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Pathological set with loss of regularity for nonlinear Schrödinger equations

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Abstract. We consider the mass-supercritical, defocusing, nonlinear Schrödinger equation. We prove loss of regularity in arbitrarily short times for regularized initial data belonging to a dense set of any fixed Sobolev space for which the nonlinearity is supercritical. The proof relies on the construction of initial data as a superposition of disjoint bubbles at different scales. We get an approximate solution with a time of existence bounded from below, provided by the compressible Euler equation, which enjoys zero speed of propagation. Introducing suitable renormalized modulated energy functionals, we prove spatially localized estimates which make it possible to obtain the loss of regularity.

1. Introduction

We consider the defocusing nonlinear Schrödinger equation, for $m \in \mathbb{N}$:

$$i\partial_t \psi + \frac{1}{2}\Delta\psi = |\psi|^{2m}\psi, \quad \psi_{|t=0} = f_0, \tag{1.1}$$

where $x \in \mathbb{R}^d$ and $f_0 \in H^s(\mathbb{R}^d)$. We assume that the nonlinearity is L^2-supercritical,

$$s_c = \frac{d}{2} - \frac{1}{m} > 0,$$

and we suppose that $0 < s < s_c$: the nonlinearity is H^s-supercritical. For technical reasons, we also assume $s \le 2$ (which is no further restriction when $d \le 4$). Our goal is to improve [13, Theorem 1.4], recalled below, and prove some loss of regularity in the spirit of [24]. We emphasize the fact that our analysis remains valid for compact geometries, typically for (1.1) on the torus $\mathbb{T}^d$.

1.1. Context

It is known that when $s \ge s_c$, then the Cauchy problem (1.1) is locally well posed in $H^s(\mathbb{R}^d)$ (see [15]), whereas when $s < s_c$, then the Cauchy problem is ill posed, as established initially in [22]. In [16], the notion of norm inflation was introduced, and proven

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in the case of (1.1): there exists a sequence of initial data $(f_0^h)_{h\in(0,1]}$ in $\mathcal{S}(\mathbb{R}^d)$ going to zero in H^s, but such that the corresponding maximal solutions $(\psi^h)_{h\in(0,1]}$ are defined on $[0,t^h]$ for some $t^h \to 0$ and $\psi^h(t^h)$ goes to infinity in H^s:

$$\|f_0^h\|_{H^s} \xrightarrow[h\to 0]{} 0, \quad \|\psi^h(t^h)\|_{H^s} \xrightarrow[h\to 0]{} +\infty,$$

as established initially in [9]. It turns out that this norm inflation mechanism also occurs around any initial data, as proven initially in [30] in the case of the wave equation (see also [32, 33]), and more recently in [34] for a fourth-order Schrödinger equation.

The question of norm inflation becomes more delicate when replacing the sequence $(f_0^h)_h$ of initial data by the sequence $(\iota_h * f_0)_h$, where the approximate identity ι_h is given by

$$\iota_h(x) = \frac{1}{h^d}\iota\Big(\frac{x}{h}\Big),$$

with $\iota \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} \iota(x)\,\mathrm{d}x = 1$. In this case, it has been shown in [13, Theorem 1.4], in the case $d = 3$, that there exists a dense set of functions $f_0 \in H^s(\mathbb{R}^3)$ (called the *pathological set*) such that norm inflation holds for this sequence of regularized initial data:

$$\|\iota_{h_k} * f_0 - f_0\|_{H^s} \xrightarrow[k\to\infty]{} 0, \quad \|\psi^{h_k}(t^{h_k})\|_{H^s} \xrightarrow[k\to\infty]{} +\infty.$$

The construction of such a pathological set was first evidenced in [29] for the wave equation, then extended to Schrödinger equations in [13] by removing a finite speed of propagation argument. The result is also valid on $\mathbb{T}^3$.

On the other hand, ill-posedness for the range of exponents $s < s_c$ was strengthened in [1] as a loss of regularity result (extending the cubic case from [14]; see also [31]). More precisely, there exist a sequence of initial data $(f_0^h)_{h\in(0,1]} \subset \mathcal{S}(\mathbb{R}^d)$, global weak solutions $(\psi^h)_{h\in(0,1]}$, and $t^h \to 0$, such that

$$\|f_0^h\|_{H^s} \xrightarrow[h\to 0]{} 0, \quad \|\psi^h(t^h)\|_{H^\sigma} \xrightarrow[h\to 0]{} +\infty, \quad \forall \sigma > \frac{s}{1+m(s_c-s)}.$$

This result is an analogue of the original loss of regularity theorem from Lebeau [24] concerning energy-supercritical wave equations ($s_c > 1$),

$$(\partial_{tt} - \Delta)u + u^{2m+1} = 0,$$

which is as follows. There exist f_0 in H^s and $t^{h_k} \to 0$ such that the solution ψ to the wave equation satisfies that for every $\sigma > I(s)$,

$$\|\psi(t^{h_k})\|_{H^\sigma} \xrightarrow[k\to +\infty]{} +\infty.$$

The exponent $I(s)$ is given by $I(s) = \frac{s}{1+m(s_c-s)}$ when $s \geq s_{\mathrm{sob}}$,

$$s_{\mathrm{sob}} = \frac{dm}{2m+2} \quad \text{being such that } \dot{H}^{s_{\mathrm{sob}}}(\mathbb{R}^d) \hookrightarrow L^{2m+2}(\mathbb{R}^d),$$

and $I(s) = 1$ when $s \leq s_{\text{sob}}$ (note that $I(s_{\text{sob}}) = 1$). The result from [24] uses in a crucial fashion the property that (weak) solutions to the nonlinear wave equation enjoy finite speed of propagation. It is mostly because of this that the result of [1] concerns a sequence of initial data (one bubble) rather than some fixed data (superposition of disjoint bubbles) like in [24]. In this paper, we prove a result which is essentially the same as in [24], by looking at the sequence of initial data $(\iota_h * f_0)_h$ regularized by convolution. Of course there is no finite speed of propagation for (1.1). Instead, our argument takes advantage of a finite propagation speed property at the level of compressible Euler equations, which naturally appears in the WKB analysis of the semiclassical version of (1.1).

1.2. Main results

Our first result is in the spirit of [32, Theorem 1.33], and states that the main result from [1] is valid not only at the origin, but near any initial datum in $H^s(\mathbb{R}^d)$:

Theorem 1.1 (Loss of regularity near any initial datum). *Let $0 < s < s_c$, with $s \leq 2$. For any $f_0 \in H^s(\mathbb{R}^d)$, there exist a sequence $f_{0,k} \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and times $t_k \to 0$ as $k \to \infty$, such that*

$$\|f_0 - f_{0,k}\|_{H^s} \underset{k\to\infty}{\longrightarrow} 0, \quad \|\psi_k(t_k)\|_{H^\sigma} \underset{k\to\infty}{\longrightarrow} +\infty, \quad \forall \sigma > \frac{s}{1+m(s_c-s)},$$

where for any (fixed) k, $\psi_k \in L^\infty(\mathbb{R}; H^1 \cap L^{2m+2})$ solves (1.1) with initial datum $f_{0,k}$.

As a corollary of this result, in the energy-supercritical case $s_c > 1$, taking $s = s_{\text{sob}}$ yields $f_{0,k}$ converging to f_0 in the energy space $H^1 \cap L^{2m+2}$, while ψ_k is instantaneously unbounded in H^σ for any $\sigma > 1$, since $I(s_{\text{sob}}) = 1$. Like in [29], one may ask how the above phenomenon depends on the approximating sequence $(f_{0,k})_k$, and consider a general approximate identity. Our main result is as follows, and somehow unites the results from [1, 14, 31] on the one hand, and [13] on the other hand:

Theorem 1.2 (Loss of regularity for regularized data). *Let $0 < s < s_c$, with $s \leq 2$. There exists a dense pathological set of initial data f_0 in $H^s(\mathbb{R}^d)$ such that the following holds. For $h > 0$, let ψ^h be the solution to (1.1) with initial data $\iota_h * f_0$. There exist a sequence of parameters $h_k \to 0$ and times $t_k \to 0$ as $k \to \infty$, such that*

$$\|\iota_{h_k} * f_0 - f_0\|_{H^s} \underset{k\to\infty}{\longrightarrow} 0, \quad \|\psi^{h_k}(t_k)\|_{H^\sigma} \underset{k\to\infty}{\longrightarrow} +\infty, \quad \forall \sigma > \frac{s}{1+m(s_c-s)}.$$

Remark 1.3. We will see in the proof that this result is also valid in $\mathbb{T}^d$.

Remark 1.4. Note that, as mentioned in [13, Theorem 1.4 and Proposition 2.10], whenever the Cauchy problem is globally well posed in $H^k(\mathbb{R}^d)$ for some $k > \frac{d}{2}$, then the set of initial data f_0 satisfying Theorem 1.2 contains a dense G_δ set.

The pathological set of initial data, on which norm inflation or loss of regularity happens, is the counterpart of the set of initial data such that probabilistic well-posedness holds, initiated by Bourgain [7, 8] for the cubic Schrödinger equation on $\mathbb{T}^2$, then developed by Burq and Tzvetkov [11,12] for the cubic wave equation on manifolds, and by Burq, Thomann, and Tzvetkov [10] for the nonlinear Schrödinger equation on $\mathbb{R}$. Indeed, as mentioned in [13, Theorem 1.3], the generic well-posedness result from Bényi, Oh, and Pocovnicu [4] on the cubic Schrödinger equation on $\mathbb{R}^3$ implies that for $\frac{1}{4} < s < s_c$, there exist a nondegenerate probability measure μ supported on $H^s(\mathbb{R}^3)$ and a dense set $\Sigma \subset H^s(\mathbb{R}^3)$ with full μ-measure such that the following holds. For every $f_0 \in \Sigma$, the solution ψ^h to (1.1) with initial data $\iota_h * f_0$ is well defined up to some time $T(f_0)$ and converges to some limiting distributional solution ψ to (1.1) with initial data f_0 on $[0, T(f_0)]$:

$$\|\iota_h * f_0 - f_0\|_{H^s} \xrightarrow[h\to 0]{} 0, \quad \|\psi^h - \psi\|_{L^\infty([0,T],H^s(\mathbb{R})^3)} \xrightarrow[h\to 0]{} 0.$$

This result was improved in [5], with the lower bound $s > \frac{1}{5}$, and in [28] with the lower bound $s > \frac{1}{7}$. Consequently, the full-measure set Σ must be disjoint from the pathological set. In particular, if the pathological set contains a dense G_δ set, this cannot be the case for Σ. This conflict between full measure (probability 1 for initial data) and density (of the pathological set) is also striking in view of the results of [23] (see also [17, 28] where the initial Sobolev regularity is lowered), since for the cubic Schrödinger equation on $\mathbb{R}^4$ (which is an energy-critical equation), it is proved that for all initial data in $H^s(\mathbb{R}^4)$ with $\frac{5}{6} < s < 1$, the nonlinear evolution of the randomization of f is almost surely global in time and stable in the sense that it is asymptotically linear (scattering).

Remark 1.5. For $d \geq 5$, it was proven in [27] that for certain values of m at least, concerning energy-supercritical cases ($s_c > 1$), there exists a finite co-dimensional manifold of smooth initial data with spherical symmetry such that the solution blows up in finite time $T_* > 0$, with

$$\|\psi(t)\|_{L^\infty(\mathbb{R}^d)} \xrightarrow[t\to T_*]{} +\infty.$$

A common feature of our analysis with the approach in [27] is the use of hydrodynamical formulations, measuring a strong interaction between the phase and the amplitude for the solution of (1.1), an aspect already at the origin of the results in [14]. In particular, it is proven in [27] that there exists $1 < \sigma < s_c$ such that

$$\|\psi(t)\|_{H^\sigma(\mathbb{R}^d)} \xrightarrow[t\to T_*]{} +\infty.$$

Our main result concerns the whole range $\sigma > \frac{s}{1+m(s_c-s)}$, which, for $s_c > 1$ and $s = s_{\mathrm{sob}}$, corresponds exactly to $\sigma > 1$.

1.3. Scheme of the proof

The general strategy mixes ideas from [1] and [13]. In [1], and like in [9, 16], the general data are of the form

$$f_0^h(x) = h^{s-d/2} a\Big(\frac{x}{h}\Big).$$

Introducing $\varepsilon = h^{m(s_c-s)}$ and the change of unknown function

$$u^\varepsilon(t,x) = h^{d/2-s}\psi^h(h^2\varepsilon t, hx),$$

the family of functions $(u^\varepsilon)_{0<\varepsilon\le 1}$ solves the semiclassical version of (1.1),

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = |u^\varepsilon|^{2m}u^\varepsilon,$$

with initial data $u^\varepsilon_{|t=0} = a$ independent of ε. The solution u^ε becomes instantaneously ε-oscillatory, in the sense that there exists $\tau > 0$ (independent of ε) such that

$$\|u^\varepsilon(\tau)\|_{\dot H^1} \approx \frac{1}{\varepsilon}.$$

In principle, this property can be shown thanks to WKB analysis, as in [14] for the cubic case $m = 1$, and in [31] for analytic data (see [14] for a discussion regarding the differences between the approach of [9, 16], where the Laplacian is neglected for very short time, and [14]). However the proof in [1] is cheaper in the sense that it merely requires the use of modulated energy functionals, without justifying WKB analysis. Using various interpolation estimates and the property $\|u^\varepsilon(\tau)\|_{L^2} = \|a\|_{L^2}$ (independent of ε), one infers

$$\|u^\varepsilon(\tau)\|_{\dot H^\sigma} \approx \frac{1}{\varepsilon^\sigma}, \quad \forall\sigma \ge 0.$$

The result follows when using the scaling to go back to ψ^h, with $t^h = h^2\varepsilon\tau$.

The proof of Theorem 1.1 can be viewed as a consequence of the proof of Theorem 1.2, as we explain in Appendix B, so we now focus on the proof of the latter.

Instead of starting from one concentrating data, we start from a superposition of such bubbles, like in [24] initially, along a sequence $h_k \to 0$ as $k \to +\infty$ (see Section 2). As pointed out above, we also regularize this sum of bubbles, like in [13]. We then adapt the modulated energy analysis along the bubble corresponding to the scale h_k. The more direct approach is interesting under the constraint $s < s_{\text{sob}}$ (Section 4). It is improved in Section 5 by considering a renormalized modulated energy. However, unlike in the case of a single bubble recalled above, this is not enough to conclude directly, as the L^2-norm of the rescaled function u^ε is not uniformly bounded, due to the (initial) bubbles corresponding to $\ell < k$. In the spirit of [24], we pick bubbles with initial pairwise disjoint supports. Using a finite propagation speed for the approximate bubbles involved in the modulated energy functionals, due to Makino, Ukai, and Kawashima [26] (see Section 3), we prove spatially localized estimates in Section 6, and then conclude. In an appendix, we provide an alternative proof of the most delicate result of Section 6, which does not use Fourier analysis, and is thus more flexible for different geometries.

