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The distribution of negative eigenvalues of Schrödinger operators on asymptotically hyperbolic manifolds

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Abstract. We study the asymptotic behavior of the counting function of negative eigenvalues of Schrödinger operators with real valued potentials which decay at infinity on asymptotically hyperbolic manifolds. We establish conditions on the rate of decay of the potential that determine if there are finitely or infinitely many negative eigenvalues. In the latter case, they may only accumulate at zero and we obtain the asymptotic behavior of the counting function of eigenvalues in an interval $(-\infty, -E)$ as $E \to 0$.

1. Introduction

We are concerned with the following type of problem. Let (X, g) be a non-compact complete C^∞ Riemannian manifold and let Δ_g be its (positive) Laplacian. Suppose that V is a real valued potential such that $V < 0$ near infinity and the corresponding Schrödinger operator $H = \Delta_g + V$ is self-adjoint. Furthermore, suppose its point spectrum $\sigma_{\mathrm{p}}(H) \subset (-E_0, 0)$ and the eigenvalues only accumulate at zero. The problem is to find conditions on V which determine whether the point spectrum is finite or infinite and if it is infinite, find the asymptotic behavior of the number of eigenvalues (counted with multiplicity) in an interval $(-E_0, -E)$ as $E \downarrow 0$.

In the Euclidean case, it has been shown, see for example [26], that if V is bounded and if that near infinity $V(z) \leq -C|z|^{-2+\delta}$, $\delta > 0$, then H has infinitely many eigenvalues, while if $V \geq -C|z|^{-2-\delta}$, $\delta > 0$, there are finitely many eigenvalues. The threshold decay of $V(z)$ for H to have finitely or infinitely many eigenvalues is therefore $V(z) \sim F(\omega)|r|^{-2}$, $r = |z|$, $z = r\omega$, $\omega \in \mathbb{S}^{n-1}$. Moreover, an application of Hardy's inequality shows that if $V(z) \sim -cr^2$, there are finitely many eigenvalues when $c < (n-1)^2/4$ and infinitely many if $c > (n-1)^2/4$. Precise results on the asymptotics of the counting function of eigenvalues as $E \to 0$, in the case where

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$V(z) = r^{-2}(F(\omega) + \mathcal{E}(r,\omega))$ with $\mathcal{E}(r,\omega) = o((\log r)^{-1-\varepsilon})$ as $r \to \infty$, were obtained by Kirsch and Simon [16] and Hassell and Marshal [12]. We are not aware of similar results for asymptotically Euclidean manifolds.

While this problem has been well studied in Euclidean space, it seems that it has not been studied as much in hyperbolic space. We will work in the class of asymptotically hyperbolic manifolds in the sense of [9, 20], for which the hyperbolic space serves as a model. Akutagawa and Kumura [2] have established bounds on the potential, similar to those in Euclidean space, which determine if the discrete spectrum is finite or infinite (as in the first two items of Corollary 1.3 below), but they do not establish bounds on the counting function of negative eigenvalues and do not consider the cases of critical decay, as in our Theorems 1.2 and 1.4. We should also mention the work of Mazzeo and McOwen [19]. While the problems considered in [19] are somewhat the opposite of the ones we study here, there are some similarities.

The Poincaré model of the hyperbolic space $\mathbb{H}^{n+1}$ with constant curvature -1 is given by the Euclidean ball of radius one:

$$\mathbb{B}^{n+1} = \{z \in \mathbb{R}^{n+1} : |z| < 1\} \quad \text{equipped with the metric } g_0(z) = \frac{4dz^2}{(1-|z|^2)^2}. \tag{1.1}$$

The closure of $\mathbb{B}^{n+1}$ is a compact C^∞ manifold with boundary, $g_0 \in C^\infty(\mathbb{B}^{n+1})$, but it is singular at $\{|z| = 1\} = \partial\mathbb{B}^{n+1}$. The function $\varsigma(z) = 1 - |z|^2 \in C^\infty(\overline{\mathbb{B}^{n+1}})$, is a defining function of $\partial\mathbb{B}^{n+1}$ in the sense that $\varsigma(z) \geq 0$, $\varsigma^{-1}(0) = \partial\mathbb{B}^{n+1}$, and $d\varsigma(z) \neq 0$ if $|z| = 1$. Moreover, $\varsigma^2 g_0 = 4dz^2$ is a C^∞ Riemannian metric on the closure $\overline{\mathbb{B}^{n+1}}$. Following [9,20,21], one can extend this notion to any C^∞ Riemannian manifold with boundary.

Throughout this paper, $\overset{\circ}{X}$ will denote the interior of a C^∞ compact manifold X with boundary ∂X of dimension $n+1$. We say that $\varsigma \in C^\infty(X)$ is a *defining function of* ∂X, or a *boundary defining function,* if $\varsigma > 0$ in $\overset{\circ}{X}$, $\varsigma = 0$ at ∂X and $d\varsigma \neq 0$ at ∂X. We assume $\overset{\circ}{X}$ is equipped with a C^∞ Riemannian metric g such that $G = \varsigma^2 g$ is non-degenerate at ∂X and so $(\overline{X}, G)$ is a C^∞ compact Riemannian manifold with boundary. According to [17], the sectional curvature of $(\overset{\circ}{X}, g)$ converges to $-|d\varsigma|_G$ along any curve that goes towards ∂X. The manifold $(\overset{\circ}{X}, g)$ is called an *asymptotically hyperbolic manifold,* or *AHM,* if $|d\varsigma|_G = 1$ at ∂X. One might relax this assumption if X has more than one boundary component and $\partial X = Y_1 \sqcup Y_2 \ldots \sqcup Y_N$ and $|d\varsigma|_G = \kappa_j$ at Y_j, κ_j is a constant.

The hyperbolic space serves as a model for this class of manifolds, and its quotients by certain discrete groups of fractional linear transformations having a geometrically finite fundamental domain without cusps at infinity are also examples of such manifolds, see [1, 20, 23, 24]. In fact, our results apply for manifolds with more than one boundary component and with different (constant) asymptotic curvatures at each end. One important example from mathematical physics where this occurs is the stationary model of the de Sitter–Schwarzschild model of black holes discussed in [27]. In this case, the manifold is $\mathring{X} = (a,b) \times \mathbb{S}^n$, and the asymptotic curvatures are different on both ends.

If $(\mathring{X}, g)$ is an AHM and $\varsigma \in C^\infty(X)$ is a boundary defining function, then the conformal metric $G = \varsigma^2 g$ is non-degenerate up to ∂X, but the metric $\varsigma^2 G|_{\partial X} = h_0$ obviously depends on the choice of ς. In fact, given any two boundary defining functions ς and $\tilde{\varsigma}$, we must have $\varsigma = a(z)\tilde{\varsigma}$, with $a > 0$. If $\widetilde{G} = \tilde{\varsigma}^2 g$, then $G = a^2 \widetilde{G}$, and so $G|_{\partial X} = (a^2 \widetilde{G})|_{\partial X}$, and hence $\varsigma^2 g$ determines a conformal class of metrics at ∂X. As shown in [11, 15], given a representative h_0 of the class $[\varsigma^2 g|_{\partial X}]$, there exists $e > 0$ and a unique boundary defining function x defined on a collar neighborhood U of ∂X and a map $\Psi \colon [0,e) \times \partial X \to U$ such that

$$\Psi^* g = \frac{dx^2}{x^2} + \frac{h(x)}{x^2}, \quad h(0) = h_0, \tag{1.2}$$

where $h(x)$ is in $C^\infty([0,e))$ with values on the space of Riemannian metrics on ∂X. Of course, x can be extended (non-uniquely) from the collar neighborhood U to a boundary defining function $x \in C^\infty(X)$.

For example, in the case of the hyperbolic space (1.1), the geodesic distance with respect to the origin is given by

$$r = \log\Big(\frac{1+|z|}{1-|z|}\Big),$$

and using polar coordinates (r,θ), $\theta = z/|z|$, with respect to this distance r, the metric g_0 is given by

$$g_0 = dr^2 + (\sinh r)^2 d\theta^2,$$

where $d\theta^2$ is the standard metric on the sphere. If we set $x = e^{-r}$, then

$$g_0 = \frac{dx^2}{x^2} + \frac{(1-x^2)^2}{4}\frac{d\theta^2}{x^2}. \tag{1.3}$$

While x is not smooth on X because $|z|$ is not C^∞ at $\{z = 0\}$, it is C^∞ near $\partial\mathbb{B}^{n+1}$, and one can modify it in the interior of X to satisfy the definition of a boundary defining function and still keep (1.3) near $\partial\mathbb{B}^{n+1}$.

Let Δ_g denote the (positive) Laplace operator on an AHM $(\mathring{X}, g)$. We know from [17, 18], that the spectrum of Δ_g, denoted by $\sigma(\Delta_g)$, satisfies

$$\sigma(\Delta_g) = \sigma_{\mathrm{pp}}(\Delta_g) \cup \sigma_{\mathrm{ac}}(\Delta_g),$$

where $\sigma_{\mathrm{pp}}(\Delta_g)$ consists of a finite number of eigenvalues in $(0, n^2/4)$ with finite multiplicity and $\sigma_{\mathrm{ac}} = [n^2/4, \infty)$ is the absolutely continuous spectrum. There are no eigenvalues in $[n^2/4, \infty)$, see for example [4, 18]. We shall work with $\Delta_g - n^2/4$

which has continuous spectrum $[0,\infty)$. Let $V \in L^\infty(X)$ be real valued and such that $V(z) < 0$ near ∂X and $V(z) \to 0$ as $z \to \partial X$. We shall denote

$$H_0 = \Delta_g - \frac{n^2}{4} \quad \text{and} \quad H = \Delta_g - \frac{n^2}{4} + V. \tag{1.4}$$

We will show that under the assumptions on the rate of decay of the potential $V(z)$ as $z \to \partial X$, the point spectrum of H consists of eigenvalues of finite multiplicity contained in some interval $[-E_0, 0)$, which only possibly accumulate at zero. Its essential spectrum $\sigma_{\text{ess}}(H) = [0,\infty)$ and it has no embedded eigenvalues, including the bottom of $\sigma_{\text{ess}}(H)$. We want to count negative eigenvalues of H and other operators, and so for an operator T and for $E \geq 0$, we define the counting function

$$N_E(T) = \#\{\mu \in (-\infty, -E) \cap \sigma_{\text{pp}}(T),\, E \geq 0 \text{ counted with multiplicity}\}. \tag{1.5}$$

In coordinates for which (1.2) is valid, the Laplacian with respect to g is given by

$$\Delta_g = -(x\partial_x)^2 - nx\partial_x - x^2 A(x,y)\partial_x + x^2 \Delta_{h(x)}, \tag{1.6}$$

where $\Delta_{h(x)}$ is the Laplacian with respect to the metric $h(x)$ on ∂X, $|h|$ is the volume element of the metric h and $A = \partial_x \log|h|/2$. It is convenient to define $\rho \stackrel{\text{def}}{=} -\log x$, and so

$$\Delta_g = -\partial_\rho^2 - n\partial_\rho - e^{-\rho}\mathcal{A}(\rho,y)\partial_\rho + e^{-2\rho}\Delta_{\tilde{h}(\rho)}, \tag{1.7}$$

where $\mathcal{A}(\rho,y) = A(e^{-\rho},y)$ and $\tilde{h}(\rho) = h(e^{-\rho})$.

Throughout the paper, $(\mathring{X}, g)$ will denote a $n+1$-dimensional asymptotically hyperbolic manifold. Its closure X is a C^∞ manifold with boundary, and $x \in C^\infty(X)$ is a boundary defining function such that (1.2) holds for $x \in [0,\epsilon)$. We define $\rho \stackrel{\text{def}}{=} -\log x$. We assume that $V \in L^\infty(X)$ *is real valued,* and that H_0 and H and $N_E(H)$ will be defined as in (1.4) and (1.5) respectively.

Theorem 1.1. *Suppose that there exists $\rho_0 > 0$ such that for $\rho \in (\rho_0, \infty)$*

$$V(e^{-\rho}, y) = -c\rho^{-2+\delta} + O(\rho^{-2+\delta}(\log\rho)^{-\varepsilon}), \tag{1.8}$$

with $c > 0$, $\varepsilon > 0$, and $\delta < 2$, as $\rho \to \infty$. If $\delta < 0$, then $N_0(H) < \infty$, but if $\delta \in (0,2)$, then $N_0(H) = \infty$ and

$$\log\log N_E(H) = \frac{1}{2-\delta}\log E^{-1} + O(1), \quad \text{as } E \to 0. \tag{1.9}$$

Notice that if $\tilde{x} = e^{\varphi(x,y)}x$, $\varphi \in C^\infty$, is another boundary defining function, then

$$\tilde{\rho} = -\log\tilde{x} = -\varphi - \log x = \rho + O(1), \quad \text{as } \rho \to \infty,$$

so (1.8) does not depend on the choice of x.

In the threshold case $\delta = 0$, we have the following.

Theorem 1.2. *Suppose that there exists* ρ_0 *such that for all* $\rho \in (\rho_0, \infty)$,

$$V(e^{-\rho}, y) = -c\rho^{-2} + o(\rho^{-2}(\log\rho)^{-\varepsilon}), \quad c > 0,\ \varepsilon > 0,\ \textit{as } \rho \to \infty. \tag{1.10}$$

If $c < 1/4$, *then* $N_0(H) < \infty$, *but if* $c > 1/4$, *then* $N_0(H) = \infty$ *and*

$$\log\log N_E(H) = -\frac{1}{2}\log E + O(1) \quad \textit{as } E \to 0. \tag{1.11}$$

Our proofs in fact give somewhat more precise upper and lower bounds for $N_E(H)$, and (1.9) and (1.11) are used to unify these bounds and provide the asymptotic behavior of iterated logarithms of $N_E(H)$. The methods we use do not allow us to treat the case where c is a function of y. However, we can use Theorems 1.1 and 1.2 to prove the following.

Corollary 1.3. *Suppose that there exists* $\rho_0 > 0$ *such that for all* $\rho \in (\rho_0, \infty)$,

$$-c_1\rho^{-2+\delta} \le V(e^{-\rho}, y) \le -c_2\rho^{-2+\delta};$$

then we can say that

(1) *if* $\delta < 0$, *then* $N_0(H) < \infty$;
(2) *if* $\delta \in (0, 2)$, *then* $N_0(H) = \infty$ *and* (1.9) *holds;*
(3) *if* $\delta = 0$ *and* $c_1 < 1/4$, *then* $N_0(H) < \infty$;
(4) *if* $\delta = 0$ *and* $c_2 > 1/4$, *then* $N_0(H) = \infty$ *and* (1.11) *holds.*

We can say more in the threshold case $c = 1/4$. For ρ large enough, we define

$$\log_{(j)}\rho = \log\log\ldots\log\rho \quad j \text{ times}.$$

Theorem 1.4. *Suppose that there exists* $\rho_0 > 0$ *such that for all* $\rho \in (\rho_0, \infty)$,

$$V(e^{-\rho}, y) = -\frac{1}{4}\rho^{-2} - c_1\rho^{-2}(\log\rho)^{-2} + O(\rho^{-2}(\log\rho)^{-2}(\log\rho)^{-\varepsilon}), \quad \varepsilon > 0.$$

If $c_1 < 1/4$, *then* $N_0(H) < \infty$ *and if* $c_1 > 1/4$, $N_0(H) = \infty$ *and*

$$\log_{(3)} N_E(H) = \log_{(2)}(E^{-1}) + O(1) \quad \textit{as } E \to 0. \tag{1.12}$$

In fact, this process keeps going indefinitely and the result holds if for some ρ_0 *large, and* $\rho \in (\rho_0, \infty)$, *the potential has an expansion of the form*

$$V(e^{-\rho}, y) = V_0(\rho) + O(\mathcal{G}_N(\rho)(\log\rho)^{-\varepsilon}), \quad \varepsilon > 0, \tag{1.13a}$$

where, for some $N \in \mathbb{N}$,

$$V_0(\rho) = -\frac{1}{4}\rho^{-2} - \frac{1}{4}\sum_{j=1}^{N-1}\mathcal{G}_j(\rho) + c_N\mathcal{G}_N(\rho), \tag{1.13b}$$

c_N is a constant, and

$$\mathcal{G}_{(j)}(\rho) = \rho^{-2}(\log\rho)^{-2}(\log\log\rho)^{-2}\dots(\log_{(j)}\rho)^{-2}. \tag{1.13c}$$

The existence of infinitely many eigenvalues depends on whether $c_N < 1/4$, or $c_N > 1/4$. If $c_N < 1/4$, there are only finitely many eigenvalues, but if $c_N > 1/4$,

$$\log_{(N+2)} N_E(H) = \log_{(N+1)}(E^{-1}) + O(1) \quad \text{as } E \to 0. \tag{1.14}$$

Notice that one cannot hope to take $N = \infty$ in the definition of $V_0(\rho)$ because the denominators will be equal to zero at points of the form

$$\rho = e^{e^{e^{\cdot^{\cdot^{\cdot}}}}}.$$

As in the case of Theorems 1.1 and 1.2, our proofs actually give better upper and lower bounds for $N_E(H)$ and this formulation is used to unify these bounds.

As we have already mentioned, the metric g induces a conformal structure at ∂X and this is reflected in the choice of the boundary defining function x. Since $N_E(H)$ does not depend on this choice, its asymptotic behavior in principle could reveal some invariants of the conformal structure of the metric induced by g at ∂X. However, our methods are not refined enough to detect that. This dependence will not affect the main term of the asymptotic behavior of $N_E(H)$ and the contributions of the boundary structure will be hidden among the terms of the $O(1)$ part of the estimates above and these are very hard to track.

1.1. The strategy of the proofs

The methods used in the proof of Theorems 1.1, 1.2, and 1.4 are the Dirichlet–Neumann bracketing and the Sturm oscillation theorem, which are standard for this type problems.

For x as in (1.2) and $x_0 \in [0, e)$, let

$$X_\infty = \{z \in X : x(z) \le x_0\}, \quad X_0 = \{z \in X : x(z) \ge x_0\}. \tag{1.15}$$

So (X_0, g) and $(X_\infty, x^2 g)$ are C^∞ compact Riemannian manifolds with boundary. We will define $\Upsilon_0^\bullet$, and $\Upsilon_\infty^\bullet$, to be the restrictions of the operator H to X_0 and X_∞ with Dirichlet ($\bullet = D$) and Neumann ($\bullet = N$) boundary conditions. Since (X_0, g) is a compact C^∞ Riemannian manifold with boundary, it is well known, see for example [26], that

$$\sigma(\Upsilon_0^D) = \{\tilde\lambda_1 < \tilde\lambda_2 \le \tilde\lambda_3 \dots\}, \quad \tilde\lambda_j \in \mathbb{R}, \tilde\lambda_j \to \infty, \tag{1.16a}$$

$$\sigma(\Upsilon_0^N) = \{\tilde\mu_1 < \tilde\mu_2 \le \tilde\mu_3 \dots\}, \quad \tilde\mu_j \in \mathbb{R}, \tilde\mu_j \to \infty. \tag{1.16b}$$

We will show that $\sigma_{\rm ess}(\Upsilon_\infty^\bullet)=[0,\infty)$, $\bullet=D,N$, with no embedded eigenvalues, and their point spectra satisfy

$$\sigma_{\rm pp}(\Upsilon_\infty^D)=\{\lambda_1<\lambda_2\leq\lambda_3\ldots\},\quad \lambda_j<0,$$

is either finite or $\lambda_j\to 0,$ as $j\to\infty$;

$$\sigma_{\rm pp}(\Upsilon_\infty^N)=\{\mu_1<\mu_2\leq\mu_3\ldots\},\quad \mu_j<0,$$

is either finite or $\mu_j\to 0$ as $j\to\infty$.

