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Algebraic Geometry. – *The value semigroup of a plane curve singularity with several branches*, by Marco D'Anna, Félix Delgado de la Mata, Lorenzo Guerrieri, Nicola Maugeri and Vincenzo Micale, communicated on 14 March 2025.

Abstract. – We present a constructive procedure, based on the notion of Apéry set, to obtain the value semigroup of a plane curve singularity from the value semigroup of its blow-up and vice-versa. In particular, we give a blow-down process that allows us to reconstruct a plane algebroid curve form its blow-up, even if it is not local. Then, we characterize numerically all the possible multiplicity trees of plane curve singularities, obtaining in this way a constructive description of all their value semigroups.

Keywords. – good semigroup, Apéry set, plane curve singularity, multiplicity tree.

Mathematics Subject Classification 2020. – 14H20 (primary); 14H50, 13H15, 20M14 (secondary).

1. Introduction

An algebroid branch is a ring of the form $\mathcal{O} = K[\![X_1, \dots, X_n]\!]/P$, where K is an algebraically closed field and P is a height $n-1$ prime ideal. Algebroid branches naturally appear in the study of curve singularities, as completions of the local rings associated with a singular point of an algebraic curve, with one branch in that point. Since Zariski [19], a classical tool to study and classify singularities is given by the value semigroup associated with an algebroid branch: in fact, the integral closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ in its quotient field is a DVR isomorphic to $K[\![t]\!]$. Hence, every non-zero element $g \in \mathcal{O}$ has a value $v(g) := \mathrm{ord}_t(g) \in \mathbb{N}$ and the set of values of its elements constitute a numerical semigroup $v(\mathcal{O}) = S$, i.e., a submonoid of $\mathbb{N}$ with finite complement in it. The knowledge of the value semigroup gives much information on the ring $\mathcal{O}$; for example, its smallest non-zero value is the multiplicity $e(\mathcal{O})$ of the singularity, and from the value semigroup one can easily compute the degree of singularity (i.e. the length $l_{\mathcal{O}}(\overline{\mathcal{O}}/\mathcal{O})$), or one can check the Gorenstein and the complete intersection properties.

Another classical invariant to classify a branch singularity is given by the sequence of multiplicities of the successive blow-ups of $\mathcal{O}$, $(e(\mathcal{O}), e(\mathcal{O}^{(1)}), e(\mathcal{O}^{(2)}), \dots)$ (see e.g. [19]). Two algebroid branches are said formally equivalent it they share the same sequence of multiplicities; in general, the value semigroups and the multiplicity sequence are independent criteria of equisingularity.

If we want to consider a curve singularity with d branches, we have to deal with algebroid curves, i.e. rings of the form $\mathcal{O} = K[\![X_1, \dots, X_n]\!]/P_1 \cap \cdots \cap P_d$, where the P_i are pairwise distinct prime ideals of height $n-1$ and determine the branches. In this case, the integral closure of $\mathcal{O}$ in its total ring of fractions is a product of DVRs, $\overline{\mathcal{O}} \cong K[\![t_1]\!] \times \cdots \times K[\![t_d]\!]$, where $K[\![t_i]\!]$ is the integral closure $\overline{\mathcal{C}_i}$ of the i-th branch $\mathcal{C}_i := K[\![X_1, \dots, X_n]\!]/P_i$ and the set of values $S = v(\mathcal{O})$ is a submonoid of $\mathbb{N}^d$ (here $v(g) = (v_1(g), \dots, v_d(g)) \in \mathbb{N}^d$, where v_i is the valuation of the i-th branch). The projections S_i of S on the coordinate axes are the value semigroups of each branch.

Again, as for the one-branch case, the value semigroup gives much information on the singularity. However, while any numerical semigroup is the value semigroup of a one-branch singularity, there is no characterization of the semigroups appearing as value semigroups of algebroid curves with $d > 1$ branches.

In this article, we consider the case of plane curve singularities. When we have only one branch, there are classical characterizations for the possible numerical semigroups that are value semigroups of an algebroid plane branch (that now is a ring of the form $\mathcal{O} = K[\![X, Y]\!]/(F)$, with F irreducible). Moreover, it is well known that the value semigroup and the multiplicity sequence of an algebroid branch become two equivalent criteria of equisingularity (see [19]), and in fact it is possible to reconstruct the multiplicity sequence from the value semigroup and vice-versa. More precisely, in [1], Apéry considered a particular generating set of $v(\mathcal{O})$, called the Apéry set, and showed that one can compute the Apéry set of the value semigroup $v(\mathcal{B}(\mathcal{O}))$ of the blow-up of $\mathcal{O}$ from that of $v(\mathcal{O})$, and vice-versa. This is the reason why, for plane branches, the value semigroup and the multiplicity sequence are two equivalent sets of invariants. In [4], it has been shown how to use Apery's result to easily obtain the value semigroup from the multiplicity sequence and vice-versa. It is worth noticing that, if we instead consider plane analytic branches, these invariants (value semigroup or multiplicity sequence) determine the topological class of the branch (see again [19]).

If we want to study a plane curve singularity with $d > 1$ branches, we have to deal with plane algebroid curves, i.e. rings of the form $\mathcal{O} = K[\![X, Y]\!]/(H_1 \cdots H_d)$, where the H_i are irreducible and pairwise coprime. In this case, two plane algebroid curves $\mathcal{O} = K[\![X, Y]\!]/(H_1 \cdots H_d)$ and $\mathcal{Q} = K[\![X, Y]\!]/(G_1 \cdots G_d)$ are formally equivalent if (after a renumbering of the branches) the branches $\mathcal{C}_i = K[\![X, Y]\!]/(H_i)$ and $\mathcal{D}_i = K[\![X, Y]\!]/(G_i)$ have the same multiplicity sequence for $i = 1, \dots, d$ and if the intersection multiplicities $[\mathcal{C}_i, \mathcal{C}_j] := l_{\mathcal{O}}(K[\![X, Y]\!]/(H_i, H_j))$ and $[\mathcal{D}_i, \mathcal{D}_j] := l_{\mathcal{Q}}(K[\![X, Y]\!]/(G_i, G_j))$ (where l denotes the length of a module over a ring) are the same for all pairs (i, j), $i \neq j$. Waldi has shown in [18] that two plane algebroid curves are formally equivalent if and only if they have the same value semigroup.

Hence, it is natural to ask whether it is possible to characterize the value semigroup of a plane singularity with more than one branch and to investigate how to reconstruct it by the multiplicity sequences and the intersection multiplicities of its branches and vice-versa.

The problem of the computation of the semigroup of values for $d > 1$ (and as a consequence of its characterization in some terms) from the semigroups of each branch together with the intersection multiplicities between a pair of branches was resolved in [13] following the next inductive way. Assume that one knows the semigroups of less than d branches, i.e. the semigroups S_J of the proper subset of branches corresponding to $J \subset \{1, \ldots, d\}$, $\#J < d$. Then, one can compute S from the subsemigroups $\{S_J \mid \#J = d-1\}$ and a finite set of elements $B = \{\beta^1, \ldots, \beta^m\} \subset S$ (the generalization of maximal contact values, i.e. of the minimal set of generators of the case $d = 1$). The set B can be computed explicitly from the semigroups S_i, $i = 1, \ldots, d$, and the intersection multiplicities of pairs of branches. It must be noticed that this way was made for the case $d = 2$ by García in [15] and Bayer in [6].

However, the above description is not easy, as it among other things demands inductively the computation of the projections S_J; moreover, it is not established in terms of the resolution process, which is a very natural way to understand the plane curve singularities.

This different approach to the problem was addressed and solved in [5] for the two-branches case and for characteristic 0. In that paper, the authors use two main tools: firstly, they show how to encode the data that determine formal equivalence in a tree, which they call multiplicity tree; secondly, they define the Apéry set of the value semigroup (which is now an infinite set) and make a partition of it in "levels", describing them as value sets of particular elements of the algebroid curve. Then, they show that, in case $\mathcal{O}$ and its blow-up $\mathcal{B}(\mathcal{O})$ are both local, the levels of the Apéry sets of their value semigroups can be obtained one from the other. Using these tools and a result of Garcia [15] (that holds only in the two-branch case), they show how to obtain the value semigroup from the multiplicity tree and vice-versa; this fact, together with a numerical description of the admissible multiplicity trees, gives a constructive characterization of the value semigroups of a plane singularity with two branches.

The aim of this paper is to generalize this approach to any number of branches, without restrictions on the characteristic. There are two main problems that arise. The first one is the fact that the definition of the partition of the Apéry set given in [5] does not work in more than two branches and in the non-local case. This problem has been addressed and solved in [11, 16, 17], where a new definition of the levels of the Apéry set, which works well in general, has been given; moreover, in [17], the authors show that this new definition agrees with the old one in the two-branch local case.

The second problem derives from the fact that when blowing up the algebroid curve, at some point (i.e. when at least two branches have different tangents) the blow-up is no more local. Our aim is to obtain a procedure to obtain the Apéry set of $v(\mathcal{O})$ from the Apéry set of the value semigroup $v(\mathcal{B}(\mathcal{O}))$ of its blow-up and vice-versa; to do this, we can make use of the new definition of levels of the Apéry set that holds also in the non-local case. Moreover, we also need to show, for any number of branches, that the levels of the Apéry set can be obtained as value sets of particular sets of elements of $\mathcal{B}(\mathcal{O})$, also in the non-local case. And since $\mathcal{B}(\mathcal{O})$ is not local, we cannot anymore present it as a quotient of $K[\![X, Y]\!]$, as it was done in [5].

Hence, our main task is to prove Theorems 4.3 and 4.4, where we show in the general case (i.e. for any number of branches, in the semilocal case and with no restrictions on the characteristic) how to describe the levels of the Apéry set. After doing that, we can give the searched procedure (see Theorem 4.15). In order to obtain it, we prove at ring level a procedure that, starting by a product $\mathcal{V}$ of local rings of plane algebroid curves, produces a local ring $\mathcal{U}$ of a plane algebroid curve, such that $\mathcal{B}(\mathcal{U}) = \mathcal{V}$ (Proposition 4.12). So, we have a sort of blow-down process that reverses the blow-up: in fact, if we start by a plane algebroid curve $\mathcal{O}$, we blow it up and then blow $\mathcal{B}(\mathcal{O})$ down; we get again $\mathcal{O}$ (Proposition 4.14).

Now, in order to obtain a constructive characterization of the value semigroup of a plane curve singularity, it remains to characterize numerically the admissible multiplicity trees of a curve singularity with any number of branches; this is classically known for the one-branch case, it was done for the two-branches case and characteristic 0 in [5], and here it is generalized for any number of branches without restriction on the characteristic (see Proposition 5.14). Using this last result, we can summarize in Theorem 5.15 the equivalence of the following sets of data:

(1) the semigroup of values S of $\mathcal{O}$;

(2) the multiplicity tree $\mathcal{T}(R)$ of R;

(3) the set $E = \{\underline{e}^i = (e^i_0, e^i_1, \ldots);\ i = 1, \ldots, d\}$ of the multiplicity sequences of the branches $\{\mathcal{C}_i \mid 1 \le i \le d\}$ plus the splitting numbers $\{k_{i,j}\}$ between pairs of branches C_i, C_j; $1 \le i < j \le d$.

We now briefly describe the structure of the paper. Section 2 is devoted to the basic definitions about good semigroups; in particular, in Definition 2.1, we recall the partition of the Apéry set in levels, fixing the notation in a more convenient way with respect to previous papers. Then, we show that this partition works well when both $\mathcal{O}$ and $\mathcal{B}(\mathcal{O})$ are both local, generalizing the arguments of [5] (see Propositions 2.2, 2.4 and Theorem 2.5).

Section 3 is very technical and contains some new results on the Apéry set, when the semigroup is not local. These results will allow us to find particular elements in the Apéry set, keeping the control on the levels (see e.g. Remark 3.4 and Lemmas 3.7 and 3.8).

In Section 4, we extend [5, Theorem 4.1] to the case where the blow-up of the coordinate ring of a plane curve is not local. In the first part of the section, we describe the levels of the Apéry set of the value semigroup of a semilocal ring R as sets of values of specific subsets of R (see Theorems 4.3 and 4.4). In the second part, we describe the blow-down process (Proposition 4.12) and how the levels of the Apéry set of the value semigroup behave when passing from the ring of a plane curve to its blow-up and vice-versa (Theorem 4.15).

Finally, in Section 5, we give a characterization of the admissible multiplicity trees of a plane singularity for any number of branches and independently of the characteristic. To this aim we have to recall the Hamburger–Noether expansion in the one-branch case and, using it, we can generalize the results for the two-branches case proved in [5] for characteristic zero. With an inductive argument, we can give the requested characterization for any number of branches (Proposition 5.14), which leads to Theorem 5.15 and to a constructive characterization of the admissible value semigroups of a plane curve singularity.

2. Preliminaries on algebroid curves

To work with value semigroups of algebroid curves, we will use the more general concept of good semigroup, introduced in [3]. Let $\leq$ denote the standard component-wise partial ordering in $\mathbb{N}^d$. Given two elements $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_d), \boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_d) \in \mathbb{N}^d$, the element $\boldsymbol{\delta}$ such that $\delta_i = \min(\alpha_i, \beta_i)$ for every $i = 1, \dots, d$ is called the *infimum* of the set $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ and will be denoted by $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$.

Let S be a submonoid of $(\mathbb{N}^d, +)$. We say that S is a *good semigroup* if

(G1) for every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in S$, $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \in S$;

(G2) given two elements $\boldsymbol{\alpha}, \boldsymbol{\beta} \in S$ such that $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$ and $\alpha_i = \beta_i$ for some $i \in \{1, \dots, d\}$, then there exists $\boldsymbol{\varepsilon} \in S$ such that $\varepsilon_i > \alpha_i = \beta_i$ and $\varepsilon_j \geq \min\{\alpha_j, \beta_j\}$ for each $j \neq i$ (and if $\alpha_j \neq \beta_j$, the equality holds);

(G3) there exists an element $\boldsymbol{c} \in S$ such that $\boldsymbol{c} + \mathbb{N}^d \subseteq S$.

A good semigroup is said to be *local* if $\mathbf{0} = (0, \dots, 0)$ is its only element with a zero component.

By (G1), it is always possible to define the element $\boldsymbol{c} := \min\{\boldsymbol{\alpha} \in \mathbb{Z}^d \mid \boldsymbol{\alpha} + \mathbb{N}^d \subseteq S\}$; this element is called a *conductor* of S. We set $\boldsymbol{\gamma} := \boldsymbol{c} - \mathbf{1}$.

A subset $E \subseteq \mathbb{N}^d$ is a *relative ideal* of S if $E + S \subseteq E$ and there exists $\boldsymbol{\alpha} \in S$ such that $\boldsymbol{\alpha} + E \subseteq S$. A relative ideal E contained in S is simply called an ideal. An ideal E satisfying properties (G1), (G2) is called a *good ideal* (notice that all ideals satisfy (G3) by definition). The minimal element $\boldsymbol{c}_E$ such that $\boldsymbol{c}_E + \mathbb{N}^d \subseteq E$ is called the *conductor* of E. As for S, we set $\boldsymbol{\gamma}_E := \boldsymbol{c}_E - \mathbf{1}$.

We denote by $\boldsymbol{e} = (e_1, e_2, \dots, e_d)$ the minimal element of S such that $e_i > 0$ for all $i \in \{1, \dots, d\}$. The set $\boldsymbol{e} + S$ is a good ideal of S and its conductor is $\boldsymbol{c} + \boldsymbol{e}$. Similarly, for every $\boldsymbol{\omega} \in S$, the principal good ideal $E = \boldsymbol{\omega} + S$ has conductor $\boldsymbol{c}_E = \boldsymbol{c} + \boldsymbol{\omega}$.

Let $\mathcal{O}$ be an algebroid curve with d branches. The value semigroup $S = v(\mathcal{O})$ is a local good semigroup contained in $\mathbb{N}^d$ [3]. In this case, the sum of the coordinates of the element $\boldsymbol{e}$ is the multiplicity of the curve. Non-local good semigroups may appear as value semigroups of semilocal rings obtained from algebroid curves after blow-ups. General results on good semigroups and value semigroups of curve singularities appear in many papers, e.g. [3, 8–18].

Given a non-zero divisor $x \in \mathcal{O}$, set $\boldsymbol{\omega} = (\omega_1, \dots, \omega_d) = v(x)$ and consider the good ideal $E = \boldsymbol{\omega} + S$. The set $\mathbf{Ap}(S, \boldsymbol{\omega}) = S \setminus E$ is called the Apéry set of S with respect to $\boldsymbol{\omega}$. Often we will consider the case $\boldsymbol{\omega} = \boldsymbol{e}$, and then we simply write $\mathbf{Ap}(S) = \mathbf{Ap}(S, \boldsymbol{e})$. This set has useful applications in the study of the quotient ring $\mathcal{O}/(x)$. In the case of algebroid branches, $\boldsymbol{\omega} \in \mathbb{N}$ and $\mathbf{Ap}(S, \boldsymbol{\omega})$ is a finite set of cardinality $\boldsymbol{\omega}$. Apéry sets of numerical semigroups and their properties are very well known. For an extensive treatment of numerical semigroups and semigroup rings, the reader may consult the monography [2]. In the case $d \geq 2$, $\mathbf{Ap}(S, \boldsymbol{\omega})$ is infinite, but it can be canonically partitioned in $N = \omega_1 + \cdots + \omega_d$ sets, as proved in [16, Theorem 4.4].

We recall the definition of this partition, which can be defined analogously for any set $A \subseteq S$ that is the complement of some proper good ideal. For this we need to recall several technical definitions that allow us to work combinatorially on a good semigroup.

Given a set $U \subseteq \{1, \dots, d\}$ and an element $\boldsymbol{\alpha} \in \mathbb{N}^d$, we define the following sets:

$$\Delta^S_U(\boldsymbol{\alpha}) = \{\boldsymbol{\beta} \in S \mid \beta_i = \alpha_i \text{ for } i \in U \text{ and } \beta_j > \alpha_j \text{ for } j \notin U\},$$

$$\widetilde{\Delta}^S_U(\boldsymbol{\alpha}) = \{\boldsymbol{\beta} \in S \mid \beta_i = \alpha_i \text{ for } i \in U \text{ and } \beta_j \geq \alpha_j \text{ for } j \notin U\} \setminus \{\boldsymbol{\alpha}\},$$

$$\Delta^S_i(\boldsymbol{\alpha}) = \{\boldsymbol{\beta} \in S \mid \beta_i = \alpha_i \text{ and } \beta_j > \alpha_j \text{ for } j \neq i\},$$

$$\Delta^S(\boldsymbol{\alpha}) = \bigcup_{i=1}^{d} \Delta^S_i(\boldsymbol{\alpha}).$$

In particular, for $S = \mathbb{N}^d$, we set $\Delta_U(\boldsymbol{\alpha}) := \Delta^{\mathbb{N}^d}_U(\boldsymbol{\alpha})$ and $\widetilde{\Delta}_U(\boldsymbol{\alpha}) := \widetilde{\Delta}^{\mathbb{N}^d}_U(\boldsymbol{\alpha})$. In general, we denote by $\widehat{U}$ the set $\{1, \dots, d\} \setminus U$.

Given any subset $A \subseteq S$, we say that an element $\boldsymbol{\alpha} \in A$ is a *complete infimum* in A if there exist $\boldsymbol{\beta}^{(1)}, \dots, \boldsymbol{\beta}^{(r)} \in A$, with $r \geq 2$, satisfying the following properties:

(1) $\boldsymbol{\beta}^{(j)} \in \Delta^S_{F_j}(\boldsymbol{\alpha})$ for some non-empty set $F_j \subsetneq \{1, \dots, d\}$.

(2) For every distinct $j, k \in \{1, \dots, r\}$, $\boldsymbol{\alpha} = \boldsymbol{\beta}^{(j)} \wedge \boldsymbol{\beta}^{(k)}$.