2. Pathological set of initial data

2.1. Definition

From now on, we fix $0 < s < s_c$ such that $s \leq 2$. Following [13,29], we consider pathological initial data as a superposition of bubbles displaying norm inflation at different scales, of the form

$$f_0 = \varphi_0 + \sum_{k=k_0}^{\infty} \varphi_k,$$

for some $k_0 \geq 1$. The background $\varphi_0 \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ automatically satisfies that $\mathbb{R}^d \setminus \operatorname{supp}(\varphi_0)$ is a large set; in particular it contains a ball $\bar{B}(0, r_0)$. In order to extend our result to $\mathbb{T}^d$, one should assume that $\mathbb{T}^d \setminus \operatorname{supp}(\varphi_0)$ contains an open set $B(0, r_0)$. As such, we will choose bubbles φ_k with pairwise disjoint supports inside $\bar{B}(0, r_0)$.

We fix some parameter $M > 1$, and define the scale

$$h_k := e^{-M^k}.$$

Each bubble φ_k is a rescaling of a profile $\alpha_k \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ at scale $h_k > 0$:

$$\varphi_k(x) = \frac{1}{|\log(h_k)|} h_k^{s-\frac{d}{2}} \alpha_k\Big(\frac{x}{h_k}\Big). \tag{2.1}$$

We assume for simplicity that

$$\alpha_k(x) = \alpha\Big(x - \frac{x_k}{h_k}\Big)$$

for some fixed profile $\alpha \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, where the position $x_k \in \mathbb{R}^d$ will be characterized later on.

The radial case can be handled by considering instead, for instance, $\alpha_k(x) = \alpha(|x|^2 - r_k^2/h_k^2)$ with $\alpha \in \mathcal{C}_c^\infty(\mathbb{R}_+)$.

The logarithmic factor aims at guaranteeing the convergence of the above series in H^s, but its presence can be forgotten to grasp the main ideas and details of the computations.

Let ψ_k solve (1.1) with regularized initial data defined from f_0 as

$$\psi_{k|t=0} := \iota_{h_k/100} * f_0.$$

We note that for any fixed k, we can find such a global-in-time solution, $\psi_k \in L^\infty(\mathbb{R}; H^1 \cap L^{2m+2})$, as a weak solution, from [19]. If $s_c \leq 1$, ψ_k is actually the unique, global, mild solution, and it is smooth for all time by propagation of regularity. If $s_c > 1$, as there exists a unique local solution to (1.1) in $H^N(\mathbb{R}^d)$ for $N > d/2$ by standard arguments, at least locally in time, the weak solution is the unique, smooth solution. Note that this means that for any k, ψ_k remains smooth locally in time, on a time interval which may shrink to $\{0\}$ as k goes to infinity, in agreement with the result from [27].

We expect that this regularized solution will display a norm inflation in H^σ around the kth bubble at some time t_k to be defined in the next paragraph.

2.2. Semiclassical form

In order to show a loss of regularity result in the spirit of Lebeau [24], we rather consider the rescaled equation from [1] following [14]: consider

$$\varepsilon_k = h_k^{m(s_c-s)} |\log(h_k)|^m \xrightarrow[k\to\infty]{} 0, \tag{2.2}$$

where this limit stems from the assumption $s < s_c$. Rescale the function ψ as

$$u_k(t,x) = |\log(h_k)| h_k^{\frac{d}{2}-s} \psi_k(h_k^2 \varepsilon_k t, h_k x).$$

The equation satisfied by u_k is the semiclassical Schrödinger equation

$$i\varepsilon_k \partial_t u_k + \frac{\varepsilon_k^2}{2} \Delta u_k = |u_k|^{2m} u_k, \quad u_{k|t=0} = u_{0,k}. \tag{2.3}$$

According to the definition, the initial data $u_{0,k}$ is given by the following formula.

Lemma 2.1 (Rescaled initial data). *One can write*

$$u_{0,k}(x) = \jmath * \Big(\varphi_{0,k} + \sum_{\ell=k_0}^{\infty} \varphi_{\ell,k} \Big),$$

where $\jmath = \iota_{1/100}$,

$$\varphi_{0,k}(x) = |\log(h_k)| h_k^{d/2-s} \varphi_0(h_k x),$$

and for $\ell \geq k_0$,

$$\varphi_{\ell,k}(x) = \frac{|\log(h_k)|}{|\log(h_\ell)|} \Big(\frac{h_k}{h_\ell}\Big)^{d/2-s} \alpha_\ell\Big(\frac{h_k}{h_\ell} x\Big).$$

Proof. We have

$$u_{0,k}(x) = \mathcal{R}_k(\iota_{h_k/100} * f_0),$$

where $\mathcal{R}_k$ is the scaling transformation

$$\mathcal{R}_k f(x) = |\log(h_k)| h_k^{d/2-s} f(h_k x).$$

We first write this initial data in a more convenient way. When $k \geq 1$, $\ell \geq k_0$, and $h > 0$ we have

$$\begin{aligned} \mathcal{R}_k(\iota_h * \varphi_\ell)(x) &= \frac{|\log(h_k)|}{|\log(h_\ell)|} \Big(\frac{h_k}{h_\ell}\Big)^{d/2-s} \int \frac{1}{h^d} \iota\Big(\frac{y}{h}\Big) \alpha_\ell\Big(\frac{h_k}{h_\ell} x - \frac{y}{h_\ell}\Big)\, \mathrm{d}y \\ &= \frac{|\log(h_k)|}{|\log(h_\ell)|} \Big(\frac{h_k}{h}\Big)^{d} \Big(\frac{h_k}{h_\ell}\Big)^{d/2-s} \int \iota\Big(\frac{h_k}{h} y\Big) \alpha_\ell\Big(\frac{h_k}{h_\ell}(x-y)\Big)\, \mathrm{d}y. \end{aligned}$$

When $h = h_k/100$, then h_k/h is independent of k, so that

$$\mathcal{R}_k(\iota_{h_k/100} * \varphi_\ell) = \jmath * \varphi_{\ell,k}.$$

The same argument works in the case $\ell = 0$. ∎

2.3. Norms of the bubbles

We now estimate the Sobolev norms of the initial data.

Lemma 2.2 (Sobolev norms of the initial bubbles). *Let $s' \geq 0$. When $\ell < k$, we have*

$$\|\jmath * \varphi_{\ell,k}\|_{\dot{H}^{s'}} \lesssim \frac{|\log(h_k)|}{|\log(h_\ell)|}\left(\frac{h_k}{h_\ell}\right)^{s'-s},$$

whereas when $\ell > k$, there holds

$$\|\jmath * \varphi_{\ell,k}\|_{\dot{H}^{s'}} \lesssim \frac{|\log(h_k)|}{|\log(h_\ell)|}\left(\frac{h_\ell}{h_k}\right)^{s}.$$

As a consequence, when $\ell < k$, then $\|\jmath * \varphi_{\ell,k}\|_{\dot{H}^{s'}}$ is large when $s' < s$ but small when $s' > s$, and when $\ell > k$, then $\|\jmath * \varphi_{\ell,k}\|_{\dot{H}^{s'}}$ is small for every s'.

Proof of Lemma 2.2. When $k > \ell$, we have $h_k/h_\ell \to 0$, so the initial data $\jmath * \varphi_{\ell,k}$ spread as $k \to \infty$. When $k < \ell$, the initial data $\jmath * \varphi_{\ell,k}$ rather concentrate, but the convolution prevents the growth of Sobolev norms.

More precisely, letting $s' \geq 0$, we then have

$$\|\varphi_{\ell,k}\|_{\dot{H}^{s'}} \lesssim \frac{|\log(h_k)|}{|\log(h_\ell)|}\left(\frac{h_k}{h_\ell}\right)^{s'-s}.$$

Note that in L^∞, we get a small norm when $\ell < k$ and $s < s_c < d/2$:

$$\|\varphi_{\ell,k}\|_{L^\infty} \lesssim \frac{|\log(h_k)|}{|\log(h_\ell)|}\left(\frac{h_k}{h_\ell}\right)^{d/2-s}.$$

We now remark that the L^1-norms of $\jmath$ and its derivatives are bounded independently of k. Using the Young inequality, we deduce that if $0 \leq \beta \leq s'$, then

$$\|\jmath * \varphi_{\ell,k}\|_{\dot{H}^{s'}} \leq \|\jmath\|_{W^{1,\beta}}\|\varphi_{\ell,k}\|_{\dot{H}^{s'-\beta}} \lesssim \frac{|\log(h_k)|}{|\log(h_\ell)|}\left(\frac{h_k}{h_\ell}\right)^{s'-\beta-s}.$$

As a consequence, when $\ell < k$, choosing $\beta = 0$ we get the first inequality of the statement, and when $\ell > k$, we choose $\beta = s'$ so that we get the second inequality of the statement. ∎

From these estimates, one can deduce upper bounds on the Sobolev norms of $u_{0,k}$:

$$\|u_{0,k}\|_{\dot{H}^{s'}} \lesssim \sum_{\ell<k} \frac{|\log(h_k)|}{|\log(h_\ell)|}\left(\frac{h_k}{h_\ell}\right)^{s'-s} + \sum_{\ell\geq k} \frac{|\log(h_k)|}{|\log(h_\ell)|}\left(\frac{h_\ell}{h_k}\right)^{s}.$$

In the case $s' = s$, the sum over bubbles $\ell < k$ is convergent, but a logarithmic factor remains: $\sum_{\ell<k} \frac{|\log(h_k)|}{|\log(h_\ell)|} \lesssim |\log(h_k)|$. Therefore, we get the upper bounds

$$\|u_{0,k}\|_{\dot{H}^{s'}} \lesssim \begin{cases} \left(\dfrac{h_k}{h_{k-1}}\right)^{s'-s} + 1 & \text{if } s' > s, \\ |\log(h_k)| & \text{if } s' = s, \\ h_k^{s'-s} + 1 & \text{if } s' < s. \end{cases}$$

The logarithmic unboundedness in the case $s' = s$ is essentially irrelevant for the rest of this paper, as in case it appears, it is always multiplied by a positive power of h_k.

For $s' = 0$, using the conservation of mass, we deduce the estimate

$$\|u_k(t)\|_{L^2} \lesssim h_k^{-s}. \tag{2.4}$$

As pointed out in the introduction, this estimate is in sharp contrast with the case of a single bubble considered in [1]: this bound above, which is sharp, shows that the L^2-norm of the initial data is not uniformly bounded in k. This forces us to adapt the arguments from [1] at several stages: modulated energy estimates, and interpolation steps to estimate u_k in homogeneous Sobolev spaces.

One can also estimate the semiclassical energy of u_k, which is (formally) conserved by the flow of (2.3):

$$E_k(t) = \frac{\varepsilon_k^2}{2}\|\nabla u_k(t)\|_{L^2}^2 + \frac{1}{m+1}\|u_k(t)\|_{L^{2m+2}}^{2m+2}.$$

Strictly speaking, for weak solutions (a case we may have to consider, if $s_c > 1$, as explained in Section 2.1), the energy is not necessarily conserved, but it is a nonincreasing function of time; see [19]. Let $0 < s \le s_c$. From the Sobolev embedding $\dot{H}^{s_{\text{sob}}} \hookrightarrow L^{2m+2}$, we get

$$E_k(t) \le E_k(0) \lesssim \left(\varepsilon_k^2\Big(\frac{h_k}{h_{k-1}}\Big)^{2(1-s)} + \Big(\frac{h_k}{h_{k-1}}\Big)^{2(m+1)(s_{\text{sob}}-s)}\right)|\log(h_k)| + 1.$$

In view of (2.2) and the algebraic relation

$$ms_c + 1 = (m+1)s_{\text{sob}},$$

the first term on the right-hand side is controlled by the second one, and

$$E_k(t) \lesssim \Big(\frac{h_k}{h_{k-1}}\Big)^{2(m+1)(s_{\text{sob}}-s)}|\log(h_k)| + 1. \tag{2.5}$$

Note that E_k stays bounded when $s < s_{\text{sob}}$, but tends to infinity otherwise.

Remark 2.3 (WKB analysis). We emphasize that due to the unboundedness, in $L^2(\mathbb{R}^d)$, of $u_{0,k}$, and possibly also $\varepsilon_k \nabla u_{0,k}$ (if $s > s_{\text{sob}}$), WKB analysis for (2.3) is not obvious at all. Typically in the cubic case $m = 1$, where the beautiful idea of Grenier [20] makes it possible to justify WKB analysis in Sobolev spaces, the limit $k \to \infty$ in (2.3) is unclear. This is so even in Zhidkov spaces $X^s(\mathbb{R}^d)$ ($f \in X^s$, $s > 1$, if $f \in L^\infty$ and $\nabla f \in H^{s-1}$), where WKB is justified in [2]. We have seen above that $\varepsilon_k \nabla u_{0,k}$ need not be bounded uniformly in k. We will bypass this difficulty by considering suitable modulated energy functionals, in Sections 4 and 5.

3. Analysis of semiclassical bubbles

In view of Lemma 2.1, and of the semiclassical analysis from [1], we introduce the hydrodynamical system associated to the initial mode ℓ, at scale k:

$$\begin{cases} \partial_t \phi_{\ell,k} + \frac{1}{2}|\nabla \phi_{\ell,k}|^2 + |a_{\ell,k}|^{2m} = 0, & \phi_{\ell,k|t=0} = 0, \\ \partial_t a_{\ell,k} + \nabla \phi_{\ell,k} \cdot \nabla a_{\ell,k} + \frac{1}{2} a_{\ell,k} \Delta \phi_{\ell,k} = 0, & a_{\ell,k|t=0} = \jmath * \varphi_{\ell,k}. \end{cases} \tag{3.1}$$

We use the convention to denote $(\phi_{k,k}, a_{k,k}) = (\phi_k, a_k)$.

