Elemente der Mathematik 80 (2025), 45–58
DOI 10.4171/EM/528

Zero-free regions for lacunary polynomials with positive coefficients

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1 Introduction

It is known (e.g., [3] (1923) and [1] (1954), and proved in Section 2) that a polynomial of degree n with nonnegative coefficients cannot have zeros in a sector around the real axis with an angle of $2\pi/n$. Simple examples of such polynomials are the cyclotomic polynomials of prime order n (a cyclotomic polynomial Φ_n of order n is a divisor of $x^n - 1$, but not of $x^m - 1$ for any $m < n$), e.g., $\Phi_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.

However, when polynomials with nonnegative coefficients are lacunary, i.e., when they have many consecutive zero coefficients, there can be several additional zero exclusion sectors that are not captured by the aforementioned result. Our purpose here is to derive such exclusion sectors. As an illustration, consider first the polynomial

$$2z^{10} + 3z^9 + z^8 + 4z^7 + 5z^6 + 3z^5 + z^4 + 2z^3 + 8z^2 + z + 4$$

with positive coefficients and no gaps between its powers. Its zeros are given by $-1.4384 \pm 0.1939i$, $-0.6686 \pm 0.8906i$, $0.5230 \pm 1.1272i$, $0.8559 \pm 0.5747i$, $-0.0220 \pm 0.6826i$,

Ein Polynom mit nichtnegativen Koeffizienten kann keine positiven Nullstellen haben, aber auch keine Nullstellen in einem gewissen Sektor um die positive reelle Achse. Wenn ein solches Polynom jedoch grosse Lücken zwischen aufeinanderfolgenden Potenzen aufweist, dann lassen sich zusätzliche nullstellenfreie Sektoren finden. In der vorliegenden Arbeit wird gezeigt, wie man solche Sektoren erhält und wie man die Anzahl der Nullstellen bestimmt, die zwischen diesen Sektoren liegen. Die geometrische Natur des Ansatzes liefert zudem Schranken für den Betrag der Nullstellen in einem gegebenen Sektor, die tendenziell besser sind als die Schranken, die für alle Nullstellen gelten.

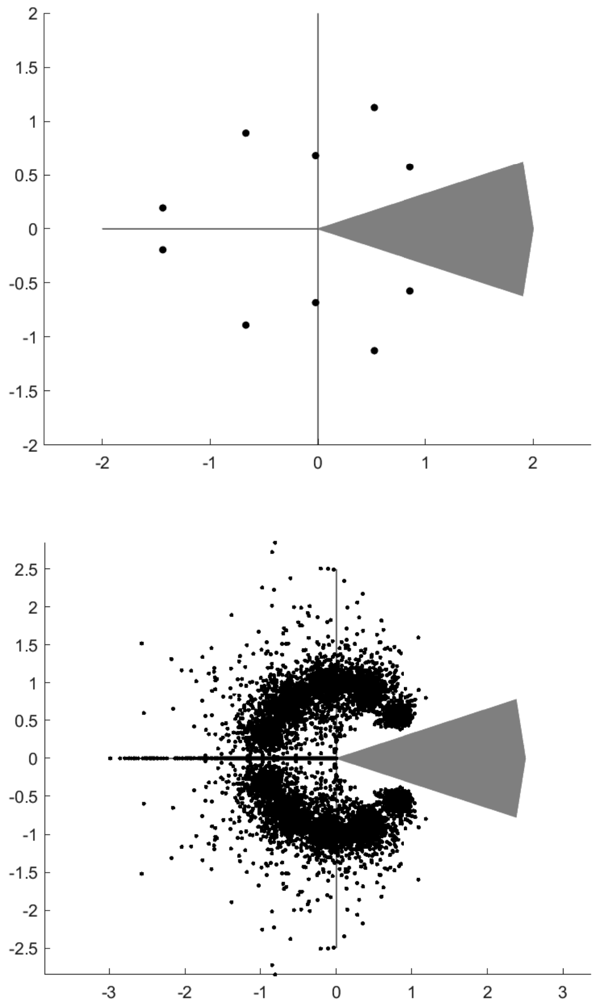

Figure 1. Zero exclusion sectors for polynomials of the form $\sum_{j=0}^{10} a_j z^j$, with $a_j > 0$.

which are excluded from the shaded sector on the top in Figure 1, where the black dots represent the zeros. This sector only depends on the nonnegativity of the coefficients, i.e., it is unaffected by the magnitude of the coefficients. This is shown on the bottom in Figure 1, where the zeros of 1000 polynomials of the form $\sum_{j=0}^{10} a_j z^j$, with coefficients a_j uniformly randomly distributed in $(0, 10)$, are plotted as black dots, clearly showing that they are excluded from the shaded exclusion sector.

