (N, α) admits a compatible almost complex structure J which is cylindrical. Then it is easy to see that, for $g_J := \omega(\cdot, J\cdot)$, the condition (GB2) fails at the negative end so that the triple (ω, J, g_J) is not GB.

Definition 3.3. A symplectic manifold is said to be *strongly geometrically bounded* (SGB) if there is a compatible almost complex structure J for which $(\omega, J, g_{\omega,J})$ is a geometrically bounded triple.

Lemma 3.4. *Leaves of symplectic foliations on closed manifolds are* SGB.

Proof. This can be proven with a polar decomposition argument as in the non-foliated case (cf., for instance, Proposition 12.3 in [5]); we give a sketch of the argument. Let $\omega \in \Omega^2(\mathcal{F})$ be the leafwise symplectic form, and fix an auxiliary Riemannian metric g on the ambient manifold M. By non-degeneracy of both ω and g on $T\mathcal{F}$, there is an endomorphism $\psi: T\mathcal{F} \to T\mathcal{F}$ such that $g(\psi\cdot,\cdot) = \omega$. One can then explicitly check that ψ is skew-symmetric, and that the composition $\psi \circ \psi^*$ of ψ with its g-adjoint $\psi^* = -\psi$ is symmetric and positive-definite. In particular, it admits a square root $\phi := \sqrt{\psi \circ \psi^*}$, which is also symmetric and positive-definite. It follows that the endomorphism $J := \phi^{-1} \circ \psi$ of $T\mathcal{F}$ is an ω-compatible leafwise almost complex structure. We then define the Riemannian metric $g_{\omega,J} := \omega(\cdot, J\cdot)$ on $T\mathcal{F}$. The compactness of the ambient manifold then implies $(\omega, J, g_{\omega,J})$ is a SGB triple, as desired. ∎

Apart from (universal covers of) closed symplectic manifolds, standard examples in the literature of GB manifolds are (twisted) cotangent bundles and Liouville completions of symplectic fillings.

Definition 3.5. An *exhaustion of contact type* on a symplectic manifold (W, ω) is a collection of compact sets $\mathcal{K} = \{K_n\}_{n \in \mathbb{N}}$ such that

(i) $K_0 = \emptyset$, $W = \bigcup_{n \in \mathbb{N}} K_n$ and $K_n \subset \mathrm{int} K_{n+1}$;

(ii) $(K_n, \omega|_{K_n})$ is a symplectic domain with (smooth) boundary of convex contact type.

The main class of examples admitting an exhaustion of contact type is given by Liouville manifolds of infinite type.

The rest of this section deals with the proof of Theorem 1.2, which states that admitting an exhaustion of contact type is, up to homotopy, a sufficient condition for being SGB. The proof uses two ingredients, given in the following two lemmas. On the compact sets $K_{n+1} \setminus \mathcal{O}p(K_n)$, where $\mathcal{O}p(K_{n-1})$ denotes an arbitrarily small open neighborhood of K_{n-1}, we can rescale g to satisfy any bound on the curvature and injectivity radius. The resulting metrics are then glued by interpolating. To ensure that the interpolation preserves the SGB condition, we show that we can insert an arbitrarily large part of the symplectization of ∂K_n. This is where Definition 3.5 is used.

Lemma 3.6. *Let (M, g) be a Riemannian manifold and let $k > 0$ be a constant. Then*

$$\mathrm{sec}(kg, M) = k^{-1} \mathrm{sec}(g, M) \quad \textit{and} \quad \mathrm{inj}(kg, M) = \sqrt{k}\, \mathrm{inj}(g, M).$$

In particular, this lemma shows that if Theorem 3.3 is satisfied for some g, then it is satisfied for all $e^c g$, with $c \geq 0$ a real number.

Recall that given a contact manifold $(M, \xi = \ker\alpha)$, we can form the symplectization

$$(M \times \mathbb{R}, \omega = \mathrm{d}(e^t\alpha)).$$

Since $(\xi, \mathrm{d}\alpha|_\xi)$ is a symplectic vector bundle, we can find a compatible almost complex structure J_ξ, which in turn gives a metric $g_\xi = \mathrm{d}\alpha|_\xi(\cdot, J_\xi\cdot)$. Then,

$$\omega = \mathrm{d}(e^t\alpha), \quad g := e^t(g_\xi + \alpha\otimes\alpha + \mathrm{d}t\otimes\mathrm{d}t), \quad J := J_\xi + R\otimes\mathrm{d}t - \partial_t\otimes\alpha, \tag{3.1}$$

is a compatible triple on $M \times \mathbb{R}$.

