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A generalization of the dimension and radius of a subcategory of modules and its applications

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Abstract – Let R be a commutative noetherian local ring and denote by mod R the category of finitely generated R-modules. In this paper, we give some evaluations of the singular locus of R and annihilators of Tor and Ext from a viewpoint of the finiteness of dimensions/radii of full subcategories of mod R. As an application, we recover a theorem of Dey and Takahashi when R is Cohen–Macaulay. Moreover, we obtain the divergence of the dimensions of specific full subcategories of mod R in the non-Cohen–Macaulay case.

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1. Introduction

Let R be a commutative noetherian local ring. Denote by mod R the category of finitely generated R-modules. The notions of the dimension and radius of a full subcategory of mod R were introduced by Dao and Takahashi [6, 7]. These notions are abelian analogs of the dimension of a triangulated category introduced by Rouquier [14], and it has turned out that they can connect several notions appearing in commutative algebra and representation theory.

The concept of a resolving subcategory was introduced by Auslander and Bridger [2] in the 1960s and has been studied widely and deeply so far; see [1, 3, 12, 17–19] for instance. Moreover, the finiteness of dimension/radius of a resolving subcategory of mod R also has been studied; see [6, 7, 13, 15] for example. Roughly speaking, the

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dimension/radius of a full subcategory of mod R is the least number of extensions to build up the subcategory from one object, up to finite direct sums, direct summands, and syzygies. (The precise definition will be recalled in Definition 2.5.) It is closely related to several well-studied notions, including the Rouquier dimension of the stable category of maximal Cohen–Macaulay modules and the (finite or countable) Cohen–Macaulay representation type.

For full subcategories $\mathcal{X}$, $\mathcal{Y}$ of mod R, we denote by $\operatorname{Ann}\operatorname{Tor}(\mathcal{X},\mathcal{Y})$ (respectively, $\operatorname{Ann}\operatorname{Ext}(\mathcal{X},\mathcal{Y})$) the intersection of the ideals $\operatorname{ann}_R \operatorname{Tor}_i^R(X,Y)$ (respectively, $\operatorname{ann}_R \operatorname{Ext}_R^i(X,Y)$) of all positive integers i and objects $X \in \mathcal{X}$, $Y \in \mathcal{Y}$. When R is Cohen–Macaulay, the relationships among $\mathrm{CM}_i(R)$, $\operatorname{Ann}\operatorname{Tor}(\mathcal{X},\mathcal{Y})$, $\operatorname{Ann}\operatorname{Ext}(\mathcal{X},\mathcal{Y})$, and the singular locus of R have been studied by Dey and Takahashi [9]. In this paper we shall unify the notions of dimension and radius. To be more precise, we introduce the *radius of* $\mathcal{X}$ *in* $\mathcal{Y}$, denoted $\operatorname{radius}(\mathcal{X},\mathcal{Y})$, which measures how many extensions are necessary to build $\mathcal{X}$ up in $\mathcal{Y}$ up to finite direct sums, direct summands, and syzygies. This concept generalizes the radius and dimension of $\mathcal{X}$ in the following way; see Remark 2.6:

$$\dim \mathcal{X} = \operatorname{radius}(\mathcal{X},\mathcal{X}) \quad \text{and} \quad \operatorname{radius}\mathcal{X} = \operatorname{radius}(\mathcal{X}, \operatorname{mod} R).$$

The main result of this paper is the following theorem, which extends a theorem of Dey and Takahashi [9] to the non-Cohen–Macaulay case. (For more details, see Corollary 3.13.)

Theorem 1.1 (Theorem 3.8). *Let R be a noetherian local ring with residue field k and $\mathcal{X}$, $\mathcal{Y}$ be full subcategories of* mod R *with* $\operatorname{radius}(\mathcal{X},\mathcal{Y}) < \infty$. *Assume either that $\mathcal{X}$ is closed under extensions and contains $\Omega_R^n k$ for some $n \geq 0$, or that $\mathcal{X}$ contains the nth syzygy category of finite length modules for some $n \geq 0$. Then both of the closed subsets* $\mathrm{V}(\operatorname{Ann}\operatorname{Tor}(\mathcal{X},\mathcal{Y}))$ *and* $\mathrm{V}(\operatorname{Ann}\operatorname{Ext}(\mathcal{X},\mathcal{Y}))$ *of* Spec R *contain the singular locus of R and are contained in the nonfree locus of $\mathcal{Y}$.*

Denote by $\mathrm{C}(R)$ the full subcategory of mod R consisting of R-modules M such that the inequality $\operatorname{depth} M_{\mathfrak{p}} \geq \operatorname{depth} R_{\mathfrak{p}}$ holds for all prime ideals $\mathfrak{p}$ of R. This resolving subcategory was introduced by Takahashi [20], and it coincides with the full subcategory $\mathrm{CM}(R)$ of maximal Cohen–Macaulay R-modules when R is Cohen–Macaulay. As an application of the above theorem, we relate the finiteness of dimensions/radii of specific resolving subcategories to the dimensions of the closed subsets defined by the annihilators of Tor and Ext, and singular locus of R.

Corollary 1.2 (Corollary 3.11). *Let R be a noetherian local ring and n a nonnegative integer. Assume that* $\operatorname{radius}(\mathrm{C}_0(R), \operatorname{mod}_n R) < \infty$. *Then the dimensions of* *the closed subsets* $\mathrm{V}(\mathrm{Ann}\,\mathrm{Tor}(\mathrm{C}_n(R), \mathrm{C}_n(R)))$*,* $\mathrm{V}(\mathrm{Ann}\,\mathrm{Ext}(\mathrm{C}_n(R), \mathrm{C}_n(R)))$*, and the singular locus of* R *are less than or equal to* n*.*

Here, $\mathrm{mod}_n\, R$ is the full subcategory of $\mathrm{mod}\, R$ consisting of modules M such that the dimension of nonfree locus of M is less than or equal to n, and $\mathrm{C}_n(R) = \mathrm{C}(R) \cap \mathrm{mod}_n\, R$. As a consequence of the above corollary, we obtain the divergence of the dimension of $\mathrm{C}_i(R)$ for $i = 0, \dots, \dim R - 1$ when R is non-Cohen–Macaulay. The following result is included in Corollary 3.12.

Corollary 1.3. *Let* R *be a* d*-dimensional non-Cohen–Macaulay local ring. Then the dimension of* $\mathrm{C}_i(R)$ *is infinite for all* $0 \le i \le d - 1$*.*

The organization of this paper is as follows. In Section 2 we state basic definitions and their properties for later use. In Section 3 we give the proofs of the theorem and corollaries stated above.

