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Poincaré series of curves on rational surface singularities

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Abstract. For a reducible curve singularity embedded in a rational surface singularity the Poincaré series is computed. Here the Poincaré series is defined by the multi-index filtration on the local ring defined by orders of a function on the branches of the curve. The method of the computations is based on the notion of the integral with respect to the Euler characteristic over the projectivization of the ring of functions (notion similar to, and inspired by, the notion of motivic integration). For the case of the $\mathbf{E}_8$ surface singularity it appears that the Poincaré series coincides with the Alexander polynomial of the corresponding link.

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In [3] and [4] there was computed the Poincaré series (in several variables) of the multi-index filtration on the ring of germs of functions of two variables defined by orders of a function on the branches of a reducible plane curve singularity $(C, 0) \subset (\mathbb{C}^2, 0)$. It was shown that this Poincaré series coincides with the Alexander polynomial (in several variables) of the link of the singularity $(C, 0)$. In [5] there was computed the Poincaré series of the multi-index filtration on the ring of germs of functions on a rational surface singularity $(\mathcal{S}, 0)$ defined by the multiplicities of a function along components of the exceptional divisor of a resolution of the singularity $(\mathcal{S}, 0)$. The method of the computations is based on the notion of the integral with respect to the Euler characteristic over the projectivization of the ring of functions. This notion is similar to (and inspired by) the notion of motivic integration.

The Poincaré series of a plane curve singularity is computed in terms of an embedded resolution of the curve. The answer is tightly connected with the Poincaré series of the set of divisorial valuations corresponding to the resolution. A generalization of this approach for a twisted (i.e., non plane) curve would be to consider the curve being embedded into a surface singularity and to use its embedded resolution. The Poincaré series of the set of divisorial valuations of a resolution of a surface singularity is well understood only for rational ones. Therefore it is natural to consider curves

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on them. Here we apply the mentioned method of computing the Poincaré series to a (reducible) curve on a rational surface singularity.

It appears that curves on the $\mathbf{E}_8$ surface singularity have special properties. The link of the surface singularity $\mathbf{E}_8$ is a homology sphere. Therefore, for a curve on the $\mathbf{E}_8$ surface singularity, there is defined the Alexander polynomial (in several variables) of the corresponding link. We show that it also coincides with the Poincaré series of the curve singularity.

1. Poincaré series of a curve on a rational surface singularity

Let $(C,0)$ be a (in general, reducible) germ of a curve and let $C = \bigcup_{j=1}^r C_j$ be its decomposition into irreducible components (with a fixed numbering). Let $\varphi_j\colon (\mathbb{C},0) \to (C,0)$ be an uniformization of the branch C_j, $j = 1,\dots,r$. For a germ $g \in \mathcal{O}_{C,0}$, let $w_j = w_j(g)$ be the power of the leading term in the power series decomposition of the germ $g \circ \varphi_j\colon (\mathbb{C},0) \to \mathbb{C}$: $g \circ \varphi_j(\tau) = a \cdot \tau^{w_j} +$ terms of higher degree ($a \neq 0$). If $g \circ \varphi_j(\tau) \equiv 0$ we assume $w_j(g)$ to be equal to $+\infty$. Let $\underline{w}(g) := (w_1(g),\dots,w_r(g))$ (we call $\underline{w}(g)$ the value of the function g on the curve C). For $\underline{w} = (w_1,\dots,w_r) \in \mathbb{Z}^r$, let $J(\underline{w}) = \{g \in \mathcal{O}_{C,0} : w_j(g) \geq w_j,\ j = 1,\dots,r\}$ ($J(\underline{w})$ is an ideal in $\mathcal{O}_{C,0}$), and let $c(\underline{w}) := \dim J(\underline{w})/J(\underline{w}+\underline{1})$, where $\underline{1} = (1,\dots,1)$, $L_C(t_1,\dots,t_r) = \sum_{\underline{w}\in\mathbb{Z}^r} c(\underline{w}) \cdot \underline{t}^{\underline{w}}$ (here $\underline{t}^{\underline{w}} = t_1^{w_1}\dots t_r^{w_r}$, pay attention that the sum is over all $\underline{w}$ in $\mathbb{Z}^r$, not only over positive ones).