Now consider the lacunary polynomial $2z^{10} + 3z^9 + 5$, having a large gap in its powers between the constant term and z^9. Its zeros are $-1.3591 \pm 0.0676i$, $-0.7742 \pm$

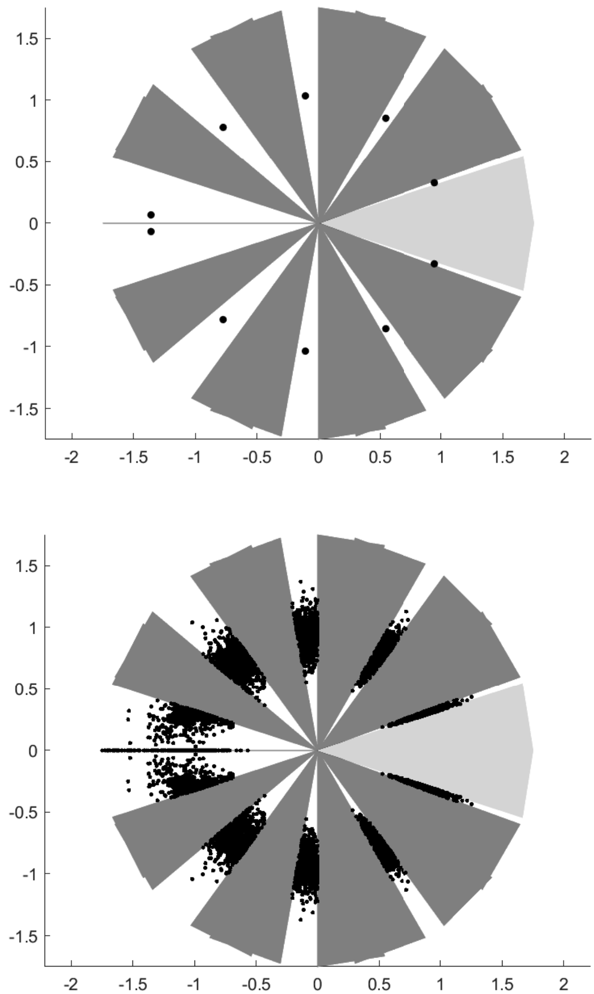

Figure 2. Zero exclusion sectors for polynomials of the form $a_2 z^{10} + a_1 z^9 + a_0$, with $a_j > 0$.

$0.7805i$, $-0.1076 \pm 1.0358i$, $0.9460 \pm 0.3285i$, and $0.5448 \pm 0.8546i$, which, as will be shown later, must be excluded from several (shaded) sectors shown on the top in Figure 2, where, as before, the black dots represent the zeros. That these sectors are independent of the magnitudes of the coefficients is demonstrated on the bottom in Figure 2, where the zeros of 1000 polynomials of the form $a_2 z^{10} + a_1 z^9 + a_0$, with coefficients a_j uniformly randomly distributed in $(0, 10)$, plotted as black dots, are shown to be excluded from the shaded sectors. In contrast, the aforementioned result from [1, 3] would only yield the single exclusion sector shaded in lighter grey.

In the next section, we derive exclusion sectors for lacunary polynomials with nonnegative coefficients, and determine the number of zeros in between exclusion sectors, as well as bounds on the magnitudes of such zeros.

To avoid trivial situations, we will assume that all polynomials are *simple*, i.e., that their powers are not multiples of the same integer $q \geq 2$. Polynomials that are not simple can easily be simplified since

$$a_0 + \sum_{j=1}^{k} a_j z^{qm_j} = a_0 + \sum_{j=1}^{k} a_j y^{m_j},$$

where $y = z^q$. Throughout, the *convex hull* of the points $s_1, \ldots, s_m$, which is the smallest convex set containing these points, is denoted by $\mathrm{Conv}\{s_1, \ldots, s_m\}$. The argument of a complex number z is denoted by $\arg(z)$.

2 Exclusion sectors

For future reference, we begin by formally stating and proving that a polynomial of degree n with nonnegative coefficients cannot have zeros in a sector around the real axis with an angle of $2\pi/n$. Of the several ways to prove this result, the geometric approach in [2] is preferable as it will also be useful further on.

Theorem 2.1 (Obrechkoff 1923, Cowling and Thorn 1954). *The zeros of the polynomial $p(z) = a_k z^{n_k} + a_{k-1} z^{n_{k-1}} + \cdots + a_1 z^{n_1} + a_0$, with $n_k > n_{k-1} > \cdots > n_1 > n_0 = 0$, $a_j > 0$, and $k \geq 2$, are excluded from the sector*

$$\left\{ z \in \mathbb{C} : |\arg(z)| \leq \frac{\pi}{n_k} \right\}.$$

Proof. (See [2].) Let ζ be a zero of p with $\zeta = \rho e^{i\varphi}$, $\rho > 0$, and $0 < \varphi \leq \pi$. With $\Phi = e^{i\varphi}$ and $\sigma = \sum_{j=0}^{k} a_j \rho^{n_j}$, we have

$$p(\zeta) = 0 \iff \frac{1}{\sigma} p(\zeta) = \frac{a_0}{\sigma} + \sum_{j=1}^{k} \frac{a_j \rho^{n_j}}{\sigma} \Phi^{n_j} = 0.$$

