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An inverse problem for the fractionally damped wave equation

Li Li and Yang Zhang

Abstract. We consider an inverse problem for a Westervelt type nonlinear wave equation with fractional damping. This equation arises in nonlinear acoustic imaging and we show the forward problem is locally well posed. We prove that the smooth coefficient of the nonlinearity can be uniquely determined, based on the knowledge of the source-to-solution map and a priori knowledge of the coefficient, in an arbitrarily small subset of the domain. Our approach relies on a second order linearization as well as the unique continuation property of the spectral fractional Laplacian.

1. Introduction

Ultrasound waves are widely used in medical imaging. The propagation of high-intensity ultrasound waves are modeled by nonlinear wave equations; see [22]. Nonlinear ultrasound waves play an important role in diagnostic and therapeutic medicine, for example, see [3, 13, 20]. On the other hand, damping effects naturally exist for wave equations in many fields of physics and engineering, for instance, see [1].

In this paper, we consider a nonlinear wave equation of Westervelt type with a damping term, given by

$$\partial_t^2(u - \kappa u^2) - \Delta u + Du = f.$$

Here we focus on the space-fractional damping $D = \partial_t(-\Delta)^s$, which models the case when the damping is frequency-dependent and obeys an empirical power law, see [6, 21, 27, 45, 46]. To define the fractional Laplacian on the bounded domain, we consider the spectral fractional Laplacian $(-\Delta)^s$ for $0 < s < 1$, i.e., the fractional power of the Dirichlet Laplacian $-\Delta = (-\Delta)_\Omega$ (the restriction of the Laplacian to the functions satisfying the homogeneous Dirichlet boundary condition on $\partial\Omega$). This spectral fractional Laplacian with Dirichlet boundary condition corresponds to the infinitesimal generator of the so-called subordinate stopped Brownian motion at the boundary, see [40]. Similarly, one can consider the spectral fractional Laplacian with

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Neumann boundary condition, which corresponds to the reflected Brownian motion at the boundary. For more details about its definition, see Section 2.1.

More explicitly, let Ω be a bounded domain with smooth boundary. Suppose W is an arbitrary nonempty and open subset of Ω, which is known. Suppose $\Omega \setminus W$ contains the region of interest that remains unknown. We consider the problem

$$\begin{aligned} \partial_t^2(u-\kappa(x,t)u^2) - \Delta u + \partial_t(-\Delta)^s u &= f, \quad (x,t) \in \Omega \times (0,T), \\ u &= 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(0) = \partial_t u(0) &= 0, \quad x \in \Omega, \end{aligned} \tag{1}$$

where u is the pressure field of the acoustic waves, f is a source supported in $W \times (0,T)$, and $\kappa \in C^\infty(\bar{\Omega} \times [0,T])$ is the coefficient of the nonlinearity.

We will prove that (1) is locally well posed at least for sufficiently regular and small f, see Section 3. Then we can define the source-to-solution map

$$L_{\kappa,W}\colon f \to u|_{W\times(0,T)}, \quad f \in C_c^\infty(W \times (0,T)). \tag{2}$$

The goal is to determine κ in the whole domain $\Omega \times (0,T)$ based on the knowledge of $L_{\kappa,W}$ and the a priori knowledge of κ in $W \times (0,T)$. The following theorem is our main result.

Theorem 1.1. *Let $\kappa_1, \kappa_2 \in C^\infty(\bar{\Omega} \times [0,T])$. Suppose we have $\kappa_1 = \kappa_2$ in $W \times (0,T)$. Then*

$$L_{\kappa_1,W} = L_{\kappa_2,W} \tag{3}$$

implies $\kappa_1 = \kappa_2$ in $\Omega \times (0,T)$.

We emphasise that we are able to determine the coefficient κ depending on both x and t, based on the knowledge of the source-to-solution map and a priori knowledge of κ, with an arbitrarily small choice of W. We will see that this is mainly due to the nonlocal features of the spectral fractional Laplacian. We remark that the assumption $\kappa_1 = \kappa_2$ in $W \times (0,T)$ in the statement is necessary since the value of κ in W cannot be determined from the equation in (1) and the information on f, u in W. The reason is that $(-\Delta)^s$ is nonlocal, so the value of $(-\Delta)^s u$ in W relies on the value of u outside W.

To the best of our knowledge, Theorem 1.1 is the first rigorous unique determination result for the Calderón type inverse problem in the setting of fractionally damped wave, and no such strong partial data unique determination results have been obtained for integer-order wave models in the existing literature.

We also remark that the theorem above can be extended to more general models, although we restrict ourselves to the Dirichlet Laplacian and a nonlinearity of power two in (1). In fact, the spectral and semigroup definitions of the fractional operator in Section 2.1 and the heat kernel estimate used in the proof of Proposition 4.1 work for general elliptic operators in divergence form. Hence, the main theorem still holds true if we replace the Dirichlet Laplacian by such operators in (1). In addition, we can consider a higher order power type nonlinearity instead of u^2 in (1). Once we show the well-posedness of the corresponding forward problem, we can derive an equation (similar to (27)) involving products of solutions of the linear equations based on the multiple-fold linearization technique. This will enable us to use the density result for linear equations (Runge approximation) to uniquely determine the variable coefficient.

1.1. Connection with earlier literature

The inverse problem of determining the nonlinear coefficient from the Dirichlet-to-Neumann map without damping is studied in [2] for the Westervelt equation and in [49] for a more general nonlinear model. In [14], the authors consider the reconstruction of the nonlinear coefficient using high frequency waves for the Westervelt equation. In [51], the recovery of both a general nonlinearity and a weakly damping term from the Dirichlet-to-Neumann map is studied. The main idea is to use multi-fold linearization and interaction of distorted plane waves. In this case, the nonlinearity helps to solve the inverse problem, as is first shown in [32]. Other damped or attenuated models have been studied in [17, 23–28].

The rigorous mathematical study of (Calderón type) inverse problems for space-fractional equations was initiated in [18] where the authors considered the exterior Dirichlet problem

$$((-\Delta)^s + q)u = 0 \quad \text{in } \Omega, \quad u|_{\Omega_e} = g,$$

where $(-\Delta)^s$ is the fractional Laplacian in $\mathbb{R}^n$ and $\Omega_e := \mathbb{R}^n \setminus \bar{\Omega}$. They defined the associated Dirichlet-to-Neumann map

$$\Lambda_q : g \to (-\Delta)^s u|_{\Omega_e}.$$

Rather than constructing complex geometrical optics solutions (which have been used for solving the classical Calderón problem), the authors exploited the nonlocal features of the fractional operator to uniquely determine the potential q in Ω from partial knowledge of Λ_q. We refer readers to [8, 9, 33, 34, 36–39] for further unique determination results for fractional operators based on the knowledge of the Dirichlet-to-Neumann map. In particular, the Calderón problem for linear and semilinear fractional wave equations have been studied in [29, 30].