The Poincaré series of the multi-index filtration defined by $\underline{w}(\bullet)$ (for short the Poincaré series of the curve $(C,0)$) is the power series (in fact a polynomial for $r \geq 2$):

$$P_C(t_1,\dots,t_r) = \frac{L_C(t_1,\dots,t_r) \cdot \prod_{j=1}^r (t_j - 1)}{t_1 \dots t_r - 1}.$$

Remark. If the curve $(C,0)$ is embedded into an ambient space $(X,0)$, in the definition of the Poincaré series $P_C(t_1,\dots,t_r)$, one can use the ring $\mathcal{O}_{X,0}$ of germs of functions on $(X,0)$ instead of $\mathcal{O}_{C,0}$ above.

From now on let the curve singularity $(C,0)$ be embedded into a rational surface singularity $(\mathcal{S},0)$. In [5] there was defined the notion of the integral with respect to the Euler characteristic over the projectivization $\mathbb{P}\mathcal{O}_{\mathcal{S},0}$ of the ring of germs of the functions on the surface $(\mathcal{S},0)$ (see also [2], [4]). Just as in [4], [5] one can show that

$$P_C(t_1,\dots,t_r) = \int\limits_{\mathbb{P}\mathcal{O}_{\mathcal{S},0}} \underline{t}^{\underline{w}(g)} d\chi,$$

where $\underline{t}^{\underline{w}(g)}$ is a function on $\mathbb{P}\mathcal{O}_{\mathcal{S},0}$ with the values in the abelian group (with respect to the addition) $\mathbb{Z}[[t_1,\dots,t_r]]$, t_i^{∞} is assumed to be equal to 0.

Let $\pi\colon (\mathcal{X},\mathcal{D}) \to (\mathcal{S},0)$ be an embedded resolution of the curve $C \subset \mathcal{S}$, i.e., a resolution of the surface singularity $(\mathcal{S},0)$ such that $\pi^{-1}(C)$ is a normal crossing divisor. Let $\widetilde{C}_j$ be the strict transform of the component C_j of the curve C ($j = 1,\dots,r$). Let E_i, $i = 1,\dots,s$, be the irreducible components of the exceptional divisor $\mathcal{D} = \pi^{-1}(0)$. All the components E_i are isomorphic to the projective line $\mathbb{CP}^1$. Let $-k_i$ be the self intersection number $E_i \circ E_i$ of the component E_i. Let $M = -(E_i \circ E_\ell)$ be minus the intersection matrix of the components of the exceptional divisor $\mathcal{D}$. The matrix M has numbers k_i on the diagonal, other entries of M are equal to -1 or 0.

For a function $g \in \mathcal{O}_{\mathcal{S},0}$, $g \neq 0$, let $v_i(g)$ ($i = 1,\dots,s$) be the multiplicity of the lifting $g \circ \pi$ of the function g along the component E_i ($v_i(\bullet)$ is the divisorial valuation corresponding to the component E_i). Let $\underline{v}(g) = (v_1(g),\dots,v_s(g))$. The set $S_{\mathcal{S}}$ of points of the lattice $\mathbb{Z}^s_{\geq 0}$ of the form $\underline{v}(g)$ is a subsemigroup of $\mathbb{Z}^s_{\geq 0}$ (the semigroup of divisorial valuations of the resolution π). The semigroup $S_{\mathcal{S}}$ consists of all $\underline{v} \in \mathbb{Z}^s_{\geq 0}$ such that $\underline{v}M \geq 0$ (i.e., such that

$$k_i v_i - \sum_{\ell : E_i \cap E_\ell = pt} v_\ell \geq 0 \tag{1}$$

for $i = 1,\dots,s$; see e.g [8]). Let $m = \det(M)$, $A = M^{-1} = (a_{ij})$, and let $\underline{a}_i = (a_{i1},\dots,a_{is})$ (note that the determinant m does not depend on the resolution π of $(\mathcal{S},0)$ and that ma_{ij} are integers). The set $\underline{v}M \geq 0$ in $\mathbb{R}^s$ is the simplicial cone generated by the vectors $\underline{a}_i$, $i = 1,\dots,s$. One can show (see [8]) that $\underline{a}_i > 0$, $i = 1,\dots,s$.