(3) $\bigcap_{k=1}^{r} F_k = \emptyset$.

In this case, we write $\boldsymbol{\alpha} = \boldsymbol{\beta}^{(1)} \,\tilde{\curlywedge}\, \boldsymbol{\beta}^{(2)} \cdots \tilde{\curlywedge}\, \boldsymbol{\beta}^{(r)}$.

Furthermore, given $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_d)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_d)$ in $\mathbb{N}^d$, we say that $\boldsymbol{\alpha} \leq\leq \boldsymbol{\beta}$ if and only if either $\boldsymbol{\alpha} = \boldsymbol{\beta}$ or $\alpha_i < \beta_i$ for every $i \in \{1, \dots, d\}$. In the second case, we say that $\boldsymbol{\beta}$ *dominates* $\boldsymbol{\alpha}$ and use the notation $\boldsymbol{\alpha} \ll \boldsymbol{\beta}$.

The partition of $\mathbf{Ap}(S, \boldsymbol{\omega})$ is defined in the following way.

Definition 2.1. Let $A = \mathbf{Ap}(S, \boldsymbol{\omega})$. Set

$$B^{(1)} := \{\boldsymbol{\alpha} \in A : \boldsymbol{\alpha} \text{ is maximal with respect to } \leq\leq\},$$
$$C^{(1)} := \{\boldsymbol{\alpha} \in B^{(1)} : \boldsymbol{\alpha} = \boldsymbol{\beta}^{(1)} \,\tilde{\curlywedge} \cdots \tilde{\curlywedge}\, \boldsymbol{\beta}^{(r)} \text{ for } 1 < r \leq d \text{ and } \boldsymbol{\beta}^{(k)} \in B^{(1)}\},$$
$$D^{(1)} := B^{(1)} \setminus C^{(1)}.$$

For $i > 1$, assume that $D^{(1)}, \dots, D^{(i-1)}$ have been defined and set inductively

$$B^{(i)} := \left\{\boldsymbol{\alpha} \in A \setminus \left(\bigcup_{j<i} D^{(j)}\right) : \boldsymbol{\alpha} \text{ is maximal with respect to } \leq\leq \right\},$$
$$C^{(i)} := \{\boldsymbol{\alpha} \in B^{(i)} : \boldsymbol{\alpha} = \boldsymbol{\beta}^{(1)} \,\tilde{\curlywedge} \cdots \tilde{\curlywedge}\, \boldsymbol{\beta}^{(r)} \text{ for } 1 < r \leq d \text{ and } \boldsymbol{\beta}^{(k)} \in B^{(i)}\},$$
$$D^{(i)} := B^{(i)} \setminus C^{(i)}.$$

By construction, $D^{(i)} \cap D^{(j)} = \emptyset$, for any $i \neq j$ and, since the set $S \setminus A = \boldsymbol{\omega} + S$ has a conductor, there exists $N \in \mathbb{N}_+$ such that $A = \bigcup_{i=1}^{N} D^{(i)}$. As in [16], we prefer to enumerate the sets in this partition in increasing order setting $A_i := D^{(N-i)}$. Hence, $A = \bigcup_{i=0}^{N-1} A_i$. We call the sets A_i the *levels* of A.

Notice that in the previous works [16, 17], the levels are enumerated from 1 to N. In this paper, we prefer to shift them and start from 0 in order to adapt our notation to the one in [5].

In [16, Theorem 4.4], it is proved that the number of levels of the Apéry set $\mathbf{Ap}(S, \boldsymbol{\omega})$ is equal to $\sum_{i=1}^{d} \omega_i$.

We recall that if $\boldsymbol{\alpha}, \boldsymbol{\beta} \in A$, $\boldsymbol{\alpha} \ll \boldsymbol{\beta}$, and $\boldsymbol{\alpha} \in A_i$, then $\boldsymbol{\beta} \in A_j$ for some $j > i$. Moreover, the last level of the partition is $A_{N-1} = \Delta(\boldsymbol{\gamma}_E) = \Delta^S(\boldsymbol{\gamma}_E)$ (here, $E = \boldsymbol{\omega} + S$). If S is local, then $A_0 = \{\mathbf{0}\}$.

Other basic properties of the Apéry set and its partition in levels are listed in [16, Lemma 2.3].

In [5], a slightly different partition in levels is defined for the Apéry set, only in the case of plane algebroid curves with two branches. However, it is proved in [17, Proposition 5.1] that in the case of Apéry sets of plane algebroid curves, the partition in [5] coincides with the one given in Definition 2.1. For this reason, since in this article we deal with plane curves, the results in [5] can be used as starting point of the inductive arguments in our work, even if we work with a partition in levels defined in a different way.

In the introduction of [5], it is mentioned that all the results in that paper until Theorem 4.1 can be proved analogously for arbitrary $d \geq 2$. We discuss this fact more specifically, showing first a way to present a plane algebroid curve as a finite module over a power series ring in one variable. The following extends the content of [5, Discussion, page 6] and is independent of the characteristic of the base field.

Proposition 2.2. *Let $\mathcal{O} = K[\![X,Y]\!]/I$ be an algebroid plane curve with d branches. Then, we can always write*

$$\mathcal{O} = K[\![x]\!] + K[\![x]\!]y + K[\![x]\!]y^2 + \cdots + K[\![x]\!]y^{e-1},$$

where $v(x) = (e_1, \dots, e_d) = \min(v(\mathcal{O}) \setminus \{(0,\dots,0)\})$, $e_1 + \cdots + e_d = e$.

Proof. We can assume $I = (H_1 \cdots H_d)$ with $H_1, \dots, H_d$ irreducible elements and pairwise coprime. Let us denote $\mathcal{O}$ also by $K[\![x,y]\!]$, where $x = X + I$ and $y = Y + I$. If the d branches defined by $H_1, \dots, H_d$ have all the same tangent, we can assume it is $Y = 0$ and, according to Weierstrass' Preparation Theorem, we can assume that $H_j = Y^{e_j} + \sum_{i=0}^{e_j-1} a_i(X)Y^i$ where e_j is the minimal power such that H_j contains a pure power aY^{e_j}, with $a \in K \setminus \{0\}$, and $a_i(X)$ are all non-invertible power series in $K[\![X]\!]$. Thus, $H_1 \cdots H_d = Y^e + \sum_{i=0}^{e-1} c_i(X)Y^i$ where $e = e_1 + \cdots + e_d$ is the multiplicity of the curve and $c_i(X)$ are all non-invertible.

If instead the tangents of the d-branches are not all the same, we can assume that at least one is $Y = 0$ and, as above, $H_j = Y^{e_j} + \sum_{i=0}^{e_j-1} a_i(X)Y^i$ for each branch H_j with tangent $Y = 0$. Then, for each branch H_k with a tangent different from $Y = 0$, if we write it as $H_k(X+Y, Y)$, we get a term Y^{e_k} where e_k is the minimal degree of the non-zero terms of H_k. Hence, after applying the substitution $X = X + Y$ and Weierstrass' Preparation Theorem, we get again $H_1 \cdots H_d = Y^e + \sum_{i=0}^{e-1} c_i(X)Y^i$ where $e = e_1 + \cdots + e_d$ is the multiplicity of the curve and $c_i(X)$ are all non-invertible.

It is clear that, in both cases, we can express $\mathcal{O}$ as a $K[\![x]\!]$-module minimally generated by $1, y, y^2, \dots, y^{e-1}$, with $v(x) = (e_1, \dots, e_d)$ and $e_1 + \cdots + e_d = e$. ∎

Remark 2.3. Let us keep the same notations of the previous proposition. Let $F, G \in \mathcal{O}$ be two elements such that $\mathcal{O}$ is a $K[\![F]\!]$-module minimally generated by the elements $1, G, G^2, \dots, G^{N-1}$, with $N = n_1 + \cdots + n_d$ and $v(F) = (n_1, \dots, n_d)$. Hence, $\mathcal{O} \cong K[\![X,Y]\!]/(\Phi)$, where $\Phi(X,Y) = Y^N + \sum_{i=0}^{N-1} b_i(X)Y^i$ comes from the relation of dependence of G over $K[\![F]\!]$ in degree N. Indeed, there is a surjective homomorphism

$\varphi : K[\![X,Y]\!] \to \mathcal{O}$, mapping X to F and Y to G, whose kernel contains (Φ). Now, since $K[\![X,Y]\!]$ is a 2-dimensional UFD, $\ker\varphi$ has to be the intersection of d principal prime ideals $P_1,\dots,P_d$; hence, $P_i = (H_i)$ and $\ker\varphi = (H_1\cdots H_d)$. Moreover, $H_1\cdots H_d$ divides Φ, so it has to be of the form $Y^j + \psi(X,Y)$, with $j \le N$, and since $\mathcal{O}$ is minimally generated by $1, G, G^2,\dots,G^{N-1}$ as $K[\![F]\!]$-module, then $j = N$ and $(H_1\cdots H_d) = (\Phi)$.

Notice that the classes $x = X + I$, $y = Y + I \in \mathcal{O}$ always satisfy the condition requested for F and G. Hence, by Proposition 2.2, we can always assume that $\mathcal{O} = K[\![x]\!] + K[\![x]\!]y + K[\![x]\!]y^2 + \cdots + K[\![x]\!]y^{e-1}$, where $v(x) = (e_1,\dots,e_d) = \min(v(\mathcal{O}\setminus\{(0,\dots,0)\})$, $e_1+\cdots+e_d = e$. Moreover, up to replacing y with $y+\alpha x$ (with $\alpha\in K$), we can choose y in such a way that $v(y) = (r_1,\dots,r_d)$ with $r_i > e_i$ for those indices i such that H_i has tangent $Y = 0$ and $r_j \ge e_j$ for the remaining indexes.

As consequences of Proposition 2.2, we can state the two following results (with the same identical proofs) [5, Proposition 3.8 and Theorem 4.1].

Let $\mathcal{O} = K[\![x]\!] + K[\![x]\!]y + K[\![x]\!]y^2 + \cdots + K[\![x]\!]y^{e-1}$ be a plane curve expressed as in Proposition 2.2. The element $\boldsymbol{e} = (e_1,\dots,e_d)$ is as usual the minimal element of $v(\mathcal{O})$ having all components distinct from zero. Set $R_0 = K$ and for $i = 1,\dots,e-1$,

$$R_i = K[\![x]\!] + K[\![x]\!]y + \cdots + K[\![x]\!]y^i.$$

Similarly, set $T_0 = K$, and for $i = 1,\dots,e-1$,

$$T_i = \big\{y^i + \phi \mid \phi \in R_{i-1} \text{ and } v(y^i+\phi) \notin v(R_{i-1})\big\}.$$

Proposition 2.4. *Let A_i denote the levels of* $\mathbf{Ap}(v(\mathcal{O}))$. *Then, for $i = 0,\dots,e-1$, $A_i = v(T_i)$.*

Theorem 2.5. *Let $\mathcal{B}(\mathcal{O})$ denote the blow-up of $\mathcal{O}$ and suppose $\mathcal{B}(\mathcal{O})$ to be also local. Let A'_i denote the levels of* $\mathbf{Ap}(v(\mathcal{B}(\mathcal{O})),\boldsymbol{e})$. *Then, for $i = 0,\dots,e-1$, one has $A'_i = A_i - i\boldsymbol{e}$.*

The aim of the next sections is to extend Theorem 2.5 to the case where the blow-up of $\mathcal{O}$ is not local. In this case it is no more true that $\mathcal{B}(\mathcal{O})$ can be presented as a quotient of $K[\![X,Y]\!]$, so we cannot apply Proposition 2.2 and Remark 2.3. To proceed in this direction, we will need to consider the levels of the Apéry set of non-local good semigroups.

3. Preliminary results on good semigroups

In this section, we prove several technical results on good semigroups that will be needed in Section 4. The proofs often require the combinatorial methods developed in the previous works [16, 17]. We start by recalling the main result of [17, Section 4], restated with the new notation, renumbering the levels of the Apéry set (or more in general of the complement of a good ideal) starting from 0 rather than from 1.

Along the section $S \subseteq \mathbb{N}^d$ will denote an arbitrary good semigroup (not necessarily local) and $A = S \setminus E = \bigcup_{i=0}^{N-1} A_i$ the complement of a good ideal E, partitioned in levels as in Definition 2.1. If S is numerical, $A = \{w_0, \dots, w_{N-1}\}$ is finite and we set $A_i = \{w_i\}$.

We define a *level function* $\lambda : S \to \{0, \dots, N\}$ in the following way:

- If $\boldsymbol{\alpha} \in A_i$, $\lambda(\boldsymbol{\alpha}) = i$.
- If $\boldsymbol{\alpha} \notin A$, $\lambda(\boldsymbol{\alpha}) = 1 + \max\{i$ such that $\boldsymbol{\alpha} > \boldsymbol{\theta}$ for some $\boldsymbol{\theta} \in A_i\}$.

THEOREM 3.1 ([17, Theorem 4.5]). *Let $S = S_1 \times S_2$ be a direct product of two arbitrary good semigroups. Let $E \subsetneq S$ be a good ideal and set $A := S \setminus E$. Then, given $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}) \in A$ ($\boldsymbol{\alpha}^{(i)} \in S_i$, for $i = 1, 2$), the level of $\boldsymbol{\alpha}$ in A is equal to*

$$\lambda(\boldsymbol{\alpha}^{(1)}) + \lambda(\boldsymbol{\alpha}^{(2)}).$$

We recall that two elements $\boldsymbol{\alpha}, \boldsymbol{\beta} \in S$ are *consecutive* if $\boldsymbol{\alpha} < \boldsymbol{\beta}$ and there are no elements $\boldsymbol{\delta} \in S$ such that $\boldsymbol{\alpha} < \boldsymbol{\delta} < \boldsymbol{\beta}$. The function λ has the following property.

LEMMA 3.2. *Let S be any good semigroup and let $\boldsymbol{\alpha} \in S$. Let $E \subsetneq S$ be a good ideal and set $A := S \setminus E$. Then, for $j < N$, $\lambda(\boldsymbol{\alpha}) \le j$ if and only if there exists $\boldsymbol{\beta} \in A_j$ such that $\boldsymbol{\alpha} \le \boldsymbol{\beta}$. In particular, if $\boldsymbol{\theta} \in S$ and $\boldsymbol{\alpha} \le \boldsymbol{\theta}$, then $\lambda(\boldsymbol{\alpha}) \le \lambda(\boldsymbol{\theta})$.*

PROOF. If $\boldsymbol{\alpha} \in A$, this is straightforward. Suppose $\boldsymbol{\alpha} \in E$ and set $\lambda(\boldsymbol{\alpha}) = h$. Let $\boldsymbol{\theta} \in A$ be a maximal element such that $\boldsymbol{\theta} < \boldsymbol{\alpha}$. By the definition of λ, $\boldsymbol{\theta} \in A_{h-1}$. Now, if there exists $\boldsymbol{\beta} \in A_j$ such that $\boldsymbol{\alpha} \le \boldsymbol{\beta}$, it follows that $j \ge h - 1$. If $j \ge h$, we are done. If $j = h - 1$, by [17, Lemma 2.8], we get $\boldsymbol{\alpha} \in A_{h-1}$ and this is a contradiction.

Conversely, if $\lambda(\boldsymbol{\alpha}) = h = N$, then clearly $\boldsymbol{\theta} \in A_{N-1}$ and there are no elements of A larger than or equal to $\boldsymbol{\alpha}$. Thus, we suppose $h < N$ and prove that we can find $\boldsymbol{\beta} \in A_h$ such that $\boldsymbol{\alpha} \le \boldsymbol{\beta}$. Clearly, no elements of A_h are smaller than $\boldsymbol{\alpha}$. Let $\boldsymbol{\beta} \in A_h$ be such that the element $\boldsymbol{\delta} = \boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ is the maximal possible. If $\boldsymbol{\delta} = \boldsymbol{\alpha}$, we are done; hence, suppose by way of contradiction that $\boldsymbol{\delta} < \boldsymbol{\alpha}$. By the assumption $\lambda(\boldsymbol{\alpha}) = h$, we also have $\boldsymbol{\delta} < \boldsymbol{\beta}$. We can fix coordinates saying that $\boldsymbol{\alpha} \in \Delta_U^S(\boldsymbol{\delta})$ and $\boldsymbol{\beta} \in \Delta_V^S(\boldsymbol{\delta})$ with $V \supseteq \widehat{U}$. We need to produce an element $\boldsymbol{\theta} \in A_h$ such that $\boldsymbol{\theta} \wedge \boldsymbol{\alpha} > \boldsymbol{\delta}$. We can do it proceeding exactly as in Cases 1 and 2 of the proof of [17, Proposition 2.10], noticing that $\boldsymbol{\alpha} \in E$ and therefore if $\boldsymbol{\delta}$ and $\boldsymbol{\beta}$ are consecutive, we cannot have $\boldsymbol{\delta} \in A$ by [17, Theorem 2.7] (for the convenience of the reader, we are adopting here the same notation of that proof, except for the fact that the index of the level of $\boldsymbol{\beta}$ is shifted by one). Since in this way we find a contradiction, we must have $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = \boldsymbol{\alpha}$ and $\boldsymbol{\beta} > \boldsymbol{\alpha}$. ■

The next lemma proves the existence of ascending sequences of elements, one for each level, satisfying some extra condition on their respective positions.

Lemma 3.3. *Let S be an arbitrary good semigroup. Let $E \subsetneq S$ be a good ideal and set $A := S \setminus E$. Then, for every $i > j \geq 0$ and $\boldsymbol{\alpha} \in A_i$, there exists $\boldsymbol{\beta} \in A_j$ such that $\boldsymbol{\beta} < \boldsymbol{\alpha}$, and if $\boldsymbol{\alpha} \in \Delta^S_U(\boldsymbol{\beta})$, then $\widetilde{\Delta}^S_{\widehat{U}}(\boldsymbol{\beta}) \subseteq A$.*

Proof. Observe that if there exists $\boldsymbol{\beta} \in A_j$ such that $\boldsymbol{\alpha} \gg \boldsymbol{\beta}$, the thesis is satisfied since $U = \emptyset$ and $\widetilde{\Delta}^S_{\widehat{U}}(\boldsymbol{\beta}) = \{\boldsymbol{\beta}\} \subseteq A$. First, let us consider the case $j = i - 1$. This case will also provide a base for an induction on i. By [17, Proposition 2.10], there exists $\boldsymbol{\beta} \in A_{i-1}$ such that $\boldsymbol{\beta} < \boldsymbol{\alpha}$. We can assume that there are no other elements in A_{i-1} between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Let $\boldsymbol{\theta} \in S$ be an element consecutive to $\boldsymbol{\beta}$ such that $\boldsymbol{\beta} < \boldsymbol{\theta} \leq \boldsymbol{\alpha}$. Hence, $\boldsymbol{\theta} \in \Delta^S_H(\boldsymbol{\beta})$ with $H \supseteq U$ and $\widetilde{\Delta}^S_{\widehat{U}}(\boldsymbol{\beta}) \subseteq \widetilde{\Delta}^S_{\widehat{H}}(\boldsymbol{\beta})$. If by way of contradiction $\widetilde{\Delta}^S_{\widehat{U}}(\boldsymbol{\beta}) \nsubseteq A$, by [16, Theorem 2.8], the element $\boldsymbol{\theta} \in A_{i-1}$. In particular, $\boldsymbol{\theta} < \boldsymbol{\alpha}$, and this contradicts the fact that no elements between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are in A_{i-1}.