This equation is of the form $\gamma_0 \cdot 1 + \gamma_1 \cdot \Phi^{n_1} + \cdots + \gamma_{k-1} \cdot \Phi^{n_{k-1}} + \gamma_k \cdot \Phi^{n_k}$, where $\gamma_j > 0$ and $\sum_{j=0}^{k} \gamma_j = 1$, which shows that, for ζ to be a zero of p, the origin has to lie in the relative interior of $\mathrm{Conv}(S)$, where $S = \{1, \Phi^{n_1}, \ldots, \Phi^{n_k}\}$, illustrated in Figure 3 for $k = 4$, which is clearly impossible when $0 < n_k \varphi \leq \pi$ or $0 < \varphi \leq \pi/n_k$. The rest of the theorem follows because the zeros of p occur in complex conjugate pairs. ■

We now consider zero exclusion sectors for a polynomial with nonnegative coefficients, exhibiting a large gap between two consecutive powers of the variable. The following theorem quantifies the meaning of "large" and derives the resulting exclusion sectors. Subsequent theorems derive results about the number of zeros in a sector and bounds on the magnitudes of zeros in a sector.

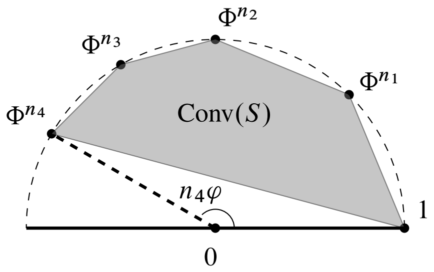

Figure 3. Convex hull of powers of a zero.

Theorem 2.2. *Let $p(z) = a_0 + \sum_{j=1}^{k} a_j z^{n_j}$, with $a_j > 0$, $n_k > n_{k-1} > \cdots > n_1 > n_0 = 0$, and positive integer k, be simple, let ℓ be a nonnegative integer, and let $0 < \alpha \le 1$.*

(1) *If $k \ge 2$, then the following holds.*

(a) *If $n_1 \ge \frac{(2\ell+1)n_k}{2(\ell+1)}$, then p has no nonreal zeros in each of the two sectors*

$$\frac{(2\ell+1)\pi}{n_1} \le |\varphi| \le \min\left\{\frac{\pi}{n_k - n_1}, \frac{2(\ell+1)\pi}{n_1}\right\}.$$

(b) *If $n_1 \ge \frac{(2\ell+1)n_k}{2\ell+\alpha+1}$, then p has no nonreal zeros in each of the two sectors*

$$\frac{(2\ell+\alpha+1)\pi}{n_k} \le |\varphi| \le \min\left\{\frac{\alpha\pi}{n_k - n_1}, \frac{(2\ell+\alpha+2)\pi}{n_k}\right\}.$$

(2) *If $k \ge 3$, $1 \le s \le k-2$, then the following holds.*

(a) *If $n_{s+1} \ge \max\{2(\ell+1)n_s, \frac{(2\ell+1)n_k+n_s}{2(\ell+1)}\}$, then p has no nonreal zeros in each of the two sectors*

$$\frac{(2\ell+1)\pi}{n_{s+1} - n_s} \le |\varphi| \le \min\left\{\frac{2(\ell+1)\pi}{n_{s+1}}, \frac{\pi}{n_k - n_{s+1}}\right\}.$$

(b) *If $n_{s+1} \ge \max\{\frac{(2\ell+\alpha+1)n_s}{\alpha}, \frac{(2\ell+\alpha+1)n_k}{2\ell+\alpha+2}\}$, then p has no nonreal zeros in each of the two sectors*

$$\frac{(2\ell+\alpha+1)\pi}{n_{s+1}} \le |\varphi| \le \min\left\{\frac{\alpha\pi}{n_s}, \frac{(2\ell+\alpha+2)\pi}{n_k}\right\}.$$

(3) *If $k \ge 2$, then the following holds.*

(a) *If $n_k \ge 2(\ell+1)n_{k-1}$, then p has no nonreal zeros in each of the two sectors*

$$\frac{(2\ell+1)\pi}{n_k - n_{k-1}} \le |\varphi| \le \min\left\{\frac{\pi}{n_{k-1}}, \frac{2(\ell+1)\pi}{n_k - n_{k-1}}\right\}.$$

(b) *If $n_k \ge \frac{(2\ell+\alpha+1)n_{k-1}}{\alpha}$, then p has no nonreal zeros in each of the two sectors*

$$\frac{(2\ell+\alpha+1)\pi}{n_k} \le |\varphi| \le \min\left\{\frac{\alpha\pi}{n_{k-1}}, \frac{(2\ell+\alpha+2)\pi}{n_k}\right\}.$$