Following [26, Chapter XIII], we will show that

$$\Upsilon_0^N\oplus\Upsilon_\infty^N\leq H\leq\Upsilon_0^D\oplus\Upsilon_\infty^D,$$

and it follows that for any $E<0$,

$$N_E(\Upsilon_0^N)+N_E(\Upsilon_\infty^N)\leq N_E(H)\leq N_E(\Upsilon_0^D)+N_E(\Upsilon_\infty^D),$$

where $N_E(T)$ is the counting function defined in (1.5) for the operators $T=\Upsilon_0^\bullet$ and $\Upsilon_\infty^\bullet$ instead of H. It follows from (1.16) that there exists $N^\#>0$ such that for any $E<0$, $N_E(\Upsilon_0^N)<N^\#$ and $N_E(\Upsilon_0^D)<N^\#$. We will show that if V satisfies the hypotheses of either one of the Theorems 1.1, 1.2, or 1.4, then for $E<0$, $N_E(\Upsilon_\infty^\bullet)$, $\bullet=N,D$, both have either finitely many eigenvalues or both infinitely many eigenvalues. In case both have infinitely many eigenvalues, the corresponding counting function of their eigenvalues have the same asymptotic behavior as $E\searrow 0$, and therefore it gives the asymptotic behavior of $N_E(H)$.

2. The spectrum of *H*

For the lack of suitable references, we will briefly discuss some properties of the spectrum of H. First, we recall some results about of the spectrum of H_0 from [18,20]. Let x be a boundary defining function for which (1.2) holds in a collar neighborhood of ∂X. In these coordinates, the Laplacian Δ_g is given by (1.7), so Δ_g is a zero differential operator in the sense of [20], and we define the zero-Sobolev spaces of order k as in [20]. Let $\mathcal{V}(\partial X)$ denote the Lie algebra of C^∞ vector fields on X which are equal to zero at ∂X. In coordinates (x,y), these vector fields are spanned by $\{x\partial_x, x\partial_{y_j},\ 1\leq j\leq n\}$ over the C^∞ functions. Let

$$\mathcal{H}_0^k(X)=\{u\in L^2(X): W_1W_2\ldots W_m u\in L^2(X),\ W_j\in\mathcal{V}(\partial X),\ m\leq k\}. \quad (2.1)$$

We know from [20] that Δ_g with domain $\mathcal{H}_0^2(X)\subset L^2(X)$ is a self-adjoint operator, we also know from [4, 17, 20] that its spectrum consists of an absolutely continuous part $\sigma_{\rm ac}(\Delta_g)=[n^2/4,\infty)$ and finitely many eigenvalues in the point spectrum

$\sigma_{\rm pp}(\Delta_g) \subset (0, n^2/4)$. There are no eigenvalues in $[0,\infty)$, see [4, 18]. As above, we set $H_0 = \Delta_g - n^2/4$, and so the resolvent

$$\begin{gathered} R_{H_0}(\lambda) = (H_0 - \lambda)^{-1} \colon L^2(X) \mapsto \mathcal{H}_0^2(X), \\ \text{provided } \lambda \in \mathbb{C} \setminus ([0,\infty) \cup \{\lambda_1, \lambda_2, \dots, \lambda_N\}), \end{gathered}$$

where $\lambda_j \in (-n^2/4, 0)$, is an eigenvalue of finite multiplicity of H_0. By definition, the resolvent set of H_0 is

$$\rho(H_0) = \mathbb{C} \setminus ([0,\infty) \cup \{\lambda_1, \dots, \lambda_N\}), \quad \lambda_j \text{ is an eigenvalue of } H_0.$$

To analyze the spectrum of H, we begin by observing that for $\lambda \in \rho(H_0)$,

$$\Big(\Delta_g + V - \frac{n^2}{4} - \lambda\Big) R_{H_0}(\lambda) = \mathrm{I} + V R_{H_0}(\lambda),$$

Since

$$R_{H_0}(\lambda) \colon L^2(X) \to \mathcal{H}_0^2(X) \text{ is a bounded operator for } \lambda \in \rho(H_0),$$

then if $\chi_j \in C_0^\infty(\mathring{X})$, $\chi_j(z) = 1$ in the region $x(z) > 1/j$ and $\chi_j(z) = 0$ if $x(z) < 1/(j+1)$, it follows that

$$\chi_j(z) V(z) R_{H_0}(\lambda) \colon L^2(X) \to H_c^2(\mathring{X}) \hookrightarrow L_c^2(\mathring{X}),$$

where the subindex c indicates compact support. Notice that since supports are compact, we can use either $\mathcal{H}_0^2(X)$ or the standard Sobolev space $H^2(X)$. It follows from Rellich's embedding theorem that for fixed j,

$$\chi_j(z) V(z) R_{H_0}(\lambda) \colon L^2(X) \to L_c^2(\mathring{X})$$

is a compact operator. Since $V \in L^\infty(X)$ and $V(z) \to 0$ as $z \to \partial X$, it follows that

$$\|\chi_j(z) V(z) R_{H_0}(\lambda) - V(z) R_{H_0}(\lambda)\|_{\mathcal{L}(L^2(X))} \to 0 \quad \text{as } j \to \infty$$

in the operator norm, and so we conclude that $V R_{H_0}(\lambda) \colon L^2(X) \to L^2(X)$ is a compact operator, provided $\lambda \in \rho(H_0)$. We also know that for $\operatorname{Im} \lambda << 0$, the operator norm of $R_{H_0}(\lambda)$ is less than or equal to $1/\operatorname{Im}(\lambda)$, see for example [25, Theorem VI.8], and therefore $(\mathrm{I} + V R_{H_0}(\lambda))^{-1}$ is bounded for $\operatorname{Im} \lambda \ll 0$, and $|\operatorname{Re} \lambda| > 1$. Then, the Fredholm theorem, see for example [25, Theorem VI.14], guarantees that with the exception of a countable set of points, which are poles of $R_H(\lambda)$,

$$R_H(\lambda) = R_{H_0}(\lambda)(\mathrm{I} + V R_{H_0}(\lambda))^{-1} \quad \text{for } \lambda \in \rho(H_0).$$

Moreover, the poles of $R_H(\lambda)$ in $\rho(H_0)$ consist of a countable set $\{\mu_j,\ j \in \mathbb{N}\} \subset (-\infty, 0)$ such that μ_j are eigenvalues of H with finite multiplicity. This set is either finite or infinite. If there are infinitely may eigenvalues, they accumulate only at zero.

Finally, since $VR_{H_0}(\lambda)$ is compact for $\lambda \in \rho(H_0)$, it follows that the operator

$$V: L^2(X) \to L^2(X), \quad f \mapsto V(z)f$$

is relatively compact with respect to H_0 and it follows from Weyl's Theorem, see [13, Theorem 14.6] that $\sigma_{\mathrm{ess}}(H) = \sigma(H) \setminus \sigma_{\mathrm{pp}}(H) = \sigma_{\mathrm{ess}}(H_0) = [0,\infty)$. Therefore, we have the following.

Theorem 2.1. *Let $(\mathring{X}, g)$ be an AHM, let $V \in L^\infty(X)$ be real valued and suppose that $V(z) \to 0$ as $z \to \partial X$. Then, $\sigma_{\mathrm{ess}}(H) = \sigma(H) \setminus \sigma_{\mathrm{pp}}(H) = [0,\infty)$. There are no embedded eigenvalues. Moreover, the resolvent*

$$R_H(\lambda) = (H-\lambda)^{-1}: L^2(X) \to \mathcal{H}_0^2(X) \text{ is bounded for } \lambda \in \mathbb{C} \setminus ([0,\infty) \cup \mathcal{D}),$$

where $\mathcal{D} = \{\mu_1, \mu_2, \ldots\} \subset (-\infty, 0)$, with $\mu_{j+1} \geq \mu_j$, is a bounded discrete set which consists of eigenvalues of H of finite multiplicity and only possibly accumulate at 0.

The fact that there are no embedded eigenvalues is due to Mazzeo [18], see also [3, 4]. We also have the following.

Theorem 2.2. *Let $(\mathring{X}, g)$ be an AHM, let x be a boundary defining function such that (1.2) holds. Let X_∞ be as in (1.15) and let $\Upsilon_\infty^\bullet$, $\bullet = N, D$ denote the operator H with Dirichlet or Neumann boundary conditions in X_∞. If $V(z) \to 0$ as $z \to \partial X$, according to either (1.8), (1.10), or (1.13), then $\sigma_{\mathrm{ess}}(\Upsilon_\infty^\bullet) = [0,\infty)$. Moreover, there are no embedded eigenvalues.*

The fact that $\sigma_{\mathrm{ess}}(\Upsilon_\infty^\bullet) = [0,\infty)$ is a consequence of Theorem 2.1 and [7, Proposition 2.1]; see also [3, Theorem 9.43]. The proofs of these results are actually for the Dirichlet boundary conditions, but they work for Neumann conditions as well. The results of [3, 4, 18] also guarantee that there are no embedded eigenvalues in these cases. The point is to show that if there were eigenfunctions in L^2, they would decay exponentially and a Carleman estimate shows that they are actually equal to zero. The argument takes place in a neighborhood of ∂X and thus also works for $\Upsilon_\infty^\bullet$.

3. Dirichlet–Neumann bracketing

The operator H as the unique self-adjoint operator on $L^2(X)$ whose quadratic form is the closure of

$$\begin{aligned} Q(\varphi,\psi) &= \langle \nabla_g \phi, \nabla_g \psi \rangle_{L_g^2(X)} + \Big\langle \Big(V - \frac{n^2}{4}\Big)\phi, \psi \Big\rangle_{L_g^2(X)} \\ &= \int_X g^{ij}\partial_i\phi\partial_j\bar{\psi} d\,\mathrm{vol}_g + \int_X \Big(V - \frac{n^2}{4}\Big)\phi\bar{\psi} d\,\mathrm{vol}_g, \quad \text{with } \phi, \psi \in C_0^\infty(\mathring{X}). \end{aligned}$$

The domain of the quadratic form Q is $\mathcal{H}_0^1(X) \times \mathcal{H}_0^1(X)$, defined in (2.1).

Let x be a boundary defining function such that (1.2) holds and let X_0 and X_∞ be as defined in (1.15). We consider the quadratic forms to be the closure of

$$Q^D(X_0)(\varphi,\psi) = Q(\varphi,\psi) \quad \text{with } \varphi,\psi \in C_0^\infty(\mathring{X}_0), \tag{3.1}$$
$$Q^N(X_0)(\varphi,\psi) = Q(\varphi,\psi) \quad \text{with } \varphi,\psi \in C^\infty(X_0), \tag{3.2}$$

where

$$\partial_x\varphi|_{\{x=x_0\}} = \partial_x\psi|_{\{x=x_0\}} = 0.$$

It turns out that the domains of these quadratic forms are

- $\mathcal{D}(Q^D(X_0)) = H_0^1(X_0) \times H_0^1(X_0)$, where

$$H_0^1(X_0) = \overline{C_0^\infty(\mathring{X}_0)}$$

with the $H^1_{\text{loc}}(\mathring{X})$ norm;

- $\mathcal{D}(Q^N(X_0)) = \bar{H}^1(X_0) \times \bar{H}^1(X_0)$, where

$$\bar{H}^1(X_0) = \{\varphi \in L^2(X_0) : \exists f \in H^1_{\text{loc}}(\mathring{X}),\ f = \varphi \text{ in } X_0\}.$$

The self-adjoint operators corresponding to $Q^D(X_0)$ and $Q^N(X_0)$ are defined to be the operator H respectively with Dirichlet or Neumann boundary conditions, which we denote by Υ_0^D and Υ_0^N respectively.

Similarly, we define the quadratic forms

$$Q^D(X_\infty) = Q(\varphi,\psi), \quad \varphi,\psi \in C_0^\infty(\mathring{X}_\infty), \tag{3.3a}$$
$$Q^N(X_\infty) = Q(\varphi,\psi), \quad \varphi,\psi \in C^\infty(X_\infty) \cap \mathcal{H}_0^2(X_\infty), \tag{3.3b}$$

where

$$\partial_x\varphi|_{\{x=x_0\}} = \partial_x\psi|_{\{x=x_0\}} = 0. \tag{3.3c}$$

The domains of their closures are

- $\mathcal{D}(Q^D(X_\infty)) = \mathcal{W}_0^1(X_\infty) \times \mathcal{W}_0^1(X_\infty)$, where

$$\mathcal{W}_0^1(X_\infty) = \overline{C_0^\infty(\mathring{X}_\infty)}$$

with the $\mathcal{H}_0^1(X)$ norm;

- $\mathcal{D}(Q^N(X_\infty)) = \bar{\mathcal{H}}^1_0(X_\infty) \times \bar{\mathcal{H}}^1_0(X_\infty)$, where

$$\bar{\mathcal{H}}^1_0(X_\infty) = \{\varphi \in L^2(X_\infty) : \exists f \in \mathcal{H}_0^1(X),\ f = \varphi \text{ in } X_\infty\}.$$

The corresponding self-adjoint operators are defined to be Υ_∞^D and Υ_∞^N, which are the operator H with Dirichlet or Neumann boundray conditions on X_∞.

We follow [26, Section XIII.15] and define the direct sum of self-adjoint operators: if A_j, $j = 1, 2$, are self-adjoint operators acting on Hilbert spaces $\mathcal{L}_j$, $j = 1, 2$, with domains $\mathcal{D}(A_j)$, $j = 1, 2$, let

$$\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$$

and

$$A_1 \oplus A_2(\phi_1, \phi_2) = (A_1\phi_1, A_2\phi_2), \quad \phi_j \in \mathcal{D}(A_j),\ j = 1, 2.$$

It is proved in [26, Section XII.15] that

(1) $A_1 \oplus A_2$ is self adjoint;

(2) the associated quadratic forms satisfy $Q(A_1 \oplus A_2) = Q(A_1) \oplus Q(A_2)$;

(3) if $N(\lambda, A) = \dim P_{(-\infty,\lambda)}(A)$, then

$$N(\lambda, A_1 \oplus A_2) = N(\lambda, A_1) + N(\lambda, A_2). \tag{3.4}$$

In our case, we have four natural operators $\Upsilon^\bullet(X_0)$ and $\Upsilon^\bullet(X_\infty)$, $\bullet = D, N$. Notice that

$$H_0^1(X_0) \oplus \mathcal{W}_0^1(X_\infty) \subset \mathcal{H}_0^1(X),$$

and that, for $\varphi \in C_0^\infty(\mathring{X}_0)$, $\psi \in C_0^\infty(\mathring{X}_\infty)$,

$$Q^D(X_0)(\varphi, \varphi) + Q^D(X_\infty)(\psi, \psi) = Q(H)(\varphi, \varphi) + Q(H)(\psi, \psi).$$

On the other hand, if $\varphi \in \mathcal{H}_0^1(X)$,

$$\varphi|_{X_0} \in \bar{\mathcal{H}}_0^1(X_0) \quad \text{and} \quad \varphi|_{X_\infty} \in \bar{\mathcal{H}}_0^1(X_\infty),$$

this means that

$$\mathcal{H}_0^1(X_0) \subset \bar{\mathcal{H}}_0^1(X) \oplus \bar{\mathcal{H}}_0^1(X_\infty),$$

and, for $\varphi \in \mathcal{H}_0^1(X)$,

$$Q^N(X_0)(\varphi, \varphi) + Q^N(X_\infty)(\varphi, \varphi) = Q(H)(\varphi, \varphi).$$

If A and B are self-adjoint operators defined on a Hilbert space $\mathcal{L}$ and $Q(A)$ and $Q(B)$ are their corresponding quadratic forms with domains $\mathcal{D}(Q(A))$ and $\mathcal{D}(Q(B))$,

$$Q(A)(\phi, \phi) \geq M\|\phi\|, \quad \phi \in \mathcal{D}(Q(A))$$

and

$$Q(B)(\psi, \psi) \geq M\|\psi\|, \quad \psi \in \mathcal{D}(Q(B)),$$

we say that

$$A \leq B \quad \text{if } \mathcal{D}(Q(B)) \subset \mathcal{D}(Q(A)) \text{ and } Q(A)(\varphi, \varphi) \leq Q(B)(\varphi, \varphi),\ \varphi \in \mathcal{D}(Q(B)).$$

This translates into the following.

Proposition 3.1. *Let* X_0, X_∞, $\mathcal{H}^D(X_\bullet)$ *and* $\mathcal{H}^N(X_\bullet)$, $\bullet = 0, \infty$, *be defined as above, then*

$$\Upsilon_0^N \oplus \Upsilon_\infty^N \leq H \leq \Upsilon_0^D \oplus \Upsilon_\infty^D.$$

It follows from (3.4) that if $E > 0$,

$$N_E(\Upsilon_0^N) + N_E(\Upsilon_\infty^N) \leq N_E(H) \leq N_E(\Upsilon_0^D) + N_E(\Upsilon_\infty^D). \tag{3.5}$$

Since Υ_0^D and Υ_0^N are Schrödinger operators with C^∞ potentials on compact manifolds with boundary, their spectra satisfy (1.16) and so they have finitely many eigenvalues less than zero and do not contribute to the asymptotic behavior of the counting function $N_E(H)$ in case there are infinitely many eigenvalues. We will show that the point spectra of Υ_∞^D and Υ_∞^N are jointly either finite or infinite. In the latter case, we will show that both have the same asymptotic behavior as E goes to zero, and so the asymptotic behavior of $N_E(H)$ as $E \to 0$ is determined by that of $N_E(\Upsilon_\infty^\bullet)$, $\bullet = D, N$.