By induction, after finding $\boldsymbol{\beta} \in A_{i-1}$ satisfying the thesis, taking $j < i - 1$, we can find $\boldsymbol{\delta} \in A_j$ such that $\boldsymbol{\beta} \in \Delta^S_V(\boldsymbol{\delta})$ and $\widetilde{\Delta}^S_{\widehat{V}}(\boldsymbol{\delta}) \subseteq A$. It follows that $\boldsymbol{\alpha} \in \Delta^S_H(\boldsymbol{\delta})$ with $H \subseteq U \cap V$. Since $\widehat{H} \supseteq \widehat{U} \cup \widehat{V} \supseteq \widehat{V}$, we get $\widetilde{\Delta}^S_{\widehat{H}}(\boldsymbol{\delta}) \subseteq \widetilde{\Delta}^S_{\widehat{V}}(\boldsymbol{\delta}) \subseteq A$. This concludes the proof. ∎

Remark 3.4. The proof of Lemma 3.3 shows that, starting from an element $\boldsymbol{\alpha}^{(N-1)} \in A_{N-1}$, we can find a chain of elements

$$\mathbf{0} = \boldsymbol{\alpha}^{(0)} < \boldsymbol{\alpha}^{(1)} < \cdots < \boldsymbol{\alpha}^{(j)} < \cdots < \boldsymbol{\alpha}^{(N-2)} < \boldsymbol{\alpha}^{(N-1)}$$

such that for every $j = 0, \ldots, N-1$, $\boldsymbol{\alpha}^{(i)} \in A_i$, and for every $k < j$, if $\boldsymbol{\alpha}^{(j)} \in \Delta^S_U(\boldsymbol{\alpha}^{(k)})$ for some $U \neq \emptyset$, then $\widetilde{\Delta}^S_{\widehat{U}}(\boldsymbol{\alpha}^{(j)}) \subseteq A$.

All the results from now until the end of the section are very technical and use the notion of subspaces of a good semigroup introduced in [16]. The only result needed in the next sections is the statement of Lemma 3.8.

Let $S \subseteq \mathbb{N}^d$ be an arbitrary good semigroup and let $A = \bigcup_{i=0}^{N-1}$ be its Apéry set with respect to a non-zero element $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_d)$. Set as usual $E = S \setminus A$ and denote the conductor of E by $\boldsymbol{c_E} = (c_1, \ldots, c_d) = \boldsymbol{\gamma} + \boldsymbol{\omega} + \mathbf{1}$.

The following definition and properties are taken from [16, Section 3].

We recall the next useful fact which describes the behavior of the levels of the Apéry set for large elements.

Proposition 3.5 ([16, Proposition 2.9]). *Let $\boldsymbol{c}$ be the conductor of $E = \boldsymbol{\omega} + S$, let $\boldsymbol{\delta} \geq \boldsymbol{c}$, and let $\boldsymbol{\alpha} \in \mathbb{N}^d$ be such that $\boldsymbol{\alpha} \nleq \boldsymbol{\delta}$ and $\boldsymbol{\theta} = \boldsymbol{\alpha} \wedge \boldsymbol{\delta}$. Let $U = \{i : \alpha_i < \delta_i\}$. Then, the following conditions are equivalent:*

(1) $\boldsymbol{\alpha} \in A_j$;

(2) $\widetilde{\Delta}_U(\boldsymbol{\alpha}) \cup \{\boldsymbol{\alpha}\} \subset A_j$;

(3) $\widetilde{\Delta}_U(\boldsymbol{\theta}) \cup \{\boldsymbol{\theta}\} \subset A_j$.

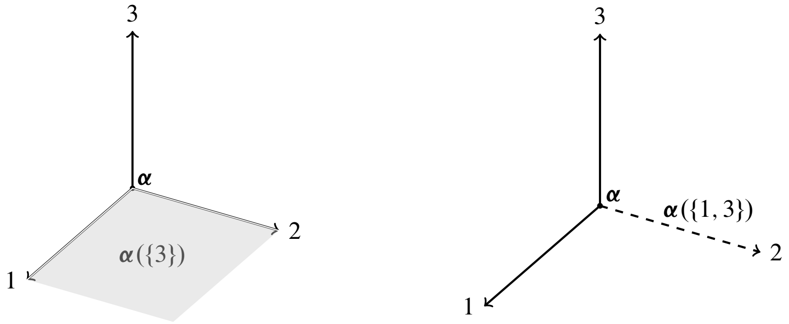

FIGURE 1. In the figure on the left is represented the plane $\alpha(\{3\})$ which is a subspace of dimension 2. In the figure on the right, the dashed line $\alpha(\{1,3\})$ represents a subspace of dimension 1.

In particular, as a consequence, if $\boldsymbol{\delta} = \boldsymbol{c}$*, the Apéry set* $A = \mathbf{Ap}(S, \boldsymbol{\omega})$ *and its levels* A_j *depend only on the finite subset* $\{\boldsymbol{\alpha} \in A : \boldsymbol{\alpha} \leq \boldsymbol{c}\}$.

DEFINITION 3.6. Pick a non-empty set $U \subseteq \{1, \dots, d\}$. For $\boldsymbol{\alpha} \in \mathbb{N}^d$ such that $\alpha_j = c_j$ for all $j \in \widehat{U}$, define

$$\boldsymbol{\alpha}(U) = \widetilde{\Delta}_U(\boldsymbol{\alpha}) \cup \{\boldsymbol{\alpha}\}.$$

We say that $\boldsymbol{\alpha}(U)$ is a U*-subspace* (or simply a subspace) of $\mathbb{N}^d$ (see Figure 1). We have the following:

- If $\boldsymbol{\alpha} \in E$, then $\boldsymbol{\alpha}(U) \subseteq E$, and in this case we say that it is a U-subspace of E, or that $\boldsymbol{\alpha}(U) \in E(U)$.
- If $\boldsymbol{\alpha} \in A$, then $\boldsymbol{\alpha}(U) \subseteq A$, and in this case we say that it is a U-subspace of A. In particular, if $\boldsymbol{\alpha} \in A_i$, the subspace $\boldsymbol{\alpha}(U) \subseteq A_i$, and we write shortly that $\boldsymbol{\alpha}(U) \in A_i(U)$.

Observe that if $\boldsymbol{\delta}(V)$ is a subspace, $U \supseteq V$ and $\boldsymbol{\alpha} \in \widetilde{\Delta}_V(\boldsymbol{\delta})$, then $\boldsymbol{\alpha}(U) \subseteq \boldsymbol{\delta}(V)$.

The dimension of a subspace is defined according to its intuitive geometric representation. We say that $\boldsymbol{\alpha}(U)$ has a dimension equal to the cardinality of $\widehat{U}$. Indeed, the subspaces of dimension zero are points, those of dimension one are lines, those of dimension two are planes, and so on.

The proof of the following lemma is based on the part of the argument used to prove [16, Theorem 4.4].

LEMMA 3.7. *Fix an index* $i \in \{1, \dots, d\}$*. Let* V *be a non-empty set of indexes not containing* i *and set* $W := V \cup \{i\}$*. Choose a subspace of the form* $\boldsymbol{\theta}(V)$ *contained in* A *such that* $\boldsymbol{\theta}$ *is a minimal element for which a subspace of* A *of such form exists. Then, there exist* ω_i *distinct subspaces of the form* $\boldsymbol{\beta}^{(0)}(W), \dots, \boldsymbol{\beta}^{(\omega_i-1)}(W) \subseteq \bigcup_{l<\lambda(\boldsymbol{\theta})} A_l$ *such that the coordinates* $\boldsymbol{\beta}_i^{(0)}, \dots, \boldsymbol{\beta}_i^{(\omega_i-1)}$ *form a complete system of residues modulo* ω_i.

To help the reader, we add separately the proof in the case $d = 2$, and then the proof of the general case.

Proof of Lemma 3.7 in the case $d = 2$. First, set $i = 1$. Clearly, by the definition of the conductor $\boldsymbol{c_E} = (c_1, c_2)$ of the good ideal E, there are infinitely many elements $\boldsymbol{\alpha} \in E$ such that $\alpha_2 = c_2$. Thus, for every $j = 0, \dots, \omega_1 - 1$, we can find a unique minimal element $\boldsymbol{\alpha}^{(j)} \in E$ such that $\alpha_1^{(j)} \equiv j \mod \omega_1$ and $\alpha_2^{(j)} = c_2$. Hence, for every j, there exists $n_j \geq 1$ such that $\boldsymbol{\alpha}^{(j)} - n_j \boldsymbol{\omega} \in A$. For $\gamma \in S$, set $H_1(\gamma) = \{\boldsymbol{\delta} \in S | \delta_1 = \gamma_1\}$. If $H_1(\boldsymbol{\alpha}^{(j)} - n_j \boldsymbol{\omega}) \cap E \neq \emptyset$, we can continue subtracting multiples of $\boldsymbol{\omega}$ to some element in $H_1(\boldsymbol{\alpha}^{(j)}) \cap E$ until we find an element $\boldsymbol{\beta}^{(j)} \in A$ such that $\beta_1^{(j)} \equiv \alpha_1^{(j)} \equiv j \mod \omega_1$ and $H_1(\boldsymbol{\beta}^{(j)}) \subseteq A$. Without loss of generality, we can assume $\boldsymbol{\beta}^{(j)}$ to be the minimal element of $H_1(\boldsymbol{\beta}^{(j)})$. Now let $\boldsymbol{\theta} \in A$ be the minimal element of S such that $\theta_1 = c_1$ and $\Delta_2(\boldsymbol{\theta}) \subseteq A$. We show that $\lambda(\boldsymbol{\beta}^{(j)}) < \lambda(\boldsymbol{\theta})$ for every j. Indeed, by the minimality of $\boldsymbol{\beta}^{(j)}$ in $H_1(\boldsymbol{\beta}^{(j)})$, using property (G1), we must have $\beta_2^{(j)} \leq \theta_2$, and by the construction of $\boldsymbol{\beta}^{(j)}$, we must have $\beta_1^{(j)} < c_1 = \theta_1$. Using that $\Delta_1^S(\boldsymbol{\beta}^{(j)}) \subseteq H_1(\boldsymbol{\beta}^{(j)}) \subseteq A$, we get the inequality $\lambda(\boldsymbol{\beta}^{(j)}) < \lambda(\boldsymbol{\theta})$ by [11, Lemma 2 (3)] together with the definition of levels. ■

Proof of Lemma 3.7 for arbitrary d. Relabeling the indexes, we can assume that $W = \{1, \dots, i\}$ and $V = \{1, \dots, i-1\}$. Denoting by $l = \lambda(\boldsymbol{\theta})$, we have that $\boldsymbol{\theta}(V) \subseteq A_l$; hence, it is clear that there exist infinitely many W-subspaces contained in level A_l (a space of dimension j contains infinitely many spaces of dimension $j-1$). Among them, for every $j = 1, \dots, w_i$, there exist subspaces $\boldsymbol{\theta}^j(W) \in A_l(W)$ minimal with respect to the property of having $\boldsymbol{\theta}_i^j \equiv j \mod w_i$.

For each j, we show that $\widetilde{\Delta}_i^E(\boldsymbol{\theta}^j(W)) \neq \emptyset$. Indeed, after fixing $\boldsymbol{\theta}^j(W)$, using the fact that there are infinitely many W-subspaces contained in $\boldsymbol{\theta}(V)$, we can find $\boldsymbol{\theta}'(W) \in A_l(W)$ such that $\theta_i' > \theta_i^j$ (observe that since they are in the same level necessarily $\theta_h' = \theta_h^j$ for some $h < i$). Now, if we assume $\widetilde{\Delta}_i^E(\boldsymbol{\theta}^j(W)) = \emptyset$, applying [16, Theorem 3.7] to $\boldsymbol{\theta}^j(W)$ and $\boldsymbol{\theta}'(W)$, we can write

$$\boldsymbol{\theta}^j(W) = \boldsymbol{\theta}'(W) \mathbin{\widetilde{\curlywedge}} \boldsymbol{\alpha}^1(W) \mathbin{\widetilde{\curlywedge}} \cdots \mathbin{\widetilde{\curlywedge}} \boldsymbol{\alpha}^r(W),$$

where $\boldsymbol{\alpha}^m(W) \in \widetilde{\Delta}_i^S(\boldsymbol{\theta}^j(W)) \subseteq A(W)$ and we may assume $\boldsymbol{\alpha}^m(W)$ to be consecutive to $\boldsymbol{\theta}^j(W)$ for all $m \in 1, \dots, r$. By [16, Theorem 3.9.1], for every m, $\boldsymbol{\alpha}^m(W) \in A_j(W)$ implies that $\boldsymbol{\theta}^j(W)$ has to be in a lower level. This is a contradiction (for a graphical representation, see Figure 2 (a)).

Hence, we can set $\boldsymbol{\tau}^j(W)$ to be a minimal element in $\widetilde{\Delta}_i^E(\boldsymbol{\theta}^j(W))$. We define $\overline{\boldsymbol{\omega}}$ such that $\overline{\omega_k} = \omega_k$ if $k \in W$ and $\overline{\omega_k} = c_k$ otherwise, and, starting from $\boldsymbol{\tau}^j(W)$ and subtracting multiples of $\overline{\boldsymbol{\omega}}(W)$, we find a unique $m_j \geq 1$ such that $\boldsymbol{\tau}^j(W) - m_j \overline{\boldsymbol{\omega}}(W) =: \boldsymbol{\beta}^j(W) \in A(W)$ (see Figure 2 (b)).

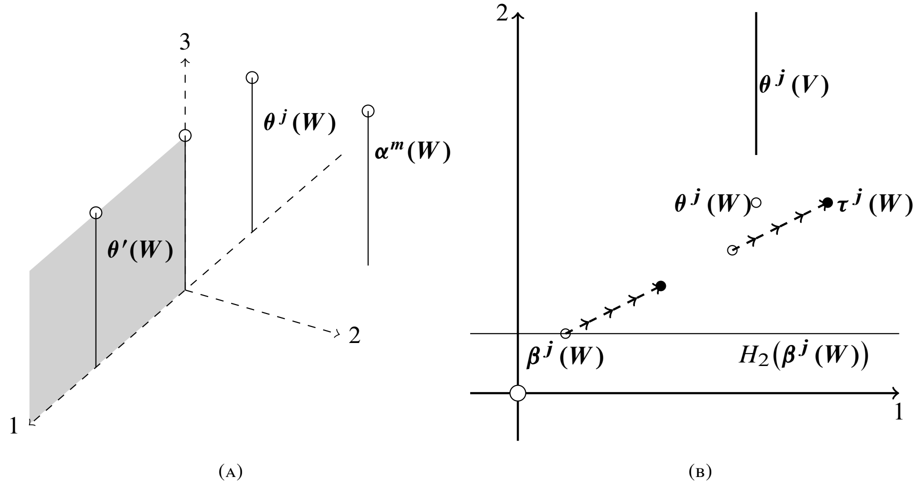

FIGURE 2. (A) We have $d = 3$, $U = \{1, 2\}$; $\boldsymbol{\theta}^j(W), \boldsymbol{\theta}'(W), \boldsymbol{\alpha}^m(W)$ are lines. (B) This is a perspective from "above" of the case $d = 3$, $U = \{1, 2\}$, $V = \{1\}$. In this case, $\boldsymbol{\theta}^j(V)$ is a plane contained in A; $\boldsymbol{\theta}^j(W), \boldsymbol{\tau}^j(W), \boldsymbol{\beta}^j(W)$ are lines.

Consider now the set $H_i(\boldsymbol{\alpha}^j(W)) = \{\boldsymbol{\beta}(U) \subseteq S | \beta_i = \alpha_i^j\}$. In the case this set contains some subspace of E, starting by one of these subspaces and subtracting multiples of $\overline{\boldsymbol{\omega}}(U)$, we can repeat the process and, after changing names, we can finally assume to have a collection of subspaces $\boldsymbol{\beta}^1(W), \ldots, \boldsymbol{\beta}^{w_k}(W) \in A(W)$ such that for every j, $\beta_i^j \equiv \theta_i^j \equiv j \mod w_i$ and $H_i(\boldsymbol{\beta}^j(W)) \subseteq A(W)$. We can further replace $\boldsymbol{\beta}^j(W)$ by another subspace, and assume that $\boldsymbol{\beta}^j(W)$ is the minimal W-subspace in the set $H_i(\boldsymbol{\beta}^j(W))$ (this minimal subspace is well defined by property (G1); see the results in [16, Sections 3 and 4]).

To conclude, notice that for every j, the level of $\boldsymbol{\beta}^j(W)$ has to be strictly lower than l since $\boldsymbol{\theta}^j(U)$ has been chosen to be the minimal in A_l having k-th component congruent to j modulo w_j. ■

LEMMA 3.8. *Let S and A be defined as above. Then, it is possible to find a sufficiently large element $\boldsymbol{\eta} \gg \boldsymbol{\gamma} + \boldsymbol{\omega}$ such that, given any index i and any element $\boldsymbol{\alpha} \in A$ such that $\alpha_i \geq \eta_i$, there exists $\boldsymbol{\delta} \in \Delta_i^E(\boldsymbol{\alpha})$ such that $\boldsymbol{\delta} = m\boldsymbol{\omega} + \boldsymbol{\beta}$, with $m \geq 1$, $\boldsymbol{\beta} \in A$, and $\lambda(\boldsymbol{\beta}) < \lambda(\boldsymbol{\alpha})$.*

PROOF. Fixing a coordinate i, we want to find an element $\boldsymbol{\eta}(i) \gg \boldsymbol{\gamma} + \boldsymbol{\omega}$ such that if $\alpha_i \geq \eta(i)_i$, then there exists $\boldsymbol{\delta} \in \Delta_i^E(\boldsymbol{\alpha})$ of the required form. Then, we can simply define $\boldsymbol{\eta}$ as the minimal element of S that is larger than or equal to all the elements $\boldsymbol{\eta}(1), \ldots, \boldsymbol{\eta}(d)$ with respect to the partial ordering $\leq$.

Let V be a non-empty set of indexes not containing i and set $W := V \cup \{i\}$. Given the minimal subspace of the form $\boldsymbol{\theta}(V)$ contained in A, by Lemma 3.7, we can find ω_i distinct subspaces of the form $\boldsymbol{\beta}^{(0)}(W), \dots, \boldsymbol{\beta}^{(\omega_i-1)}(W) \subseteq \bigcup_{l<\lambda(\boldsymbol{\theta})} A_l$ such that the coordinates $\boldsymbol{\beta}_i^{(0)}, \dots, \boldsymbol{\beta}_i^{(\omega_i-1)}$ form a complete system of residues modulo ω_i. For every $j = 0, \dots, \omega_i - 1$, define $\boldsymbol{\tau}^{(j)} := \boldsymbol{\beta}^{(j)} + m_j\boldsymbol{\omega}$ where m_j is the minimal positive integer such that $\boldsymbol{\beta}^{(j)} + m_j\boldsymbol{\omega} \gg \boldsymbol{\gamma} + \boldsymbol{\omega}$. Then, set $\boldsymbol{\eta}(V)$ equal to the element $\boldsymbol{\tau}^{(j)}$ which has the largest i-coordinate. Finally, set $\boldsymbol{\eta}(i)$ to be the minimal element of S larger than or equal to all the elements $\boldsymbol{\eta}(V)$ for every V not containing i.