3.1. Cauchy problem and zero speed of propagation

Discarding the dependence upon the parameters ℓ and k, the set of equations (3.1) can be written in a universal way,

$$\begin{cases} \partial_t \phi + \frac{1}{2}|\nabla \phi|^2 + |a|^{2m} = 0, & \phi_{|t=0} = 0, \\ \partial_t a + \nabla \phi \cdot \nabla a + \frac{1}{2} a \Delta \phi = 0, & a_{|t=0} = a_{\mathrm{init}}, \end{cases} \tag{3.2}$$

with the common feature that $a_{\mathrm{init}} \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. The following result will be of constant use in the rest of this paper:

Lemma 3.1. *Let $a_{\mathrm{init}} \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. There exist $T > 0$ and a unique solution $(\phi, a) \in \mathcal{C}([0,T]; H^\infty(\mathbb{R}^d))$ to (3.2). Moreover, (ϕ, a) remains compactly supported for $t \in [0,T]$, and*

$$\operatorname{supp} \phi(t,\cdot),\ \operatorname{supp} a(t,\cdot) \subset \operatorname{supp} a_{\mathrm{init}}.$$

Proof. This result is a consequence of the analysis from [26], whose main ideas we recall. We make a change of unknown function (ϕ, a) to

$$(V, A) = (\nabla \phi, a^m).$$

It solves

$$\begin{cases} \partial_t V + V \cdot \nabla V + \nabla(|A|^2) = 0, & V_{|t=0} = 0, \\ \partial_t A + V \cdot \nabla A + \frac{m}{2} A \operatorname{div} V = 0, & A_{|t=0} = (a_{\mathrm{init}})^m, \end{cases} \tag{3.3}$$

which turns out to be a symmetric hyperbolic system, with a constant symmetrizer. Indeed, denote $U = (\mathrm{Re}(A), \mathrm{Im}(A), V)^{\mathsf{T}}$: the system (3.3) becomes

$$\partial_t U + \sum_{j=1}^{d} M_j(U) \partial_j U = 0, \tag{3.4}$$

where

$$\sum_{j=1}^{d} M_j(U)\xi_j = \begin{pmatrix} V\cdot\xi & 0 & \frac{m}{2}\operatorname{Re}(A)\xi^{\mathsf{T}} \\ 0 & V\cdot\xi & \frac{m}{2}\operatorname{Im}(A)\xi^{\mathsf{T}} \\ 2\operatorname{Re}(A)\xi & 2\operatorname{Im}(A)\xi & V\cdot\xi\,\mathrm{I}_d \end{pmatrix}.$$

Hence the matrices M_j belong to the set $\mathcal{M}_{d+2}(\mathbb{R})$ of $(d+2)\times(d+2)$ matrices, and the matrices SM_j belong to the set $\mathcal{S}_{d+2}(\mathbb{R})$ of $(d+2)\times(d+2)$ symmetric matrices, with

$$S = \begin{pmatrix} m\,\mathrm{I}_d & 0 \\ 0 & 4\,\mathrm{I}_2 \end{pmatrix}.$$

Local existence in $H^\sigma(\mathbb{R}^d)$ with $\sigma > d/2+1$ is then standard for (3.4) hence for (3.3); see e.g. [25]. We emphasize tame estimates, which show that the lifespan T is independent of $\sigma > d/2+1$. Let $\Lambda = (1-\Delta)^{\frac{1}{2}}$: by symmetry,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle S\Lambda^\sigma U, \Lambda^\sigma U\rangle = 2\langle S\Lambda^\sigma \partial_t U, \Lambda^\sigma U\rangle.$$

In view of (3.4),

$$\langle S\Lambda^\sigma \partial_t U, \Lambda^\sigma U\rangle = -\sum_{j=1}^{d}\langle S\Lambda^\sigma (M_j(U)\partial_j U), \Lambda^\sigma U\rangle.$$

Write

$$\langle S\Lambda^\sigma (M_j(U)\partial_j U), \Lambda^\sigma U\rangle = \langle SM_j(U)\partial_j \Lambda^\sigma U, \Lambda^\sigma U\rangle - \langle S[M_j(U),\Lambda^\sigma]\partial_j U, \Lambda^\sigma U\rangle.$$

Since SM_j is symmetric,

$$\langle SM_j(U)\partial_j \Lambda^\sigma U, \Lambda^\sigma U\rangle = -\frac{1}{2}\langle \Lambda^\sigma U, \partial_j(SM_j(U))\Lambda^\sigma U\rangle.$$

As SM_j is linear in its argument, we readily infer

$$|\langle SM_j(U)\partial_j \Lambda^\sigma U, \Lambda^\sigma U\rangle| \lesssim \|S\partial_j M_j(U)\|_{L^\infty}\|U\|_{H^\sigma}^2 \lesssim \|\nabla U\|_{L^\infty}\|U\|_{H^\sigma}^2.$$

By commutator estimates (see [21]), we have

$$|\langle S[M_j(U),\Lambda^\sigma]\partial_j U, \Lambda^\sigma U\rangle| \lesssim \|\nabla U\|_{L^\infty}\|U\|_{H^\sigma}^2.$$

We infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle S\Lambda^\sigma U, \Lambda^\sigma U\rangle \lesssim \|\nabla U\|_{L^\infty}\langle S\Lambda^\sigma U, \Lambda^\sigma U\rangle.$$

To return to the initial unknown (ϕ, a), we integrate the first equation from (3.2) with respect to time:

$$\phi(t) = -\frac{1}{2}\int_0^t |V(\tau)|^2\,\mathrm{d}\tau - \int_0^t |A(\tau)|^2\,\mathrm{d}\tau.$$

We check that $V = \nabla\phi$, as

$$\partial_t(\nabla\phi - V) = \nabla\partial_t\phi - \partial_t V = 0.$$

We go back to a by now viewing the second equation in (3.2) as a transport equation with smooth coefficients. This leads to the local existence result of the lemma.

We now move to the zero propagation speed property, established initially in [26, Theorem 2]. We see that there exists $C \geq 0$ such that for every j,

$$|M_j(U)| \leq C|U|.$$

Fix (t, x) in $[0, T] \times \mathbb{R}^d$. Since the solution is well defined on $[0, t]$, we can define

$$C'(t, x) := \sup_{t' \in [0,t], 1 \leq j \leq d} |\partial_j U_k(t', x)| < +\infty.$$

Using the equation, for $t' \in [0, t]$ there holds

$$|\partial_t U_k(t', x)| \leq dCC'(t, x)|U_k(t', x)|.$$

Using the Gronwall lemma, we deduce that $U_k(t, x) = 0$ if and only if $U_k(0, x) = 0$, hence

$$\operatorname{supp}(A(t), V(t)) = \operatorname{supp}(A, V)_{|t=0} = \operatorname{supp} a_{\text{init}}.$$

We go back to the unknown (ϕ, a) as described above. ∎

Remark 3.2 (Lifespan of the bubbles without convolution). Introduce bubbles without the initial convolution,

$$\begin{cases} \partial_t \breve{\phi}_{\ell,k} + \frac{1}{2}|\nabla\breve{\phi}_{\ell,k}|^2 + |\breve{a}_{\ell,k}|^{2m} = 0, & \breve{\phi}_{\ell,k|t=0} = 0, \\ \partial_t \breve{a}_{\ell,k} + \nabla\breve{\phi}_{\ell,k} \cdot \nabla\breve{a}_{\ell,k} + \frac{1}{2}\breve{a}_{\ell,k}\Delta\breve{\phi}_{\ell,k} = 0, & \breve{a}_{\ell,k|t=0} = \varphi_{\ell,k}. \end{cases} \tag{3.5}$$

Using the convention $(\breve{\phi}_{\ell,\ell}, \breve{a}_{\ell,\ell}) = (\breve{\phi}_\ell, \breve{a}_\ell)$, we check the algebraic relations

$$\begin{aligned} \breve{\phi}_{\ell,k}(t, x) &= \frac{\varepsilon_k}{\varepsilon_\ell}\breve{\phi}_\ell\Big(\Big(\frac{\varepsilon_k h_k^2}{\varepsilon_\ell h_\ell^2}\Big)t, \frac{h_k}{h_\ell}x\Big), \\ \breve{a}_{\ell,k}(t, x) &= \frac{|\log h_k|}{|\log h_\ell|}\Big(\frac{h_k}{h_\ell}\Big)^{d/2-s}\breve{a}_\ell\Big(\Big(\frac{\varepsilon_k h_k^2}{\varepsilon_\ell h_\ell^2}\Big)t, \frac{h_k}{h_\ell}x\Big), \end{aligned} \tag{3.6}$$

where we recall the notation

$$\varepsilon_k = h_k^{m(s_c - s)}|\log h_k|^m.$$

Using a virial computation, showing that if a global smooth solution exists for (3.5), then it is dispersive, the authors in [26] show that T, in Lemma 3.1, is necessarily finite. In view of (3.6), this implies that for $\ell < k$, $(\breve{\phi}_{\ell,k}, \breve{a}_{\ell,k})$ remains smooth on $[0, T_{\ell,k}]$, with

$$T_{\ell,k} = \frac{\varepsilon_\ell h_\ell^2}{\varepsilon_k h_k^2}T \xrightarrow[k\to\infty]{} +\infty,$$

while for $\ell > k$, $(\breve{\phi}_{\ell,k}, \breve{a}_{\ell,k})$ remains smooth on $[0, T_{\ell,k}]$ for the same expression of $T_{\ell,k}$, which now goes to zero as k goes to infinity.

Remark 3.3 (On low modes). The scaling (3.6) also shows that for $\ell < k$ and $t \approx 1$, nonlinear effects are negligible in (3.5), a remark reminiscent of the strategy adopted in the proof of [13, Proposition 2.6], where the *linear* evolution of "low modes" is considered. Actually, in this régime, even the linear evolution of the initial data is negligible: the low modes are essentially constant in time, a remark which is exploited in the introduction of a renormalized modulated energy functional in Section 5.

3.2. Superposition principle

Let us assume that the points x_ℓ are chosen so that the profiles $\jmath * \varphi_{\ell,k}$ have disjoint supports. We recall that $\alpha_k(x) = \alpha(x - \frac{x_k}{h_k})$. Assuming that α and $\jmath = \iota(\cdot/100)$ are supported in $\bar{B}(0, r_1)$ for some small constant $r_1 > 0$, we have

$$\operatorname{supp}(\jmath * \varphi_{\ell,k}) \subset \bar{B}\Big(\frac{x_\ell}{h_k}, 2r_1\frac{h_\ell}{h_k}\Big).$$

Choosing the points x_ℓ sufficiently far away from each other so that

$$|x_\ell - x_{\ell+1}| > 4r_1 h_\ell,$$

the bubbles are therefore disjoint.

In this case, we may use a nonlinear superposition principle. Introduce the following intermediate approximate solution with initial data including all the scales $\ell \leq k$:

$$\begin{cases} \partial_t\tilde{\phi}_k + \frac{1}{2}|\nabla\tilde{\phi}_k|^2 + |\tilde{a}_k|^{2m} = 0, & \tilde{\phi}_{k|t=0} = 0, \\ \partial_t\tilde{a}_k + \nabla\tilde{\phi}_k\cdot\nabla\tilde{a}_k + \frac{1}{2}\tilde{a}_k\Delta\tilde{\phi}_k = 0, & \tilde{a}_{k|t=0} = \sum_{\ell\leq k}\jmath * \varphi_{\ell,k}. \end{cases}$$

Lemma 3.1 implies that

$$(\tilde{\phi}_k, \tilde{a}_k) = \sum_{\ell\leq k}(\phi_{\ell,k}, a_{\ell,k}),$$

where each bubble is given by (3.1). In view of the above analysis, we also introduce $(\tilde{V}_k, \tilde{A}_k) = (\nabla\tilde{\phi}_k, \tilde{a}_k^m)$, which solves

$$\begin{cases} \partial_t\tilde{V}_k + \tilde{V}_k\cdot\nabla\tilde{V}_k + \nabla(|\tilde{A}_k|^2) = 0, & \tilde{V}_{k|t=0} = 0, \\ \partial_t\tilde{A}_k + \tilde{V}_k\cdot\nabla\tilde{A}_k + \frac{m}{2}\tilde{A}_k\operatorname{div}\tilde{V}_k = 0, & \tilde{A}_{k|t=0} = \Big(\sum_{\ell\leq k}\jmath * \varphi_{\ell,k}\Big)^m. \end{cases} \tag{3.7}$$

Since the functions $\jmath * \varphi_{\ell,k}$ have pairwise disjoint supports, we may also write

$$\tilde{A}_{k|t=0} = \sum_{\ell\leq k}(\jmath * \varphi_{\ell,k})^m.$$

Like above, the zero speed of propagation from Lemma 3.1 then implies the relation

$$(\widetilde{V}_k, \tilde{A}_k) = \sum_{\ell \le k} (V_{\ell,k}, A_{\ell,k}),$$

where each $(V_{\ell,k}, A_{\ell,k})$ solves the same system as $(\widetilde{V}_k, \tilde{A}_k)$, with initial datum

$$(V_{\ell,k}, A_{\ell,k})_{|t=0} = (0, (\jmath * \varphi_{\ell,k})^m).$$

3.3. Refined estimates for the bubbles

Like Lemma 3.1, the following lemma will be crucial for the rest of this paper:

Lemma 3.4 (Uniform estimates for the bubbles). *There exist $C > 0$ and $T > 0$ such that the smooth solutions $(\widetilde{V}_k, \tilde{A}_k)$ have a lifespan T_k which is uniformly bounded from below: $T_k \ge T > 0$. Moreover, for every $t \in [0, T]$, for every $k \ge 1$ and for every nonnegative integer σ, there holds*

$$\| |\nabla|^\sigma \widetilde{V}_k(t)\|_{L^2} + \| |\nabla|^\sigma \tilde{A}_k(t)\|_{L^2} \le C \| |\nabla|^\sigma \tilde{A}_k(0)\|_{L^2} \lesssim 1 + \Big(\frac{h_k}{h_{k-1}}\Big)^{\sigma+1+m(s_c-s)-\frac{d}{2}}.$$

Note that, in particular, the Sobolev norms are bounded whenever $\sigma + 1 \ge \frac{d}{2}$. Considering the equation satisfied by $(\widetilde{V}_k, \tilde{A}_k)$, this implies that for every $\varepsilon > 0$,

$$\| |\nabla|^\sigma \widetilde{V}_k(t)\|_{L^\infty} + \| |\nabla|^\sigma \tilde{A}_k(t)\|_{L^\infty} \lesssim \| |\nabla|^\sigma \tilde{A}_k(0)\|_{L^\infty} \lesssim 1 + \Big(\frac{h_k}{h_{k-1}}\Big)^{\sigma+1+m(s_c-s)+\varepsilon}.$$

Given that $\frac{h_k}{h_{k-1}} < 1$, one can replace ε by 0 in the latter upper bound.

