Rev. Mat. Iberoamericana **18** (2002), 41–97

The infinite Brownian loop on a symmetric space

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Abstract

The infinite Brownian loop $\{B_t^0, t \geq 0\}$ on a Riemannian manifold $\mathbb{M}$ is the limit in distribution of the Brownian bridge of length T around a fixed origin 0, when $T \to +\infty$. It has no spectral gap. When $\mathbb{M}$ has nonnegative Ricci curvature, B^0 is the Brownian motion itself. When $\mathbb{M} = G/K$ is a noncompact symmetric space, B^0 is the relativized Φ_0–process of the Brownian motion, where Φ_0 denotes the basic spherical function of Harish–Chandra, i.e. the K–invariant ground state of the Laplacian. In this case, we consider the polar decomposition $B_t^0 = (K_t, X_t)$, where $K_t \in K/M$ and $X_t \in \bar{\mathfrak{a}}_+$, the positive Weyl chamber. Then, as $t \to +\infty$, K_t converges and $d(0, X_t)/t \to 0$ almost surely. Moreover the processes $\{X_{tT}/\sqrt{T}, t \geq 0\}$ converge in distribution, as $T \to +\infty$, to the intrinsic Brownian motion of the Weyl chamber. This implies in particular that $d(0, X_{tT})/\sqrt{T}$ converges to a Bessel process of dimension $D = \operatorname{rank} \mathbb{M} + 2j$, where j denotes the number of positive indivisible roots. An ingredient of the proof is a new estimate on Φ_0.

1. Introduction

In order to get some insight into the geometry at infinity of a Riemannian manifold $\mathbb{M}$, it is natural to look at the asymptotic properties of its Brownian motion $\{B_t, t \geq 0\}$, i.e. the Markov process with generator $\Delta/2$, where Δ is the Laplace Beltrami operator on $\mathbb{M}$. On manifolds with a spectral gap this can be disappointing. Consider the two following examples of manifolds with

2000 Mathematics Subject Classification: (Primary) 43A85, 53C35, 58G32, 60J60; (Secondary) 22E30, 43A90, 58G11, 60H30, 60F17.

Keywords: Brownian bridge, central limit theorem, ground state, heat kernel, quotient limit theorem, relativized process, Riemannian manifold, spherical function, symmetric space, Weyl chamber.

nonpositive curvature, either $\mathbb{M}$ is the universal cover of a compact manifold with negative curvature or $\mathbb{M}$ is a Riemannian symmetric space of the noncompact type. In these cases the asymptotic behavior of the Brownian motion is completely understood. Let us consider for instance the distance $d(x, B_t)$ from a fixed point $x \in \mathbb{M}$. Then there exists $\ell > 0, \sigma > 0$ such that $\frac{1}{t} d(x, B_t) \to \ell$ almost surely and $\frac{1}{\sigma\sqrt{t}}(d(x, B_t) - t\ell)$ converges in distribution to a Gaussian law $N(0,1)$ when $t \to \infty$ (see Virtser [54], Orihara [45], Malliavin & Malliavin [38], Taylor [53], Babillot [4] for symmetric spaces, Pinsky [47], Ledrappier [35] for manifolds). The dependence of this result on the geometry in the large of $\mathbb{M}$ is rather poor. This can be intuitively explained: since B_t goes to infinity with a linear rate it goes too fast to be able to see the geometry of $\mathbb{M}$.

On the other hand, the so-called central local limit theorem is more precise. It associates to a symmetric space $\mathbb{M}$ an integer $D \geq 3$ with the following property: let m be the Riemannian measure and $\lambda_0 > 0$ be the spectral gap, then there is a positive function φ on $\mathbb{M}$ such that, as $t \to +\infty$,

$$\mathbb{P}(B_t \in C) \sim \frac{e^{-t\lambda_0}}{t^{D/2}} \int_C \varphi \, dm$$

for any compact set C with negligible boundary (see [6, 7]). This integer D is equal to $d + 2j$, where d is the rank of $\mathbb{M}$ and j is the number of positive indivisible roots. It depends only on the geometry of the Weyl chamber. Thus it is natural to look for a random process on $\mathbb{M}$ such that the asymptotic behavior of its paths is clearly related to D. Intuitively, this process should be connected with the Brownian motion, but should go to infinity slowly and have no spectral gap. This has led us to introduce the infinite Brownian loop (I.B.L.), which is roughly speaking the limit of the Brownian motion constrained to come back to its starting point at a very large time. We will show that the behavior of the radial part of this process at infinity is the same as the one of the Brownian motion in a D-dimensional Euclidean space.

Firstly, it is worth defining the infinite Brownian loop on a general Riemannian manifold $\mathbb{M}$. We fix a point $a \in \mathbb{M}$. The Brownian bridge $B^{(L)}$ around a of length $L > 0$ is the Brownian motion $\{B_t, 0 \leq t \leq L\}$ on $\mathbb{M}$ conditioned by $B_0 = B_L = a$.

Definition 1.1 *The infinite Brownian loop (B_t^0) around a is, when it exists, the limit in distribution of the Brownian bridge $B^{(L)}$ as $L \to +\infty$.*

For any $T > 0$, the process $\{B_t^0, 0 \leq t \leq T\}$ is the limit of $\{B_t^{(L)}, 0 \leq t \leq T\}$ when L tends to infinity. Thus, this infinite Brownian loop can be seen as the limit of the beginning of the Brownian bridge.

We will first show the following theorem. Let $p_t(x,y)$ be the heat kernel and λ_0 be the bottom of the spectrum of $-\Delta/2$ on $L^2(\mathbb{M},m)$ where m is the Riemannian measure.

Theorem 1.2 *On a Riemannian manifold $\mathbb{M}$, the infinite Brownian loop B^0 around $a \in \mathbb{M}$ exists if and only if the following limit exists:*

$$\lim_{t\to+\infty} \frac{p_t(x,a)}{p_t(a,a)} = \varphi(x).$$

In this case φ is of class C^2, $(\Delta + 2\lambda_0)\varphi = 0$ and B^0 is the relativized φ-process, i.e. the Markov process starting from $a \in \mathbb{M}$ with semigroup P_t^0 given by

$$P_t^0 f(x) = e^{\lambda_0 t} \int \frac{p_t(x,y)\varphi(y)}{\varphi(x)} f(y)\, dm(y),$$

for any measurable $f : \mathbb{M} \mapsto \mathbb{R}_+$. Its generator is $\Delta^0/2$ where

$$\Delta^0 f = \frac{1}{\varphi}\Delta(f\varphi) + 2\lambda_0 f = \Delta f + 2\nabla \log\varphi \cdot \nabla f.$$

A positive solution of $(\Delta + 2\lambda_0)\varphi = 0$ is usually called a ground state. The relativized φ-process is a generalized h-process in the sense of Doob. The processes relativized by a ground state were introduced on general Riemannian manifolds by Sullivan in [51] and [52]. In general there are many positive ground states. The interesting feature of the infinite Brownian loop is that it chooses in a canonical way one of them, which is arguably in a sense the most symmetric one. According to Davies [15], the idea of studying the heat kernel by using Δ^0 goes back to Nelson [43] and Gross [25], at least when there is a unique ground state. It plays a major role in Davies & Simon [16] for instance.

Often, the infinite Brownian loop is the Brownian motion itself. This is the case when $\mathbb{M} = \mathbb{R}^n$ or more generally when the Ricci curvature is nonnegative. This follows from the following proposition and from Li & Yau [37]. Notice also that the hypotheses of this proposition are fulfilled for recurrent manifolds. This generalizes Theorem 28 in Davies [15] as conjectured by himself.

Proposition 1.3 *If $\lambda_0 = 0$ and if the positive harmonic functions are constant, then*

$$\lim_{t\to+\infty} \frac{p_{t+s}(x,y)}{p_t(a,a)} = 1$$

and the infinite Brownian loop is the Brownian motion itself.

The purpose of this paper is to study in details a class of manifolds with nonpositive curvature, namely symmetric spaces $\mathbb{M} = G/K$ associated with a noncompact semisimple Lie group G. The infinite Brownian loop in this case is the relativized φ-process, where φ is the basic spherical function Φ_0 of Harish-Chandra, i.e. the unique K-invariant ground state. Its generator is

$$\frac{\Delta^0}{2} = \frac{\Delta}{2} + \nabla \log \Phi_0 \cdot \nabla.$$

We will show that its asymptotic behavior depends on the geometry of $\mathbb{M}$ in a more interesting way that the Brownian motion itself. Let us recall the generalized polar decomposition of $\mathbb{M}$ associated with a Cartan decomposition $G = KAK$ and with a Weyl chamber $\mathfrak{a}_+$ in $\mathfrak{a}$. Let M be the centralizer of A in K. For any $x \in \mathbb{M}$ we define $\dot{k}(x) \in K/M$ and $C(x) \in \bar{\mathfrak{a}}_+$ by the fact that if $k(x) \in K$ is a representative of $\dot{k}(x)$ then

$$k(x) \exp C(x).o = x$$

where o is the origin in $\mathbb{M}$, i.e. the class K in G/K, $C(x)$ is called the radial part of x. The main result of this paper is the following. Since the action of G is transitive on $\mathbb{M}$, there is no restriction to study the I.B.L. only around o.