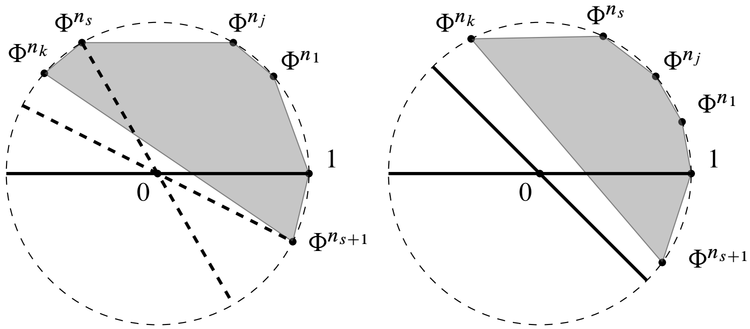

Figure 4. Convex hull of powers of a zero for Theorem 2.2.

Proof. Let ζ be a zero of p with $\zeta = \rho e^{i\varphi}$, $\rho > 0$, and $0 < \varphi \leq \pi$. As in the proof of Theorem 2.1, with $\Phi = e^{i\varphi}$, we have that $p(\zeta) = 0$ is equivalent to requiring that the origin lies in the relative interior of $\text{Conv}\{1, \Phi^{n_1}, \ldots, \Phi^{n_k}\}$. We consider only $\varphi > 0$, as the corresponding results for $\varphi < 0$ follow from the fact that the zeros occur in complex conjugate pairs.

We start with the proof of part (2) of the theorem. We distinguish two cases that preclude ζ from being a nonreal zero of p, illustrated by Figure 4.

The first case (left side of Figure 4) corresponds to a situation where $n_s\varphi \leq \pi$ and $n_{s+1}\varphi$ is much larger, namely, at least as large as $n_s\varphi + \pi$, but less than 2π, so that the origin cannot lie in the interior of the convex hull depicted on the left in Figure 4 as long as $n_k\varphi \leq n_{s+1}\varphi + \pi$. In other words, and taking into account that angles are determined up to a multiple of 2π, ζ cannot be a zero of p, i.e., the origin cannot strictly lie in $\text{Conv}\{1, \Phi^{n_1}, \ldots, \Phi^{n_k}\}$, if

$$\begin{aligned} n_s\varphi &\leq \pi, \\ n_{s+1}\varphi &\geq n_s\varphi + \pi + 2\ell\pi, \\ n_{s+1}\varphi &\leq 2\pi + 2\ell\pi, \\ n_k\varphi &\leq n_{s+1}\varphi + \pi, \end{aligned}$$

where ℓ is a nonnegative integer, unless all the vertices of the convex hull lie at 1 and -1, which is only possible when $\varphi = \pi$ since p is simple. This means that there can be no nonreal zeros of p in the sector

$$\frac{(2\ell+1)\pi}{n_{s+1} - n_s} \leq \varphi \leq \min\left\{\frac{\pi}{n_s}, \frac{2(\ell+1)\pi}{n_{s+1}}, \frac{\pi}{n_k - n_{s+1}}\right\}, \tag{1}$$

provided that the upper bound is not smaller than the lower bound. We now show that this follows immediately from the assumptions of the theorem. We have

$$\frac{(2\ell+1)\pi}{n_{s+1} - n_s} \leq \frac{2(\ell+1)\pi}{n_{s+1}} \iff n_{s+1} \geq 2(\ell+1)n_s,$$

$$\frac{(2\ell+1)\pi}{n_{s+1}-n_s} \le \frac{\pi}{n_k-n_{s+1}} \iff n_{s+1} \ge \frac{(2\ell+1)n_k+n_s}{2(\ell+1)}.$$

The requirement $n_{s+1} \ge 2(\ell+1)n_s$ implies that

$$\frac{\pi}{n_s} \ge \frac{2(\ell+1)\pi}{n_{s+1}}$$

so that π/n_s in (1) is superfluous. The conditions on n_s, n_{s+1}, and n_k therefore ensure that the sector in (1) is well defined, and part (2) (a) of the theorem follows.

In the second case (right side of Figure 4), for similar reasons as in the first case, ζ cannot be a zero of p if

$$\begin{aligned} n_s\varphi &\le \alpha\pi, \\ n_{s+1}\varphi &\ge \alpha\pi+\pi+2\ell\pi, \\ n_k\varphi &\le \alpha\pi+2\pi+2\ell\pi, \end{aligned}$$

unless, as in the previous case, $\varphi=\pi$. The polynomial p can therefore not have nonreal zeros in the sector

$$\frac{(2\ell+\alpha+1)\pi}{n_{s+1}} \le \varphi \le \min\left\{\frac{\alpha\pi}{n_s}, \frac{(2\ell+\alpha+2)\pi}{n_k}\right\},$$

provided, once again, that the upper bound is not smaller than the lower bound. That this is indeed the case follows directly from the assumptions of the theorem, analogously as before.