Remark. Let $n_i = n_i(\underline{v})$ be the left hand sides of the inequalities (1), i.e., $\underline{n} = (n_1,\dots,n_s) = \underline{v}M$ (in particular $n_i(\underline{a}_j) = \delta_{ij}$). One can easily see that, for $\underline{v} = \underline{v}(g) \in S_{\mathcal{S}}$, $n_i(\underline{v})$ is equal to the intersection number of the strict transform of the curve $\{g = 0\}$ with the component E_i of the exceptional divisor. Let $\underline{v}(\underline{n}) := \underline{n}M^{-1}$.

Let $\overset{\circ}{E}_i$ (respectively $\overset{\bullet}{E}_i$) be the "smooth part" of the component E_i in the total transform of the curve C (respectively in the exceptional divisor $\mathcal{D}$), i.e., E_i minus intersection points with all other components of the total transform $\pi^{-1}(C)$ of the curve C (respectively of the exceptional divisor $\mathcal{D}$). The divisorial valuations v_i define a multi-index filtration on the ring $\mathcal{O}_{\mathcal{S},0}$ of functions on the surface singularity $(\mathcal{S},0)$. Let $P_{\mathcal{S},\pi}(T_1,\dots,T_s)$ be the Poincaré series of this filtration defined in the same way as above (it depends on the surface singularity $\mathcal{S}$ and on its resolution π).

For a fractional power series $Q(T_1,\dots,T_s) \in \mathbb{Z}[[T_1^{1/m},\dots,T_s^{1/m}]]$, let $\operatorname{Int} Q(T_1,\dots,T_s)$ be its "integer part", i.e., the sum of all the monomials from $Q(T_1,\dots,T_s)$ with integer exponents. The main result of [5] (formulated in somewhat different terms) is the following.

Theorem 1. *For a rational surface singularity* $(\mathcal{S}, 0)$ *and a resolution* $\pi : (\mathcal{X}, \mathcal{D}) \to (\mathcal{S}, 0)$, *one has*

$$P_{\mathcal{S},\pi}(T_1, \dots, T_s) = \mathrm{Int} \prod_{i=1}^{s} (1 - \underline{T}^{\underline{a}_i})^{-\chi(\overset{\bullet}{E}_i)}.$$

Now we give a formula for the Poincaré series of the curve $(C, 0) \subset (\mathcal{S}, 0)$.

Theorem 2. *One has*

$$P_C(t_1, \dots, t_r) = \Big(\mathrm{Int} \prod_{i=1}^{s} (1 - \underline{T}^{\underline{a}_i})^{-\chi(\overset{\circ}{E}_i)} \Big) \Big|_{T_i \mapsto \prod_{j : \widetilde{C}_j \cap E_i = pt} t_j}$$

(*in the substitution above* $\prod_{j \in \emptyset} t_j$ *is supposed to be equal to* 1).

Proof. For a topological space X, let $S^n X = X^n / S_n$ be the nth symmetric power of the space X. Let

$$Y = \bigcup_{\underline{n} \in \mathbb{Z}^s_{\geq 0}} \Big(\prod_{i=1}^{s} S^{n_i} \overset{\circ}{E}_i \Big) = \prod_{i=1}^{s} \Big(\bigcup_{n=0}^{\infty} S^n \overset{\circ}{E}_i \Big)$$

and

$$Y_0 = \bigcup_{\underline{n} \in \mathbb{Z}^s_{\geq 0} : \underline{v}(\underline{n}) \in \mathbb{Z}^s} \Big(\prod_{i=1}^{s} S^{n_i} \overset{\circ}{E}_i \Big).$$

