Recent progress on the problem of soliton resolution

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Dispersive partial differential equations are evolution equations whose solutions decay in large time due to the fact that various frequencies propagate with distinct velocities. In some cases, there exist special solutions called solitons*, which do not change their shape as time passes. The* soliton resolution conjecture *predicts that solitons are the only obstruction to the decay of solutions. More precisely, every solution eventually decomposes into a superposition of solitons and a decaying term called* radiation*. We discuss the conjecture in the context of the wave maps equation, which is the analog of the wave equation for sphere-valued maps.*[1]

1 The phenomenon of dispersion

This section is devoted to standard introductory material. For a comprehensive introduction to the topic, the reader can consult for instance [27, 44].

1.1 The wave equation

Consider the *wave equation* in dimension $1+2$,

$$c^{-2}\partial_t^2\psi(t,x_1,x_2) = \partial_{x_1}^2\psi(t,x_1,x_2) + \partial_{x_2}^2\psi(t,x_1,x_2), \tag{W}$$

where $(t,x_1,x_2) \in \mathbb{R}^{1+2}$. The positive number $c > 0$ is the wave speed. Let us assume for simplicity that ψ is real-valued, but it could just as well be vector-valued. We will always write $\mathbb{R}^{1+2}$ instead of $\mathbb{R}^3$ in order to stress that one deals with one time dimension and two space dimensions. Equation (W) is equivalent to requiring that ψ is a *critical point* of the Lagrangian

$$\mathscr{L}(\psi) \coloneqq \frac{1}{2}\int_{\mathbb{R}^{1+2}} \big(c^{-2}(\partial_t\psi)^2 - (\partial_{x_1}\psi)^2 - (\partial_{x_2}\psi)^2\big). \tag{1}$$

The precise meaning of this assertion is the following. Let $\zeta\colon \mathbb{R}^{1+2} \to \mathbb{R}$ be a smooth compactly supported function and $\psi_\varepsilon \coloneqq \psi + \varepsilon\zeta$, where ε is a small real number. We then have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\mathscr{L}(\psi_\varepsilon) &= \int_{\mathbb{R}^{1+2}} (c^{-2}\partial_t\psi\partial_t\zeta - \partial_{x_1}\psi\partial_{x_1}\zeta - \partial_{x_2}\psi\partial_{x_2}\zeta) \\ &= -\int_{\mathbb{R}^{1+2}} \zeta(c^{-2}\partial_t^2\psi - \partial_{x_1}^2\psi - \partial_{x_2}^2\psi),\end{aligned} \tag{2}$$

where the last step is integration by parts. The left-hand side can be interpreted as the directional derivative of $\mathscr{L}$ at ψ in the direction ζ. Hence, we see that all the directional derivatives vanish if and only if ψ satisfies (W).

Equation (W) appears in many physical contexts, the most familiar being the evolution in time of a small disturbance of the membrane of a drum. It should be understood that the membrane extends to infinity and occupies the whole horizontal plane, and $\psi(t,x_1,x_2)$ is the vertical displacement at time t of the element of the membrane whose horizontal coordinates are (x_1,x_2).

Recalling that the Lagrangian density is the difference of the kinetic and the potential energy densities, from the form of (1) we find that the total energy is given by

$$\begin{aligned}E(\psi) &\coloneqq E_{\mathrm{kinetic}} + E_{\mathrm{potential}} \\ &= \frac{1}{2}\int_{\mathbb{R}^2} c^{-2}(\partial_t\psi)^2 + \frac{1}{2}\int\big((\partial_{x_1}\psi)^2 + (\partial_{x_2}\psi)^2\big),\end{aligned} \tag{3}$$

and is a conservation law (a quantity independent of time).