Lemma 3.7. *Let $(M, \xi = \ker\alpha)$ be a contact manifold, and let (ω, g, J) be the associated compatible triple on $M \times (-\varepsilon, \varepsilon)$ from (3.1). Then, for any constants $a < b$, there exist a compatible triple $(\tilde\omega, \tilde g, \tilde J)$ and a homotopy of symplectic forms ω_s, $s \in [0, 1]$, on $M \times (-\varepsilon, \varepsilon)$ satisfying*

- (i) *$\omega_0 = \omega$ and $\omega_1 = \tilde\omega$;*
- (ii) *on $M \times (-\varepsilon, -\varepsilon/3)$, we have $(\tilde\omega, \tilde g, \tilde J) = (e^a\omega, e^a g, J)$ and $\omega_s = e^{sa}\omega$;*
- (iii) *on $M \times (\varepsilon/3, \varepsilon)$, we have $(\tilde\omega, \tilde g, \tilde J) = (e^b\omega, e^b g, J)$ and $\omega_s = e^{sb}\omega$;*
- (iv) *the metric $\tilde g$ satisfies*

$$\sec(\tilde g, M \times (-\varepsilon, \varepsilon)) \le e^{-a} \sup_{p\in M\times(-\varepsilon,\varepsilon)} \sec_p(g, M \times (-\varepsilon, \varepsilon)),$$

$$\mathrm{inj}(\tilde g, M \times (-2\varepsilon/3, 2\varepsilon/3)) \ge e^{a/2} \min(\varepsilon/3, \mathrm{inj}(g|_{M\times\{0\}})),$$

where $\mathrm{inj}(\tilde g, M\times(-2\varepsilon/3, 2\varepsilon/3))$ means the injectivity radius of $\tilde g$ considering geodesic balls centered at points in $M\times(-2\varepsilon/3, 2\varepsilon/3)$ and contained in $M\times(-\varepsilon, \varepsilon)$, the domain of definition of $\tilde g$.

Notice that there is an overlap between the regions where $\tilde g$ is a multiple of g and where there is a positive lower bound on the injectivity radius for $\tilde g$. This will be important in the proof of Theorem 1.2 below.

Proof of Lemma 3.7. Let (ω, g, J) be the compatible triple on $M \times \mathbb{R}$ as defined in equation (3.1). We let $\psi\colon (-\varepsilon, \varepsilon) \to (a - \varepsilon, b + \varepsilon)$ be a smooth diffeomorphism such that

$$\psi(t) = \begin{cases} t + a & \text{for } t \in (-\varepsilon, -\varepsilon/3), \\ t + b & \text{for } t \in (\varepsilon/3, \varepsilon). \end{cases} \tag{3.2}$$

This induces a diffeomorphism $\Psi\colon M \times (-\varepsilon, \varepsilon) \to M \times (a - \varepsilon, b + \varepsilon)$, and we claim that

$$(\tilde\omega, \tilde g, \tilde J) := (\Psi^*\omega, \Psi^*g, \Psi^*J)|_{M\times(-\varepsilon,\varepsilon)}$$

satisfies the desired properties.

Using the explicit form of ψ near the boundary, Conditions (ii) and (iii) are easily checked.

For condition (iv), note that $M \times (a - \varepsilon, b + \varepsilon)$ can be covered by c-translates of $M \times (-\varepsilon, \varepsilon)$ for $c \ge a$. Moreover, it follows from equation (3.1) that $\tau_c^* g = e^c g$, where $\tau_c(t) = t + c$ denotes translation by c. Using Lemma 3.6, this implies that

$$\sec(g, M \times (a - \varepsilon, b + \varepsilon)) \le e^{-a} \max_{p\in M\times(-\varepsilon,\varepsilon)} \sec_p(g, M \times (-\varepsilon, \varepsilon)).$$

On the other hand, since $\tilde{g} = \Psi^* g$, the curvature of $\tilde{g}$ on $M \times (-\varepsilon, \varepsilon)$ is the same as that of g on $M \times (a - \varepsilon, b + \varepsilon)$. Together this implies the first inequality of condition (iv).

Next we show that the injectivity radius is bounded. For each point $p \in M \times (-\varepsilon, \varepsilon)$, let $\rho(p)$ denote the maximal radius of a g-geodesic ball centered at p and entirely contained in $M \times (a - \varepsilon, b + \varepsilon)$. On $M \times (a - \varepsilon, a - \varepsilon/3)$, we have $\tilde{g} = e^a g$, so that

$$\min_{p \in M \times \{a - 2\varepsilon/3\}} \rho(p) \geq e^{a/2} \min(\varepsilon/3, \operatorname{inj}(g|_{M \times \{0\}})).$$

Now, according to Theorem 3.6 and the fact that $\tau_c^* g = e^c g$ as pointed out above, ρ is at least $\min_{p \in M \times \{a - 2\varepsilon/3\}} \rho(p)$ on all of $M \times [a - 2\varepsilon/3, b + 2\varepsilon/3]$; this proves the desired bound on the injectivity radius.