Proof of Lemma 3.4. The proof relies on the symmetry of the hyperbolic system (3.7). Let $N > 1 + d/2$ and $\sigma \le N$ be integers, and let D^σ denote the family of differential operators in space, of order σ. In particular, the notation $\|D^\sigma U\|_{L^2}$ stands for

$$\|D^\sigma U\|_{L^2} = \sum_{|\beta|=\sigma} \|\partial_x^\beta U\|_{L^2}.$$

As in the framework recalled in the proof of Lemma 3.1, denote

$$U_k = (\mathrm{Re}(\tilde{A}_k), \mathrm{Im}(\tilde{A}_k), \widetilde{V}_k)^{\mathsf{T}}.$$

We observe that it solves (3.4). First, we examine its initial data. In view of Section 2.3, since σ is an integer, and the initial bubbles have pairwise disjoint supports, the Leibniz rule and Hölder inequality yield

$$\|D^\sigma U_k(0)\|_{L^2} = \sum_{\ell \le k} \|D^\sigma (\jmath * \varphi_{\ell,k})^m\|_{L^2} \lesssim 1 + \Big(\frac{h_k}{h_{k-1}}\Big)^{\sigma-ms+(m-1)\frac{d}{2}}.$$

We rewrite the last power as

$$\sigma - ms + (m-1)\frac{d}{2} = \sigma + 1 - \frac{d}{2} + m(s_c - s).$$

In particular, for $d \le 2$, $U_k(0)$ is uniformly bounded in $H^N(\mathbb{R}^d)$ (we choose $\sigma = 0$), and for $d \le 4$, $\nabla U_k(0)$ is uniformly bounded in $H^{N-1}(\mathbb{R}^d)$ (we choose $\sigma = 1$). We start by proving Lemma 3.4 in the case $d \le 4$. For $1 \le n \le d$, $\partial_n U$ solves

$$\partial_t \partial_n U_k + \sum_{j=1}^d M_j(U_k)\partial_j \partial_n U_k = -\sum_{j=1}^d M_j(\partial_n U_k)\partial_j U_k.$$

We proceed classically, like in the proof of Lemma 3.1, by viewing the term on the right-hand side as a perturbative (semilinear) term, and compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle S\Lambda^{N-1}\partial_n U_k, \Lambda^{N-1}\partial_n U_k\rangle = &-\sum_{j=1}^d \langle S\Lambda^{N-1}(M_j(U_k)\partial_j\partial_n U_k), \Lambda^{N-1}\partial_n U_k\rangle \\ &-\sum_{j=1}^d \langle S\Lambda^{N-1}(M_j(\partial_n U_k)\partial_j U_k), \Lambda^{N-1}\partial_n U_k\rangle.\end{aligned}$$

Using the fact that $H^{N-1}(\mathbb{R}^d)$ is a Banach algebra, and since M_j is linear in its argument, the last term is controlled by $\|\nabla U_k\|^3_{H^{N-1}}$. For the first term on the right-hand side, we proceed exactly like in the proof of Lemma 3.1, and write

$$\begin{aligned}&\langle S\Lambda^{N-1}(M_j(U_k)\partial_j\partial_n U_k), \Lambda^{N-1}\partial_n U_k\rangle \\ &\quad = \langle S(M_j(U_k)\partial_j\Lambda^{N-1}\partial_n U_k), \Lambda^{N-1}\partial_n U_k\rangle \\ &\qquad + \langle S([\Lambda^{N-1}, M_j(U_k)]\partial_j\partial_n U_k), \Lambda^{N-1}\partial_n U_k\rangle.\end{aligned}$$

By symmetry, the first term on the right-hand side is controlled by

$$\|\nabla U_k\|_{L^\infty}\|\nabla U_k\|^2_{H^{N-1}},$$

and by Kato–Ponce commutator estimates, the last term is estimated similarly:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle S\Lambda^{N-1}\partial_n U_k, \Lambda^{N-1}\partial_n U_k\rangle &\lesssim \|\nabla U_k\|_{L^\infty}\|\nabla U_k\|^2_{H^{N-1}} + \|\nabla U_k\|^3_{H^{N-1}} \\ &\lesssim \|\nabla U_k\|^3_{H^{N-1}},\end{aligned}$$

where we have used the Sobolev embedding. Summing over $n \in \{1, \dots, d\}$, and recalling the equivalence of norms

$$\langle S\Lambda^{N-1}\nabla U_k, \Lambda^{N-1}\nabla U_k\rangle \sim \|\nabla U_k\|^2_{H^{N-1}},$$

we infer that there exists $T > 0$ independent of k such that $(\nabla U_k)_k$ is uniformly bounded in $L^\infty([0,T]; H^{N-1})$, and Lemma 3.4 follows in the case $d \le 4$.

When $d \geq 5$, we replace the above quantity $\|\nabla U_k\|_{H^{N-1}}$ by $\|D^{\sigma_0} U_k\|_{H^{N-\sigma_0}}$ for $\sigma_0 > 1$:

$$\|U_k\|_{\sigma_0,N} := \|D^{\sigma_0} U_k\|_{H^{N-\sigma_0}} = \sum_{\sigma=\sigma_0}^{N} \|D^{\sigma} U_k\|_{L^2}.$$

Three conditions are required:

- The initial norm $\|U_k(0)\|_{\sigma_0,N}$ is bounded uniformly in k.
- The norm $\|\cdot\|_{\sigma_0,N}$ controls the L^∞-norm of ∇U_k,

$$\|\nabla U_k\|_{L^\infty} \lesssim \|U_k\|_{\sigma_0,N}.$$

- This norm controls each term in the analogue of the above energy estimates.

As we saw above, the first condition is fulfilled for $\sigma_0 \geq \frac{d}{2} - 1$, and we choose

$$\sigma_0 = \begin{cases} \dfrac{d}{2} - 1 & \text{if } d \text{ is even,} \\ \left[\dfrac{d}{2}\right] & \text{if } d \text{ is odd.} \end{cases} \tag{3.8}$$

The second condition is satisfied, in view of the following elementary result, valid in any space dimension:

Lemma 3.5. *Let $d \geq 1$, σ_0 be given by (3.8), and $K > \frac{d}{2}$. There exists $C = C(d, K)$ such that for every $f \in H^\sigma(\mathbb{R}^d)$,*

$$\|f\|_{L^\infty} \leq C \|D^{\sigma_0} f\|_{H^{K-\sigma_0}}.$$

Proof. We use the inverse Fourier transform to infer

$$\|f\|_{L^\infty} \lesssim \int_{\mathbb{R}^d} |\hat{f}(\xi)|\, \mathrm{d}\xi.$$

The Cauchy–Schwarz inequality yields

$$\left(\int_{\mathbb{R}^d} |\hat{f}(\xi)|\, \mathrm{d}\xi\right)^2 \lesssim \left(\int_{\mathbb{R}^d} |\xi|^{2\sigma_0} \langle \xi \rangle^{2(K-\sigma_0)} |\hat{f}(\xi)|^2\, \mathrm{d}\xi\right)\left(\int_{\mathbb{R}^d} \frac{1}{|\xi|^{2\sigma_0} \langle \xi \rangle^{2(K-\sigma_0)}}\, \mathrm{d}\xi\right).$$

Since $2K > d$, the last integral is convergent at infinity, and our definition of σ_0 makes it convergent near zero, as we always have $d - 1 - 2\sigma_0 > -1$. ■

Resume the strategy presented for $d \leq 4$: the equation satisfied by $D^{\sigma_0} U_k$ is of the form

$$\partial_t D^{\sigma_0} U_k + \sum_{j=1}^{d} M_j(U_k) \partial_j D^{\sigma_0} U_k = -\sum_{j=1}^{d} [D^{\sigma_0}, M_j(U_k)] \partial_j U_k.$$

When considering

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle S \Lambda^{N-\sigma_0} D^{\sigma_0} U_k, \Lambda^{N-\sigma_0} D^{\sigma_0} U_k \rangle,$$

the terms coming from $M_j(U_k)\partial_j D^{\sigma_0}U_k$ (the left-hand side above) are treated like before, by using symmetry and Kato–Ponce commutator estimates, as $D^{\sigma_0}U_k$ is estimated in the inhomogeneous Sobolev space $H^{N-\sigma_0}$. To control the terms coming from the right-hand side, by the Leibniz rule and the Cauchy–Schwarz inequality, we have to estimate a combination of terms of the form

$$\|D^{\sigma_1}U_k D^{\sigma_2}U_k\|_{L^2}, \quad \sigma_1,\sigma_2 \geq 1,\ \sigma_0+1 \leq \sigma_1+\sigma_2 \leq N+1.$$

Let us fix such exponents σ_1, σ_2. We have

$$\|D^{\sigma_1}U_k D^{\sigma_2}U_k\|_{L^2} \leq \|D^{\sigma_1}U_k\|_{L^p}\|D^{\sigma_2}U_k\|_{L^q}, \quad \frac{1}{p}+\frac{1}{q}=\frac{1}{2}.$$

In view of the Sobolev embedding $\dot{H}^{d(1/2-1/p)}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$, we infer

$$\|D^{\sigma_1}U_k D^{\sigma_2}U_k\|_{L^2} \leq \|D^{\sigma_1}U_k\|_{\dot{H}^{d/q}}\|D^{\sigma_2}U_k\|_{\dot{H}^{d/p}}.$$

By symmetry of the roles, we assume that $\sigma_1 \geq \sigma_2$. In everything that follows we assume that $\frac{d}{q}$ is an integer. We get

$$\begin{aligned}\|D^{\sigma_1}U_k D^{\sigma_2}U_k\|_{L^2} &\leq \|D^{\sigma_1+d/q}U_k\|_{L^2}\|D^{\sigma_2+[d/p]}U_k\|_{\dot{H}^{d/p-[d/p]}}\\ &\leq \|D^{\sigma_1+d/q}U_k\|_{L^2}\|D^{\sigma_2+[d/p]}U_k\|_{H^1}.\end{aligned}$$

From a bootstrap argument based on the control of $\|\nabla U_k(t)\|_{L^\infty}$ uniformly in k and $t \in [0,T]$, it suffices to check that we may find such a p and a q, satisfying in addition

$$\sigma_0 \leq \sigma_1+[d/q] \leq N, \quad \sigma_0 \leq \sigma_2+[d/p] \leq N-1.$$

We proceed as follows:

- If $\sigma_1 \geq \sigma_0$, we choose $q=\infty$ and $p=2$. We get

  $$\begin{aligned}&\sigma_1+[d/q]=\sigma_1 \in [\sigma_0, N],\\ &\sigma_2+[d/p] \geq [d/2] \geq \sigma_0.\end{aligned}$$

  Moreover, since $2\sigma_2 \leq \sigma_1+\sigma_2 \leq N+1$, we have

  $$\sigma_2+[d/p] \leq \frac{N+1}{2}+\frac{d}{2},$$

  which is bounded above by $N-1$ when N is chosen large enough.
- Otherwise, we have $\sigma_2 \leq \sigma_1 \leq \sigma_0-1$. If $d=1$ we have $\sigma_0=0$, hence this case does not occur. Let $q \in [2,\infty)$ such that $\frac{d}{q}$ is an integer and

  $$\sigma_1+\frac{d}{q}=\sigma_1+[d/q]=\sigma_0.$$

In particular, we check that $\sigma_1 + [d/q] \in [\sigma_0, N]$. Then $\sigma_1 + \sigma_2 \geq \sigma_0 + 1$, hence

$$\frac{d}{p} = \frac{d}{2} - \frac{d}{q} = \frac{d}{2} - \sigma_0 + \sigma_1 \geq \frac{d}{2} + 1 - \sigma_2.$$

Therefore, we have

$$\sigma_2 + \frac{d}{p} \geq \frac{d}{2} + 1.$$

In particular,

$$\sigma_2 + [d/p] \geq \frac{d}{2} \geq \sigma_0.$$

Finally,

$$\sigma_2 + [d/p] \leq \sigma_1 + \frac{d}{2} \leq \sigma_0 - 1 + \frac{d}{2},$$

which is bounded above by $N-1$ when N is chosen large enough. ∎

4. Modulated energy estimate

We now use the approximate bubbles analyzed in the previous section in order to establish some information regarding the actual solution u_k to (2.3). We emphasize that even in the case of a single bubble, $\tilde{a}_k e^{i\tilde{\phi}_k/\varepsilon}$ must not be expected to approximate u_k in L^2, due to phase modulations; see [14]. But even if phase modulations are taken into account, the unboundedness of $u_{0,k}$ in L^2 is an issue to justify WKB analysis; see Remark 2.3. We follow the strategy of [1, Theorem 4.1], and introduce a modulated energy functional

$$H_k(t) = \frac{1}{2}\|(\varepsilon_k\nabla - i\widetilde{V}_k)u_k\|_{L^2}^2 + \int_{\mathbb{R}^d}(F(|u_k|^2) - F(\tilde{\rho}_k) - (|u_k|^2 - \tilde{\rho}_k)f(\tilde{\rho}_k))\,\mathrm{d}x,$$

where $\tilde{\rho}_k = |\tilde{a}_k|^2$, $f(y) = y^m$, and

$$F(y) = \int_0^y f(z)\,\mathrm{d}z = \frac{y^{m+1}}{m+1}.$$

Denote the kinetic part by

$$K_k(t) = \frac{1}{2}\|(\varepsilon_k\nabla - i\widetilde{V}_k)u_k\|_{L^2}^2$$

and the potential part by

$$P_k(t) = \int_{\mathbb{R}^d}(F(|u_k|^2) - F(\tilde{\rho}_k) - (|u_k|^2 - \tilde{\rho}_k)f(\tilde{\rho}_k))\,\mathrm{d}x.$$

We know from [1] that the potential part is bounded from below by

$$P_k(t) \geq c\int_{\mathbb{R}^d}\big|\,|u_k|^2 - \tilde{\rho}_k\big|(|u_k|^{2m} + (\tilde{\rho}_k)^m)\,\mathrm{d}x,$$

for some $c > 0$ depending only on m. In particular, H_k is the sum of two nonnegative terms.

We first detail the case $s < s_{\text{sob}}$ (with $s \leq 2$), for the sake of clarity. When $t = 0$, we get, in view of the fact that $\widetilde{V}_k(0) = 0$ and of Section 2.3,

$$K_k(0) = \frac{1}{2}\|\varepsilon_k \nabla u_{0,k}\|_{L^2}^2 \lesssim \varepsilon_k^2 h_k^{2(1-s)}|\log(h_k)|.$$

Under the assumption $s < s_{\text{sob}}$, $K_k(0) = \mathcal{O}(\varepsilon_k^2)$ as $k \to \infty$. Indeed, we note the algebraic relation

$$\varepsilon_k^2 h_k^{2(1-s)}|\log(h_k)| = |\log h_k|^{2m+1} h_k^{2m(s_c-s)+2(1-s)} = |\log h_k|^{2m+1} h_k^{(2m+2)(s_{\text{sob}}-s)}.$$

As for the initial potential part, we first use the fact that P_k corresponds to the beginning of a Taylor expansion,

$$P_k = \frac{1}{2}\int_{\mathbb{R}^d} (|u_k|^2 - \tilde{\rho}_k)^2 \int_0^1 f'(\tilde{\rho}_k + \theta(|u_k|^2 - \tilde{\rho}_k))\, \mathrm{d}\theta\, \mathrm{d}x.$$

Considering this quantity at time $t = 0$, and using the fact that the supports of the initial bubbles $\jmath * \varphi_{\ell,k}$ are pairwise disjoint,

$$\operatorname{supp} \jmath * \varphi_{\ell,k} \cap \operatorname{supp} \jmath * \varphi_{\ell',k} = \emptyset, \quad \forall \ell \neq \ell',$$

we obtain the rough bound

$$0 \leq P_k(0) \lesssim \sum_{\ell > k} \int_{\mathbb{R}^d} |\jmath * \varphi_{\ell,k}|^4 \times \sum_{\ell'} |\jmath * \varphi_{\ell',k}|^{2m-2} \lesssim \sum_{\ell > k} \|\jmath * \varphi_{\ell,k}\|_{L^{2m+2}}^{2m+2}.$$

In view of the embedding $\dot{H}^{s_{\text{sob}}} \hookrightarrow L^{2m+2}$, we infer, thanks to Lemma 2.2, and since the map $z \mapsto M^{z^2} e^{-M^z}$ is integrable on $[1, \infty)$ and decreasing for $z \geq z_0 \gg 1$,

$$P_k(0) \lesssim \left(\frac{|\log(h_k)|}{|\log(h_{k+1})|}\right)^{2m+2} \left(\frac{h_{k+1}}{h_k}\right)^{(2m+2)s} |\log(h_k)|.$$

Choosing $M > 1$ sufficiently large depending both on $s_c - s$ and on s, we have in particular $P_k(0) = \mathcal{O}(\varepsilon_k^2)$. In the rest of the paper, we will use Lemma 3.4 several times. The notation $\tilde{h}_k = h_k / h_{k-1}$ would lighten the formulas (note that $\tilde{h}_k \to 0$ as $k \to \infty$). We will instead keep the notation h_k, which is legitimate up to increasing the value of M.