Now we can pick $\boldsymbol{\alpha} \in A$ and suppose that $\alpha_i \geq \eta(i)_i$. Since $\boldsymbol{\alpha}$ has at least one coordinate larger than the conductor, it belongs to an infinite subspace of A of the form $\boldsymbol{\theta}'(V)$ with $i \notin V$. In particular, V is non-empty and $\alpha_k \leq \gamma_k + \omega_k$ for all $k \in V$. Fixing this set V, we can take the elements $\boldsymbol{\beta}^{(j)}$ and $\boldsymbol{\tau}^{(j)}$ defined previously. Clearly, $\alpha_i \equiv \beta_i^{(j)}$ modulo ω_i for some j. Hence, there exists $m \geq 1$ such that

$$\alpha_i = \beta_i^{(j)} + m\omega_i \geq \eta(i)_i \geq \tau_i^{(j)} = \beta_i^{(j)} + m_j\omega_i.$$

Set $\boldsymbol{\delta} := \boldsymbol{\beta}^{(j)} + \boldsymbol{\varepsilon} + m\boldsymbol{\omega}$ where $\boldsymbol{\varepsilon}$ is an element of $\mathbb{N}^d$ such that $\varepsilon_k = 0$ for $k \in V \cup \{i\}$, and $\varepsilon_k > \alpha_k$ for the other coordinates. Notice that with these assumptions, $\boldsymbol{\beta}^{(j)} + \boldsymbol{\varepsilon} \in \boldsymbol{\beta}^{(j)}(W) \subseteq A$ and $\boldsymbol{\delta} \in S$ since it is larger than the conductor (notice that $m \geq m_j$). Observe that $\delta_i = \alpha_i$ and, since a subspace is all contained in the same level, observe also that $\lambda(\boldsymbol{\beta}^{(j)} + \boldsymbol{\varepsilon}) = \lambda(\boldsymbol{\beta}^{(j)}) < \lambda(\boldsymbol{\theta}) \leq \lambda(\boldsymbol{\theta}') = \lambda(\boldsymbol{\alpha})$. Furthermore, for $k \in V$, we have $\delta_k > \gamma_k + \omega_k \geq \alpha_k$, and for $k \notin W$, we have $\delta_k > \alpha_k$ by the definition of $\boldsymbol{\varepsilon}$. In conclusion, we obtain $\boldsymbol{\delta} \in \Delta_i^E(\boldsymbol{\alpha})$. ■

4. Semilocal rings associated with plane curves

In this section, we extend [5, Theorem 4.1] to the case where the blow-up of the coordinate ring of a plane curve is not local. In the first part of the section, we describe the level of the Apéry set of the value semigroup of a semilocal ring R as sets of values of specific subsets of R. In the second part, we describe how the levels of the Apéry set of the value semigroup behave when passing from the ring of a plane curve to its blow-up and vice-versa.

4.1. *The Apéry set of the value semigroup of a semilocal ring*

Let $R \cong \mathcal{O}_1 \times \cdots \times \mathcal{O}_c$ be a direct product of local rings $\mathcal{O}_j$ associated with plane algebroid curves defined over an infinite field K. For every $j = 1, \dots, c$, let $S_j \subseteq \mathbb{N}^{d_j}$ denote the value semigroup of $\mathcal{O}_j$. For every j, S_j is a local good semigroup (or a numerical semigroup). The value semigroup of R is $S = S_1 \times \cdots \times S_c \subseteq \mathbb{N}^d$ where $d = d_1 + \cdots + d_c$.

Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_d)$ be an element of S such that $\omega_i > 0$ for every $i = 1, \dots, d$. Let A be the Apéry set of S with respect to $\boldsymbol{\omega}$ and set $N := \omega_1 + \cdots + \omega_d$. The set A can be partitioned as $\bigcup_{i=0}^{N-1} A_i$. Let $F \in R$ be an element of value $\boldsymbol{\omega}$.

LEMMA 4.1. *Let $h_1, \dots, h_t \in R$ with $t \leq N-1$ be such that for every j,*

- $v(h_j) = \boldsymbol{\alpha}_j \in A$,
- $\boldsymbol{\alpha}_j < \boldsymbol{\alpha}_{j+1}$,
- *if $\boldsymbol{\alpha}_k \in \Delta_U^S(\boldsymbol{\alpha}_j)$ for some $k > j$ and $U \neq \emptyset$, then $\widetilde{\Delta}_{\widehat{U}}^S(\boldsymbol{\alpha}_j) \subseteq A$.*

Then, the images of $h_1, \dots, h_t$ modulo (F) are linearly independent over $K \cong \frac{K[\![F]\!]}{(F)}$.

PROOF. Call $\overline{h_j}$ the image of h_j modulo (F). Suppose $\sum_{j=1}^t a_j \overline{h_j} = 0$ for some $a_k \in K$ not all equal to zero. Then, $H := \sum_{j=1}^t a_j h_j \in (F)R$ and therefore $v(H) \notin A$. It follows that at least two coefficients a_j are non-zero, and without loss of generality, we can assume $a_1, a_2 \neq 0$. Clearly, $\boldsymbol{\alpha}_1 \not\ll \boldsymbol{\alpha}_2$; otherwise, we would have $v(H) = v(a_1 h_1) = \boldsymbol{\alpha}_1 \in A$. Thus, $\boldsymbol{\alpha}_2 \in \Delta_U^S(\boldsymbol{\alpha}_1)$ for some $U \neq \emptyset$. Since $\boldsymbol{\alpha}_2 \leq \boldsymbol{\alpha}_j$ for $j > 2$, it follows that $v(H) \in \widetilde{\Delta}_{\widehat{U}}^S(\boldsymbol{\alpha}_1) \subseteq A$. This is a contradiction. ■

SETTING 4.2. Let R, A, and F be defined as above. For an element $G \in R$ not divisible by F, set $R_0 = K$, and for $i = 1, \dots, N-1$,

$$R_i = K[\![F]\!] + K[\![F]\!]G + \cdots + K[\![F]\!]G^i. \tag{4.1}$$

Similarly, set $T_0 = K$, and for $i = 1, \dots, N-1$,

$$T_i = \left\{G^i + \phi \mid \phi \in R_{i-1} \text{ and } v(G^i + \phi) \notin v(R_{i-1})\right\}. \tag{4.2}$$

We want to prove that we can find G in such a way that $R = R_{N-1}$ and the equality $v(T_i) = A_i$ holds for every i. More precisely, we will prove the two following theorems.

THEOREM 4.3. *Adopt the notation of Setting 4.2. Then, there exists $G \in R$ such that*

$$R = K[\![F]\!] + K[\![F]\!]G + \cdots + K[\![F]\!]G^{N-1}. \tag{4.3}$$

THEOREM 4.4. *Adopt the notation of Setting 4.2 and define G as in Theorem 4.3. Then, for every $i = 0, \dots, N-1$,*

$$A_i = v(T_i).$$

REMARK 4.5. In the case where $R = \mathcal{O}_1$ is local, these results follow by Proposition 2.2, Remark 2.3, and Proposition 2.4.

By Remark 4.5, the results of the two theorems hold in particular in the case $d = 1$. Hence, to prove Theorems 4.3 and 4.4, we can work by induction on d, assuming that R is not local. It is sufficient then, slightly changing the notation, to assume that $R \cong \mathcal{O}_1 \times \mathcal{O}_2$ with $\mathcal{O}_1$ not necessarily local and $\mathcal{O}_2$ local. The value semigroup of R will be denoted by $S = S_1 \times S_2$ with $S_i \subseteq \mathbb{N}^{d_i}$ and $d = d_1 + d_2$.

We can thus write $F = (F_1, F_2)$ and $\boldsymbol{\omega} = (\boldsymbol{\omega}^{(1)}, \boldsymbol{\omega}^{(2)})$. Also $A^{(i)}$ will denote the Apéry set of S_i with respect to $\boldsymbol{\omega}^{(i)} \in S_i$ (the projection of $\boldsymbol{\omega}$ with respect to the coordinates in S_i). The number of levels of $A^{(i)}$ is equal to N_i, where N_i is the sum of the coordinates of $\boldsymbol{\omega}^{(i)}$.

For $h = (h_1, h_2) \in R$, we let $v(h) = (v^{(1)}(h_1), v^{(2)}(h_2))$ denote the value of h in the semigroup S.

The next proposition explains how to construct the power series G in the ring R.

PROPOSITION 4.6. *Adopt the notation of Setting 4.2. Then, there exists $G \in R$ such that, for every $j = 0, \dots, N-1$ and $\boldsymbol{\alpha} \in A_j$, we can find $\phi \in R_{j-1}$ such that $v(G^j + \phi) = \boldsymbol{\alpha}$.*

PROOF. We divide the proof in three parts. First, we prove the result for elements of the form $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^{(1)}, \mathbf{0})$ with $\boldsymbol{\alpha}^{(1)} \in A^{(1)}$, then we consider elements of the form $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^{(1)}, \mathbf{0})$ with $\boldsymbol{\alpha}^{(1)} \notin A^{(1)}$, and by analogy we obtain the same results also for all the elements of the form $\boldsymbol{\alpha} = (\mathbf{0}, \boldsymbol{\alpha}^{(2)})$ with $\boldsymbol{\alpha}^{(2)} \in S_2$ (our proof is independent of whether S_i is local or not). Finally, we will deal with the case $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)})$ with $\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)} \neq \mathbf{0}$.

As mentioned in the above paragraph, by induction on d, we can assume that Theorems 4.3 and 4.4 hold for S_1 and S_2 with respect to the elements F_1 and F_2. Hence, for $i = 1, 2$, there exists $G_i \in \mathcal{O}_i$ such that

$$\mathcal{O}_i = K[\![F_i]\!] + K[\![F_i]\!]G_i + \cdots + K[\![F_i]\!]G_i^{N_i-1}.$$

Before we treat each one of the described cases, we prove the next statement.

LEMMA 4.7. *Let L be a finite set of elements of the form $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^{(1)}, \mathbf{0})$, $\boldsymbol{\alpha}^{(1)} \in A_j^{(1)}$, $j \leq N_1 - 1$. Then, for all but finitely many choices of $\beta \in K$, we have*

$$v\big(G^j + \phi(F, G)\big) = \boldsymbol{\alpha}$$

for some $\phi \in R_{j-1}$ and $G = (G_1, \beta + G_2)$.

PROOF OF THE LEMMA. Let $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^{(1)}, \mathbf{0})$ with $\boldsymbol{\alpha}^{(1)} \in A_j^{(1)}$. Using the fact that both Theorems 4.3 and 4.4 hold for S_1, we can find $\phi(F_1, G_1) \in \mathcal{O}_1$ of degree at most $j-1$ in G_1 such that $\boldsymbol{\alpha}^{(1)} = v^{(1)}(G_1^j + \phi(F_1, G_1))$. Clearly, since K is infinite, for all but finitely many elements $\beta \in K$, the value $v^{(2)}$ of $(\beta + G_2)^j + \phi(F_2, \beta + G_2)$ is equal to the zero element of S_2. For all these choices of β, we have $\boldsymbol{\alpha} = v(G^j + \phi(F, G))$. Hence, fixing any finite set L, consisting of elements of the form $(\boldsymbol{\alpha}^{(1)}, \mathbf{0})$ with $\boldsymbol{\alpha}^{(1)} \in A^{(1)}$, we can choose the element $\beta \in K$ in such a way that all the elements in L satisfy the thesis of this lemma. ■

Modifying G as $(\beta + G_1, G_2)$, we can clearly obtain the analogous result, for infinitely many choices of the same β, for a finite set L' consisting of elements of the form $(\mathbf{0}, \boldsymbol{\alpha}^{(2)})$ with $\boldsymbol{\alpha}^{(2)} \in A^{(2)}$.

Let us now prove the proposition, considering the different described cases for $\boldsymbol{\alpha} \in S$.

Case 1. $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^{(1)}, \mathbf{0})$ with $\boldsymbol{\alpha}^{(1)} \in A_j^{(1)}$ (or analogously $\boldsymbol{\alpha} = (\mathbf{0}, \boldsymbol{\alpha}^{(2)})$ with $\boldsymbol{\alpha}^{(2)} \in A_j^{(2)}$).

For $j = 0$, the result is clear since we must have $\boldsymbol{\alpha} = \mathbf{0} = v(1)$. By induction, we can assume that $\boldsymbol{\alpha}^{(1)} \in A_j^{(1)}$ for $j > 0$, and the thesis holds for any $\boldsymbol{\beta}^{(1)} \in A_k^{(1)}$ with $k < j$.

Choosing the element $\boldsymbol{\eta}$ for the semigroup S_1 according to Lemma 3.8, by Lemma 4.7, we can assume also that the thesis holds for all the elements $(\boldsymbol{\alpha}^{(1)}, \mathbf{0})$ with $\boldsymbol{\alpha}^{(1)} \in A^{(1)}$ and $(\boldsymbol{\alpha}^{(1)}, \mathbf{0}) \leq (\boldsymbol{\eta}, \mathbf{0})$ (these elements form obviously a finite set).

Thus, we can assume that the element $\boldsymbol{\alpha}$ is such that $\alpha_i^{(1)} > \eta_i$ for some i. Let $\boldsymbol{\theta} = \boldsymbol{\alpha}^{(1)} \wedge \boldsymbol{\eta}$, $U = \{i : \alpha_i^{(1)} \geq \eta_i\}$, and $V = \{i : \alpha_i^{(1)} < \eta_i\} = I_1 \setminus U$. Then, one has (see Proposition 3.5) $\boldsymbol{\alpha}^{(1)} \in \widetilde{\Delta}_V^{\mathbb{N}^{d_1}}(\boldsymbol{\theta}) \subseteq A_j^{(1)}$ and also

$$\widetilde{\Delta}_V^{\mathbb{N}^{d_1}}(\boldsymbol{\alpha}^{(1)}) \subseteq \widetilde{\Delta}_V^{\mathbb{N}^{d_1}}(\boldsymbol{\theta}) \subseteq A_j^{(1)}.$$

Note that every element of $\widetilde{\Delta}_V^{\mathbb{N}^{d_1}}(\boldsymbol{\theta}) \cup \{\boldsymbol{\theta}\}$ (in particular $\boldsymbol{\alpha}^{(1)}$) satisfies the assumptions of Lemma 3.8 choosing any index $i \in U$.

Now, let us prove the next lemma.

Lemma 4.8. *Let $\boldsymbol{\varepsilon} \in \widetilde{\Delta}_V^{\mathbb{N}^{d_1}}(\boldsymbol{\theta}) \subset A_j^{(1)}$. Then, there exists $(\boldsymbol{\delta}, m\boldsymbol{\omega}^{(2)}) \in \Delta_U^E(\boldsymbol{\varepsilon}, \mathbf{0})$ with $m \geq 1$, such that $(\boldsymbol{\delta}, m\boldsymbol{\omega}^{(2)}) = v(\psi)$ for some $\psi \in R_{j-1}$.*

Proof of the Lemma. Let $i \in U$. By Lemma 3.8, there exists an element $\boldsymbol{\delta}^{(i)} \in \Delta_i^{E_1}(\boldsymbol{\varepsilon})$ such that $\boldsymbol{\delta}^{(i)} = m_i\boldsymbol{\omega}^{(1)} + \boldsymbol{\beta}^{(i)}$, with $m_i \geq 1$ and $\boldsymbol{\beta}^{(i)} \in A_{k_i}^{(1)}$ with $k_i < j$. By the inductive hypothesis on j, we know that $(\boldsymbol{\beta}^{(i)}, \mathbf{0}) = v(\Phi_i)$ with $\Phi_i \in R_{j-1}$. Since $\boldsymbol{\omega} \gg \mathbf{0}$, we get

$$m_i\boldsymbol{\omega} + (\boldsymbol{\beta}^{(i)}, \mathbf{0}) = (\boldsymbol{\delta}^{(i)}, m_i\boldsymbol{\omega}^{(2)}) = v(F^{m_i}\Phi_i) \in v(R_{j-1}) \cap \Delta_i^E\big((\boldsymbol{\varepsilon}, \mathbf{0})\big).$$

Setting $m = \min_{i \in U}\{m_i\}$, we consider the infimum

$$\bigwedge_{i \in U} (\boldsymbol{\delta}^{(i)}, m_i\boldsymbol{\omega}^{(2)}) = (\boldsymbol{\delta}, m\boldsymbol{\omega}^{(2)}) \in \Delta_U^E\big((\boldsymbol{\varepsilon}, \mathbf{0})\big).$$

For some choice of elements $z_i \in K$, we know that $(\boldsymbol{\delta}, m\boldsymbol{\omega}^{(2)}) = v(\sum_{i\in U} z_i F^{m_i}\Phi_i)$. Set $\psi := \sum_{i\in U} z_i F^{m_i}\Phi_i \in R_{j-1}$. Note that if $j \in V$, then $\delta_j^{(i)} > \varepsilon_j$ for all $i \in U$ and therefore $\delta_j > \varepsilon_j$; on the other hand, if $j \in U$, then $\delta_j = \varepsilon_j$. ■

Now, let us apply Lemma 4.8 to the element $\boldsymbol{\varepsilon} = \boldsymbol{\theta}$. Since $\boldsymbol{\theta} \leq \boldsymbol{\eta}$, we know that $(\boldsymbol{\theta}, \mathbf{0}) = v(G^j + \phi_{\boldsymbol{\theta}})$ for some $\phi_{\boldsymbol{\theta}} \in R_{j-1}$. Let us fix an index $k \in U$. Since $\delta_k = \theta_k$, we can choose $t_k \in K$ such that $v(G^j + \phi_{\boldsymbol{\theta}} + t_k\psi) = (\boldsymbol{\theta}', \mathbf{0}) > (\boldsymbol{\theta}, \mathbf{0})$ with $\theta'_k > \theta_k$. Note that if $j \in V$, then $\theta'_j = \theta_j$; hence, $\boldsymbol{\theta}' \in \widetilde{\Delta}_V^{\mathbb{N}^{d_1}}(\boldsymbol{\theta})$.

Iterating this process, replacing each time $\boldsymbol{\theta}$ by $\boldsymbol{\theta}'$ and possibly using the other indices $k \in U$, we can find an element $\boldsymbol{\theta}' \in \widetilde{\Delta}_V^{\mathbb{N}^{d_1}}(\boldsymbol{\theta})$ with arbitrarily large coordinates with respect to the indices in U such that $(\boldsymbol{\theta}', \mathbf{0}) = v(G^j + \phi_{\boldsymbol{\beta}})$ for some $\phi_{\boldsymbol{\beta}} \in R_{j-1}$.

Going back to the element $\boldsymbol{\alpha}^{(1)} \in \widetilde{\Delta}_V^{\mathbb{N}^{d_1}}(\boldsymbol{\theta})$, in particular, we can find $\boldsymbol{\beta} \geq \boldsymbol{\alpha}^{(1)}$ such that $\boldsymbol{\beta} \in \widetilde{\Delta}_V^{\mathbb{N}^{d_1}}(\boldsymbol{\theta})$ and $(\boldsymbol{\beta}, \mathbf{0}) = v(G^j + \phi_{\boldsymbol{\beta}})$ with $\phi_{\boldsymbol{\beta}} \in R_{j-1}$. Explicitly, we can say that $\boldsymbol{\beta} \in \Delta_W^{S_1}(\boldsymbol{\alpha}^{(1)})$ with $W \supseteq V$.

Furthermore, by Lemma 4.8 applied to the element $\boldsymbol{\varepsilon} = \boldsymbol{\alpha}^{(1)}$, we can construct an element $(\boldsymbol{\delta}', \boldsymbol{\tau}) \in \Delta_U^E((\boldsymbol{\alpha}^{(1)}, \mathbf{0}))$ such that $(\boldsymbol{\delta}', \boldsymbol{\tau}) = v(\psi')$ with $\psi' \in R_{j-1}$ (and $\boldsymbol{\tau} \gg \mathbf{0}$). It is easy to observe that $(\boldsymbol{\alpha}^{(1)}, \mathbf{0}) = (\boldsymbol{\beta}, \mathbf{0}) \wedge (\boldsymbol{\delta}', \boldsymbol{\tau})$. Thus, we can choose $z \in K$ such that $v(G^j + \phi_{\boldsymbol{\beta}} + z\psi') = (\boldsymbol{\alpha}^{(1)}, \mathbf{0})$. This shows that $(\boldsymbol{\alpha}^{(1)}, \mathbf{0})$ is the value of some element of the form $G^j + \phi$ with $\phi \in R_{j-1}$ and completes the proof of Case 1.

Case 2. $(\boldsymbol{\alpha}^{(1)}, \mathbf{0}) \in A_j$ with $\boldsymbol{\alpha}^{(1)} \notin A^{(1)}$ (or analogously $\boldsymbol{\alpha} = (\mathbf{0}, \boldsymbol{\alpha}^{(2)}) \in A_j$ with $\boldsymbol{\alpha}^{(2)} \notin A_j^{(2)}$).