Remark 1.1. By an appropriate choice of units, one can assume that $c = -1$, which we will always do in the sequel. We will also write

$$x \coloneqq (x_1,x_2), \qquad \nabla \coloneqq (\partial_{x_1},\partial_{x_2}), \tag{4}$$

$$\Delta \coloneqq \partial_{x_1}^2 + \partial_{x_2}^2, \qquad \mathrm{d}x \coloneqq \mathrm{d}x_1\mathrm{d}x_2. \tag{5}$$

Mechanical intuition suggests that in order to determine the evolution in time of the disturbance of the membrane we need to specify the *initial conditions* consisting of the initial positions $\psi(0,x)$ and the initial velocities $\partial_t\psi(0,x)$ of all the elements of the drum. One can indeed prove that for any such initial conditions, equation (W) has a unique solution for all time, which moreover depends continuously on the initial conditions (in an appropriate sense that we will not make precise here), which is referred to as *global well-posedness*.

[1] This note is based on the talk given by the author at the 9th European Congress of Mathematics.

Having once more recourse to the intuition from mechanics, we can expect that, if the membrane is initially perturbed only in a bounded region and flat elsewhere, then this disturbance will propagate in various directions, resulting in a decay of its amplitude, namely

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^2} |\psi(t,x)| = 0, \tag{6}$$

which is referred to as *radiative behavior*.

1.2 A few generalities on linear dispersive PDEs

Since it is hard to give a rigorous definition of a linear dispersive PDE which would cover all the interesting cases, we limit ourselves to the following heuristic definition.

Definition 1.2. A linear PDE is called *dispersive* if

(i) it is an evolution equation: it involves the time variable t and the space variable x,
(ii) various frequencies propagate with distinct velocities.

Examples of linear dispersive PDEs include:

- the wave equation (W) and its analogs in higher space dimensions,
- the Schrödinger equation,
- the Klein–Gordon equation,

but the list could be made longer. All these examples are *time-reversible* and have a *conserved energy*, and yet smooth localized initial conditions lead to *radiative behavior* as $t \to \infty$.

Remark 1.3. Radiative behavior crucially depends on the fact that the spatial domain is the whole Euclidean space (or in any case that it is unbounded).

Remark 1.4. In some sense, for a linear dispersive PDE, the trivial solution $\{\psi \equiv 0\}$ is the *global attractor* of the flow.

2 The nonlinear setting: dispersion, solitons, soliton resolution

For a comprehensive introduction to the topic of this section, the reader can consult the monographs [42] and [36].

2.1 Wave maps

Wave maps are nonlinear, geometric analogs of linear waves in the case of maps taking values in a Riemannian manifold, rather than in a Euclidean space. We consider here wave maps $\Psi\colon \mathbb{R}^{1+2} \to \mathbb{S}^2 \subset \mathbb{R}^3$. In classical mechanics, a constrained mechanical system is obtained from the same Lagrangian as for the ambient system, see [1, Chapter 4]. Following this principle and recalling (1), we say that a map $\Psi\colon \mathbb{R}^{1+2} \to \mathbb{S}^2 \subset \mathbb{R}^3$ is a *wave map* if it is a critical point of the Lagrangian

$$\mathscr{L}(\Psi) = \frac{1}{2}\int_{\mathbb{R}^{1+2}} \big(|\partial_t\Psi|^2 - |\partial_{x_1}\Psi|^2 - |\partial_{x_2}\Psi|^2\big), \tag{7}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^3$. Similarly, as in Section 1.1, we can consider $\Psi_\varepsilon = \Psi + \varepsilon Z$, where $Z\colon \mathbb{R}^{1+2} \to \mathbb{R}^3$ is smooth and compactly supported. In order not to violate at main order the condition that Ψ_ε takes values in $\mathbb{S}^2$, it is necessary and sufficient to require that

$$Z(t,x) \perp \Psi(t,x) \qquad \text{for all } (t,x). \tag{8}$$

As in Section 1.1, we have

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\mathscr{L}(\Psi_\varepsilon) = -\int_{\mathbb{R}^{1+2}} Z\cdot(\partial_t^2\Psi - \Delta\Psi). \tag{9}$$