Proposition 4.1 (Modulated energy estimate). *Let $s < s_{\text{sob}}$ with $s \leq 2$. Then for every $t \in [0, T]$, where T is given by Lemma 3.4, there holds*

$$H_k(t) \lesssim \varepsilon_k^2 + h_k^{2(m+1)(s_{\text{sob}}-s)}.$$

Proof. We estimate the time derivative of H_k.

We follow line by line the proof of [1, Theorem 4.1], taking into account the fact that the L^2-norm of u_k has the upper bound

$$\|u_k\|_{L^2} = \|u_{0,k}\|_{L^2} \lesssim h_k^{-s},$$

hence it is not necessarily bounded as $k \to \infty$. Similarly, low-order Sobolev norms of $\widetilde{V}_k$ may grow to infinity as $k \to \infty$.

Since we only consider weak solutions u_k, some integrations by parts may not make sense. For simplicity, the following proof will hence be formal only, but can be fully justified by working on a sequence $u_k^{(n)} \in \mathcal{C}(\mathbb{R}, H^2)$ of global strong solutions to the equation

$$i\varepsilon_k \partial_t u_k^{(n)} + \frac{\varepsilon_k^2}{2}\Delta u_k^{(n)} = f_n(|u_k^{(n)}|^2); \quad u_{k|t=0}^{(n)} = u_{0,k},$$
$$f_n(y) = \frac{y^m}{1+(\delta_n y)^m}, \quad \delta_n \to 0^+.$$

Indeed, as $n \to \infty$, the sequence $(u_k^{(n)})_n$ will converge to a weak solution to (2.3) with initial data $u_{0,k}$. In this case, one should replace F by $F_n(y) = \int_0^y f_n(z)\,\mathrm{d}z$ in the formula for the modulated energy $H_k^{(n)}$, and $G(y) = \int_0^y z f'(z)\,\mathrm{d}z$ by $G_n(y) = \int_0^y z f_n'(z)\,\mathrm{d}z$.

In order to estimate $H_k(t)$, we use the Madelung transform and switch to the hydrodynamic variables

$$\rho = |u_k|^2, \quad J = \operatorname{Im}(\varepsilon_k \bar{u}_k \nabla u_k),$$

satisfying the equation

$$\begin{cases} \partial_t \rho + \operatorname{div}(J) = 0, \\ \partial_t J_j + \dfrac{\varepsilon_k^2}{4} \displaystyle\sum_{p=1}^d \partial_p (4\operatorname{Re}(\partial_j \bar{u}_k \partial_p u_k) - \partial_{pj}\rho) + \partial_j G(\rho) = 0. \end{cases}$$

As in [1], we write the time derivative of the kinetic energy using the hydrodynamic variables, then proceed by integration by parts. We get the formula

$$\frac{\mathrm{d}}{\mathrm{d}t} H_k = -\frac{1}{4} I - I' - \int (G(\rho) - G(\tilde{\rho}_k) - (\rho - \tilde{\rho}_k) G'(\tilde{\rho}_k)) \operatorname{div} \widetilde{V}_k,$$

where

$$I = \varepsilon_k^2 \int \nabla(\operatorname{div} \widetilde{V}_k) \cdot \nabla |u_k|^2,$$
$$I' = \operatorname{Re} \sum_{p,j} \int \partial_j (\widetilde{V}_k)_p (\varepsilon_k \partial_j u_k - i(\widetilde{V}_k)_j u_k) \overline{(\varepsilon_k \partial_p u_k - i(\widetilde{V}_k)_p u_k)}.$$

We note that $\|\nabla \widetilde{V}_k\|_{L^\infty}$ is uniformly bounded, hence $I' = \mathcal{O}(K_k)$.

The only new difficulty, compared to [1], is to show that $I = \mathcal{O}(K_k + \varepsilon_k^2)$. We have

$$I = \varepsilon_k \int \nabla(\operatorname{div} \widetilde{V}_k) \cdot (\bar{u}_k(\varepsilon_k \nabla - i\widetilde{V}_k)u_k + u_k \overline{(\varepsilon_k \nabla - i\widetilde{V}_k)u_k}).$$

We note that if we proceed like in [1], we invoke the Cauchy–Schwarz and then the Young inequalities to get the upper bound

$$|I| \leq \|(\varepsilon_k \nabla - i\widetilde{V}_k)u_k\|_{L^2}^2 + \varepsilon_k^2 \|u_k\|_{L^2}^2,$$

which may go to infinity as $k \to \infty$ since we only know that $\|u_k\|_{L^2} \lesssim h_k^{-s}$.

Instead, we decompose

$$|I| \leq \varepsilon_k \int |\nabla(\operatorname{div} \widetilde{V}_k)| \times |u_k| \times |(\varepsilon_k \nabla - i\widetilde{V}_k)u_k| = \mathrm{II} + \mathrm{III},$$

where

$$\mathrm{II} = \varepsilon_k \int |\nabla(\operatorname{div} \widetilde{V}_k)| \times (|u_k| - \sqrt{\tilde{\rho}_k}) \times |(\varepsilon_k \nabla - i\widetilde{V}_k)u_k|,$$
$$\mathrm{III} = \varepsilon_k \int |\nabla(\operatorname{div} \widetilde{V}_k)| \times \sqrt{\tilde{\rho}_k} \times |(\varepsilon_k \nabla - i\widetilde{V}_k)u_k|.$$

Then, using the Hölder inequality,

$$|\mathrm{II}| \leq \varepsilon_k \|\nabla \operatorname{div} \widetilde{V}_k\|_{L^{2+\frac{2}{m}}} \| |u_k| - \sqrt{\tilde{\rho}_k}\|_{L^{2m+2}} \|(\varepsilon_k \nabla - i\widetilde{V}_k)u_k\|_{L^2}.$$

We know thanks to Lemma 3.4 and Sobolev embedding, that for N sufficiently large,

$$\begin{aligned}\|\nabla \operatorname{div} \widetilde{V}_k\|_{L^{2+\frac{2}{m}}} &\lesssim \|\nabla \operatorname{div} \widetilde{V}_k\|_{H^N} = \|\nabla \operatorname{div} \widetilde{V}_k\|_{L^2} + \| |\nabla|^N \nabla \operatorname{div} \widetilde{V}_k\|_{L^2} \\ &\lesssim 1 + h_k^{3+m(s_c-s)-\frac{d}{2}}.\end{aligned}$$

Rewriting

$$3 + m(s_c - s) - \frac{d}{2} = (m-1)\Big(\frac{d}{2} - s\Big) + 2 - s,$$

we see that this power is nonnegative provided that $s \leq 2$. Moreover, the last two factors are estimated respectively by

$$\begin{aligned}\| |u_k| - \sqrt{\tilde{\rho}_k}\|_{L^{2m+2}}^{2m+2} &= \int \big| |u_k| - \sqrt{\tilde{\rho}_k}\big|^{2m+2} \\ &\lesssim \int \big| |u_k|^2 - \tilde{\rho}^k\big|^2 (|u_k|^{2m} + (\tilde{\rho}_k)^m) \lesssim P_k\end{aligned}$$

and

$$\|(\varepsilon_k \nabla - i\widetilde{V}_k)u_k\|_{L^2}^2 = 2K_k.$$

Therefore,

$$|\mathrm{II}| \lesssim \varepsilon_k P_k^{\frac{1}{2m+2}} \sqrt{K_k} \lesssim K_k + \varepsilon_k^2 P_k^{\frac{1}{m+1}} \lesssim K_k + P_k + \varepsilon_k^{2\frac{m+1}{m}},$$

where we have used the Young inequality twice. Finally, using a similar strategy, we obtain the estimate

$$|\mathrm{III}| \leq \varepsilon_k \|\nabla \operatorname{div} \widetilde{V}_k\|_{L^2} \|\sqrt{\tilde{\rho}_k}\|_{L^\infty} \|(\varepsilon_k \nabla - i \widetilde{V}_k) u_k\|_{L^2}.$$

In view of Lemma 3.1, (3.6), Lemma 3.4, and Sobolev embedding, we have

$$\|\tilde{\rho}_k\|_{L^\infty([0,T];L^\infty)} \lesssim 1.$$

To estimate $\nabla \operatorname{div} \widetilde{V}_k$, that is $\nabla^3 \tilde{\phi}_k$, in L^2, we invoke Lemma 3.4 like above, to have, since $s \leq 2$,

$$\|\nabla \operatorname{div} \widetilde{V}_k\|_{L^\infty([0,T];L^2)} \lesssim 1.$$

Therefore,

$$|\mathrm{III}| \lesssim \varepsilon_k \sqrt{K_k} \lesssim \varepsilon_k^2 + K_k.$$

We then conclude as in the proof of [1, Theorem 4.1] that

$$\frac{\mathrm{d}H_k}{\mathrm{d}t} \lesssim H_k + \varepsilon_k^2,$$

and the proposition follows from a Gronwall argument. ∎

5. Renormalized modulated energy estimate

When $s \geq s_{\mathrm{sob}}$, the approach from Proposition 4.1 is not satisfying since the modulated energy functional is large even at time $t = 0$, due to the kinetic part. It turns out that this initial value is primarily responsible for the failure of the approach: we renormalize the modulated energy functional by removing the initial bubbles $\ell < k$ (those which have large lower-order Sobolev norms). More precisely, we introduce

$$\widetilde{H}_k(t) = \widetilde{K}_k(t) + P_k(t),$$

where we denote the renormalized kinetic part by

$$\widetilde{K}_k(t) = \frac{1}{2}\|(\varepsilon_k \nabla - i \widetilde{V}_k) u_k - \varepsilon_k \nabla \varphi\|_{L^2}^2, \quad \varphi = \sum_{\ell<k} J * \varphi_{\ell,k},$$

and we leave the potential part unchanged. With this choice, at $t = 0$,

$$\widetilde{H}_k(0) \lesssim \varepsilon_k^2.$$

Note that this strategy somehow meets the approach followed in the proof of [13, Proposition 2.6], as noted in Remark 3.3.

Proposition 5.1 (Renormalized modulated energy). *Let $0 < s < s_c$, with $s \leq 2$. For every $t \in [0, T]$, there holds*

$$\widetilde{H}_k(t) \lesssim \varepsilon_k^2.$$

The rest of this section is devoted to the proof of Proposition 5.1. To emphasize the new terms compared to Section 4, we develop

$$\begin{aligned}\widetilde{K}_k(t) &= K_k(t) - L_k(t) + \|\varepsilon_k \nabla\varphi\|_{L^2}^2,\\ L_k(t) &= 2\,\mathrm{Re}\langle(\varepsilon_k\nabla - i\widetilde{V}_k)u_k, \varepsilon_k\nabla\varphi\rangle,\end{aligned}$$

and then we evaluate the time derivative of the two components $K_k(t)$ and $L_k(t)$ separately ($\nabla\varphi$ is obviously time independent).

5.1. Localized Sobolev norms of the WKB ansatz

Let $\chi_k \in C_c^\infty(\mathbb{R}^d)$ such that χ_k localizes in space around the kth bubble $x \approx \frac{x_k}{h_k}$:

$$\begin{aligned}\chi_k &\equiv 1 \quad \text{on } \operatorname{supp}(\jmath * \varphi_{k,k}) = \operatorname{supp}(\jmath * \alpha_k),\\ \chi_k &\equiv 0 \quad \text{on } \operatorname{supp}(\jmath * \varphi_{\ell,k}) \text{ for } \ell \neq k.\end{aligned}$$

On the support of $(1-\chi_k)$, we will use the fact that the Sobolev norms of $(\widetilde{V}_k, \tilde{A}_k)$ are proportional to the restriction of the Sobolev norms of its initial data $(0, (\tilde{\rho}_k)^m)$ to the support of $(1-\chi_k)$, hence the higher-order Sobolev norms are decaying in k. More precisely we establish the following lemma.

Lemma 5.2 (Localized Sobolev norms of the approximate phase). *For every $n \geq 0$, there holds*

$$\|(1-\chi_k)|\nabla|^n \widetilde{V}_k\|_{L^\infty} \lesssim \varepsilon_k^2 h_k^{3+n}.$$

Proof. We note that at time $t = 0$, we have $|\nabla|^n \widetilde{V}_{k|t=0} = 0$. Hence we use equation (3.7) satisfied by $\widetilde{V}_k$:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla|^n \widetilde{V}_k = -|\nabla|^n(\widetilde{V}_k \cdot \nabla\widetilde{V}_k + \nabla(|\tilde{A}_k|^2)).$$

Then we note that on the support of $(1-\chi_k)$, the Sobolev norms of $|\tilde{A}_k|^2$ are proportional to the Sobolev norms of the restriction of its initial data $(\tilde{\rho}_k)^m$ according to Lemma 3.4. Therefore, the equality right below Lemma 3.4 taking into account the localization implies the estimate

$$\|(1-\chi_k)|\nabla|^n\nabla(|\tilde{A}_k|^2)\|_{L^\infty} \lesssim h_k^{1+n+2m(\frac{d}{2}-s)} \lesssim h_k^{3+n}\varepsilon_k^2.$$

Let us now treat the convective part. Similarly, this term vanishes at time $t = 0$, hence we compute one more time derivative:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla|^n(\widetilde{V}_k\cdot\nabla\widetilde{V}_k) = &-|\nabla|^n\big((\widetilde{V}_k\cdot\nabla\widetilde{V}_k + \nabla(|\tilde{A}_k|^2))\cdot\nabla\widetilde{V}_k\big)\\ &-|\nabla|^n\big(\widetilde{V}_k\cdot\nabla(\widetilde{V}_k\cdot\nabla\widetilde{V}_k + \nabla(|\tilde{A}_k|^2))\big).\end{aligned}$$

The terms involving $\tilde{A}_k$ are small thanks to the same argument as before. Regarding the terms involving only $\widetilde{V}_k$, they are of the form

$$\| |\nabla|^n ((\widetilde{V}_k \cdot \nabla \widetilde{V}_k) \cdot \nabla \widetilde{V}_k)\|_{L^\infty} \lesssim h_k^{2+n} h_k^{3m(\frac{d}{2}-s)} \lesssim h_k^{5+n} \varepsilon_k^3,$$

hence the lemma after integration in time. ∎

5.2. Linear part

We compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} L_k(t) &= 2\,\mathrm{Re}\langle (\varepsilon_k \nabla - i\widetilde{V}_k)\partial_t u_k, \varepsilon_k \nabla\varphi\rangle + 2\,\mathrm{Re}\langle -i\partial_t \widetilde{V}_k u_k, \varepsilon_k \nabla\varphi\rangle \\ &= -\mathrm{Re}\langle i\varepsilon_k \Delta u_k, (\varepsilon_k \nabla - i\widetilde{V}_k)\varepsilon_k \nabla\varphi\rangle + 2\,\mathrm{Re}\langle i|u_k|^{2m} u_k, (\varepsilon_k \nabla - i\widetilde{V}_k)\cdot\nabla\varphi\rangle \\ &\quad - 2\,\mathrm{Re}\langle i\partial_t \widetilde{V}_k u_k, \varepsilon_k \nabla\varphi\rangle.\end{aligned}$$

In the subsequent estimates, we leave out the dependence of ε_k upon $\log(h_k)$ (see (2.2)) to lighten the notation, since, as pointed out before, the logarithmic correction is irrelevant when we deal with open algebraic conditions.