Convention. Throughout this paper, all rings are commutative noetherian local rings, all modules are finitely generated, and all subcategories are full and strict, that is, closed under isomorphism. Let R be a (commutative noetherian local) ring with maximal ideal $\mathfrak{m}$ and residue field k.

2. Preliminaries

In this section, we recall various definitions and basic properties needed later (see [6, 7] for instance). We begin with the notion of additive closures and syzygies of modules.

Definition 2.1. Let $\mathcal{X}$ be a subcategory of $\mathrm{mod}\, R$ and M an R-module.

(1) We denote by $\mathrm{add}_R\, \mathcal{X}$ the *additive closure of* $\mathcal{X}$, namely, the subcategory of $\mathrm{mod}\, R$ consisting of direct summands of finite direct sums of modules in $\mathcal{X}$. When $\mathcal{X}$ consists of a single module X, we simply denote $\mathrm{add}_R\{X\}$ by $\mathrm{add}_R\, X$. Note that $\mathrm{add}_R\, R$ consists of (finitely generated) free R-modules.

(2) Take a minimal free resolution $\cdots \xrightarrow{\delta_{n+1}} F_n \xrightarrow{\delta_n} F_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} M \to 0$ of M. Then for each $n \ge 0$, the image of δ_n is called the *nth syzygy of* M and denoted by $\Omega_R^n M$.

(3) For a nonnegative integer n, we denote by $\Omega_R^n \mathcal{X}$ the subcategory of $\mathrm{mod}\, R$ consisting of R-modules M such that there exists an exact sequence of the form $0 \to M \to F_{n-1} \to \cdots \to F_0 \to X \to 0$, where $X \in \mathcal{X}$ and $F_i \in \mathrm{add}_R\, R$ for all $i \ge 0$. We call it the *nth syzygy category* of $\mathcal{X}$.

Remark 2.2. Let R be a noetherian local ring.

(1) Let $\mathcal{X}$, $\mathcal{Y}$ be subcategories of $\operatorname{mod} R$ and n a nonnegative integer. If one has $\mathcal{X} \subseteq \mathcal{Y}$, then one has $\operatorname{add}_R \mathcal{X} \subseteq \operatorname{add}_R \mathcal{Y}$, and $\Omega_R^n \mathcal{X} \subseteq \Omega_R^n \mathcal{Y}$.

(2) Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} R$ and n a nonnegative integer. Then one has

$$\Omega_R^n \mathcal{X} = \begin{cases} \mathcal{X} & \text{if } n = 0, \\ \{\Omega_R^n X \oplus R^{\oplus m} \mid X \in \mathcal{X}, m \geq 0\} & \text{if } n > 0. \end{cases}$$

In particular, if $\mathcal{X}$ contains 0 and $n > 0$, then $\Omega_R^n \mathcal{X}$ contains free R-modules. Note that both $\Omega_R^n \{X\}$ and $\Omega_R^n X$ are not equal for any R-modules X in general.

Now we recall the definitions of a resolving subcategory and the resolving closure of a subcategory of $\operatorname{mod} R$.

Definition 2.3. Let R be a noetherian local ring.

(1) A subcategory $\mathcal{X}$ of $\operatorname{mod} R$ is *resolving* if $\mathcal{X}$ contains R and is closed under direct summands, extensions, and kernels of epimorphisms in $\operatorname{mod} R$. The last two closure properties say that for any exact sequence $0 \to L \to M \to N \to 0$ of R-modules with $N \in \mathcal{X}$, we have the equivalence $L \in \mathcal{X} \Leftrightarrow M \in \mathcal{X}$.

(2) For a subcategory $\mathcal{X}$ of $\operatorname{mod} R$, we denote by $\operatorname{res} \mathcal{X}$ the *resolving closure* of $\mathcal{X}$, namely, the smallest resolving subcategory of $\operatorname{mod} R$ containing $\mathcal{X}$. When $\mathcal{X}$ consists of a single module X, we simply denote $\operatorname{res}\{X\}$ by $\operatorname{res} X$.

Remark 2.4. Let $\mathcal{X}$ be a resolving subcategory of $\operatorname{mod} R$. Then $\mathcal{X}$ contains (finitely generated) free R-modules and closed under syzygies.

Next we state the definition of the radius of a subcategory in another subcategory. This is a common generalization of the notions of the dimension and radius of a subcategory introduced by Dao and Takahashi [6, 7].

Definition 2.5. Let $\mathcal{X}$, $\mathcal{Y}$ be subcategories of $\operatorname{mod} R$ and M an R-module.

(1) We denote by $[\mathcal{X}]$ the additive closure of the subcategory consisting of R and all modules of the form $\Omega_R^i X$, where $i \geq 0$ and $X \in \mathcal{X}$. When $\mathcal{X}$ consists of a single module X, we simply denote $[\{X\}]$ by $[X]$.

(2) We denote by $\mathcal{X} \circ \mathcal{Y}$ the subcategory of $\operatorname{mod} R$ consisting of R-modules Z such that there exists an exact sequence $0 \to X \to Z \to Y \to 0$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We set $\mathcal{X} \bullet \mathcal{Y} = [[\mathcal{X}] \circ [\mathcal{Y}]]$.

(3) We inductively define the *ball of radius r centered at $\mathcal{X}$* as follows:

$$[\mathcal{X}]_r = \begin{cases} [\mathcal{X}] & \text{if } r = 1, \\ [\mathcal{X}]_{r-1} \bullet [\mathcal{X}] & \text{if } r > 1. \end{cases}$$

For convention, we set $[\mathcal{X}]_0 = 0$. If $\mathcal{X}$ consists of a single module G, then we simply denote $[\{G\}]_r$ by $[G]_r$ and call it the *ball of radius r* centered at G. Note that the operator "$\bullet$" is associative; see [6, Proposition 2.2].

(4) We define the *radius* of $\mathcal{X}$ in $\mathcal{Y}$, denoted by $\operatorname{radius}(\mathcal{X}, \mathcal{Y})$, as the infimum of the nonnegative integers n such that there exists a ball of radius $n+1$ centered at a module containing $\mathcal{X}$ and contained in $\mathcal{Y}$, that is,

$$\operatorname{radius}(\mathcal{X}, \mathcal{Y}) = \inf\{n \geq 0 \mid \mathcal{X} \subseteq [G]_{n+1} \subseteq \mathcal{Y} \text{ for some } G \in \operatorname{mod} R\}.$$

Remark 2.6. Let $\mathcal{X}$, $\mathcal{Y}$ be subcategories of $\operatorname{mod} R$.