This quantity vanishes for all Z satisfying (8) if and only if

$$\partial_t^2\Psi(t,x) - \Delta\Psi(t,x) = \mu(t,x)\Psi(t,x) \tag{10}$$

for some $\mu(t,x) \in \mathbb{R}$. Differentiating twice the identity $\Psi\cdot\Psi = 1$, we obtain

$$\Psi\cdot(\partial_t^2\Psi - \Delta\Psi) = -|\partial_t\Psi|^2 + |\partial_{x_1}\Psi|^2 + |\partial_{x_2}\Psi|^2, \tag{11}$$

so we can write the wave map equation $\mathbb{R}^{1+2} \to S^2$ as

$$\partial_t^2\Psi - \Delta\Psi = -\big(|\partial_t\Psi|^2 - |\partial_{x_1}\Psi|^2 - |\partial_{x_2}\Psi|^2\big)\Psi. \tag{WM}$$

Similarly, as in the linear case, the *total energy*

$$E(\Psi) := \int_{\mathbb{R}^2}\Big(\frac{1}{2}|\partial_t\Psi|^2 + \frac{1}{2}|\partial_{x_1}\Psi|^2 + \frac{1}{2}|\partial_{x_2}\Psi|^2\Big) \tag{12}$$

is a conservation law for wave maps.

Equation (WM) has trivial constant in space-time solutions $\Psi(t,x) = \omega_0 \in \mathbb{S}^2$. If we linearize around such a solution, that is, if we write $\Psi = \omega_0 + \varepsilon\Phi$ with $\varepsilon \ll 1$ and plug into (WM), at main order we obtain $\partial_t^2\Phi - \Delta\Phi = 0$; thus, each component of Φ satisfies the wave equation (W), which indicates that small perturbations of a constant solution should exhibit a radiative behavior, so that the whole wave map should converge to a constant. It was proved in the works of Tataru [43] and Tao [41] that, in an appropriate sense, this is indeed the case.

For the study of the long-time behavior of solutions of (WM), *criticality* is an important (and helpful) property. Let $\lambda > 0$ and consider

$$\Psi_\lambda(t,x) := \Psi(t/\lambda, x/\lambda). \tag{13}$$

It is clear from (WM) (or from the Lagrangian) that Ψ is a wave map if and only if Ψ_λ is a wave map. Moreover,

$$E(\Psi_\lambda) = E(\Psi). \tag{14}$$

For this reason, equation (WM) is called *energy critical*, and its solutions *critical wave maps*.

Remark 2.1. In general, a problem is *subcritical* if it becomes a "small data problem" when rescaling (zooming) to a small region. It is called *supercritical* if such a zoom makes it large. It is called *critical* if the size of the data remains unchanged.

2.2 Harmonic maps

One might wonder if every, not necessarily small, solution of (WM) has radiative behavior. The answer is "no" for a simple reason: there exist non-trivial static (time-independent) solutions. Namely, inserting $\Psi(t,x) = \omega(x)$ into (WM), we obtain the *critical harmonic map equation*:

$$-\Delta\omega = |\nabla\omega|^2\omega, \qquad \omega\colon \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3. \tag{HM}$$

Its solutions are called *harmonic maps* $\mathbb{R}^2 \to \mathbb{S}^2$. It was proved by Eells and Wood [13], and Hélein [17] that harmonic maps of finite energy correspond to *rational functions* $\mathbb{S}^2 \to \mathbb{S}^2$ and their complex conjugates (we identify $\mathbb{R}^2$ with $\mathbb{S}^2$ using the stereographic projection).

Remark 2.2. It was proved by Krieger, Schlag and Tataru [26], Rodnianski and Sterbenz [32], and Raphaël and Rodnianski [31], that solutions of large energy can even cease to exist in finite time. Equation (WM) is thus locally well-posed, but not globally well-posed.

2.3 A few generalities on nonlinear dispersive PDEs

A nonlinear PDE is called dispersive if it is related to a linear dispersive PDE. Most frequently, "related" means "obtained through linearization around trivial solutions," like in the case of wave maps discussed above.

Nonlinear dispersive PDEs appear frequently in physics, for example in the study of water waves and nonlinear optics, see [44, Chapters 12, 13, 16, 17]. Typical examples are Hamiltonian systems, which, in particular are *time-reversible* and have a *conserved energy*.

One is often interested in the *dynamical behavior* of solutions of a given nonlinear PDE, by which we mean their asymptotic description as time becomes large (for solutions defined for all time; if they are not, one studies the limit as the time tends to the maximal time of existence of the solution). Among the most common questions of this type is the *problem of stability*, which can be formulated as follows.