- First, we note that

$$|\langle i\varepsilon_k \Delta u_k, \varepsilon_k^2 \Delta\varphi\rangle| \le \|\varepsilon_k \nabla u_k\|_{L^2} \|\varepsilon_k^2 \nabla\Delta\varphi\|_{L^2} \lesssim \varepsilon_k^3 h_k^{4-2s},$$

where we have used (2.5) to estimate $\varepsilon_k \nabla u_k$, and the first part of Lemma 2.2 with $s' = 3$ to estimate $\nabla\Delta\varphi$.

- We now estimate $\langle i|u_k|^{2m} u_k, \varepsilon_k \Delta\varphi\rangle$. First, we decompose

$$\langle i|u_k|^{2m} u_k, \varepsilon_k \Delta\varphi\rangle = \langle i(|u_k|^{2m} - |\tilde{a}_k|^{2m}) u_k, \varepsilon_k \Delta\varphi\rangle + \langle i|\tilde{a}_k|^{2m} u_k, \varepsilon_k \Delta\varphi\rangle.$$

The first term on the right-hand side has the upper bound

$$\left|\langle i(|u_k|^{2m} - |\tilde{a}_k|^{2m}) u_k, \varepsilon_k \Delta\varphi\rangle\right| \lesssim \| |u_k|^{2m} - |\tilde{a}_k|^{2m} \|_{L^{\frac{2m+2}{2m}}} \|u_k\|_{L^{2m+2}} \|\varepsilon_k \Delta\varphi\|_{L^{2m+2}}.$$

This implies, in view of (2.5), the embedding $\dot{H}^{s_{\mathrm{sob}}} \hookrightarrow L^{2m+2}$, Lemma 2.2, and the Young inequality,

$$\begin{aligned}\left|\langle i(|u_k|^{2m} - |\tilde{a}_k|^{2m}) u_k, \varepsilon_k \Delta\varphi\rangle\right| &\lesssim P_k^{\frac{2m}{2m+2}} h_k^{\frac{d}{2}-s-\frac{d}{2m+2}} \varepsilon_k h_k^{2+\frac{d}{2}-s-\frac{d}{2m+2}} \\ &\lesssim P_k + \varepsilon_k^{m+1} h_k^{2(m+1)+2(m+1)(\frac{d}{2}-s)-d} \\ &\lesssim P_k + \varepsilon_k^{m+1} h_k^{2(m+1)+2m(\frac{d}{2}-s)-2s}.\end{aligned}$$

We note that $2m(\frac{d}{2}-s) - 2s = 2 + 2m(s_c - s) - 2s$, so $h_k^{2m(\frac{d}{2}-s)-2s} \lesssim \varepsilon_k^2 h_k^{2-2s}$. Let us now investigate the second term $\langle i|\tilde{a}_k|^{2m} u_k, \varepsilon_k \Delta\varphi\rangle$:

$$\left|\langle i|\tilde{a}_k|^{2m} u_k, \varepsilon_k \Delta\varphi\rangle\right| \lesssim \varepsilon_k \|\tilde{a}_k\|_{L^\infty}^{2m} \|u_k\|_{L^2} \|\Delta\varphi\|_{L^2} \lesssim \varepsilon_k h_k^{2m(\frac{d}{2}-s)+2-2s} \lesssim \varepsilon_k^3 h_k^{4-2s}.$$

We conclude that when $t \lesssim 1$,

$$\left|\langle i|u_k|^{2m}u_k, \varepsilon_k\Delta\varphi\rangle\right| \lesssim P_k + \varepsilon_k^3 h_k^{4-2s}.$$

- Then we note that

$$|\langle\varepsilon_k\Delta u_k, \widetilde{V}_k\cdot\varepsilon_k\nabla\varphi\rangle| = \varepsilon_k^2\left|\langle u_k, \Delta(\widetilde{V}_k\cdot\nabla\varphi)\rangle\right|.$$

As a consequence, since $\nabla\varphi = (1-\chi_k)\nabla\varphi$, using the conservation of the L^2-norm and Lemma 5.2,

$$|\langle\varepsilon_k\Delta u_k, \widetilde{V}_k\cdot\varepsilon_k\nabla\varphi\rangle| \lesssim \varepsilon_k^2 h_k^{3-2s}\varepsilon_k^2 h_k^3.$$

- Moreover, (2.5) and Lemma 5.2 yield

$$\begin{aligned}\left|\langle i|u_k|^{2m}u_k, i\widetilde{V}_k\cdot\nabla\varphi\rangle\right| &\lesssim \|u_k\|_{L^{2m+2}}^{2m+1}\|(1-\chi_k)\widetilde{V}_k\|_{L^\infty}\|\nabla\varphi\|_{L^{2m+2}}\\ &\lesssim \varepsilon_k^2 h_k^{2-2s+1}\varepsilon_k^2 h_k^3.\end{aligned}$$

- Finally, we estimate $\langle i\partial_t\widetilde{V}_k u_k, \varepsilon_k\nabla\varphi\rangle$ on the support of φ: Using equation (3.7) satisfied by $\widetilde{V}_k$ and Lemma 5.2, we have the estimate

$$\|\partial_t\widetilde{V}_k\|_{L^\infty} \lesssim \varepsilon_k^2 h_k^4,$$

and so

$$|\langle i\partial_t\widetilde{V}_k u_k, \varepsilon_k\nabla\varphi\rangle| \lesssim \varepsilon_k^3 h_k^{5-2s}.$$

- In conclusion,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}L_k(t)\right| \lesssim P_k(t) + \varepsilon_k^3 h_k^{4-2s}.$$

5.3. Nonlinear part

We now also replace K_k by $\widetilde{K}_k$ in the upper bounds for estimating $\frac{\mathrm{d}K_k}{\mathrm{d}t}$ that appear in the proof of [1, Theorem 4.1].

- First, we treat the following problematic term evidenced in the proof of Proposition 4.1,

$$I = \varepsilon_k^2\int\nabla(\operatorname{div}\widetilde{V}_k)\cdot\nabla(|u_k|^2).$$

The same argument as in Section 4 would lead to the inequalities

$$\left|\varepsilon_k\int\nabla(\operatorname{div}\widetilde{V}_k)\cdot(\bar{u}_k(\varepsilon_k\nabla u_k - i\widetilde{V}_k u_k - \varepsilon_k\nabla\varphi))\right| \lesssim \varepsilon_k(1+h_k^{2-s})(P_k^{\frac{1}{2m+2}}+1)\sqrt{\widetilde{K}_k},$$

$$\left|\varepsilon_k^2\int\nabla(\operatorname{div}\widetilde{V}_k)\cdot(\bar{u}_k\nabla\varphi)\right| \lesssim \varepsilon_k^2 h_k^{3-2s}(P_k^{\frac{1}{2m+2}}+1).$$

However, since we want to tackle exponents s that may be larger than $\frac{3}{2}$, we refine the argument slightly in order to improve the estimate. Indeed, as we wish s to be arbitrarily close to s_c, the power of ε_k is essentially useless, and we need the remaining power of h_k to be nonnegative.

Lemma 5.2 taken with two derivatives leads to

$$\|(1-\chi_k)\nabla \operatorname{div} \widetilde{V}_k\|_{L^\infty} \lesssim \varepsilon_k^2 h_k^5,$$

and applying this estimate in the formula for

$$I = \varepsilon_k \int \nabla(\operatorname{div} \widetilde{V}_k) \cdot \left(\bar{u}_k(\varepsilon_k\nabla - i\widetilde{V}_k)u_k + u_k\overline{(\varepsilon_k\nabla - i\widetilde{V}_k)u_k}\right),$$

we get

$$|I| \lesssim \widetilde{H}_k + \varepsilon_k^2 h_k^{4-2s}.$$

- We also need to estimate the second problematic term from the proof of Proposition 5.1,

$$I' := -\operatorname{Re}\sum_{i,j}\int \partial_j(\widetilde{V}_k)_i(\varepsilon_k\partial_j u_k - i(\widetilde{V}_k)_j u_k)\overline{(\varepsilon_k\partial_i u_k - i(\widetilde{V}_k)_i u_k)}.$$

Again we decompose $(\varepsilon_k\nabla - i\widetilde{V}_k)u_k = (\varepsilon_k\nabla - i\widetilde{V}_k)u_k - \varepsilon_k\nabla\varphi + \varepsilon_k\nabla\varphi$ to get

$$\begin{aligned}|I'| &\lesssim \widetilde{K}_k + 2\sum_{i,j}\int |\partial_j(\widetilde{V}_k)_i|\,|\varepsilon_k\partial_j\varphi|\,|\varepsilon_k\partial_i u_k - i(\widetilde{V}_k)_i u_k| \\ &\quad + \sum_{i,j}\int |\partial_j(\widetilde{V}_k)_i|\,|\varepsilon_k\partial_j\varphi|\,|\varepsilon_k\partial_i\varphi|.\end{aligned}$$

The second term on the right-hand side can be estimated from the first and third terms on the right-hand side. Finally, Lemma 5.2 implies

$$\|(1-\chi_k)\partial_j\widetilde{V}_k\|_{L^\infty} \lesssim \varepsilon_k^2 h_k^4,$$

so that, since $(1-\chi_k) = 1$ on the support of φ,

$$\int |\partial_j(\widetilde{V}_k)_i|\,|\varepsilon_k\partial_j\varphi|\,|\varepsilon_k\partial_i\varphi| \lesssim \varepsilon_k^2 h_k^{6-2s}.$$

This leads to

$$|I'| \lesssim \widetilde{K}_k + \varepsilon_k^2 h_k^{6-2s}.$$

- In conclusion, we get

$$\left|\frac{\mathrm{d}\widetilde{H}_k}{\mathrm{d}t}\right| \lesssim \widetilde{H}_k + \varepsilon_k^2 h_k^{4-2s},$$

hence Proposition 5.1 by integration.

6. Localized estimates and conclusion

6.1. Localized mass estimate

Like in Section 5, let $\chi_k \in C_c^\infty(\mathbb{R}^d)$ such that χ_k localizes in space around the kth bubble $x \approx \frac{x_k}{h_k}$:

$$\begin{aligned} \chi_k &\equiv 1 \quad \text{on } \operatorname{supp}(\jmath * \varphi_{k,k}) = \operatorname{supp}(\jmath * \alpha_k), \\ \chi_k &\equiv 0 \quad \text{on } \operatorname{supp}(\jmath * \varphi_{\ell,k}) \text{ for } \ell \neq k. \end{aligned}$$

More precisely, we choose a cutoff function χ independent of k, and a scaling parameter $R_k \gtrsim 1$ (which will be crucial in the alternative argument presented in the appendix), and set

$$\chi_k(x) = \chi\Big(\frac{x - x_k}{R_k}\Big).$$

We recall that in view of Lemma 3.1, for $\ell \leq k$ and $t \in [0, T]$,

$$\operatorname{supp} \nabla\phi_{\ell,k}(t,\cdot) \cup \operatorname{supp} \rho_{\ell,k}(t,\cdot) \subset \operatorname{supp} \jmath * \varphi_{\ell,k}.$$

We consider the mass localized near the bubble living at scale k,

$$M_k(t) := \|\chi_k u_k(t)\|_{L^2}^2,$$

and show that it is bounded.