(1) There is an ascending chain

$$0 \subseteq [\mathcal{X}] \subseteq [\mathcal{X}]_2 \subseteq \cdots \subseteq [\mathcal{X}]_n \subseteq \cdots \subseteq \operatorname{res} \mathcal{X}$$

of subcategories of $\operatorname{mod} R$, and the equality $\bigcup_{i \geq 0} [\mathcal{X}]_i = \operatorname{res} \mathcal{X}$ holds.

(2) The notions of dimensions and radii of subcategories introduced by Dao and Takahashi [6,7] are expressed as follows:

- $\dim \mathcal{X} = \operatorname{radius}(\mathcal{X}, \mathcal{X}) = \inf\{n \geq 0 \mid X = [G]_{n+1} \text{ for some } G \in \mathcal{X}\}$.
- $\operatorname{radius} \mathcal{X} = \operatorname{radius}(\mathcal{X}, \operatorname{mod} R) = \inf\{n \geq 0 \mid \mathcal{X} \subseteq [G]_{n+1} \text{ for some } G \in \operatorname{mod} R\}$.

(3) Let $\mathcal{X}'$ be another subcategory of $\operatorname{mod} R$. If one has $\mathcal{X} \subseteq \mathcal{X}'$, then one has $\operatorname{radius}(\mathcal{X}, \mathcal{Y}) \leq \operatorname{radius}(\mathcal{X}', \mathcal{Y})$.

(4) Let $\mathcal{Y}'$ be another subcategory of $\operatorname{mod} R$. If one has $\mathcal{Y} \subseteq \mathcal{Y}'$, then one has $\operatorname{radius}(\mathcal{X}, \mathcal{Y}) \geq \operatorname{radius}(\mathcal{X}, \mathcal{Y}')$.

(5) Let $\mathcal{X}$, $\mathcal{Y}$ be subcategories of $\operatorname{mod} R$. If $\mathcal{Y}$ is resolving, then we have

$$\operatorname{radius}(\mathcal{X}, \mathcal{Y}) = \inf\{n \geq 0 \mid \mathcal{X} \subseteq [G]_{n+1} \text{ for some } G \in \mathcal{Y}\}.$$

Recall that a subset W of $\operatorname{Spec} R$ is *specialization-closed* if $\mathfrak{p} \subseteq \mathfrak{q}$ are prime ideals of R and W contains $\mathfrak{p}$, then $\mathfrak{q}$ belongs to W.

Definition 2.7. Let $(R, \mathfrak{m})$ be a noetherian local ring.

(1) Let Φ be a subset of $\operatorname{Spec} R$. The *dimension* of Φ, denoted by $\dim \Phi$, is defined as the supremum of nonnegative integers n such that there exists a chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ of prime ideals in Φ.

(2) We denote by $\mathrm{Sing}(R)$ the *singular locus* of R, namely, the set of prime ideals $\mathfrak{p}$ of R such that $R_{\mathfrak{p}}$ is not regular. We say that R is an *isolated singularity* if $\mathrm{Sing}(R) \subseteq \{\mathfrak{m}\}$, or equivalently, if $\mathrm{Sing}(R)$ has dimension at most zero. Note that $\mathrm{Sing}(R)$ is a specialization-closed subset of $\operatorname{Spec} R$.

(3) Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} R$ and M an R-module. Denote by $\mathrm{NF}(M)$ the *nonfree locus* of M, namely, the set of prime ideals $\mathfrak{p}$ of R such that $M_{\mathfrak{p}}$ is not a free $R_{\mathfrak{p}}$-module. In addition, the *nonfree locus* of $\mathcal{X}$ is defined by $\mathrm{NF}(\mathcal{X}) = \bigcup_{X \in \mathcal{X}} \mathrm{NF}(X)$.

Remark 2.8. We make several remarks concering the above definitions.

(1) A subset W of $\operatorname{Spec} R$ is specialization-closed if and only if W is a union of closed subsets of $\operatorname{Spec} R$ in the Zariski topology.

(2) If W is a specialization-closed subset of $\operatorname{Spec} R$, then one has $\dim W = \sup\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in W\}$.

(3) Let $\mathcal{X} \subseteq \mathcal{Y}$ be subcategories of $\operatorname{mod} R$. Then one has $\mathrm{NF}(\mathcal{X}) \subseteq \mathrm{NF}(\mathcal{Y})$.

(4) Let M be an R-module. Then one has $\mathrm{NF}(M) = \operatorname{Supp} \operatorname{Ext}^1_R(M, \Omega_R M)$ by [16, Proposition 2.10]. Hence $\mathrm{NF}(M)$ is a closed subset of $\operatorname{Spec} R$.

(5) Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} R$. Then for any positive integer n, the equalities

$$\mathrm{NF}(\mathcal{X}) = \mathrm{NF}([\mathcal{X}]_n) = \mathrm{NF}(\operatorname{res} \mathcal{X})$$

hold. This follows from [16, Corollary 3.6].

(6) Let n be a nonnegative integer and W a specialization-closed subset of $\operatorname{Spec} R$. Let W_n be the set of prime ideals $\mathfrak{p}$ of R such that $\dim R/\mathfrak{p} \leq n$. Then one has $W \subseteq W_n \Leftrightarrow \dim W \leq n$.

We close this section by defining some specific subcategories of $\operatorname{mod} R$, and state basic properties of them. Notation is based on [5, 20].

Definition 2.9. Let n be a nonnegative integer.

(1) Denote by $\operatorname{mod}_n R$ the subcategory of $\operatorname{mod} R$ consisting of R-modules M such that the inequality $\dim \mathrm{NF}(M) \leq n$ holds.

(2) Denote by $\mathrm{Deep}(R)$ the subcategory of $\operatorname{mod} R$ consisting of R-modules M such that the inequality $\operatorname{depth} M \geq \operatorname{depth} R$ holds. We set $\mathrm{Deep}_n(R) = \mathrm{Deep}(R) \cap \operatorname{mod}_n R$.

(3) Denote by $\mathrm{C}(R)$ the subcategory of $\operatorname{mod} R$ consisting of R-modules such that the inequality $\operatorname{depth} M_{\mathfrak{p}} \geq \operatorname{depth} R_{\mathfrak{p}}$ holds for all prime ideals $\mathfrak{p}$ of R. We set $\mathrm{C}_n(R) = \mathrm{C}(R) \cap \operatorname{mod}_n R$.

