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Short note

On integral inequalities with applications to the logarithmic mean

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Abstract. This short note aims to introduce and derive a sequence of integral inequalities based on the well-established Radon inequality. In particular, it includes a generalization of the Chebyshev and Dunkel integral inequalities. As a special case, these inequalities readily yield the well-known Arithmetic-Logarithmic-Geometric Mean Inequality.

The purpose of this short note is to present and derive a sequence of integral inequalities based on the well-known Radon inequality [1, Chapter 18, Rademacher-Rotation, p. 259]. By applying some simple substitutions in Radon's inequality, we obtain the following main result:

$$\left[\frac{b-a}{\int_a^b \left(\frac{1}{f(x)}\right)^{\frac{1}{p-1}}\,dx}\right]^{p-1} \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \left[\frac{\int_a^b f^p(x)\,dx}{b-a}\right]^{\frac{1}{p}}, \quad x \in [a,b], \tag{1}$$

where $f\colon [a,b] \to (0,\infty)$ is a continuous function and $p > 1$ or $p < 0$ and $a < b$.

Observe that, in our inequality, the first inequality serves as a generalization of the Chebyshev integral inequality [1], while the second represents the Dunkel integral inequality [3], which is obtained here through a simple substitution in Radon's inequality.

From these inequalities, we can easily deduce the classic Arithmetic-Logarithmic-Geometric Mean Inequality, which is formulated as

$$\sqrt{ab} \le \frac{b-a}{\ln b - \ln a} \le \frac{a+b}{2}. \tag{2}$$

Furthermore, we can also deduce a refinement of (2), as noted in [2], which is presented as follows:

$$(ab)^{\frac{1}{4}}\frac{\sqrt{a}+\sqrt{b}}{2} \le \frac{b-a}{\ln b - \ln a} \le \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2. \tag{3}$$

The proof of (1) is based on Radon's inequality. Since Radon's inequality is not widely known, we will first state and prove it before proceeding further.

Proposition 1 (Radon's inequality). *If F and G are positive and continuous functions on $[a,b]$, then the following holds:*

$$\int_a^b \frac{F^p(x)}{G^{p-1}(x)}\,dx \geq \frac{\left(\int_a^b F(x)\,dx\right)^p}{\left(\int_a^b G(x)\,dx\right)^{p-1}}, \quad p>1 \text{ or } p<0, \tag{4}$$

with equality occurring if and only if F and G are proportional. Conversely, for $0<p<1$, this inequality is reversed.

Proof. For $p>1$, observe that

$$\left[\left(\int_a^b G(x)\,dx\right)^{\frac{p-1}{p}}\left(\int_a^b \frac{F^p(x)}{G^{p-1}(x)}\,dx\right)^{\frac{1}{p}} \geq \int_a^b F(x)\,dx\right]$$
$$\Longleftrightarrow \left[\int_a^b \frac{F^p(x)}{G^{p-1}(x)}\,dx \geq \frac{(\int_a^b F(x)\,dx)^p}{(\int_a^b G(x)\,dx)^{p-1}}\right].$$

Now, applying Hölder's inequality to $\int_a^b F(x)\,dx$, we obtain

$$\int_a^b F(x)\,dx = \int_a^b G^{\frac{1}{s}}(x)\frac{F(x)}{G^{\frac{1}{s}}(x)}\,dx \leq \left(\int_a^b G(x)\,dx\right)^{\frac{1}{s}}\left(\int_a^b \left(\frac{F(x)}{G^{\frac{1}{s}}(x)}\right)^t dx\right)^{\frac{1}{t}},$$

where $\frac{1}{s}+\frac{1}{t}=1$. Just set $s=\frac{p}{p-1}$ and $t=p$ into this inequality to obtain (4). A parallel reasoning applies for $p<0$. Specifically, by setting $q=1-p>1$, we observe that

$$\left[\int_a^b G(x)\,dx \leq \left(\int_a^b F(x)\,dx\right)^{\frac{q-1}{q}}\left(\int_a^b \frac{G^q(x)}{F^{q-1}(x)}\,dx\right)^{\frac{1}{q}}\right]$$
$$\Longleftrightarrow \left[\int_a^b \frac{G^q(x)}{F^{q-1}(x)}\,dx \geq \frac{\left(\int_a^b G(x)\,dx\right)^q}{\left(\int_a^b F(x)\,dx\right)^{q-1}}\right].$$

Thus, we obtain inequality (4) by applying Hölder's inequality to the integral

$$\int_a^b G(x)\,dx = \int_a^b F^{\frac{1}{s}}(x)\frac{G(x)}{F^{\frac{1}{s}}(x)}\,dx$$

by choosing $\frac{1}{s}+\frac{1}{t}=1$, where $s=\frac{q}{q-1}$ and $t=q$. The same reasoning is applicable in the opposite direction when $0<p<1$. ∎

In the next proposition, we will state and prove our main result with the help of Radon's inequality.