Suppose $\boldsymbol{\alpha}^{(1)}$ to be non-zero. By Theorem 3.1 also in this case, we have $\lambda(\boldsymbol{\alpha}^{(1)}) = j > 0$. By the definition of λ, we can write $\boldsymbol{\alpha}^{(1)} = m\boldsymbol{\omega}^{(1)} + \boldsymbol{\theta}$ for $m \geq 1$ and $\boldsymbol{\theta} \in A_k^{(1)}$ with $k < j$. If $j < N_1$, then by Lemma 3.2, there exists $\boldsymbol{\delta} \in A_j^{(1)}$ such that $\boldsymbol{\alpha}^{(1)} < \boldsymbol{\delta}$. As a consequence of Case 1, we know that $(\boldsymbol{\delta}, \mathbf{0}) = v(G^j + \phi)$ with $\phi \in R_{j-1}$ and $(\boldsymbol{\theta}, \mathbf{0}) = v(G^k + \psi)$ with $\psi \in R_{k-1}$. Since K is infinite, we can find a non-zero constant $z \in K$ such that $(\boldsymbol{\alpha}^{(1)}, \mathbf{0}) = v(G^j + \phi + zF^m(G^k + \psi))$. The result now follows since by construction $\phi + zF^m(G^k + \psi) \in R_{j-1}$. If instead $j = N_1$, we use the fact that we can express $G_1^{N_1} = \sum_{i=0}^{N_1-1} h_i(F_1)G_1^i$ and the choice of the element $\beta \in K$ can be made in such a way that

$$v^{(2)}\left((\beta + G_2)^{N_1} - \sum_{i=0}^{N_1-1} h_i(F_2)(\beta + G_2)^i\right) = 0.$$

Thus, writing again $(\boldsymbol{\theta}, \mathbf{0}) = v(G^k + \psi)$ with $\psi \in R_{k-1}$, we obtain

$$\boldsymbol{\alpha} = (\boldsymbol{\alpha}^{(1)}, \mathbf{0}) = v\left(G^{N_1} - \sum_{i=0}^{N_1-1} h_i(F)G^i + F^m(G^k + \psi)\right).$$

As before, $\sum_{i=0}^{N_1-1} h_i(F)G^i + F^m(G^k + \psi) \in R_{j-1}$.

In both Cases 1 and 2, we get the same results for the elements of the form $\boldsymbol{\alpha} = (\mathbf{0}, \boldsymbol{\alpha}^{(2)})$. Indeed, we can proceed in the same way working over the components corresponding to S_2 and replacing G by $G - (\beta, \beta)$ in all the formulas (again the choice of β at the beginning of the proof can be made generic enough to satisfy all the needed conditions). We finally consider the general case.

CASE 3. $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}) \in A_j$ with $\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)} \neq 0$.

We can say that $\lambda(\boldsymbol{\alpha}^{(1)}, \mathbf{0}) = k$, $\lambda(\mathbf{0}, \boldsymbol{\alpha}^{(2)}) = l$ with $k, l \geq 1$. By Theorem 3.1, $k + l = j$. By what was proved in the previous cases, $(\boldsymbol{\alpha}^{(1)}, \mathbf{0}) = v(G^k + \Phi)$ and $(\mathbf{0}, \boldsymbol{\alpha}^{(2)}) = v(G^l + \Psi)$ for opportune choices of $\Phi \in R_{k-1}$ and $\Psi \in R_{l-1}$. It follows that $\boldsymbol{\alpha} = v((G^k + \Phi)(G^l + \Psi)) = v(G^j + \xi)$ with $\xi \in R_{j-1}$.

This concludes the proof of the proposition. ■

We prove now Theorem 4.3.

PROOF OF THEOREM 4.3. We know that R is a $K[\![F]\!]$−module and, since the quotient ring $\frac{R}{(F)R}$ is a K-vector space of dimension $N_1 + N_2$, the ring R is minimally generated as module over $K[\![F]\!]$ by $N = N_1 + N_2$ elements. For $H \in R$, denote by $\bar{H}$ the image of H in the quotient $\frac{R}{(F)R}$.

Let G be defined as in Proposition 4.6. To prove the theorem, we need to show that

$$\bar{1}, \bar{G}, \bar{G}^2, \ldots, \bar{G}^{N_1+N_2-1}$$

are linearly independent over K. We use now Remark 3.4 to construct a sequence of elements of S

$$\mathbf{0} = \boldsymbol{\alpha}^{(0)} < \boldsymbol{\alpha}^{(1)} < \cdots < \boldsymbol{\alpha}^{(j)} < \cdots < \boldsymbol{\alpha}^{(N-2)} < \boldsymbol{\alpha}^{(N-1)},$$

such that $\boldsymbol{\alpha}^{(i)} \in A_i$ and, for every $k < j$, if $\boldsymbol{\alpha}^{(j)} \in \Delta_U^S(\boldsymbol{\alpha}^{(k)})$ for some $U \neq \emptyset$, then

$$\widetilde{\Delta}_{\hat{U}}^S(\boldsymbol{\alpha}^{(k)}) \subseteq A.$$

By Proposition 4.6, $\boldsymbol{\alpha}^{(j)}$ is the value of an element of the form $h_j := G^j + \phi$ with $\phi \in R_{j-1}$. The elements $h_0, \ldots, h_{N-1}$ satisfy the hypothesis of Lemma 4.1. Thus, their images modulo (F) are linearly independent over K. By the definition of the subsets R_j, it follows that also $\bar{1}, \bar{G}, \bar{G}^2, \ldots, \bar{G}^{N_1+N_2-1}$ are linearly independent over K. This proves the theorem. ■

Before proving Theorem 4.4, we need to prove several lemmas.

LEMMA 4.9. *Take the notation of Setting 4.2. Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in v(T_i)$ for some $i = 0, \ldots, N-1$. If $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$, then $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are incomparable with respect to the partial order relation $\leq\leq$.*

PROOF. Write $\boldsymbol{\alpha} = v(G^i + \phi)$ and $\boldsymbol{\beta} = v(G^i + \psi)$ for $\phi, \psi \in R_{i-1}$. If by way of contradiction $\boldsymbol{\alpha} \ll \boldsymbol{\beta}$, we would have $\boldsymbol{\alpha} = v(G^i + \phi - G^i - \psi) = v(\phi - \psi)$ and this would contradict the definition of T_i. ■

LEMMA 4.10. *Let R be the local ring of a plane curve and let S be its value semigroup. Let the elements F, G and the subsets R_i, T_i be defined as in Setting 4.2 and Remark 4.5. For $j \leq N - 1$, let $\phi = \sum_{k=0}^{j} a_k(F)G^k \in R$ be a power series not divisible by F. Then, $\lambda(v(\phi)) \leq j$.*

PROOF. In the case $j = 0$, ϕ is a power series in $K[\![F]\!]$ not divisible by F and $v(\phi) = \mathbf{0}$. It follows that $\lambda(v(\phi)) = 0$. Thus, we can argue by induction and assume the thesis true for all the power series having degree in G strictly smaller than j. Since ϕ is not divisible by F, at least one of the series $a_k(F)$ has a non-zero constant term. Thanks to the fact that the ring R is local, we can use Weierstrass' Preparation Theorem to write $\phi = u(F, G)(G^h + \psi)$ with $h \leq j$, $\psi \in R_{j-1}$, and $v(u(F, G)) = \mathbf{0}$. If $h < j$, we can conclude by the inductive hypothesis. From this we can reduce to the case where $\phi = G^j + \psi$ with $\psi \in R_{j-1}$. Now set $\boldsymbol{\alpha} = v(G^j + \psi)$. By way of contradiction, suppose $\lambda(\boldsymbol{\alpha}) > j$. Hence, by the definition of λ and by Lemma 3.3, we can find $\boldsymbol{\beta} \in A_j$ such that $\boldsymbol{\alpha} > \boldsymbol{\beta}$. Since R is local, by Remark 4.5, we know that Theorem 4.4 holds for R and we get $A_j = v(T_j)$. Hence, we can find $\xi \in R_{j-1}$ such that $G^j + \xi \in T_j$ and $v(G^j + \xi) = \boldsymbol{\beta}$. Set $\boldsymbol{\delta} = v(\psi - \xi)$ and observe that $\boldsymbol{\alpha} = v(G^j + \xi + \psi - \xi)$. By the definition of T_j, $\boldsymbol{\beta} \neq \boldsymbol{\delta}$. For any component i such that $\delta_i \neq \beta_i$, we get $\min(\delta_i, \beta_i) = \alpha_i \geq \beta_i$ and thus $\alpha_i = \beta_i$. This implies that $\boldsymbol{\beta} < \boldsymbol{\delta}$. Now if $\boldsymbol{\beta} \ll \boldsymbol{\delta}$, we get the contradiction $\boldsymbol{\alpha} = \boldsymbol{\beta}$. Therefore, there exists a non-empty set of indices U such that $\boldsymbol{\alpha} \in \Delta_U^S(\boldsymbol{\beta})$ and $\boldsymbol{\delta} \in \widetilde{\Delta}_{\widehat{U}}^S(\boldsymbol{\beta})$. Now if $\boldsymbol{\alpha} \in E$, clearly $\widetilde{\Delta}_{\widehat{U}}^S(\boldsymbol{\beta}) \subseteq A$ by property (G1) since $\boldsymbol{\delta} \wedge \boldsymbol{\alpha} = \boldsymbol{\beta} \notin E$. If $\boldsymbol{\alpha} \in A$, then $\boldsymbol{\alpha} \in A_l$ with $l > j$ and we can use Lemma 3.3 to choose $\boldsymbol{\beta}$ in such a way that $\widetilde{\Delta}_{\widehat{U}}^S(\boldsymbol{\beta}) \subseteq A$. In any case, $\boldsymbol{\delta} \in v(R_{j-1}) \cap A$ and therefore F does not divide $\psi - \xi$. By the inductive hypothesis, $\lambda(\boldsymbol{\delta}) \leq j - 1$ implying that $\boldsymbol{\delta} \in A_k$ with $k < j$. This is a contradiction since $\boldsymbol{\delta} > \boldsymbol{\beta}$. ■

LEMMA 4.11. *Adopt the notation of Setting 4.2 and let G be defined as in the proof of Theorem 4.3. Let $G^j + \phi \in T_j$. Suppose that $v(G^j + \phi) \in A_j$ and there exists $u = u(F, G) \in R$ of degree k in G such that $v(u) = (\mathbf{0}, \boldsymbol{\theta})$ and $v(uG^j + u\phi) \in A_h$. Then, $h \leq j + k$.*

PROOF. Let Y_1 and Y_2 be the components of $uG^j + u\phi$ with respect to the direct product $\mathcal{O}_1 \times \mathcal{O}_2$. We recall that, from what is written right after Remark 4.5, we can assume $\mathcal{O}_2$ to be local. By Theorem 3.1, $h = \lambda(v^{(1)}(Y_1)) + \lambda(v^{(2)}(Y_2))$. Similarly, write $j = j_1 + j_2$ where j_1 and j_2 are the values of the function λ applied to the two components of $G^j + \phi$. Since the first component of u has value zero, we get

$\lambda(v^{(1)}(Y_1)) = j_1$. We need to prove that $\lambda(v^{(2)}(Y_2)) \leq k + j_2$. Applying Lemma 4.10 to the second component of u in the local ring $\mathcal{O}_2$, we get $\lambda(\boldsymbol{\theta}) \leq k$. Thus, it is sufficient to prove that if $\boldsymbol{\alpha}, \boldsymbol{\beta} \in S_2$, $\lambda(\boldsymbol{\alpha}) = i$, and $\lambda(\boldsymbol{\beta}) = k$, then $\lambda(\boldsymbol{\alpha} + \boldsymbol{\beta}) \leq i + k$. Since the maximal value of $\lambda(\boldsymbol{\delta})$ for $\boldsymbol{\delta} \in S_2$ is N_2, we can reduce to assume $i + k < N_2$. By Lemma 3.2, we can replace $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ by $\boldsymbol{\alpha}' \in A_i$ and $\boldsymbol{\beta}' \in A_k$ such that $\boldsymbol{\alpha} \leq \boldsymbol{\alpha}'$ and $\boldsymbol{\beta} \leq \boldsymbol{\beta}'$ (in particular, $\lambda(\boldsymbol{\alpha} + \boldsymbol{\beta}) \leq \lambda(\boldsymbol{\alpha}' + \boldsymbol{\beta}')$). Hence, let us assume that $\boldsymbol{\alpha} \in A_i$ and $\boldsymbol{\beta} \in A_k$. By assumption on R_2, we can find $G_2^i + \psi_{i-1}$ and $G_2^k + \psi_{k-1}$ having values respectively equal to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Then, $\boldsymbol{\alpha} + \boldsymbol{\beta} = v(G_2^{i+k} + \psi)$ for some ψ having degree at most $i + k - 1$ in G_2. To conclude, we can now apply Lemma 4.10 at the element $G_2^{i+k} + \psi \in R_2$. ∎

We are now ready to prove Theorem 4.4.

PROOF OF THEOREM 4.4. Starting from the fact that $A_0 = \{(0,0)\} = v(K) = v(T_0)$, we prove that $A_j = v(T_j)$ for every $j = 0, \ldots, N-1$ by induction. Fixing $j > 0$, assume that $A_k = v(T_k)$ for all $k < j$. Thanks to Proposition 4.6, we know that for every $\boldsymbol{\alpha} \in A_j$, there exists $\phi \in R_{j-1}$ such that $\boldsymbol{\alpha} = v(G^j + \phi)$. Thus, we only need to prove that, given $\phi \in R_{j-1}$, the following conditions are equivalent:

(i) $v(G^j + \phi) \in A_j$.

(ii) $G^j + \phi \in T_j$.

Let us prove (i)⇒(ii). Assume by way of contradiction $G^j + \phi \notin T_j$ and set $\boldsymbol{\alpha} = v(G^j + \phi)$. Hence, there exists $H \in R_{j-1}$ such that $v(H) = \boldsymbol{\alpha}$. Write $H = H(F,G) = \sum_{k=0}^{j-1} a_k(F)G^k$. Since $v(H) \in A$, H is not divisible by F and thus at least one of the power series $a_k(F)$ has a non-zero constant term. We can apply Weierstrass' Preparation Theorem on the power series $\sum_{k=0}^{j-1} a_k(x)y^k$ in the local formal power series ring $K[\![x,y]\!]$. This gives $H(x,y) = u(x,y)(\sum_{k=0}^{h-1} b_k(x)y^k + y^h)$ for $h \leq j-1$ and $u(x,y)$ with a non-zero constant term. Mapping to the ring R, we obtain $H = u(F,G)(\sum_{k=0}^{h-1} b_k(F)G^k + G^h)$, where still $u := u(F,G)$ has a non-zero constant term but is not necessarily a unit. In particular, by the definition of F and G, we know that $v(u) = (0,a)$ for some $a \in S_2$. Set $G^h + \psi = \sum_{k=0}^{h-1} b_k(F)G^k + G^h$. Clearly, since $(0,a) + v(G^h + \psi) \in A$, then also $\boldsymbol{\beta} := v(G^h + \psi) \in A$. Possibly iterating the same process finitely many times, replacing $G^j + \phi$ by $G^h + \psi$, we can reduce to the case where $G^h + \psi \in T_h$ (eventually, $R_0 = T_0$). By the inductive hypothesis, we get $\boldsymbol{\beta} \in A_h$. The division argument of Weierstrass' Preparation Theorem implies that $u = (u_1, u_2)$ is a polynomial in G of degree $j - 1 - h$. By Lemma 4.11, we obtain $\boldsymbol{\alpha} = v(H) = v(uG^h + u\psi) \in A_i$ with $i \leq (j-1-h) + h = j-1$. This contradicts the assumption of having $\boldsymbol{\alpha} = v(G^j + \phi) \in A_j$.

We prove now (ii)⇒(i). Let $\boldsymbol{\alpha} = v(G^j + \phi)$ and suppose first that $\boldsymbol{\alpha} \notin A$. Hence, we can write $\boldsymbol{\alpha} = m\boldsymbol{\omega} + \boldsymbol{\delta}$ with $\boldsymbol{\delta} \in A_h$ and $m \geq 1$. If $h < j$, by the inductive hypothesis, we can find $G^h + \psi \in T_h$ such that $\boldsymbol{\delta} = v(G^h + \psi)$. It follows that $\boldsymbol{\alpha} = v(F^m(G^h + \psi))$, and this contradicts the definition of T_j. If instead $h \geq j$, there exists $\boldsymbol{\beta} \in A_j$ such that $\boldsymbol{\beta} \leq \boldsymbol{\delta}$. Hence, $\boldsymbol{\alpha} \gg \boldsymbol{\beta}$ and Proposition 4.6 together with the implication (i)$\Rightarrow$(ii) allows us to find $G^j + \psi \in T_j$ such that $\boldsymbol{\beta} = v(G^j + \psi)$. This yields a contradiction by Lemma 4.9.

Suppose then $\boldsymbol{\alpha} \in A_h$ for some h. By the inductive hypothesis, since the sets $v(T_i)$ are disjoint by definition, we must have $h \geq j$. If $h > j$, by Lemma 3.3, we can find $\boldsymbol{\beta} \in A_j$ such that $\boldsymbol{\beta} < \boldsymbol{\alpha}$. As before, we can find $G^j + \psi \in T_j$ such that $\boldsymbol{\beta} = v(G^j + \psi)$. If $\boldsymbol{\alpha} \gg \boldsymbol{\beta}$, we conclude as previously using Lemma 4.9. Otherwise, we have $\boldsymbol{\alpha} \in \Delta^S_F(\boldsymbol{\beta})$ and we can use Lemma 3.3 to assume also that $\widetilde{\Delta}^S_{\widehat{F}}(\boldsymbol{\beta}) \subseteq A$. From this we get

$$\boldsymbol{\delta} := v(G^j + \phi - G^j - \psi) \in \widetilde{\Delta}^S_{\widehat{F}}(\boldsymbol{\beta}) \subseteq A.$$

In particular, $\boldsymbol{\delta} \in v(R_{j-1}) \cap A$. To conclude, we prove that $v(R_{j-1}) \cap A \subseteq \bigcup_{l=0}^{j-1} A_l$. This will show that $\boldsymbol{\delta} \in A_l$ with $l < j$ in contradiction with the fact that $\boldsymbol{\delta} \geq \boldsymbol{\beta}$. For $\boldsymbol{\delta} \in v(R_{j-1}) \cap A$, arguing as in the proof of implication (i)$\Rightarrow$(ii), we write $\boldsymbol{\delta} = v(\sum_{k=0}^{i-1} a_k(F)G^k)$ and use Weierstrass' Preparation Theorem to get

$$\sum_{k=0}^{j-1} a_k(F)G^k = u(F,G)(G^s + \xi)$$

such that $s < j$, $v(u(F,G)) = (0,a)$, and $G^s + \xi \in T_s$. The same argument used previously shows that $\boldsymbol{\beta} \in A_l$ with $l \leq j-1$. ∎

4.2. *Apéry's theorem for semilocal blow-ups of plane algebroid curves*

Let K be an infinite field and let $\mathcal{V}$ be a product of local rings of plane algebroid curves defined over K. Then, it is well known that

- $\mathcal{V} \cong \mathcal{V}_1 \times \cdots \times \mathcal{V}_c \subseteq \overline{\mathcal{V}} \cong K[\![t_1]\!] \times \cdots \times K[\![t_d]\!]$ with $(\mathcal{V}_i, \mathbf{m}_i)$ local rings, and $\overline{\mathcal{V}}$ a finite $\mathcal{V}$-module.
- $\mathcal{V}$ is reduced.
- $\mathcal{V}_i/\mathbf{m}_i \cong K$.