It remains to construct the homotopy of symplectic forms ω_s, $s \in [0, 1]$. Define

$$\psi_s : (-\varepsilon, \varepsilon) \to \mathbb{R}, \quad t \mapsto (1 - s)t + s\psi(t),$$

with ψ as in equation (3.2). As before, this induces a diffeomorphism

$$\Psi_s : M \times (-\varepsilon, \varepsilon) \to M \times (sa - \varepsilon, sb + \varepsilon),$$

and it is easily verified that $\omega_s := \Psi_s^* \omega$ is the desired homotopy. ∎

Proof of Theorem 1.2. By assumption, the boundaries ∂K_n, $n \in \mathbb{N}$, are hypersurfaces of contact type. Hence we can fix collar neighborhoods

$$(\partial K_n \times (-\varepsilon_n, \varepsilon_n), \omega = \mathrm{d}(e^t \alpha_n)). \tag{3.3}$$

On these neighborhoods, equation (3.1) defines a compatible triple $(\omega_n = \mathrm{d}(e^t \alpha_n), g_n, J_n)$, and we extend this to a compatible triple (ω, g, J) on M.

Using Lemma 3.6, we choose a strictly increasing sequence of constants $\{k_n\}_{n \in \mathbb{N}}$ such that

$$\sec(e^{k_n} g) < 1, \quad \operatorname{inj}(e^{k_n} g) > 1 \quad \text{on } K_n \setminus \operatorname{int}(K_{n-1}) \tag{3.4}$$

and

$$e^{k_n} \min(\varepsilon_n/2, \operatorname{inj}(g|_{\partial K_n})) > 1.$$

For each of the neighborhoods in equation (3.3), we apply Lemma 3.7 with $a = k_n$ and $b = k_{n+1}$.

The resulting compatible triple (ω', g', J') has bounded injectivity radius, since the regions where the injectivity radius are bounded by Lemma 3.7 and equation (3.3), respectively, have non-trivial overlap. The other conditions of Definition 3.3 are easily verified.

Lastly, note that Theorem 3.7 also provides a homotopy of symplectic forms ω_s from ω to ω' on each of the neighborhoods $\partial K_n \times (-\varepsilon, \varepsilon)$. Moreover, the homotopy ω_s is a linear interpolation between ω and ω' on the boundaries of the given neighborhoods $\partial K_n \times (-\varepsilon_n, \varepsilon_n)$. Hence, we can extend ω_s to the whole M by setting $\omega_s = e^{sk_n} \omega$ on $K_n \setminus K_{n-1}$, showing that ω is homotopic to ω' through symplectic forms. ∎

4. Manifolds not diffeomorphic to symplectic leaves

In this section, we prove Theorem 1.3 and Theorem 1.5. The core of the proof consists of a volume argument inspired by [3]. However, the implementation is slightly different for proper and non-proper leaves. Therefore, we separate the argument into two propositions.

Proposition 4.1. *Let W^{2n} be an open manifold with a finite number of ends. Suppose that at least one of the ends has a neighborhood of the form $N \times [0,\infty)$, where N has trivial π_1 and H^2. Then W is not diffeomorphic to a proper leaf of a symplectic foliation.*

Proposition 4.2. *Let W^{2n} be an open manifold with a finite number k of ends. Suppose that, for $i = 1,\dots,k$, the i-th end has a neighborhood of the form $N_i \times [0,\infty)$, where N_i has trivial π_1 and H^2. If there is only one end,* additionally *assume that $W \setminus N_1 \times (0,\infty)$ has non-trivial π_1 or non-trivial H^2. Then W is not diffeomorphic to a non-proper leaf of a symplectic foliation.*

Proof of Theorem 1.3. It is enough to combine Theorems 4.1 and 4.2 above. ■

Proof of Proposition 4.1. Suppose by contradiction that W is diffeomorphic to a proper leaf L of a symplectic foliation $(M,\mathcal{F},\omega)$. Let e be the end of L having a neighborhood of the form $\mathcal{N} := N \times [0,\infty)$, as in the assumptions of Theorem 4.2. According to Theorem 2.3 and Theorem 2.1, e spirals onto a closed leaf L_∞. Let us assume, for the moment, that the associated projection $\pi\colon \mathcal{N} \to L_\infty$ (as in Definition 2.2) maps $N \times \{0\}$ to an *embedded* submanifold $N_\infty \subset L_\infty$; we will explain how to deal with the general case at the end of the proof.

Since $H^1(N) = 0$, Reeb stability allows us to find a foliated subset U, which we identify with $N \times [0,1]^2$, such that N_∞ corresponds to $N \times \{(0,0)\}$,

$$\mathcal{F}|_U = \bigcup_{z\in[0,1]} N \times [0,1] \times \{z\}, \quad L_\infty \cap U = N \times [0,1] \times \{0\}, \quad \text{and}$$
$$\mathcal{N} \cap U = \bigcup_{k\in\mathbb{N}} N \times [0,1] \times \{1/k\}.$$

Moreover, there exist differential forms $\alpha \in \Omega^1(\mathcal{F}|_U)$ and $\beta \in \Omega^1(\mathcal{N})$ satisfying

$$\omega|_U = \mathrm{d}\alpha, \quad \omega|_{\mathcal{N}} = \mathrm{d}\beta, \quad \alpha|_{\mathcal{N}\cap U} = \beta|_{\mathcal{N}\cap U}. \tag{4.1}$$