Elements of Y and of Y_0 are represented by collections of points (finite sets of points with multiplicities) of the smooth part $\overset{\circ}{\mathcal{D}} = \bigcup_{i=1}^{s} \overset{\circ}{E}_i$ of the exceptional divisor $\mathcal{D}$ (for elements of Y_0 with an additional condition). For a function $g \in \mathcal{O}_{\mathcal{S},0}$ such that the strict transform of its zero level curve $\{g = 0\}$ intersects the exceptional divisor $\mathcal{D}$ only at smooth points (i.e., at points of $\overset{\circ}{\mathcal{D}}$), the collection of intersection points counted with multiplicities (intersection numbers) belongs to Y_0. Moreover, a divisor on $\mathcal{X}$ which intersects the exceptional divisor $\mathcal{D}$ only at smooth points is the strict transform of the zero level curve of a function if and only if the collection of the intersection points of this divisor with $\mathcal{D}$ (counted with multiplicities) belongs to Y_0.

Let $\underline{v}$ be a function on Y with values in $\mathbb{Q}^s_{\geq 0}$ which is equal to $\sum_{i=1}^{s} n_i \underline{a}_i$ on the component of Y with the number $\underline{n}$. The values of the function $\underline{v}$ on the space $Y_0 \subset Y$ belong to $\mathbb{Z}^s_{\geq 0}$ and, moreover, $\underline{v}^{-1}(\mathbb{Z}^s_{\geq 0}) = Y_0$. One has

$$\int_Y \underline{T}^{\underline{v}} d\chi = \prod_{i=1}^{s} (1 - \underline{T}^{\underline{a}_i})^{-\chi(\overset{\circ}{E}_i)}$$

(see, e.g., [4]; this follows from the formula $1+\chi(X)t+\chi(S^2X)t^2+\chi(S^3X)t^3+\cdots = (1-t)^{-\chi(X)}$, where $\chi(X)$ is the Euler characteristic of the space X),

$$\int_{Y_0} \underline{T}^{\underline{v}} d\chi = \operatorname{Int} \prod_{i=1}^{s} (1 - \underline{T}^{\underline{a}_i})^{-\chi(\overset{\circ}{E}_i)}.$$

Let a map $\mathbb{Z}^s_{\geq 0} \to \mathbb{Z}^r_{\geq 0}$ ($\mathbb{Q}^s_{\geq 0} \to \mathbb{Q}^r_{\geq 0}$) be defined by $(v_1, \dots, v_s) \mapsto (w_1, \dots, w_r)$ with $w_j = v_{i(j)}$, where $i = i(j)$ is the number of the component E_i of the exceptional divisor $\mathcal{D}$ which intersects the strict transform $\widetilde{C}_j$ of the component C_j of the curve C. One has

$$\int_{Y_0} \underline{t}^{\underline{w}(\underline{v})} d\chi = \int_{Y_0} \underline{T}^{\underline{v}} d\chi \,\Big|_{T_i \mapsto \prod_{j:\widetilde{C}_j \cap E_i = pt} t_j}$$

(and the same for the integrals over Y).

For a function $g \in \mathcal{O}_{\mathcal{S},0}$ such that the strict transform of its zero level curve $\{g = 0\}$ intersects the exceptional divisor $\mathcal{D}$ only at smooth points (and thus the collection of them belongs to the space Y_0) the value $\underline{w}(g)$ of the function g on the curve C is equal to $\underline{w}(\underline{v}(g))$.