Theorem 1.4 *Consider the infinite Brownian loop (B_t^0) around o on the symmetric space $\mathbb{M}$. Then, as $T \to +\infty$,*

a. *Almost surely, $\dot{k}(B_T^0)$ converges in K/M and $\frac{1}{T} C(B_T^0) \to 0$.*

b. *The processes $\{\frac{1}{\sqrt{T}} C(B_{tT}^0), t \geq 0\}$ converge in distribution to the intrinsic Brownian motion of the Weyl chamber $\mathfrak{a}_+$.*

c. *$\{\dot{k}(B_{tT}^{(0)}), t > 0\}$ and $\{\frac{1}{\sqrt{T}} C(B_{tT}^0), t \geq 0\}$ are asymptotically independent.*

The intrinsic Brownian motion of the Weyl chamber $\mathfrak{a}_+$ has several equivalent definitions which will be given in Section 3. Roughly speaking, it is the usual Euclidean Brownian motion inside the Weyl chamber with Dirichlet conditions on the walls, conditioned to have infinite lifetime, starting from 0.

The local central limit theorem of [7] appears to be the "local" version of the "central limit theorem" given by Theorem 1.4. We will also obtain the following corollary.

Corollary 1.5 *As $T \to +\infty$, $\frac{1}{T} d(x, B_T^0)$ converges almost surely to 0 and the processes $\{\frac{1}{\sqrt{T}} d(x, B_{tT}^0), t \geq 0\}$ converge in distribution to the Bessel process of dimension D.*

These results show that the I.B.L. is deeply connected with a process of dimension D. In a sense, for the infinite Brownian loop D plays the role of a dimension at infinity of the manifold. This integer D is called the pseudo-dimension of $\mathbb{M}$ by Cowling, Giulini & Meda in [14]. Another interpretation is available: we will see that the situation is almost trivial when the group of isometries G of $\mathbb{M}$ is complex. In this case $D = \dim \mathbb{M}$, and $C(B_t^0)$ is equal to the intrinsic Brownian motion of the Weyl chamber $\mathfrak{a}_+$ (without any normalization). Thus Theorem 1.4 shows that complex groups are models for the general situation. The asymptotic behavior of the radial part of the I.B.L. is the same for all the symmetric spaces with the same Weyl chamber. It is the behavior of the radial part of the I.B.L. for the unique complex group corresponding to this Weyl chamber. For instance $\mathrm{Sl}(2, \mathbb{C})$ is the complex group associated to all rank-one symmetric spaces, and the limit process is the 3-dimensional Bessel process.

One can also consider simultaneously the two ends of the Brownian bridge $\{B^{(L)}, 0 \leq t \leq L\}$ around a. More generally, given two points $a, b \in \mathbb{M}$, it is also interesting to look at the two ends of the Brownian bridge $\{B_t^{(L,a,b)}, 0 \leq t \leq L\}$ which is the Brownian motion $\{B_t, 0 \leq t \leq L\}$ conditioned by $B_0 = a, B_L = b$.

Definition 1.6 *The double-ended infinite Brownian loop* $(B_t^0, \tilde{B}_t^0)$ *from a to b is, when it exists, the limit in distribution of* $\{(B_t^{(L,a,b)}, B_{L-t}^{(L,a,b)}), 0 \leq t \leq L\}$ *as* $L \to +\infty$.

We will describe its asymptotic behavior on symmetric spaces in Theorem 7.1. We will see that the two ends B^0 and $\tilde{B}^0$ are not asymptotically independent, and that the asymptotic behavior of the first one does depend on the extremity b of the other one. In a sense, this reflects the importance of boundaries for these manifolds with nonpositive curvature.

This paper is organized as follows. In Section 2 we present some general considerations on the infinite Brownian loop and the double-ended infinite Brownian loop and we prove Theorem 1.2. In Section 3 we present the intrinsic Brownian motion of the Weyl chamber, considered by Biane [5] and Grabiner [23] (after Dyson [19]). We show its relation with the I.B.L. on symmetric spaces associated with complex groups. Theorem 1.4, which is the heart of this paper, is proved in Section 5. The almost sure behavior of the "angular" part follows from the description of the Martin boundary at the bottom of the spectrum given by Guivarc'h, Ji & Taylor in [27]. For each fixed $t > 0$, the convergence of the distribution of $C(B_{tT}^0)/\sqrt{T}$, as $T \to +\infty$, follows easily from the asymptotics of the heat kernel in Anker & Ji [2] for instance. But the behavior of the process itself is more difficult since its generator becomes singular on the walls of the Weyl chamber. To study this process, we use on the one hand stochastic calculus and in particular Girsanov's theorem and, on the other hand, a new estimate on the Harish–Chandra's function Φ_0 . Specifically we establish in the appendix (Section 8) the boundedness of the derivative $\mathcal{E}\log(\delta^{1/2}\Phi_0)$ with respect to the Euler operator $\mathcal{E}$, where δ is the density function in the Cartan decomposition. Its proof follows Harish–Chandra's induction argument, i.e. the analysis along faces of the Weyl chamber is performed by reduction to symmetric subspaces of lower rank. In Section 6, we describe the behavior of all the relativized processes at the bottom of spectrum. We shall see that they satisfy a theorem similar to Theorem 1.4, as a straightforward application of our study of the I.B.L.. Section 7 is devoted to the double-ended infinite Brownian loop.

Notice that the generalization to arbitrary simply connected symmetric space of the results of this paper is straightforward.

Let us indicate some open questions related to this work. As mentioned above, Ledrappier [35] has proved the central limit theorem on the universal covering of a compact manifold with negative curvature. In that context the local limit theorem is not known. One can ask for conditions ensuring that the I.B.L. exists and then that its normalized distance to a fixed point converges to a three dimensional Bessel process. In a recent work, Hamenstädt studies the set of ground states on these manifolds in [28].

Let us consider again the Brownian bridge $\{B_t^{(L)}, 0 \leq t \leq L\}$ of length L. In this paper we let $L \to +\infty$ and then look at the normalized process. It is also natural to first normalize and then let $L \to +\infty$. More precisely one may study the behavior of $\{B_{tL}^{(L)}/\sqrt{L}, 0 \leq t \leq 1\}$ as $L \to +\infty$. With physical motivations, this question has been considered in Nechaev & Sinaï [42], Nechaev, Grosberg & Vershik [41], Nechaev [40], Letchikov [36]. It is shown in [9] that in the rank-one case, the limit of the radial part is the Brownian excursion. It is natural to conjecture that in the general case the limit will be the excursion in the Weyl chamber, i.e., roughly speaking, the Brownian motion in the Weyl chamber conditioned to be at 0 at the times 0 and 1, and inside the chamber between 0 and 1. It is straightforward to check that this conjecture is true when G is complex, see Remark 4.6. This problem will be dealt with in a future work (see [10]).

2. The infinite Brownian loop on a Riemannian manifold

Let $\mathbb{M}$ be a Riemannian manifold, not necessarily complete, Δ its Laplace Beltrami operator and let $\{B_t, t \geq 0\}$ be the associated Brownian motion on $\mathbb{M}$. By definition (B_t) is the minimal Markov process on $\mathbb{M}$ with generator $\Delta/2$. It takes its values in the set $\hat{C}(\mathbb{R}_+, \mathbb{M} \cup \{\infty\})$ of continuous paths in the Alexandrov compactification $\mathbb{M} \cup \{\infty\}$ of $\mathbb{M}$ which remain in ∞ once they meet it. Its transition semigroup P_t has a smooth symmetric positive density $p_t(x, y)$ with respect to the Riemannian measure m, when $t > 0$. We denote by λ_0 the bottom of the spectrum of $-\Delta/2$ on $L^2(\mathbb{M}, m)$. The purpose of this section is to prove Theorem 1.2 and its analogue for the double-ended I.B.L. (see Proposition 2.6).

Let us establish some preliminary results. The following lemma occurs in the proof of Theorem 25 in Davies [15].

Lemma 2.1 *For any $f \in L^2$ and any $x \in \mathbb{M}$,*

$$\lim_{s \to +\infty} \frac{\langle P_{s+t} f, f \rangle}{\langle P_s f, f \rangle} = \lim_{s \to +\infty} \frac{p_{s+t}(x, x)}{p_s(x, x)} = e^{-\lambda_0 t}.$$

The next theorem is of general interest. It is inspired by a quotient theorem of Guivarc'h on Lie groups [26] and by Collet, Martinez & San Martin [11].

Theorem 2.2 *For all $a \in \mathbb{M}$, the family of functions*

$$\{(t, x, y) \mapsto \frac{p_{s+t}(x, y)}{p_s(a, a)}, \quad s \geq 1\}$$

is relatively compact for the topology of $C^{1,2}$ uniform convergence on compact subsets of $\mathbb{R}_+ \times \ \mathbb{M}^2$. As $s \to +\infty$, each limit point Ψ satisfies

$$\Psi(t, x, y) = e^{-\lambda_0 t} \psi(x, y)$$

where

$$(\Delta_x + 2\lambda_0)\psi = (\Delta_y + 2\lambda_0)\psi = 0.$$

Proof. Let $s_n \to +\infty$, $s_n \geq 1$, and let

$$u_n(t, x, y) = \frac{p_{s_n+t}(x, y)}{p_{s_n}(a, a)}.$$

It follows from the local parabolic Harnack inequality of Moser [39] that for each compact set K in $\mathbb{M}$, there exists $R > 0$ such that

$$p_{s_n+t}(x, y) \leq R \, p_{s_n+t}(a, a)$$

for any $n \in \mathbb{N}, t \geq 0, x, y \in K$ (see Theorem 10 of Davies [15]). Since $s \mapsto p_s(a,a)$ is non increasing (see [15]), this implies that

$$u_n(t,x,y) \leq R. \tag{2.1}$$

Since $s_n \geq 1$, u_n is a solution of the heat equation on $(-1,+\infty) \times \mathbb{M}^2$:

$$\left(4\frac{\partial}{\partial t} - \Delta_x - \Delta_y\right) u_n = 0.$$

It thus follows from the Schauder parabolic estimates (see, e.g., Theorem 3.3.5 in Friedman [21]) and from (2.1) that for each $0 < \alpha < 1$ and each compact set K_0 of $\mathbb{R}_+ \times \mathbb{M}^2$, there is $C_0 > 0$ and $C_0' > 0$ such that (using local coordinates and D^i equal first to $\frac{\partial}{\partial x_i}$ and then to $\frac{\partial}{\partial y_i}$),

$$\begin{aligned} \|\frac{\partial u_n}{\partial t}\|_{\alpha,K_0} \quad + \quad & \|u_n\|_{\alpha,K_0} + \sum_i \|D^i u_n\|_{\alpha,K_0} + \sum_{i,j} \|D^i D^j u_n\|_{\alpha,K_0} \\ & \leq C_0 \sup_{(t,x,y)\in K_0} u_n(t,x,y) \leq C_0', \end{aligned} \tag{2.2}$$

where $\|\cdot\|_{\alpha,K_0}$ is the Hölder norm of order α on K_0 for the distance

$$\tilde{d}((t,x,y),(t',x',y')) = (d(x,x')^2 + d(y,y')^2 + |t-t'|)^{1/2}.$$

This implies (by a diagonal argument) that there is a subsequence n_k such that the functions $(t,x,y) \mapsto u_{n_k}(t,x,y)$ and their derivatives (up to the first order in t and second order in x, y) converge uniformly on the compact subsets of $\mathbb{R}_+ \times \mathbb{M}^2$. Let $\Psi(t,x,y)$ be the limit of this subsequence. For each $x \in \mathbb{M}$, $(t,y) \mapsto \Psi(t,x,y)$ is a smooth solution of the heat equation:

$$\left(\frac{\partial}{\partial t} - \frac{\Delta_y}{2}\right)\Psi = 0. \tag{2.3}$$

Let f and g be two continuous functions with compact support on $\mathbb{M}$ and let $r_k = 1/p_{s_{n_k}}(a,a)$. Then

$$\begin{aligned} \lim_{k\to+\infty} r_k \langle P_{s_{n_k}+t} f, f\rangle &= \lim_{k\to+\infty} \iint f(x)f(y)u_{n_k}(t,x,y)dm(x)dm(y) \\ &= \iint f(x)f(y)\Psi(t,x,y)\,dm(x)\,dm(y), \end{aligned}$$

hence if $\iint f(x)f(y)\Psi(0,x,y)dm(x)dm(y) \neq 0$,

$$\lim_{k\to+\infty} \frac{\langle P_{s_{n_k}+t} f, f\rangle}{\langle P_{s_{n_k}} f, f\rangle} = \frac{\iint f(x)f(y)\Psi(t,x,y)dm(x)dm(y)}{\iint f(x)f(y)\Psi(0,x,y)dm(x)dm(y)}.$$

It follows from Lemma 2.1 that

$$\iint f(x)f(y)\Psi(t,x,y)dm(x)dm(y) = e^{-\lambda_0 t}\iint f(x)f(y)\Psi(0,x,y)dm(x)dm(y)$$

and by polarization

$$\iint f(x)g(y)\Psi(t,x,y)dm(x)dm(y) = e^{-\lambda_0 t}\iint f(x)g(y)\Psi(0,x,y)dm(x)dm(y)$$

hence $\Psi(t,x,y) = e^{-\lambda_0 t}\psi(x,y)$ if $\psi(x,y) = \Psi(0,x,y)$. Now $\psi(x,y) = \psi(y,x)$ and

$$(2\frac{\partial}{\partial t} - \Delta_y)\Psi(t,x,y) = -e^{-\lambda_0 t}(2\lambda_0 + \Delta_y)\psi(x,y)$$

hence $(2\lambda_0 + \Delta_y)\psi(x,y) = (2\lambda_0 + \Delta_y)\psi(y,x) = 0$ by (2.3). ■

We will also use the following lemma, which is Theorem 4.1.1 of Pinsky [49] adapted to our setting. It introduces the notion of relativized φ-process which is coincides with the notion of h-process due to Doob when $\lambda_0 = 0$.

Lemma 2.3 *Let φ be a positive C^2 function on $\mathbb{M}$ such that $(\Delta+2\lambda_0)\varphi = 0$. Consider the second order elliptic operator L^φ defined by*

$$L^\varphi f = \frac{1}{2\varphi}\Delta(f\varphi) + \lambda_0 f = \frac{1}{2}\Delta f + \nabla \log\varphi \cdot \nabla f.$$

The semigroup of the minimal Markov process associated with L^φ has the transition semigroup (P_t^φ) defined by

$$P_t^\varphi f(x) = \int \frac{e^{\lambda_0 t} p_t(x,y)\varphi(y)}{\varphi(x)} f(y)\, dm(y),$$

for all measurable $f : \mathbb{M} \to \mathbb{R}_+$ and all $x \in \mathbb{M}$. We call it the relativized φ-process.

The relativized φ-process takes its values in $\hat{C}(\mathbb{R}_+, \mathbb{M} \cup \{\infty\})$, see Azencott [3] or Ikeda & Watanabe [33], Theorem 5.1.1. It is nonexploding and thus $C(\mathbb{R}_+, \mathbb{M})$-valued if and only if $P_t\varphi = e^{-\lambda_0 t}\varphi$ for some $t > 0$.

For a fixed point $a \in \mathbb{M}$, the Brownian bridge of length L around a is intuitively the Brownian motion $\{B_t, 0 \le t \le L\}$ conditioned by $\{B_0 = B_L = a\}$. It is rigorously defined as the non-homogeneous Markov process $\{B^{(L)}, 0 \le t \le L\}$ on $\mathbb{M}$ with generator

$$\frac{\Delta}{2} + \nabla(\log p_{L-t}(\cdot, a)) \cdot \nabla$$

starting from a. Its transition kernel $P_{s,t}^{(L)}$ is given when $0 < s < t < L$ by

$$P_{s,t}^{(L)} f(x) = \int_{\mathbb{M}} \frac{p_s(a,x)p_{t-s}(x,y)p_{L-t}(y,a)}{p_L(a,a)} f(y)\, dm(y). \tag{2.4}$$

We equip $\hat{C}(\mathbb{R}_+, \mathbb{M} \cup \{\infty\})$ with the topology of uniform convergence on compact sets. We define the infinite Brownian loop around a as the limit of $B^{(L)}$ in distribution in $\hat{C}(\mathbb{R}_+, \mathbb{M} \cup \{\infty\})$ as $L \to +\infty$, when this limit exists.

Proof of Theorem 1.2. Let $\mathcal{F}_t$ be the σ-algebra generated by the evaluation maps ω_s, $0 \le s \le t$, on $\hat{C}(\mathbb{R}_+, \mathbb{M} \cup \{\infty\})$. We denote by $\mathbb{P}_a$ the distribution of the Brownian motion B starting at a and by $\mathbb{P}_a^{(L)}$ the distribution of the Brownian bridge $B^{(L)}$ around a. The distribution $\mathbb{Q}$ of the infinite Brownian loop around a is the weak limit of $\mathbb{P}_a^{(L)}$ as $L \to +\infty$. By definition, this means that for any $t > 0$ and any $\mathcal{F}_t$-measurable continuous bounded function F on $\hat{C}(\mathbb{R}_+, \mathbb{M} \cup \{\infty\})$, $\int F\, d\mathbb{P}_a^{(L)}$ converges to $\int F\, d\mathbb{Q}$, i.e.

$$\lim_{L \to +\infty} \mathbb{E}_a \left[F \frac{p_{L-t}(B_t, a)}{p_L(a,a)} \right] = \int F\, d\mathbb{Q} \tag{2.5}$$

since it follows from (2.4) that

$$\int F\, d\mathbb{P}_a^{(L)} = \mathbb{E}_a \left[F \frac{p_{L-t}(B_t, a)}{p_L(a,a)} \right].$$

Let us first suppose that the I.B.L. around a exists. By Theorem 2.2, the set of functions $x \mapsto p_L(x,a)/p_L(a,a)$ is relatively compact for the topology of uniform convergence on compact sets. Let φ be a limit point of this set as L tends to infinity along a suitable sequence (L_n). It follows from Lemma 2.1 that, for all $t \ge 0$

$$\lim_{n \to +\infty} \frac{p_{L_n}(x,a)}{p_{L_n+t}(a,a)} = \lim_{n \to +\infty} \frac{p_{L_n}(x,a)}{p_{L_n}(a,a)} \frac{p_{L_n}(a,a)}{p_{L_n+t}(a,a)} = e^{\lambda_0 t} \varphi(x).$$

We deduce from (2.5) that for any continuous function $f : \mathbb{M} \mapsto \mathbb{R}$ with compact support,

$$\int f(\omega_t)\, d\mathbb{Q}(\omega) = \lim_{n \to +\infty} \mathbb{E}_a \left[f(B_t) \frac{p_{(L_n+t)-t}(B_t, a)}{p_{L_n+t}(a,a)} \right] = \mathbb{E}_a \left[f(B_t) \varphi(B_t) e^{\lambda_0 t} \right].$$

This formula determines φ, hence the functions $p_L(\cdot, a)/p_L(a,a)$ do indeed converge to φ as $L \to +\infty$.

Conversely, let us suppose that $p_L(\cdot, a)/p_L(a,a)$ converges to some function φ and let us show that the I.B.L. converges to the relativized φ-process. It follows from Lemma 2.1 and Theorem 2.2 that $p_{L-t}(x,a)/p_L(a,a)$ converges to $e^{\lambda_0 t}\varphi(x)$ uniformly on compact sets and that $(\Delta + 2\lambda_0)\varphi = 0$.