Indeed, $H^2(N) = 0$ implies that $H^2(\mathcal{F}|_U) = 0$, so that we find a leafwise primitive $\alpha \in \Omega^1(\mathcal{F}|_U)$ for $\omega|_U$. Similarly, $H^2(\mathcal{N}) = 0$, and we find $\tilde\beta \in \Omega^1(\mathcal{N})$ such that $\omega|_{\mathcal{N}} = \mathrm{d}\tilde\beta$. On the intersection $U \cap \mathcal{N}$, the difference of the two primitives is a closed form which is exact since $H^1(U \cap \mathcal{N}) = 0$. That is, there exists a function $\tilde f \in C^\infty(U \cap \mathcal{N})$ so that

$$(\alpha - \tilde\beta)|_{U\cap\mathcal{N}} = \mathrm{d}\tilde f.$$

Since $U \cap L$ is closed in L, we can extend $\tilde f$ to a function f on L. Taking

$$\beta := \tilde\beta + \mathrm{d}f,$$

we obtain the desired differential forms satisfying equation (4.1).

Choose now the sequence of submanifolds

$$N_m = N \times \{1/2\} \times \{1/m\} \subset U, \quad \text{for } m > 0;$$

notice that, by the definition of U, $N_m \subset \mathcal{N}$ as well. Notice that, as we are dealing with the case where N_∞ is embedded in L_∞, N_∞ can be seen as the juncture of the spiraling of the end e around L_∞. In particular, up to modifying the projection map π of the spiraling of e onto L_∞ in such a way that it coincides with the projection onto the first two factors $N \times [0,1]$ on $U = N \times [0,1]^2$ (notice that this can be achieved up to shrinking U and up to passing to a subsequence of the N_m's; cf. Theorem 2.2), we can assume that all the lifts N_m project down via π onto the juncture N_∞. In particular, all the N_m's are separating the end e. Denote then by K_m the compact subset of $\mathcal{N} = N \times [0,\infty)$ bounded by N_m and N_1. Observe that the closure P of the unbounded subset $N \times [0,\infty) \setminus N_1$ inside the leaf is given by the union $\bigcup_{m\geq 1} K_m$, which we think of as "limit" of K_m for $m \to \infty$.

As U is compact and ω is continuous, the ω-volume of each of the plaques $N \times [0,1] \times \{z\}$ of U is bounded from below by a strictly positive constant. Since P intersects U in infinitely many of these plaques and $K_m \subset K_{m+1}$, we have

$$\lim_{m\to\infty} \int_{K_m} \omega^n = \int_P \omega^n = \infty.$$

Using Stokes' theorem and the fact that $\alpha = \beta$ on $\mathcal{N} \cap U$, we then get

$$\lim_{m\to\infty} \int_{N_m} \alpha \wedge \omega^{n-1} = \lim_{m\to\infty} \int_{N_m} \beta \wedge \mathrm{d}\beta^{n-1} = \infty.$$

On the other hand, because of the continuity of α and ω on U, and the choice of $N_m = N \times \{1/2\} \times \{1/m\} \subset U$, we must have

$$\lim_{m\to\infty} \int_{N_m} \alpha \wedge \omega^{n-1} = \lim_{m\to\infty} \int_{N\times\{1/2\}\times\{1/m\}} \alpha \wedge \omega^{n-1} = \int_{N_\infty} \alpha \wedge \omega^{n-1} \in \mathbb{R}.$$

We have thus reached a contradiction, proving that W cannot be diffeomorphic to a proper symplectic leaf as desired.

It remains to explain how to deal with the case where the projection of $N \times \{0\}$ under π is *not* embedded in L_∞. In general, $N \times \{0\}$ is contained in the union $B_i \cup \cdots \cup B_{i+k}$ for some $i \in \mathbb{N}$ and $k > 1$ (the projection is embedded precisely when $k = 1$), see Theorem 2.2 for notation. Let $p\colon L_{\infty,k} \to L_\infty$ be the k-cover obtained by cutting L_∞ along a juncture, gluing k copies of this cut manifold one after the other, and gluing the two remaining free boundary components to create a closed manifold $L_{\infty,k}$. Next, extend p to an immersion

$$\tilde{p} : L_{\infty,k} \times (-\varepsilon,\varepsilon) \to \mathcal{O}p(L_\infty),$$

for some $\varepsilon > 0$, where $\mathcal{O}p(L_\infty)$ is an open neighborhood of L_∞, and consider the pullback (symplectic) foliation.

Each connected component of the preimage of the neighborhood $\mathcal{N}$ (which we may assume to be contained in the image of $\tilde{p}$ up to taking a smaller neighborhood of e of the same form $N \times [0,\infty)$) is diffeomorphic to $N \times [0,\infty)$, and spirals onto $L_{\infty,k}$. However, since the induced projection is injective on $B_i \cup \dots \cup B_{i+k}$, now $N \times \{0\}$ does project to an embedded submanifold of $L_{\infty,k}$. Then, the argument previously described shows that $N \times [0,\infty)$ cannot be (part of) a leaf in the pullback foliation on $L_{\infty,k} \times (-\varepsilon, \varepsilon)$, thus giving the desired contradiction. ∎

Proof of Proposition 4.2. As L is non-proper, there is an end e of L such that any distinguished neighborhood $\mathcal{N} := N \times [0,\infty)$ of e accumulates onto itself. Indeed, the limit set of a leaf is the union of the limit sets of its ends, and each of the latter is a saturated set; cf., for instance, Lemmas 4.3.5 and 4.3.7 in [4].

