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# *Short note* Jacobsthal analogues of two Fibonacci–Lucas identities and a generalization

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## 1 Two Fibonacci–Lucas identities

The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ and its companion Lucas sequence $2, 1, 3, 4, 7, 11, 18, \ldots$ are defined by the same recurrence relation but with different initial conditions

$$F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n \quad \text{and} \quad L_0 = 2,\ L_1 = 1,\ L_{n+2} = L_{n+1} + L_n.$$

In [14], B. Sury proved the nice Fibonacci–Lucas relation

$$2^{n+1} F_{n+1} = \sum_{i=0}^{n} 2^i L_i \tag{1}$$

using a polynomial identity. Kwong uses in [9] an alternate proof via generating functions, and earlier, Benjamin and Quinn [2] proved (1) through colored tilings (see also [3, Identity 236]). It should be noted that, after the publication of the proof of identity (1) in [14], many extensions and generalizations of (1) appeared in the mathematical literature (see [1, 5–7, 10–13], etc.).

One of the first extensions of identity (1) is Martinjak's [11] Fibonacci–Lucas identity

$$(-1)^n F_{n+1} = \sum_{i=0}^{n} (-1)^i 2^{n-i} L_{i+1}. \tag{2}$$

The goal of this note is to obtain analogues of these identities for the Jacobsthal sequence and its companion the Jacobsthal–Lucas sequence and to extend (1) to generalized Fibonacci–Lucas numbers.

## 2 The Jacobsthal analogues

The Jacobsthal and the Jacobsthal–Lucas sequences $0, 1, 1, 3, 5, 11, 21, 43, \ldots$ and $2, 1, 5, 7, 17, 31, 65, \ldots$, respectively, are defined by

$$J_0 = 0,\ J_1 = 1,\ J_{n+2} = J_{n+1} + 2J_n \quad \text{and} \quad j_0 = 2,\ j_1 = 1,\ j_{n+2} = j_{n+1} + 2j_n.$$

As we shall prove, the Jacobsthal analogue of identity (2) has the following two versions, one for even and one for odd terms:

$$(-1)^n J_{2(n+1)} = \sum_{i=0}^{n} (-1)^i 2^{n-i} j_{2i+1}, \tag{3a}$$

$$(-1)^n J_{2n+1} = -2^n + \sum_{i=0}^{n} (-1)^i 2^{n-i} j_{2i}, \tag{3b}$$

and the Jacobsthal analogue of identity (1) is

$$2^{n+1} J_{n+1} = \sum_{i=0}^{n} 2^i j_i. \tag{4}$$

A striking similarity between these Fibonacci–Lucas identities and their Jacobsthal analogues (especially between (1) and (4)) can be recognized. In what follows, we shall present two significantly different proofs of identities (3a)–(3b) and (4), one proof characterized by its simplicity, and the other one by its specificity.

## 3 A simple proof of Jacobsthal analogues (3a) and (3b)

We begin with the simple proofs of the Jacobsthal analogues (3a) and (3b).

In the case of Fibonacci and Lucas numbers, the basic connected identity is $L_{n+1} = F_n + F_{n+2}$. The corresponding connected identity between Jacobsthal and Jacobsthal–Lucas numbers is the following one:

$$j_{n+1} = 2J_n + J_{n+2}.$$

Thus, using this identity and rearranging the parentheses, we obtain

$$\begin{aligned}
&\sum_{i=0}^{n} (-1)^i 2^{n-i} j_{2i+1} \\
&\quad = 2^n j_1 - 2^{n-1} j_3 + 2^{n-2} j_5 - \cdots + (-1)^{n-1} 2 j_{2n-1} + (-1)^n j_{2n+1} \\
&\quad = 2^n J_1 - 2^{n-1}(2J_2 + J_4) + 2^{n-2}(2J_4 + J_6) - \cdots \\
&\qquad\quad + (-1)^{n-1} 2(2J_{2n-2} + J_{2n}) + (-1)^n (2J_{2n} + J_{2n+2}) \\
&\quad = 2^n (J_1 - J_2) - 2^{n-1}(J_4 - J_4) + \cdots + (-1)^{n-1} 2(J_{2n} - J_{2n}) + (-1)^n J_{2n+2} \\
&\quad = (-1)^n J_{2(n+1)}
\end{aligned}$$

since $j_1 = J_1 = J_2 = 1$.

We proceed in a similar way in the case of (3b):

$$\begin{aligned}\sum_{i=0}^{n}(-1)^i 2^{n-i} j_{2i} &= 2^n j_0 + \sum_{i=1}^{n}(-1)^i 2^{n-i}(2J_{2i-1} + J_{2i+1}) \\ &= 2^n(j_0 - J_1) + \sum_{i=1}^{n-1}(-1)^i 2^{n-i}(J_{2i+1} - J_{2i+1}) + (-1)^n J_{2n+1} \\ &= 2^n + (-1)^n J_{2n+1}.\end{aligned}$$

## 4 A proof of the Jacobsthal analogue (4)

We give now a proof of identity (4) using Euler's Telescoping Lemma.

In 2011, Bhatnagar [4] reformulated an identity used by Euler in his proof of the Pentagon Number Theorem. This new version was called by Bhatnagar "Euler's Telescoping Lemma".