We can always assume that $\mathcal{V} \cong \mathcal{V}_1 \times \mathcal{V}_2$ with $\mathcal{V}_1$ not necessarily local and $\mathcal{V}_2$ local. By Theorems 4.3 and 4.4, we can write

$$\mathcal{V} = K[\![F]\!] + K[\![F]\!]G + \cdots + K[\![F]\!]G^{N-1},$$

where F is any element of $\mathcal{V}$ of value $\boldsymbol{\omega} = (\boldsymbol{\omega}^{(1)}, \boldsymbol{\omega}^{(2)})$ with $\boldsymbol{\omega}^{(i)} \gg \mathbf{0}$, and G defined according to the proof of Proposition 4.6.

PROPOSITION 4.12. *The ring $\mathcal{U} = K[\![F]\!] + K[\![F]\!]H + \cdots + K[\![F]\!]H^{N-1}$ with $H = G \cdot F$ is the local ring of a plane algebroid curve and its blow-up, $\mathcal{B}(\mathcal{U})$, is equal to $\mathcal{V}$.*

PROOF. We first show that $\mathcal{U}$ is local with maximal ideal (F, H). The element $G^N \in \mathcal{V}$ satisfies a relation $G^N = a_0(F) + a_1(F)G + \cdots + a_{N-1}(F)G^{N-1}$. Hence,

$$(*) \qquad H^N = a_0(F)F^N + a_1(F)F^{N-1}H + \cdots + a_{N-1}(F)FH^{N-1}.$$

Let $\varphi : K[\![x]\!][y] \to \mathcal{U}$ be the surjective homomorphism defined by $\varphi(x) = F$ and $\varphi(y) = H$. Since $\mathcal{V}$ is minimally generated as $K[\![F]\!]$-module by $\{1, G, \dots, G^{N-1}\}$, then necessarily N is the minimal integer such that the powers $1, H, H^2, \dots, H^N$ are linearly dependent over $K[\![F]\!]$. Hence,

$$f = y^N - a_0(x)x^N - a_1(x)x^{N-1}y - \cdots - a_{N-1}(x)xy^{N-1}$$

is an irreducible element of $K[\![x]\!][y]$ and therefore $\ker\varphi = (f)$, and $\mathcal{U} \cong \frac{K[\![x]\!][y]}{(f)}$. Let $\mathfrak{m}$ be a maximal ideal of $K[\![x]\!][y]$ containing (f). Then, $\mathfrak{m} \cap K[\![x]\!] = (x)$ and $\mathfrak{m} \supseteq (x)$. Hence,

$$\frac{K[\![x]\!][y]}{\mathfrak{m}} \cong \frac{K[\![x]\!][y]/(x)}{\mathfrak{m}/(x)} \cong \frac{K[y]}{\mathfrak{m}/(x)}$$

and $\frac{\mathfrak{m}}{(x)} \supseteq (\bar{f})$, where $\bar{f}$ denotes the image of f in $\frac{K[\![x]\!][y]}{(x)}$. But now it is easy to observe that $\bar{f} = \bar{y}^N$. From this we get $\frac{\mathfrak{m}}{(x)} = (\bar{y})$; hence, $\mathfrak{m} = (x, y)$. By the isomorphism

$$\mathcal{U} \cong \frac{K[\![x]\!][y]}{(f)},$$

we conclude that the only maximal ideal of $\mathcal{U}$ is (F, H).

Let us now prove that $\mathcal{U}$ and $\mathcal{V}$ have the same field of fractions; that is, $Q(\mathcal{U}) = Q(\mathcal{V})$. One inclusion is trivial as $\mathcal{U} \subseteq \mathcal{V}$. Given $g \in \mathcal{V}$, we observe that $F^n g \in \mathcal{U}$. Thus, given $g/h \in Q(\mathcal{V})$, we get $g/h = (F^N g)/(F^N h) \in Q(\mathcal{U})$.

We note then also that $\overline{\mathcal{U}} = \overline{\mathcal{V}}$. Indeed, we have the following chains of inclusions:

$$K[\![F]\!] \subseteq \mathcal{U} \subseteq \mathcal{V} \subseteq K[\![t_1]\!] \times \cdots \times K[\![t_d]\!],$$

where the second and the third inclusions are integral as $\mathcal{V}$ is a finite $K[\![F]\!]$-module and $K[\![t_1]\!] \times \cdots \times K[\![t_d]\!] \cong \overline{\mathcal{V}}$. Hence, $\overline{\mathcal{U}} = \overline{\mathcal{V}}$ and $\overline{\mathcal{U}}$ is a finite $\mathcal{U}$-module.

Finally, since F is an element of minimal value in (F, H), we have $\mathcal{B}(\mathcal{U}) = \mathcal{U}[\frac{H}{F}] = \mathcal{U}[G] = \mathcal{V}$. ■

REMARK 4.13. Let $\mathcal{O}$ be the ring of a plane algebroid curve. Then, by Proposition 2.2, $\mathcal{O} = k[\![x]\!] + k[\![x]\!]y + k[\![x]\!]y^2 + \cdots + k[\![x]\!]y^{N-1}$, where $v(x) = (e_1, \dots, e_d) = \min(v(\mathcal{O} \setminus \{\mathbf{0}\})$ and $N = e_1 + \cdots + e_d = e$ is the multiplicity of $\mathcal{O}$. Let $\mathcal{B}(\mathcal{O})$ be the blow-up ring of $\mathcal{O}$ and suppose $\mathcal{B}(\mathcal{O})$ to be semilocal. By Theorems 4.3 and 4.4, choosing $\boldsymbol{\omega} = (e_1, \dots, e_d)$, we can write

$$\mathcal{B}(\mathcal{O}) = K[\![F]\!] + K[\![F]\!]G + \dots + K[\![F]\!]G^{N-1}$$

for opportune choices of F and G. Since F can be any element of $\mathcal{B}(\mathcal{O})$ of value $\boldsymbol{\omega}$, we can choose $F = x$ and get

$$\mathcal{B}(\mathcal{O}) = K[\![x]\!] + K[\![x]\!]G + \dots + K[\![x]\!]G^{N-1}.$$

Finally, by Proposition 4.12, we have that the local ring

$$\mathcal{U} = K[\![x]\!] + K[\![x]\!]xG + \dots + K[\![x]\!]x^{N-1}G^{N-1}$$

is the ring an algebroid curve and $\mathcal{B}(\mathcal{U}) = \mathcal{B}(\mathcal{O})$.

Proposition 4.14. *The rings $\mathcal{O}$ and $\mathcal{U}$ considered in Remark 4.13 are equal.*

Proof. We need to prove that $y \in \mathcal{U}$ and $xG \in \mathcal{O}$. We know that $y/x \in \mathcal{O}[y/x] = \mathcal{B}(\mathcal{O}) = \mathcal{B}(\mathcal{U})$ and $v(y/x)$ is in the Apéry set of $v(\mathcal{B}(\mathcal{O}))$ with respect to $\boldsymbol{\omega}$. Hence, by Theorem 4.4, $y/x = G^j + \phi(x, G)$ for some $\phi(x, G)$ of degree at most $j-1$ in G. We claim that $j = 1$; that is, $v(y/x)$ is in the first level of $\mathbf{Ap}(v(\mathcal{B}(\mathcal{O})), \boldsymbol{\omega})$.

Indeed, as recalled before Proposition 4.12, $\mathcal{B}(\mathcal{O}) \subseteq K[\![t_1]\!] \times \dots \times K[\![t_d]\!]$ and we can write $\mathcal{B}(\mathcal{O}) = C_1 \times C_2$ where C_1 and C_2 are the natural projection of $\mathcal{B}(\mathcal{O})$ over the sets of indexes $I_1 = \{i \in \{1, \dots, d\} \mid v(y)_i = e_i\}$ and $I_2 = \{i \in \{1, \dots, d\} \mid v(y)_i > e_i\}$, respectively. Both sets I_1 and I_2 are non-empty since we assumed $\mathcal{B}(\mathcal{O})$ to be not local.

Thus, observe that $v(y/x) = (\mathbf{0}, \boldsymbol{\beta}) \in v(C_1) \times v(C_2)$. Observe that C_2 is local and generated as module by the powers of the image of y/x. By Proposition 2.4, this implies that $\boldsymbol{\beta}$ is in the first level of the Apéry set of $v(C_2)$. Theorem 3.1 yields $j = \lambda(v(y/x)) = 1$ and therefore $y/x = G + \phi(x, G)$. It follows that $y = xG + x\phi(x, G)$ and $\mathcal{O} = \mathcal{U}$ as $\phi(x, G) \in K[\![F]\!] \subseteq \mathcal{O} \cap \mathcal{U}$. ∎

Theorem 4.15. *Let $\mathcal{O}$ be the ring of a plane algebroid curve and suppose its blow-up ring $\mathcal{B}(\mathcal{O})$ to be not local. Let $\boldsymbol{\omega}$ be the minimal non-zero element of $\mathcal{O}$. Let A_i and A'_i denote the i-th levels of the Apéry sets with respect to $\boldsymbol{\omega}$ of $v(\mathcal{O})$ and of $v(\mathcal{B}(\mathcal{O}))$, respectively. Then, $A_i = A'_i + i\boldsymbol{\omega}$.*

Proof. We can describe $\mathcal{O}$ and $\mathcal{U}$ according to the notation used in Remark 4.13. Furthermore, denote by $\mathcal{O}_i$ the $K[\![x]\!]$-submodule of $\mathcal{O}$ generated by $1, y, y^2, \dots, y^i$ and, similarly, denote by $\mathcal{U}_i$ the $K[\![x]\!]$-submodule of $\mathcal{U}$ generated by $1, xG, x^2G^2 \dots, x^iG^i$. For the ring $\mathcal{B}(\mathcal{O})$, we adopt the notation of Theorems 4.3 and 4.4 setting $R = \mathcal{B}(\mathcal{O})$ and defining the subsets R_i as for those theorems.

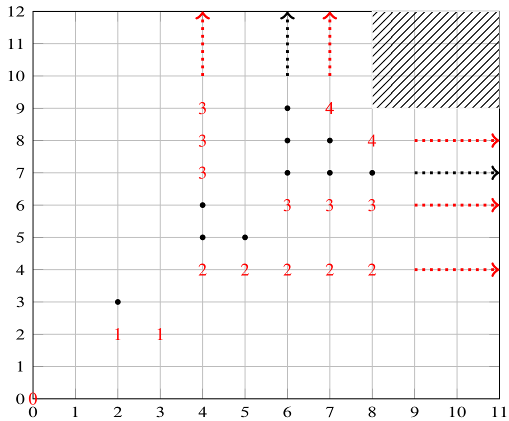

FIGURE 3. The Apéry set of the semigroup $v(\mathcal{O}'_2)$.

Thus, by [5, Proposition 3.8] and Proposition 4.14, we have

$$\begin{aligned} A_i &= \left\{v(y^i + \phi_{i-1}) \mid \phi_{i-1} \in \mathcal{O}_{i-1} \text{ and } v(y^i + \phi_{i-1}) \notin v(\mathcal{O}_{i-1})\right\} \\ &= \left\{v(x^i G^i + \psi_{i-1}) \mid \psi_{i-1} \in \mathcal{U}_{i-1} \text{ and } v(x^i G^i + \psi_{i-1}) \notin v(\mathcal{U}_{i-1})\right\}. \end{aligned}$$

By Theorem 4.4, we have

$$A'_i = \left\{v(G^i + \varphi_{i-1}) \mid \varphi_{i-1} \in R_{i-1} \text{ and } v(G^i + \varphi_{i-1}) \notin v(R_{i-1})\right\}.$$

Hence, in order to prove the theorem, we can use exactly the same proof of [5, Theorem 4.1]. ■

EXAMPLE 4.16. Let us consider the ring

$$\mathcal{O} = \frac{K[\![X,Y]\!]}{(X^5 - Y^2) \cap (X^7 + X^5 + 3X^4Y - Y^3) \cap (X^5 - X^2 + 2XY - Y^2)}$$

of a plane algebroid curve, which is parametrized by

$$\mathcal{O} = K[\![(t^2, u^3, v^2), (t^5, u^5 + u^7, v^2 + v^5)]\!].$$

If we compute the blow-up, we obtain

$$\begin{aligned} \mathcal{O}' := \mathcal{B}(\mathcal{O}) &= K[\![(t^2, u^3, v^2), (t^5, u^2 + u^4, 1 + v^3)]\!] \\ &= K[\![(t^2, u^3), (t^3, u^2 + u^4)]\!] \times K[\![(v^2, v^3)]\!]. \end{aligned}$$

If we denote $\mathcal{O}'_1 := K[\![(t^2, u^3), (t^3, u^2 + u^4)]\!]$ and $\mathcal{O}'_2 := K[\![(v^2, v^3)]\!]$, we have that the Apéry set of the semigroup $v(\mathcal{O}'_1)$ with respect to the element 2 is the set $\{0, 3\}$ and $\lambda(0) = 0$, $\lambda(2) = 1$, $\lambda(3) = 1$, $\lambda(4) = 2$. The Apéry set of $\mathcal{O}'_2$ with respect to the element (2,3) is depicted in Figure 3.

Using the method described in [17, Theorem 4.5], we can determine the levels of the Apéry set A' of the ring $\mathcal{O}'$ with respect to the element $\boldsymbol{\omega} = (2,3,2)$. In this case, $\boldsymbol{\gamma} = (5,5,1)$ and we have that

$$\begin{aligned}
A'_0 &= \{(0,0,0)\},\\
A'_1 &= \{(0,0,2),(0,0,3),(2,2,0),(3,2,0)\},\\
A'_2 &= \{(0,0,\infty),(2,2,2),(2,2,3),(3,2,2),(3,2,3),(4,4,0),\\
&\qquad (2,3,0),(5,4,0),(6,4,0),(7,4,0),(\infty,4,0)\},\\
A'_3 &= \{(2,2,\infty),(3,2,\infty),(2,3,3),(4,4,2),(4,4,3),(5,4,2),(5,4,3),(6,4,2),\\
&\qquad (6,4,3),(7,4,2),(7,4,3),(\infty,4,2),(\infty,4,3),(4,5,0),(5,5,0),(4,6,0),\\
&\qquad (4,7,0),(4,8,0),(4,\infty,0),(6,6,0),(7,6,0),(\infty,6,0)\},\\
A'_4 &= \{(4,4,\infty),(5,4,\infty),(6,4,\infty),(7,4,\infty),(\infty,4,\infty),(4,5,3),(5,5,3),\\
&\qquad (4,6,3),(4,7,2),(4,7,3),(4,8,2),(4,8,3),(4,\infty,2),(4,\infty,3),(6,6,2),\\
&\qquad (6,6,3),(7,6,2),(7,6,3),(\infty,6,2),(\infty,6,3),(\infty,8,0),(7,\infty,0)\},\\
A'_5 &= \{(4,7,\infty),(4,8,\infty),(4,\infty,\infty),(6,6,\infty),(7,6,\infty),(\infty,6,\infty),(6,7,3),\\
&\qquad (7,7,3),(\infty,7,3),(6,8,3),(7,8,3),(6,\infty,3),(\infty,8,2),(\infty,8,3),\\
&\qquad (7,\infty,2),(7,\infty,3),(\infty,\infty,0)\},\\
A'_6 &= \{(\infty,8,\infty),(7,\infty,\infty),(\infty,\infty,3)\},
\end{aligned}$$

where, by convention, we say that an element of the form $\boldsymbol{\alpha} = (\alpha_1,\alpha_2,\alpha_3)$ with $\alpha_i = \infty$ belongs to the set A'_k if all the elements $\boldsymbol{\beta}$ with $\beta_i > \gamma_i + \omega_i$ and $\boldsymbol{\beta}_j = \alpha_j$, $j \neq i$, belong to the set A'_k.

Hence, using Theorem 4.15, we can compute the levels of the Apéry set of the semigroup $v(\mathcal{O})$ with respect to the multiplicity $(2,3,2)$ using the formula $A_i = A'_i + i(2,3,2)$, for $i \in \{0,\ldots,6\}$.

5. Multiplicity trees of plane curve singularities

Let $R \cong \mathcal{O}_1 \times \cdots \times \mathcal{O}_c$ be a direct product of local rings $\mathcal{O}_j$ $(1 \le j \le c)$ each one associated with a reduced plane algebroid curve defined over an algebraically closed field K. Let us denote by $\mathcal{C}_1,\ldots,\mathcal{C}_d$ (resp. $\nu_1,\ldots,\nu_d$) the branches of R (resp. its valuations). For $j = 1,\ldots,c$, let $S_j \subseteq \mathbb{N}^{d_j}$ denote the value semigroup of $\mathcal{O}_j$. For every j, S_j is a local good semigroup (a numerical semigroup if $d_j = 1$). The value semigroup of R is $S = S_1 \times \cdots \times S_c \subseteq \mathbb{N}^d$ where $d = d_1 + \cdots + d_c$. In this section, it will be useful to identify each semigroup $S_j \subset \mathbb{N}^{d_j}$ as a the subsemigroup of S: $S_j = \{0\} \times \cdots \times \{0\} \times S_j \times \{0\} \times \cdots \times \{0\}$.

The *fine multiplicity* of $\mathcal{O}_j$ is the minimal value $\nu(x) \in S$ for $x \in \mathcal{O}_j$ not unit. Notice that the identification of S_j inside S implies that $\nu_i(x) \neq 0$ if and only if ν_i is a valuation of $\mathcal{O}_j$.

The local rings $\mathcal{O}_1, \ldots, \mathcal{O}_c$ will be called the rings (or the points following a more classical terminology) in the 0-neighborhood of R. Let $R^{(1)} \cong \mathcal{O}_1^{(1)} \times \cdots \times \mathcal{O}_c^{(1)}$ denote the ring in the first neighborhood of R, i.e. the ring produced after the blowing-up of R. Notice that each ring $\mathcal{O}_i^{(1)}$ is the product of a finite number of local rings: the local rings (points) of the first-neighborhood of $\mathcal{O}_i$. All the local rings of the ring $R^{(1)}$ constitute the rings (or points) of the first-neighborhood of R.

Recursively, for $j \geq 2$, let $R^{(j)} \cong \mathcal{O}_1^{(j)} \times \cdots \times \mathcal{O}_c^{(j)}$ denote the ring in the j-th neighborhood of R, i.e. the ring produced after j blowing-ups of R or equivalently the ring in the first neighborhood of $R^{(j-1)}$. As in the case $j = 1$, the ring $R^{(j)}$ is the product of a finite number of local rings: the local rings (or the points) of the j-neighborhood of R. Notice that for j big enough, $R^{(j)} \simeq \overline{R} \simeq K[\![t_1]\!] \times \cdots \times K[\![t_d]\!]$.

The whole set of local rings of the successive neighborhoods is encoded as the set of vertices $\mathcal{N}$ of (infinite) graph $\mathcal{T}$ in such a way that two vertices corresponding to local rings $\mathcal{O}$ and $\mathcal{O}'$ are connected by an edge if one of them is in the first neighborhood of the other. Thus, $\mathcal{T}$ is the disjoint union of c graphs $\mathcal{T}_1, \ldots, \mathcal{T}_c$, $\mathcal{T}_i$ being the graph corresponding to the local ring $\mathcal{O}_i$. Each $\mathcal{T}_i$ is a tree with root in the vertex corresponding to $\mathcal{O}_i$ and such that the j-th level of $\mathcal{T}_i$ consists of the vertices corresponding to the rings of the j-neighborhood of $\mathcal{O}_i$.

The multiplicity graph of R is the graph $\mathcal{T}$ with the additional information of the fine multiplicity of each local ring attached as a weight of the corresponding vertex. Although it is a tree only if $c = 1$, we will refer to it as the *multiplicity tree* of R and we denote it by $\mathcal{T}(R)$ or simply $\mathcal{T}$.

The purpose of this section is the characterization of the admissible multiplicity trees of a plane curve singularity (not necessarily local) over an algebraically closed field of arbitrary characteristic and to prove the equivalence between the multiplicity tree, the semigroup of values, and the suitable sequences of multiplicities of each branch, together with the splitting numbers (equivalent to the intersection multiplicities) between a pair of branches.

The case $d \leq 2$ with characteristic zero has been treated in [5]; however, the extension to any algebraically closed field is made convenient to be included here for the sake of completeness. All the proofs of the results for $d = 2$ (and characteristic zero) can be found in the above reference.