3.1. Model operators on X_∞

Our analysis in this section will be restricted to a small-enough collar neighborhood of $\partial X = \{x = 0\}$ where (1.2) holds and so Δ_g is given by (1.6). Since $h(x)$ is a C^∞ one-parameter family of tensors on ∂X, we may write, in local coordinates,

$$h(x) = \sum_{j,k=1}^{n} h_{jk}(x,y)dy_j dy_k, \qquad h_{jk}(x,y) \in C^\infty \tag{3.6a}$$

and

$$h_{jk}(x,y) = h_{jk}(0,y) + x\tilde{h}_{jk}(x,y), \quad \tilde{h}_{jk}(x,y) \in C^\infty. \tag{3.6b}$$

and so the corresponding quadratic forms for H on X_∞ with Dirichlet or Neumann boundary conditions defined in (3.3) may be written as

$$\begin{aligned} Q^\bullet(X_\infty)(\varphi,\varphi) = \int_0^{x_0}\int_{\partial X} \Big(|x\partial_x \varphi|^2 + \sum_{j,k=1}^{n} h^{jk}(x,y)(x\partial_{y_j}\varphi)(x\partial_{y_k}\bar{\varphi}) \\ + \Big(V - \frac{n^2}{4}\Big)|\varphi|^2\Big) \frac{\sqrt{|h(x)|}}{x^{n+1}} dy dx, \end{aligned} \tag{3.7}$$

with

- $\varphi \in \mathcal{W}_0^1(X_\infty)$ if $\bullet = D$,
- $\varphi \in \bar{\mathcal{H}}_0^1(X_\infty)$, if $\bullet = N$, and
- $(h^{jk}) = (h_{jk})^{-1}$.

We want to show that the quadratic forms $Q^\bullet(X_\infty)$, $\bullet = D, N$, can be bounded from above and below by quadratic forms associated with the product metric where $h(x)$ is replaced by $h(0)$ and $V(x, y)$ is replaced by potentials which depend on x only. If $|h(x)| = |\det h_{jk}(x, y)|$ is the volume element of the metric $h(x)$ and $(h^{jk}(x, y))(h_{jk}(x, y))^{-1}$ is the inverse of the matrix $(h_{jk}(x, y))$, we deduce from (3.6) that there exists $\gamma > 0$ and x_0 small such that for $x \in (0, x_0)$, $(1 - \gamma x) > 1/2$ and for all $\xi \in \mathbb{R}^n$,

$$\begin{aligned}(1-\gamma x)^{1/2}\sum_{j,k=1}^{n} h^{jk}(0,y)\xi_j\xi_k \le \sum_{j,k=1}^{n} h^{jk}(x,y)\xi_j\xi_k \\ \le (1+\gamma x)^{1/2}\sum_{j,k=1}^{n} h^{jk}(0,y)\xi_j\xi_k,\end{aligned} \tag{3.8a}$$

and

$$(1-\gamma x)^{1/2}\sqrt{|h(0)|} \le \sqrt{|h(x)|} \le (1+\gamma x)^{1/2}\sqrt{|h(0)|}. \tag{3.8b}$$

Recall that Theorems 1.1, 1.2, and 1.4 require that, if $x = e^{-\rho}$, the potential $V(x, y)$ satisfies

$$-V_0(\rho) - aV_1(\rho) \le V(x,y) \le -V_0(\rho) + aV_1(\rho), \quad \text{where } a > 0 \text{ is a constant,}$$

and $V_0(\rho)$ and $V_1(\rho)$ satisfy, for $\rho \in (\rho_0, \infty)$, one of the following three assumptions:

$$\begin{aligned} V_0(\rho) &= c\rho^{-2+\delta}, \\ V_1(\rho) &= \rho^{-2+\delta}(\log\rho)^{-\varepsilon}, \quad \varepsilon > 0, \end{aligned} \tag{3.9a}$$

in Theorem 1.1;

$$\begin{aligned} V_0(\rho) &= c\rho^{-2}, \\ V_1(\rho) &= \rho^{-2}(\log\rho)^{-\varepsilon}, \quad \varepsilon > 0, \end{aligned} \tag{3.9b}$$

in Theorem 1.2; and

$$\begin{aligned} V_0(\rho) &= \frac{1}{4}\rho^{-2} - \frac{1}{4}\sum_{j=1}^{N}\mathcal{G}_j(\rho) + c_N\mathcal{G}_N(\rho), \\ V_1(\rho) &= \mathcal{G}_N(\rho)(\log\rho)^{-\varepsilon}, \quad \varepsilon > 0, \end{aligned} \tag{3.9c}$$

with $\mathcal{G}_N(\rho)$ defined by (1.13), in Theorem 1.4.

We will prove the following result.

Proposition 3.2. *Let $Q^\bullet(X_\infty)$, $\bullet = N, D$ be as in (3.7). Then, for x_0 small enough and γ given by (3.8),*

$$\mathcal{Q}^\bullet_-(X_\infty)(\varphi,\varphi) \le Q^\bullet(X_\infty)(\varphi,\varphi) \le \mathcal{Q}^\bullet_+(X_\infty)(\varphi,\varphi), \tag{3.10}$$

where $\mathcal{Q}^\bullet_\pm(X_\infty)$ are the quadratic forms defined by

$$\begin{aligned} &\mathcal{Q}^\bullet_\pm(X_\infty)(\varphi,\varphi) \\ &= \int_0^{x_0}\int_{\partial X} (1\pm\gamma x)\Big(|x\partial_x\varphi|^2 + \sum_{j,k=1}^n h^{jk}(0,y)(x\partial_{y_j}\varphi)(x\partial_{y_k}\bar\varphi)\Big)\frac{\sqrt{|h(0)|}}{x^{n+1}}dydx \\ &\quad + \int_0^{x_0}\int_{\partial X}\Big(-V_0(-\log x) \pm aV_1(-\log x) + xW^\pm(x) - \frac{n^2}{4}\Big)|\varphi|^2 \\ &\qquad\qquad \times (1\pm\gamma x)^{n+1}\frac{\sqrt{|h(0)|}}{x^{n+1}}dydx, \end{aligned}$$

with

- $\varphi \in \mathcal{W}^1_0(X_\infty)$, *if* $\bullet = D$,
- $\varphi \in \bar{\mathcal{H}}^1_0(X_\infty)$, *if* $\bullet = N$, *and*
- $(h^{jk}(0)) = (h_{jk}(0))^{-1}$,

where

$$\begin{aligned} W^\pm(x) &= \Big(-V_0(-\log x) \pm aV_1(-\log x) - \frac{n^2}{4}\Big)F_\pm(x), \\ F_\pm(x) &= \frac{1}{x}\Big(\frac{(1\pm\gamma x)^{1/2}}{(1\pm\gamma x)^{n+1}} - 1\Big). \end{aligned}$$

Moreover, $\mathcal{Q}^\bullet_\pm(X_\infty)$ are respectively associated with the operators

$$\mathcal{M}_\pm = \Delta_{g_+} - V_0(-\log x) \ \pm aV_1(-\log x) + xW^\pm(x) - \frac{n^2}{4}, \tag{3.11}$$

where

$$g_\pm = (1\pm\gamma x)^{\frac{2}{n-1}}\Big(\frac{dx^2}{x^2} + \frac{h(0)}{x^2}\Big)$$

with $\bullet = D, N$, for Dirichlet or Neumann boundary conditions.

Proof. We observe that because of (3.8),

$$\begin{aligned} &(1-\gamma x)\big(|x\partial_x\varphi|^2 + h^{jk}(0,y)(x\partial_{y_j}\varphi)(x\partial_{y_k}\bar\varphi)\big)\sqrt{|h(0)|} \\ &\quad\le \big(|x\partial_x\varphi|^2 + h^{jk}(x,y)(x\partial_{y_j}\varphi)(x\partial_{y_k}\bar\varphi)\big)\sqrt{|h(x)|} \\ &\quad\le (1+\gamma x)\big(|x\partial_x\varphi|^2 + h^{jk}(0,y)(x\partial_{y_j}\varphi)(x\partial_{y_k}\bar\varphi)\big)\sqrt{|h(0)|}, \end{aligned}$$

and

$$\Big(V(x,y)-\frac{n^2}{4}\Big)(1-\gamma x)^{1/2}\sqrt{|h(0)|}\le\Big(V(x,y)-\frac{n^2}{4}\Big)\sqrt{|h(x)|}$$
$$\le\Big(V(x,y)-\frac{n^2}{4}\Big)(1+\gamma x)^{1/2}\sqrt{|h(0)|}.$$

We may write

$$(1\pm\gamma x)^{1/2}=(1\pm\gamma x)^{n+1}(1+xF_{\pm}(x)),$$

where

$$F_{\pm}(x)=\frac{1}{x}\Big(\frac{(1\pm\gamma x)^{1/2}}{1(\pm\gamma x)^{n+1}}-1\Big),$$

and so

$$\Big(V(x,y)-\frac{n^2}{4}\Big)(1\pm\gamma x)^{1/2}=V(x,y)\Big(1\pm\frac{1}{2}x\Big)^{n+1}(1+xF_{\pm}(x)).$$

Therefore,

$$\begin{aligned}&\big(-V_0(\log x)-aV_1(-\log x)\big)(1-xF_{-}(x))(1+\gamma x)^{n+1}\sqrt{|h(0)|}\\ &\quad\le\Big(V(x,y)-\frac{n^2}{4}\Big)\sqrt{|h(x)|}\\ &\quad\le\big(-V_0(\log x)+aV_1(-\log x)\big)(1+xF_{+}(x))(1+\gamma x)^{n+1}\sqrt{|h(0)|}.\end{aligned}$$

This proves (3.10). Notice that for $g_{\pm}$ as in (3.11),

$$\sqrt{|g_{\pm}|}=\frac{(1\pm\gamma x)^{(n+1)/(n-1)}}{x^{n+1}}|\sqrt{|h(0)|}$$

and

$$\begin{aligned}\Delta_{g_{\pm}}=&-\frac{x^{n+1}}{(1\pm\gamma x)^{n+1}\sqrt{|h(0)|}}\partial_x\Big(\frac{(1\pm\gamma x)}{x^{n+1}}x^2\sqrt{|h(0)|}\partial_x\Big)\\ &-\frac{x^{n+1}}{(1\pm\gamma x)^{n+1}\sqrt{|h(0)|}}\partial_{y_j}\Big(\frac{(1\pm\gamma x)}{x^{n+1}}\sqrt{|h(0)|}x^2h^{jk}(0)\partial_{y_k}\Big),\end{aligned}$$

and so, the quadratic forms associated with $g_{\pm}$ define the operators $\mathcal{M}_{\pm}^{\bullet}$, as claimed. This ends the proof of the proposition. ■

We have shown that

- the domains of $\mathcal{Q}_{\pm}^{D}(X_\infty)$ and $Q^D(X_\infty)$ are the same and equal to $\mathcal{W}_0^1(X_\infty)$;
- the domains of $\mathcal{Q}_{\pm}^{N}(X_\infty)$ and $Q^N(X_\infty)$ are the same and equal to $\overline{\mathcal{H}}_0^1(X_\infty)$;

- and, moreover,

$$\begin{aligned} &\mathcal{Q}_-^D(X_\infty)(\varphi,\varphi) \le Q^D(X_\infty)(\varphi,\varphi) \le \mathcal{Q}_+^D(X_\infty)(\varphi,\varphi), \quad \varphi \in \mathcal{W}_0^1(X_\infty),\\ &\mathcal{Q}^N(X_\infty)(\varphi,\varphi) \le Q^N(X_\infty)(\varphi,\varphi) \le \mathcal{Q}^N(X_\infty)(\varphi,\varphi), \quad \varphi \in \bar{\mathcal{H}}_0^1(X_\infty). \end{aligned} \tag{3.12}$$

Notice that the $L^2(X_\infty)$ spaces defined with respect to g or $g_\pm$ are the same, but are equipped with different, but equivalent norms, and there are constants $C_j^\pm$, $j = 1, 2$ such that

$$\begin{aligned} C_1^- \|\varphi\|_{L^2_{g_-}} &\le \|\varphi\|_{L^2_g} \le C_2^- \|\varphi\|_{L^2_{g_-}},\\ C_1^+ \|\varphi\|_{L^2_{g_+}} &\le \|\varphi\|_{L^2_g} \le C_2^+ \|\varphi\|_{L^2_{g_+}}. \end{aligned} \tag{3.13}$$

If we put together (3.12) and (3.13), we obtain

$$\begin{aligned} &\frac{1}{C_2^-}\frac{\mathcal{Q}_-^D(X_\infty)(\varphi,\varphi)}{\langle\varphi,\varphi\rangle_{L^2_{g_-}}} \le \frac{Q^D(X_\infty)(\varphi,\varphi)}{\langle\varphi,\varphi\rangle_{L^2_g}} \le \frac{1}{C_1^+}\frac{\mathcal{Q}_+^D(X\infty)(\varphi,\varphi)}{\langle\varphi,\varphi\rangle_{L^2_{g_+}}}, \quad \varphi \in \mathcal{W}_0^1(X_\infty),\\ &\frac{1}{C_2^-}\frac{\mathcal{Q}_-^N(X_\infty)(\varphi,\varphi)}{\langle\varphi,\varphi\rangle_{L^2_{g_-}}} \le \frac{Q^N(X_\infty)(\varphi,\varphi)}{\langle\varphi,\varphi\rangle_{L^2_g}} \le \frac{1}{C_1^+}\frac{\mathcal{Q}_+^N(X_\infty)(\varphi,\varphi)}{\langle\varphi,\varphi\rangle_{L^2_{g_+}}}, \quad \varphi \in \bar{\mathcal{H}}_0^1(X_\infty). \end{aligned}$$

We will use $\mathcal{M}_\pm^\bullet$ to indicate the operator $\mathcal{M}_\pm$ with $\bullet = D, N$. We also remark that one may extend the metrics $g_\pm$ to the manifold X, so that it becomes an AHM, and as a consequence of Theorem 2.2, we obtain

$$\sigma_{\text{ess}}(\mathcal{M}_\pm^\bullet) = [0,\infty), \quad \bullet = D, N. \tag{3.14}$$

We now appeal to the following characterization of the eigenvalues of a self-adjoint operator, see for example [8, p. 1543] or [28, Theorem 3].

Theorem 3.3. *Let H be a separable Hilbert space with inner product $\langle u, v\rangle$ and let A be a self-adjoint operator corresponding to a semi-bounded quadratic form Q with domain $\mathcal{D}(Q)$. Suppose that the essential spectrum of A satisfies $\sigma_{\text{ess}}(A) = [0,\infty)$ and that its point spectrum satisfies*

$$\sigma_{\text{pp}}(A) = \{\lambda_1 \le \lambda_2 \le \cdots\}.$$

For $u \in \mathcal{D}(Q)$, $u \neq 0$, let $R(u) = Q(u,u)/\langle u,u\rangle$ denote the Rayleigh quotient, and for $n \in \mathbb{N}$, let

$$\mu_n = \inf\{\max\{R(u), u \in \mathcal{B} \text{ such that } \mathcal{B} \subset \mathcal{D}(Q) \text{ is a subspace with } \dim\mathcal{B} = n\}\}.$$

Then, $\mu_n \le 0$. If $\mu_n = 0$, then A has at most $n-1$ eigenvalues $\lambda_j < 0$, counted with multiplicity. If $\mu_n < 0$, then $\mu_n = \lambda_n$ is the n-th eigenvalue of A counted with multiplicity.

The important aspect of this characterization is that there is no orthogonality required, as would be the case if we switched the order of max and min. As a consequence of Theorem 2.2 and Theorem 3.3 and (3.14), we arrive at the following.

Proposition 3.4. *Let $\lambda_j(\Upsilon_\infty^D)$ and $\lambda_j(\mathcal{M}_\pm^D)$ denote the eigenvalues of the operators Υ_∞ and $\mathcal{M}^\pm$, with Dirichlet boundary conditions. Similarly, let $\mu_j(\Upsilon_\infty^N)$ and $\mu_j(\mathcal{M}_\pm^N)$ denote the eigenvalues of these operators with Neumann boundary conditions. If $\mathcal{M}_-^\bullet$ has finitely many eigenvalues, so do $\mathcal{M}_+^\bullet$ and $\Upsilon_\infty^\bullet$, $\bullet = D, N$. If $\mathcal{M}_+^\bullet$ has infinitely many eigenvalues, so do $\mathcal{M}_-^\bullet$ and $\Upsilon_\infty^\bullet$, $\bullet = D, N$ and if $C_j^\pm$, $j = 1, 2$, are as defined in (3.13), then, for all j,*

$$\frac{1}{C_2^-}\lambda_j(\mathcal{M}_-^D) \leq \lambda_j(\Upsilon_\infty^D) \leq \frac{1}{C_1^+}\lambda_j(\mathcal{M}_+^D),$$
$$\frac{1}{C_2^-}\mu_j(\mathcal{M}_-^N) \leq \mu_j(\Upsilon_\infty^N) \leq \frac{1}{C_1^+}\mu_j(\mathcal{M}_+^N).$$

In particular, in the case there exist infinitely many eigenvalues, then for all $E < 0$,

$$N_{C_2^- E}(\mathcal{M}_-^\bullet) \leq N_E(\Upsilon_\infty^\bullet) \leq N_{C_1^+ E}(\mathcal{M}_+^\bullet), \quad \bullet = D, N, \tag{3.15}$$

where $N_E(T)$ is the counting function of negative eigenvalues defined in (1.5).

4. The asymptotic behavior of $N_E(\mathcal{M}_\pm^\bullet)$, $\bullet = D, N$ as $E \to 0$

We will show that under the hypotheses of either one of Theorems 1.1, 1.2 or 1.4, and if $\rho_0 = -\log x_0$ is large enough, we have two possibilities: either both $\mathcal{M}_\pm^D$ and $\mathcal{M}_\pm^N$ have no negative eigenvalues, or both have infinitely many. In the latter case, the iterated logarithms of the counting functions $N_E(\mathcal{M}_\pm^\bullet)$, $\bullet = D, N$, defined in (1.5), have the same asymptotic behavior $E \to 0$, according to the asymptotic behavior of the potential as in Theorems 1.1, 1.2, and 1.4. More precisely, we will prove the following.

Proposition 4.1. *Let $\mathcal{M}_\pm^\bullet$, $\bullet = D, N$, be defined as above and let $N_E(\mathcal{M}_\pm^\bullet)$ denote the corresponding counting function of eigenvalues.*

(T.1) *Suppose that $V_0(\rho)$ and $V_1(\rho)$ satisfy (1.8). If ρ_0 is large enough and $\delta < 0$, then $\mathcal{M}_\pm^\bullet$ has no negative eigenvalues, but if $\delta \in (0, 2)$, then $N_E(\mathcal{M}_\pm^\bullet)$ satisfies (1.9).*

(T.2) *Suppose that $V_0(\rho)$ and $V_1(\rho)$ satisfy (1.10). If ρ_0 is large enough and $c < 1/4$, then $\mathcal{M}_\pm^\bullet$ has no negative eigenvalues, but if $c > 1/4$, then $N_E(\mathcal{M}_\pm^\bullet)$ satisfies (1.11).*

(T.3) *Suppose that* $V_0(\rho)$ *and* $V_1(\rho)$ *satisfy* (1.13). *If* ρ_0 *is large enough and* $c_N < 1/4$, *then* $\mathcal{M}^\bullet_\pm$ *has no negative eigenvalues, but if* $c_N > 1/4$ *then* $N_E(\mathcal{M}^\bullet_\pm)$ *satisfies* (1.14).