For part (3), corresponding to $s=k-1$, the two cases illustrated in Figure 4 are similar, but simpler. In the first case (on the left in Figure 4), the polynomial p cannot have nonreal zeros if

$$\begin{aligned} n_{k-1}\varphi &\le \pi, \\ n_k\varphi &\ge n_{k-1}\varphi+\pi+2\ell\pi, \\ n_k\varphi &\le n_{k-1}\varphi+2\pi+2\ell\pi, \end{aligned}$$

i.e., if

$$\frac{(2\ell+1)\pi}{n_k-n_{k-1}} \le \varphi \le \min\left\{\frac{\pi}{n_{k-1}}, \frac{2(\ell+1)\pi}{n_k-n_{k-1}}\right\}.$$

Part (3) (a) then follows since the sector is well defined under the assumptions of the theorem. For the second case (on the right in Figure 4), we have that p cannot have nonreal zeros if

$$\begin{aligned} n_{k-1}\varphi &\le \alpha\pi, \\ n_k\varphi &\ge \alpha\pi+\pi+2\ell\pi, \\ n_k\varphi &\le \alpha\pi+2\pi+2\ell\pi, \end{aligned}$$

i.e., if

$$\frac{(2\ell+\alpha+1)\pi}{n_k} \le \varphi \le \min\left\{\frac{\alpha\pi}{n_{k-1}}, \frac{(2\ell+\alpha+2)\pi}{n_k}\right\},$$

which is well defined, as before, under the assumptions of the theorem. Part (3) (b) then follows.

For part (1), corresponding to $s = 0$, we consider the reverse polynomial

$$p^{\#}(\zeta) = \zeta^{n_k} p(1/\zeta) = a_0\zeta^{n_k} + a_1\zeta^{n_k-n_1} + \cdots + a_{k-1}z^{n_k-n_{k-1}} + a_k,$$

whose zeros are the reciprocals of those of p, and we exploit the fact that these zeros are excluded from the same sectors. We can therefore apply part (3) of the theorem to the polynomial

$$a_k + \sum_{j=1}^{k} a_{k-j}\zeta^{\nu_j}, \quad \text{with } \nu_j = n_k - n_{k-j} \text{ and } s = k-1,$$

which directly leads to the exclusion sectors in part (1). This completes the proof. ∎

To summarize, Theorem 2.2 derives exclusion sectors when a large gap exists between n_s and n_{s+1}, where s ranges from 0 to n_{k-1}. Part (1) of the theorem deals with $s = 0$, part (2) deals with $1 \le s \le k-2$, and part (3) treats the case $s = k-1$. The sectors are independent of the magnitudes of the coefficients.

The conditions on the powers of the polynomial in Theorem 2.2 impose obvious upper bounds on the nonnegative integer ℓ. As just one example, let us take the case (2) (a) of Theorem 2.2:

$$n_{s+1} \ge 2(\ell+1)n_s \implies \ell \le \frac{n_{s+1}}{2n_s} - 1.$$

Similar upper bounds on ℓ can be obtained for the other cases of the theorem.

Determining the exclusion sectors requires almost no computational effort and, in fact, could even be determined by hand. Although beyond our scope here, possible extensions of Theorem 2.2 could be to incorporate more than one gap in the powers of the variable, or to allow different sign restrictions on the coefficients, or to consider complex coefficients.

The converse of the Theorem 2.2 is not true, namely, zeros being excluded from certain sectors does not imply that the polynomial is lacunary. Consider, e.g., the polynomials $4z^6 + 5z^5 + 3z^4 + z^3 + z^2 + z + 4$, with zeros $-1.0377 \pm 0.5171i$, $-0.2402 \pm 0.9751i$, and $0.6529 \pm 0.5580i$, and $4z^6 + 5z^5 + 4$, with zeros $-1.1890 \pm 0.3402i$, $-0.1614 \pm 0.9177i$, and $0.7254 \pm 0.4763i$. In both cases, the zeros are very similar with wide exclusion sectors separating them. However, one polynomial is lacunary and the other is not. It is possible that a converse could be formulated stating that, given certain sectors, there must exist a lacunary polynomial whose zeros are excluded from them. However, proving such a converse, if true, requires a separate study.