Note that we set $\operatorname{depth} 0 = \infty$ from the viewpoint of [4, Definition 1.2.6].

REMARK 2.10. Let R be a d-dimensional local ring.

(1) If R is Cohen–Macaulay, then the subcategories $\operatorname{Deep}(R)$ and $\mathrm{C}(R)$ coincide with the category $\operatorname{CM}(R)$ of maximal Cohen–Macaulay R-modules.

(2) All the subcategories $\operatorname{mod}_n R$, $\operatorname{Deep}(R)$, $\operatorname{Deep}_n(R)$, $\mathrm{C}(R)$, $\mathrm{C}_n(R)$ are resolving subcategories of $\operatorname{mod} R$.

(3) Since the modules in $\operatorname{mod}_0 R$ are locally free on the punctured spectrum of R, the equality $\operatorname{Deep}_0(R) = \mathrm{C}_0(R)$ holds. The relationships among subcategories defined above are as follows:

$$\begin{array}{ccccccccc}
\operatorname{mod}_0 R & \subseteq & \operatorname{mod}_1 R & \subseteq \cdots \subseteq & \operatorname{mod}_d R & = & \operatorname{mod} R \\
\text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} \\
\operatorname{Deep}_0(R) & \subseteq & \operatorname{Deep}_1(R) & \subseteq \cdots \subseteq & \operatorname{Deep}_d(R) & = & \operatorname{Deep}(R) \\
\| & & \text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} \\
\mathrm{C}_0(R) & \subseteq & \mathrm{C}_1(R) & \subseteq \cdots \subseteq & \mathrm{C}_d(R) & = & \mathrm{C}(R).
\end{array}$$

3. Proof of main theorem

The main purpose of this section is to state and prove Theorem 3.8. First of all, we introduce some notation for convenience.

DEFINITION 3.1. Let $\mathcal{X}$, $\mathcal{Y}$ be subcategories of $\operatorname{mod} R$. We define $\operatorname{Ann}\operatorname{Tor}(\mathcal{X}, \mathcal{Y})$ and $\operatorname{Ann}\operatorname{Ext}(\mathcal{X}, \mathcal{Y})$ by

$$\operatorname{Ann}\operatorname{Tor}(\mathcal{X}, \mathcal{Y}) = \left\{a \in R \;\middle|\; a \operatorname{Tor}_i^R(X, Y) = 0 \text{ for all } X \in \mathcal{X},\, Y \in \mathcal{Y}, \text{ and } i \geq 1\right\},$$
$$\operatorname{Ann}\operatorname{Ext}(\mathcal{X}, \mathcal{Y}) = \left\{a \in R \;\middle|\; a \operatorname{Ext}_R^i(X, Y) = 0 \text{ for all } X \in \mathcal{X},\, Y \in \mathcal{Y}, \text{ and } i \geq 1\right\}.$$

When $\mathcal{X}$ (respectively, $\mathcal{Y}$) consists of a single module X (respectively, Y), we simply denote them by $\operatorname{Ann}\operatorname{Tor}(X, \mathcal{Y})$ and $\operatorname{Ann}\operatorname{Ext}(X, \mathcal{Y})$ (respectively, $\operatorname{Ann}\operatorname{Tor}(\mathcal{X}, Y)$ and $\operatorname{Ann}\operatorname{Ext}(\mathcal{X}, Y)$).

REMARK 3.2. Let $\mathcal{X} \subseteq \mathcal{X}'$ and $\mathcal{Y} \subseteq \mathcal{Y}'$ be subcategories of $\operatorname{mod} R$. Then the following inclusion relations hold:

- $\mathrm{V}(\operatorname{Ann}\operatorname{Tor}(\mathcal{X}, \mathcal{Y})) \subseteq \mathrm{V}(\operatorname{Ann}\operatorname{Tor}(\mathcal{X}', \mathcal{Y}'))$.
- $\mathrm{V}(\operatorname{Ann}\operatorname{Ext}(\mathcal{X}, \mathcal{Y})) \subseteq \mathrm{V}(\operatorname{Ann}\operatorname{Ext}(\mathcal{X}', \mathcal{Y}'))$.

We need the following lemma which is proved by Dao and Takahashi. Denote by $\operatorname{fl} R$ the subcategory of $\operatorname{mod} R$ consisting of modules of finite length.

Lemma 3.3 ([7, Corollary 4.4]). *Let R be a d-dimensional local ring and $a \in R$, $n \geq 0$. Then the following hold:*

(1) *Suppose that $a\operatorname{Tor}_i^R(\mathrm{fl}\,R, \mathrm{fl}\,R) = 0$ for all $n - 2d \leq i \leq n$. Then one has*

$$a^{2^{2d}} \operatorname{Tor}_n^R(\mathrm{mod}\,R, \mathrm{mod}\,R) = 0.$$

(2) *Suppose that $a\operatorname{Ext}_R^i(\mathrm{fl}\,R, \mathrm{fl}\,R) = 0$ for all $n \leq i \leq n + 2d$. Then one has*

$$a^{2^{2d}} \operatorname{Ext}_R^n(\mathrm{mod}\,R, \mathrm{mod}\,R) = 0.$$

The lemma below is shown similarly to the proof of [7, Proposition 4.5].

Lemma 3.4. *Let X, Y be R-modules and F a minimal free resolution of X. For any R-module M and nonnegative integer i, we set $\mathfrak{a}_M^i = \operatorname{ann}_R \operatorname{Ext}_R^i(Y, M)$. Then for all $i, j \geq 0$, one has*

$$\mathfrak{a}_X^i \supseteq \mathfrak{a}_{\Omega_R^j X}^{i+j} \mathfrak{a}_{F_0}^i \mathfrak{a}_{F_1}^{i+1} \cdots \mathfrak{a}_{F_{j-1}}^{i+j-1}.$$

Proof. Fix $l \geq 0$. Since F is a minimal free resolution of X, there exists an exact sequence

$$0 \to \Omega_R^{l+1} X \to F_l \to \Omega_R^l X \to 0$$

of R-modules. Applying the functor $\operatorname{Hom}_R(Y, -)$, one has an exact sequence

$$\operatorname{Ext}_R^i(Y, F_l) \to \operatorname{Ext}_R^i(Y, \Omega_R^l X) \to \operatorname{Ext}_R^{i+1}(Y, \Omega_R^{l+1} X).$$

This yields an inclusion $\mathfrak{a}_{\Omega_R^l X}^i \supseteq \mathfrak{a}_{\Omega_R^{l+1} X}^{i+1} \mathfrak{a}_{F_l}^i$ of ideals. Take $l = 0, \dots, j-1$; then we have

$$\mathfrak{a}_X^i \supseteq \mathfrak{a}_{\Omega_R^j X}^{i+j} \mathfrak{a}_{F_0}^i \mathfrak{a}_{F_1}^{i+1} \cdots \mathfrak{a}_{F_{j-1}}^{i+j-1}. \qquad \blacksquare$$

The next proposition says that the annihilations of Tor and Ext are controlled by the annihilations of Tor and Ext on the syzygies of modules of finite length. The restriction to the case where R is Cohen–Macaulay recovers [7, Proposition 4.5].