We first suppose that the relativized φ-process does not explode. In this case $P_t\varphi = e^{-\lambda_0 t}\varphi$, thus

$$\mathbb{E}_a[e^{\lambda_0 t}\varphi(B_t)] = e^{\lambda_0 t} P_t\varphi(a) = \varphi(a) = 1.$$

Therefore, for any $L > t$,

$$\mathbb{E}_a\left[\frac{p_{L-t}(B_t,a)}{p_L(a,a)}\right] = 1 = \mathbb{E}_a[e^{\lambda_0 t}\varphi(B_t)].$$

Hence $\frac{p_{L-t}(B_t,a)}{p_L(a,a)}$ converges to $e^{\lambda_0 t}\varphi(B_t)$ in $L^1(\mathbb{P}_a)$ by Scheffe's theorem. Thus, for any $\mathcal{F}_t$-measurable bounded function F,

$$\lim_{L\to+\infty} \mathbb{E}_a\left[F\frac{p_{L-t}(B_t,a)}{p_L(a,a)}\right] = \mathbb{E}_a\left[Fe^{\lambda_0 t}\varphi(B_t)\right] = \int F\, d\mathbb{Q},$$

and (2.5) holds.

If there is explosion, the proof is more delicate (there is no explosion on symmetric spaces, thus this proof will not be used in other sections). Let $\mathbb{Q}$ be the distribution on $\hat{C}(\mathbb{R}_+, \mathbb{M} \cup \{\infty\})$ of the relativized φ-process. Let $\zeta(\omega) = \inf\{t \geq 0; \omega_t = \infty\}$. For all stopping time σ, if F is a bounded $\mathcal{F}_\sigma$-measurable bounded random variable

$$\mathbb{E}_a\left[Fe^{\lambda_0\sigma}\varphi(B_\sigma)\ ;\ \sigma < \zeta\right] = \mathbb{Q}[F; \sigma < \zeta]. \tag{2.6}$$

We first verify this formula when σ takes its values in a finite set $\{t_k, 1 \leq k \leq n\}$. In this case,

$$\begin{aligned}\mathbb{E}_a\left[Fe^{\lambda_0\sigma}\varphi(B_\sigma); \sigma < \zeta\right] &= \sum_{i=1}^n \mathbb{E}_a\left[Fe^{\lambda_0 t_i}\varphi(B_{t_i}); \sigma = t_i, B_{t_i} \in \mathbb{M}\right] \\ &= \sum_{i=1}^n \mathbb{Q}[F; \sigma = t_i, \omega_{t_i} \in \mathbb{M}] = \mathbb{Q}[F; \sigma < \zeta].\end{aligned}$$

One then uses the fact that every stopping time σ is the limit of a decreasing sequence (σ_n) of such finite valued stopping times (notice that $e^{\lambda_0\sigma_n}\varphi(B_{\sigma_n})$ is a supermartingale and thus converges to $e^{\lambda_0\sigma}\varphi(B_\sigma)$ in L^1). Actually this formula is well known and characterizes $\mathbb{Q}$ as the Föllmer's measure associated with the supermartingale $e^{\lambda_0 t}\varphi(B_t)$ under $\mathbb{P}_a$ (see Föllmer [20] or Dellacherie & Meyer [18], XVI.2.29), however we will only use (2.6).

Let $T > 0$. We have to prove that, for any bounded $\mathcal{F}_T$-measurable uniformly continuous function Ψ on $\hat{C}(\mathbb{R}_+, \mathbb{M} \cup \{\infty\})$,

$$\lim_{L \to +\infty} \mathbb{E}_a^{(L)}[\Psi] = \mathbb{Q}[\Psi]. \tag{2.7}$$

Let K be some compact set in $\mathbb{M}$ with $a \in K$. Define

$$\ell = \sup\{t \leq T \mid B_t \in K\}.$$

Let also $B_s^{(t)} = B_{s \wedge t}$. When $\ell < u \leq T$, B_u is in K^c, hence $\left|\Psi\left(B^{(\ell)}\right) - \Psi(B)\right|$ is uniformly controlled by the distance of ∞ to K and can be made arbitrary small by an appropriate choice of K. Thus it remains to prove that

$$\lim_{L \to +\infty} \mathbb{E}_a^{(L)}[J_\ell] = \mathbb{Q}[J_\ell] \tag{2.8}$$

where $J_t = \Psi\left(B^{(t)}\right)$. Let (V_t) be some adapted bounded increasing process. Let us first show that

$$\lim_{L \to +\infty} \mathbb{E}_a^{(L)}[V_\ell] = \mathbb{Q}[V_\ell]. \tag{2.9}$$

For $s \in [0, T]$, let

$$\sigma(s) = \inf\{t \geq s \mid B_t \in K\},$$

one has,

$$\{s \leq \ell\} = \{\sigma(s) \leq T\}.$$

Using Theorem 2.2 we can choose a $C > 0$ such that

$$\frac{p_{L-t}(x, a)}{p_L(a, a)} \leq C$$

for all $t \in [0, T], x \in K, L \geq 2T$. We have

$$\begin{aligned}
\mathbb{E}_a^{(L)}[V_\ell] &= \mathbb{E}_a\left[\tfrac{p_{L-T}(B_T, a)}{p_L(a,a)} \int_{[0,T]} 1_{\{s \leq \ell\}}\, dV_s\right] \\
&= \mathbb{E}_a\left[\int_{[0,T]} \mathbb{E}_a\left[\tfrac{p_{L-T}(B_T, a)}{p_L(a,a)} 1_{\{\sigma(s) \leq T\}} \mid \mathcal{F}_{\sigma(s)}\right] dV_s\right] \\
&= \mathbb{E}_a\left[\int_{[0,T]} \tfrac{p_{L-\sigma(s)}\left(B_{\sigma(s)}, a\right)}{p_L(a,a)} 1_{\{\sigma(s) \leq T\}}\, dV_s\right] \\
&\leq C\, \mathbb{E}_a\left[\int_{[0,T]} 1_{\{\sigma(s) \leq T\}}\, dV_s\right] = C\mathbb{E}_a[V_\ell].
\end{aligned}$$

Thus

$$\mathbb{E}_a^{(L)}[V_\ell] \leq C\, \mathbb{E}_a[V_\ell]. \tag{2.10}$$

On the other hand, using Lebesgue's theorem,

$$\mathbb{E}_a^{(L)}\left[V_\ell\right] = \mathbb{E}_a\left[\int_{[0,T]} \frac{p_{L-\sigma(s)}\left(B_{\sigma(s)},a\right)}{p_L(a,a)} 1_{\{\sigma(s)\le T\}}\, dV_s\right] \underset{L\to+\infty}{\longrightarrow} \mathbb{E}_a\left[\int_{[0,T]} e^{\lambda_0\sigma(s)}\varphi\left(B_{\sigma(s)}\right) 1_{\{\sigma(s)\le T\}}\, dV_s\right].$$

Now, let v be the right-continuous inverse of V. One has, using (2.6),

$$\begin{aligned}
&\mathbb{E}_a\left[\int_{[0,T]} e^{\lambda_0\sigma(s)}\,\varphi\left(B_{\sigma(s)}\right) 1_{\{\sigma(s)\le T\}}\, dV_s\right] \\
&= \int_0^{+\infty} \mathbb{E}_a\left[e^{\lambda_0\sigma(v(y))}\varphi\left(B_{\sigma(v(y))}\right)\,;\,\sigma\left(v\left(y\right)\right)\le T\right]\, dy \\
&= \int_0^{+\infty} \mathbb{E}_a\left[e^{\lambda_0\sigma(v(y))}\varphi\left(B_{\sigma(v(y))}\right)\,;\,\sigma\left(v\left(y\right)\right)\le T, \sigma\left(v\left(y\right)\right)<\zeta\right]\, dy \\
&= \int_0^{+\infty} \mathbb{Q}\left[\,\sigma\left(v\left(y\right)\right)\le T, \sigma\left(v\left(y\right)\right)<\zeta\right]\, dy \\
&= \mathbb{Q}\left[\int_0^{+\infty} 1_{\{\sigma(v(y))<\zeta,\,\sigma(v(y))\le T\}}\, dy\right] = \mathbb{Q}\left[\int_0^{+\infty} 1_{\{\sigma(s)<\zeta,\,\sigma(s)\le T\}}\, dV_s\right] \\
&= \mathbb{Q}\left[\int_0^{+\infty} 1_{\{s\le \ell\}}\, dV_s\right] = \mathbb{Q}\left[V_\ell\right],
\end{aligned}$$

since $\mathbb{Q}\left[\sigma(s)<\zeta\right] = 1$. This shows (2.9). Let us now prove (2.8). The process J is continuous, adapted and bounded. Define for $\varepsilon>0$,

$$J_t^{(\varepsilon)} = \int_{-\infty}^t J\left(t-x\right)\frac{1}{\varepsilon}\, g\left(\frac{x}{\varepsilon}\right) dx,$$

where g is a smooth nonnegative bounded function on $\mathbb{R}$, with support in $]0,1[$ such that $\int_{\mathbb{R}} g\left(x\right) dx = 1$. The map $t\to J_t^{(\varepsilon)}$ is C^1, with a bounded derivative when $0\le t\le T$. Therefore $J^{(\varepsilon)}$ is the difference of two bounded increasing processes, and it follows from (2.9) that

$$\lim_{L\to+\infty} \mathbb{E}_a^{(L)}\left[J_\ell^{(\varepsilon)}\right] = \mathbb{Q}\left[J_\ell^{(\varepsilon)}\right]. \tag{2.11}$$

Since $|J_t^{(\varepsilon)} - J_t| \le \sup_{a,b\le t,|a-b|<\varepsilon}|J_a-J_b|$, one also has for all $L>0$, by (2.10),

$$\mathbb{E}_a^{(L)}\left[|J_\ell^{(\varepsilon)} - J_\ell|\right] \le C\,\mathbb{E}_a\left[\sup_{a,b\le \ell,|a-b|<\varepsilon}|J_a-J_b|\right] \underset{\varepsilon\to0}{\to} 0,$$

thus (2.11) implies that (2.8) holds. This concludes the proof. ■

The following proposition is straightforward. Notice that it implies that the infinite Brownian loop has no spectral gap on $L^2(\varphi^2.m)$ and that it is its own I.B.L.