Then, using Reeb stability and $\pi_1(N) = 0$, we find a foliated subset $U := N \times [0,1]^2$ of M such that

$$\begin{aligned} &\mathcal{F}|_U = \bigcup_{z \in [0,1]} N \times [0,1] \times \{z\}, \\ &\mathcal{N} \cap U \supset N \times [0,1] \times \{0\} \cup \bigcup_{m \in \mathbb{N}} N \times [0,1] \times \{s_m\}, \end{aligned} \tag{4.2}$$

for a strictly decreasing sequence $\{s_m\}_{m \in \mathbb{N}}$. The submanifolds $N \times \{1/2\} \times \{s_m\}$ give a sequence of submanifolds $N_m \subset \mathcal{N}$ converging (in the ambient manifold) to $N_\infty := N \times \{1/2\} \times \{0\} \subset U$. In particular, this implies that N_∞ is non-zero in the homology of L.

Now, the same argument as in the proof of Theorem 4.1 shows that there exist differential forms $\alpha \in \Omega^1(\mathcal{F}|_U)$ and $\beta \in \Omega^1(\mathcal{N})$ satisfying

$$\omega|_U = \mathrm{d}\alpha, \quad \omega|_{\mathcal{N}} = \mathrm{d}\beta, \quad \alpha|_{\mathcal{N} \cap U} = \beta.$$

As α and ω are continuous over N_∞, we have that

$$\lim_{m \to \infty} \int_{N_m} \alpha \wedge \omega^{n-1} = \int_{N_\infty} \alpha \wedge \omega^{n-1} \in \mathbb{R}. \tag{4.3}$$

We now conclude the proof under the following assumption. The proof of the claim is given at the end of the proof.

Claim 4.3. Up to passing to a subsequence, the N_m all represent the same (non-trivial) homology class as N_∞ in $\mathcal{N}$.

The claim implies that there exist compact subsets $K_m \subset \mathcal{N}$ such that

$$\partial K_m = N_m - N_0.$$

The plaques in equation (4.2) have ω-volume bounded from below by a strictly positive constant. Therefore, the ω-volume of the K_m goes to infinity. Then, Stokes' theorem implies

$$\lim_{m \to \infty} \int_{N_m} \beta \wedge \omega^{n-1} - \int_{N_0} \beta \wedge \omega^{n-1} = \lim_{m \to \infty} \int_{K_m} \omega^n = \infty,$$

which contradicts equation (4.3) as $\alpha = \beta$ over $U \cap \mathcal{N}$, thus concluding the proof. ∎

Proof of Claim 4.3. As each N_m is connected and $H^{2n-1}(\mathcal{N}) = \mathbb{Z}$ generated by N_∞, it follows (see for instance [18]) that $[N_m] \in H^{2n-1}(\mathcal{N};\mathbb{Z})$ is equal to either $[N_\infty]$ or 0.

So, assume by contradiction that $[N_{m_0}] = 0$ in $H^{2n-1}(\mathcal{N};\mathbb{Z})$ for some $m_0 \in \mathbb{N}$; denote by C_{m_0} the compact region (contained inside $\mathcal{N}$) which is bounded by N_{m_0}. We may assume that N_{m_0} bounds C_{m_0} "from the left", by which we mean

$$C_{m_0} \cap U = N \times [1/2, 1] \times \{s_{m_0}\},$$

i.e., C_n sits on the right of N_n inside U.

We consider then the set

$$\mathcal{S} := \left\{ s \in [0,1] \;\middle|\; \begin{array}{l} N \times \{1/2\} \times \{s\} \subset U \text{ bounds from the left a submanifold} \\ \text{in a leaf, diffeomorphic to } C_{m_0}, \end{array} \right\}.$$

Notice that $\mathcal{S}$ is open and does not contain 0. Indeed, since N_{m_0} and $\mathcal{N}$ are simply connected and have trivial second cohomology, the same holds for C_{m_0}. Thus, if $s \in \mathcal{S}$, by Reeb stability there exists an open (foliated) product neighborhood around the C_s bounded by $N_s = N \times \{1/2\} \times \{s\}$ and diffeomorphic to C_{m_0}; hence, C_s can be pushed-off to nearby leaves, proving that $\mathcal{S}$ is open. Furthermore, by the assumptions of Theorem 4.2, N_∞ does not bound any compact, simply connected region without second cohomology, so that $0 \notin \mathcal{S}$.

Let s_* be the infimum of the connected component of $\mathcal{S}$ containing s_{m_0}, and notice that $s_* \notin \mathcal{S}$ and $s_* > 0$.

We now use the following theorem, due to Schweitzer [22].