Besides, there are also several unique determination results for fractional operators based on the knowledge of the source-to-solution map in the existing literature.

In [7, 16, 41], the authors considered equations involving spectral fractional operators on manifolds, and they determined the Riemannian manifold up to an isometry.

In this paper, we combine the elements in [16, 18] in the setting of fractional damping. Our source-to-solution map (2) can be viewed as an analogue of [16, (1.2)]. Our approach to proving the unique determination result is motivated by the framework established in [18]. We will see that nonlocal phenomenons will play a fundamental role in solving the inverse problem as expected.

1.2. Organization

The rest of this paper is organised in the following way. In Section 2, we will summarise the background knowledge. In Section 3, we will first show the well-posedness of a linear problem associated with (1) and obtain several regularity results. Then we will further use a fixed-point argument to show the well-posedness of (1) for small f. In Section 4, we will first prove the unique continuation property of the spectral fractional Laplacian and derive the related Runge approximation property. Then we will combine the unique continuation property and the Runge approximation property with a second order linearization technique to prove the main theorem.

2. Preliminaries

Throughout this paper we use the following notations.

- We fix the space dimension $n = 3$.
- We fix the fractional power $0 < s < 1$ and the length of the time interval $T > 0$.
- Ω denotes a bounded domain with smooth boundary.
- $c, C, C', C_1, \ldots$ denote positive constants (which may depend on some parameters).
- $\langle \cdot, \cdot \rangle$ denotes the standard L^2-distributional pairing.

2.1. Sobolev spaces and fractional operators

We use H^s to denote the standard $W^{s,2}$-type Sobolev space. Let U be an open set in $\mathbb{R}^n$. Let F be a closed set in $\mathbb{R}^n$. Then

$$H^s(U) := \{u|_U : u \in H^s(\mathbb{R}^n)\}, \quad H^s_F(\mathbb{R}^n) := \{u \in H^s(\mathbb{R}^n) : \operatorname{supp} u \subset F\},$$

$$\widetilde{H}^s(U) := \text{the closure of } C^\infty_c(U) \text{ in } H^s(\mathbb{R}^n).$$

Since Ω is a bounded domain with smooth boundary, we have the identification $\widetilde{H}^s(\Omega) = H^s_{\overline{\Omega}}(\mathbb{R}^n)$, and its dual space is $H^{-s}(\Omega)$.

The Dirichlet Laplacian $-\Delta$ is a non-negative self-adjoint operator in $\widetilde{H}^1(\Omega)$. Therefore, there exists an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions $\phi_k \in \widetilde{H}^1(\Omega)$ $(k = 1, 2, \ldots,)$ that correspond to the eigenvalues $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$.

The spectral fractional Laplacian mapping

$$\widetilde{H}^s(\Omega) = \left\{ u \in L^2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^s |\langle u, \phi_k \rangle|^2 < +\infty \right\}$$

into $H^{-s}(\Omega)$ is defined by

$$(-\Delta)^s u := \sum_{k=1}^{\infty} \lambda_k^s \langle u, \phi_k \rangle \phi_k \tag{4}$$

(see [5, Section 2.1] and [4, Section 3.1.3]). The spectral fractional Laplacian can be also equivalently defined via the semigroup approach (see [5, Lemma 2.2]). Let $U(x,t) = e^{-t(-\Delta)} u(x)$ be the solution of the parabolic problem

$$\begin{aligned} \partial_t U - \Delta U &= 0, \quad (x,t) \in \Omega \times (0,\infty), \\ U &= 0, \quad (x,t) \in \partial\Omega \times (0,\infty), \\ U|_{t=0} &= u, \quad x \in \Omega. \end{aligned}$$

Then for $u \in \widetilde{H}^s(\Omega)$,

$$(-\Delta)^s u = \frac{1}{\Gamma(-s)} \int_0^{\infty} (U - u) \frac{\mathrm{d}t}{t^{1+s}} \tag{5}$$

in $H^{-s}(\Omega)$, where $\Gamma(\cdot)$ is the standard Gamma function. More precisely, for $v \in \widetilde{H}^s(\Omega)$, we have

$$\langle (-\Delta)^s u, v \rangle = \frac{1}{\Gamma(-s)} \int_0^{\infty} (\langle U, v \rangle - \langle u, v \rangle) \frac{\mathrm{d}t}{t^{1+s}}.$$

We remark that the spectral fractional Laplacian defined here is different from the restriction of $(-\Delta_{\mathbb{R}^n})^s$ to Ω, although they enjoy several common properties (see [4, Section 2.1]).

2.2. Sets Z^m

To study the well-posedness of (1), we introduce the set $Z^m(R,T)$ consisting of u satisfying

$$u \in \bigcap_{k=0}^{m} H^{m-k}(0,T;H^k(\Omega)), \quad \|u\|_{Z^m}^2 = \sum_{k=0}^{m} \int_0^T \|\partial_t^{m-k} u(t)\|_{H^k}^2 \,\mathrm{d}t \le R^2$$

and $\partial_t^k u(0) = 0$ for $k \le m$.

The proof of [48, Claim 1] ensures that $Z^m(R,T)$ has the following property.