Example 1. Consider the polynomial

$$a_5z^{15} + a_4z^{14} + a_3z^3 + a_2z^2 + a_1z + a_0, \quad \text{where } a_j > 0,$$

exhibiting a significant gap in its powers between z^3 and z^{14}. On the top in Figure 5 are shown the zero exclusion sectors (shaded regions) for this polynomial, along with the zeros (black dots) of 1000 such polynomials whose coefficients are uniformly randomly distributed in (0, 10) as a test of the theory, namely, to demonstrate that these sectors are truly exclusion sectors, and that they only depend on the sign of the coefficients. The sector

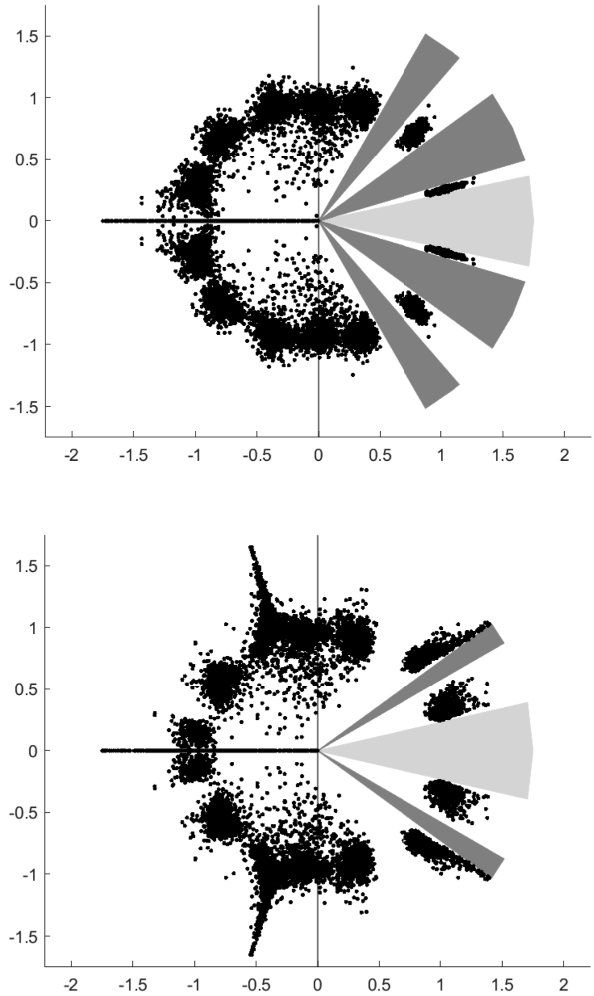

Figure 5. Exclusion sectors for Example 1.

centred around the positive real axis, shaded in lighter grey, is obtained from Theorem 2.1, while the four additional sectors are obtained from Theorem 2.2 with $s = 3$ and $\alpha = 1$. On the bottom in Figure 5, the same is done for the polynomial

$$a_5 z^{14} + a_4 z^9 + a_3 z^3 + a_2 z^2 + a_1 z + a_0,$$

for which the result is clearly less satisfactory, due to the relatively large difference between the two leading powers, creating a second gap between the powers.

Knowledge about exclusion sectors allows one to easily determine the number of zeros in certain sectors, as shown in the next theorem. We call a sector degenerate if its defining upper and lower limits coincide.

Theorem 2.3. *Let $p(z) = \sum_{j=0}^{k} a_j z^{n_j}$, with $a_j > 0$, $n_k > n_{k-1} > \cdots > n_1 > n_0 = 0$, and integer $k \geq 2$, be simple, and let θ be the lower limit of a nondegenerate exclusion sector from Theorem 2.2 with $\pi/n_k < \theta < \pi$. Then p has*

$$\left\lfloor \frac{n_k\theta - \pi}{2\pi} \right\rfloor + 1$$

zeros in each of the two open sectors defined by

$$\frac{\pi}{n_k} < |\varphi| < \theta. \tag{2}$$

Proof. The proof relies on the fact that the zeros of a polynomial are continuous functions of its coefficients. Define $q_\varepsilon(z) = a_k z^{n_k} + a_0 + \varepsilon \sum_{j=1}^{k-1} a_j z^{n_j}$ so that $q_1 = p$. The polynomials p and q_ε have the same exclusion sectors defined by Theorems 2.1 and 2.2 for $\varepsilon > 0$. The zeros of q_0 are given by

$$\left(\frac{a_0}{a_k}\right)^{1/n_k} e^{i(2j+1)\pi/n_k}, \quad j = 0, 1, \ldots, n_k - 1,$$

and when ε continuously increases from 0 to 1, the set S comprised of the zeros of q_ε that lie in the closure of the sector defined in (2) change continuously into those zeros of p that lie in the open sector, since the two exclusion sectors with upper and lower limits π/n_k and θ, respectively, cannot contain any zeros. As a result, the zeros in S that are changing with increasing ε cannot migrate outside this sector, nor can zeros that are not in S migrate into the sector.

We now determine the number of zeros of q_0 that lie in the closure of the sector defined in (2). Other than $e^{i\pi/n_k}$, that sector contains j zeros of q_0, all of which are complex because $\theta < \pi$, where j is the largest number satisfying

$$\frac{(2j+1)\pi}{n_k} \leq \theta \implies j \leq \frac{n_k\theta - \pi}{2\pi},$$

from which the proof follows since j is an integer. ∎

We remark that the sectors defined in (2) can contain additional zero exclusion sectors, and we also observe that exclusion sectors for the nonreal zeros of p, which are also exclusion sectors for the nonreal zeros of q_ε with $\varepsilon > 0$ (as defined in the above proof), cannot contain any nonreal zeros of $q_0 = a_n z^{n_k} + a_0$ in their interior since such zeros would not be able to continuously migrate out of such an exclusion sector when ε continuously increases from 0. As a result, the width (in radians) of an exclusion sector for nonreal zeros can never exceed $2\pi/n_k$ (the angle between consecutive zeros of q_0), unless q_0 has a real negative zero and the exclusion sector terminates at $\varphi = \pi$, in which case the sectors in the upper and lower complex plane are joined.