Proposition 3.5. *Let R be a d-dimensional local ring and n, m be nonnegative integers.*

(1) *If $a \in \operatorname{Ann}\operatorname{Tor}(\Omega_R^m(\mathrm{fl}\,R), \Omega_R^n(\mathrm{fl}\,R))$, then one has $a^{2^{2d}} \operatorname{Tor}_i^R(\mathrm{mod}\,R, \mathrm{mod}\,R) = 0$ for all $i > m + n + 2d$.*

(2) *If $a \in \operatorname{Ann}\operatorname{Ext}(\Omega_R^m(\mathrm{fl}\,R), \Omega_R^n(\mathrm{fl}\,R))$, then one has $a^{2^{2d}(n+1)} \operatorname{Ext}_R^i(\mathrm{mod}\,R, \mathrm{mod}\,R) = 0$ for all $i > m$.*

PROOF. We prove the assertions (1) and (2) in order.

(1) Fix $i > m + n + 2d$. Then for all $i - 2d \le l \le i$, we have $a \operatorname{Tor}_l^R(\operatorname{fl} R, \operatorname{fl} R) = a \operatorname{Tor}_{l-(m+n)}^R(\Omega_R^m(\operatorname{fl} R), \Omega_R^n(\operatorname{fl} R)) = 0$. Hence, one has $a^{2^{2d}} \operatorname{Tor}_i^R(\operatorname{mod} R, \operatorname{mod} R) = 0$ by Lemma 3.3.

(2) Let M, N be R-modules of finite length. Take a minimal free resolution F of N. By Lemma 3.4, we have the inclusion

$$\begin{aligned}(3.1)\qquad & \operatorname{ann}_R \operatorname{Ext}_R^l(\Omega_R^m M, N) \\ & \supseteq \operatorname{ann}_R \operatorname{Ext}_R^{l+n}(\Omega_R^m M, \Omega_R^n N) \prod_{k=0}^{n-1} \operatorname{ann}_R \operatorname{Ext}_R^{l+k}(\Omega_R^m M, F_k)\end{aligned}$$

of ideals for all $l > 0$. Since $\Omega_R^n N, F_0, \ldots, F_{n-1}$ belong to $\Omega_R^n(\operatorname{fl} R)$, the right-hand side of (3.1) has a^{n+1}. Hence, we have $a^{n+1} \operatorname{Ext}_R^l(\Omega_R^m M, N) = 0$. Fix $i > m$. Then for all $i \le l \le i + 2d$, one has $a^{n+1} \operatorname{Ext}_R^l(M, N) = a^{n+1} \operatorname{Ext}_R^{l-m}(\Omega_R^m M, N) = 0$. Thus by Lemma 3.3, we have $a^{2^{2d}(n+1)} \operatorname{Ext}_R^i(\operatorname{mod} R, \operatorname{mod} R) = 0$. ■

As a consequence of the above proposition, the singular locus of a local ring R is contained in closed subsets defined by the annihilators of Tor and Ext on the syzygies of R-modules of finite length. Restricting this result to the case where R is Cohen–Macaulay, we can recover [7, Proposition 4.6].

PROPOSITION 3.6. *Let R be a d-dimensional local ring. Then the following inclusion relation holds:*

$$\begin{aligned}\operatorname{Sing}(R) \subseteq & \Big(\bigcap_{m,n \ge 0} \mathrm{V}\big(\operatorname{Ann} \operatorname{Tor}(\Omega_R^m(\operatorname{fl} R), \Omega_R^n(\operatorname{fl} R))\big)\Big) \\ & \cap \Big(\bigcap_{m,n \ge 0} \mathrm{V}\big(\operatorname{Ann} \operatorname{Ext}(\Omega_R^m(\operatorname{fl} R), \Omega_R^n(\operatorname{fl} R))\big)\Big).\end{aligned}$$

PROOF. Let m, n be nonnegative integers. We shall show the following inclusion relation:

$$\operatorname{Sing}(R) \subseteq \mathrm{V}\big(\operatorname{Ann} \operatorname{Tor}(\Omega_R^m(\operatorname{fl} R), \Omega_R^n(\operatorname{fl} R))\big).$$

Let $\mathfrak{p}$ be any prime ideal of $\operatorname{Sing}(R)$ and a an element of $\operatorname{Ann} \operatorname{Tor}(\Omega_R^m(\operatorname{fl} R), \Omega_R^n(\operatorname{fl} R))$. Then by Proposition 3.5, we have $a^{2^{2d}} \operatorname{Tor}_i^R(\operatorname{mod} R, \operatorname{mod} R) = 0$ for all $i > m + n + 2d$. Hence one has $0 = a^{2^{2d}} \operatorname{Tor}_i^R(R/\mathfrak{p}, R/\mathfrak{p})_\mathfrak{p} \cong a^{2^{2d}} \operatorname{Tor}_i^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \kappa(\mathfrak{p}))$ for all $i > m + n + 2d$. If a does not belong to $\mathfrak{p}$, one has $\operatorname{Tor}_i^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \kappa(\mathfrak{p})) = 0$ for all $i > m + n + 2d$. This implies that $R_\mathfrak{p}$ is regular, and it contradicts the choice of $\mathfrak{p}$. Hence, we have $a \in \mathfrak{p}$. The assertion for Ext is also proved similarly. ■