Proposition 2.1. *Suppose $u \in Z^m(R,T)$ for some $R > 0$ and $m \ge 5$. Then $\partial_t u \in Z^{m-1}(R,T)$ with $\|\partial_t u\|_{Z^{m-1}} \le \|u\|_{Z^m}$. Moreover, we have the following estimates.*

(1) *If $v \in Z^m(R',T)$, then $\|uv\|_{Z^m} \le C\|u\|_{Z^m}\|v\|_{Z^m}$.*

(2) *If $v \in Z^{m-1}(R',T)$, then $\|uv\|_{Z^{m-1}} \le C\|u\|_{Z^m}\|v\|_{Z^{m-1}}$.*

3. Forward problem

3.1. Linear equation

We first study the well-posedness of the linear problem

$$\begin{aligned} \partial_t^2 u - \Delta u + \partial_t(-\Delta)^s u &= f, \quad (x,t) \in \Omega \times (0,T), \\ u &= 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(0) = \partial_t u(0) &= 0, \quad x \in \Omega. \end{aligned} \tag{6}$$

Proposition 3.1. *For any $f \in L^2(0,T;L^2(\Omega))$, (6) has a unique solution u satisfying*

$$u \in H^2(0,T;H^{-1}(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;\widetilde{H}^1(\Omega))$$

and $\partial_t u \in L^2(0,T;\widetilde{H}^s(\Omega))$. Moreover, for $t \in [0,T]$, we have the estimate

$$\|\partial_t u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 + \int_0^t \|\partial_t(-\Delta)^{s/2} u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \le C \int_0^t \|f(\tau)\|_{L^2}^2 \,\mathrm{d}\tau, \tag{7}$$

where C is a positive constant independent of f.

Proof. We use the Galerkin method. For $l \in \mathbb{N}$, consider the approximate solution $u_l(t)$ of the form $\sum_{k=1}^{l} u_{l,k}(t)\phi_k$ satisfying

$$\langle \partial_t^2 u_l, v\rangle + \langle \nabla u_l, \nabla v\rangle + \langle \partial_t(-\Delta)^s u_l, v\rangle = \langle f, v\rangle \tag{8}$$

for any v in the space spanned by $\phi_1, \dots, \phi_l$ and the initial conditions $u_l(0) = \partial_t u_l(0) = 0$. (The standard theory for linear ODE systems ensures that C^2-function $u_{l,k}$ can be uniquely determined.)

By choosing $v = \partial_t u_l$, we have

$$\frac{1}{2}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t u_l(t)\|_{L^2}^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_l(t)\|_{L^2}^2\Big) + \|\partial_t(-\Delta)^{s/2}u_l(t)\|_{L^2}^2 = \langle f, \partial_t u_l\rangle$$

and

$$\frac{1}{2}(\|\partial_t u_l(t)\|_{L^2}^2 + \|\nabla u_l(t)\|_{L^2}^2) + \int_0^t \|\partial_t(-\Delta)^{s/2}u_l(\tau)\|_{L^2}^2 \,\mathrm{d}\tau = \int_0^t \langle f(\tau), \partial_t u_l(\tau)\rangle \,\mathrm{d}\tau.$$

Since the first eigenvalue the Dirichlet Laplacian is strictly positive, the definition of the spectral fractional Laplacian (4) ensures the Poincaré inequality

$$\|(-\Delta)^{s/2}v\|_{L^2}^2 \ge c\|v\|_{L^2}^2, \quad v \in \widetilde{H}^s(\Omega). \tag{9}$$

Using the inequality

$$\int_0^t \langle f(\tau), \partial_t u_l(\tau)\rangle \,\mathrm{d}\tau \le \frac{2}{c}\int_0^t \|f(\tau)\|_{L^2}^2 \,\mathrm{d}\tau + \frac{c}{2}\int_0^t \|\partial_t u_l(\tau)\|_{L^2}^2 \,\mathrm{d}\tau,$$

we obtain

$$\|\partial_t u_l(t)\|_{L^2}^2 + \|\nabla u_l(t)\|_{L^2}^2 + \int_0^t \|\partial_t(-\Delta)^{s/2}u_l(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \le \frac{4}{c}\int_0^t \|f(\tau)\|_{L^2}^2 \,\mathrm{d}\tau.$$

To derive an estimate for $\partial_t^2 u_l$, we estimate (8) for $v \in \widetilde{H}^1(\Omega)$ with $\|v\|_{H^1} \le 1$ and following the same idea as before. Thus, $\{u_l\}_{l=1}^\infty$ is bounded in

$$H^2(0,T;H^{-1}(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;\widetilde{H}^1(\Omega))$$

and $\{\partial_t u_l\}_{l=1}^\infty$ is bounded in $L^2(0,T;\widetilde{H}^s(\Omega))$. Next, by using the standard compactness argument, we can find a subsequence of $\{u_l\}_{l=1}^\infty$ weakly convergent to u satisfying (6) and (7). The uniqueness of the solution directly follows from the estimate (7). ∎

To study the well-posedness of (1) later, we also need to consider the following linear problem:

$$\begin{aligned} (1-2\kappa v)\partial_t^2 u - \Delta u + \partial_t(-\Delta)^s u = f, &\quad (x,t) \in \Omega \times (0,T), \\ u = 0, &\quad (x,t) \in \partial\Omega \times (0,T), \\ u(0) = \partial_t u(0) = 0, &\quad x \in \Omega. \end{aligned} \tag{10}$$

Proposition 3.2. *Suppose $m \geq 8$, $\kappa \in C^\infty(\bar{\Omega} \times [0,T])$ and $f \in Z^{m-1}(R,T)$. There exists $r_0 > 0$ depending on κ, m, Ω, such that for any $v \in Z^{m-1}(r_0,T)$, the linear problem* (10) *has a unique solution u satisfying*

$$u \in \bigcap_{k=0}^{m} H^{m-k}(0,T;H^k(\Omega)), \quad \|u\|_{Z^m} \leq C\|f\|_{Z^{m-1}},$$

where C is a positive constant depending on $r_0, \kappa, m, T, \Omega$.

We will prove this proposition using the Galerkin method as before but with more complicated energy estimates, following the idea in [10]. To prove Proposition 3.2, we need the following embedding property, which follows from the Sobolev embedding $H^k(\Omega) \hookrightarrow C^{k-2}(\bar{\Omega})$ (where we use the assumption $n = 3$) and [15, Theorem 2 in Section 5.9.2].