Theorem 4.4 (Proposition 7.1 in [22]). *Let $(M, \mathcal{F})$ be a foliated manifold and let C be a compact manifold with boundary $\partial C = B$. If $h\colon (C \times (0,1] \cup B \times [0,1]) \to (M, \mathcal{F})$ is a foliated embedding that cannot be extended over $C \times [0,1]$, then the leaf containing $h(B \times \{0\})$ is the boundary of a generalized Reeb component whose interior is the union of the leaves meeting $h(B \times (0,1])$.*

The fact that a foliated embedding as in the hypothesis exists in our setting simply follows from a continuation argument using Reeb stability starting from C_{m_0}. Then, the result readily implies that L is either the boundary of a generalized Reeb component, in the case where N_{s_*} is in L, or in the interior of a generalized Reeb component, in the case where N_{s_*} is in a leaf L_* different from L. In both cases, L must be proper, thus giving the desired contradiction. ■

We now turn our attention to Theorem 1.5 concerning $\mathbb{R}^{2n}$ and its realizability as a symplectic leaf.

Proof of Corollary 1.5. The fact that $\mathbb{R}^{2n}$ is not diffeomorphic to a proper symplectic leaf simply follows from Theorem 4.1 above (recall that we assume $n \geq 2$ here). It is then enough to realize $(\mathbb{R}^{2n}, \omega_{\text{std}})$ as a dense leaf in some symplectic foliation; this can be done as follows.

Start from a foliation of $\mathbb{R}^{2n+1}$ by affine hyperplanes $\mathbb{R}^{2n}$, pairwise parallel to each other, and each directed by a $2n$-dimensional vector subspace of irrational and $\mathbb{Q}$-linearly independent slopes. In other words, the directing vector subspaces are defined by $z =$

$\sum_i (a_i x_i + b_i y_i)$, where we use coordinates $(x_i, y_i, z) \in \mathbb{R}^{2n+1}$ and the $(a_i, b_i) \in \mathbb{R}^{2n}$ are linearly independent over $\mathbb{Q}$. Now, one can pull back, to each of these affine subspaces, the standard symplectic form on $\mathbb{R}^{2n}$ by the natural projection $\mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$; this gives a leafwise symplectic form on this irrational foliation on $\mathbb{R}^{2n+1}$. Now, notice that both the foliation and the leafwise symplectic structure are invariant under the natural action of $\mathbb{Z}^{2n+1}$ by translation on each coordinate (x_i, y_i, z) of $\mathbb{R}^{2n+1}$. What is more, because of the $\mathbb{Q}$-linear independence condition on the coefficients a_i and b_i, the resulting leaves are just symplectomorphic to $(\mathbb{R}^{2n}, \omega_{\mathrm{std}})$, and each of them is dense in $\mathbb{T}^{2n+1}$, as desired. ∎

Remark 4.5. The first claim in Theorem 1.5 is false for $n = 1$. Indeed, the Reeb foliation on S^3 contains proper leaves diffeomorphic to $\mathbb{R}^2$. The second claim however does hold also for $n = 1$, with the same proof as above.

5. Examples of symplectic non-leaves via blowups

The aim of the section is to prove Theorem 1.6 and Theorem 1.7.

Proof of Theorem 1.6. Fix an exhaustion of contact type $\{K_m\}_{m\in\mathbb{N}}$ of (W, ω) (as in Definition 3.5). Let $l_m \geq 1$ be the number of connected components of $W \setminus K_m$. Using Theorem 1.2, we find a symplectic form ω' on W which away from the K_m equals a very large rescaling of ω; these rescaling factors are in particular taken so big that there exists, for every $m \geq 1$ and $1 \leq l \leq l_m$, a ball

$$B_{m,l} \subset K_m \setminus K_{m-1},$$

in the l-th connected component of $W \setminus K_m$, such that $(B_{m,l}, \tilde{\omega}|_{B_{m,l}})$ is a standard symplectic ball of radius m.

We perform a blowup of weight m at $B_{m,l}$ for each $m \geq 1$ and $1 \leq l \leq l_m$, as defined in Theorem 7.1.21 of [17]. Notice that, by the explicit formulas for the blown-up symplectic form, the complex projective $(n-1)$-space $C_{m,l}$ resulting, as a divisor, from the blowup at the ball $B_{m,l}$ with weight m has the following properties:

(i) the complex projective $(n-1)$-space $C_{m,l}$ resulting from the blowup at the ball $B_{m,l}$ with weight m has ω'-volume $m^{2(n-1)}\pi^{n-1}$;

(ii) a "transverse self intersection of $C_{m,l}$" (i.e., a transverse intersection of $C_{m,l}$ with a generic small pushoff of itself) is homologous to a complex projective $(n-2)$-space standardly embedded into $C_{m,l} \simeq \mathbb{CP}^{n-1}$. To see this note that the homology class of the self intersection is Poincaré dual to the Euler class of the normal bundle of the divisor. Consequently, the ω'-volume of the self intersection equals $m^{2(n-2)}\pi^{n-2}$.