Remark 3.7. Let R be as in Proposition 3.6. Fix an integer $n \geq 0$. Since R is noetherian, the ascending chain of ideals

$$\begin{aligned}\operatorname{Ann}\operatorname{Tor}(\operatorname{fl} R, \Omega_R^n(\operatorname{fl} R)) &\subseteq \operatorname{Ann}\operatorname{Tor}(\Omega_R(\operatorname{fl} R), \Omega_R^n(\operatorname{fl} R)) \\ &\subseteq \operatorname{Ann}\operatorname{Tor}(\Omega_R^2(\operatorname{fl} R), \Omega_R^n(\operatorname{fl} R)) \subseteq \cdots\end{aligned}$$

of R stabilizes. Hence so does the descending chain of closed subsets

$$\begin{aligned}\mathrm{V}\big(\operatorname{Ann}\operatorname{Tor}(\operatorname{fl} R, \Omega_R^n(\operatorname{fl} R))\big) &\supseteq \mathrm{V}\big(\operatorname{Ann}\operatorname{Tor}(\Omega_R(\operatorname{fl} R), \Omega_R^n(\operatorname{fl} R))\big) \\ &\supseteq \mathrm{V}\big(\operatorname{Ann}\operatorname{Tor}(\Omega_R^2(\operatorname{fl} R), \Omega_R^n(\operatorname{fl} R))\big) \supseteq \cdots\end{aligned}$$

of $\operatorname{Spec} R$. Thus, there exists an integer $s_n \geq 0$ such that

$$\bigcap_{m \geq 0} \mathrm{V}\big(\operatorname{Ann}\operatorname{Tor}(\Omega_R^m(\operatorname{fl} R), \Omega_R^n(\operatorname{fl} R))\big) = \mathrm{V}\big(\operatorname{Ann}\operatorname{Tor}(\Omega_R^{s_n}(\operatorname{fl} R), \Omega_R^n(\operatorname{fl} R))\big).$$

The equalities of ideals and closed subsets for $\operatorname{Tor}(\Omega_R^n(\operatorname{fl} R), -)$ and $\operatorname{Ext}(-, \Omega_R^n(\operatorname{fl} R))$ also hold.

Let $\mathcal{X}, \mathcal{Y}$ be subcategories of $\operatorname{mod} R$. Assume that $\mathcal{Y}$ contains 0. Then by Lemma 3.4, we have $\operatorname{Ann}\operatorname{Ext}(\mathcal{X}, \Omega_R^m \mathcal{Y}) \subseteq \sqrt{\operatorname{Ann}\operatorname{Ext}(\mathcal{X}, \mathcal{Y})}$ for all $m \geq 0$. Thus, we have an ascending chain of closed subsets

$$\mathrm{V}(\operatorname{Ann}\operatorname{Ext}(\mathcal{X}, \mathcal{Y})) \subseteq \mathrm{V}(\operatorname{Ann}\operatorname{Ext}(\mathcal{X}, \Omega_R \mathcal{Y})) \subseteq \mathrm{V}(\operatorname{Ann}\operatorname{Ext}(\mathcal{X}, \Omega_R^2 \mathcal{Y})) \subseteq \cdots$$

of $\operatorname{Spec} R$. In particular, one has

$$\bigcap_{m \geq 0} \mathrm{V}\big(\operatorname{Ann}\operatorname{Ext}(\Omega_R^n(\operatorname{fl} R), \Omega_R^m(\operatorname{fl} R))\big) = \mathrm{V}(\operatorname{Ann}\operatorname{Ext}(\Omega_R^n(\operatorname{fl} R), \operatorname{fl} R)).$$

Now we have reached the main result of this section.

Theorem 3.8. *Let R be a local ring with residue field k and $\mathcal{X}, \mathcal{Y}$ be subcategories of* $\operatorname{mod} R$ *with* $\operatorname{radius}(\mathcal{X}, \mathcal{Y}) < \infty$. *Assume either that $\mathcal{X}$ is closed under extensions and contains $\Omega_R^n k$ for some $n \geq 0$, or that $\mathcal{X}$ contains $\Omega_R^n(\operatorname{fl} R)$ for some $n \geq 0$. Then the following inclusion relations hold:*

- $\operatorname{Sing}(R) \subseteq \mathrm{V}(\operatorname{Ann}\operatorname{Tor}(\mathcal{X}, \mathcal{Y})) \subseteq \operatorname{NF}(\mathcal{Y})$.
- $\operatorname{Sing}(R) \subseteq \mathrm{V}(\operatorname{Ann}\operatorname{Ext}(\mathcal{X}, \mathcal{Y})) \subseteq \operatorname{NF}(\mathcal{Y})$.

Proof. Let $m = \operatorname{radius}(\mathcal{X}, \mathcal{Y})$. Then there exists an R-module G such that $\mathcal{X} \subseteq [G]_{m+1} \subseteq \mathcal{Y}$. If $\mathcal{X}$ contains $\Omega_R^n k$ and is closed under extensions, then we have $\Omega_R^n(\operatorname{fl} R) \subseteq \operatorname{add}_R(\mathcal{X} \cup \{R\}) \subseteq [G]_{m+1} \subseteq \mathcal{Y}$. On the other hand, if $\mathcal{X}$ contains $\Omega_R^n(\operatorname{fl} R)$, then we have $\Omega_R^n(\mathrm{fl}\, R) \subseteq \mathrm{add}_R\, \mathcal{X} \subseteq [G]_{m+1} \subseteq \mathcal{Y}$. Hence, under either assumption, the following inclusion relations hold:

$$\begin{aligned}
\mathrm{Sing}(R) &\subseteq \mathrm{V}\big(\mathrm{Ann}\,\mathrm{Tor}(\Omega_R^n(\mathrm{fl}\, R), \Omega_R^n(\mathrm{fl}\, R))\big) && \text{(by Proposition 3.6)}\\
&\subseteq \mathrm{V}(\mathrm{Ann}\,\mathrm{Tor}(\mathrm{add}_R\, \mathcal{X}, \mathcal{Y}))\\
&= \mathrm{V}(\mathrm{Ann}\,\mathrm{Tor}(\mathcal{X}, \mathcal{Y}))\\
&\subseteq \mathrm{V}(\mathrm{Ann}\,\mathrm{Tor}([G]_{m+1}, \mathrm{mod}\, R))\\
&= \mathrm{V}(\mathrm{Ann}\,\mathrm{Tor}(G, \mathrm{mod}\, R)) && \text{(by [7, Lemma 5.3])}\\
&= \mathrm{NF}(G) && \text{(by [7, Proposition 5.1])}\\
&= \mathrm{NF}([G]_{m+1})\\
&\subseteq \mathrm{NF}(\mathcal{Y}).
\end{aligned}$$

The assertion for Ext is proved similarly. ∎

REMARK 3.9. If R is a quasi-excellent local ring, then $\mathrm{mod}\, R$ has a strong generator by [8, Corollary 3.12]. Hence there exist integers $s, n \geq 0$ and an R-module G such that $\Omega_R^s(\mathrm{mod}\, R) \subseteq [G]_n$. Thus, one has $\mathrm{Sing}(R) \subseteq \mathrm{NF}(G)$ by Theorem 3.8.