Proposition 3.3. *Suppose that l, k are positive integers and that $k \geq 2$. If $u \in H^l(0,T;H^k(\Omega))$. Then,*

$$u \in C^{l-1}([0,T];H^k(\Omega)) \quad \textit{and} \quad u \in C^{l-1}([0,T];C^{k-2}(\bar{\Omega}))$$

with the estimates

$$\sum_{j=0}^{l-1} \sup_{t\in[0,T]} \|\partial_t^j u(t)\|_{C^{k-2}(\bar{\Omega})} \leq C \sum_{j=0}^{l-1} \sup_{t\in[0,T]} \|\partial_t^j u(t)\|_{H^k(\Omega)}$$
$$\leq C'\|u\|_{H^l(0,T;H^k(\Omega))}.$$

Proof of Proposition 3.2. In the following, we write $a(t) = 1 - 2\kappa v$ and let C_1, C_2 be generic positive constants only depending on κ, m, Ω. With $\kappa \in C^\infty([0,T] \times \bar{\Omega})$, we choose r_0 small enough such that a satisfies

$$\begin{aligned} &\frac{1}{2} \leq a(t,x) \leq \frac{3}{2}, && \text{for any } (t,x) \in [0,T] \times \Omega, \\ &\sup_{t\in[0,T]} \sum_{k=1}^{m-3} \|\partial_t^k a(t)\|_{C^{m-3-k}(\bar{\Omega})} \leq C_1 r_0, && \text{for any } t \in [0,T]. \end{aligned} \tag{11}$$

As in the proof of Proposition 3.1, we consider the Galerkin approximation method and construct a sequence of approximate solutions $u_i(t)$ given by

$$u_i(t) = \sum_{k=1}^{i} u_{i,k}(t)\phi_k,$$

which satisfy

$$\langle a(t)\partial_t^{l+1}u_i, w\rangle + \Big\langle \sum_{j=2}^{l} \partial_t^{l+1-j} a(t)\partial_t^j u_i, w\Big\rangle - \langle \partial_t^{l-1}\Delta u_i, w\rangle + \langle \partial_t^l(-\Delta)^s u_i, w\rangle$$
$$= \langle \partial_t^{l-1} f(t,x), w\rangle \tag{12}$$

for any $t \in [0,T]$ and any w in the space spanned by $\phi_1,\dots,\phi_n$. Note that the initial conditions are $\partial_t^l u_i(0) = 0$ for $l \le m$, since we are given $u_i(0,x) = \partial_t u_i(0,x) = \partial_t^k f(0,x) = 0$ for $k \le m-1$. Here we differentiate the equation $l-1$ times with respect to t and note that when $l = 1$, we do not have the second term. There exists a unique solution $u_{i,k}(t)$ to the ODE obtained from the equation above. We derive energy estimates for u_i in the following.

Step 1. We set $w = \partial_t^l u_i$ in (12) and we integrate it with respect to t. We estimate each term below. From the first term, we have

$$\begin{aligned}
&\int_0^t \langle a(\tau)\partial_t^{l+1}u_i(\tau), \partial_t^l u_i(\tau)\rangle \, d\tau \\
&\quad = \frac{1}{2}\langle a(t)\partial_t^l u_i(t), \partial_t^l u_i(t)\rangle - \frac{1}{2}\int_0^t \langle \partial_t a(\tau)\partial_t^l u_i(\tau), \partial_t^l u_i(\tau)\rangle \, d\tau \\
&\quad \ge \frac{1}{4}\|\partial_t^l u_i(t)\|_{L^2}^2 - C_1 r_0 \int_0^t \|\partial_t^l u_i(\tau)\|_{L^2}^2 \, d\tau.
\end{aligned}$$

Next, we estimate

$$\begin{aligned}
-\int_0^t \langle \partial_t^{l-1}\Delta u_i(\tau), \partial_t^l u_i(\tau)\rangle \, d\tau &= \int_0^t \langle \partial_t^{l-1}\nabla u_i(\tau), \partial_t^l \nabla u_i(\tau)\rangle \, d\tau \\
&= \frac{1}{2}\|\partial_t^{l-1}\nabla u_i(t)\|_{L^2}^2, \\
\int_0^t \langle \partial_t^l(-\Delta)^s u_i(\tau), \partial_t^l u_i(\tau)\rangle \, d\tau &= \int_0^t \|\partial_t^l(-\Delta)^{s/2}u_i(\tau)\|_{L^2}^2 \, d\tau,
\end{aligned}$$

$$\int_0^t \langle \partial_t^{l-1} f(\tau), \partial_t^l u_i(\tau)\rangle \, d\tau \le \int_0^t \frac{1}{C_1 r_0}\|\partial_t^{l-1} f(\tau)\|_{L^2}^2 + C_1 r_0 \|\partial_t^l u_i(\tau)\|_{L^2}^2 \, d\tau.$$

To get an estimate for the second term in (12), we consider three different cases, when $l \le m-2$, $l = m-1$ and $l = m$. For the first case, we have $l+1-j \le m-3$ for any

$j \geq 2$, which implies$a \in C^{l+1-j}([0,T]; C(\overline{\Omega}))$ with (11) by Proposition 3.3. Then we have

$$\sum_{j=2}^{l} \int_0^t \langle \partial_t^{l+1-j} a(\tau) \partial_t^j u_i(\tau), \partial_t^l u_i(\tau) \rangle \, \mathrm{d}\tau$$
$$\leq C_1 r_0 \Big(l \int_0^t \|\partial_t^l u_i(\tau)\|_{L^2}^2 \, \mathrm{d}\tau + \sum_{j=2}^{l-1} \int_0^t \|\partial_t^j u_i(\tau)\|_{L^2}^2 \, \mathrm{d}\tau \Big), \tag{13}$$

for $l = 1, \ldots, m-2$. We summarise over l to have

$$\sum_{l=1}^{m-2} \frac{1}{4} \|\partial_t^l u_i(t)\|_{L^2}^2 + \frac{1}{2} \|\partial_t^{l-1} \nabla u_i\|_{L^2}^2 + \int_0^t \|\partial_t^l (-\Delta)^{s/2} u_i\|_{L^2}^2 \, \mathrm{d}\tau$$
$$\leq \frac{1}{C_1 r_0} \sum_{l=1}^{m-2} \int_0^t \|\partial_t^{l-1} f\|_{L^2}^2 \, \mathrm{d}\tau + 2m C_1 r_0 \sum_{l=1}^{m-2} \int_0^t \|\partial_t^l u_i(\tau)\|_{L^2}^2 \, \mathrm{d}\tau.$$

By choosing r_0 small enough to satisfy the Poincaré inequality (9), i.e.,

$$r_0 = \frac{c}{4mC_1},$$

we have

$$\sum_{l=1}^{m-2} \|\partial_t^l u(t)\|_{L^2}^2 + \|\partial_t^{l-1} \nabla u_i\|_{L^2}^2 + \int_0^t \|\partial_t^l (-\Delta)^{s/2} u_i\|_{L^2}^2 \, \mathrm{d}\tau \leq C_2 \|f\|_{Z^{m-1}}.$$

In particular, this implies that

$$\sum_{l=1}^{m-2} \int_0^t \|\partial_t^l u_i\|_{L^2}^2 \, \mathrm{d}\tau + \int_0^t \|\partial_t^{l-1} \nabla u_i\|_{L^2}^2 \, \mathrm{d}\tau \leq C_2 T \|f\|_{Z^{m-1}}. \tag{14}$$

When $l = m-1$ and $l = m$, we need different inequalities instead of (13) (since we cannot control the L^∞-bound of $\partial_t^{m-2} a$ and $\partial_t^{m-1} a$) using estimates for $\partial_t^2 u_i$, $\partial_t^3 u_i$ in $H^2(\Omega)$. We will deal with these two cases in Step 3.