Bhatnagar, by examining the finite version of the product $\prod_1^\infty(1 + x_i)$, that is,

$$\begin{aligned}&(1 + x_1)(1 + x_2)\cdots(1 + x_n) \\ &\quad = (1 + x_1) + (1 + x_1)x_2 + \cdots + (1 + x_1)\cdots(1 + x_{n-1})x_n,\end{aligned}$$

in short,

$$\prod_{i=1}^{n}(1 + x_i) = 1 + x_1 + \sum_{i=2}^{n}(1 + x_1)\cdots(1 + x_{i-1})x_i,$$

and by setting $1 + x_i = \frac{y_i}{z_i}$ $(z_i \neq 0)$ obtains the following lemma.

**Lemma 1** (Euler's Telescoping Lemma [4, Equation (3.4)], [5, Equation (2.1)]).

$$\prod_{i=1}^{n}\frac{y_i}{z_i} = \frac{y_1}{z_1} + \sum_{i=2}^{n}\frac{y_1 y_2\cdots y_{i-1}}{z_1 z_2\cdots z_{i-1} z_i}(y_i - z_i) \quad (z_i \neq 0). \tag{5}$$

Now, it is straightforward to check that the Jacobsthal analogue (4) of identity (1) is obtained when $y_i = 2J_{i+1}$ and $z_i = J_i$. Thus,

$$\prod_{i=1}^{n}\frac{y_i}{z_i} = 2^n J_{n+1} = \frac{2^{n+1}J_{n+1}}{2} \quad (\text{since } z_1 = J_1 = 1),$$

and

$$\begin{aligned}\frac{y_1}{z_1} + \sum_{i=2}^{n}\frac{y_1 y_2\cdots y_{i-1}}{z_1 z_2\cdots z_{i-1} z_i}(y_i - z_i) &= 2 + \sum_{i=2}^{n}2^{i-1}(2J_{i+1} - J_i) \\ &= 1 + 1 + \sum_{i=2}^{n}2^{i-1}j_i = 1 + \sum_{i=1}^{n}2^{i-1}j_i \\ &= \frac{\sum_{i=0}^{n}2^i j_i}{2}\end{aligned}$$

since $2J_{i+1} - J_i = J_{i+1} + (J_{i+1} - J_i) = J_{i+1} + 2J_{i-1} = j_i$. The proof is now complete.

## 5 A generalization of identity (1)

Let $a$ and $b$ be two fixed real numbers. Kalman and Mena [8] denote by $\mathcal{R}(a,b)$ the set of all sequences $\{A_n\}_{n\geq 0}$ with initial terms $A_0$, $A_1$ and all succeeding terms given by

$$A_{n+2} = aA_{n+1} + bA_n.$$

The space $\mathcal{R}(a,b)$ is a subspace of $\mathbb{R}^\infty$ and every subspace $\mathcal{R}(a,b)$ contains two distinguished elements called by Kalman and Mena [8] the $(a,b)$-Fibonacci sequence denoted $\{F_n^{a,b}\}_{n\geq 0}$, and the $(a,b)$-Lucas sequence denoted $\{L_n^{a,b}\}_{n\geq 0}$. The initial terms of the first sequence are $F_0^{a,b} = 0$, $F_1^{a,b} = 1$, and the second starts with $L_0^{a,b} = 2$, $L_1^{a,b} = a$. For example, the ordinary Fibonacci and Lucas sequences are the $(1,1)$-Fibonacci and the $(1,1)$-Lucas sequences, respectively. The Jacobsthal sequence is the $(1,2)$-Fibonacci sequence and the Jacobsthal–Lucas sequence is the $(1,2)$-Lucas sequence.

Now, the fundamental connectivity between the $(a,b)$-Fibonacci numbers and the $(a,b)$-Lucas numbers is given by (see [8, Equation (5)])

$$L_{n+1}^{a,b} = bF_n^{a,b} + F_{n+2}^{a,b}.$$

Since

$$\begin{aligned}
2F_{i+1}^{a,b} - F_i^{a,b} &= F_{i+1}^{a,b} + (F_{i+1}^{a,b} - aF_i^{a,b}) + (a-1)F_i^{a,b} \\
&= F_{i+1}^{a,b} + bF_{i-1}^{a,b} + (a-1)F_i^{a,b} = L_i^{a,b} + (a-1)F_i^{a,b}
\end{aligned}$$

for $i \geq 1$, using Euler's Telescoping Lemma (5) with $y_i = 2F_{i+1}^{a,b}$ and $z_i = F_i^{a,b}$, and following step by step the proof of the Jacobsthal analogue (4), we obtain the following generalization:

$$2^{n+1}F_{n+1}^{a,b} = \sum_{i=0}^{n} 2^i [L_i^{a,b} + (a-1)F_i^{a,b}]. \tag{6}$$

## 6 Two examples

The Pell numbers $P_n$ and the Pell–Lucas numbers $Q_n$ are the $(2,1)$-Fibonacci and the $(2,1)$-Lucas numbers, respectively. Following (6), the Pell analogue of the Fibonacci–Lucas identity (1) is

$$2^{n+1}P_{n+1} = \sum_{i=0}^{n} 2^i (Q_i + P_i).$$

The Mersenne numbers $M_n = 2^n - 1$ are the $(3,-2)$-Fibonacci numbers and the Mersenne–Lucas numbers $m_n = 2^n + 1$ are the $(3,-2)$-Lucas numbers. Following (6), the Mersenne analogue of the Fibonacci–Lucas identity (1) is

$$2^{n+1}M_{n+1} = \sum_{i=0}^{n} 2^i (m_i + 2M_i).$$

Other examples and extensions can be given and we encourage the reader to find them using Euler's Telescoping Lemma.

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