These results, together with equations (3.15), (3.5), and (1.16) respectively prove Theorems 1.1, 1.2, and 1.4.

We will consider the Dirichlet and Neumann eigenvalue problems in $X_\infty = \{x \leq x_0\}$ for the operators $\mathcal{M}^\bullet_\pm$, $\bullet = D, N$ defined above. We will drop the $\pm$ sub-indices and work with $\gamma, a \in \mathbb{R}$. We will assume that x_0 is small enough so that $x|\gamma| < 1/2$ for $x \in (0, x_0)$. We will work with the metric

$$\mathcal{G} = (1+\gamma x)^{\frac{2}{n-1}}\Big(\frac{dx^2}{x^2} + \frac{h(0)}{x^2}\Big).$$

We find that

$$\Delta_{\mathcal{G}} = -x^{n+1} f^{-(n+1)}\partial_x(f^{n-1}x^{1-n}\partial_x) + x^2 f^{-2}\Delta_{h(0)},$$

where

$$f(x) = (1+\gamma x)^{1/(n-1)}.$$

To get rid of the factor $n^2/4$ in $\mathcal{M}_\pm$, we observe that

$$x^{-n/2}\Big(\Delta_{\mathcal{G}} - \frac{n^2}{4}\Big)x^{n/2} = -f^{-(n+1)}x\partial_x(f^{n-1}x\partial_x) + x^2 f^{-1}\Delta_{h(0)} - xA(x),$$

where

$$A(x) = -\frac{n}{2} f^{-(n+1)}(x)\partial_x(f^{n-1}(x)) + \frac{n^2}{4x}(f^2(x)-1).$$

So, we study the eigenvalue problems corresponding to the operators $x^{-n/2}\mathcal{M}_\pm x^{n/2}$, which are of the form

$$\big(-f^{-(n+1)}(f^{n-1}x\partial_x)^2 + x^2 f^{-2}\Delta_{h(0)} + U(x) + x\mathcal{W}(x) + E\big)u^\bullet = 0,$$

with $E > 0$, $\bullet = D, N$,

$$u^D(x_0, y) = 0 \quad \text{or} \quad \partial_x u^N(x_0, y) = 0,$$

where

$$U(x) = V_0(-\log x) + aV_1(-\log x)$$

and

$$\mathcal{W}(x) = \Big(U(x) - \frac{n^2}{4}\Big)F(x) - A(x), \quad F \in C^\infty.$$

We multiply the equation by f^{n+1}, and use that for x small

$$f^{n+1}(x) = 1 + x\tilde{f}(x), \quad \tilde{f} \in C^\infty,$$

and we arrive at

$$\big(-(f^{n-1}x\partial_x)^2 + x^2 f^{n-1}\Delta_{h(0)} + U(x) + x\widetilde{\mathcal{W}}(x) + E\big)u^\bullet = 0, \quad E > 0, \bullet = D.N, \tag{4.1}$$

with

$$u^D(x_0, y) = 0 \quad \text{or} \quad \partial_x u^N(x_0, y) = 0$$

and

$$\widetilde{\mathcal{W}} = \mathcal{W} + \tilde{f}(U + x\mathcal{W} + E), \quad \tilde{f} \in C^\infty.$$

Noice that $\widetilde{\mathcal{W}}(x)$ has a term of the form $Ex\tilde{f}(x)$, $f \in C^\infty$. So, one should keep in mind that it depends on E, but it will not affect the estimates below because this term goes to zero if $E \to 0$.

Since $f^{n-1} = 1 + \gamma x$, we define r to be such that

$$\frac{dr}{r} = \frac{dx}{x(1+\gamma x)},$$
$$r = 0 \quad \text{if } x = 0,$$

and, therefore,

$$r = \frac{x}{1+\gamma x}$$

and so

$$x = \frac{r}{1-\gamma r} = r + r^2 X(r), \quad X \in C^\infty([0, r_0]),\ r_0 \text{ small enough.}$$

Therefore, after the change of variables, equation (4.1) becomes

$$\big(-(r\partial_r)^2 + r^2(1 + rF(r))\Delta_{h(0)} + U(x(r)) + x(r)\widetilde{\mathcal{W}}(x(r)) + E\big)u^\bullet = 0, \tag{4.2}$$

with

$$u^D(r_0, y) = 0 \quad \text{or} \quad \partial_r u^N(r_0, y) = 0,$$

where $E > 0$, $\bullet = D, N$, and $F \in C^\infty([0, r_0])$.

We will need the following fact.

Lemma 4.2. *Suppose that $x = x(r) = r + r^2 X(r)$, with $X(r) \in C^\infty([0, r_0])$ and r_0 is small enough. Let $\rho = -\log x$ and let $\mathcal{G}_{(j)}(\rho)$ be as defined in (1.13); then,*

$$\log_{(j)}(x(r)^{-1}) = \log_{(j)}(r^{-1})\Big(1 + \frac{r}{(\log r)(\log_{(j)} r^{-1})}\mathcal{T}_j(r)\Big), \tag{4.3a}$$

with $\mathcal{T}_j(r) \in C^\infty((0,r_0]) \cap C^0([0,r_0])$ *and, for* $\alpha \in \mathbb{R}$,

$$(\log_{(j)}(x(r))^{-1})^\alpha = (\log_{(j)}(r^{-1}))^\alpha\Big(1 + \frac{r}{(\log r^{-1})(\log_{(j)} r^{-1})} T_{j,\alpha}(r)\Big), \quad (4.3b)$$

with $T_{j,\alpha} \in C^\infty((0,r_0]) \cap C^0([0,r_0])$.

As a consequence, we find that if $\mathcal{G}_{(j)}$ *is given by* (1.13)*, then*

$$\mathcal{G}_{(j)}(\log(x(r)^{-1})) = \mathcal{G}_{(j)}(\log(r^{-1}))\Big(1 + \frac{r}{(\log r)(\log_{(2)} r^{-1})}\mathcal{T}_j(r)\Big), \quad (4.4)$$

where $\mathcal{T}_j \in C^\infty((0,r_0]) \cap L^\infty([0,r_0])$. *If* $V_0(\rho)$ *and* $V_1(\rho)$, $\rho = -\log x$, *are given by one of the alternatives of* (3.9)*, and* $\varrho = -\log r$, *then*

$$\begin{aligned} V_0(\rho) &= V_0(\varrho) + e^{-\varrho}\mathcal{V}_0(\rho), \quad \mathcal{V}_0 \in C^\infty([\varrho_0,\infty)) \cap L^\infty([\varrho_0,\infty)),\\ V_1(\rho) &= V_1(\varrho) + e^{-\varrho}\mathcal{V}_1(\varrho), \quad \mathcal{V}_1 \in C^\infty([\varrho_0,\infty) \cap L^\infty([\varrho_0,\infty)). \end{aligned} \quad (4.5)$$

Proof. The point is that since for t small, $\log(1+t) = tf(t)$, $f \in C^\infty$, and we find that

$$\begin{aligned} \log x(r) &= \log(r(1 + rX(r))) = \log r + \log(1 + rX(r))\\ &= \log r + rX(r)f(rX(r)) = (\log r)\Big(1 + \frac{r}{\log r}T(r)\Big), \end{aligned}$$

where $T(r) = X(r)f(rX(r)) \in C^\infty([0,r_0])$.

Similarly if $\alpha \in \mathbb{R}$, we have that $(1+t)^\alpha = 1 + tf_\alpha(t)$, $f_\alpha \in C^\infty$, and so

$$\begin{aligned} &(-\log x(r))^\alpha = (-\log r)^\alpha\Big(1 + \frac{r}{\log r}T(r)\Big)^\alpha = (-\log r)^\alpha\Big(1 + \frac{r}{\log r}T_\alpha(r)\Big),\\ &T_\alpha = T(r)f_\alpha\Big(\frac{rT(r)}{\log r}\Big) \in C^\infty((0,r_0]) \cap L^\infty([0,r_0]). \end{aligned}$$

Using the same ideas, we obtain

$$\begin{aligned} \log_{(2)}(x(r)^{-1}) &= \log\Big(\log(r^{-1})\Big(1 + \frac{r}{\log r}T(r)\Big)\Big)\\ &= \log_{(2)}(r^{-1}) + \log\Big(1 + \frac{r}{\log r}T(r)\Big)\\ &= (\log_{(2)}(r^{-1}))\Big(1 + \frac{r}{(\log r)(\log_{(2)} r^{-1})}T_1(r)\Big), \end{aligned}$$

where $T_1 \in C^\infty((0,r_0]) \cap L^\infty([0,r_0])$. Using induction, we find that

$$\log_{(j)}(x(r)^{-1}) = (\log_{(j)}(r^{-1}))\Big(1 + \frac{r}{(\log r)(\log_{(j)} r^{-1})}T_j(r)\Big),$$

where $T_j \in C^\infty((0,r_0]) \cap L^\infty([0,r_0])$, and this proves the first equation in (4.3). The second equation in (4.3) and the equations (4.4) and (4.5) follow directly from (4.3).

■

If we replace $r = e^{-\varrho}$, $\varrho \in [\varrho_0, \infty)$ in (4.2) (we have used $\rho = -\log x$ before, and here we are using $\varrho = -\log r$, and these are not the same) and to simplify the notation we use $\mathcal{X}(\varrho) = \widetilde{W}(x(e^{-\varrho}))$ and $B(\varrho) = F(e^{-\varrho})$, then equation (4.1) becomes

$$\mathcal{M}u^\bullet = -Eu^\bullet, \quad \bullet = D, N,\ u^D(\varrho_0, y) = 0 \text{ or } \partial_\varrho u^N(\varrho_0, y) = 0, \tag{4.6}$$

where

$$\mathcal{M} = -\partial_\varrho^2 + q(\varrho)e^{-2\varrho}\Delta_{h(0)} + U(e^{-\varrho}) + e^{-\varrho}\mathcal{X}(\varrho), \quad q(\varrho) = 1 + e^{-\varrho}B(\varrho),$$

$B, \mathcal{X} \in L^\infty([\varrho_0, \infty)) \cap C^\infty([\varrho_0, \infty))$.

We decompose $u^\bullet(\varrho, y)$ in Fourier series with respect to the eigenfunctions of the Laplacian $\Delta_{h(0)}$ on ∂X:

$$u^\bullet(\varrho, y) = \sum_{j=0}^{\infty} u_j^\bullet(\varrho)\psi_j(y), \quad u_j^\bullet(\varrho) = \langle u^\bullet(\varrho, y), \psi_j(y)\rangle_{L^2(\partial X, h(0))},$$

where

$$\Delta_{h(0)}\psi_j = \zeta_j\psi_j, \quad 0 = \zeta_0 < \zeta_1 < \zeta_2 \le \zeta_3 \le \cdots.$$

Let

$$\mathcal{Y}_j = \text{ the eigenspace corresponding to } \zeta_j \tag{4.7a}$$

and define

$$m(\zeta_j) = \dim \mathcal{Y}_j = \text{ the multiplicity of } \zeta_j, \tag{4.7b}$$

and so we have that

$$L^2(X_\infty) = \bigoplus_{j=1}^{\infty} \mathcal{Y}_j \quad \text{and} \quad \mathcal{M}^\bullet = \bigoplus_{j=1}^{\infty} \mathcal{M}_j^\bullet, \quad \bullet = D, N, \tag{4.8}$$

where

$$\mathcal{M}_j = -\left(\frac{d}{d\varrho}\right)^2 + e^{-2\varrho}q(\varrho)\zeta_j + U(e^{-\varrho}) + e^{-\varrho}\mathcal{X}(\varrho).$$

For each j, $\mathcal{M}_j^\bullet$, $\bullet = D, N$, are self-adjoint operators and $\sigma_{\text{ess}}(\mathcal{M}_j^\bullet) = [0, \infty)$, see for example [10]. We prove in Appendix A, for the convenience of the reader, that they have no eigenvalues in $[0, \infty)$, so they have only negative eigenvalues which only possibly accumulate at zero. It also follows from (4.8) that if $E > 0$,

$$\dim P_{(-\infty,-E)}(\mathcal{M}^\bullet) = \sum_{j=1}^{\infty} m(\zeta_j) \dim P_{(-\infty,-E)}(\mathcal{M}_j^\bullet), \quad \bullet = D, N,$$

or in other words,

$$N_E(\mathcal{M}^\bullet) = \sum_{j=1}^{\infty} m(\zeta_j) N_E(\mathcal{M}_j^\bullet), \quad \bullet = D, N, \tag{4.9}$$

where as above, $N_E(T)$ denotes the counting function (1.5).

The eigenfunctions $\phi_j^\bullet(\varrho, E)$, $\bullet = D, N$, with eigenvalue $-E$ satisfy

$$(\mathcal{M}_j^\bullet + E)\phi_j^\bullet = 0,$$
$$\phi_j^D(\varrho_0, E) = 0,\ \partial_\varrho \phi_j^D(\varrho_0, E) = 1 \quad \text{or} \quad \phi_j^N(\varrho_0, E) = 1,\ \partial_\varrho \phi_j^N(\varrho_0, E) = 0;$$

just notice that by dividing the equation by a constant, we can always assume either the function or its derivative is equal to one at ϱ_0. For $E > 0$, we will consider the Cauchy problems

$$(\mathcal{M}_j^\bullet + E)u_j^\bullet(\varrho, E) = 0, \quad \bullet = D, N, \quad \text{on } (\varrho_0, \infty), \tag{4.10a}$$

with boundary conditions

$$u_j^D(\varrho_0, E) = 0,\ \partial_\varrho u_j^D(\varrho_0, E) = 1 \quad \text{or} \quad u_j^N(\varrho_0, E) = 1,\ \partial_\varrho u_j^N(\varrho_0, E) = 0, \tag{4.10b}$$

which have unique solutions in X_∞. Even though $u_j^\bullet(\varrho, E)$ exist for every E, for $\varrho \in (\varrho_0, \infty)$, they are not necessarily eigenfunctions because they may not be in L^2. In fact, it follows from Theorem A.2 that a solution $u_j^\bullet(\varrho, E)$ of (4.10) is an eigenfunction if and only if $u_j^\bullet(\varrho, E) \sim Ce^{-\varrho\sqrt{E}}$ as $\varrho \to \infty$. The key point of the proof Proposition 4.1 is the following.

Proposition 4.3. *Let $u_j^\bullet(\varrho, E)$, $\bullet = D, N$ be the unique solutions of the corresponding Cauchy problems in (4.10). Let $Z_j^\bullet(E)$ denote the number of its zeros, which are different from $\varrho = \varrho_0$ in the case $\bullet = D$. If $E < 0$, then*

$$N_E(\mathcal{M}_j^\bullet) = Z_j^\bullet(E), \quad \bullet = D, N, \tag{4.11}$$

and, as a consequence of (4.9), we have

$$N_E(\mathcal{M}^\bullet) = \sum_{j=1}^{\infty} m(\zeta_j) Z_j^\bullet(E), \quad \bullet = D, N. \tag{4.12}$$

This result is somewhat well known and its proof is essentially, but not quite, the same as the proof of [26, Theorem XIII.8]. For the convenience of the reader, we provide the details in Appendix B.

4.1. The set up of the problems

Now, we have to count the zeros of solutions of (4.10). We set up the general type of problem for $U(e^{-\varrho}) = -V_0(\varrho) + aV_1(\varrho)$ with $V_j(\varrho)$, $j = 0, 1$, satisfying one of the alternatives in (3.9). The arguments we use do not depend on the boundary condition, so we work with the Dirichlet problem in (4.10). We consider the problem

$$\Big(-\frac{d^2}{d\varrho^2} - V_0(\varrho) + e^{-2\varrho}(1 + B(\varrho)e^{-\varrho})\mu + E + aV_1(\varrho) + e^{-\varrho}\mathcal{X}(\varrho)\Big)u = 0, \tag{4.13a}$$

with $E > 0$,

$$u(\varrho_0) = 0, \quad \partial_\varrho u(\varrho_0) = 1, \tag{4.13b}$$

where $B(\varrho), \mathcal{X}(\varrho) \in C^\infty([\varrho_0, \infty)) \cap L^\infty([\varrho_0, \infty))$. We will denote

$$\mathcal{R}(\varrho) = aV_1(\varrho) + e^{-\varrho}\mathcal{X}(\varrho) \quad \text{and} \quad \mathcal{P}(\varrho) = \mu e^{-2\varrho}(1 + e^{-\varrho}B(\varrho)) + E, \tag{4.14}$$

and (4.13) is reduced to

$$\begin{gathered}\Big(-\frac{d^2}{d\varrho^2} - V_0(\varrho) + \mathcal{P}(\varrho) + \mathcal{R}(\varrho)\Big)u = 0, \quad E > 0,\\ u(\varrho_0) = 0.\end{gathered} \tag{4.15}$$

We will deal with each one of the cases (1.8), (1.10), and, in general, (1.13) separately.