Proposition 2.4 *Let φ be the ground state associated with the I.B.L. around a and let p_t^0 be the density of its semigroup with respect to the measure $\varphi^2.m$. Then $p_t^0(x,y) = p_t^0(y,x) = \frac{e^{\lambda_0 t} p_t(x,y)}{\varphi(x)\varphi(y)}$ for all $x, y \in \mathbb{M}$ and*

$$\lim_{t \to +\infty} \frac{p_{t+s}^0(b,a)}{p_t^0(a,a)} = 1$$

for all $s > 0$ and $b \in \mathbb{M}$.

Let us give two examples. Consider first a complete model manifold $\mathbb{M}$ in the sense of Grigor'yan [24], also called a spherically symmetric manifold with a pole. Without loss of generality, one can take $\mathbb{M} = \mathbb{R}^n$ with a Riemannian metric which can be written in polar coordinates as

$$ds^2 = dr^2 + \sigma^2(r) d\theta^2$$

where $d\theta^2$ is the standard metric on the sphere $\mathbb{S}^{n-1}$. Necessarily the function σ is a smooth positive function on $\mathbb{R}_+$ and $\sigma(0) = 0, \sigma'(0) = 1$.

Corollary 2.5 *On a complete model manifold, the I.B.L. around 0 exists. It is equal to its Brownian motion when $\lambda_0 = 0$.*

Proof. The heat kernel $p_t(x, 0)$ depends only on $r = d(0, x)$. Therefore, if for some $t_k \to +\infty$,

$$\varphi(x) = \lim_{k \to +\infty} \frac{p_{t_k}(x,0)}{p_{t_k}(0,0)},$$

then φ is radial: there is a smooth positive function f on $\mathbb{R}_+$ such that $\varphi(x) = f(d(0,x))$. It follows from Theorem 2.2 that $(\Delta + 2\lambda_0)\varphi = 0$. In polar coordinates the Laplace Beltrami operator Δ is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + (n-1)\frac{\sigma'}{\sigma}\frac{\partial}{\partial r} + \frac{1}{\sigma^2}\Delta_\theta$$

(see [24]), hence $f'' + (n-1)\frac{\sigma'}{\sigma} f' = 2\lambda_0 f$. By the theorems 1.2 and 2.2, we are reduced to proving that all the smooth positive solutions of this equation are proportional. If g is another solution and $h = (g/f)'$, then

$$h' + [\log(f^2 \sigma^{n-1})]' h = 0.$$

Thus there is a $c \geq 0$ such that $h = cf^{-2}\sigma^{1-n}$. This function is smooth only if $c = 0$, i.e. when f and g are proportional. When $\lambda_0 = 0$, $g \equiv 1$ is a solution, hence $\varphi \equiv 1$. ■

Another interesting example is the Brownian motion of $\mathbb{M} = \mathbb{R}_+^*$. This is the usual Brownian motion on $\mathbb{R}_+$ killed when it reaches 0. If follows from the reflexion principle that

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}}\left[\exp(-\frac{(x-y)^2}{2t}) - \exp(-\frac{(x+y)^2}{2t})\right]$$

which implies that, for all $x, y, a, b > 0$, $s > 0$,

$$\lim_{t\to+\infty} \frac{p_{t+s}(x,y)}{p_t(a,b)} = \frac{xy}{ab}.$$

In this case $\lambda_0 = 0$ and $\varphi(x) = x$. Thus B^0 is the 3-dimensional Bessel process. The intrinsic Brownian motion in a Weyl chamber considered in Section 3 is a generalization of this example to some other cones.

One can also consider the simultaneous behavior of the two ends of the Brownian bridge $\{B^{(L)}, 0 \leq t \leq L\}$ around a as $L \to \infty$. More generally, given two points $a, b \in \mathbb{M}$, it is interesting to look at the two ends of the Brownian bridge $\{B_t^{(L,a,b)}, 0 \leq t \leq L\}$ which is the Brownian motion $\{B_t, 0 \leq t \leq L\}$ conditioned by $B_0 = a, B_L = b$. We first observe that if F and G are two bounded $\mathcal{F}_t$-measurable functions on $\hat{C}(\mathbb{R}_+, \mathbb{M} \cup \{\infty\})$, then, as soon as $L > 2t$,

$$\mathbb{E}\left[F(B_s^{(L,a,b)}, s \leq t) G(B_{L-s}^{(L,a,b)}, s \leq t)\right]$$
$$= \mathbb{E}\left[F(B_s, s \leq t)\frac{p_{L-2t}(B_t, \tilde{B}_t)}{p_L(a,b)} G(\tilde{B}_s, s \leq t)\right],$$

where B and $\tilde{B}$ are two independent copies of the Brownian motion on $\mathbb{M}$, such that $B_0 = a, \tilde{B}_0 = b$ (in other words $(B, \tilde{B})$ is the Brownian motion on $\mathbb{M}^2$ starting from (a, b)). Using this representation, the proof of the following proposition is the same as the proof of Theorem 1.2.

Proposition 2.6 *As $L \to +\infty$, the processes $\{(B_t^{(L,a,b)}, B_{L-t}^{(L,a,b)}), 0 \leq t \leq L\}$ converge in distribution if and only if the limit*

$$\psi(x,y) = \lim_{t\to+\infty} \frac{p_t(x,y)}{p_t(a,b)} \tag{2.12}$$

exists for all $x, y \in \mathbb{M}$. The function ψ is a ground state for the Brownian motion on $\mathbb{M}^2$. The limit process $(B^0, \tilde{B}^0)$ is the relativized ψ-process, starting from (a, b). We call it the double-ended infinite Brownian loop from a to b.

Notice that the two ends B^0 and $\tilde{B}^0$ are independent if and only if one can write ψ as a product (i.e. $\psi(x,y) = \varphi(x)\varphi(y)$). Recently, Collet, Martinez & San Martin [11] have given very interesting examples of domains in $\mathbb{R}^n$ where (2.12) holds and where ψ is not a product.

Proposition 1.3 is a consequence of:

Corollary 2.7 *Suppose that there is a unique ground state, i.e. a unique C^2 positive solution φ of $(\Delta + 2\lambda_0)\varphi = 0$, up to a multiplicative constant. Then*

$$\lim_{s\to+\infty} \frac{p_{s+t}(x,y)}{p_s(a,b)} = e^{-\lambda_0 t}\frac{\varphi(x)\varphi(y)}{\varphi(a)\varphi(b)} \tag{2.13}$$

for all $x,y,a,b \in \mathbb{M}$. The infinite Brownian loop is the relativized φ-process and the double-ended infinite Brownian loop is given by two independent copies of the relativized φ-process.

Proof. By Theorem 2.2, the set of functions $\{(x,y) \mapsto p_s(x,y)/p_s(a,a), s \geq 1\}$ is relatively compact. Let ψ be a limit point of this set as $s \to +\infty$. For each $y \in \mathbb{M}$, $x \mapsto \psi(x,y)$ is a ground state, hence there exists $c(y) > 0$ such that $\psi(x,y) = c(y)\varphi(x)$. Since $y \mapsto \psi(x,y)$ is also a ground state, there exists $\gamma > 0$ such that $c(y) = \gamma\varphi(y)$. Thus $\psi(x,y) = \gamma\varphi(x)\varphi(y)$. Noticing that $\psi(a,a) = 1$ we obtain that $\psi(x,y) = \frac{\varphi(y)\varphi(x)}{\varphi(a)\varphi(a)}$. One concludes easily the proof by using Lemma 2.1. ∎

Example 1. Suppose that $\mathbb{M}$ is a λ_0-recurrent manifold, i.e.

$$\int_1^\infty e^{\lambda_0 t} p_t(x,y)\,dt = +\infty$$

for some x, y in $\mathbb{M}$. Then there is a unique ground state (see Theorem 4.3.4 in Pinsky [49]). Hence the corollary holds in this situation. This generalizes Theorem 28 in Davies [15].

Example 2. If $\mathbb{M}$ has a nonnegative Ricci curvature it follows from Li & Yau [37] that the hypothesis of the corollary holds true with $\lambda_0 = 0$. This was already proved by Davies (see [15], Theorem 27). The same result is obviously true for compact manifolds. In these cases the I.B.L. is the Brownian motion itself.

Example 3. If $\mathbb{M}$ is a bounded connected open set in $\mathbb{R}^n$ with smooth boundary, then it is well known that there is a unique ground state. Hence the corollary holds. The I.B.L. is the same as the Euclidean Brownian motion in $\mathbb{M}$ killed at the boundary and conditioned to have an infinite lifetime, see Pinsky [48]. This is also the intrinsic process considered in Davies & Simon

[16, 17]. Some unbounded domains are studied in Collet, Martinez & San Martin [11, 12].

Example 4. It follows from Sullivan [52], Example 8.4, that the (double-ended) I.B.L. does not explode when the geometry of $\mathbb{M}$ is bounded. On the other hand, Pinchover has given in [46] an example of an exploding Brownian motion on a Riemannian manifold such that $\lambda_0 = 0$ and such that the only positive harmonic functions are constant. In this case the I.B.L. is the Brownian motion itself and thus is exploding.

Davies [15] has conjectured that

$$\lim_{t\to+\infty} \frac{p_t(x,y)}{p_t(a,a)}$$

exists on any Riemannian manifold. This amounts to the existence of the double-ended I.B.L.