Problem. *Do small solutions of a nonlinear dispersive PDE exhibit radiative behavior? In other words, does the flow restricted to small solutions have a trivial attractor, like in the linear case (see Remark 1.4)?*

The intuitive reasoning is that small solutions should behave in the same way as the solutions of the linearized problem, which have radiative behavior as we saw in Section 1.2.

2.4 Solitons and soliton resolution

The notion of a soliton is somewhat controversial, see [28, Section 1.5]. We adopt the following definition.

Definition 2.3. A soliton is a solution of an evolution PDE which does not change its shape in the course of time (it can, however, change its position).

Harmonic maps from Section 2.2 are examples of solitons for (WM). Solitons moving at constant velocity can be obtained using the Lorentz invariance of (WM).

Solitons do not exhibit radiative behavior. The problem of *soliton resolution* is to prove that they are the only obstruction to radiative behavior. However, it would be too naive to expect that every solution is either radiative or a soliton. Rather, one expects that every solution eventually decomposes into a superposition of solitons which interact sufficiently weakly (for example, they could travel with distinct velocities). Such superpositions are called *multisoliton configurations*.

Problem (Soliton resolution conjecture). *For a given nonlinear dispersive PDE, does every solution converge in large time to the sum of a multisoliton and a radiative term? In other words, does the flow have a simple global attractor related to multisoliton configurations?*

The soliton resolution is inspired by

- numerical simulations, see Fermi, Pasta and Ulam [14], Zabusky and Kruskal [45],
- the theory of *completely integrable systems*, see Segur and Ablowitz [34], Eckhaus and Schuur [12],
- analogous *elliptic* and *parabolic* problems (*bubbling*), see Sacks and Uhlenbeck [33], Struwe [39].

Remark 2.4. Even with such a vague formulation, the soliton resolution is not expected to hold for all nonlinear dispersive PDEs. For example, the sine-Gordon equation has so-called breather solutions, which do not fall into the regime of soliton resolution.

Remark 2.5. Strictly speaking, soliton resolution only provides an "upper bound" on the global attractor, in the sense that it does not say anything on the types of multisoliton configurations which can be realized by the evolution.

Our main goal is to provide an example of "natural" nonlinear dispersive PDEs for which we can prove soliton resolution. Even though we cannot provide a full description of the global attractor, we will see that it contains configurations consisting of more than one soliton.

3 Soliton resolution for equivariant energy-critical wave maps

3.1 Equivariant maps

The governing PDE will be obtained from (WM) by restricting the flow to a certain subclass of all the maps $\mathbb{R}^2 \to \mathbb{S}^2$ which is preserved by the flow. They are called *equivariant maps* and are defined in the following way. We fix $k \in \{1, 2, \ldots\}$ and consider maps of the form

$$\begin{aligned} &\Psi(t, r\cos\theta, r\sin\theta) \\ &\quad = \big(\sin(\psi(t,r))\cos(k\theta), \sin(\psi(t,r))\sin(k\theta), \cos(\psi(t,r))\big), \end{aligned} \tag{15}$$

where $r > 0$ and $\psi(t,r) \in \mathbb{R}$. Plugging this expression into (WM), we find the *scalar equation*

$$\partial_t^2\psi(t,r) - \partial_r^2\psi(t,r) - \frac{1}{r}\partial_r\psi(t,r) + \frac{k^2}{2r^2}\sin(2\psi(t,r)) = 0, \tag{WM_k}$$

where $r > 0$ is the radial coordinate. Note that ψ and $\psi + 2\pi\ell$ represent the same map Ψ for any $\ell \in \mathbb{Z}$.

Remark 3.1. In the non-geometric context, it is common to consider spherically symmetric solutions. Equivariant solutions are analogous objects in the geometric setting of (WM). More generally, whenever symmetries of a given equation lead to invariance of a certain class of states, it is a well-known technique to study the restriction of the system to this subclass.