Lemma 6.1 (Localized mass estimate). *There exists $C > 0$ such that for every $t \in [0, T]$ and $k \geq 1$,*

$$M_k(t) \leq C M_k(0) \lesssim 1.$$

Proof. At $t = 0$, we have

$$M_k(0) = \|\jmath * \alpha_k\|_{L^2} = \|\jmath * \alpha\|_{L^2}.$$

Let us now estimate the time derivative of the localized mass. Using the equation satisfied by u_k, (2.3), we get

$$\frac{\mathrm{d}M_k}{\mathrm{d}t} = 2\operatorname{Re}\int \chi_k^2 \bar{u}_k \partial_t u_k = \varepsilon_k \operatorname{Re}\int \chi_k^2 \bar{u}_k i\Delta u_k = 2\varepsilon_k \operatorname{Im}\int \chi_k \bar{u}_k \nabla\chi_k \cdot \nabla u_k.$$

Given that $\nabla\chi_k = 0$ on the support of $\tilde{\phi}_k$ and of φ, we have

$$\frac{\mathrm{d}M_k}{\mathrm{d}t} = 2\operatorname{Im}\int \chi_k \bar{u}_k \nabla\chi_k \cdot ((\varepsilon_k \nabla u_k - i\nabla\tilde{\phi}_k)u_k - \varepsilon_k \nabla\varphi).$$

The Cauchy–Schwarz and the Young inequalities yield

$$\frac{\mathrm{d}M_k}{\mathrm{d}t} \lesssim \|\nabla\chi_k\|_{L^\infty}(\widetilde{H}_k + M_k),$$

where $\widetilde{H}_k$ is small, from Proposition 5.1. The Gronwall lemma implies that for $t \in [0, T]$,

$$M_k(t) \lesssim M_k(0) + o_k(1) \lesssim M_k(0). \qquad \blacksquare$$

6.2. Lower and upper bounds on the $\dot{H}^1$ norm

Lemma 6.2 (Bounds on the local energy). *There exist $C > 0$ and $\tau \in [0, T]$ such that for every $k \geq 1$,*

$$\frac{1}{C} \leq \|(\varepsilon_k \nabla)(\chi_k u_k)(\tau)\|_{L^2} \leq C.$$

Proof. We develop

$$(\varepsilon_k \nabla)(\chi_k u_k) = \varepsilon_k u_k \nabla \chi_k + \chi_k \varepsilon_k \nabla u_k.$$

Let $\tilde{\chi} \in \mathcal{C}_c^\infty$ such that $\tilde{\chi} \geq |\nabla \chi|$ on $\operatorname{supp}(\chi)$, and such that

$$\tilde{\chi}_k = \tilde{\chi}\Big(\frac{x - x_k}{R_k}\Big)$$

also satisfies

$$\begin{aligned} \tilde{\chi}_k &\equiv 1 \quad \text{on } \operatorname{supp}(\jmath * \varphi_{k,k}),\\ \tilde{\chi}_k &\equiv 0 \quad \text{on } \operatorname{supp}(\jmath * \varphi_{\ell,k}), \ell \neq k. \end{aligned}$$

Then Lemma 6.1 applied with $\tilde{\chi}_k$ instead of χ_k yields

$$\|(\nabla \chi_k) u_k(t)\|_{L^2} \lesssim \|\tilde{\chi}_k u_k(t)\|_{L^2} \lesssim 1.$$

Moreover, since φ and χ have disjoint supports, we have the inequality

$$\|\chi_k \varepsilon_k \nabla u_k\|_{L^2} \gtrsim \|\chi_k \widetilde{V}_k u_k\|_{L^2} - \|\chi_k((\varepsilon_k \nabla - i\widetilde{V}_k)u_k - \varepsilon_k \nabla \varphi)\|_{L^2}.$$

Now we argue like in [1, Lemma 5.3]: in view of the coupling in (3.2), there exists $\tau \in [0, T]$ such that

$$\|a(\tau)\nabla \phi(\tau)\|_{L^2} \gtrsim 1,$$

hence, for every k,

$$\|\chi_k \widetilde{V}_k(\tau)\tilde{a}_k(\tau)\|_{L^2} = \|a_k(\tau)\nabla \phi_k(\tau)\|_{L^2} \gtrsim 1.$$

Using that $|\tilde{a}_k|^2 - |u_k|^2$ is small in L^{m+1} thanks to the renormalized modulated energy estimate from Proposition 5.1, we deduce that

$$\|\chi_k \widetilde{V}_k u_k\|_{L^2} \gtrsim 1.$$

Using Proposition 5.1 to estimate the term $\|\chi_k((\varepsilon_k \nabla - i\widetilde{V}_k)u_k - \varepsilon_k \nabla \varphi)\|_{L^2}$ (which is controlled by the renormalized kinetic term), we deduce that

$$\|\chi_k \varepsilon_k \nabla u_k(\tau)\|_{L^2} \gtrsim 1.$$

Note that also

$$\|\chi_k \widetilde{V}_k(t)\tilde{a}_k(t)\|_{L^2} \lesssim 1, \quad \forall t \in [0, T],$$

hence we know that there is also an upper bound with the same order:

$$\|\chi_k \varepsilon_k \nabla u_k(t)\|_{L^2} \lesssim 1, \quad \forall t \in [0, T],$$

hence the lemma. ∎

6.3. Higher-order Sobolev norms

In this section, we fix $0 < \sigma < 2$, and prove that the homogeneous Sobolev norm $\dot{H}^\sigma$ of the solution u_k at time τ, provided by Lemma 6.2, grows like $\varepsilon_k^{-\sigma}$ as $k \to +\infty$. First, in view of Lemmas 6.1 and 6.2, we have, by interpolation, for $\sigma > 1$,

$$1 \lesssim \|(\varepsilon_k \nabla)(\chi_k u_k(\tau))\|_{L^2} \leq \|\chi_k u_k(\tau)\|_{L^2}^{1-1/\sigma} \| \, |\varepsilon_k \nabla|^\sigma (\chi_k u_k(\tau))\|_{L^2}^{1/\sigma}$$
$$\lesssim \| \, |\varepsilon_k \nabla|^\sigma (\chi_k u_k(\tau))\|_{L^2}^{1/\sigma}.$$

For the case $0 < \sigma < 1$, we invoke the following result:

Lemma 6.3 ([1, Lemma 5.1]). *There exists a constant K such that, for all $\varepsilon \in]0,1]$, for all $\sigma \in [0,1]$, for all $u \in H^1(\mathbb{R}^n)$, and for all $v \in W^{1,\infty}(\mathbb{R}^n)$,*

$$\| \, |v|^\sigma u\|_{L^2} \leq \| \, |\varepsilon D_x|^\sigma u\|_{L^2} + \|(\varepsilon\nabla - iv)u\|_{L^2}^\sigma \|u\|_{L^2}^{1-\sigma} + \varepsilon^{\sigma/2} K(1 + \|\nabla v\|_{L^\infty})\|u\|_{L^2}.$$

Applying this lemma to $u = \chi_k u_k$, $v = \nabla \phi_k$, and noticing that $\chi_k \nabla \varphi \equiv 0$ so we may invoke Proposition 5.1, we infer, thanks to Lemma 6.1,

$$\| \, |\varepsilon_k \nabla|^\sigma (\chi_k u_k(\tau))\|_{L^2} \gtrsim 1, \quad \forall \sigma > 0. \tag{6.1}$$

Proposition 6.4 (Norm inflation). *Let $0 < \sigma \leq 2$. Then, with $\tau \in [0,T]$ as in Lemma 6.2,*

$$\| \, |\varepsilon_k \nabla|^\sigma u_k(\tau)\|_{L^2} \gtrsim 1.$$

Roughly speaking, the above result consists in getting rid of the cutoff function in (6.1). In the case $\sigma = 2$, this is direct from the Leibniz formula:

$$\varepsilon_k^2 \Delta(\chi_k u_k) = \varepsilon_k^2 \chi_k \Delta u_k + 2\varepsilon_k^2 \nabla \chi_k \cdot \nabla u_k + \varepsilon_k^2 u_k \Delta \chi_k.$$

The L^2-norm of the second term is $\mathcal{O}(\varepsilon_k)$ from Lemma 6.2, and the last term is $\mathcal{O}(\varepsilon_k^2)$ from Lemma 6.1; hence, in view of (6.1),

$$\|\varepsilon_k^2 \Delta u_k(\tau)\|_{L^2} \geq \|\varepsilon_k^2 \chi_k \Delta u_k(\tau)\|_{L^2} + \mathcal{O}(\varepsilon_k) \gtrsim 1.$$

In the case $\sigma = 1$, the proof follows the same lines. We expand

$$\varepsilon_k \nabla(\chi_k u_k) = \varepsilon_k \chi_k \nabla u_k + \varepsilon_k u_k \nabla \chi_k.$$

The left-hand side is of order 1 thanks to Lemma 6.2, and the second term on the right-hand side is $\mathcal{O}(\varepsilon_k)$ thanks to Lemma 6.1. We deduce that

$$\|\varepsilon_k \nabla u_k(\tau)\|_{L^2} \geq \|\varepsilon_k \chi_k \nabla u_k(\tau)\|_{L^2} \gtrsim 1.$$

To prove Proposition 6.4 in the case $0 < \sigma < 1$ or $1 < \sigma < 2$, we first recall a characterization of the homogeneous Sobolev norms, based on the seminal work [6], which then makes it possible to easily extend the above Leibniz formula to the case of fractional derivatives.

Lemma 6.5. *Let $\sigma \in (0,1) \cup (1,2)$, $\sigma < d/2$. The following equivalence holds:*

$$\|u\|^2_{\dot{H}^\sigma(\mathbb{R}^d)} \approx \iint_{\mathbb{R}^{2d}} |u(x+y) + u(x-y) - 2u(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}y}{|y|^{d+2\sigma}}.$$

In the case $x \in \mathbb{T}^d$, we have a similar result:

$$\|u\|^2_{\dot{H}^\sigma(\mathbb{T}^d)} \approx \iint_{\mathbb{T}^d \times [-1,1]^d} |u(x+y) + u(x-y) - 2u(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}y}{|y|^{d+2\sigma}}.$$

Proof. From the Plancherel equality in x, the quantity on the right-hand side is equal to

$$4 \iint_{\mathbb{R}^{2d}} |\cos(y \cdot \xi) - 1|^2 |\hat{u}(\xi)|^2 \frac{\mathrm{d}\xi\,\mathrm{d}y}{|y|^{d+2\sigma}}.$$

By the homogeneous change of variable $z = y|\xi|$ followed by the change of variable $w = Rz$, where R is the rotation sending $\xi/|\xi|$ to $(1,0,\dots,0)$ in $\mathbb{R}^d$, the integral in y is equal to

$$\begin{aligned}\int_{\mathbb{R}^d} |\cos(y \cdot \xi) - 1|^2 \frac{\mathrm{d}y}{|y|^{d+2\sigma}} &= |\xi|^{2\sigma} \int_{\mathbb{R}^d} \left|\cos\left(z \cdot \frac{\xi}{|\xi|}\right) - 1\right|^2 \frac{\mathrm{d}z}{|z|^{d+2\sigma}} \\ &= |\xi|^{2\sigma} \int_{\mathbb{R}^d} |\cos(w_1) - 1|^2 \frac{\mathrm{d}w}{|w|^{d+2\sigma}}.\end{aligned}$$

This quantity is finite whenever $\sigma \in (0,2)$.

When $x \in \mathbb{T}^d$, the method of proof is similar (see e.g. [3] for the case $\sigma \in (0,1)$), by replacing Fourier transforms with Fourier series. Using the Parseval equality in x, we come up with

$$\int_{[-1,1]^d} |\cos(y \cdot n) - 1|^2 \frac{\mathrm{d}y}{|y|^{d+2\sigma}} =: \zeta_n,$$

for $n \in \mathbb{Z}^d \setminus \{0\}$. Again, by a homogeneous change of variable, we get

$$|n|^{2\sigma} \int_{[-1,1]^d} |\cos(z_1) - 1|^2 \frac{\mathrm{d}z}{|z|^{d+2\sigma}} \le \zeta_n \le |n|^{2\sigma} \int_{\mathbb{R}^d} |\cos(z_1) - 1|^2 \frac{\mathrm{d}z}{|z|^{d+2\sigma}},$$

hence the equivalence of norms, as both integrals are finite. ∎

Note that for $\delta > 0$, the integral analyzed in the proof of the lemma, when restricted to the region $\{|y| > \delta\}$, is controlled by

$$\int_{|y|>\delta} |\cos(y \cdot \xi) - 1|^2 \frac{\mathrm{d}y}{|y|^{d+2\sigma}} \le 4 \int_{|y|>\delta} \frac{\mathrm{d}y}{|y|^{d+2\sigma}} \lesssim \delta^{-2\sigma},$$

and so

$$\iint_{|y|>\delta} |u(x+y) + u(x-y) - 2u(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}y}{|y|^{d+2\sigma}} \lesssim \delta^{-2\sigma} \|u\|^2_{L^2}.$$

We apply these properties to the function $\chi_k u_k$: for $\delta > 0$ to be fixed later, we have

$$\varepsilon_k^{2\sigma} \|\chi_k u_k\|_{\dot H^\sigma}^2 \lesssim \left(\frac{\varepsilon_k}{\delta}\right)^{2\sigma} \|\chi_k u_k\|_{L^2}^2 + \varepsilon_k^{2\sigma} \iint_{|y|<\delta} |(\chi_k u_k)(x+y)+(\chi_k u_k)(x-y)-2(\chi_k u_k)(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}y}{|y|^{d+2\sigma}}.$$

We leave out the index k in order to lighten the notation. For the last integral, we use a discrete form of the Leibniz formula:

$$\begin{aligned}(\chi u)(x+y)+(\chi u)(x-y)-2(\chi u)(x) &= \chi(x)(u(x+y)+u(x-y)-2u(x)) \\ &\quad + (\chi(x+y)-\chi(x))(u(x+y)-u(x)) \\ &\quad + (\chi(x)-\chi(x-y))(u(x)-u(x-y)) \\ &\quad + u(x)(\chi(x+y)+\chi(x-y)-2\chi(x)).\end{aligned}$$

The integral corresponding to the first term on the right-hand side is obviously estimated by

$$\|\chi\|_{L^\infty}^2 \|u\|_{\dot H^\sigma}^2.$$

The second and third terms are similar, and the corresponding integrals are actually equal, through the change of variable $y \mapsto -y$. We choose $\delta > 0$ (independent of k) such that

$$\forall \ell \neq k, \quad (\operatorname{supp}\chi + B(0,3\delta)) \cap \operatorname{supp} a_{\ell,k} = \emptyset,$$

and we pick another cutoff function $\chi_1 \in C_0^\infty(\mathbb{R}^d;[0,1])$ such that $\chi_1 \equiv 1$ on $\operatorname{supp}\chi + B(0,\delta)$, and

$$\operatorname{supp}\chi_1 \subset \operatorname{supp}\chi + B(0,3\delta).$$

We thus have

$$(\chi(x+y)-\chi(x))(u(x+y)-u(x)) = (\chi(x+y)-\chi(x))((\chi_1 u)(x+y)-(\chi_1 u)(x)).$$

We infer

$$\begin{aligned}&\iint_{|y|<\delta} |(\chi(x+y)-\chi(x))(u(x+y)-u(x))|^2 \frac{\mathrm{d}x\,\mathrm{d}y}{|y|^{d+2\sigma}} \\ &\quad \lesssim \|\nabla\chi\|_{L^\infty}^2 \iint_{|y|<\delta} |(\chi_1 u)(x+y)-(\chi_1 u)(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}y}{|y|^{d+2\sigma-2}} \\ &\quad \lesssim \|\nabla\chi\|_{L^\infty}^2 \|\chi_1 u\|_{\dot H^{\max(0,\sigma-1)}}^2,\end{aligned}$$

where we have used the more standard characterization of the $\dot H^{\sigma-1}$-norm (note that $\sigma - 1 < 1$) analogous to the one given in Lemma 6.5.