We now apply Theorem 1.2 again, and thus get a SGB symplectic manifold (W', ω'). This can moreover be done in such a way that the following modifications of the above properties are satisfied: there is a strictly increasing sequence of real positive numbers $c_m > 0$ such that, for every $m \geq 1$ and $1 \leq l \leq l_m$,

(i') $C_{m,l}$ has ω'-volume

$$c_m^{n-1} m^{2(n-1)} \pi^{n-1}; \tag{5.1}$$

(ii') the transverse self intersection of $C_{m+1,l}$ has ω'-volume

$$c_m^{n-2} m^{2(n-2)} \pi^{n-2}.$$

We claim that this implies that (W', ω') is not symplectomorphic to a leaf L of a symplectic foliation. The argument for this claim splits into two cases, depending on whether L is a proper or a non-proper leaf:

Proper leaf case.

Firstly assume by contradiction that (W', ω') is symplectomorphic to a *proper* symplectic leaf. Then, according to Theorem 2.3 and Theorem 2.5, the ends of W' are symplectically almost periodic. As such, there exists a fundamental system of neighborhoods $\{h^k(U)\}_{k \in \mathbb{N}}$ as in Definition 2.4. In particular, if C is one of the blown-up complex projective spaces entirely contained in U, then for any $\varepsilon > 0$ there is an $N > 0$ such that the ω'-symplectic volume of all the $h^n(C)$, for $n > N$, differ by at most ε. Now, the assumption that there is a compact $K \subset W$ such that $H_{2n-2}(W \setminus K; \mathbb{Z}) = \{0\}$ guarantees that the integer homology H_{2n-2} of W' minus a sufficiently big compact K' is generated by the classes of the complex projective spaces coming from the blowups. Hence, the ω'-volumes of the $h^n(C)$'s must differ, for n big enough, by at least 1 due to the choice of the blowup weights, so we arrive at a contradiction.

Non-proper leaf case.

Secondly, assume by contradiction that (W', ω') is symplectomorphic to a *non-proper leaf* L of a symplectic foliation. We fix one of the $C_{m,l} \cong \mathbb{CP}^{n-1}$ resulting from the blowups, and denote it by X_∞; as this will later serve as "limit submanifold", we denote its indices by m_∞ and l_∞, i.e., we have

$$X_\infty = C_{m_\infty, l_\infty}.$$

We also let $Y_\infty \subset X_\infty$ denote the submanifold $\mathbb{CP}^{n-2} \subset \mathbb{CP}^{n-1} \cong X_\infty$, whose ω'-volume equals $c_\infty^{n-2} m_\infty^{2(n-2)} \pi^{n-2}$ according to the volume formula in (5.1). As such, we have that

$$[X_\infty] \cap [X_\infty] = [Y_\infty] \in H_{2n-2}(W', \mathbb{Z}), \tag{5.2}$$

where the left-hand side denotes the (homological) self-intersection of X_∞.

Let $V \subset L$ be a small tubular neighborhood of X_∞. In particular, V is simply connected, since X_∞ is. By Reeb stability, there exists a foliation chart $V \times (-1, 1)$ of the ambient manifold in which V corresponds to $V \times \{0\}$. As L is non-proper and has finitely many ends, one of its ends e intersects $V \times (-1, 1)$ in infinitely many plaques $V \times \{t_j\}$, with $t_j \to 0$ for $j \to \infty$ (by the argument given in the proof of Theorem 4.2).

This yields infinite sequences

$$X_j := X_\infty \times \{t_j\}, \quad Y_j := Y_\infty \times \{t_j\}, \quad j \in \mathbb{N},$$

of submanifolds of W'. Note that the normal bundle of $Y_j \subset X_j$ canonically identified with the normal bundle of $Y_\infty \subset X_\infty$, so that (5.2) implies

$$[X_j] \cap [X_j] = [Y_j] \in H_{2n-2}(W', \mathbb{Z}). \tag{5.3}$$

Note that, for $j \to \infty$, the X_j (respectively, Y_j) escape to infinity in the end e of $L \cong W'$, while converging to X_∞ (respectively, Y_∞) inside the ambient manifold.

On the one hand, this means that

$$\int_{[X_j]\cap[X_j]} (\omega')^{n-2} = \int_{[Y_j]} (\omega')^{n-2} \xrightarrow{j\to\infty} \int_{[Y_\infty]} (\omega')^{n-2} = c_\infty^{n-2} m_\infty^{2(n-2)} \pi^{n-2}. \tag{5.4}$$

On the other hand, we know that there exists a compact $K' \subset W'$ such that $H_{2n-2}(W' \setminus K'; \mathbb{Z})$ is a free abelian group generated by the $[C_{m,l}]$. By choosing an even bigger compact K'', we may assume that the ω'-volume of all $C_{m,l}$ in $W \setminus K''$ is much bigger than that of X_∞. Since for j sufficiently large we have $X_j \subset W \setminus K''$, we deduce that there are an integer $k \geq 1$ and $a_{1,j}, \ldots, a_{k,j} \in \mathbb{Z}$ such that

$$[X_j] = \sum_{i=1}^{k} a_{ij} [C_{m_i,l_i}] \in H_{2n-2}(W' \setminus K''; \mathbb{Z}).$$