Step 2. We would like to derive higher-order regularity estimates at least for $l \leq m-3$. More explicitly, for $l = 1, \ldots, m-2$, we rewrite (12) as

$$\langle \partial_t^{l-1} (-\Delta) u_i, w \rangle + \langle \partial_t^l (-\Delta)^s u, w \rangle$$
$$= -\langle a(t) \partial_t^{l+1} u, w \rangle - \sum_{j=2}^{l} \langle \partial_t^{l+1-j} a(t) \partial_t^j u, w \rangle + \langle \partial_t^{l-1} f(t,x), w \rangle,$$

where we set $w = \partial_t^{l-1}(-\Delta)^k u_i$, for non-negative integer k satisfying $k + l \le m - 2$. It follows that $a \in C^{l-1}([0,T]; C^k(\bar{\Omega}))$ with (11). Following the same idea as before, we can prove that

$$\int_0^t \|\partial_t^{l-1} u_i(\tau)\|_{H^{k+1}}^2 \,\mathrm{d}\tau \le C \int_0^t \|\partial_t^{l+1} u_i(\tau)\|_{H^{k-1}}^2 + \sum_{j=2}^{l} \|\partial_t^j u_i(\tau)\|_{H^k}^2 + \|\partial_t^{l-1} f(\tau)\|_{H^k}^2 \,\mathrm{d}\tau,$$

for $l = 1, \ldots, m-2$ and each k satisfying $k \le m - l - 2$. Note that in (14) we have the estimates for $\|\partial_t^l u_i\|_{L^2}$ and $\|\partial_t^{l-1} u_i\|_{H^1}$ when $l = 1, \ldots, m-2$. Setting $k = 1$, we have

$$\int_0^t \|\partial_t^{l-1} u_i(\tau)\|_{H^2}^2 \,\mathrm{d}\tau \le C_2 T \|f\|_{Z^{m-1}}, \quad \text{for } l = 1, \ldots, m-3.$$

In particular, with $m \ge 8$, this estimate implies $\partial_t^{j+1} u_i \in L^2(0,T; H^2(\Omega))$ and therefore $\partial_t^j u_i \in H^1(0,T; H^2(\Omega))$, for $j = 2, 3$. By Proposition 3.3, we have

$$\partial_t^j u_i \in C([0,T]; C(\bar{\Omega})) \quad \text{with } \|\partial_t^j u_i\|_{C([0,T];C(\bar{\Omega}))}^2 \le C_2 T \|f\|_{Z^{m-1}}, \; j = 2, 3. \tag{15}$$

Further, we can use an inductive procedure to show

$$\sum_{l=1}^{m-1-k} \int_0^t \|\partial_t^{l-1} u_i(\tau)\|_{H^k}^2 \,\mathrm{d}\tau \le C_2 T \|f\|_{Z^{m-1}}, \quad \text{for } k = 0, \ldots, m-3, \tag{16}$$

following the same idea as before.

Step 3. We would like to finish Step 1 and consider (12) for $l > m - 2$. When $l = m - 1$, we write the left-hand side of (13) as

$$\begin{aligned}
&\sum_{j=3}^{m-1} \int_0^t \langle \partial_t^{m-j} a(\tau) \partial_t^j u_i(\tau), \partial_t^{m-1} u_i(\tau) \rangle \,\mathrm{d}\tau + \int_0^t \langle \partial_t^{m-2} a(\tau) \partial_t^2 u_i(\tau), \partial_t^{m-1} u_i(\tau) \rangle \,\mathrm{d}\tau \\
&\le C_1 r_0 \sum_{j=3}^{m-2} \int_0^t \|\partial_t^j u_i(\tau)\|_{L^2}^2 \,\mathrm{d}\tau + (m-2) \int_0^t \|\partial_t^{m-1} u_i(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \\
&\quad + \frac{1}{C_1 r_0} \|\partial_t^2 u_i\|_{C([0,T];C(\bar{\Omega}))}^2 \int_0^t \|\partial_t^{m-2} a(\tau)\|_{L^2}^2 \,\mathrm{d}\tau + C_1 r_0 \int_0^t \|\partial_t^{m-1} u_i(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \\
&\le C_2 \|f\|_{Z^{m-1}},
\end{aligned}$$

where we use (15) and the inequality

$$\|\partial_t^{m-2} a\|_{L^2(0,T;L^2(\Omega))} \le C_2 \|v\|_{Z^{m-1}} \le C_2 r_0.$$

When $l = m$, we similarly estimate the left-hand side of inequality (13) based on the L^∞-boundedness of $\partial_t^3 u_i$. This proves a complete version of (14), i.e.,

$$\sum_{l=0}^{m} \int_0^t \|\partial_t^l u_i\|_{L^2}^2 \,\mathrm{d}\tau + \int_0^t \|\partial_t^{l-1} \nabla u_i\|_{L^2}^2 \,\mathrm{d}\tau \le C_2 T \|f\|_{Z^{m-1}}. \tag{17}$$

Step 4. From (17), we conclude that $\{u_i\}_{n=1}^{\infty}$ is bounded in $H^m(0,T;L^2(\Omega))$ and also in $H^{m-1}(0,T;\widetilde{H}^1(\Omega))$, with the desired estimates. Using the standard compactness argument, we can extract a subsequence which converges weakly to the solution

$$u \in H^m(0,T;L^2(\Omega)) \cap H^{m-1}(0,T;\widetilde{H}^1(\Omega)).$$

At last, we would like to show such u is in $H^k([0,T];H^{m-k}(\Omega))$ for $k = 0,\dots,m$ by an inductive procedure. Following the same proof of (17) and passing to limits as $i \to +\infty$, we know that this statement holds true for $k = m$ and $k = m-1$. Then we prove by induction. The key point is to combine the estimate for a in (11) with the regularity of u in (16) to derive an estimate for $\partial^\alpha(a(t)\partial_t^{l+1} u_i(t))$ and $\sum_{j=2}^{l} \partial^\alpha(\partial_t^{l+1-j} a(t) \partial_t^j u_i(t))$, where α is an multi-index with $|\alpha| \le k$ and $l \le k$. We follow the same idea as before and conclude that

$$\int_0^t \|\partial_t^l u(\tau)\|_{H^{m-k}}^2 \,\mathrm{d}\tau \le C_2 T \|f\|_{Z^{m-1}}, \quad \text{for } 0 \le l \le k \le m. \qquad \blacksquare$$

3.2. Nonlinear equation

Based on Proposition 3.2, we can use a fixed-point argument to show the well-posedness of (1) for small f.