One can check that under the substitution (15), the Lagrangian (7) becomes

$$\mathscr{L} := \pi \iint \Big((\partial_t\psi)^2 - (\partial_r\psi)^2 - \frac{k^2\sin(\psi)^2}{r^2}\Big)\, r\mathrm{d}r\,\mathrm{d}t. \tag{16}$$

Its critical points are thus k-equivariant wave maps, a fact that can easily be checked directly. The kinetic energy and the potential energy are

$$E_{\text{kinetic}} := \pi \int_0^\infty (\partial_t\psi)^2\, r\mathrm{d}r, \tag{17}$$

$$E_{\text{potential}} := \pi \int_0^\infty \Big((\partial_r\psi)^2 + \frac{k^2\sin(\psi)^2}{r^2}\Big)\, r\mathrm{d}r. \tag{18}$$

Their sum is the total energy, and it is a conserved quantity.

We always consider strong solutions of finite energy (that is, strong limits of sequences of smooth solutions in the topology induced by the energy, locally uniformly in time). Their existence and uniqueness for any finite-energy initial conditions was obtained in [15,35]. It can be deduced from Strichartz estimates for the wave equation, see for example [5, Section 2] in the case $k \in \{1, 2\}$. If $k \geq 3$ is large, Strichartz estimates from [30] can be applied, see [19, Section 2]. Finite-energy solutions of (WM_k) are not guaranteed to exist for all time. We denote $(T_-, T_+) \subset \mathbb{R}$ the maximal time interval on which the solution exists.

3.2 Multibubble (multisoliton) configurations

Recalling the discussion from Section 2.2, the only k-equivariant harmonic maps correspond to rational functions $az^k, az^{-k}, a\bar{z}^k$ and $a\bar{z}^{-k}$, with $a > 0$. In order to represent these maps in the context of the scalar equation (WM_k), it is convenient to denote

$$Q_\lambda(r) := 2\arctan\Big(\frac{r^k}{\lambda^k}\Big), \qquad \lambda > 0. \tag{19}$$

Then the stationary solutions of (WM_k) are

$$\begin{aligned} &Q_\lambda + 2\pi\ell, && -Q_\lambda + 2\pi\ell, \\ &\pi + Q_\lambda + 2\pi\ell, && \pi - Q_\lambda + 2\pi\ell, \end{aligned} \tag{20}$$

for any $\lambda > 0$ and $\ell \in \mathbb{Z}$. In this context of equation (WM_k), solitons are also called *bubbles*. Note that, with our notational conventions, λ is the spatial scale of the bubble Q_λ. For a given number of bubbles M, an integer m, and scales $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_M$, we define a *multibubble configuration* by

$$Q(\lambda_1, \ldots, \lambda_M) := m\pi + \sum_{j=1}^{M} \pm Q_{\lambda_j} \tag{21}$$

(in the notation, we skip the dependence on m, which is not going to be essential here). One should think of the scales as satisfying $\lambda_1 \ll \cdots \ll \lambda_M$, so that each bubble is separated in scale from all the others. Figure 1 shows a multibubble configuration with $(\lambda_1, \lambda_2, \lambda_3) = (\frac{1}{10}, \frac{1}{2}, 5)$.

3.3 Soliton resolution

Our main result can be formulated as follows.

Theorem 3.2 (Jendrej–Lawrie [21]). *Let ψ be a solution of* (WM_k) *defined for all $t \in (0, \infty)$. As $t \to \infty$, $(\psi(t,\cdot), \partial_t\psi(t,\cdot))$ decomposes into a superposition of*

- *multibubble configuration,*
- *radiation, corresponding to a solution of the linear wave equation* (W),
- *remainder whose energy converges to* 0*.*

Remark 3.3. We also prove a similar result in the case of a finite maximal time of existence of the solution.

Remark 3.4. The case $k = 1$ was settled by Duyckaerts, Kenig, Martel and Merle [8] using a different approach (so-called channels of energy, see below).