4.2. Proof of item (T.1) of Proposition 4.1

In this case,

$$V_0(\varrho) = c\varrho^{-2+\delta} \quad \text{and} \quad V_1(\varrho) = a\varrho^{-2+\delta}(\log\varrho)^{-\varepsilon}.$$

We multiply equation (4.15) by $\varrho^{2-\delta}$, and notice that

$$\varrho^{-(1-\delta/2)/2}\Big(\varrho^{2-\delta}\frac{d^2}{d\varrho^2}\Big)\varrho^{(1-\delta/2)/2} = \Big(\varrho^{1-\delta/2}\frac{d}{d\varrho}\Big)^2 - \frac{1}{4}\Big(1 - \frac{\delta^2}{4}\Big)\varrho^{-\delta}.$$

So if $u(\varrho) = \varrho^{(1-\delta/2)/2}w(\varrho)$, then (4.15) becomes

$$\begin{gathered}\Big(-\Big(\varrho^{1-\delta/2}\frac{d}{d\varrho}\Big)^2 - c + \frac{1}{4}\Big(1 - \frac{\delta^2}{4}\Big)\varrho^{-\delta} + \mathcal{E}(\varrho)\Big)w = 0,\\ w(\varrho_0) = 0,\end{gathered} \tag{4.16}$$

where

$$\mathcal{E}(\varrho) = \varrho^{2-\delta}(\mathcal{P}(\varrho) + \mathcal{R}(\varrho)),$$

$$\mathcal{R}(\varrho) = a\varrho^{-2+\delta}(\log\varrho)^{-\varepsilon} + e^{-\varrho}\mathcal{X}(\varrho), \quad \varepsilon > 0,$$
$$\mathcal{P}(\varrho) = \mu e^{-2\varrho}\big(1 + e^{-\varrho}B(\varrho)\big) + E.$$

Set

$$t = \frac{2}{|\delta|}\sqrt{c}\varrho^{\delta/2},$$

and in this case (4.16) becomes

$$\begin{gathered}\Big(-\frac{d^2}{dt^2} - 1 + \frac{1}{c}\mathcal{E}(\varrho(t)) + \frac{1}{4c}\Big(1-\frac{\delta^2}{4}\Big)\Big(\frac{t|\delta|}{2\sqrt{c}}\Big)^{-2}\Big)w = 0, \quad t = \sqrt{c}\,\frac{2}{|\delta|}\varrho^{\delta/2}, \\ w(t_0) = 0.\end{gathered} \tag{4.17}$$

The case $\delta < 0$. Since $\mathcal{X}(\varrho) \in L^\infty$, it follows from the assumptions on the decay of $V_0(\varrho)$ and $V_1(\varrho)$ that $\varrho^{2-\delta}\mathcal{R}(\varrho) \to 0$ as $\varrho \to \infty$. Since $\mathcal{P}(\varrho) > 0$ for ϱ large, there exists $t_0 > 0$ independent of E and μ such that for $t < t_0$,

$$\begin{aligned} U(t) = & -1 + \frac{1}{c}\mathcal{E}(\varrho(t)) + \frac{1}{4c}\Big(1-\frac{\delta^2}{4}\Big)\Big(\frac{t|\delta|}{2\sqrt{c}}\Big)^{-2} \\ & > -1 + \frac{1}{4c}\varrho^{2-\delta}\mathcal{R}(\varrho) + \frac{1}{4}\Big(1-\frac{\delta^2}{4}\Big)\Big(\frac{t|\delta|}{2\sqrt{c}}\Big)^{-2} > 0,\end{aligned}$$

and so $w(t)$ is a solution of a differential equation

$$\begin{gathered}\frac{d^2w}{dt^2} = U(t)w, \quad t < t_0, \quad \text{with } U(t) > 0, \\ w(t_0) = 0.\end{gathered} \tag{4.18}$$

We have three possibilities for $w'(t_0)$: $w'(t_0) = 0$, $w'(t_0) > 0$, or $w'(t_0) < 0$. If $w'(t_0) = 0$, then by uniqueness, $w(t) = 0$ for $t < t_0$. If $w'(t_0) > 0$, since $w(t)$ is C^∞, there exists $t_1 < t_0$ such that $w'(t) > 0$ for $t_1 < t \leq t_0$ and so $w(t) < 0$ for $t_1 < t \leq t_0$ and therefore, $w''(t) < 0$ for $t_1 < t \leq t_0$, and so, $w'(t) < w'(t_0) < 0$ for $t_1 < t < t_0$ and $w(t) < 0$ for $t_1 < t < t_0$. Repeating this argument, we conclude that $w(t) < 0$ for all $t < t_0$. If $w'(t_0) < 0$, since $-w(t)$ also solves the equation, then $w(t) > 0$ for all $t < t_0$. Therefore, we conclude that either $w(t) = 0$ for all $t > t_0$ or $w(t)$ has no zeros for $t > t_0$.

In this case, we conclude from (4.12) that for this choice of ϱ_0,

$$N_E(\mathcal{M}^\bullet) = 0, \quad \bullet = N, D. \tag{4.19}$$

The case $0 < \delta < 2$. In this case, in view of the discussion above, the set of zeros of the solution w is contained in the set

$$\Big\{\varrho > \varrho_0 : -1 + \frac{1}{c}\mathcal{E}_1(\varrho(t)) \leq 0\Big\}$$

where

$$\mathcal{E}_1(\varrho) = \varrho^{2-\delta}\Big(\mathcal{P}(\varrho) + \mathcal{R}(\varrho) + \frac{1}{4}\Big(1 - \frac{\delta^2}{4}\Big)\varrho^{-2}\Big).$$

We want to obtain upper and lower bounds on the number of zeros of the solution $w(t)$ of (4.17) and consequently upper bounds on the number of eigenvalues of $\mathcal{M}^\bullet$, $\bullet = N, D$. We begin by taking ϱ_0 large enough, and independently of μ and E, such that

$$\Big|\frac{1}{c}\varrho^{2-\delta}\mathcal{R}(\varrho) + \frac{1}{4c}\Big(1 - \frac{\delta^2}{4}\Big)\varrho^{-\delta}\Big| \leq \frac{1}{2} \quad \text{for all } \varrho > \varrho_0. \tag{4.20}$$

Next, we obtain upper bounds for the number of zeros of $w(\rho(t))$, in $t \in [t_0, \infty)$ where t_0 corresponds to ρ_0. Notice that, for this choice of ϱ_0, since E is small,

$$\begin{aligned}
&\Big\{\varrho > \varrho_0 : -1 + \frac{1}{c}\mathcal{E}_1(\varrho(t)) \leq 0\Big\} \\
&\quad \subset \Big\{\varrho \geq \varrho_0 : -1 \leq \frac{1}{c}\mathcal{E}_1(\varrho) \leq \frac{3}{2}\Big\} \\
&\quad \subset \Big\{\varrho \geq \varrho_0 : \frac{\varrho^{2-\delta}}{c}(E + \mu e^{-2\varrho}(1 + B(\varrho)e^{-\varrho})) \leq 2\Big\},
\end{aligned}$$

and one may yet taken ϱ_0 larger if necessary, such that $1 + e^{-\varrho}B(\rho) \geq 1/2$, and so

$$\begin{aligned}
&\Big\{\varrho \geq \varrho_0 : \frac{\varrho^{2-\delta}}{c}(E + \mu e^{-2\varrho}(1 + B(\varrho)e^{-\varrho})) \leq 2\Big\} \\
&\quad \subset \Big\{\varrho \geq \varrho_0 : \frac{\varrho^{2-\delta}}{c}\Big(E + \frac{1}{2}\mu e^{-2\varrho}\Big) \leq 2\Big\}.
\end{aligned} \tag{4.21}$$

Also, if ϱ_0 is large,

$$\Big\{\varrho \geq \varrho_0 : \frac{\varrho^{2-\delta}}{c}\Big(E + \frac{1}{2}\mu e^{-2\varrho}\Big) \leq 2\Big\} = [\varrho_L, \varrho_U],$$

and this is because if $F(\varrho) = \varrho^{2-\delta}(E + \mu e^{-2\varrho}/2)$; then,

$$\begin{aligned}
&F'(\varrho) + F''(\varrho) \\
&\quad = (2-\delta)\varrho^{-\delta}\Big(E + \frac{1}{2}\mu e^{-2\varrho}\Big)(\varrho + 1 - \delta) + 4\mu\varrho^{1-\delta}e^{-2\varrho}(\varrho - 2(2-\delta)) > 0,
\end{aligned}$$

and therefore, if $F'(\varrho) = 0$, then $F''(\varrho) > 0$, so if $F(\varrho)$ has a critical point, it is a local minimum. We then observe that

$$\begin{aligned}
&\Big\{\varrho \geq \varrho_0 : \frac{\varrho^{2-\delta}}{c}\Big(E + \frac{1}{2}\mu e^{-2\varrho}\Big) \leq 2\Big\} \\
&\quad = [\varrho_L, \varrho_U] \subset \Big\{\varrho \geq \varrho_0 : \frac{E}{c}\varrho^{2-\delta} \leq 2\Big\} \cap \Big\{\varrho \geq \varrho_0 : \frac{\mu}{2c}\varrho^{2-\delta}e^{-2\varrho} \leq 2\Big\}.
\end{aligned}$$

If we take ϱ_{u_δ} such that

$$\frac{E}{c}\varrho_{u_\delta}^{2-\delta}=2,$$

so

$$\varrho_{u_\delta}=\Big(\frac{2c}{E}\Big)^{1/(2-\delta)},$$

then $\varrho_U\leq \varrho_{u_\delta}$ and we will get an upper bound on the number of zeros of $w(\varrho(t))$ in the interval $[t_0,t_{u_\delta}]$, where $\varrho_0=\varrho(t_0)$ and $\varrho_{u_\delta}=\varrho(t_{u_\delta})$. This gives an upper bound on the zeros of $w(t)$ in $[t_0,\infty)$.

Since $\varrho^{2-\delta}$ is an increasing function, then

$$\frac{E}{c}\varrho^{2-\delta}\leq 2 \quad \text{for } \varrho\leq \varrho_{u_\delta},$$

but in view of (4.21) if $\varrho\leq\varrho_{u_\delta}$ and $\varrho^{2-\delta}(E+\mu e^{-2\varrho}/2)/c\leq 2$, only if μ satisfies

$$\mu\leq \frac{4c}{\varrho^{2-\delta}}e^{2\varrho}\leq \frac{4c}{\varrho_{u_\delta}^{2-\delta}}e^{2\varrho_{u_\delta}}=2Ee^{(2c/E)^{1/(2-\delta)}}\stackrel{\text{def}}{=}\mu_{\delta,U},$$

then

$$\frac{1}{c}\mathcal{E}_1(\varrho)\leq\frac{3}{2} \quad \text{provided } \varrho\in[\varrho_0,\varrho_{u_\delta}] \text{ and } \mu\leq\mu_{\delta,U}.$$

As usual, see for example [5, Chapter 8], to count the zeros of $w(t)$ for $t\leq t_{u_\delta}=2(\sqrt{c}/\delta)\varrho_{u_\delta}^{\delta/2}$, one sets

$$\theta(t)=\tan^{-1}\Big(\frac{w(t)}{w'(t)}\Big), \quad \text{where } w(t) \text{ satisfies (4.17).}$$

The number of zeros of w in the interval $[t_0,t_{u_\delta}]$, coincides with the number of times $\theta(t)=k\pi$, for some $k\in\mathbb{N}$. It follows from (4.17) that

$$\frac{d\theta}{dt}=\frac{(w')^2-w''w}{w^2+(w')^2}=1-\frac{1}{c}\mathcal{E}_1(t(\varrho))\Big(\frac{w^2}{w^2+(w')^2}\Big), \tag{4.22}$$

and since $-1\leq\mathcal{E}_1(t(\varrho))/c<3/2$, it follows that $|d\theta/dt|\leq 3$, and so

$$\theta(t_{u,\delta})-\theta(t_0)\leq 3(t_{u_\delta}-t_0)\leq 3t_{u_\delta}, \quad t_{u_\delta}=\frac{2\sqrt{c}}{\delta}\varrho_{u_\delta}^{\delta/2}.$$

Therefore, if $Z(w)$ denotes the number of zeros of $w(t)$, then

$$Z(w)\leq\frac{3t_{u_\delta}}{\pi}=\frac{6\sqrt{c}}{\pi\delta}\Big(\frac{2c}{E}\Big)^{\delta/(2(2-\delta))}.$$

So, we conclude that for E small,

$$Z_j^\bullet(E)\leq\frac{6\sqrt{c}}{\pi\delta}\Big(\frac{2}{cE}\Big)^{\delta/(2(2-\delta))} \quad \text{for all } j.$$

In view of (4.7) and (4.12), as $E \to 0$, we have for $\bullet = D, N$,

$$N_E(\mathcal{M}^\bullet) \le \sum_{\zeta_j \le \mu_{\delta,U}} m_j(\zeta_j) Z_j^\bullet(E) \le C\Big(\frac{2}{cE}\Big)^{\delta/(2(2-\delta))} \sum_{\zeta_j \le \mu_{\delta,U}} m_j(\zeta_j)$$
$$= C\Big(\frac{2}{cE}\Big)^{\delta/(2(2-\delta))} N_{\mu_{\delta,U}}(\Delta_{h(0)}),$$

where $\mathcal{N}_\kappa(\Delta_{h(0)})$ is the number of eigenvalues of $\Delta_{h(0)}$ which are less than or equal to κ counted with multiplicity. Weyl's Law, see for example [14, Corollary 17.5.8], says that

$$\mathcal{N}_\kappa(\Delta_{h(0)}) = C_n \kappa^n + O(\kappa^{n-1}), \tag{4.23}$$

and this implies that

$$N_E(\mathcal{M}^\bullet) \le C\Big(\frac{2}{cE}\Big)^{\delta/(2(2-\delta))} \mu_{\delta,U}^n = C\Big(\frac{2}{cE}\Big)^{\delta/(2(2-\delta))} (E e^{(2/cE)^{1/(2-\delta)}})^n,$$

and we find that

$$\log\log N_E(\mathcal{M}^\bullet) \le \frac{1}{2-\delta}\log E^{-1} + O(1), \quad \text{as } E \to 0,$$

which is the upper bound of (1.9).

To obtain a similar lower bound, we will find $\varrho_1 = \varrho_1(E) < \varrho_2(E) = \varrho_2$, such that $\varrho_1(E) > \varrho_0$, for E small enough with ϱ_0 as in (4.20), and $\mu_{\delta,L}$ such that

$$\frac{\varrho^{2-\delta}}{c}(E + \frac{1}{2}\mu e^{-2\varrho}) \le \frac{1}{4} \quad \text{for } \varrho \in [\varrho_1, \varrho_2] \text{ and } \mu \le \mu_{\delta,L}$$

and this implies that $1/c\mathcal{E}_1(\varrho) \le 3/4$ and so

$$[\varrho_1, \varrho_2] \subset \Big\{\varrho \ge \rho_0 : \frac{1}{c}\mathcal{E}_1(\varrho) < 1\Big\},$$

and therefore the number of zeros of $w(t)$ in this interval is less than or equal to the number of zeros of $w(t)$ in $[\varrho_0, \infty)$. We deduce from (4.22) that

$$\theta(t_2) - \theta(t_1) \ge \frac{1}{4}(t_2 - t_1),$$

and so

$$Z_j^\bullet(E) \ge \frac{1}{4\pi}(t_2 - t_1), \quad t_j = \frac{2\sqrt{c}}{\delta}\varrho_j^{\delta/2}, \quad j = 1, 2.$$

Notice that, since $E\varrho^{2-\delta}$ is an increasing function and $\mu\varrho^{2-\delta}e^{-2\varrho}$ is an decreasing function, we choose ϱ_2 such that

$$\frac{E}{c}\varrho_2^{2-\delta} = \frac{1}{8}, \quad \varrho_2 = \Big(\frac{c}{8E}\Big)^{1/(2-\delta)},$$

and for $\varrho_1 = \varrho_2/2$, we only pick μ such that

$$\frac{\mu}{2c}\varrho_1^{2-\delta}e^{-2\varrho_1} \le \frac{1}{8};$$

this implies that

$$\mu \le \frac{c}{4\varrho_1^{2-\delta}}e^{2\varrho_1} \le \frac{c}{4\varrho_2^{2-\delta}}e^{2\varrho_2} = 2Ee^{(c/(8E))^{1/(2-\delta)}} \stackrel{\text{def}}{=} \mu_{\delta,L}.$$

Therefore, for this choice of ϱ_1 and ϱ_2, and $\mu \le \mu_{\delta,L}$ it follows that

$$\frac{1}{c}\varrho^{2-\delta}\Big(\frac{1}{2}\mu e^{-2\varrho} + E\Big) \le \frac{1}{4} \quad \text{in } \Big[\frac{1}{2}\varrho_2, \varrho_2\Big].$$

Therefore,

$$\begin{aligned} Z_j^\bullet(E) &\ge \frac{1}{4\pi}(t_2 - t_1) = \frac{2\sqrt{c}}{\pi\delta}(\varrho_2^{\delta/2} - \varrho_1^{\delta/2}) \\ &\ge \frac{2\sqrt{c}}{\pi\delta}\varrho_2^{\delta/2}\Big(1 - \Big(\frac{1}{2}\Big)^{\delta/2}\Big) \ge C\Big(\frac{c}{8E}\Big)^{\delta/(2-\delta)}. \end{aligned}$$

It follows from (4.7) and (4.12) that as $E \to 0$,

$$\begin{aligned} N_E(\mathcal{M}^\bullet) &\ge \sum_{\zeta_j \le \mu_{\delta,L}} m_j(\zeta_j)Z_j^\bullet(E) \ge C\Big(\frac{c}{4E}\Big)^{1/(2-\delta)}(\mu_{\delta,L})^n \\ &\ge C\Big(\frac{c}{8E}\Big)^{\delta/(2-\delta)}E^n e^{n(c/(8E))^{1/(2-\delta)}}, \end{aligned}$$

and this shows that

$$\log\log N_E(\mathcal{M}^\bullet) \ge \frac{1}{2-\delta}\log E^{-1} + O(1) \quad \text{as } E \to 0.$$

This ends the proof of item (T.1) of Proposition 4.1 and together with equations (3.15), (3.5), and (1.16) it also ends the proof of Theorem 1.1.

4.3. Proof of item (T.2) of Proposition 4.1

In this case

$$V_0(\varrho) = c\varrho^{-2} \quad \text{and} \quad V_1(\varrho) = \varrho^{-2}(\log\varrho)^{-\varepsilon},$$

so if $\mathcal{R}$, $\mathcal{P}$ are defined by (4.14), equation (4.15) becomes

$$\begin{gathered} \Big(-\frac{d^2}{d\varrho^2} - c\varrho^{-2} + \mathcal{R}(\varrho) + \mathcal{P}(\varrho)\Big)u = 0, \quad E > 0, \\ u(\varrho_0) = 0. \end{gathered} \tag{4.24}$$

Next, we multiply the equation by ϱ^2, set $u = \varrho^{1/2} w$, and notice that

$$\varrho^{-1/2}\Big(\varrho^2 \frac{d}{d\varrho}\Big)^2 \varrho^{1/2} = \Big(\varrho \frac{d}{d\varrho}\Big)^2 - \frac{1}{4}; \tag{4.25}$$

then, (4.24) becomes

$$\begin{gathered}\Big(-\Big(\varrho \frac{d}{d\varrho}\Big)^2 - \Big(c - \frac{1}{4}\Big) + \mathcal{E}(\varrho)\Big) w = 0, \quad \mathcal{E}(\varrho) = \varrho^2(\mathcal{R}(\varrho) + \mathcal{P}(\varrho)), \\ w(\varrho_0) = 0.\end{gathered} \tag{4.26}$$

Set $s = \log \varrho$, and (4.26) becomes

$$\begin{gathered}\Big(-\frac{d^2}{ds^2} - \Big(c - \frac{1}{4}\Big) + \mathcal{E}(\varrho(s))\Big) w = 0, \\ w(s_0) = 0, \quad \text{where } s_0 = \log(\varrho_0).\end{gathered} \tag{4.27}$$

The case $c < 1/4$. Since $\mathcal{P}(\varrho) > 0$ and $\varrho^2 \mathcal{R}(\varrho) \to 0$ as $\varrho \to \infty$, there exists $R > 0$ independent of μ and E such that

$$\Big(\frac{1}{4} - c\Big) + \mathcal{E}(\varrho) \geq \Big(\frac{1}{4} - c\Big) + \varrho^2 \mathcal{R}(\varrho) > 0, \quad \text{for } \varrho_0 > R.$$

Therefore, we have an equation as (4.18), with $U(s) = (1/4 - c) + \mathcal{E}(\varrho(s)) > 0$ and so w has no zeros, and again we conclude from (4.12) that for this choice of ϱ_0, (4.19) holds.