Proposition 3.4. *Suppose $m \ge 8$, $\kappa \in C^\infty(\overline{\Omega} \times [0,T])$ and $f \in Z^{m-1}(\rho,T)$. Then, for sufficiently small $\rho > 0$, the nonlinear problem (1) has a unique solution u satisfying*

$$u \in \bigcap_{k=0}^{m} H^{m-k}(0,T;H^k(\Omega)), \quad \|u\|_{Z^m} \le C\|f\|_{Z^{m-1}},$$

where C is a positive constant independent of f.

Proof. We consider the linearised problem

$$\begin{aligned}
&(1-2\kappa v)\partial_t^2 u - \Delta u + \partial_t(-\Delta)^s u \\
&\quad = f + 2\kappa(\partial_t v)^2 + 4(\partial_t\kappa)v\partial_t v + (\partial_t^2\kappa)v^2 \quad \text{in } \Omega\times(0,T),\\
&\qquad\qquad u = 0, \quad (x,t)\in\partial\Omega\times(0,T),\\
&\qquad\qquad u(0) = \partial_t u(0) = 0, \quad x\in\Omega.
\end{aligned}$$

For given $f \in Z^{m-1}(\rho, T)$, we consider the map

$$J : v \to u, \quad v \in Z^m(r,T).$$

The parameters ρ, r are chosen in the following to ensure J is a contraction map on $Z^m(r,T)$. First, we choose r_0 to satisfy Proposition 3.2. For any $v \in Z^m(r,T)$ with $r < r_0$, we have

$$\begin{aligned}
\|u\|_{Z^m} &\le C\|f + 2\kappa(\partial_t v)^2 + 4(\partial_t\kappa)v\partial_t v + (\partial_t^2\kappa)v^2\|_{Z^{m-1}}\\
&\le C'(\|f\|_{Z^{m-1}} + \|v\|_{Z^m}^2) \le C'(\rho + r^2),
\end{aligned} \tag{18}$$

where the second inequality comes from Proposition 2.1. Then we choose

$$r < \min\Big\{r_0, \frac{1}{(2C')}\Big\}$$

and $\rho = r/(2C')$ to ensure that J maps $Z^m(r,T)$ into itself.

Next, we introduce a weaker metric

$$d(\omega_1,\omega_2) = \sup_{s\in[0,T]} \|\omega_1(s) - \omega_2(s)\|_{H^1}^2 + \|\partial_t(\omega_1(s) - \omega_2(s))\|_{L^2}^2.$$

By [11] and [44, Theorem 2.2.2], the set Z^m equipped with d is a complete metric space. We would like to prove that J is a contraction with respect to d. Let $u_j = Jv_j$ with $v_j \in Z^m(r,T)$ for $j = 1, 2$. We write $w := u_2 - u_1$, which satisfies

$$\begin{aligned}
&(1-2\kappa v_1)\partial_t^2 w - \Delta w + \partial_t(-\Delta)^s w\\
&\quad = 2\kappa(\partial_t v_1 + \partial_t v_2)(\partial_t v_2 - \partial_t v_1) + 2\kappa(v_2 - v_1)\partial_t^2 u_2\\
&\qquad + 4(\partial_t\kappa)(v_2 - v_1)\partial_t v_1 + 4(\partial_t\kappa)v_2\partial_t(v_2 - v_1) + (\partial_t^2\kappa)(v_1+v_2)(v_1 - v_2).
\end{aligned}$$

We denote the right-hand side by I and by Proposition 2.1 we have

$$\sup_{s\in[0,T]} \|I(s)\|_{L^2} \le C(\|v_1\|_{Z^m} + \|v_2\|_{Z^m} + \|u_2\|_{Z^m})d(v_1,v_2).$$

Recall r is small enough such that $v_j \in Z^m(r,T)$ implies $u_j \in Z^m(r,T)$. By Step 1 in the proof of Proposition 3.2, we have

$$d(u_2,u_1) \le C'' r d(v_1, v_2).$$

Hence, J is a contraction with respect to d when r is sufficiently small. In this case, there exists a unique solution $\widetilde{u}$ in $Z^m(r,T)$ to the nonlinear problem (1), as the fixed point of J. Note for sufficiently small r, we have $C'\|v\|_{Z^m}^2 < \|v\|_{Z^m}/2 = \|\widetilde{u}\|_{Z^m}/2$ in (18). This implies $\|\widetilde{u}\|_{Z^m} \leq C\|f\|_{Z^{m-1}}$. ∎

4. Inverse problem

4.1. Unique continuation property

The following proposition is an analogue of the unique continuation property of the fractional Laplacian in $\mathbb{R}^n$, which was first established in [18] based on the Carleman estimates in [42]. Here we will exploit the semigroup definition of the spectral fractional Laplacian (5) and the unique continuation property of the classical parabolic operator in the proof. This idea was also used for proving [16, Theorem 1.1] and [19, Proposition 3.2].

Proposition 4.1. *Let $u \in \widetilde{H}^s(\Omega)$. Suppose*

$$(-\Delta)^s u = u = 0$$

in W. Then $u = 0$ in Ω.