Let $\underline{V} \in \mathbb{Z}^r_{\geq 0}$. Making as many additional blow-ups of intersection points of components of the total transform of the curve C as it is necessary, we can suppose that, for any $g \in \mathcal{O}_{\mathcal{S},0}$ with $\underline{w}(g) \leq \underline{V}$, the strict transform of the curve $\{g = 0\}$ intersects the exceptional divisor $\mathcal{D}$ only at smooth points. Let $\mathbb{P}\mathcal{O}_{\mathcal{S},0}(\underline{V})$ be the set $\{g \in \mathbb{P}\mathcal{O}_{\mathcal{S},0} : \underline{w}(g) \leq \underline{V}\}$. The set $\mathbb{P}\mathcal{O}_{\mathcal{S},0}(\underline{V})$ is cylindric (see, e.g., [5], Proposition 1). Let I be the map from $\mathbb{P}\mathcal{O}_{\mathcal{S},0}(\underline{V})$ to the space Y_0 which sends a class $g \in \mathbb{P}\mathcal{O}_{\mathcal{S},0}(\underline{V})$ with $\underline{w}(g) \leq \underline{V}$ to the collection of intersection points of the strict transform of the curve $\{g = 0\}$ with the exceptional divisor $\mathcal{D}$ (counted with multiplicities). One can easily see that $\underline{w}(I(g)) = \underline{w}(g)$ (in fact also $\underline{v}(I(g)) = \underline{v}(g)$). Moreover, the image $\operatorname{Im} I$ of the map I coincides with the union $Y_0^{\underline{V}}$ of all the components of the space Y_0 with $\underline{w} \leq \underline{V}$. Preimages of points of the space $Y_0^{\underline{V}}$ under the map I are complex affine spaces (see, e.g., [5] Proposition 2). Since the Euler characteristic of a complex affine space is equal to 1, the Fubini formula (applied to the map $I : \mathbb{P}\mathcal{O}_{\mathcal{S},0}(\underline{V}) \to Y_0^{\underline{V}}$) implies that

$$\int_{Y_0^{\underline{V}}} \underline{t}^{\underline{w}} d\chi = \int_{\mathbb{P}\mathcal{O}_{\mathcal{S},0}(\underline{V})} \underline{t}^{\underline{w}} d\chi.$$

Since this equation holds for any $\underline{V} \in \mathbb{Z}^r_{\geq 0}$, one has

$$\int_{Y_0} \underline{t}^{\underline{w}} d\chi = \int_{\mathbb{P}\mathcal{O}_{\mathcal{S},0}} \underline{t}^{\underline{w}} d\chi = P_C(\underline{t}). \qquad \square$$

2. Curves on the surface singularity $\mathbf{E}_8$

Let $(\mathcal{S}, 0)$ be the rational double point of the type $\mathbf{E}_8$ ($\{x^2+y^3+z^5=0\} \subset (\mathbb{C}^3, 0)$). In this case the determinant m of the intersection matrix M is equal to 1. This implies that all the vectors $\underline{a}_i$ are integer and therefore any curve on $(\mathcal{S}, 0)$ is the zero level curve of a function (i.e., each Weil divisor is a Cartier one). Therefore, for a curve $C = \bigcup_{i=1}^r C_i \subset (\mathcal{S}, 0)$, one has

$$P_C(t_1, \dots, t_r) = \prod_{i=1}^{s} (1 - \underline{T}^{\underline{a}_i})^{-\chi(\overset{\circ}{E}_i)} \Big|_{T_i \mapsto \prod_{j: \widetilde{C}_j \cap E_i = pt} t_j}. \quad (2)$$

The link $L = \mathcal{S} \cap S^5_\epsilon$ (where S^5_ϵ is the sphere of small radius ϵ centered at the origin of $\mathbb{C}^3$) is a homology 3-sphere (see, e.g., [7]). For the curve C, let $K = S^5_\epsilon \cap C$ be the corresponding link. The manifold K is the union of r circles in the homology sphere L. Therefore there is defined the Alexander polynomial of (L, K) which is a polynomial in r variables (see, e.g., [6]). Let $\Delta^{\mathcal{S},C}(t_1, \dots, t_r)$ denote the Alexander polynomial of the pair (L, K).

Let $\pi \colon (\mathcal{X}, \mathcal{D}) \to (\mathcal{S}, 0)$ be an embedded resolution of the curve $C \subset \mathcal{S}$. For a component E_i of the exceptional divisor $\mathcal{D} = \pi^{-1}(0)$, let $\widetilde{L}$ be a germ of a smooth curve transversal to the component E_i at a smooth point. Let the curve $L = \pi(\widetilde{L}) \subset (\mathcal{S}, 0)$ be defined by an equation $\{g = 0\}$. From the remark on page 3 the following statement follows.