Here, we give an easy non-Cohen–Macaulay example of a quasi-excellent local ring with s and G as in Remark 3.9.

EXAMPLE 3.10. Let K be a field and $R = K[\![x, y]\!]/(x^2, xy)$. Then R is dimension 1 and depth 0, so is not Cohen–Macaulay. We set $\mathfrak{p} = (\bar{x}) \in \mathrm{Spec}\, R$. We shall show that $\Omega_R^1(\mathrm{mod}\, R) \subseteq [R/\mathfrak{p}]_4$. Let X be an R-module. Since $\mathfrak{p}^2 = 0$, the natural exact sequence $0 \to \mathfrak{p}X \to X \to X/\mathfrak{p} \to 0$ yields that $X \in [\mathrm{mod}\, R/\mathfrak{p}]_2^R$. This implies that $\Omega_R X \in [\Omega_R(\mathrm{mod}\, R/\mathfrak{p})]_2^R$. As $R/\mathfrak{p} \cong K[\![y]\!]$ is a discrete valuation ring, we have $\Omega_{R/\mathfrak{p}}(\mathrm{mod}\, R/\mathfrak{p}) = \mathrm{add}\, R/\mathfrak{p}$. Combining this with [6, Proposition 5.3], one has $\Omega_R(\mathrm{mod}\, R/\mathfrak{p}) \subseteq [R/\mathfrak{p}]_2^R$. Hence, we conclude that $\Omega_R X \in [R/\mathfrak{p}]_4^R$.

Let R be a d-dimensional local ring. Note that all the closed subsets

$$\begin{array}{cc}
\mathrm{V}(\mathrm{Ann}\,\mathrm{Ext}(\mathrm{C}_0(R), \mathrm{C}_0(R))), & \mathrm{V}(\mathrm{Ann}\,\mathrm{Ext}(\mathrm{C}_n(R), \mathrm{C}_n(R))),\\
\mathrm{V}(\mathrm{Ann}\,\mathrm{Ext}(\mathrm{Deep}_n(R), \mathrm{Deep}_n(R))), & \mathrm{V}(\mathrm{Ann}\,\mathrm{Ext}(\mathrm{C}(R), \mathrm{C}(R)))
\end{array}$$

coincide for all $0 \leq n \leq d$ by [11, Proposition 2.4 (2)], and [9, Lemma 3.8].

COROLLARY 3.11. *Let n be a nonnegative integer. Assume that* $\mathrm{radius}(\mathrm{C}_0(R), \mathrm{mod}_n\, R) < \infty$. *Then the dimensions of* $R/\mathrm{Ann}\,\mathrm{Tor}(\mathrm{C}_n(R), \mathrm{C}_n(R))$, $R/\mathrm{Ann}\,\mathrm{Ext}(\mathrm{C}_n(R), \mathrm{C}_n(R))$, *and* $\mathrm{Sing}(R)$ *are less than or equal to n.*

Proof. We set $t = \operatorname{depth} R$ and $W_n = \{\mathfrak{p} \in \operatorname{Spec} R \mid \dim R/\mathfrak{p} \leq n\}$. Since $\mathrm{C}_0(R)$ contains $\Omega^t_R k$ and is closed under extensions, we have $\mathrm{V}(\operatorname{Ann}\operatorname{Tor}(\mathrm{C}_n(R), \mathrm{C}_n(R))) \subseteq \mathrm{V}(\operatorname{Ann}\operatorname{Tor}(\mathrm{C}_n(R), \operatorname{mod}_n(R))) \subseteq \operatorname{NF}(\operatorname{mod}_n R) \subseteq W_n$ by Theorem 3.8. Hence one has $\dim R/\operatorname{Ann}\operatorname{Tor}(\mathrm{C}_n(R), \mathrm{C}_n(R)) \leq n$. The assertions for the dimensions of $R/\operatorname{Ann}\operatorname{Ext}(\mathrm{C}_n(R), \mathrm{C}_n(R))$, and $\operatorname{Sing}(R)$ are proved similarly. ■

Denote by $\operatorname{Assh}(R)$ the set of prime ideals $\mathfrak{p}$ of R such that the equality $\dim R/\mathfrak{p} = \dim R$ holds. Note that $\operatorname{Assh}(R)$ is nonempty. The next corollary implies that there are few resolving subcategories of finite dimension which are contained in $\operatorname{mod}_{d-1} R$ when R is non-Cohen–Macaulay.

Corollary 3.12. *Let R be a d-dimensional local ring. Assume that R is not Cohen–Macaulay. Then we have* $\operatorname{radius}(\mathrm{C}_0(R), \operatorname{mod}_{d-1} R) = \infty$. *In particular, the dimensions of* $\mathrm{C}_i(R)$*, and* $\operatorname{Deep}_i(R)$ *are infinite for all* $0 \leq i \leq d-1$.

Proof. Since R is not Cohen–Macaulay, by [11, Proposition 2.6] one has $\operatorname{Assh}(R) \subseteq \mathrm{V}(\operatorname{Ann}\operatorname{Ext}(\mathrm{C}_{d-1}(R), \mathrm{C}_{d-1}(R)))$. Take an element $\mathfrak{p}$ of $\operatorname{Assh}(R)$. Then $\mathrm{V}(\operatorname{Ann}\operatorname{Ext}(\mathrm{C}_{d-1}(R), \mathrm{C}_{d-1}(R)))$ contains $\mathfrak{p}$ and is specialization-closed. Thus, we have that $\mathrm{V}(\mathfrak{p}) \subseteq \mathrm{V}(\operatorname{Ann}\operatorname{Ext}(\mathrm{C}_{d-1}(R), \mathrm{C}_{d-1}(R)))$, and this implies that $\dim R/\operatorname{Ann}\operatorname{Ext}(\mathrm{C}_{d-1}(R), \mathrm{C}_{d-1}(R)) = d$. Hence by Corollary 3.11, one has $\operatorname{radius}(\mathrm{C}_0(R), \operatorname{mod}_{d-1} R) = \infty$.

Fix $0 \leq i \leq d-1$. Since the inequalities $\dim \mathrm{C}_i(R), \dim \operatorname{Deep}_i(R) \geq \operatorname{radius}(\mathrm{C}_0(R), \operatorname{mod}_{d-1} R)$ hold, the assertion follows from the above consequence. ■

The following result is a generalization to the non-Cohen–Macaulay case of [9, Theorem 1.2].