In the following theorem, we derive a lower bound on the magnitudes of the zeros in a sector.

Theorem 2.4. *Let* $p(z) = \sum_{j=0}^{k} a_j z^{n_k}$, *with* $a_j > 0$, $n_k > n_{k-1} > \cdots > n_1 > n_0 = 0$, *and positive integer* $k \geq 2$, *be simple, and let* $\theta \in \mathbb{R}$ *with* $\theta \leq \pi/n_s$, *where* $0 \leq s \leq k-1$ *with the convention that* $\pi/n_0 = +\infty$. *Then a lower bound on the magnitudes of the zeros of* p *in the sectors*

$$\frac{\pi}{n_k} < |\varphi| \leq \theta \tag{3}$$

is given by the unique positive solution of

$$\sum_{j=s+1}^{k} a_j z^{n_j} - \cos\left(\frac{n_s\theta}{2}\right) \cdot \sum_{j=0}^{s} a_j z^{n_j} = 0. \tag{4}$$

Proof. Let $\pi/n_k < \varphi \leq \theta$, which implies that $n_s\varphi \leq \pi$. As before, one only needs to consider $\varphi > 0$. We begin with the case $s \neq 0$ and define

$$\sigma_1 = \sum_{j=0}^{s} a_j \rho^{n_j}, \qquad S_1 = \operatorname{Conv}\{1, \Phi^{n_1}, \ldots, \Phi^{n_s}\},$$

$$\sigma_2 = \sum_{j=s+1}^{k} a_j \rho^{n_j}, \quad S_2 = \operatorname{Conv}\{\Phi^{n_{s+1}}, \ldots, \Phi^{n_k}\},$$

where $\Phi = e^{i\varphi}$. Let $\zeta = \rho e^{i\varphi}$ be a zero of p with argument φ, where φ satisfies (3). Then the equation $p(\zeta) = 0$ can be written as

$$-\frac{\sigma_2}{\sigma_1}\left(\sum_{j=s+1}^{k} \frac{a_j\rho^{n_j}}{\sigma_2}\Phi^{n_j}\right) = \sum_{j=0}^{s} \frac{a_j\rho^{n_j}}{\sigma_1}\Phi^{n_j} \in S_1,$$

i.e., for ζ to be a zero of p, it must be possible for a point $x \in S_2$ to be multiplied by a negative number so that the result lies in S_1. With the help of Figure 6, one observes that this is only possible if x lies in the bottom unit semidisk and if $\sigma_2|x|/\sigma_1 \geq \Delta$, where Δ is the minimal distance between the origin and S_1.

Since $|x| \leq 1$, a necessary condition for ζ to be a zero of p is that $\sigma_2/\sigma_1 \geq \Delta$, which is equivalent to requiring that ρ satisfies

$$\sum_{j=s+1}^{k} a_j\rho^{n_j} - \Delta\sum_{j=0}^{s} a_j\rho^{n_j} \geq 0. \tag{5}$$

In other words, ρ cannot be smaller than the unique positive root of (5), which is therefore a lower bound on the magnitude of ζ.

Figure 6 shows that Δ is the height of an isosceles triangle whose equal sides are of length 1. Since the triangle's interior angle at the origin is $n_s\varphi$, we have that $\Delta = \cos(n_s\varphi/2)$. The smaller the value of Δ, the smaller the unique root of (5), and this smaller value is obtained for the largest value of φ, given by the upper bound in (3). Since we are obtaining a lower bound, we need to consider the worst case possible so that the proof follows by substituting this upper bound for φ in the expression for Δ.

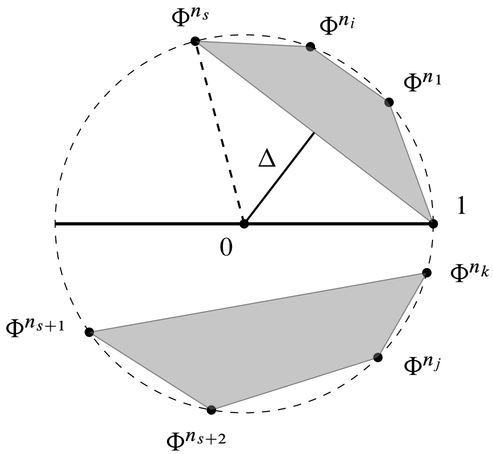

Figure 6. Convex hull of powers of a zero for Theorem 2.4.