Lemma 1. *One has* $\underline{v}(g) = \underline{a}_i$.

From [6] it follows that

$$\Delta^{\mathcal{S},C}(t_1, \dots, t_r) = \prod_{i=1}^{s} (1 - \underline{T}^{\underline{a}_i})^{-\chi(\overset{\circ}{E}_i)} \Big|_{T_i \mapsto \prod_{j: \widetilde{C}_j \cap E_i = pt} t_j} \quad (3)$$

when $r > 1$ and, for $r = 1$,

$$\Delta^{\mathcal{S},C}(t_1)/(1 - t_1) = \prod_{i=1}^{s} (1 - \underline{T}^{\underline{a}_i})^{-\chi(\overset{\circ}{E}_i)} \Big|_{T_i \mapsto \prod_{j: \widetilde{C}_j \cap E_i = pt} t_j}. \quad (4)$$

Note that the substitution in the last formula means that $T_i \mapsto 1$ if $\widetilde{C_1} \cap E_i = \emptyset$ and $T_i \mapsto t_1$ otherwise.

Remark. According to the general definition (see, e.g., [6]), the Alexander polynomial $\Delta^{\mathcal{S},C}(t_1, \dots, t_r)$ of a link is well defined only up to multiplication by monomials $\pm \underline{t}^{\underline{m}} = \pm t_1^{m_1} \dots t_r^{m_r}$ with $\underline{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$. For the link $(L, K) =$

$(\mathcal{S}, C) \cap S^5_\epsilon$, the formulae (3) and (4) fixes the choice of the Alexander polynomial in such a way that it is really a polynomial (i.e., does not contain monomials with negative exponents) and its value at the origin ($\underline{t} = 0$) is equal to 1.

Comparing (3) and (4) with (2) we have the following statement.

Theorem 3. *Let $C = \bigcup_{i=1}^r C_i$ be a curve on the $\mathbf{E}_8$ surface singularity $(\mathcal{S}, 0)$. Then, if $r > 1$, one has*

$$P_C(t_1, \dots, t_r) = \Delta^{\mathcal{S},C}(t_1, \dots, t_r),$$

and for the case $r = 1$,

$$P_C(t_1) = \Delta^{\mathcal{S},C}(t_1)/(1 - t_1).$$

Corollary. *On the $\mathbf{E}_8$ surface singularity, there are only curves the Poincaré series of which are products/ratios of cyclotomic polynomials.*

As an example, the curve given by $t \mapsto (t^3, t^4, t^5)$ does not lie on an $\mathbf{E}_8$ surface singularity.

Remark. Let $\mathcal{S}$ be an arbitrary rational surface singularity and let $\widetilde{P}_{\mathcal{S},C}(T_1, \dots, T_s) = \prod_{i=1}^s (1 - \underline{T}^{\underline{a}_i})^{-\chi(\overset{\circ}{E}_i)}$ be the fractional power series corresponding to an embedded resolution $\pi\colon (\mathcal{X}, \mathcal{D}) \to (\mathcal{S}, 0)$ of the curve $C \subset (\mathcal{S}, 0)$. Suppose that the curve C is a Cartier divisor, i.e., it is the zero level curve of a function $f \in \mathcal{O}_{\mathcal{S},0}$. Let $\zeta_f(t)$ be the zeta-function of the classical monodromy transformation of the germ $f\colon (\mathcal{S}, 0) \to (\mathbb{C}, 0)$ (see, e.g., [1]). Then one can see that

$$\zeta_f(t) = \widetilde{P}_{\mathcal{S},C}(T_1, \dots, T_s)\,_{\big|\, T_i \mapsto t^{m_i}}$$

where m_i is the intersection number of the strict transform of the curve C with the component E_i of the exceptional divisor $\mathcal{D}$, in other words m_i is the number of components of the strict transform of the curve C which intersects the component E_i.

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