Corollary 3.13. *Let R be a local ring and $n \geq 0$. Consider the following four conditions:*

(i) *The dimension of* $\mathrm{C}_n(R)$ *is finite.*

(ii) *The dimension of* $R/\operatorname{Ann}\operatorname{Ext}(\mathrm{C}_n(R), \mathrm{C}_n(R))$ *is less than or equal to n.*

(iii) *The dimension of* $R/\operatorname{Ann}\operatorname{Tor}(\mathrm{C}_n(R), \mathrm{C}_n(R))$ *is less than or equal to n.*

(iv) *The dimension of* $\operatorname{Sing}(R)$ *is less than or equal to n.*

Then the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) *hold.*

Proof. We set $W_n = \{\mathfrak{p} \in \operatorname{Spec} R \mid \dim R/\mathfrak{p} \leq n\}$ as in the proof of Corollary 3.11.

(i) $\Rightarrow$ (ii). Since one has $\dim \mathrm{C}_n(R) = \operatorname{radius}(\mathrm{C}_n(R), \mathrm{C}_n(R)) \geq \operatorname{radius}(\mathrm{C}_0(R), \operatorname{mod}_n R)$, we have $\dim(R/\operatorname{Ann}\operatorname{Ext}(\mathrm{C}_n(R), \mathrm{C}_n(R))) \leq n$ by Corollary 3.11.

(ii) ⇒ (iii). Since $\mathrm{C}_n(R)$ is closed under syzygies, the inclusion relation

$$\operatorname{Ann}\operatorname{Ext}(\mathrm{C}_n(R),\mathrm{C}_n(R)) \subseteq \operatorname{Ann}\operatorname{Tor}(\mathrm{C}_n(R),\mathrm{C}_n(R))$$

of ideals follows from [9, Proposition 3.9]. Thus, the assertion holds.

(iii) ⇒ (iv). Let $t = \operatorname{depth} R$. Since one has $\Omega^t_R(\operatorname{fl} R) \subseteq \mathrm{C}_0(R) \subseteq \mathrm{C}_n(R)$, we have $\operatorname{Sing}(R) \subseteq \mathrm{V}(\operatorname{Ann}\operatorname{Tor}(\Omega^t_R(\operatorname{fl} R), \Omega^t_R(\operatorname{fl} R))) \subseteq \mathrm{V}(\operatorname{Ann}\operatorname{Tor}(\mathrm{C}_n(R),\mathrm{C}_n(R))) \subseteq W_n$. The first inclusion follows from Proposition 3.6. Hence one has $\dim \operatorname{Sing}(R) \leq n$. We are done. ■

Remark 3.14. We make some remarks concering Corollary 3.13.

(1) Under the assumption of the condition (iv) in Corollary 3.13, one has $\mathrm{C}_n(R) = \mathrm{C}(R)$. Indeed, let W_n be as in the proof of Corollary 3.11. Then we have $\mathrm{NF}(\mathrm{C}(R)) \subseteq \operatorname{Sing}(R) \subseteq W_n$ by [20, Remark 3.3 (4)] and the hypothesis. Hence the inclusion $\mathrm{C}(R) \subseteq \operatorname{mod}_n R$ holds. Thus we obtain $\mathrm{C}_n(R) = \mathrm{C}(R)$.

(2) If R is a quasi-excellent Cohen–Macaulay local ring, then the implication (iv) ⇒ (ii) in Corollary 3.13 holds. Indeed, since R is quasi-excellent, the category $\operatorname{mod} R$ has a strong generator by [8, Corollary 3.12], and the equality $\operatorname{Sing}(R) = \mathrm{V}(\operatorname{ca}(R))$ holds by [10, Theorem 4.3]. Here, $\operatorname{ca}(R)$ is the cohomology annihilator of R, which is defined as

$$\operatorname{ca}(R) = \bigcup_{n\geq 0}\left(\bigcap_{m\geq n} \operatorname{ann}_R \operatorname{Ext}^m_R(\operatorname{mod} R, \operatorname{mod} R)\right).$$

Thus we have $\operatorname{Sing}(R) = \mathrm{V}(\operatorname{ca}(R)) = \mathrm{V}(\operatorname{Ann}\operatorname{Ext}(\mathrm{CM}(R),\mathrm{CM}(R)))$ by [11, Proposition 2.4 (3)]. In addition, if R admits a canonical module, then the implication (iv) ⇒ (i) holds by [8, Corollary 3.12] and [6, Proof of Corollary 5.9].

(3) Let $d = \dim R$. If R is not a Cohen–Macaulay ring, then we have $\dim \mathrm{C}_n(R) = \infty$ for all $0 \leq n \leq d-1$ by Corollary 3.12. On the other hand, in the case of $n = d$, the above conditions (ii), (iii), and (iv) are trivial. Therefore, the implications (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) have no meaning for a non-Cohen–Macaulay local ring.

Here is an application example of Corollary 3.12 for a non-Cohen–Macaulay local ring.

Example 3.15. Let K be a field and $R = K[\![x,y,z]\!]/(x^2, xy)$. Then R is dimension 2 and depth 1, so is not Cohen–Macaulay. Hence, we have $\dim \mathrm{C}_0(R) = \dim \mathrm{C}_1(R) = \dim \operatorname{Deep}_1(R) = \infty$ by Corollary 3.12.

Corollary 3.12 does not deal with the case where $i = d$, that is, on the finiteness of dimensions of $\mathrm{C}(R)$ and $\operatorname{Deep}(R)$. Therefore, the following question arises.

Question 3.16. Let R be a noetherian local ring. If R is not Cohen–Macaulay, then are the dimensions of the subcategories $\mathrm{C}(R)$ and $\mathrm{Deep}(R)$ infinite?

Let $d = \dim R$. If depth $R = 0$, then we have $\dim \mathrm{Deep}(R) = \dim(\mathrm{mod}\, R) = \mathrm{radius}(\mathrm{mod}\, R) = \infty$ by [6, Theorem 4.4]. On the other hand, assume that the dimension of $\mathrm{Sing}(R)$ is less than d (e.g. R is reduced), and we set $n = \dim \mathrm{Sing}(R)$. Then one has $\dim \mathrm{C}(R) = \dim \mathrm{C}_n(R) = \infty$ by Corollary 3.12. In the general case, we do not know of any examples where $\mathrm{C}(R)$ or $\mathrm{Deep}(R)$ has finite dimension.

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