As is well known, in positive characteristic, the Newton–Puiseux theorem is not valid. Therefore, in this section, we will systematically use the Hamburger–Noether expansions which are valid in arbitrary characteristic. We have chosen to include them in an almost self-contained way from Campillo's book [7, Chapter II], where the reader

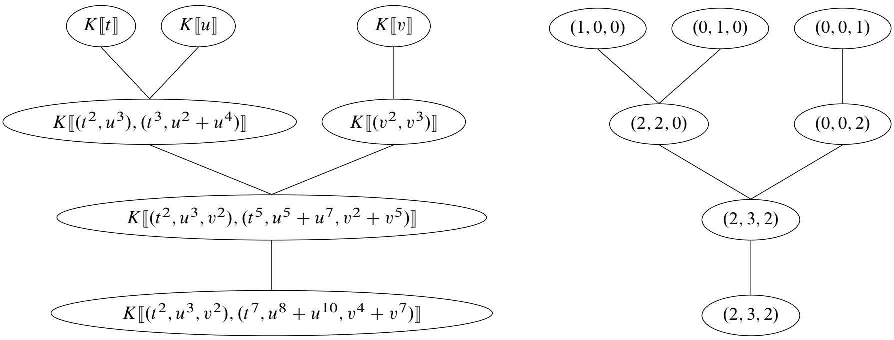

FIGURE 4. On the left is represented the blow-up tree of R and on the right the multiplicity tree of the semigroup S.

can find the precise proofs of the results we will use here. In some cases, we use some of the classical terminology of the treatment of singularities of complex plane curves since from the point of view of the resolution and the combinatorial invariants of the curves there is no substantial difference.

EXAMPLE 5.1. Let

$$\mathcal{O} = \frac{K[\![X,Y]\!]}{P_1 \cap P_2 \cap P_3}$$

be a plane algebroid curve, parametrized by

$$\mathcal{O} = K[\![(t^2,u^3,v^2),(t^7,u^8+u^{10},v^4+v^7)]\!],$$

with a semigroup of values $S := v(R)$ and multiplicity $\omega = (2,3,2)$. We can compute the blow-up and multiplicity sequence:

$$\mathcal{O}' := \mathcal{B}(\mathcal{O}) = K[\![(t^2,u^3,v^2),(t^5,u^5+u^7,v^2+v^5)]\!]$$

with a semigroup of value S' and multiplicity $\omega_1 = (2,3,2)$;

$$\mathcal{O}'' := \mathcal{B}(\mathcal{O}') = O_1'' \times O_2'' = K[\![(t^2,u^3),(t^3,u^2+u^4)]\!] \times K[\![(v^2,v^3)]\!]$$

with semigroups of values $S_1'' \times S_2'' := v(O_1'') \times v(O_2'')$ and multiplicities $\omega_{2,1} = (2,2)$ and $\omega_{2,3} = 2$;

$$\mathcal{O}''' = \mathcal{B}(\mathcal{O}'') = O_1''' \times O_2''' \times O_3''' = K[\![t]\!] \times K[\![u]\!] \times K[\![v]\!]$$

with semigroups of values $S_1''' \times S_2''' \times S_3''' := v(O_1''') \times v(O_2''') \times v(O_3''')$. In Figure 4 are represented the blow-up tree $\mathcal{T}(R)$ and the multiplicity tree of semigroup S.

We want to show how to determine the semigroups of the tree, using the multiplicity tree of the semigroup S represented in Figure 4. We have that $\mathbf{Ap}(S_1''', 2) = \mathbf{Ap}(S_2''', 2) = \{0, 1\}$; hence, we can determine the levels of Apéry set $\mathfrak{A} := \mathbf{Ap}(S_1''' \times S_2''', \omega_{2,1})$, which are

$$\mathfrak{A}_0 = \{(0,0)\}, \quad \mathfrak{A}_1 = \{(0,1),(1,0)\},$$
$$\mathfrak{A}_2 = \{(0,\infty),(1,1),(\infty,0)\}, \quad \mathfrak{A}_3 = \{(\infty,1),(1,\infty)\}.$$

Using Theorem 4.15, we have that

$$\mathbf{Ap}(S_1'', (2,2))_i = \mathfrak{A}_i + i(2,2) \quad \text{for all } i \in \{1,\dots,4\}.$$

Hence, we can determine $S_1'' = \mathbf{Ap}(S'', (2,2)) + k(2,2)$ with $k \in \mathbb{N}$. Considering $\mathbf{Ap}(S_2''', 2)$, using Theorem 4.15, we obtain $\mathbf{Ap}(S_2'', 2)_1 = \{0\}$ and $\mathbf{Ap}(S_2'', 2)_2 = \{3\}$, determining the semigroup S_2''. In Example 4.16, we showed how to compute the levels of $\mathbf{Ap}(S', (2,3,2))$ knowing the levels of $\mathbf{Ap}(S'', (2,2))$ and $\mathbf{Ap}(S'', 2)$; this Apéry set determines the semigroup S'. Using again Theorem 4.15, we can determine the levels of $\mathbf{Ap}(S, (2,3,2))$ and the semigroup S.

5.1. *Case R irreducible (i.e. $c = 1$ and $d_1 = 1$)*

Let $\mathcal{C}(= \mathcal{O})$ be a plane irreducible algebroid curve (a branch) over an algebraically closed field K and ν its valuation. The multiplicity tree is just a bamboo, so it is equivalent to the sequence of multiplicities $\underline{e} = (e_0, e_1, \dots, e_n, \dots)$ of $\mathcal{C}$. It is well known that the sequence of multiplicities $\underline{e}$ is equivalent data to the semigroup $S = \nu(\mathcal{C}) \subset \mathbb{N}$. The sequence of multiplicities of a branch must be a (not strictly) decreasing sequence satisfying also the following property.

(**Proximity**) If $e_i > e_{i+1}$, let $e_i = q_i e_{i+1} + r_i$, $r_i < e_{i+1}$ be the Euclidean division. Then, $e_{i+j} = e_{i+1}$ for $j = 1, \dots, q_i$, and if $r_i \neq 0$, then $r_i := e_{i+q_i+1} < e_{i+1}$.

We will say that a sequence of positive integers $\underline{e} = (e_0, e_1, \dots)$ is a *plane sequence* if it is a decreasing one and satisfies the Proximity relation above.

Note that, as a consequence, for each $i \geq 0$, one has that $e_i = \sum_{k=1}^{h(i)} e_{i+k}$ for a suitable $h(i) \geq 1$. The *restriction number*, $r(e_j)$, of an element e_j of the sequence $\underline{e}$ is defined as the number of sums $e_i = \sum_{k=1}^{h(i)} e_{i+k}$ in which e_j appears as a summand. One has that $1 \leq r(e_j) \leq 2$ and, following the classical terminology of the infinitely near points, if $r(e_j) = 1$, we say that $\mathcal{C}^{(j)}$ is a free point, and if $r(e_j) = 2$, $\mathcal{C}^{(j)}$ is a satellite point.

5.1.1. *Hamburger–Noether expansions*

Let K be an algebraically closed field of arbitrary characteristic, and let $\nu(g) = \mathrm{ord}_t(g)$ be the valuation defined on the ring of power series $K[\![t]\!]$.

DEFINITION 5.2. Let $x, y \in K[\![t]\!]$ be such that $\nu(y) \geq \nu(x) \geq 1$. The Hamburger–Noether (HN) expansion of $\{x, y\}$ is the finite set of expressions

$$z_{j-1} = \sum_{i=1}^{h_j} a_{ji} z_j^i + z_j^{h_j} z_{j+1}; \quad 0 \leq j \leq r, \tag{5.1}$$

where $z_{-1} = y$, $z_0 = x$, $a_{ji} \in K$, $h_r = \infty$, and $z_1, \ldots, z_r \in K[\![t]\!]$ are such that $\nu(z_0) > \nu(z_1) > \cdots > \nu(z_r) \geq 1$.

The HN expansion can be better understood from the recursive process of computation: When $\nu(y) \geq \nu(x)$, there exists a unique $a_{01} \in K$ such that $\nu((y/x) - a_{01}) > 0$ (note that $a_{01} = 0$ if and only if $\nu(y) > \nu(x)$). Let $y_1 := (y/x) - a_{01}$. If $\nu(y_1) \geq \nu(x)$, we repeat the same operation with $\{x, y_1\}$.

In this way, it is clear that we have one (and only one) of the next possibilities:

(a) After a finite number of steps, h_0, we have $a_{0,1}, \ldots, a_{0,h_0} \in K$ and $z_1 \in K[\![t]\!]$ such that $\nu(z_1) < \nu(x)$ and $y = a_{01}x + a_{02}x^2 + \cdots + a_{0\,h_0}x^{h_0} + x^{h_0} z_1$.

(b) We have an infinite series $y = a_{01}x + a_{02}x^2 + \cdots$ and the HN expansion is just this series.

Now, in case (a), the process continues with the system $\{z_1, x\}$ in a new row. After a finite number, r, of steps (a), we reach the case (b) because $\nu(z_i) < \nu(z_{i-1})$ for every i and the valuation ν is discrete.

REMARK 5.3. It is useful to write the HN expansion in a more detailed way (called reduced form). To do this, let $s_1 < s_2 < \cdots < s_g = r$ be the ordered set of indices j such that $\nu(z_j)|\nu(z_{j-1})$, and for convenience, we put also $s_0 = 0$. Then, in the row $j = s_i$, there exists the minimum k_i such that $a_{j\,k_i} \neq 0$ (note that $k_i \geq 2$ because also $\nu(z_j) < \nu(z_{j-1})$). In this way, the HN expansion (5.1) could be written as

$$\begin{aligned}
z_{-1} &= y = a_{0\,1}x + \cdots + a_{0\,h_0}x^{h_0} + x^{h_0} z_1 \\
z_0 &= x = z_1^{h_1} z_2 \\
&\;\;\vdots \\
z_{s_1-1} &= a_{s_1\,k_1} z_{s_1}^{k_1} + \cdots + a_{s_1\,h_{s_1}} z_{s_1}^{h_{s_1}} + z_{s_1}^{h_{s_1}} z_{s_1+1} \\
z_{s_1} &= z_{s_1+1}^{h_{s_1+1}} z_{s_1+2} \\
&\;\;\vdots \\
z_{s_g-1} &= a_{s_g\,k_g} z_{s_g}^{k_g} + \cdots,
\end{aligned} \tag{5.2}$$

where, for $i = 1, \ldots, g$, one has $a_{s_i\,k_i} \neq 0$.

5.1.2. *Plane curves and HN expansions*

Let $\mathcal{C} = K[\![x, y]\!] = K[\![X, Y]\!]/P$ be a plane algebroid branch over K and let $\mathbf{m} = (x, y)$ be its maximal ideal. Let $\overline{\mathcal{C}} \simeq K[\![t]\!]$ be the integral closure of $\mathcal{C}$ in its field of fractions, so the valuation ν of $\mathcal{C}$ is given by $\nu(g) = \mathrm{ord}_t(g(x(t), y(t)))$. We assume that $\nu(x) \leq \nu(y)$; i.e. x is a transversal parameter.

The *Hamburger–Noether expansion* of $\mathcal{C}$ (with respect to $\{x, y\}$) is the Hamburger–Noether expansion of $\{x, y\} \in K[\![t]\!]$. Notice that in this case, it must be $\nu(z_r) = 1$.

Let $\underline{e} = (e_0, e_1, \ldots)$ be the multiplicity sequence of $\mathcal{C}$; one has $e_0 = \nu(x)$. The blow-up of $\mathcal{C}$ is the ring $\mathcal{C}^{(1)} = \mathcal{C}[y/x] \subset \overline{\mathcal{C}}$, $\mathfrak{m}_1 = (x, y_1)$ is its maximal ideal, and $\mathcal{C}^{(1)} \simeq K[\![x, y_1]\!]$. The coefficient $a_{0\,1}$ is the coordinate on the exceptional divisor of the strict transform; i.e. $y - a_{0\,1}x$ is just the tangent to $\mathcal{C}$. The multiplicity of $\mathcal{C}^{(1)}$ is $e_1 = \min\{\nu(x), \nu(y_1)\}$ and so $e_1 = \nu(x) = e_0$ if $\nu(y_1) \geq \nu(x)$ and $e_1 = \nu(y_1)$ if $\nu(y_1) < \nu(x)$. In this way, it is clear that the process of formation of the HN expansion exactly reproduces the process of resolution of the singularity. In fact, one has that (see [7, Proposition 2.2.9]) the HN expansion of $\mathcal{C}^{(1)}$ with respect to $\{x, y_1\}$ is as follows:

(1) If $h_0 > 1$,

$$y_1 = a_{02}x + \cdots + a_{0h_0}x^{h_0-1} + x^{h_0-1}z_1,$$

$$z_{j-1} = \sum_{i=1}^{h_j} a_{ji}z_j^i + z_j^{h_j} z_{j+1}; \quad 1 \leq j \leq r.$$

(2) If $h_0 = 1$,

$$z_{j-1} = \sum_{i=1}^{h_j} a_{ji}z_j^i + z_j^{h_j} z_{j+1}; \quad 1 \leq j \leq r.$$

In particular, let $n_i = \nu(z_i)$ be the values of the elements $z_i \in K[\![t]\!]$, $0 \leq i \leq r$. Then, the multiplicity sequence $\underline{e}$ of $\mathcal{C}$ is

$$\underline{n} = (n_0, \ldots, n_0, n_1, \ldots, n_1, \ldots, n_i, \ldots, n_i, \ldots, n_r, \ldots),$$

where n_i appears h_i times.

5.1.3. *Multiplicity sequence and HN expansions*

A set of formal expressions

$$z_{j-1} = \sum_{i=1}^{h_j} a_{ji}z_j^i + z_j^{h_j} z_{j+1}; \quad 0 \leq j \leq r, \tag{5.3}$$

where $h_0, \ldots, h_{r-1}$ are positive integers, $h_r = \infty$, and $a_{ji} \in K$ are such that $a_{j1} = 0$ if $j > 0$, will be called a Hamburger–Noether type expansion.

Let us fix $r \geq 0$, and if $r \geq 1$, let $1 \leq g \leq r$. Let $h_0, \ldots, h_{r-1}$ be positive integers, $0 < s_1 < \cdots < s_g = r$, and for $i = 1, \ldots, g$, let k_i be integers such that $2 \leq k_i \leq h_{s_i}$. Let $H = (H_0, \ldots, H_r)$ be the sequence defined by $H_j = [k_i, h_{s_i}]$ if $j = s_i$ and $H_j = h_j$ otherwise. The sequence H defines an HN type expansion such that its reduced form is like (5.2) with arbitrary coefficients $a_{s_i\,k} \in K$, $0 \leq i \leq g$, $k_i \leq k \leq h_{s_i}$, $a_{s_i,k_i} \neq 0$. We say that this is an HN expansion of type H.

Lemma 5.4. *There is a bijective correspondence between plane sequences $\underline{e}$ and finite sequences H as above.*

Proof. Let $\underline{e}$ be a plane sequence. Let us write $\underline{e} = (n_0, \ldots, n_0, n_1 \ldots, n_1, \ldots, n_r, \ldots)$ in such a way that $n_i > n_{i+1}$ and let h_i be the number of repetitions of n_i ($h_r = \infty$). Let $s_1 < s_2 < \cdots < s_g$ be the indexes j, $1 \leq j \leq r$, such that $n_j | n_{j-1}$ and $k_i = n_{j-1}/n_j \geq 2$ for $j = s_i$. The proximity relation for n_{j-1} implies that $k_i \leq h_{s_i}$. Thus, we have defined a sequence $H(\underline{e})$.

Let H be a sequence defined as above. Then, H allows us to define a unique sequence of positive integers $(n_0, \ldots, n_0, n_1, \ldots)$ starting with $n_j = 1$ for $j \geq r$. Then, if $j < r$, define $n_{j-1} = h_j n_j + n_{j+1}$ if $j \neq s_i$ for all i and $n_{s_i - 1} = k_i n_{s_i}$ if $j = s_i$. Obviously, this sequence $E(H) = \underline{n}$ satisfies the proximity conditions and so is a plane sequence. It is trivial that $E(-)$ and $H(-)$ are applications inverse to each other. ∎

Proposition 5.5. *A Hamburger–Noether type expansion defines a unique plane irreducible curve $\mathcal{C} = K[\![x, y]\!]$ with $\overline{\mathcal{C}} \simeq K[\![z_r]\!]$ and whose HN sequence is the prefixed one.*

Moreover, let $\underline{e}$ be a plane sequence and let $H(\underline{e})$ be a sequence defined as above for $\underline{e}$. Then, an HN expansion of type $H(\underline{e})$ defines a unique plane irreducible curve over K such that its multiplicity sequence is $\underline{e}$.

Proof. Let $x = z_0$, $y = z_{-1}$, $t = z_r$. Performing the successive (inverse) substitutions, we have a parametrization $x = x(t)$, $y = y(t)$, and so we have a morphism $\varphi : K[\![X, Y]\!] \to K[\![t]\!]$ defined by $\varphi(X) = x(t)$, $\varphi(Y) = y(t)$. The ring $\mathcal{C} = K[\![x, y]\!] = K[\![X, Y]\!]/\ker(\varphi)$ is the ring of an irreducible algebroid plane curve. Moreover, if $K((x, y))$ is the field of fractions of $\mathcal{C}$, it is easy to see (recursively) that $z_i \in K((x, y))$ for all i, in particular $t = z_r \in K((x, y))$, and so $K((x, y)) = K((t))$, $\overline{\mathcal{C}} = K[\![t]\!]$. Obviously, the HN expansion of $\mathcal{C}$ with respect to $\{x, y\}$ is the one we started with.

The second assertion is a trivial consequence of the first one and of Lemma 5.4. ∎

Remark 5.6. The relation between a plane sequence $\underline{e}$ and the sequence $H(\underline{e})$ implies that the free points (multiplicities) are exactly as follows:

(1) the first h_0 points of multiplicity n_0 and the first one of multiplicity n_1;

(2) for each $t = 1, \dots, g-1$, the last $h_{s_t} - k_t \geq 0$ points of multiplicity n_{s_t} and the first one of multiplicity n_{s_t+1}; for $t = g$, all the points of multiplicity $n_{s_g} = 1$ but the first k_g.

As a consequence, the free points (except the first one) are in a one to one correspondence with the coefficients $\{a_{ji}\}$ of the HN expansion which are not forced to be zero. So, for any choice of

$$\{a_{s_t,i} \in K \mid 0 \leq t \leq g;\ k_t \leq i \leq h_{s_t};\ a_{s_t,k_t} \neq 0\},$$

one has a curve with multiplicity sequence $\underline{e}$.

Moreover, the Euclidean algorithm for n_{s_t} and n_{s_t+1} determines all the multiplicities n_i $(s_t + 2 \leq i \leq s_{t+1})$, the integers h_i $(s_t + 1 \leq i < s_{t+1})$, and also $k_{s_{t+1}}$, that is, all the satellite points after the free point n_{s_t+1} up to the next free point.

The rows $\{s_i : i = 0, \dots, g\}$ are called *the free rows* and the rest *the satellite rows* because of the distribution of free and satellite points.

5.2. *Case of two branches (i.e. $d = 2$)*

Let us assume first that the ring $R = \mathcal{O}$ is a local one (i.e. $c = 1$) with two branches $\mathcal{C}$ and $\mathcal{C}'$ $(d = 2)$. Let $\mathfrak{p}$, $\mathfrak{p}'$ be the minimal prime ideals of $\mathcal{O}$, and then the branch $\mathcal{C}$ is $\mathcal{C} = R/\mathfrak{p}$ and the branch $\mathcal{C}'$ is $\mathcal{C}' = R/\mathfrak{p}'$. Let $\underline{e} = (e_0, e_1, \dots)$ (resp. $\underline{e}' = (e'_0, e'_1, \dots)$) be the sequence of multiplicities of the branch $\mathcal{C}$ (resp. $\mathcal{C}'$).