The case $c > 1/4$. We set $t = \lambda s$, $\lambda = (c - 1/4)^{1/2}$ and (4.27) becomes

$$\begin{gathered}\Big(-\frac{d^2}{dt^2} - 1 + \frac{1}{\lambda^2}\mathcal{E}(\varrho(t))\Big) w = 0, \\ w(t_0) = 0, \quad \text{where } t_0 = \lambda \log(\varrho_0).\end{gathered} \tag{4.28}$$

The argument used above shows that the zeros of the solution w of (4.27) lie on the set

$$\Big\{t > t_0 : \frac{1}{\lambda^2}\mathcal{E}(\varrho(t)) < 1\Big\}$$

and as in the first case, we prove upper and lower bounds for the number of zeros of the solution $w(t)$ of (4.28) and we start by picking ϱ_0 large so that $|\varrho^2 \mathcal{R}(\varrho)| < 1/2$, for $\varrho > \varrho_0$. In this case,

$$\begin{aligned}\Big\{\varrho > \varrho_0 : \frac{1}{\lambda^2}\mathcal{E}(\varrho) < 1\Big\} &\subset \Big\{\varrho \geq \varrho_0 : \frac{\varrho^2}{\lambda^2}(\mathcal{R}(\varrho) + \mathcal{P}(\varrho)) \leq \frac{3}{2}\Big\} \\ &\subset \Big\{\varrho \geq \varrho_0 : \frac{\varrho^2}{\lambda^2}\Big(E + \frac{1}{2}\mu e^{-2\varrho}\Big) \leq 2\Big\}.\end{aligned}$$

We have the following result.

Lemma 4.4. *Let $f(\varrho) > 0$ be a C^∞ function such that $f''(\varrho) > 0$ and $f(\varrho) > f'(\varrho)$ for $\varrho > \varrho_0$. Then, the function $F(\varrho) = f(\varrho)(E + \mu e^{-2\varrho})$ is convex and therefore the set*

$$\Omega(E, \mu, C) = \Big\{ t > t_0 : f(\varrho)\Big(E + \frac{1}{2}\mu e^{-2\varrho}\Big) \leq C \Big\}$$

is either empty or is equal to an interval $[a, b]$ with $a \geq \varrho_0$.

Proof. We find that

$$F''(\varrho) = f''(\varrho)\Big(E + \frac{1}{2}\mu e^{-2\varrho}\Big) + 4\mu(f(\varrho) - f'(\varrho))e^{-2\varrho} > 0. \qquad \blacksquare$$

Therefore,

$$\begin{aligned} [a, b] &= \Big\{ \varrho \geq \varrho_0 : \frac{\varrho^2}{\lambda^2}\Big(E + \frac{1}{2}\mu e^{-2\varrho}\Big) \leq 2 \Big\} \\ &\subset \Big\{ \varrho \geq \varrho_0 : \frac{E}{\lambda^2}\varrho^2 \leq 2 \Big\} \cap \Big\{ \varrho \geq \varrho_0 : \frac{\mu}{2\lambda^2}\varrho^2 e^{-2\varrho} \leq 2 \Big\}. \end{aligned} \tag{4.29}$$

If we take ϱ_u to satisfy

$$\frac{E}{\lambda^2}\varrho_u^2 = 2,$$

then $b \leq \varrho_u$ and $(E/\lambda_2^2)\varrho^2 \leq 2$, for $\varrho \leq \varrho_u$. We count the zeros of $w(t)$ in the interval $[t_0, t_u]$ and as above one sets

$$\theta(t) = \tan^{-1}\Big(\frac{w(t)}{w'(t)}\Big), \quad \text{where } w(t) \text{ satisfies (4.28)}$$

and then (4.22) holds, and since $-1 \leq \mathcal{E}(\varrho(t)) \leq 2$, it follows that $|d\theta/d\varrho| \leq 3$ and so

$$\theta(t_u) - \theta(t_0) \leq 3t_u.$$

Therefore, if $Z(w)$ denotes the number of zeros of $w(\varrho(t))$ in an interval $[t_0, t_u]$, we have

$$Z(w) = \frac{2}{\pi}(\theta(t_u) - \theta(t_0)) \leq \frac{3t_u}{\pi}.$$

So, we conclude that for E small,

$$Z_j^\bullet(E) \leq \frac{\lambda}{\pi}\log\Big(\sqrt{\frac{\lambda^2}{E}}\Big) \leq C \log E^{-1/2} \quad \text{as } E \to 0$$

On the other hand, according to (4.29), we must also have

$$\mu \leq \frac{4\lambda^2}{\varrho^2}e^{2\varrho} \leq \frac{4\lambda^2}{\varrho_u^2}e^{2\varrho_u} = 2Ee^{2\sqrt{2\lambda^2/E}} \stackrel{\text{def}}{=} \mu_u,$$

and so

$$\frac{\varrho^2}{\lambda^2}\Big(E + \frac{1}{2}e^{-2\varrho}\mu\Big) \leq 2 \quad \text{on the interval } [\varrho_0, \varrho_u]$$

and $\mu \leq \mu_u$, and in view of (4.7) and (4.12), as $E \to 0$, we have

$$\begin{aligned} N_E(\mathcal{M}^\bullet) &\leq \sum_{\zeta_j \leq \mu_U} m_j(\zeta_j) Z_j^\bullet(E) \\ &\leq C \log E^{-1/2} \sum_{\zeta_j \leq \mu_U} m_j(\zeta_j) = C \log E^{-1/2} \mathcal{N}_{\mu_u}(\Delta_{h(0)}), \end{aligned}$$

and, because of (4.23), this implies that

$$N_E(\mathcal{M}^\bullet) \leq C\mu_u^n \log E^{-1/2}.$$

It follows from the definition of μ_u that

$$\log\log N_E(\mathcal{M}^\bullet) \leq -\frac{1}{2}\log E + O(1),$$

which gives the upper bound in (1.11).

To prove a similar lower bound for $N_E(\mathcal{M}^\bullet)$, we will find $\varrho_1 = \varrho_1(E) < \varrho_2(E) = \varrho_2$ and μ_L such that $\varrho_0 < \varrho_1(E)$ for E small enough and

$$\frac{\varrho^2}{\lambda^2}\Big(E + \frac{1}{2}\mu e^{-2\varrho}\Big) \leq \frac{1}{4} \quad \text{for } \varrho \in [\varrho_1, \varrho_2] \text{ and } \mu \leq \mu_L,$$

and so $\mathcal{E}(\varrho)/\lambda^2 < 3/4$ for $\varrho \in [\varrho_1, \varrho_2]$, and this implies that

$$[\varrho_1, \varrho_2] \subset \Big\{\rho \geq \rho_0 : \frac{1}{\lambda^2}\mathcal{E}(\varrho) < 1\Big\}.$$

It then follows from (4.22) that for $t \in [t_0, t_1]$, $d\theta/dt \geq 1/4$ and so $\theta(t_2) - \theta(t_1) \geq (t_2 - t_1)/4$ and

$$Z(\omega) \geq \frac{1}{4\pi}(t_2 - t_1), \quad t_j = \lambda \log \varrho_j,\ j = 1, 2.$$

Since $E\varrho^2$ is increasing, we pick ϱ_2 such that

$$\frac{E}{\lambda^2}\varrho_2^2 = \frac{1}{8},$$

and so $E\varrho^2/\lambda^2 \leq 1/8$, for $\varrho \leq \varrho_2$. Pick ϱ_1 such that $\varrho_1 = \varrho_2/M$, with M to be chosen; we want

$$\frac{\mu}{2\lambda^2}\varrho_1^2 e^{-2\varrho_1} \leq \frac{1}{8},$$

thus

$$\mu \leq \frac{\lambda^2}{4\varrho_1^2} e^{2\varrho_1} \leq \frac{\lambda^2}{4\varrho_2^2} e^{2\varrho_2} = 2E e^{2\sqrt{\lambda^2/(8E)}} \stackrel{\text{def}}{=} \mu_L. \tag{4.30}$$

Since $t_1 = \lambda \log \varrho_1$ and $t_2 = \lambda \log \varrho_2$, we can choose M, independently of E and μ, such that

$$Z(w) \geq \frac{1}{4\pi}(t_2 - t_1) \geq \frac{\lambda}{4\pi} \log\Big(\frac{\varrho_2}{\varrho_1}\Big) = \frac{\lambda}{4\pi} \log M > 1.$$

Again, we conclude from (4.7) and (4.12) that for μ_L as in (4.30),

$$N_E(\mathcal{M}^\bullet) \geq \sum_{\zeta_j \leq \mu_L} m_j(\mu_j) Z_j^\bullet(E) \geq C \sum_{\zeta_j \leq \mu_L} m_j(\mu_j) \geq C\mu_L^n.$$

This gives that

$$\log\log N_E(\mathcal{M}^\bullet) \geq -\frac{1}{2} \log E + O(1),$$

which is the lower bound in (1.11) and proves item (T.2) of Proposition 4.1 and together with equations (3.15), (3.5), and (1.16) it also ends the proof of Theorem 1.2.

4.4. Proof of item (T.3) of Proposition 4.1

We first consider the case $N = 1$ in (1.13) and we have equation (4.15) with

$$V_0(e^{-\varrho}) = \frac{1}{4}\varrho^{-2} + c_1 \varrho^{-2}(\log\varrho)^{-2} \quad \text{and} \quad V_1(e^{-\varrho}) = \varrho^{-2}(\log\varrho)^{-2}(\log\varrho)^{-\varepsilon},$$

and $\mathcal{R}$, $\mathcal{P}$ given by (4.14). We multiply the equation by ϱ^2, set $u = \varrho^{1/2} w$, and use (4.25), and we obtain

$$\begin{gathered} \big(-(\varrho\partial_\varrho)^2 - c_1(\log\varrho)^{-2} + \varrho^2 \mathcal{E}(\varrho)\big) w = 0, \quad \mathcal{E}(\varrho) = \mathcal{R}(\varrho) + \mathcal{P}(\varrho), \\ w(\varrho_0) = 0, \end{gathered} \tag{4.31}$$

Now, we set $\xi = \log\varrho$, multiply (4.31) by ξ^2 and set $w = \xi^{1/2} v$, use (4.25) and we obtain

$$\begin{gathered} \Big(-\Big(\xi \frac{d}{d\xi}\Big)^2 - \Big(c_1 - \frac{1}{4}\Big) + \mathcal{E}_1(\varrho(\xi))\Big) v = 0, \quad \mathcal{E}_1(\varrho) = \varrho^2(\log\varrho)^2 \mathcal{E}(\varrho), \\ v(\xi_0) = 0. \end{gathered}$$

As before, if $c_1 < 1/4$, v has no zeros for $\xi > \xi_0$, and so it follows from (4.12) that $\mathcal{M}^\bullet$, $\bullet = N, D$ have no eigenvalues.

If $c_1 > 1/4$, we set $\lambda_1 = (c_1 - 1/4)^{1/2}$ and $\tau = \lambda_1 \log\xi$, and we obtain

$$\begin{gathered} \Big(-\partial_\tau^2 - 1 + \frac{1}{\lambda_1^2} \mathcal{E}_1(\tau(\varrho))\Big) v = 0, \quad \tau = \lambda_1 \log_{(2)} \varrho, \\ v(\tau_0) = 0. \end{gathered}$$

As before, we assume that ϱ_0 is so large that

$$\left|\frac{1}{\lambda_1}^2 \varrho^2 (\log \varrho)^2 \mathcal{R}(\varrho)\right| \leq \frac{1}{2}, \quad \text{for } \varrho \geq \varrho_0,$$

and for this choice of ϱ_0, we have

$$\begin{aligned}
&\Big\{\varrho \geq \varrho_0 : \frac{1}{\lambda_1^2}\mathcal{E}_1(\varrho) < 1\Big\} \\
&\quad \subset \Big\{\varrho \geq \varrho_0 : \frac{1}{\lambda_1^2}(\varrho \log \varrho)^2 \mathcal{P}(\varrho) \leq 2\Big\} \\
&\quad \subset \Big\{\varrho \geq \varrho_0 : \frac{E}{\lambda_1^2}(\varrho \log \varrho)^2 \leq 2\Big\} \cap \Big\{\varrho \geq \varrho_0 : \frac{\mu}{2\lambda_1^2}(\varrho \log \varrho)^2 e^{-2\varrho} \leq 2\Big\}. \qquad (4.32)
\end{aligned}$$

Since $f(\varrho) = \varrho^2(\log \varrho)^2$ satisfies the hypothesis of Lemma 4.4, we deduce that

$$\Big\{\varrho \geq \varrho_0 : \frac{1}{\lambda_1^2}(\varrho \log \varrho)^2 \Big(E + \frac{1}{2}\mu e^{-2\varrho}\Big) \leq 2\Big\} = [a, b],$$

but then we must have

$$\frac{E}{\lambda_1^2}(b \log b)^2 \leq 2$$

and, in particular, if we take

$$\varrho_{u_1} = \frac{2}{\log A_1} A_1 \quad \text{where } A_1 = \sqrt{\frac{2\lambda_1^2}{E}},$$

then, for E small, and independently of μ,

$$\varrho_{u_1} \log \varrho_{u_1} = 2A_1\Big(1 + \frac{\log 2}{\log A_1} - \frac{1}{\log A_1}\log(\log A_1)\Big) \geq A_1,$$

and so $b \leq \varrho_{u_1}$ and therefore for $\varrho \leq \varrho_{u_1}$, in view of (4.32), we must have

$$\mu \leq \frac{4\lambda_1^2}{(\varrho \log \varrho)^2} e^{2\varrho} \leq \frac{4\lambda_1^2}{(\varrho_u \log \varrho_{u_1})^2} e^{2\varrho_{u_1}} \stackrel{\text{def}}{=} \mu_{U_1}. \qquad (4.33)$$

Since $\tau = \lambda_1 \log \log \varrho$, we obtain for small E,

$$\begin{aligned}
\theta(\tau_{u_1}) - \theta(\tau_0) &\leq 3(\tau_{u_1} - \tau_0) \leq 3\tau_{u_1} = 3\lambda_1 \log \log \varrho_{u_1} \\
&\leq 3\lambda_1 \log \log \varrho_{u_1} \leq C\lambda_1 \log \log E^{-1},
\end{aligned}$$

and for μ_{U_1} as in (4.33),

$$N_E(\mathcal{M}^\bullet) \leq C\lambda_1 \log \log E^{-1} \mathcal{N}_{\partial X, h_0}(\mu_{U_1}) \leq C\lambda_1 \log(\log E^{-1})(\mu_{U_1})^n.$$

It follows that

$$\log_{(3)} N_E(H) \leq \frac{1}{2}\log_{(2)} E^{-1} + O(1),$$

and this gives the upper bound in (1.12).

To obtain the lower bound, we find $\varrho_1 = \varrho_1(E) < \varrho_2(E) = \varrho_2$ and μ_L such that $\varrho_0 < \varrho_1(E)$ for E small enough and

$$\frac{\varrho^2(\log\varrho)^2}{\lambda_1^2}\Big(\frac{1}{2}\mu e^{-2\varrho} + E\Big) \leq \frac{1}{4} \quad \text{for } \varrho \in [\varrho_1, \varrho_2] \text{ and } \mu \leq \mu_L,$$

and this can be achieved if we take ϱ_1 and ϱ_2 such that

$$\frac{E}{\lambda_1^2}(\varrho_2\log\varrho_2)^2 \leq \frac{1}{8} \quad \text{and} \quad \varrho_1 = \varrho_2^{1/M},$$

with M large enough to be chosen independently of μ and E. For instance, for E small enough, take

$$\varrho_2 = \frac{1}{\log\beta_1}\beta_1, \quad \text{where } \beta_1 = \sqrt{\frac{\lambda_1^2}{8E}},$$

and, therefore,

$$\varrho_2\log\varrho_2 = \beta_1\Big(1 - \frac{1}{\log\beta_1}\log_{(2)}\beta_1\Big) < \beta_1.$$

But we also want $(\mu/(2\lambda_1^2))\varrho^2(\log\varrho)^2 e^{-2\varrho} \leq 1/8$, and so we need μ to satisfy

$$\mu \leq \frac{1}{16(\varrho_1\log\varrho_1)^2}e^{2\varrho_1} \stackrel{\text{def}}{=} \mu_{L_1}.$$

Since $\tau = \lambda_1 \log\log\varrho$, we can choose M, independent of E and μ, so that

$$\theta(\tau_2) - \theta(\tau_1) \geq \frac{1}{4\pi}(\tau_2 - \tau_1) \geq \frac{\lambda_1}{4\pi}\log\Big(\frac{\log\varrho_2}{\log\varrho_1}\Big) = \frac{\lambda_1}{4\pi}\log\big(M\big) > 1.$$

This implies that, for μ_{L_1} as above,

$$N_E(\mathcal{M}^\bullet) \geq \mathcal{N}_{\partial X, h_0}(\mu_{L_1}) \geq C(\mu_{L_1})^n,$$

and we conclude that

$$\log_{(3)} N_E(\mathcal{M}^\bullet) \geq \log_{(2)} E^{-1} + O(1),$$

which implies (1.12).