3. The intrinsic Brownian motion of a Weyl chamber

3.1. A definition of the intrinsic B.M. of a Weyl chamber

Let $\mathfrak{a}$ be an Euclidean space of dimension d equipped with a reduced root system $\Sigma_0 \subset \mathfrak{a}$, see Helgason ([31], X.3.1). Recall that a root system is reduced if the only roots proportional to a root α are α and $-\alpha$. We choose a Weyl chamber $\mathfrak{a}_+$ in $\mathfrak{a}$, i.e. a connected component of $\{x \in \mathfrak{a}\,;\, \langle \alpha, x\rangle \neq 0, \forall \alpha \in \Sigma_0\}$. This is an open convex cone. Let $\bar{\mathfrak{a}}_+$ be its closure and $\partial\mathfrak{a}_+ = \bar{\mathfrak{a}}_+ - \mathfrak{a}_+$ be its boundary. The set of positive roots is $\Sigma_0^+ = \{\alpha \in \Sigma_0\,;\, \langle\alpha, x\rangle > 0, \forall x \in \mathfrak{a}_+\}$. The function

$$\boldsymbol{\pi}(x) = \prod_{\alpha\in\Sigma_0^+} \langle \alpha, x\rangle,\ x \in \mathfrak{a}. \tag{3.1}$$

is harmonic for the Euclidean Laplacian $\Delta_{\mathfrak{a}}$ on $\mathfrak{a}$, cf. Helgason ([32], Theorem III.3.6). Biane [5] and Grabiner [23] have considered the following stochastic process in $\mathfrak{a}_+ \cup \{0\}$ which will play a major role in this paper.

Definition 3.1 *The intrinsic Brownian motion of $\mathfrak{a}_+$ is the continuous Markov process Z_t such that $Z_0 = 0$, and for all $t > 0$,*

(i) $Z_t \in \mathfrak{a}_+$ and Z_t is the relativized $\boldsymbol{\pi}$-process of the Brownian motion on $\mathfrak{a}_+$ killed at the boundary $\partial\mathfrak{a}_+$. We denote by $h_t(x,y), x, y \in \mathfrak{a}_+$, the density of its semigroup with respect to the Lebesgue measure.

(ii) The distribution of Z_t has the density $h_t(0,y) = \lim_{x\to 0} h_t(x,y)$.

By definition the generator of Z_t inside $\mathfrak{a}_+$ is

$$L^{\boldsymbol{\pi}} = \frac{1}{2}\Delta_{\mathfrak{a}} + \nabla_{\mathfrak{a}} \log \boldsymbol{\pi} \cdot \nabla_{\mathfrak{a}}. \tag{3.2}$$

where $\Delta_{\mathfrak{a}}$ is the Euclidean Laplacian on $\mathfrak{a}$ and $\nabla_{\mathfrak{a}}$ is its gradient. As noticed by Biane, $\boldsymbol{\pi}$ is, up to a multiplicative constant, the unique positive harmonic function on $\mathfrak{a}_+$ equal to 0 on the boundary. Therefore, inside $\mathfrak{a}_+$, the intrinsic Brownian motion can be interpreted as the Brownian motion in $\mathfrak{a}_+$, killed at the boundary and conditioned to go to ∞, or equivalently conditioned to remains alive. In some particular cases, this process inside $\mathfrak{a}_+$ was already considered by Dyson [19], see also Neveu [44]. The name "intrinsic" is borrowed from Davies & Simon [16], [17].

The point 0 is singular in $\mathfrak{a}_+ \cup \{0\}$. This explain why the entry distribution of the Markov process starting at 0 has to be specified by (ii). Since the finite-dimensional distributions are given, the definition determines the process. However, its existence is not completely obvious. We will show it in the next subsection by an explicit construction as a generalized Bessel process. The fact that Z_t remains in $\mathfrak{a}_+$ for all $t > 0$ is related to the following lemma that we will need later.

Lemma 3.2 *Let β_t be a Brownian motion on $\mathfrak{a}$. For any $x \in \mathfrak{a}_+$, the solution of the stochastic integral equation*

$$X_t = x + \beta_t + \int_0^t \nabla_{\mathfrak{a}} \log \boldsymbol{\pi}\,(X_s)\; ds \tag{3.3}$$

is in $\mathfrak{a}_+$ for all $t > 0$.

Proof. The function $\nabla_{\mathfrak{a}} \log \boldsymbol{\pi}$ is C^∞ on $\mathfrak{a}_+$. Thus the equation (3.3) has a unique maximal solution X in $\mathfrak{a}_+$, defined on a time interval $[0, \zeta[$ where ζ is an explosion time or the exit time from $\mathfrak{a}_+$. The function $\boldsymbol{\pi}$ is harmonic and positive on $\mathfrak{a}_+$, hence

$$\Delta_{\mathfrak{a}} \boldsymbol{\pi}^{-1} + 2\left\langle \nabla_{\mathfrak{a}} \log \boldsymbol{\pi}, \nabla_{\mathfrak{a}} \boldsymbol{\pi}^{-1} \right\rangle = 0.$$

By Ito's formula, $\boldsymbol{\pi}^{-1}(X_{t\wedge\zeta})$ is a positive local martingale, thus it converges a.s. when $t \to +\infty$. Since $\boldsymbol{\pi} = 0$ on $\partial\mathfrak{a}_+$, this implies that $\|X_t\| \to +\infty$ when $t \to \zeta$. On the other hand $\mathcal{E} \log \boldsymbol{\pi} = |\Sigma_0^+|$ where $\mathcal{E} = \sum_{i=1}^d x_i \frac{\partial}{\partial x_i}$ is the Euler operator on $\mathfrak{a}$ and $d = \dim \mathfrak{a}$. Let $D = d + 2|\Sigma_0^+|$. By Ito's formula,

$$\begin{aligned} \|X_{t\wedge\zeta}\|^2 &= \|x\|^2 + 2\int_0^{t\wedge\zeta} \langle X_s, d\beta_s\rangle + 2\int_0^{t\wedge\zeta} \mathcal{E}\,(\log \boldsymbol{\pi})\,(X_s)\; ds + (\,t\wedge\zeta)d \\ &= \|x\|^2 + 2\int_0^{t\wedge\zeta} \|X_s\|\, d\widetilde{\beta}_s + \,(t\wedge\zeta)D. \end{aligned}$$

where $\widetilde{\beta}$ is the stopped real Brownian motion

$$\widetilde{\beta}_t = \int_0^{t\wedge\zeta} \|X_s\|^{-1} \langle X_s, d\beta_s \rangle \ .$$

This shows that $\|X_{t\wedge\zeta}\|^2$ is the square of a D-dimensional Bessel process, stopped at time ζ (see Yamada [56] or Revuz & Yor [50], XI.1). Since $X_t \to +\infty$ when $t \to \zeta$, one has $\zeta = +\infty$ almost surely. ■

3.2. The intrinsic Brownian motion of a Weyl chamber as a generalized Bessel process

In this subsection we consider a complex semisimple Lie algebra $\mathfrak{g}$. Let J be the complex structure, $\mathfrak{k}$ be a compact real form of $\mathfrak{g}$ and $\mathfrak{p} = J\mathfrak{k}$. Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. The adjoint group K of $\mathfrak{k}$ is compact. We equip $\mathfrak{p}$ with the Euclidean structure given by the Killing form. For each $k \in K$, $Ad(k)\mathfrak{p} = \mathfrak{p}$ and $Ad(k)$ is a linear isometry of $\mathfrak{p}$. Let $\mathfrak{a}$ be a maximal Abelian subspace of $\mathfrak{p}$. Since $\mathfrak{g}$ is complex, the root system associated with the pair $(\mathfrak{g}, \mathfrak{a})$ is reduced, hence we denote it by Σ_0. Let $\mathfrak{a}_+$ be a Weyl chamber in $\mathfrak{a}$. We can introduce a generalized polar decomposition in $\mathfrak{p}$: for each $x \in \mathfrak{p}$ there exists $k \in K$ and $R(x) \in \bar{\mathfrak{a}}_+$ such that $x = Ad(k)R(x)$. The element $R(x)$ is uniquely determined.

Proposition 3.3 *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. If W_t is the Euclidean Brownian motion on $\mathfrak{p}$ starting from 0, then $R(W_t)$ is the intrinsic Brownian motion of $\mathfrak{a}_+$.*

Proof. Let $\Delta_\mathfrak{p}$ be the Euclidean Laplacian on $\mathfrak{p}$. Since $\mathfrak{g}$ is complex, it follows from Helgason ([32], Proposition II.3.13) that the radial part of $\frac{1}{2}\Delta_\mathfrak{p}$ on $\mathfrak{a}_+$ is the operator $L^{\boldsymbol{\pi}}$ defined by (3.2). Therefore the generator of $R(W_t)$ inside $\mathfrak{a}_+$ is $L^{\boldsymbol{\pi}}$. For all measurable $f : \mathfrak{p} \to \mathbb{R}_+$,

$$\int_\mathfrak{p} f(x)\, dx = c_0 \int_K \int_{\mathfrak{a}_+} f(Ad(k)y)\, \boldsymbol{\pi}(y)^2\, dk\, dy \tag{3.4}$$

(see Helgason, [30], Proposition X.1.17, [32], Theorem I.5.17) where dx and dy are the Lebesgue measures on $\mathfrak{p}$ and $\mathfrak{a}^+$, dk is the normalized Haar measure on K, $\boldsymbol{\pi}(y) = \prod_{\alpha\in\Sigma_0^+} \langle \alpha, y \rangle$ and $c_0 = \mathrm{Vol}(K/M)$. Let $D = \dim \mathfrak{p}$. The density of the semigroup of the Brownian motion W_t is $g_t(x_1, x_2) =$