Proof. Based on the semigroup definition (5), the assumption implies

$$\int_0^\infty \frac{U(x,t)}{t^{1+s}}\, dt = 0, \quad x \in W,$$

where

$$U(x,t) := e^{-t(-\Delta)}u(x) = \int_\Omega p_t(x,y)u(y)\,\mathrm{d}y$$

and $p_t(x,y)$ is the heat kernel associated with the Dirichlet Laplacian. The integral here (and all integrals below) should be interpreted in the distributional sense, i.e.,

$$\int_0^\infty \frac{\langle U(t),\phi\rangle}{t^{1+s}}\, dt = 0, \quad \phi \in C_c^\infty(W).$$

Based on the heat kernel estimate (see [12, Corollary 3.2.8]),

$$p_t(x,y) \leq Ct^{-n/2}e^{-|x-y|^2/(ct)}, \quad x, y \in \Omega,\ t > 0.$$

Now, we fix a nonempty $W' \subset\subset W$. Let $c' = \mathrm{dist}(W', \Omega \setminus W)$. Then we have

$$p_t(x,y) \leq Ct^{-n/2}e^{-(c')^2/(ct)} \tag{19}$$

for $x \in W'$, $y \in \Omega \setminus W$. For $m \in \mathbb{N}$, we will inductively show that

$$\int_0^\infty \frac{U(x,t)}{t^{m+s}}\,dt = 0, \quad x \in W'. \tag{20}$$

In fact, once we have shown the case m, we apply $-\Delta$ to (20). Since U solves the heat equation, we have

$$\int_0^\infty \frac{\partial_t U(x,t)}{t^{m+s}}\,dt = \int_0^\infty \frac{\Delta U(x,t)}{t^{m+s}}\,dt = 0, \quad x \in W'. \tag{21}$$

Note that for $\phi \in C_c^\infty(W')$, by (19) we have

$$\begin{aligned}\langle U(t), \phi\rangle &= \int_{W'}\int_{\Omega} p_t(x,y)u(y)\phi(x)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{W'}\int_{\Omega\setminus W} p_t(x,y)u(y)\phi(x)\,\mathrm{d}y\,\mathrm{d}x \le C' t^{-n/2} e^{-(c')^2/(ct)},\end{aligned} \tag{22}$$

so $\langle U(t), \phi\rangle / t^{m+s}$ vanishes at both 0 and $+\infty$. Hence, we can integrate by parts to derive

$$\int_0^\infty \frac{U(x,t)}{t^{m+1+s}}\,dt = 0, \quad x \in W',$$

from (21). Hence, we have verified (20).

Now, we consider the substitution $\lambda = 1/t$ and define

$$V(x,\lambda) := 1_{(0,\infty)}(\lambda)\frac{U(x,1/\lambda)}{\lambda^{-s}}.$$

Then (20) becomes

$$\int_{\mathbb{R}} V(x,\lambda)\lambda^{m-1}\,\mathrm{d}\lambda = 0, \quad x \in W',\ m \in \mathbb{N}. \tag{23}$$

Note that for each $\phi \in C_c^\infty(W')$, the function

$$\int_{\mathbb{R}} \langle V(\lambda), \phi\rangle e^{-i\xi\lambda}\,\mathrm{d}\lambda$$

is holomorphic for $\operatorname{Im}\xi < (c')^2/c$ based on (22), and (23) implies all its derivatives at $\xi = 0$ are zeros.

Hence, we conclude that the Fourier transform of $\langle V(\lambda), \phi\rangle$ is zero for $\xi \in \mathbb{R}$, so $U(x, 1/\lambda) = U(x,t) = 0$ in $W' \times (0,\infty)$. By the unique continuation property of the classical parabolic operator (see [50]), we conclude that $U = 0$ in $\Omega \times (0,\infty)$ and thus $u = 0$ in Ω. ∎

4.2. Runge approximation property

We prove a Runge approximation property based on the unique continuation property of the spectral fractional Laplacian and the well-posedness of (6) (and its dual problem). The following proposition can be viewed as a variant of the Runge approximation properties for evolutionary fractional operators established in [35, 38, 43].

Proposition 4.2. *The set*

$$S := \{u_f|_{(0,T)\times(\Omega\setminus W)} : f \in C_c^\infty(W \times (0,T))\}$$

is dense in $L^2(0,T; L^2(\Omega \setminus W))$. *Here* u_f *is the solution of* (6) *corresponding to the source* f.

Proof. By the Hahn–Banach theorem, it suffices to prove the following statement.

Let $g \in L^2(0,T; L^2(\Omega \setminus W))$. If

$$\int_0^T \int_{\Omega\setminus W} ug = 0$$

for all $u \in S$, then $g = 0$.

We consider $\tilde{g} \in L^2(0,T; L^2(\Omega))$ which is the zero extension of g, and the dual problem

$$\begin{aligned} \partial_t^2 v - \Delta v - \partial_t(-\Delta)^s v &= \tilde{g}, \quad \Omega \times (0,T) \\ v &= 0, \quad \partial\Omega \times (0,T), \\ v(T) = \partial_t v(T) &= 0, \Omega. \end{aligned} \tag{24}$$

Proposition 3.1 ensures that this problem has a unique solution v satisfying

$$v \in H^2(0,T; H^{-1}(\Omega)) \cap W^{1,\infty}(0,T; L^2(\Omega)) \cap L^\infty(0,T; \widetilde{H}^1(\Omega))$$

and $\partial_t v \in L^2(0,T; \widetilde{H}^s(\Omega))$. The assumption implies

$$0 = \int_0^T \langle \partial_t^2 v - \Delta v - (-\Delta)^s v, u\rangle \, \mathrm{d}t \tag{25}$$

for $u \in S$. Based on the initial and final conditions, we integrate by parts to obtain

$$-\int_0^T \langle \partial_t(-\Delta)^s v, u\rangle \, \mathrm{d}t = \int_0^T \langle \partial_t(-\Delta)^s u, v\rangle \, \mathrm{d}t, \quad \int_0^T \langle \partial_t^2 v, u\rangle \, \mathrm{d}t = \int_0^T \langle \partial_t^2 u, v\rangle \, \mathrm{d}t.$$

Hence, (25) implies

$$0 = \int_0^T \int_\Omega f v \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_W f v \, \mathrm{d}x \, \mathrm{d}t$$

for $f \in C_c^\infty(W \times (0,T))$ since u is the solution of (6). Hence, $v = 0$ in $W \times (0,T)$. Note that

$$\partial_t^2 v - \Delta v - \partial_t(-\Delta)^s v = 0$$

in $W \times (0,T)$ since v is the solution of (24), so $\partial_t(-\Delta)^s v = 0$ in $W \times (0,T)$. By Proposition 4.1, we have $\partial_t v = 0$ in $\Omega \times (0,T)$. We further conclude that $v = 0$ in $\Omega \times (0,T)$ based on the final conditions and thus $g = 0$. ∎

4.3. Proof of the main theorem

We are ready to prove Theorem 1.1. Our proof will heavily rely on the unique continuation property (Proposition 4.1) of the spectral fractional Laplacian and the Runge approximation property (Proposition 4.2) associated with (6), which are typical nonlocal phenomenons. To relate the nonlinear problem (1) to the linear problem (6), we will perform a second order linearization. We remark that this kind of multiple-fold linearizations have been widely applied in solving inverse problems for nonlinear equations (see for instance, [31, 32, 47]).