Next, we consider the case $N = 2$, which corresponds to

$$V_0(\varrho) = \frac{1}{4}\varrho^{-2} + \frac{1}{4}\varrho^{-2}(\log\varrho)^{-2} + c_2\varrho^{-2}(\log\varrho)^{-2}(\log_{(2)}\varrho)^{-2},$$
$$V_1(\varrho) = \varrho^{-2}(\log\varrho)^{-2}(\log_{(2)}\varrho)^{-2}(\log\varrho)^{-\varepsilon},$$

with $\mathcal{R}(\varrho)$ and $\mathcal{P}(\varrho)$ as in (4.14). This time, we set $\eta = \log_{(3)} \varrho$ in equation (4.15) and set $u = (\varrho(\log\varrho)(\log_{(2)}\varrho))^{1/2} v$, and we obtain

$$\Big(-\Big(\frac{d}{d\eta}\Big)^2 - \Big(c_2 - \frac{1}{4}\Big) + \mathcal{E}_2(\varrho(\eta))\Big) v = 0,$$

where

$$\mathcal{E}_2(\varrho) = \varrho^2(\log\varrho)^2(\log\log\varrho)^2(\mathcal{R}(\varrho) + \mathcal{P}(\varrho)), \quad v(\eta_0) = 0.$$

As before, if $c_2 < 1/4$, v has no zeros for $\mu > \mu_0$, and so $\mathcal{M}^\bullet$ has no negative eigenvalues. If $c_2 > 1/4$, we set $\lambda_2 = (c_2 - 1/4)^{1/2}$ and $\tau = \lambda_2 \eta$, and we obtain

$$\Big(-\Big(\frac{d}{d\tau}\Big)^2 - 1 + \frac{1}{\lambda_2^2}\mathcal{E}_2(\tau(\varrho))\Big) v = 0,$$
$$v(\tau_0) = 0,$$

We pick ϱ_0 large such that

$$\Big|\frac{1}{\lambda_2^2}\varrho^2(\log\varrho)^2(\log\log\varrho)^2\mathcal{R}(\varrho)\Big| \le \frac{1}{2}, \quad \text{if } \varrho \ge \varrho_0,$$

and for $\mathcal{G}_2 = (\varrho^2(\log\varrho)^2(\log\log\varrho)^2)^{-1}$, it follows that

$$\begin{aligned}
&\Big\{\varrho \ge \rho_0 : \frac{1}{\lambda_2^2}\mathcal{E}_2(\varrho) < 1\Big\} \\
&\quad \subset \Big\{\varrho \ge \varrho_0 : \frac{1}{\lambda_2^2\mathcal{G}_2(\varrho)}\Big(E + \frac{1}{2}\mu e^{-2\varrho}\Big) \le 2\Big\} \\
&\quad \subset \Big\{\varrho \ge \varrho_0 : \frac{E}{\lambda_2^2\mathcal{G}_2(\varrho)} \le 2\Big\} \cap \Big\{\varrho \ge \varrho_0 : \frac{\mu}{2\lambda_2^2\mathcal{G}_2(\varrho)} e^{-2\varrho} \le 2\Big\}.
\end{aligned}$$

and since $f(\varrho) = 1/\mathcal{G}_2(\varrho) = \varrho^2(\log\varrho)^2(\log\log\varrho)^2$ satisfies the hypothesis of Lemma 4.4, we find that

$$\Big\{\varrho \ge \varrho_0 : \frac{1}{\lambda_2^2\mathcal{G}_2(\varrho)}\Big(E + \frac{1}{2}\mu e^{-2\varrho}\Big) \le 2\Big\} = [a, b],$$

and so for $\varrho \in [a, b]$ we must have

$$\frac{E}{\lambda_2^2\mathcal{G}_2(\rho)} \le 2 \quad \text{and so } \varrho(\log\varrho)(\log\log\varrho) \le \sqrt{\frac{2\lambda_2^2}{E}} = A_2.$$

If we take

$$\varrho_{u_2} = \frac{2A_2}{\log A_2(\log\log A_2)},$$

then

$$\varrho_{u_2}(\log\varrho_{u_2})(\log\log\varrho_{u_2}) = 2A_2(1 + o(1)) \ge A_2, \quad \text{as } E \to 0,$$

and so $b \leq \varrho_{u_2}$. Therefore, for $\varrho \leq \varrho_{u_2}$, we must have

$$\mu \leq 4\lambda_2^2 \mathcal{G}_2(\varrho) e^{2\varrho} \leq 4\lambda_2^2 \mathcal{G}_2(\varrho_{u_2}) e^{2\varrho_{u_2}} \stackrel{\text{def}}{=} \mu_{u_2}.$$

It follows from (4.12) that

$$N_E(\mathcal{M}^\bullet) \leq C(\log_{(3)} \varrho_{u_2})(\mu_{u_2})^n,$$

which implies that

$$\log_{(3)} N_E(\mathcal{M}^\bullet) \leq \log_{(2)} E^{-1} + O(1) \quad \text{as } E \to 0, \tag{4.34}$$

and this implies the upper bound of (1.14) when $N = 2$.

Again, to obtain the lower bound, we find $\varrho_1 = \varrho_1(E) < \varrho_2(E) = \varrho_2$ and μ_L such that $\varrho_0 < \varrho_1(E)$ for E small enough and

$$\frac{1}{\lambda_2^2 \mathcal{G}_2(\varrho)}\Big(E + \frac{1}{2}\mu e^{-2\varrho}\Big) \leq \frac{3}{4} \quad \text{for } \varrho \in [\varrho_1, \varrho_2] \text{ and } \mu \leq \mu_L.$$

We pick ϱ_2 such that

$$\frac{E}{\lambda_2^2}(\varrho_2(\log \varrho_2)(\log\log \varrho_2))^2 \leq \frac{1}{8}.$$

For instance, we can just take

$$\varrho_2 = \frac{1}{(\log \beta_2)(\log\log \beta_2)}\beta_2, \quad \text{where } \beta_2 = \sqrt{\frac{\lambda_2^2}{8E}}.$$

We then pick ϱ_1 such that $\log \varrho_1 = (\log \varrho_2)^{1/M}$, with M to be chosen, and we need to restrict the values of μ so that

$$\mu \leq \frac{\lambda_2^2}{4}\mathcal{G}_2(\varrho_1) e^{2\varrho_1} = \mu_{L_2}.$$

Since $\tau = \lambda_2 \log_{(3)} \varrho$, we can choose M, independent of μ and E, such that

$$\theta(\tau_2) - \theta(\tau_1) \geq \frac{1}{4\pi}(\tau_2 - \tau_1) \geq \frac{C\lambda_2}{4\pi}\log\Big(\frac{\log(\log \varrho_2)}{\log(\log \varrho_1)}\Big) \geq \frac{C\lambda_2}{4\pi}\log M > 1.$$

This implies that for μ_{L_2} as above

$$N_E(\mathcal{M}^\bullet) \geq \mathcal{N}_{\partial X, h_0}(\kappa_{L2}) \geq C(\mu_{L_2})^n.$$

This implies that

$$\log_{(4)} N_E(\mathcal{M}^\bullet) \geq \log_{(3)} E^{-1} + O(1),$$

which is the lower bound (1.14) for the case $N = 2$. Because of the choice of ρ_1 we get a weaker lower bound than (4.34).

The proof in the general case in (1.13) follows the same principle. We pick ϱ_0 such that $\log_{(j)} \varrho > 1$ for $\varrho \geq \varrho_0$, and for all $j \leq N + 1$. If $\mathcal{G}_{(j)}(\varrho)$ is defined as in (1.13), we have

$$V_0(\varrho) = \frac{1}{4}\varrho^{-2} + \frac{1}{4}\sum_{j=1}^{N-1} \mathcal{G}_{(j)}j(\varrho) + c_N \mathcal{G}_{(N)}(\varrho), \quad V_1(\varrho) = \mathcal{G}_{(N)}(\varrho)(\log \varrho)^{-\varepsilon},$$

with $\mathcal{R}(\varrho)$ and $\mathcal{P}(\varrho)$ defined in (4.14).

This time, we set $\eta = \log_{(N+1)} \varrho$ in (4.15) and $u = (\mathcal{G}_N(\varrho))^{-1/2} v$; we obtain

$$\begin{gathered}\Big(-\Big(\frac{d}{d\eta}\Big)^2 - \Big(c_N - \frac{1}{4}\Big) + \mathcal{E}_N(\varrho(\eta))\Big)v = 0,\\ \mathcal{E}_N(\varrho) = (\mathcal{G}_{(N)}(\varrho))^{-1}(\mathcal{R}(\varrho) + \mathcal{P}(\varrho)),\\ v(\eta_0) = 0.\end{gathered}$$

If $c_N < 1/4$, then v has no zeros for $\eta > \eta_0$, if η_0 is large, and so $\mathcal{M}^\bullet$ has no negative eigenvalues. If $c_N > 1/4$, we set $\lambda_N = (c_N - 1/4)^{1/2}$ and $\tau = \lambda_N \eta$, and we obtain

$$\begin{gathered}\Big(-\Big(\frac{d}{d\tau}\Big)^2 - 1 + \frac{1}{\lambda_N^2}\mathcal{E}_N(\varrho(\tau))\Big)v = 0, \quad \tau = \lambda_N \log_{(N+1)} \varrho,\\ v(\varrho_0) = 0,\end{gathered}$$

Again, we follow the steps in the proof of the previous cases and pick ϱ_0 such that

$$|\mathcal{G}_N(\varrho)^{-2}\mathcal{R}(\varrho)| \leq \frac{1}{2} \quad \text{if } \varrho \geq \varrho_0,$$

and so the zeros of v will be contained in the set

$$\begin{aligned}&\Big\{\varrho \geq \varrho_0 : \frac{1}{\lambda_N^2}\mathcal{E}_N(\varrho) < 1\Big\}\\ &\quad \subset \Big\{\varrho \geq \varrho_0 : \frac{1}{\lambda_N^2(\mathcal{G}_N(\varrho))^2}\Big(E + \frac{1}{2}\mu e^{-2\varrho}\Big) \leq 2\Big\}\\ &\quad \subset \Big\{\varrho \geq \varrho_0 : \frac{E}{\lambda_N^2(\mathcal{G}_N(\varrho))^2} \leq 2\Big\} \cap \Big\{\varrho \geq \varrho_0 : \frac{\mu}{2\lambda_N^2(\mathcal{G}_N(\varrho))^2}e^{-2\varrho} \leq 2\Big\}.\end{aligned}$$

and since $1/(\mathcal{G}_N(\varrho))^2$ satisfies the hypothesis of Lemma 4.4, we find that

$$\Big\{\varrho \geq \varrho_0 : \frac{1}{\lambda_N^2(\mathcal{G}_N(\varrho))^2}\Big(E + \frac{1}{2}\mu e^{-2\varrho}\Big) \leq 2\Big\} = [a, b].$$

Since

$$\frac{1}{\mathcal{G}_N(\varrho)} = \varrho(\log\varrho)(\log\log\varrho)\dots(\log_{(N)}\varrho) \le \sqrt{\frac{2\lambda_N^2}{E}}, \quad \varrho \le \varrho_*,$$

as before, if we define

$$\varrho_{u_N} = \frac{2A_N}{\log A_N(\log\log A_N)\dots\log_{(N)} A_N}, \quad A_N = \sqrt{\frac{2\lambda_N^2}{E}},$$

then

$$\frac{1}{\mathcal{G}_N(\varrho_{u_N})} = \varrho_{u_N}(\log\varrho_{u_N})\dots\log_{(N)}\varrho_{u_N} \ge A_N,$$

and therefore, $b \le \varrho_{u_N}$. We must also have μ such that

$$\mu \le 4\lambda_N^2(\mathcal{G}_N(\varrho))^2 e^{2\varrho} = 4\mathcal{G}_N(\varrho_{u_N})e^{2\varrho_{u_N}} \stackrel{\text{def}}{=} \mu_{u_N}.$$

On the other hand,

$$\theta(\tau_2) - \theta(\tau_1) \le \frac{3}{2\pi}\tau_2 \le \frac{3}{2\pi}\lambda_N \log_{(N+1)}\varrho_{u_N},$$

and therefore

$$N_E(\mathcal{M}^\bullet) \le C(\log_{(N+1)}\varrho_{u_N})(\mu_{u_N})^n,$$

and this implies that

$$\log N_E(\mathcal{M}^\bullet) = \varrho_{u_N}\Big(2 + \frac{1}{\varrho_{u_N}}(\log\mathcal{G}_N(\varrho_{u_N}) + \log(C\log_{(N+1)}\varrho_{u_N}))\Big),$$

and so

$$\log_{(2)} N_E(\mathcal{M}^\bullet) = \log\varrho_{u_N}(1 + O(1)) \quad \text{as } E \to 0.$$

But

$$\log\varrho_{u_N} = (\log E^{-1})\Big(\frac{1}{2} + O(1)\Big) \quad \text{as } E \to 0,$$

and, therefore,

$$\log_{(2)} N_E(\mathcal{M}^\bullet) = (\log E^{-1})(C + O(1)) \text{ as } E \to 0,$$

and this implies that

$$\log_{(3)} N_E(\mathcal{M}^\bullet) \le \log_{(2)} E^{-1} + O(1), \tag{4.35}$$

which is a better bound than (1.14).

To establish the lower bound in (1.14), we need to find $\varrho_1 = \varrho_1(E) < \varrho_2(E) = \varrho_2$ and μ_L such that $\varrho_0 < \varrho_1(E)$ for E small enough and

$$\frac{1}{\lambda_N^2 \mathcal{G}_N(\varrho)}\Big(\frac{1}{2}\mu e^{-2\varrho} + E\Big) \leq \frac{3}{4} \text{ for } \varrho \in [\varrho_1, \varrho_2], \quad \mu \leq \mu_L.$$

We pick ϱ_2 such that

$$\frac{E}{\lambda_N^2 \mathcal{G}_N(\varrho_2)} \leq \frac{1}{8},$$

and we can just take

$$\varrho_2 = \frac{\beta_N}{(\log \beta_N)(\log\log \beta_N)\dots(\log_{(N)} \beta_N)}, \quad \text{where } \beta_N = \sqrt{\frac{\lambda_N^2}{8E}},$$

and we pick ϱ_1 such that

$$\log_{(N-1)} \varrho_1 = (\log_{(N-1)} \varrho_2)^{1/M},$$

where M is large enough and chosen independently of μ and E. We only consider the values of μ such that

$$\mu \leq \frac{\lambda_N^2}{4}\mathcal{G}_N(\varrho_1)e^{2\varrho_1} = \mu_{L_N}.$$

Since $\tau = \lambda_N \log_{(N+1)} \varrho$, we can choose M, independently of E and μ, large enough such that

$$\begin{aligned}\theta(\varrho_2) - \theta(\varrho_1) &\geq \frac{\lambda_N}{4\pi}(\log_{(N+1)} \varrho_2 - \log_{(N+1)} \varrho_1) \\ &= \frac{\lambda_N}{4\pi}\log\Big(\frac{\log_{(N)} \varrho_2}{\log_{(N)} \varrho_1}\Big) = \frac{\lambda_N}{4\pi}\log(M) > 1\end{aligned}$$

In view of (4.12) and (4.23), this implies that

$$N_E(\mathcal{M}^\bullet) \geq C(\mu_{L_N})^n.$$

Then, we have

$$\log N_E(\mathcal{M}^\bullet) \geq 2n\varrho_1 + \log(C\mathcal{G}_N(\varrho_1)) = \varrho_1(2n + O(1)),$$

and so

$$\log_{(2)} N_E(\mathcal{M}^\bullet) = \log \varrho_1 + O(1),$$

and we deduce that

$$\log_{N+1} N_E(\mathcal{M}^\bullet) = \log_{(N)} \varrho_1 + O(1) = \frac{1}{M}\log_{(N)} \varrho_2 + O(1),$$

and so

$$\log_{N+2} N_E(\mathcal{M}^\bullet) = \log_{(N+1)} \varrho_2 + O(1).$$

On the other hand,

$$\log \varrho_2 = (\log E^{-1})\Big(\frac{1}{2} + O(1)\Big),$$

and so

$$\log_{(j)} \varrho_2 = \log_{(j)} E^{-1} + O(1), \quad j \geq 2,$$

This implies the lower bound in (1.14) and ends the proof of item (T.3) of Proposition 4.1, and together with equations (3.15), (3.5), and (1.16) it also ends the proof of Theorem 1.4. Notice that, as in the case $N = 2$, because of the choice of ϱ_1, we get a worse lower bound than (4.35).

A. The spectrum of $\mathcal{M}_j$

We will show that if $V_0(\varrho)$ and $V_1(\varrho)$ satisfy the assumptions of either one of the Theorems 1.1, 1.2, or 1.4, the operators $\mathcal{M}_j^\bullet$ defined in (4.10) have no eigenvalues in $[0,\infty)$, $j \in \mathbb{N}$, $\bullet = D, N$. In particular, this implies that the operator $\mathcal{M}^\bullet$ defined in (4.6) with boundary conditions $\bullet = D, N$. has no eigenvalues $E \geq 0$. If it did, then each $\mathcal{M}_j^\bullet$ would have the same eigenvalue. We prove the following.

Proposition A.1. *The operators $\mathcal{M}_j^\bullet$, $j \in \mathbb{N}$, $\bullet = D, N$, $j \in \mathbb{N}$, defined in (4.8) have no eigenvalues in* $[0,\infty)$.

Proof. If E is an eigenvalue of $\mathcal{M}_j^\bullet$, then there exists $\psi \in L^2((\varrho_0,\infty))$ such that

$$\psi''(\varrho) = \big(-E + V_0(e^{-\varrho}) + aV_1(e^{-\varrho}) + e^{-\varrho}\widetilde{\mathcal{X}}(\varrho)\big)\psi(\varrho),$$

with

$$\widetilde{\mathcal{X}}(\varrho) = \mathcal{X}(\varrho) + e^{-\varrho}q(\varrho)\zeta_j.$$

Now, we appeal to [22, Theorems 2.1 and 2.4 from Section 6.2], which we state in a single theorem.

Theorem A.2. *In a given finite or infinite interval (a_1, a_2), let $f(x)$ be a positive, twice continuously differentiable function, $g(\varrho)$ a continuous real or complex function, and*

$$F(\varrho) = \int [f^{-1/4}(f^{-1/4})'' - gf^{-1/2}]d\varrho.$$

Then, in this interval, the differential equations

$$u''(\varrho) = (f(\varrho) + g(\varrho))u(\varrho) \tag{A.1a}$$

and

$$w''(\varrho) = (-f(\varrho) + g(\varrho))w(\varrho), \tag{A.1b}$$

have twice continuously differentiable solutions which in the case (A.1a) *are given by*

$$u_1(\varrho) = f^{-1/4}(\varrho)e^{\int f^{1/2}d\varrho}(1 + \varepsilon_1(\varrho)),$$
$$u_2(x) = f^{-1/4}(\varrho)e^{-\int f^{1/2}d\varrho}(1 + \varepsilon_2(\varrho)),$$

and in the case (A.1b) *are given by*

$$w_1(\varrho) = f^{-1/4}(\varrho)e^{i\int f^{1/2}d\varrho}(1 + \varepsilon_1(\varrho)),$$
$$w_2(x) = f^{-1/4}(\varrho)e^{-i\int f^{1/2}d\varrho}(1 + \varepsilon_2(\varrho)),$$

such that the error terms $\varepsilon_j(\varrho)$, $j = 1, 2$, *satisfy*

$$|\varepsilon_1(\varrho)| \le e^{\mathcal{V}_{a_1,\varrho}(F)/2} - 1 \quad \textit{and} \quad |\varepsilon_2(\varrho)| \le e^{\mathcal{V}_{\varrho,a_2}(F)/2} - 1,$$
$$\frac{1}{2}f^{-1/2}(\varrho)|\varepsilon_1'(x)| \le e^{\mathcal{V}_{a_1,\varrho}(F)/2} - 1 \quad \textit{and} \quad \frac{1}{2}f^{-1/2}(\varrho)|\varepsilon_2'(x)| \le e^{\mathcal{V}_{\varrho,a_2}(F)/2} - 1,$$

provided $\mathcal{V}_{a_1,\varrho}(F) < \infty$. *Here,* $\mathcal{V}_{\alpha,\beta}(F)$ *denotes the total variation of* F *on the interval* (α, β).