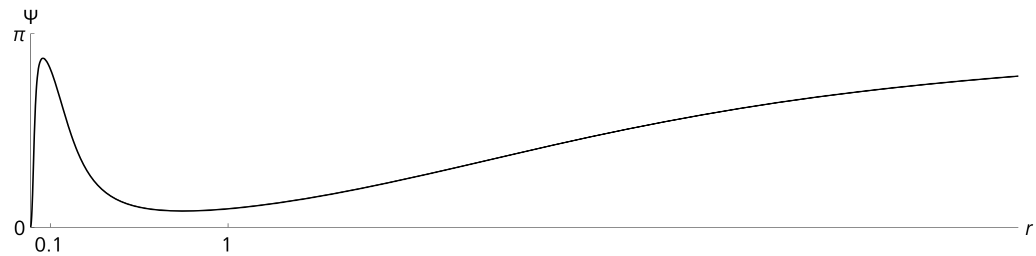

Figure 1. A multibubble configuration with three bubbles.

Remark 3.5. It was proved in [18] that for all $k \geq 2$, there exists a solution containing two bubbles. For these solutions, the radiation component vanishes. It was proved in [21] that for $k = 1$ multi-bubble solutions with vanishing radiation component do not exist. Existence of multibubble solutions with a non-vanishing radiation component is an open problem for all k.

The history of the progress on understanding the dynamical behavior of large solutions of (WM_k) is quite long. Fundamental results were obtained in [2, 3, 37, 38], see also [36, Chapter 8], the main conclusion being the *decay of energy at the self-similar scale*, which in particular excludes self-similar blow-up, but also, as proved by Struwe [40], leads to *bubbling*: if ψ is a solution of (WM_k) which blows up in finite time T_+, then there exist sequences $t_n \to T_+$ and $0 < \lambda_n \ll T_+ - t_n$ such that

$$(\psi(t_n, \lambda_n \cdot), \lambda_n \partial_t \psi(t_n, \lambda_n \cdot)) \to m\pi \pm Q, \tag{22}$$

the convergence being understood in the topology induced by the energy *locally* (on bounded sets).

The bubbling also implies that a solution whose energy is smaller than the energy of Q cannot blow up. It was proved by Côte, Kenig, Lawrie and Schlag [5] that such a solution actually has radiative behavior. Above this threshold energy, finite time blow-up can occur, as was proved in the works [26, 31, 32] already mentioned above. Important progress toward the soliton resolution conjecture was made in [6]. *Sequential* soliton resolution, that is convergence to a superposition of solitons for a sequence of times, was proved by Côte [4] for $k \in \{1, 2\}$, and Jia and Kenig [22] for $k \geq 3$. Similar results without imposing equivariant symmetry assumptions, but with a less precise description of the radiation, were obtained by Grinis [16].

In [19], continuous in time resolution was proved at the minimal possible energy level allowing for existence of a two-bubble. As a relatively simple consequence, continuous in time resolution was proved in [20] under the assumption that the solution contains at most two bubbles.

For the closely related energy-critical wave equation, scattering below the ground state energy threshold was proved by Kenig and Merle [24], establishing together with [23] the so-called *Kenig–Merle route map*. In the radially symmetric case, the soliton resolution conjecture was proved by Duyckaerts, Kenig and Merle in space dimension 3 in [9], in any odd space dimension in [10], and in dimension 4 in [8] (in collaboration with Martel). All these works used the *channels of energy* introduced in [9].

In the non-radial case, sequential soliton resolution was proved by Duyckaerts, Jia, Kenig and Merle [7].

3.4 Main ideas of the proof

Let ψ be a solution of (WM_k) defined for all $t \in (0, \infty)$. Thanks to the sequential soliton resolution results [4, 22], we know that there exists a sequence $t_n \to \infty$ such that $(\psi(t_n, \cdot), \partial_t \psi(t_n, \cdot))$ decomposes into a superposition of a multibubble configuration, radiation and a small remainder. It thus suffices to prove a *no-return lemma*: if a multibubble configuration is destroyed (we say that a *collision* takes place), it cannot recover its shape (note the analogy with non-existence of homoclinic/heteroclinic orbits). A similar idea is present in the works of Duyckaerts and Merle [11], Nakanishi and Schlag [29], Krieger, Nakanishi and Schlag [25] for a *single soliton* which is linearly unstable. In our case, interactions between solitons play a similar role as the linear instability in those works. This idea has already been used in [19] in the special case where there are only two bubbles and the radiative component vanishes.

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