Proof. For $f_1, f_2 \in C_c^\infty(W \times (0,T))$, we use $u^{(j)}_{\varepsilon_1,\varepsilon_2}$ to denote the solution of

$$\begin{aligned} \partial_t^2(u - \kappa_j(x,t)u^2) - \Delta u + \partial_t(-\Delta)^s u & \\ = \varepsilon_1 f_1 + \varepsilon_2 f_2, &\quad (x,t) \in \Omega \times (0,T), \\ u = 0, &\quad (x,t) \in \partial\Omega \times (0,T), \\ u(0) = \partial_t u(0) = 0, &\quad x \in \Omega \end{aligned} \tag{26}$$

$(j = 1,2)$ for small $\varepsilon_1, \varepsilon_2$. Then

$$\frac{\partial}{\partial \varepsilon_j}\Big|_{\varepsilon_1=\varepsilon_2=0} u^{(1)}_{\varepsilon_1,\varepsilon_2} = \frac{\partial}{\partial \varepsilon_j}\Big|_{\varepsilon_1=\varepsilon_2=0} u^{(2)}_{\varepsilon_1,\varepsilon_2} =: w_j$$

is the solution of

$$\begin{aligned} \partial_t^2 w - \Delta w + \partial_t(-\Delta)^s w = f_j, &\quad (x,t) \in \Omega \times (0,T), \\ w = 0, &\quad (x,t) \in \partial\Omega \times (0,T), \\ w(0) = \partial_t w(0) = 0, &\quad x \in \Omega. \end{aligned} \tag{27}$$

Let

$$v^{(j)} = \frac{\partial^2}{\partial \varepsilon_1 \partial \varepsilon_2}\Big|_{\varepsilon_1=\varepsilon_2=0} u^{(j)}_{\varepsilon_1,\varepsilon_2}.$$

Then, we have

$$\begin{aligned} \partial_t^2 v^{(j)} - \Delta v^{(j)} - 2\partial_t^2(\kappa_j(x,t) w_1 w_2) + \partial_t(-\Delta)^s v^{(j)} = 0, &\quad (x,t) \in \Omega \times (0,T), \\ v^{(j)} = 0, &\quad (x,t) \in \partial\Omega \times (0,T), \\ v^{(j)}(0) = \partial_t v^{(j)}(0) = 0, &\quad x \in \Omega. \end{aligned} \tag{28}$$

Assumption (3) implies

$$u^{(1)}_{\varepsilon_1,\varepsilon_2} - u^{(2)}_{\varepsilon_1,\varepsilon_2} = 0$$

in $W \times (0,T)$. Then, the assumption $\kappa_1 = \kappa_2$ in $W \times (0,T)$ implies

$$\partial_t^2(u^{(1)}_{\varepsilon_1,\varepsilon_2} - \kappa_1(x,t)(u^{(1)}_{\varepsilon_1,\varepsilon_2})^2) - \Delta u^{(1)}_{\varepsilon_1,\varepsilon_2} = \partial_t^2(u^{(2)}_{\varepsilon_1,\varepsilon_2} - \kappa_2(x,t)(u^{(2)}_{\varepsilon_1,\varepsilon_2})^2) - \Delta u^{(2)}_{\varepsilon_1,\varepsilon_2}$$

in $W \times (0,T)$. By the equation in (26), we have

$$(-\Delta)^s \partial_t (u^{(1)}_{\varepsilon_1,\varepsilon_2} - u^{(2)}_{\varepsilon_1,\varepsilon_2}) = 0$$

in $W \times (0,T)$. By Proposition 4.1, we conclude that $\partial_t u^{(1)}_{\varepsilon_1,\varepsilon_2} = \partial_t u^{(2)}_{\varepsilon_1,\varepsilon_2}$ in $\Omega \times (0,T)$. Then the initial conditions imply $u^{(1)}_{\varepsilon_1,\varepsilon_2} = u^{(2)}_{\varepsilon_1,\varepsilon_2}$ in $\Omega \times (0,T)$ and thus $v^{(1)} = v^{(2)}$ in $\Omega \times (0,T)$.

Now, by the equation in (28) we have

$$\partial_t^2((\kappa_1(x,t) - \kappa_2(x,t)) w_1 w_2) = 0 \tag{29}$$

in $\Omega \times (0,T)$. We choose $\phi \in C^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\phi \subset \overline{\Omega}$ and $\phi > 0$ in Ω. Let $\tilde{\phi}(x,t) := (t-T)^2\phi(x)$. Then $\tilde{\phi}(T) = \partial\tilde{\phi}(T) = 0$. Let (29) act on $\tilde{\phi}$. Based on the initial and final conditions, we can integrate by parts to obtain

$$\int_0^T \int_{\Omega\setminus W} w_1(x,t) w_2(x,t)(\kappa_1(x,t) - \kappa_2(x,t))\phi(x)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

By Proposition 4.2, we can choose $f_1, f_2 \in C_c^\infty(W \times (0,T))$ such that

$$w_1 \to 1, \quad w_2 \to \kappa_1(x,t) - \kappa_2(x,t)$$

in $L^2(0,T;L^2(\Omega\setminus W))$. Then, we take the limit to obtain

$$\int_0^T \int_{\Omega\setminus W} (\kappa_1(x,t)-\kappa_2(x,t))^2\phi(x)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Hence, we conclude that $\kappa_1(x,t)=\kappa_2(x,t)$ in $(\Omega\setminus W)\times(0,T)$ and thus in $\Omega\times(0,T)$. ∎

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Li Li
Yau Mathematical Sciences Center, Tsinghua University, Shuangqing Road 30, 100084 Beijing, P. R. China; lili19940301@mail.tsinghua.edu.cn

Yang Zhang
Department of Mathematics, University of California, Irvine, 340 Rowland Hall, Irvine, CA 92697, USA; yangz79@uci.edu