$(2\pi t)^{-D/2} e^{-\frac{\|x_1 - x_2\|^2}{2t}}$. Therefore if f is a bounded measurable function on $\bar{\mathfrak{a}}_+^2$,

$$\mathbb{E}[f(R(W_s), R(W_{s+t}))] = \int_{\mathfrak{p}^2} f(R(x_1), R(x_2)) g_s(0, x_1) g_t(x_1, x_2)\, dx_1 dx_2$$

$$= \int_{\mathfrak{a}_+^2} f(y_1, y_2) c_0^2 \boldsymbol{\pi}(y_1)^2 \boldsymbol{\pi}(y_2)^2 \times$$

$$\times \left\{ \int_{K^2} g_s(0, Ad(k_1)y_1) g_t(Ad(k_1)y_1, Ad(k_2)y_2)\, dk_1 dk_2 \right\} dy_1 dy_2$$

$$= \int_{\mathfrak{a}_+^2} f(y_1, y_2) c_0^2 \boldsymbol{\pi}(y_1)^2 \boldsymbol{\pi}(y_2)^2 \left\{ \int_{K^2} g_s(0, y_1) g_t(y_1, Ad(k_1^{-1}k_2)y_2)\, dk_1 dk_2 \right\} dy_1 dy_2$$

$$= \int_{\mathfrak{a}_+^2} f(y_1, y_2) g_s(0, y_1) c_0 \boldsymbol{\pi}(y_1)^2 \left\{ \int_K g_t(y_1, Ad(k)y_2) c_0 \boldsymbol{\pi}(y_2)^2\, dk \right\} dy_1 dy_2$$

This shows that the density $h_t(y_1, y_2)$ of the semigroup of $R(W_t)$ is

$$\begin{aligned} h_t(y_1, y_2) &= c_0 \boldsymbol{\pi}(y_2)^2 \int_K g_t(y_1, Ad(k)y_2)\, dk \\ &= \frac{c_0}{(2\pi t)^{D/2}} \boldsymbol{\pi}(y_2)^2 \int_K \exp\left(-\frac{\|y_1 - Ad(k)y_2\|^2}{2t}\right) dk, \end{aligned}$$

and that

$$h_s(0, y) = c_0 (2\pi s)^{-(D/2)}\, \boldsymbol{\pi}^2(y)\, e^{-\frac{\|y\|^2}{2s}} \tag{3.5}$$

Thus $h_t(0, y_2) = \lim_{y_1 \to 0} h_t(y_1, y_2)$. This also implies that, for all $t_0 > 0$ fixed, $R(W_{t_0}) \in \mathfrak{a}_+$ almost surely. For $t > t_0$, $X_t = R(W_{t_0+t})$ is a solution of (3.3), hence $R(W_t) \in \mathfrak{a}_+$ for all $t \geq t_0$, by Lemma 3.2. Since t_0 is arbitrary, $R(W_t) \in \mathfrak{a}_+$ for all $t > 0$. We have verified that $R(W_t)$ satisfies all the properties of the intrinsic Brownian motion of $\mathfrak{a}_+$. ∎

Of course, c_0 can be computed:

$$c_0 = (2\pi)^{|\Sigma_0^+|} \boldsymbol{\pi}(\rho^0)^{-1} \quad \text{where } \rho^0 = \frac{1}{2} \sum_{\alpha \in \Sigma_0^+} \alpha\,.$$

By writing that $h_1(0, x)$ is the density of a probability measure, this formula is equivalent to the relation

$$h_1(0, x) = \frac{1}{\boldsymbol{\pi}(\rho^0)(2\pi)^{d/2}} \boldsymbol{\pi}(x)^2 e^{-\frac{\|x\|^2}{2}}.$$

We will obtain it in the proof of Lemma 5.7. It can also be proved directly by applying the differential operator $\boldsymbol{\pi}(\frac{\partial}{\partial\lambda})^2\big|_{\lambda=0}$ to the classical Fourier transform

$$\int_{\mathfrak{a}} e^{-\frac{\|x\|^2}{2}}\, e^{i\,\langle\lambda,x\rangle}\, dx = (2\pi)^{\frac{d}{2}}\, e^{-\frac{\|\lambda\|^2}{2}}\,.$$

The intrinsic Brownian motion $R(W_t)$ is scale invariant: for any $T>0$, $\{\frac{1}{\sqrt{T}}R(W_{tT}), t \geq 0\}$ has the same distribution as $\{R(W_t), t \geq 0\}$. Since $R(W_t)$ is a generalized radial part of a standard Brownian motion, we can consider it as a generalized Bessel process, notice that $\|R(W_t)\| = \|W_t\|$.

Let us consider now a general Weyl chamber $\mathfrak{a}_+$ in an Euclidean space $\mathfrak{a}$ as defined in 3.1. The integer

$$D = \dim \mathfrak{a} + 2\,|\Sigma_0^+|,$$

depends only on $\mathfrak{a}_+$. It follows from Dynkin's classification that there is exactly one complex semisimple Lie algebra $\mathfrak{g}$ with Weyl chamber $\mathfrak{a}_+$ (see Helgason [31], X.3.3). Thus the above proposition gives a realization of the intrinsic Brownian motion for every Weyl chamber. When $\mathfrak{g}$ is complex, $D = \dim \mathfrak{p} = \frac{1}{2}\dim \mathfrak{g}$. Therefore we see that $\|R(W_t)\|$ is also the norm of the D-dimensional Brownian motion $\|W_t\|$. In other words (see also Grabiner [23] or the proof of Lemma 3.2)

Corollary 3.4 *The norm of the intrinsic B.M. of $\mathfrak{a}_+$ is a Bessel process of dimension D.*

For instance, when $\mathfrak{a} = \mathbb{R}$, $\mathfrak{a}_+ = \mathbb{R}_+^*$ and the infinite Brownian loop is the Bessel process of dimension 3. In this case $\boldsymbol{\pi}(x) = x$ and $\mathfrak{g} = \mathfrak{s}l(2,\mathbb{C})$.

4. The infinite Brownian loop on symmetric spaces

In this section we study the I.B.L. on a Riemannian simply connected symmetric space. Any symmetric space $\mathbb{M}$ can be decomposed as the direct product $\mathbb{M} = \mathbb{M}_1 \times \mathbb{M}_2 \times \mathbb{M}_3$, where $\mathbb{M}_1$ is of the so-called noncompact type, $\mathbb{M}_2$ is of the Euclidean type (i.e. $\mathbb{M}_2 = \mathbb{R}^d$ for some $d > 0$) and $\mathbb{M}_3$ is of the compact type (i.e. $\mathbb{M}_3$ is compact). The metric is the product metric, hence the Brownian motion W on $\mathbb{M}$ can be written as $W = (B, B', B'')$, where B, B', B'' are three independent Brownian motions. The processes B' and B'' are their own I.B.L., hence the infinite Brownian loop on $\mathbb{M}$ is $W^{(0)} = (B^{(0)}, B', B'')$, where $B^{(0)}$ is the I.B.L. of B. Thus we are reduce to studying only the noncompact type component.

From now on, let us consider a symmetric space $\mathbb{M}$ of the noncompact type. By definition, one can write $\mathbb{M} = G/K$, where G is a semisimple noncompact connected group with finite center and K is a maximal compact subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition. We choose a maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. We equip it with the Euclidean structure given by the Killing form and use it to identify $\mathfrak{a}$ with its dual. Let Σ be the root system of $(\mathfrak{g}, \mathfrak{a})$, $\mathfrak{a}_+$ an open Weyl chamber and Σ^+ the corresponding set of positive roots. The set of indivisible roots $\Sigma_0 = \{\alpha \in \Sigma; \frac{1}{2}\alpha \notin \Sigma\}$ is a reduced root system. We set $\Sigma_0^+ = \Sigma^+ \cap \Sigma_0$ and $|\Sigma_0^+| = \operatorname{Card} \Sigma_0^+$.

Let us recall the polar decomposition on $\mathbb{M}$. We choose $o = K$ to be the origin in $\mathbb{M}$. Let M be the centralizer of A in K. For any $x \in \mathbb{M}$, let $\dot{k}(x) \in K/M$ and $C(x) \in \bar{\mathfrak{a}}_+$ be such that

$$k(x) \exp C(x).o = x.$$

where $k(x) \in K$ is a representative of $\dot{k}(x)$. Such a decomposition always exists. The (generalized) radial component $C(x)$ is uniquely determined. It is also the case for $\dot{k}(x)$ provided $C(x) \in \mathfrak{a}_+$. Let m_α be the multiplicity of the root α and

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha. \tag{4.1}$$

Although we do not need it, let us recall the asymptotic behavior of the Brownian motion on $\mathbb{M}$, see Virtser [54], Orihara [45], Malliavin & Malliavin [38], Taylor [53], Babillot [4] (the convergence in distribution in $C(\mathbb{R}_+, \mathfrak{a})$ is not explicitly stated in these papers however it follows immediately from the approach given in Babillot [4]).