Proposition 2. *Let $b>a$, $p>1$ or $p<0$ and let $f\colon[a,b]\to(0,\infty)$ be a continuous function on $[a,b]$. Then*

$$\left[\frac{b-a}{\int_a^b \left(\frac{1}{f(x)}\right)^{\frac{1}{p-1}}dx}\right]^{p-1} \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \left[\frac{\int_a^b f^p(x)\,dx}{b-a}\right]^{\frac{1}{p}}, \quad x\in[a,b].$$

Proof. By substituting $F(x) = f$ and $G(x) = 1$ into Radon's inequality (4), we obtain

$$\int_a^b f^p(x)\,dx \geq \frac{\left(\int_a^b f(x)\,dx\right)^p}{(b-a)^{p-1}},$$

which corresponds to the right-hand side of the desired inequality. Additionally, if we substitute

$$F(x) = 1 \quad \text{and} \quad G(x) = \frac{1}{f^{\frac{1}{p-1}}(x)}$$

into (4), we arrive at

$$\int_a^b f(x)\,dx \geq \frac{(b-a)^p}{\left[\int_a^b \left(\frac{1}{f(x)}\right)^{\frac{1}{p-1}}\,dx\right]^{p-1}},$$

which provides the right-hand side of the desired inequality. ■

Remark 1. In particular, the special case of Chebyshev integral inequality is given by

$$\int_a^b [f(x)]^n\,dx \geq \frac{1}{(b-a)^{n-1}}\left[\int_a^b f(x)\,dx\right]^n, \tag{5}$$

where f is a nonnegative integrable function and monotonic on $[a,b]$ and n is a positive integer.

Thus, the first inequality in (1) can be considered an extension of the Chebyshev integral inequality for the class of positive continuous functions to any real exponent $p > 1$ or $p < 0$.

Also, the second inequality in (1) is just the Dunkel integral inequality [3]:

$$\int_a^b [f(x)]^{p-1}\,dx\left[\int_a^b \frac{1}{f(x)}\,dx\right]^{p-1} \geq (b-a)^p, \quad p > 1 \text{ or } p < 0. \tag{6}$$

As a direct consequence of Proposition 2, we obtain the Arithmetic-Logarithmic-Geometric Mean Inequality.

Corollary 1. *For positive real numbers $b > a$, we have*

$$G(a,b) \leq L(a,b) \leq A(a,b),$$

where $L(a,b) = \frac{b-a}{\ln b - \ln a}$, $A(a,b) = \frac{a+b}{2}$ and $G(a,b) = \sqrt{ab}$.

Proof. By substituting $f(x) = \frac{1}{x}$ and choosing $p = 2$ in Proposition 2, we derive the following expressions:

$$\frac{b-a}{\int_a^b \frac{1}{f(x)}\,dx} = \frac{2}{a+b}, \quad \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{\ln b - \ln a}{b-a},$$

$$\sqrt{\frac{1}{b-a}\int_a^b f^2(x)\,dx} = \frac{1}{\sqrt{ab}}.$$

Thus, we recover the well-known result $G(a,b) \leq L(a,b) \leq A(a,b)$. ■

Additionally, we have the following corollary.

Corollary 2. *For positive real numbers $b > a$, we have (3), i.e.,*

$$(ab)^{\frac{1}{4}} \frac{\sqrt{a}+\sqrt{b}}{2} \leq \frac{b-a}{\ln b - \ln a} \leq \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2.$$

Proof. By substituting $f(x) = \frac{1}{x}$ with $p = \frac{1}{2}$ into the reverse of the right-hand side of Proposition 2, we derive the first inequality of (3):

$$\frac{b-a}{\ln b - \ln a} \leq \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2.$$

Next, by substituting $f(x) = \frac{1}{x^2}$ with $p = \frac{1}{2}$ into the same reversed inequality, we obtain the following:

$$\frac{(b-a)^2}{ab} \geq (\ln b - \ln a)^2. \tag{7}$$

We can then replace a by $\sqrt{a}$ and b by $\sqrt{b}$ in this inequality to yield

$$4\frac{(\sqrt{b}-\sqrt{a})^2}{\sqrt{ab}} \geq (\ln b - \ln a)^2. \tag{8}$$

From this, we can derive

$$2\frac{\sqrt{b}-\sqrt{a}}{(ab)^{\frac{1}{4}}} \geq \ln b - \ln a. \tag{9}$$

Thus, we establish the second inequality in (3). ∎

An immediate consequence of this finding is a refinement of the second inequality in (3):

$$(ab)^{\frac{1}{4}} \frac{\sqrt{a}+\sqrt{b}}{2} \leq L(a,b).$$

This inequality can be further enhanced as follows.

Corollary 3. *For positive real numbers $b > a$, we have*

$$\left[\frac{3(b-a)}{b^3-a^3}\right]^{-\frac{1}{2}} \leq L(a,b) = \frac{b-a}{\ln b - \ln a}.$$

Proof. The inequality follows directly by setting $f(x) = \frac{1}{x^{1/p}}$ with $p = -\frac{1}{2}$ into the right-hand side of (1). Note that

$$\left[\frac{3(b-a)}{b^3-a^3}\right]^{-\frac{1}{2}} > (ab)^{\frac{1}{4}} \frac{\sqrt{a}+\sqrt{b}}{2}.$$ ∎

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