Moreover, because the ω'-volume of all the $C_{m,l}$'s in $W \setminus K''$ is much bigger than that of X_∞ as previously pointed out, we also have

$$c_{m_i}^{n-1} m_i^{2(n-1)} \pi^{n-1} \gg c_\infty^{n-1} m_\infty^{2(n-1)} \pi^{n-1} \quad \text{for all } i.$$

Computing its self-intersection, we obtain

$$[X_j] \cap [X_j] = \sum_{i=1}^{k} a_{ij}^2 [C_{m_i,l_i}] \cap [C_{m_i,l_i}],$$

since the $C_{m,l}$ are pairwise disjoint. Moreover, the intersection $[C_{m_i,l_i}] \cap [C_{m_i,l_i}]$ can be represented by the submanifold

$$\mathbb{CP}^{n-2} \subset \mathbb{CP}^{n-1} = C_{m_i,l_i},$$

whose ω'-symplectic volume equals $c_{m_i}^{n-2} m_i^{2(n-2)} \pi^{n-2}$. We conclude that

$$\int_{[X_j]\cap[X_j]} (\omega')^{n-2} = \sum_{i=1}^{k} a_i^2 c_{m_i}^{n-2} m_i^{2(n-2)} \pi^{n-2} \gg c_\infty^{n-2} m_\infty^{2(n-2)} \pi^{n-2}.$$

This contradicts the conclusion in equation (5.4), so that (W', ω') cannot be a non-proper leaf of a symplectic foliation. ∎

Proof of Corollary 1.7. Let W be the smooth manifold obtained by complex blowup of $\mathbb{C}^n$ at infinitely many points, as in the statement. The fact that W admits a symplectic form ω for which (W, ω) is not symplectomorphic to a symplectic leaf follows from Theorem 1.6. We now want to realize as a symplectic leaf a symplectic blowup (W, ω') of ω_{std} on $\mathbb{C}^n$ for which the resulting complex spaces C_j's are all of the same ω'-volume.

For this, we start from the example of symplectic foliation $(\mathcal{F}, \omega)$ on $\mathbb{T}^{2n+1}$ by dense leaves, all symplectomorphic to $(\mathbb{R}^{2n}, \omega_{\text{std}})$, as in the proof of Corollary 1.5 in Section 4.

Recall that such foliation is transverse to the second $\mathbb{S}^1$ factor of $\mathbb{T}^{2n+1} = \mathbb{T}^{2n} \times \mathbb{S}^1$, and that the projection onto the first factor is a local symplectomorphism onto $(\mathbb{T}^{2n}, \omega_T)$, where ω_T is the symplectic structure on $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ induced from $(\mathbb{R}^{2n}, \omega_{\mathrm{std}})$.

Now, consider any transverse curve γ, for instance $\gamma(z) = (x_i^0, y_i^0, z)$ for any choice of $(x_i^0, y_i^0) \in \mathbb{T}^{2n}$. As the restriction of the projection onto the first factor $\mathbb{T}^{2n+1} = \mathbb{T}^{2n} \times \mathbb{S}^1 \to \mathbb{T}^{2n}$ to each leaf is a local symplectomorphism, γ admits, for $\delta > 0$ sufficiently small, a symplectically foliated neighborhood of the form $(B_\delta^{2n} \times \mathbb{S}^1, \omega_{\mathrm{std}}^B)$, where B_δ^{2n} is the ball of radius δ in $\mathbb{R}^{2n}$, $\mathcal{F}$ is of the form $B_\delta \times \{\theta\}$ with $\theta \in \mathbb{S}^1$, and the leafwise symplectic form ω_{std}^B is just given by the restriction of ω_{std} on $\mathbb{R}^{2n}$ to B_δ^{2n}. This allows to perform an $\mathbb{S}^1$-equivariant symplectic blowup construction in this local model $(B_\delta^{2n} \times \mathbb{S}^1, \omega_{\mathrm{std}}^B)$ (as in Theorem 7.1.21 of [17]), in such a way that the origin of the B_δ^{2n} factor is replaced by a complex projective $(n-1)$-space of a certain symplectic volume ε (the same for every $\theta \in \mathbb{S}^1$). We denote the result of this blowup by $(X \times \mathbb{S}^1, \omega_X)$. This glues well to $(\mathbb{T}^{2n+1}, \mathcal{F}, \omega) \setminus (B_\delta^{2n}, \omega_{\mathrm{std}}^B)$ in order to give a symplectically foliated manifold $(M, \mathcal{G}, \Omega)$. Notice that, as the leaves $\mathbb{R}^{2n}$ of $(\mathbb{T}^{2n+1}, \mathcal{F})$ are all dense and γ intersects each of them in infinitely many points, the leaves of $\mathcal{G}$ are also dense and are just smoothly obtained by (complex-)blowing up $\mathbb{R}^{2n}$ at infinitely many points. Lastly, the restriction of the symplectic form to each leaf is a symplectic form ω on W such that all the C_j's obtained from blowing up each point have same ω-volume ε. This concludes the proof. ∎

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