Finally, the last integral is equal to

$$\iint_{|y|<\delta} |(\chi_1 u)(x)|^2 |\chi(x+y)+\chi(x-y)-2\chi(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}y}{|y|^{d+2\sigma}} \lesssim \|\chi\|_{W^{2,\infty}}^2 \|\chi_1 u\|_{L^2}^2,$$

since, for $\delta \leq 1$, using that $2\sigma - 3 < 1$,

$$\int_{|y|<\delta} \frac{\mathrm{d}y}{|y|^{d+2\sigma-4}} \lesssim \int_0^1 \frac{r^{d-1}}{r^{d+2\sigma-4}} \,\mathrm{d}r < \infty.$$

We conclude that

$$1 \lesssim \varepsilon^{2\sigma} \|\chi u(\tau)\|_{\dot{H}^\sigma}^2 \lesssim \varepsilon^{2\sigma} \|u(\tau)\|_{\dot{H}^\sigma}^2 + \underbrace{\varepsilon^{2\sigma}(\|\chi_1 u(\tau)\|_{L^2}^2 + \|\chi_1 u(\tau)\|_{\dot{H}^{\max(0,\sigma-1)}}^2)}_{=\mathcal{O}(\varepsilon^{2\sigma})},$$

hence

$$\varepsilon^{2\sigma} \|u\|_{\dot{H}^\sigma}^2 \gtrsim 1, \quad \forall \sigma \in (0,1) \cup (1,2).$$

6.4. Conclusion

We can now go back to the original function

$$\psi_k(t,x) = h_k^{s-\frac{d}{2}} \frac{1}{|\log(h_k)|} u_k\Big(\frac{t}{\varepsilon_k h_k^2}, \frac{x}{h_k}\Big).$$

This function satisfies

$$\| \, |\nabla|^\sigma \psi_k(\varepsilon_k h_k^2 t)\|_{L^2} = h_k^{s-\sigma} |\log(h_k)|^{-1} \| \, |\nabla|^\sigma u_k(t)\|_{L^2}.$$

Let $t_k = \varepsilon_k h_k^2 \tau$, where τ stems from Lemma 6.2: $t_k \to 0$ as $k \to +\infty$. In view of Proposition 6.4, we know that when $0 < \sigma \leq 2$,

$$\| \, |\nabla|^\sigma \psi_k(t_k)\|_{L^2} \gtrsim h_k^{s-\sigma} \varepsilon_k^{-\sigma} |\log(h_k)|^{-1}.$$

Given the choice $\varepsilon_k = h_k^{m(s_c-s)} |\log(h_k)|^m$, the lower bound goes to infinity as k goes to infinity provided that

$$s < \sigma(1 + m(s_c - s)).$$

Finally, the density of initial data f_0 in the pathological set is a direct consequence of [13, Proposition 2.10].

The proof is readily adapted to the periodic case $x \in \mathbb{T}^d$: the main constraint in the construction of the initial data f_0 on $\mathbb{R}^d$ was that φ_0 and all the φ_k had disjoint supports. In the periodic case, this condition is easily fulfilled provided that the support of φ_0 does not cover the whole of $\mathbb{T}^d$, as mentioned in Section 2.1. Then we can resume the previous proof step by step, and conclude thanks to the second part of Lemma 6.5.

A. Alternative proof of localized estimates

In this appendix, we provide an alternative proof of Proposition 6.4. It has the drawback of being longer than in Section 6.3, but the advantage of being more flexible, and easily adapted to the case of geometries different from $\mathbb{R}^d$ or $\mathbb{T}^d$.

A.1. Tuning the size of cutoff functions

We provide a proof that relies on the sizes of the cutoff R_k around the bubble k, as introduced in Section 6. We first show that up to some conditions on R_k, norm inflation occurs. In a second step, we prove that such conditions on R_k can be met even on compact manifolds.

Lemma A.1 (Size of the cutoff functions). *Let $\sigma \in (0,1) \cup (1,2)$. There exist $C, C' > 0$ such that for every $t \lesssim 1$ and $k \geq 1$, assuming that*

$$\begin{cases} R_k^\sigma \geq C' h_k^{-s} \varepsilon_k^\sigma & \text{if } 0 < \sigma < 1, \\ R_k^{\sigma-1} \geq C' h_k^{1-s} \varepsilon_k^\sigma h_{k-1}^{(m+1)(s_{\mathrm{sob}}-s)} & \text{if } 1 < \sigma < 2, \end{cases}$$

then there holds

$$\| |\varepsilon_k \nabla|^\sigma u_k \|_{L^2} \gtrsim 1.$$

The conclusion of the above lemma corresponds to that of Proposition 6.4, and from that the proof of norm inflation is inferred like in Section 6.4.

Proof of Lemma A.1. Recall that in view of (6.1),

$$\| |\varepsilon_k \nabla|^\sigma (\chi_k u_k(\tau)) \|_{L^2} \gtrsim 1, \quad \forall \sigma > 0.$$

We first treat the case $0 < \sigma < 1$, and write

$$|\nabla|^\sigma (\chi_k u_k) = [|\nabla|^\sigma, \chi_k] u_k + \chi_k |\nabla|^\sigma u_k.$$

Using the commutator estimate recalled in Lemma A.3 below,

$$\| [|\nabla|^\sigma, \chi_k] u_k \|_{L^2} \lesssim \frac{\|u_k\|_{L^2}}{R_k^\sigma}.$$

Given the nonlocalized estimate (2.4) on the L^2-norm of u_k, the upper bound is less than $\frac{\varepsilon_k^{-\sigma}}{2C}$ provided that, for some $C' > 0$, we have

$$R_k^\sigma \geq C' h_k^{-s} \varepsilon_k^\sigma.$$

In this case we conclude that

$$\| \chi_k |\nabla|^\sigma u_k(\tau) \|_{L^2} \gtrsim \varepsilon_k^{-\sigma}.$$

In the case $1 < \sigma < 2$, we expand

$$\begin{aligned} |\nabla|^\sigma (\chi_k u_k) &= |\nabla|^{\sigma-1} (\nabla \chi_k u_k + \chi_k \nabla u_k) \\ &= |\nabla|^{\sigma-1} (\nabla \chi_k u_k) + [|\nabla|^{\sigma-1}, \chi_k] \nabla u_k + \chi_k |\nabla|^\sigma u_k. \end{aligned}$$

Using interpolation, we have

$$\| |\nabla|^{\sigma-1}(\nabla\chi_k u_k)\|_{L^2} \lesssim \|\nabla\chi_k u_k\|_{L^2}^{2-\sigma}\|\nabla(\nabla\chi_k u_k)\|_{L^2}^{\sigma-1}.$$

Thanks to Lemmas 6.1 and 6.2 applied to the cutoff function $\nabla\chi_k$, this term is uniformly bounded in k when $t\in[0,T]$. Concerning the second term in the expansion, we apply the commutator estimate from Lemma A.3 with $\alpha=\sigma-1$ and get

$$\|[|\nabla|^{\sigma-1},\nabla\chi_k]\nabla u_k\|_{L^2} \lesssim \frac{\|\nabla u_k\|_{L^2}}{R_k^{\sigma-1}}.$$

As a consequence of the energy estimate (2.5), if

$$R_k^{\sigma-1} \geq C' h_k^{1-s}\varepsilon_k^\sigma h_{k-1}^{(m+1)(s_{\mathrm{sob}}-s)},$$

for some $C'>0$ large enough, we get

$$\|\chi_k|\nabla|^\sigma u_k(\tau)\|_{L^2} \gtrsim \varepsilon_k^{-\sigma}.$$

In the case $\sigma=2$, the argument presented in Section 6.3, based on the Leibniz formula and localized estimates in L^2 and $\dot H^1$, respectively, needs no modification. ∎

We now check that the condition on the size of R_k from the above lemma can be realized with a suitable choice of positions $\frac{x_k}{h_k}$ for the bubbles.

Lemma A.2 (Fitting the bubbles on the torus). *One can fix x_k such that there exists R_k satisfying $\bar B(\frac{x_k}{h_k},R_k)\cap\operatorname{supp}(\jmath*\varphi_{\ell,k})=\emptyset$ for $\ell\neq k$ and*

$$\begin{cases} R_k^\sigma \geq C' h_k^{-s}\varepsilon_k^\sigma & \text{if } 0<\sigma<1,\\ R_k^{\sigma-1} \geq C' h_k^{1-s}\varepsilon_k^\sigma h_{k-1}^{(m+1)(s_{\mathrm{sob}}-s)} & \text{if } 1<\sigma<2.\end{cases}$$

Under these conditions, it is possible to construct a cutoff χ_k of radius R_k around the kth bubble located around position $\frac{x_k}{h_k}$ in the rescaled variables for u_k.

Proof of Lemma A.2. We recall that in the original variables, we have

$$\psi_k(t,x) = h_k^{s-\frac{d}{2}}|\log(h_k)|^{-1}u\Big(\frac{t}{\varepsilon_k h_k^2},\frac{x}{h_k}\Big).$$

In the original variables, the cutoff χ_k corresponds to a cutoff of size $h_k R_k$ around position x_k. Moreover, $h_k=e^{-M^k}$ for some parameter $M>1$. As a consequence, it is possible to fit all the bubbles in the torus if there exists $\delta>0$ such that

$$h_k R_k \lesssim h_k^\delta.$$

In the case $0<\sigma<1$, we combine this condition with the lower bound and R_k and we deduce that this is possible if

$$h_k^{\sigma(\delta-1)} \gtrsim c\varepsilon_k^\sigma h_k^{-s}.$$

Given the formula $\varepsilon_k = h_k^{m(s_c-s)}|\log(h_k)|^m$, it is enough that

$$\sigma(\delta - 1) < \sigma m(s_c - s) - s,$$

that is,

$$\sigma(1 + m(s_c - s) - \delta) > s.$$

We check that when

$$\sigma(1 + m(s_c - s)) > s,$$

then it is always possible to choose $\delta > 0$ small enough so that this condition is satisfied.

In the case $1 < \sigma \leq 2$, this condition is compatible with the lower bound on R_k if

$$h_k^{(\sigma-1)(\delta-1)} \gtrsim c\varepsilon_k^{\sigma} h_k^{1-s} h_{k-1}^{(m+1)(s_{\text{sob}}-s)}.$$

Given the formula for ε_k, it is enough that

$$(\sigma - 1)(\delta - 1) < \sigma m(s_c - s) + 1 - s,$$

that is,

$$\sigma(1 + m(s_c - s) - \delta) > s - \delta.$$

We check that when

$$\sigma(1 + m(s_c - s)) > s,$$

then it is always possible to choose $\delta > 0$ small enough so that this condition is satisfied. ∎

A.2. Commutator estimate

Lemma A.3 (Commutator estimate). *For every* $0 < \alpha < 1$, *there holds*

$$\|[|\nabla|^\alpha, \chi_k] f\|_{L^2} \lesssim \frac{\|\chi\|_{W^{1,\infty}}}{R_k^\alpha} \|f\|_{L^2}.$$

We give an elementary proof of this result, which is adapted from [18, Lemma D.1].

Proof. The operator $|\nabla|^\alpha$ is the convolution operator with the tempered distribution $\kappa = \mathcal{F}^{-1}(|\xi|^\alpha)$, which is homogeneous of degree $-\alpha - d$ and even. We deduce that there exists a universal constant c such that

$$|\nabla|^\alpha f(x) = \kappa * f(x) = c \int_{\mathbb{R}^d} \frac{f(y) - f(x)}{|y - x|^{\alpha+d}} \, \mathrm{d}y.$$

As a consequence,

$$[|\nabla|^\alpha, \chi_k] f = c \int_{\mathbb{R}^d} \frac{\chi_k(y) - \chi_k(x)}{|y - x|^{\alpha+d}} f(y) \, \mathrm{d}y.$$

We split the integral between the parts $|y-x| \le R_k$ and $|y-x| > R_k$. First, using the fact that χ_k is Lipschitz continuous,

$$\frac{|\chi_k(y)-\chi_k(x)|}{|y-x|^{\alpha+d}} \le \frac{\|\nabla\chi_k\|_{L^\infty}}{|y-x|^{\alpha+d-1}},$$

so that

$$\begin{aligned}\left\|\int_{|y-x|\le R_k}\frac{\chi_k(y)-\chi_k(x)}{|y-x|^{\alpha+d}}f(y)\,\mathrm{d}y\right\|_{L^2_x} &\lesssim \|\nabla\chi_k\|_{L^\infty}\left\|\frac{\mathbf{1}_{|x|\le R_k}}{|x|^{\alpha+d-1}} * f\right\|_{L^2}\\ &\lesssim \frac{\|\nabla\chi\|_{L^\infty}}{R_k}\left\|\frac{\mathbf{1}_{|x|\le R_k}}{|x|^{\alpha+d-1}}\right\|_{L^1}\|f\|_{L^2}\\ &\lesssim \frac{\|\nabla\chi\|_{L^\infty}}{R_k^\alpha}\left\|\frac{\mathbf{1}_{|x|\le 1}}{|x|^{\alpha+d-1}}\right\|_{L^1}\|f\|_{L^2}.\end{aligned}$$

The term $\|\frac{\mathbf{1}_{|x|\le 1}}{|x|^{\alpha+d-1}}\|_{L^1}$ is finite since $\alpha+d-1<d$. Then we have

$$\begin{aligned}\left\|\int_{|y-x|> R_k}\frac{\chi_k(y)-\chi_k(x)}{|y-x|^{\alpha+d}}f(y)\,\mathrm{d}y\right\|_{L^2_x} &\lesssim \|\chi_k\|_{L^\infty}\left\|\frac{\mathbf{1}_{|x|> R_k}}{|x|^{\alpha+d}} * f\right\|_{L^2}\\ &\lesssim \|\chi\|_{L^\infty}\left\|\frac{\mathbf{1}_{|x|> R_k}}{|x|^{\alpha+d}}\right\|_{L^1}\|f\|_{L^2}\\ &\lesssim \frac{\|\chi\|_{L^\infty}}{R_k^\alpha}\left\|\frac{\mathbf{1}_{|x|> 1}}{|x|^{\alpha+d}}\right\|_{L^1}\|f\|_{L^2}.\end{aligned}$$

The L^1-norm is finite since $\alpha+d>d$. ■

B. Proof of Theorem 1.1

As mentioned in the introduction, the proof of Theorem 1.2 is readily adapted in order to prove Theorem 1.1. Let $f_0 \in H^s(\mathbb{R}^d)$: first, there exists a sequence $g_{0,k} \in \mathcal{C}_c^\infty(R^d)$ such that

$$\|f_0 - g_{0,k}\|_{H^s} \underset{k\to\infty}{\longrightarrow} 0.$$

We then complement the initial datum $g_{0,k}$ with a single bubble,

$$f_{0,k} = g_{0,k} + \varphi_k,$$

where φ_k is like in the rest of the paper, as introduced in Section 2.1, with the requirement that for any k, the supports of $g_{0,k}$ and φ_k are disjoint. In view of the logarithmic factor $|\log(h_k)|^{-1}$ in (2.1),

$$\|f_0 - f_{0,k}\|_{H^s} \le \|f_0 - g_{0,k}\|_{H^s} + \|\varphi_k\|_{H^k} \underset{k\to\infty}{\longrightarrow} 0.$$

The proof of Theorem 1.2 can then be repeated, by rescaling the unknown function as in Section 2.2: the initial datum $u_{0,k}$ is the sum of the profile α (possibly shifted in space), and a unique "low mode" stemming from $g_{0,k}$. The modulated energy estimate from Section 5 remains valid (note that the low mode stemming from $g_{0,k}$ must be incorporated into the renormalized modulated energy, in the spirit of the proof of [32, Theorem1.33], and as in [13, 29]), and we conclude like in Section 6.

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