When $s = 0$, equation (4) becomes

$$-a_0 + \sum_{j=1}^{k} a_j z^{n_j} = 0.$$

This is a well-known result, usually attributed to Cauchy, namely, that the unique positive root of this equation is a lower bound on the magnitude of all the zeros of p. This completes the proof. ∎

Remarks. An upper bound can be obtained by applying Theorem 2.4 to the reverse polynomial $p^{\#}(z) = z^{n_k} p(1/z)$, whose zeros are the reciprocals of those of p. The real polynomial equation (4) that produces the bound can easily be solved for its unique positive root using a simple iterative method such as, e.g., Newton's method, which, in this case, is guaranteed to converge from any point to the right of that root.

Example 2. Consider the polynomial $5z^{26} + 5z^{24} + 9z^{23} + 10z^5 + 10z^2 + 2z + 9$. On the top in Figure 7 are shown the exclusion sector from Theorem 2.1 in lighter grey, and the exclusion sectors from Theorem 2.2 with $s = 3$ and $\alpha = 1$ in darker grey. The black dots represent the zeros of the polynomial. The disjoint sectors H_1 and H_2 in between the exclusion sectors from Theorem 2.2 are defined by

$$H_1 = \{\varphi : 0.1208 < \varphi < 0.1745\} \quad \text{and} \quad H_2 = \{\varphi : 0.3625 < \varphi < 0.5236\}.$$

According to Theorem 2.3, there are two zeros in $H_1 \cup H_2$ and a single zero in H_1, which means that H_2 also contains a single zero. The lower bounds on the magnitudes of the zeros that lie in H_1 and H_2 from Theorem 2.4 (the zeros $1.0109 + 0.1409i$ with magnitude 1.0207, and $0.9252 + 0.4070i$ with magnitude 1.0108, respectively) are given by 1.0022 and 0.9462, respectively. The lower bound from Cauchy's result, mentioned at the end

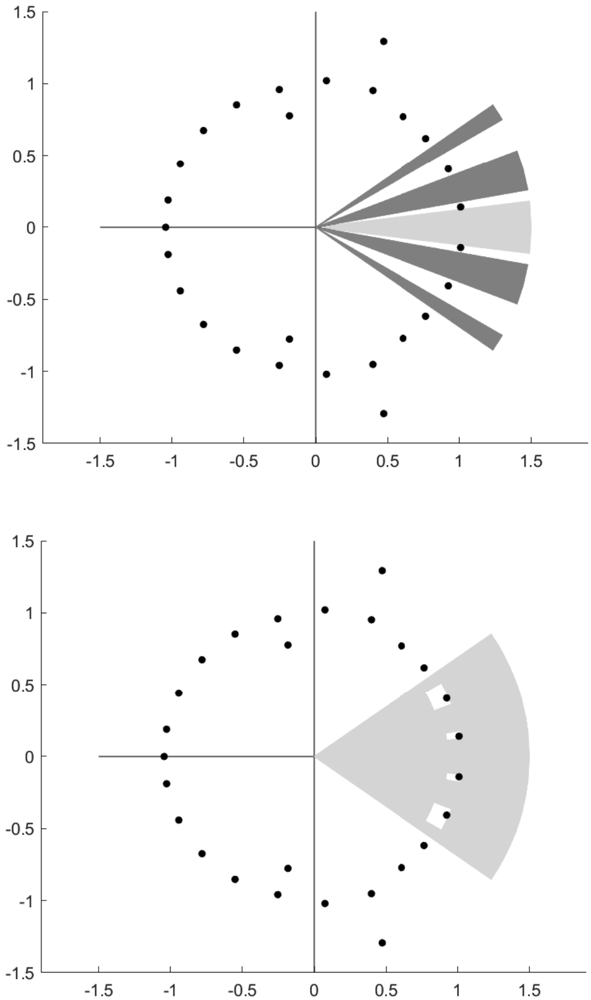

Figure 7. Exclusion sectors and bounds for Example 2.

of the proof of Theorem 2.4, which is valid for the magnitudes of all zeros, is 0.73406, a worse bound. The corresponding upper bounds obtained by applying Theorem 2.4 to the reverse polynomial are given by 1.0393 and 1.0245, respectively, which is significantly better than the Cauchy bound (see [4, Definition 8.1.2]), given by 1.4872. The latter is an upper bound on all the zeros of p, optimal over all bounds dependent only on the moduli of the coefficients. The exclusion region that takes into consideration the aforementioned bounds is shown on the bottom in Figure 7, shaded in lighter grey to make the zeros more visible.

Conclusion

We have derived zero exclusion regions for lacunary polynomials with positive coefficients, as well as results about the number of zeros in between such regions and bounds on their magnitudes. The geometric approach used here could conceivably be extended to obtain similar results for real polynomials with different sign restrictions on their coefficients, as well as polynomials with complex coefficients. We also mention that it is sometimes possible to cast a polynomial into a desired form with an appropriate transformation of the variable.

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