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Short note

How many stones can we place in empty baskets?

Cezar Lupu

Abstract. In this short note, we derive a precise estimate on the number of stones which can be placed in empty baskets. Although the problem posed is combinatorial in nature, the precise estimate involves some techniques arising in analytic number theory such as average sum of arithmetic functions.

Dedicated to the memory of my best friend, Octavian Ganea

1 Introduction

At the 1st International Mathematics Summer Camp (IMSC23) competition which was held at the end of the camp that took place in Beijing between June 18th and July 6th 2023, the following problem was proposed by Kaarel Hänni from Estonia.

Problem 2, IMSC23. There are $n!$ empty baskets in a row, labelled $1, 2, \ldots, n!$. John first puts a stone in every basket. John then puts two stones in every second basket. John continues similarly until he has put n stones into every nth basket. In other words, for each $i = 1, 2, \ldots, n$, John puts i stones into the baskets labelled $i, 2i, 3i, \ldots, n!$. Let x_i be the number of stones in basket i after all these steps. Show that

$$n! \cdot n^2 \leq \sum_{i=1}^{n!} x_i^2 \leq n! \cdot n^2 H_n.$$

Here H_n stands of the nth harmonic number. As one can see, the double inequality is not sharp enough. In this sense, we prove the following refined version of the above inequality which is given by the following theorem.

Theorem 1. *We have the following:*

$$\frac{1}{n!} \sum_{i=1}^{n!} x_i^2 = \frac{6n^2 \log n}{\pi^2} + O(n^2).$$

Based on this theorem, it is quite clear that the right-hand side inequality from the problem is closer to our estimate since $H_n \sim \log n$, $n \to \infty$. In another direction, it turns out that Theorem 1 will reduce to a number theoretical estimate which will involve the greatest common divisor of two positive integers. For that matter, we will employ the average sum of Euler's totient function which is a standard tool in any introductory analytic number theory course.

2 Proof of Theorem 1

First, let us observe that, at the ith step, John places a total of $i\frac{n!}{i} = n!$ stones into the baskets. Hence, after all n steps, there are $n!\,n$ stones in the baskets. Now, since we can write $x_i = \sum_{j|i} j$, we have

$$\sum_i x_i^2 = \sum_i \sum_{j,k=1,\, j|i,\, k|i}^{n} jk.$$

Furthermore, for fixed j and k, we have jk appearing in the index i term of the first sum if and only if j divides i and k divides i. This is valid if and only if $\mathrm{lcm}(j,k) \mid i$. The number of such $i \in \{1,\dots,n!\}$ is exactly $\frac{n!}{\mathrm{lcm}(j,k)}$, where lcm is the least common multiple. Switching the order of summation to sum first by (j,k), we obtain

$$\sum_i x_i^2 = \sum_{j,k=1}^{n} jk\frac{n!}{\mathrm{lcm}(j,k)} = n!\sum_{j,k=1}^{n} \gcd(j,k).$$

Claim. *We claim that*

$$\sum_{i,j=1}^{n} \gcd(i,j) = \frac{n^2\log n}{\zeta(2)} + O(n^2).$$

Proof of the claim. Indeed, we write

$$\sum_{i,j=1}^{n} \gcd(i,j) = \sum_{k\le n} k \sum_{1\le i,j\le n,\, (i,j)=k} 1 = \sum_{k\le n} k \sum_{1\le i,j\le \frac{n}{k},\, (i,j)=1} 1.$$

Furthermore, this is equal to

$$\sum_{i,j=1}^{n} \gcd(i,j) = \sum_{k\le n} k\Big[\Big(2\sum_{1\le i\le j\le \frac{n}{k},\, (i,j)=1} 1\Big] - 1\Big) = 2\sum_{k\le n} k\sum_{j\le \frac{n}{k}} \varphi(j) - \sum_{k\le n} k.$$

By Theorem 2 (see in the appendix), it follows that

$$\begin{aligned}\sum_{i,j=1}^{n} \gcd(i,j) &= 2\sum_{k\le n} k\Big[\frac{3}{\pi^2}\Big(\frac{n}{k}\Big)^2 + O\Big(\frac{n}{k}\log\frac{n}{k}\Big)\Big] + O(n^2)\\ &= \frac{6n^2}{\pi^2}\sum_{k\le n}\frac{1}{k} + O\Big(\sum_{k\le n} n(\log n - \log k)\Big) + O(n^2)\\ &= \frac{n^2}{\zeta(2)}\sum_{k\le n}\frac{1}{k} + O\Big(\sum_{k\le n} n\log n - n\sum_{k\le n}\log k\Big) + O(n^2).\end{aligned}$$

As a direct application of the Abel summation formula, we have

$$\sum_{k\le n}\frac{1}{k}=\log n+\gamma+O\Big(\frac{1}{n}\Big)\quad\text{and}\quad\sum_{k\le n}\log k=n\log n-n+O(\log n).$$

Putting everything together, we obtain

$$\begin{aligned}\sum_{i,j=1}^{n}\gcd(i,j)&=\frac{n^2\log n}{\zeta(2)}+O(n^2)+O\Big(n^2\log n-n\sum_{k\le n}\log k\Big)\\&=\frac{n^2\log n}{\zeta(2)}+O(n^2).\end{aligned}$$ ∎

Now, going back to the proof of the main theorem, we have finally

$$\frac{1}{n!}\sum_i x_i^2=\sum_{j,k=1}^{n}\gcd(j,k)=\frac{n^2\log n}{\zeta(2)}+O(n^2),$$

and the proof ends here. ∎

Remark. The right-hand side inequality from the competition problem can be obtained as follows. First, note that, for some fixed $i\in\{1,2,\ldots,n\}$ to be $\gcd(j,k)$, it is necessary (but not sufficient) that $i\mid j$ and $i\mid k$. Hence, for a fixed $i\in\{1,2.\ldots,n\}$, the number of such pairs $(j,k)\in\{1,2.\ldots,n\}\times\{1,2.\ldots,n\}$ is at most $\frac{n^2}{i^2}$, and this gives us

$$\sum_i x_i^2\le n!\sum_{i=1}^{n}i\frac{n^2}{i^2}=n!\,n^2\sum_{i=1}^{n}\frac{1}{i}.$$

Appendix

Given an arithmetic function $f(n)$, one of the main problems is to study the asymptotic behaviour of $f(n)$ when n is large enough. Since $f(n)$ does not behave regularly in general, we would study the behaviour of the arithmetic mean,

$$M(f(n))=\frac{1}{n}\sum_{k=1}^{n}f(k)$$

for n large enough. This implies that we need to study the partial sum $\sum_{k=1}^{n}f(k)$. Sometimes, it is more convenient to replace the upper index n by an arbitrary positive real number x and to consider the sum

$$\sum_{n\le x}f(n).$$

In this sense, we have the following asymptotic estimate for Euler's totient function which is defined as

$$\varphi(n)=\sum_{1\le k\le n,\,(k,n)=1}1=\sum_{1\le k\le n}\Big\lfloor\frac{1}{(k,n)}\Big\rfloor.$$

Theorem 2. *For all $x > 1$, we have*

$$\sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x \log x).$$

The proof of Theorem 2 is a standard application of the so-called Dirichlet hyperbola method, and it can be found in any textbook in analytic number theory such as [1].

References

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Cezar Lupu
Yanqi Lake Beijing Institute of Mathematical Sciences and Applications (BIMSA),
Yau Mathematical Sciences Center (YMSC), Tsinghua University, Beijing, China
lupucezar@gmail.com, lupucezar@bimsa.cn