The *splitting number* of $\mathcal{O}$ is defined as the biggest positive integer k such that $\mathcal{O}^{(k)}$ is local. Thus, one has that $\mathcal{O}^{(k)}$ is local and $\mathcal{O}^{(k+1)} \simeq \mathcal{C}^{(k+1)} \times \mathcal{C}'^{(k+1)}$. The multiplicity tree of $\mathcal{O}$ is the result of identifying the bamboos of both branches $\mathcal{C}$ and $\mathcal{C}'$ up to level k, and the weights on the trunk are the fine multiplicities of $\mathcal{O}^{(j)}$, for $j \leq k$, i.e. $\{(e_j, e'_j); j = 0, \dots, k\}$. After the splitting level k, i.e. for $j \geq k+1$, the weights are the fine multiplicity of $\mathcal{C}^{(j)}$: $(m(\mathcal{C}^{(j)}), 0) = (e_j, 0)$ and the one of $\mathcal{C}'^{(j)}$: $(0, m(\mathcal{C}'^{(j)})) = (0, e'_j)$.

Notice that if R is not local (i.e. $d = 2$ and $c = 2$), then the splitting number is defined as $k = -1$.

The intersection multiplicity of $\mathcal{C}$ and $\mathcal{C}'$ is given by the Noether formula $[\mathcal{C}, \mathcal{C}'] = \sum_{j=0}^{k} e_j e'_j$ (an easy consequence of the equality

$$[\mathcal{C}, \mathcal{C}'] = e_0 e'_0 + [\mathcal{C}^{(1)}, \mathcal{C}'^{(1)}];$$

see [7, Remark 2.3.2 (iv)]). Thus, if one fixes both sequences of multiplicities $\underline{e}$ and $\underline{e}'$, then the splitting number k is equivalent to the intersection multiplicity. As a consequence, one has that the semigroup of values S is equivalent data to the multiplicity tree.

The splitting number (for a fixed pair of plane sequences $\underline{e}$ and $\underline{e}'$) is not an arbitrary one.

DEFINITION 5.7. We will say that an integer $k \geq -1$ is admissible if $k = -1$ or $k \geq 0$ and it satisfies the following properties:

(1) $e_{i-1} = e_i$ if and only if $e'_{i-1} = e'_i$ for $i = 1, \ldots, k-1$.

(2) $r(e_j) = r(e'_j)$ for all $j \leq k$.

(3) If $e_{k-1} > e_k$, then $e'_{k-1} = e'_k$.

(4) If $r(e_k) = r(e'_k) = r(e_{k+1}) = r(e'_{k+1}) = 2$ and if $e_{k-1} = e_k$, then $e'_{k-1} > e'_k$.

Notice that $k = -1$ is always admissible for any pair of plane sequences.

PROPOSITION 5.8. *Let $k \geq 0$ be an integer with the properties* (1) *and* (2) *of Definition 5.7. Then, k is admissible if and only if either k is maximal with the conditions* (1) *and* (2) *or $r(e_{k+1}) = r(e'_{k+1}) = 1$.*

PROOF. Let us assume that k is admissible and that the conditions (1) and (2) are also true for $k+1$. In particular (see property (3)), $e_{k-1} = e_k$ and $e'_{k-1} = e'_k$. Moreover, $r(e_{k+1}) = r(e'_{k+1})$ and (see property (4)) if it is equal to 2, one reaches a contradiction. Thus, we have proved that if k is not maximal, then $r(e_{k+1}) = r(e'_{k+1}) = 1$.

Let us show the sufficient condition. Firstly, note that the condition $e_{k-1} > e_k$ implies that $r(e_{k+1}) = 2$. So, if $e_{k-1} > e_k$ and also $e'_{k-1} > e'_k$, then $r(e_{k+1}) = r(e'_{k+1}) = 2$ and k is forced to be maximal. But obviously, this is not the case because (1) and (2) are also true for $k+1$. This proves property (3).

To prove property (4), the hypothesis $r(e_{k+1}) = r(e'_{k+1}) = 2$ implies that k must be maximal with properties (1) and (2). So, if $e_{k-1} = e_k$, then $e'_{k-1} > e'_k$ and the proof is finished. ■

As a consequence, the properties of the definition can be expressed in a somewhat simpler form in the following way.

DEFINITION 5.9. We will say that an integer $k \geq -1$ is admissible if $k = -1$ or $k \geq 0$ and it satisfies the following properties:

(1) $e_{i-1} = e_i$ if and only if $e'_{i-1} = e'_i$ for $i = 1, \ldots, k-1$.

(2) $r(e_j) = r(e'_j)$ for all $j \leq k$.

(3) Either k is maximal with the conditions 1 and 2 or $r(e_{k+1}) = r(e'_{k+1}) = 1$.

REMARK 5.10. Notice that if $e_{i-1} > e_i$, then $r(e_{i+1}) = 2$. As a consequence, if k is admissible, then the following hold:

(1) If k is not maximal with properties (1) and (2), then $r(e_{k+1}) = r(e'_{k+1}) = 1$; i.e. both are free points and then $(e_{k-1}, e'_{k-1}) = (e_k, e'_k)$. However, it is possible to have $(e_{k-1}, e'_{k-1}) = (e_k, e'_k)$ and $r(e_{k+1}) \neq r(e'_{k+1})$.

(2) The situation $e_{k-1} > e_k$ and $e'_{k-1} > e'_k$ is not possible. In particular, e_k and e'_k cannot be simultaneously terminal free points.

5.2.1. Intersection multiplicities with HN expansions

Let $\mathcal{O} \simeq K[\![x, y]\!]$ be the local ring of a plane curve with two branches, $\mathcal{C}$ and $\mathcal{C}'$, let $\underline{e} = (e_0, e_1, \ldots)$ and $\underline{e}' = (e'_0, e'_1, \ldots)$ be the multiplicity sequences of $\mathcal{C}$ and $\mathcal{C}'$. Assume that x is a transversal parameter for $\mathcal{C}$ and $\mathcal{C}'$. Let $z_0 = z'_0 = x$, $z_{-1} = z'_{-1} = y$, and let

$$\begin{aligned} z_{j-1} &= \sum_{i=1}^{h_j} a_{ji} z_j^i + z_j^{h_j} z_{j+1}; \quad 0 \le j \le r, \\ z'_{j-1} &= \sum_{i=1}^{h'_j} a'_{ji} (z'_j)^i + (z'_j)^{h'_j} z'_{j+1}; \quad 0 \le j \le r' \end{aligned} \tag{5.4}$$

be the HN expansions of $\mathcal{C}$ and $\mathcal{C}'$ with respect to x, y.

Let s be the largest integer such that $h_j = h'_j$ for all $j < s$ and $a_{ji} = a'_{ji}$ for $j < s$ and $i \le h_j$. Let $t \le \min\{h_s + 1, h'_s + 1\}$ be the largest integer for which $a_{si} = a'_{si}$ for all $i < t$.

Note that if $t < \min\{h_s + 1, h'_s + 1\}$, then $a_{st} \neq a'_{st}$, in particular $s = s_q$ for some $0 \le q \le \min\{g, g'\}$. Otherwise, $t = \min\{h_s + 1, h'_s + 1\}$ and necessarily $h_s \neq h'_s$.

Proposition 5.11. *With the above notations, let $S = \sum_0^{s-1} h_j n_j n'_j$. Then, one has the following:*

(1) *The splitting number k between $\mathcal{C}$ and $\mathcal{C}'$ is equivalent data to the pair (s, t); in fact,*
$$k = h_0 + h_1 + \cdots + h_{s-1} + t - 1.$$

(2) *The intersection multiplicity $[\mathcal{C}, \mathcal{C}']$ is as follows:*
   (a) *If $t < \min\{h_s + 1, h'_s + 1\}$, then $[\mathcal{C}, \mathcal{C}'] = S + t n_s n'_s$.*
   (b) *If $t = h'_s + 1 < h_s + 1$, then $[\mathcal{C}, \mathcal{C}'] = S + h'_s n_s n'_s + n'_{s+1} n_s$.*
   (c) *If $t = h_s + 1 < h'_s + 1$, then $[\mathcal{C}, \mathcal{C}'] = S + h_s n_s n'_s + n_{s+1} n'_s$.*

Proof. One has that $k = 0$ if and only if $a_{01} \neq a'_{01}$. Hence, this situation is equivalent to $(s, t) = (0, 1)$ and the equality follows. The case $k > 0$ is equivalent to $a_{01} = a'_{01}$ and the proof follows by induction using the expression of the HN expansion of the strict transform of a branch in terms of the one of $\mathcal{C}$.

The equality of the intersection multiplicity is a consequence of the expression for the splitting number or can be proved also by induction using that $[\mathcal{C}, \mathcal{C}'] = n_0 n'_0 + [\mathcal{C}^{(1)}, \mathcal{C}'^{(1)}]$ (see [7, Remark 2.3.2 and Proposition 2.3.3]). ∎

PROPOSITION 5.12. *Let $\underline{e}$, $\underline{e}'$ be two plane sequences and let $k \geq -1$ be an admissible number for them. Let $\mathcal{C}$ be a branch with multiplicity sequence $\underline{e}$. Then, there exists a branch $\mathcal{C}'$ with multiplicity sequence $\underline{e}'$ and such that k is the splitting number of the curve with branches $\mathcal{C}$ and $\mathcal{C}'$. In particular, k is the splitting number of a pair of branches with multiplicity sequences $\underline{e}$ and $\underline{e}'$ if and only if k is admissible.*

PROOF. The case $k = -1$ is trivial. Let $\mathcal{C}$ be a branch with multiplicity sequence $\underline{e}$ and HN expansion

$$z_{j-1} = \sum_{i=1}^{h_j} a_{ji} z_j^i + z_j^{h_j} z_{j+1}, \quad 0 \leq j \leq r,$$

and let $k \geq 0$ be an admissible number for $\underline{e}$ and $\underline{e}'$. Let

$$z'_{j-1} = \sum_{i=1}^{h'_j} A'_{ji} (z'_j)^i + (z'_j)^{h'_j} z'_{j+1}, \quad 0 \leq j \leq r',$$

be an HN type expansion for $H(\underline{e}')$ in which we see the symbols $\{A'_{ij}\}$ as parameters to be determined. If $k = 0$, it suffices to fix $A'_{01} = a'_{01} \in K$ such that $a'_{01} \neq a_{01}$. If $k > 0$, then we fix $A'_{01} = a_{01}$. Now let $\widetilde{\underline{e}} = (e_1, \ldots)$ and $\widetilde{\underline{e}}' = (e'_1, \ldots)$ and let $\mathcal{C}^{(1)}$ be the strict transform of $\mathcal{C}$ by one blowing-up. The multiplicity sequence of $\mathcal{C}^{(1)}$ is $\widetilde{\underline{e}}$, $\widetilde{\underline{e}}'$ is a plane sequence, and $k-1$ is an admissible number for $\widetilde{\underline{e}}$ and $\widetilde{\underline{e}}'$. By the induction hypothesis, there exists a branch D with multiplicity sequence $\widetilde{\underline{e}}'$ and splitting number with $\mathcal{C}^{(1)}$ equal to $k-1$. The HN expansion of D completed with $A'_{01} = a_{01}$ provides a branch $\mathcal{C}'$ with multilicity sequence $\underline{e}'$ and such that its splitting number with $\mathcal{C}$ is k. ∎

5.3. *General case*

Let $R \cong \mathcal{O}_1 \times \cdots \times \mathcal{O}_c$ be a direct product of local rings $\mathcal{O}_j$ $(1 \leq j \leq c)$, each one associated with a reduced plane algebroid curve defined over an algebraically closed field K. Let us denote by $\mathcal{C}_1, \ldots, \mathcal{C}_d$ the branches of R. Let $\mathcal{T}$ be the multiplicity tree of R. Take the notations given at the beginning of the section. For each branch $\mathcal{C}_i$, $i = 1, \ldots, d$, one has its corresponding branch $\mathcal{T}^i$ of $\mathcal{T}$ (i.e. a maximal completely ordered subtree of $\mathcal{T}$) and so the sequence $\underline{e}^i = (e_0, e_1, \ldots)$ of multiplicities of $\mathcal{C}_i$. For $i, j \in \{1, \ldots, d\}$, let $k_{i,j} + 1$ be the length of the trunk of the subtree of $\mathcal{T}$ given by $\mathcal{C}_i$ and $\mathcal{C}_j$, so $k_{i,j}$ is just the splitting number of $\mathcal{C}_1 \cup \mathcal{C}_2$. The fact that $\mathcal{T}$ is the disjoint union of c trees implies some restrictions on the set of integers $\{k_{i,j}\}$:

(5.5) Given $i, j, t \in \{1, \ldots, d\}$, if one has that $k_{j,t} > k_{j,i}$, then $k_{i,t} = k_{i,j}$.

Note that the condition (5.5) above is enough to construct a graph $\mathcal{T}(\{\underline{e}^i\}, \{k_{i,j}\})$ by joining the d sequences of integers $\{\underline{e}^i; i = 1, \dots, d\}$ with the help of the splitting vertices indicated by $\{k_{i,j}\}$. (Pay attention that the graph is a tree if and only if $k_{i,j} \geq 0$ for any i, j.)

LEMMA 5.13. *Let $E = \{\underline{e}^i = (e^i_0, e^i_1, \dots);\ i = 1, \dots, d\}$ be a set of sequences of positive integers and $\{k_{i,j} \geq -1\}$, $i, j \in \{1, \dots, d\}$, $i \neq j$, an indexed set of integers with $k_{i,j} = k_{j,i}$ and satisfying property* (5.5)*. Then, there exists a weighted graph $\mathcal{T} = \mathcal{T}(\{\underline{e}^i\}, \{k_{i,j}\})$ such that the set of maximal completely ordered subgraphs of $\mathcal{T}$, $\{\mathcal{T}^1, \dots, \mathcal{T}^d\}$, coincides with E and for $i, j \in \{1, \dots, d\}$, the length of the trunk of $\mathcal{T}^i \cup \mathcal{T}^j \subset \mathcal{T}$ is $k_{i,j} + 1$.*

PROOF. The proof is easy by induction on the number of branches d. Otherwise, we can define directly the graph in the following way. For each integer $t \geq 0$, let $i \sim_t j$ if and only if $k_{i,j} \geq t$. If $k_{i,j}, k_{j,s} \geq t$, then by (5.5), one has $k_{i,s} \geq \min\{k_{i,j}, k_{j.s}\} \geq t$. Thus, the relation $\sim_t$ is an equivalence relation. For each equivalence class J_t, we can take a vertex with weight $m(J_t) = (m_1, \dots, m_d) \in \mathbb{N}^d$ defined as $m_i = e^i_t$ if $i \in J_t$ and $m_i = 0$ otherwise. Notice that if $t \geq \ell$ and $i \sim_t j$, then $i \sim_\ell j$. Hence, the result is the disjoint union of c trees, each one with root in one of the equivalence classes of $\sim_0$; in particular, it is a tree if and only if $k_{i,j} \geq 0$ for all $i, j \in \{1, \dots, d\}$. ■

Adding to the lemma the conditions of plane sequences and the admissibility, one has the following result.

PROPOSITION 5.14. *Let $\{\underline{e}^i = (e^i_0, e^i_1, \dots);\ i = 1, \dots, d\}$ be a set of sequences of non-negative integers and $\{k_{i,j} \geq -1\}$, $i, j \in \{1, \dots, r\}$, $i \neq j$, an indexed set of integers satisfying property* (5.5)*. Let $\mathcal{T} = \mathcal{T}(\{\underline{e}^i\}, \{k_{i,j}\})$ be the weighted graph constructed in Lemma* 5.13*. Then, there exists a plane curve with multiplicity tree $\mathcal{T}$ if and only if*

(1) *for $i = 1, \dots, r$, $\underline{e}^i$ is a plane sequence;*

(2) *for $i, j \in \{1, \dots, r\}$, $i \neq j$, $k_{i,j} = k_{j,i}$ is an admissible splitting number between the sequences $\underline{e}^i$ and $\underline{e}^j$.*

PROOF. We will proceed by induction on the number of branches d. Notice that the case $d \leq 2$ is already known. Moreover, if there exists i, j such that $k_{i,j} = -1$, then the result is trivial because we can separate the set of branches $\{1, \dots, d\}$ in two parts I, J such that $\#I, \#J < d$ and $k_{i,j} = -1$ for $i \in I$ and $j \in J$. So, we can assume that $k_{i,j} \geq 0$ for any pair i, j; i.e. the searched ring R must be a local one.

Let us fix a branch $i = 1$ and let $I = \{2, \dots, d\}$. Let us assume that $k_{1,2} \geq k_{1,i}$ for all $i \geq 2$. Let $\mathcal{T}'$ be the sub-graph of $\mathcal{T}$ defined by the sequences $\{\underline{e}^i : 2 \leq i \leq d\}$ and integers $\{k_{i,j} \mid i, j \geq 2\}$. By the induction hypothesis, there exists a reduced curve $\mathcal{C}'$ consisting of the branches $\mathcal{C}_2, \dots, \mathcal{C}_d$ such that the multiplicity tree of $\mathcal{C}'$ is $\mathcal{T}'$.

Without loss of generality, we can assume that $\mathcal{C}_i$ is given by an HN parametrization $\varphi_i : (X,Y) \mapsto (x(t_i), y(t_i))$ in such a way that if $\varphi' : K[\![X,Y]\!] \to K[\![t_2]\!] \times \cdots \times K[\![t_d]\!]$, $\varphi'(f) = (\varphi_2(f), \ldots, \varphi_d(f))$, then $R' = K[\![X,Y]\!]/\ker(\varphi)$ is the local ring of $\mathcal{C}'$.

Let $\varphi_1 : (X,Y) \mapsto (x(t_1), y(t_1))$ be an HN parametrization of a branch $\mathcal{C}_1$ such that its multiplicity sequence coincides with $\underline{e}^1$ and the splitting number with $\mathcal{C}_2$ is equal to $k_{1,2}$ (see Proposition 5.12). Since $k_{1,2} \geq k_{1,i}$ for all $i \geq 3$, it is clear that the splitting number of $\mathcal{C}_1$ and $\mathcal{C}_i$ is equal to $k_{1,i}$. Now consider the map $\varphi : K[\![X,Y]\!] \to K[\![t_1]\!] \times K[\![t_2]\!] \times \cdots \times K[\![t_d]\!]$ given by $\varphi = (\varphi_1, \ldots, \varphi_d)$ and let $R = K[\![X,Y]\!]/\ker(\varphi)$. Then, the multiplicity tree of R coincides with $\mathcal{T}$ and the proof is finished. ∎

As a consequence, one also has the following theorem.

THEOREM 5.15. *Let $R \cong \mathcal{O}_1 \times \cdots \times \mathcal{O}_c$ be a direct product of local rings $\mathcal{O}_j$ ($1 \leq j \leq c$), each one associated with a reduced plane algebroid curve defined over an algebraically closed field K. Let $\mathcal{C}_1, \ldots, \mathcal{C}_d$ be the set of branches of R. The following elements are equivalent:*

(1) *The semigroup of values S of R.*

(2) *The semigroups S_i, $1 \leq i \leq d$, of each branch and the set of intersection multiplicities $\{[\mathcal{C}_i, \mathcal{C}_j] \mid 1 \leq i < j \leq d\}$ between pairs of branches.*

(3) *The multiplicity tree $\mathcal{T}(R)$ of R.*

(4) *The set $E = \{\underline{e}^i = (e_0^i, e_1^i, \ldots);\ i = 1, \ldots, d\}$ of the multiplicity sequences of the branches $\{\mathcal{C}_i \mid 1 \leq i \leq d\}$ plus the splitting numbers $\{k_{i,j}\}$ between pairs of branches $\mathcal{C}_i$, $\mathcal{C}_j$; $1 \leq i < j \leq d$.*

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