We first show that one cannot have an eigenvalue $E > 0$. We will consider the case of Theorem 1.1, $V_0(e^{-\rho}) = c\rho^{-2+\delta}$ and $V_1(\rho) = \rho^{-2+\delta}(\log \rho)^{-\varepsilon}$, the other cases are very similar. We apply Theorem A.2 with

$$-f(\rho) = -E - c\rho^{-2+\delta} + c_1\rho^{-2+\delta}(\log \rho)^{-\varepsilon}$$

and

$$g(\rho) = e^{-\rho}\mathcal{V}(\rho)$$

Then, on the interval $[\rho_1, \infty)$, with ρ_1 large,

$$w_1(\rho) = f^{-1/4}e^{i\sqrt{f(\rho)}}(1 + \varepsilon_1(\rho)), \quad w_2(\rho) = E^{-1/4}e^{-i\sqrt{f(\rho)}}(1 + \varepsilon_2(\rho)).$$

But

$$(f^{-1/4})''f^{-1/4} = -\frac{c}{4}(3-\delta)(2-\delta)E^{-3/2}\rho^{-4+\delta}(1 + o(1))$$

and

$$|g(\rho)| \le Ce^{-\rho},$$

and so

$$\mathcal{V}_{\rho_1,\rho}(F)(\rho) = \int_{\rho_1}^{\rho} |F'(s)|ds \le \int_{\rho_1}^{\rho} (M_1e^{-s} + M_2E^{-3/2}s^{-4+\delta})ds$$
$$\le C_1(e^{-\rho_1} + e^{-\rho}) + C_2E^{-3/2}(\rho_1^{-3+\delta} + \rho^{-3+\delta}),$$

and so

$$\varepsilon_1(\rho) \leq e^{\mathcal{V}_{\rho_1,\rho}(F)(\rho)/2} - 1 \leq C((e^{-\rho_1} + e^{-\rho}) + E^{-3/2}(\rho_1^{-3+\delta} + \rho^{-3+\delta})), \quad \delta \geq 0,$$

and hence for ρ_1 large, $1 + \varepsilon_1(\rho) \sim c_1 + c_2\rho^{-3+\delta}$ and therefore $w_1 \notin L^2([\rho_1, \infty)$. A similar analysis works to estimate $\mathcal{E}_2$. Since there are constants C_1 and C_2 such that $\psi(E, \rho) = C_1 w_1(\rho) + C_2 w_2(\rho)$, it follows that $\psi \notin L^2([\rho_1, \infty))$ and so ψ cannot be an eigenfunction.

When $E = 0$, and $\delta > 0$, we apply the same argument with $-f = -c\rho^{-2+\delta}$ and we obtain

$$(f^{-1/4})'' f^{-1/4} = c_1 \rho^{-1-\delta/2},$$

and so we find that for ρ_1 large, $1 + \varepsilon_j(\rho) \sim c_1 + c_2\rho^{-\delta}$, and so there are no eigenfunctions with $E \geq 0$.

The last case does not quite apply when $\delta = 0$ and we use an argument as in the proof of Hardy's inequality in [6]. We will prove the following.

Lemma A.3. *Suppose $u \in L^2([\rho_0, \infty))$, $h(\rho)$ is continuous and $h(\rho) = o(1)$ as $\rho \to \infty$ and*

$$u''(\rho) = \rho^{-2}\Big(-\frac{1}{4} + h(\rho)\Big)u \quad \text{in } (\rho_0, \infty), \qquad c > 0, \tag{A.2}$$

then $u(\rho) = 0$ on $[\rho_0, \infty)$.

Proof of the lemma. Since $u \in L^2([\rho_0, \infty))$, by using equation (A.2) and the Cauchy–Schwarz inequality, we find that $|u'(\rho)| \leq C\rho^{-3/2}$ and hence $|u(\rho)| \leq C\rho^{-1/2}$, and the equation gives $|u''(\rho)| \leq C\rho^{-5/2}$. Therefore, if $\alpha \in (1, 2)$, $\lambda = (\alpha - 1)/2 > 0$ and $\rho_1 > \rho$,

$$\begin{aligned}\int_{\rho_1}^{\infty} \rho^{\alpha}(u'(\rho))^2 d\rho &= \int_{\rho_1}^{\infty} \rho^{\alpha}(\rho^{-\lambda}(\rho^{\lambda}u)' - \lambda\rho^{-1}u)^2 d\rho \\ &\geq \lambda^2 \int_{\rho_1}^{\infty} \rho^{\alpha-2}(u(\rho))^2 d\rho - \lambda \int_{\rho_1}^{\infty} ((\rho^{\lambda}u)^2)' d\rho\end{aligned}$$

and since $\lambda < 1/2$, we deduce that for $\alpha \in (1, 2)$,

$$\frac{(\alpha - 1)^2}{4} \int_{\rho_1}^{\infty} \rho^{\alpha-2}(u(\rho))^2 d\rho \leq \int_{\rho_1}^{\infty} \rho^{\alpha}(u'(\rho))^2 d\rho.$$

We apply the same argument to the second derivative, and use that $|u'(\rho)| \leq C\rho^{-3/2}$; then, for $\alpha \in (1, 4)$,

$$\frac{(\alpha-1)^2}{4}\int_{\rho_1}^{\infty} \rho^{\alpha-2}(u'(\rho))^2 d\rho \leq \int_{\rho_1}^{\infty} \rho^{\alpha}(u''(\rho))^2 d\rho.$$

We combine these two estimates and we obtain, for $\alpha \in (3, 4)$,

$$\frac{(\alpha-1)^2}{4}\frac{(\alpha-3)^2}{4}\int_{\rho_1}^{\infty} \rho^{\alpha-4}(u(\rho))^2 d\rho \leq \int_{\rho_1}^{\infty} \rho^{\alpha}(u''(\rho))^2 d\rho.$$

This equation implies that

$$\frac{(\alpha-1)^2}{4}\frac{(\alpha-3)^2}{4}\int_{\rho_1}^{\infty} \rho^{\alpha-4}(u(\rho))^2 d\rho \leq \int_{\rho_1}^{\infty} \Big(-\frac{1}{4}+h(\rho)\Big)^2 \rho^{\alpha-4}(u(\rho))^2\, d\rho.$$

Pick ρ_1 large and $\alpha = 4-\varepsilon$ with ε small and this implies that $u(\rho) = 0$ on $[\rho_1, \infty)$. Then, $u = 0$ on (ρ_0, ∞) by uniqueness. ∎

This ends the proof of Proposition A.1. ∎

B. Proof of Proposition 4.3

We follow the arguments used in the proof of [26, Theorem XIII.8]. We have already established that $\sigma_{\mathrm{ess}}(\mathcal{M}_j^{\bullet}) = [0, \infty)$, $\bullet = D, N$ and that there are no eigenvalues in the essential spectrum. Then, one needs to prove three lemmas.

Lemma B.1. *Let $V(\rho) \in C^{\infty}(I)$, $I \subset \mathbb{R}$ open, and let $E \in \mathbb{R}$. Let $u(\rho, E)$, not identically zero, satisfy*

$$u''(\rho, E) = (V(\rho) - E)u(\rho, E) \quad \text{on } I.$$

If $a_0 = a_0(E_0) \in I$ is such that $u(a_0, E_0) = 0$, then there exists $\delta > 0$ and a C^{∞} function $a(E)$ defined for $|E - E_0| < \delta$ such that $a(E_0) = a_0$ and $u(a(E), E) = 0$.

Proof. We know from the existence and uniqueness and stability theorems for ordinary differential equations that $u(\rho, E)$ is a C^{∞} function and since $u(\rho, E)$ is not identically zero, if $u(a_0, E_0) = 0$, then $\partial_{\rho}u(a_0, E_0) \neq 0$. The implicit function theorem then guarantees that there exists a C^{∞} function $a(E)$ defined on an interval $|E - E_0| < \delta$ such that $a(E_0) = a_0$ and $u(a(E), E) = 0$. ∎

Lemma B.2. *As above, let* $\bullet = D, N$. *Let* $\mathcal{M}_j$ *be the operators defined in* (4.10). *Let* V_0 *and* V_1 *satisfy the hypotheses of either Theorem* 1.1, 1.2, *or* 1.4. *The following statements about* $Z_j^\bullet(E)$ *hold true:*

(1) *if* $-E < 0$, *then* $Z_j^\bullet(E) < \infty$;

(2) *if* $Z_j^\bullet(E_0) \geq m$, *there exists* $\delta > 0$ *so that* $Z_j^\bullet(E) \geq m$ *for* $|E - E_0| < \delta$;

(3) $-E_0 < -E$, *then* $Z_j^\bullet(E) \geq Z_j^\bullet(E_0)$;

(4) *if* $-E_0$ *is an eigenvalue of* $\mathcal{M}_j^\bullet$, *and* $-E_0 < -E$, *then* $Z_j^\bullet(E) \geq Z_j^\bullet(E_0) + 1$;

(5) *if* $k > j$ *and* $\mu_k > \mu_j$, *then* $Z_j^\bullet(E) \geq Z_k^\bullet(E)$;

(6) *if* $k > j$, $-E$ *is an eigenvalue, and* $\mu_k > \mu_j$, *then* $Z_j^\bullet(E) \geq Z_k^\bullet(E) + 1$.

Proof. We have already shown that item (1) is true. Lemma B.1 says that if $\rho_1 < \rho_2 < \cdots < \rho_{m-1} < \rho_m \in (\rho_0, \infty)$ are such that $u_j^\bullet(\rho_j, E_0) = 0$, then there exist $\delta > 0$ and C^∞ functions $r_j(E)$ defined in $|E - E_0| < \delta$ such that $r_j(E_0) = \rho_j$ and that $u_j^\bullet(r_j(E), E) = 0$, and therefore $Z_j^\bullet(E) \geq m$.

To prove item (3), we first consider the Dirichlet problem. This is the standard form of the Sturm oscillation theorem. Let $\rho_0 < \rho_1 < \cdots < \rho_n$ be the zeros of $u_j^D(\rho, E_0)$. We claim that $u_j^D(\rho, E)$ has a zero in each of the intervals (ρ_j, ρ_{j+1}). To see that, suppose that $u_j^D(\rho, E)$ does not have a zero in this interval. By possibly multiplying the functions by -1, we may assume that $u_j^D(\rho, E) > 0$ and $u_j^D(\rho, E_0) > 0$ in (ρ_j, ρ_{j+1}). In this case, $u_j'(\rho_j, E_0) > 0$ and $u_j'(\rho_{j+1}, E_0) < 0$. Therefore,

$$\begin{aligned} I^D &= \int_{\rho_m}^{\rho_{m+1}} [(u_j^D)'(\rho, E_0) u_j^D(\rho, E) - u_j^D(\rho, E_0)(u_j^D)'(\rho, E)]' d\rho \\ &= (u_j^D)'(\rho_{m+1}, E_0) u_j^D(\rho_{m+1}, E) - (u_j^D)'(\rho_m, E_0) u_j^D(\rho_m, E) \leq 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} I^D &= \int_{\rho_m}^{\rho_{m+1}} [(u_j^D)''(\rho, E_0) u_j^D(\rho, E) - u_j^D(\rho, E_0)(u_j^D)''(\rho, E)] d\rho \\ &= (E_0 - E) \int_{\rho_m}^{\rho_{m+1}} u_j^D(\rho, E_0) u_j^D(\rho, E)\, d\rho > 0. \end{aligned}$$

If E_0 is an eigenvalue, we claim that $u_j^D(\rho, E)$ also has a zero in (ρ_n, ∞). To see that, we apply the same idea, but now one needs to justify the convergence of the integral from ρ_n to ∞. We appeal again to Theorem A.2. If we take $f = E_0$, and $f = E$, the solutions of (4.10) for ρ large are of the form

$$u_j^D(\rho, E_0) = E_0^{-1/4}\big(C_1 e^{-\rho\sqrt{E_0}}(1+\varepsilon_1(\rho)) + C_2 e^{\rho\sqrt{E_0}}(1+\varepsilon_2(\rho))\big),$$
$$u_j^D(\rho, E) = E^{-1/4}\big(\widetilde{C}_1 e^{-\rho\sqrt{E}}(1+\varepsilon_1(\rho)) + \widetilde{C}_2 e^{\rho\sqrt{E}}(1+\varepsilon_2(\rho))\big).$$

Since E_0 is an eigenvalue, $u_j^D(\rho, E_0) \in L^2((\rho_0, \infty))$ and $C_2 = 0$. Since $E_0 > E$, then integrals will involve terms of the type $e^{\rho(\sqrt{E}-\sqrt{E_0})}O(1)$, which will converge if $E_0 > E$.

As for the Neumann problem, the same argument applies with the exception of the interval (ρ_0, ρ_1). In this case, we know from the assumptions made in (4.10) that

$$(u_j^N)'(\rho_0, E_0) = (u_j^N)'(\rho_0, E) = 0, \quad u_j^N(\rho_0, E_0) = u_j^N(\rho_0, E) = 1,$$

and we also know that $u_j^N(\rho_1, E_0) = 0$. In this case, we would have $u_j^N(\rho, E) > 0$ and $u_j^N(\rho, E_0) > 0$ in (ρ_0, ρ_1), and so we would have

$$(u_j^N)'(\rho_1, E_0) \le 0 \quad \text{and} \quad u_j^N(\rho_1, E) \ge 0$$

and, therefore,

$$I^N = \int_{\rho_0}^{\rho_1} [(u_j^N)'(\rho, E_0)u_j^N(\rho, E) - u_j^N(\rho, E_0)(u_j^N)'(\rho, E)]' d\rho$$
$$= (u_j^N)'(\rho_1, E_0)u_j^N(\rho_1, E) \le 0.$$

As above,

$$I^N = (E_0 - E)\int_{\rho_0}^{\rho_1} u_j^N(\rho, E_0)u_j^N(\rho, E)\, d\rho > 0.$$

The same argument can be used to show that $Z_j^\bullet(E) \ge Z_k^\bullet(E)$, provided $j > k$ and $\mu_k > \mu_j$. In this case, we suppose that $\rho_1 < \rho_2 < \cdots < \rho_n$ are the zeros of $u_k^D(\rho, E)$ and we want to show that $u_j(\rho, D)$ has a zero in (ρ_m, ρ_{m+1}). We assume there are no zeros of $u_j(\rho, E)$ in (ρ_m, ρ_{m+1}) and we may assume that $u_k^D(\rho, E) > 0$ and $u_j^D(\rho, E) > 0$ on ρ_m, ρ_{m+1} and that

$$(u_k^D)'(\rho_m, E) > 0, \quad (u_k^D)'(\rho_{m+1}, E) < 0,$$

and

$$u_j^D(\rho_m, E) \ge 0, \qquad u_j^D(\rho_{m+1}, E) \ge 0.$$

Then,

$$I^D = \int_{\rho_m}^{\rho_{m+1}} [(u_k^D)'(\rho, E)u_j^D(\rho, E) - u_k^D(\rho, E)(u_j^D)'(\rho, E)]' d\rho$$
$$= (u_k^D)'(\rho_{m+1}, E)u_j^D(\rho_{m+1}, E) - (u_k^D)'(\rho_m, E)u_j^D(\rho_m, E) \leq 0.$$

On the other hand,

$$I^D = \int_{\rho_m}^{\rho_{m+1}} (\mu_k - \mu_j)e^{-2\rho}u_k^D(\rho, E)u_j^D(\rho, E)\, d\rho > 0.$$

The same argument works for the Neumann problem and to prove item (6). ■

Lemma B.3. *Let $-\lambda_{j,k}^\bullet$, $k = 1, 2, \ldots,$ denote the eigenvalues of $\mathcal{M}_j^\bullet$, $\bullet = D, N$. The following facts hold.*

(I) *The eigenvalues have multiplicity one.*

(II) *If $E \geq 0$, $m \in \mathbb{N}$ and $Z_j^\bullet(E) \geq m$, then $-\lambda_{j,m} < -E$. In particular,*

$$N_E(\mathcal{M}_j^\bullet) \geq Z_j^\bullet(E).$$

(III) $Z_j^\bullet(\lambda_{j,k}^\bullet) = k - 1.$

Proof. The eigenvalues are simple by the uniqueness theorem for ordinary differential equations. By dividing an eigenfunction $\psi^\bullet(\rho)$ by a constant, one may assume it will satisfy $\psi^D(\rho_0) = 0$ and $(\psi^D)'(\rho_0) = 1$ or $(\psi^N)'(\rho_0) = 0$ and $\psi^N(\rho_0) = 1$, and one cannot have two different solutions with the same Cauchy data.

We will show that there exist at least m eigenvalues $\lambda_{j,k}^\bullet$ which are less than $-E$. Let $u_j^\bullet(\rho, E)$ be the solution of (4.10) and let $\rho_1 < \rho_2 < \cdots < \rho_M$, $M \geq m$, and $\rho_0 < \rho_1$, denote its zeros (not equal to ρ_0 in case $\bullet = D$), and let

$$\psi_k^\bullet(\rho) = \begin{cases} u_j^\bullet(\rho, E) & \text{if } \rho_k \leq \rho \leq \rho_{k+1},\ k = 0, 1, \ldots, M-1, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $\langle \psi_i^\bullet, \psi_k^\bullet \rangle = 0$, if $i \neq k$. Let $\mathcal{U}$ be the M-dimensional subspace spanned by $\psi_k^\bullet$. If $\psi^\bullet = \sum_{j=1}^{m_0} a_k \psi_k^\bullet$, one can check that

$$\langle \mathcal{M}_j^\bullet \psi^\bullet, \psi^\bullet \rangle = -E\langle \psi^\bullet, \psi^\bullet \rangle;$$

it follows from the min-max principle, see [26, Theorem XIII.2], that $-\lambda_{j,M}^\bullet \leq -E$ and, in particular, $\lambda_{j,m}^\bullet \leq -E$. It follows from Lemma B.2 (2) that if $\varepsilon > 0$ is small enough, $Z_j^\bullet(E + \varepsilon) \geq m$, and we have shown that in fact $-\lambda_m(\mathcal{M}_j^\bullet) \leq -E - \varepsilon < -E$. This proves item (II).

We have $Z_j^\bullet(\lambda_{j,1}^\bullet) \geq 0$. Suppose that $Z_j^\bullet(\lambda_{j,k-1}^\bullet) \geq k-2$. It follows from Lemma B.2 (5) that $Z_j^\bullet(\lambda_{j,k}^\bullet) \geq Z_j^\bullet(\lambda_{j,k-1}^\bullet) + 1 \geq k-1$. On the other hand, notice that if $Z_j^\bullet(\lambda_{j,k}^\bullet) > k-1$, then by item (II), $-\lambda_{j,k} < -\lambda_{j,k}$. So, $Z_j^\bullet(-\lambda_{j,k}^\bullet) \leq k-1$. This proves item (III). ∎

Now, we can prove (4.11). We know from Lemma B.3 (II) that $N_E(\mathcal{M}_j^\bullet) \geq Z_j^\bullet(E)$. Since $Z_j^\bullet(E) < \infty$ if $-E < 0$, suppose that $N_E(\mathcal{M}_j^\bullet) > Z_j^\bullet(E) = m$; then, by definition, this implies that $-\lambda_{j,m+1}^\bullet \leq -E$, so Lemma B.2 (4) and Lemma B.3 (III) imply that

$$Z_j^\bullet(E) \geq Z_j^\bullet(\lambda_{j,m+1}^\bullet) + 1 = m + 1.$$

This proves (4.11).

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