Theorem 4.1 *Let B_t be the Brownian motion on $\mathbb{M}$. Then*

- *a. Almost surely, $\dot{k}(B_t)$ converges in K/M and* $\lim_{t\to+\infty} \frac{C(B_t)}{t} = \rho$.
- *b. In distribution in $C(\mathbb{R}_+, \mathfrak{a})$, the processes $\left\{\frac{C(B_{tT}) - tT\rho}{\sqrt{T}}, t \geq 0\right\}$ converge to the Euclidean Brownian motion in $\mathfrak{a}$ when $T \to +\infty$.*

For any $g \in G$, we denote by $H[g]$ the $\mathfrak{a}$-component of g in the Iwasawa decomposition $G = K(\exp \mathfrak{a})N$. Let us consider the basic spherical function of Harish–Chandra

$$\Phi_0(g) = \int_{K/M} e^{-\langle \rho, H[g^{-1}k]\rangle} d\nu(\dot{k}), \quad g \in G, \tag{4.2}$$

where ν is the unique K-invariant probability measure on K/M. This function is K-biinvariant, thus it defines a K-invariant function on $\mathbb{M}$, also denoted Φ_0, by the formula

$$\Phi_0(g.o) = \Phi_0(g), \qquad \text{for all } g \in \mathbb{M}.$$

Let Δ be the Laplace Beltrami operator on $\mathbb{M}$. The bottom of the L^2-spectrum of $-\frac{1}{2}\Delta$ is

$$\lambda_0 = \frac{1}{2}\,\|\rho\|^2. \tag{4.3}$$

There exist many ground states, but the only one which is K-invariant is Φ_0. We first consider the I.B.L., the double-ended I.B.L. will be dealt with in section 7. Since G acts on M transitively by isometries, it suffices to consider the Brownian motion starting from o.

Proposition 4.2 *Let p_t be the heat kernel of $\Delta/2$. For all $g, h \in G$,*

$$\lim_{t\to+\infty} \frac{p_t(g.o, h.o)}{p_t(o,o)} = \Phi_0(g^{-1}h.o).$$

The infinite Brownian loop B^0 on $\mathbb{M}$ around o is the relativized Φ_0-process of the Brownian motion. Its generator is $\Delta^0/2$ where

$$\Delta^0 f = \Delta f + 2\nabla \log \Phi_0 \cdot \nabla f. \tag{4.4}$$

Proof. The behavior of the quotients of p_t follows from the precise estimates of Anker & Ji [2] or from the local limit theorem in [6], (see also Guivarc'h [26]). But it is also easy to prove it directly: the set of functions $x \mapsto p_t(x,o)/p_t(o,o)$, when $t \geq 1$, is relatively compact and each limit point φ is a solution of $(2\lambda_0 + \Delta)\varphi = 0$ by Theorem 2.2. Since $p_t(k.x, o) = p_t(x, o)$ for all $k \in K$, φ is also invariant under K and satisfies $\varphi(o) = 1$. The function Φ_0 is the only one having these properties, hence $p_t(x,o)/p_t(o,o)$ must converge to $\Phi_0(x)$. By invariance under isometry, $p_t(g.o, h.o)/p_t(o.o) = p_t(h^{-1}g.o, o)/p_t(o,o)$ converges to $\Phi_0(h^{-1}g.o)$. The description of the I.B.L. follows from Theorem 1.2. ∎

For $x \in \mathfrak{a}$, let

$$\delta(x) = \prod_{\alpha\in\Sigma^+} \sinh^{m_\alpha}\langle\alpha, x\rangle. \tag{4.5}$$

One has the following decomposition of the Riemannian measure m, see [30], Theorem X.1.17: if $f : \mathbb{M} \to \mathbb{R}_+$ is measurable

$$\int_{\mathbb{M}} f(z)\,dm(z) = \int_{K/M\times\mathfrak{a}_+} f(ke^x.o)\,\tilde{\delta}(x)d\nu(\dot{k})\,dx$$

where $\tilde{\delta} = \mathrm{Vol}(K/M)\delta$. Thus the radial component $C(B_t)$ of the Brownian motion B on $\mathbb{M}$ has a semigroup with the density q_t with respect to the Lebesgue measure on $\mathfrak{a}_+$ given by

$$q_t(x,y) = \tilde{\delta}\,(y) \int_{K/M} p_t\,(e^x.o, ke^y.o)\; d\nu(\dot{k}) \tag{4.6}$$

This can be verified as in the proof of Proposition 3.3. The radial part $\mathrm{Rad}(\Delta)$ of Δ on $\mathfrak{a}_+$ is defined by: for any smooth function f on $\mathfrak{a}_+$,

$$[\mathrm{Rad}(\Delta)f] \circ C = \Delta(f \circ C).$$

It follows from Helgason [32], II.3, Remark 1, that

$$\mathrm{Rad}(\Delta) = \Delta_{\mathfrak{a}} + 2\,\nabla_{\mathfrak{a}} \log \delta^{1/2} \cdot \nabla_{\mathfrak{a}}. \tag{4.7}$$

We define $\varphi_0 : \mathfrak{a} \to \mathbb{R}$ by, for $x \in \mathfrak{a}$,

$$\varphi_0(x) = \Phi_0(e^x.o).$$

Corollary 4.3 *The radial part $C(B^0)$ on $\bar{\mathfrak{a}}_+$ of the I.B.L. on $\mathbb{M}$ around o is a continuous Markov process such that $C(B_0^0) = 0$, and such that for all $t > 0$, $C(B_t^0)$ is inside $\mathfrak{a}_+$ and has the generator*

$$\frac{1}{2}\,\mathrm{Rad}(\Delta^0) = \frac{1}{2}\Delta_{\mathfrak{a}} + \nabla_{\mathfrak{a}} \log(\delta^{1/2}\varphi_0) \cdot \nabla_{\mathfrak{a}}.$$

The semigroup of $C(B^0)$ has the density q_t^0 with respect to the Lebesgue measure on $\mathfrak{a}_+$ given by

$$q_t^0\,(x,y) = e^{\frac{1}{2}\|\rho\|^2 t}\,\tilde{\delta}\,(y)\,\frac{\varphi_0\,(y)}{\varphi_0\,(x)} \int_{K/M} p_t\,(e^x.o, ke^y.o)\;d\nu(\dot{k}),$$

for all $x \in \mathfrak{a}_+ \cup \{0\}, y \in \mathfrak{a}_+$, where $(p_t)_{t>0}$ is the heat kernel of $\frac{1}{2}\Delta$ on $\mathbb{M}$.

Proof. It is well known that the radial part $C(B_t)$ of the Brownian motion B on $\mathbb{M}$, starting from o, is for all $t > 0$ in $\mathfrak{a}_+$ and has the generator $\frac{1}{2}\,\mathrm{Rad}(\Delta)$ (see, for instance, Orihara [45], Taylor [53]). The function Φ_0 is K-invariant. Thus the radial part $C(B^0)$ of the relativized Φ_0-process is the relativized φ_0-process of $C(B)$. This implies that for all $t > 0$, the distribution of $\{C(B_s^0), s \le t\}$ is equivalent to the distribution of $\{C(B_s), s \le t\}$ (the Radon Nikodym derivative is $e^{\lambda_0 t}\Phi_0(B_t)$). Therefore $C(B_s^0) \in \mathfrak{a}_+$ for all $s > 0$ and its generator is (see, e.g., Lemma 2.3 and (4.7))

$$\frac{1}{2}\Delta_{\mathfrak{a}} + \nabla_{\mathfrak{a}} \log \delta^{1/2} \cdot \nabla_{\mathfrak{a}} + \nabla_{\mathfrak{a}} \log \varphi_0 \cdot \nabla_{\mathfrak{a}} = \frac{1}{2}\Delta_{\mathfrak{a}} + \nabla_{\mathfrak{a}} \log(\delta^{1/2}\varphi_0) \cdot \nabla_{\mathfrak{a}}.$$

The expression of q_t^0 follows from Lemma 2.3 and from (4.6). ∎

Notice that the I.B.L. is nonexploding. When G is complex $\varphi_0 = \delta^{-1/2}\boldsymbol{\pi}$ (cf. Theorem IV.4.7 in Helgason [32]). As stated next, in this case $C(B^0)$ is elementary and part (b) of Theorem 1.4 is trivial.

Corollary 4.4 *Assume that G is complex. Then the radial part in $\mathfrak{a}_+$ of the I.B.L. on $\mathbb{M}$ around o coincides with the intrinsic Brownian motion of the Weyl chamber.*

Remark 4.5 Let us pull back B^0 via the diffeomorphism $\exp : \mathfrak{p} \mapsto \mathbb{M}$. It follows from the corollary that $\exp^{-1}(B^0)$ has the same radial part on $\mathfrak{a}$ as the Euclidean Brownian motion on $\mathfrak{p}$ when G is complex. However these two processes are different on $\mathfrak{p}$ since the K/M-component of the first one converges a.s., which is not the case for second one.

Remark 4.6 One deduces immediately from the corollary that the radial part of the Brownian bridge on $\mathbb{M}$ around o is the generalized radial component on $\mathfrak{a}_+$ of the Euclidean Brownian bridge on $\mathfrak{p}$ around 0. It is natural to consider this radial process as the Brownian excursion in the Weyl chamber.

5. Asymptotic behavior of the normalized I.B.L. on a symmetric space

On a symmetric space of noncompact type $\mathbb{M}$, we first consider the infinite Brownian loop around o This particular case contains all essential difficulties. It will be also the major step towards the double-ended I.B.L. which will be dealt with in Section 7.

5.1. Asymptotic behavior of the K/M-component of the I.B.L.

We recall that ν is the K-invariant probability measure on K/M.

Proposition 5.1 *Let B^0 be the infinite Brownian loop around o on $\mathbb{M}$. Almost surely, $\dot{k}(B^0_t)$ converges to a random variable with distribution ν on K/M.*

Proof. We will use the description of the Martin boundary of $\Delta_{\lambda_0} = \frac{1}{2}\Delta + \lambda_0$ given in Guivarc'h, Ji & Taylor [27]. Let G_{λ_0} be the Green kernel of Δ_{λ_0} and let K_{λ_0} be the corresponding Martin kernel with base point o, namely

$$G_{\lambda_0}(x,y) = \int_0^{+\infty} e^{\lambda_0 t} p_t(x,y)\, dt, \qquad K_{\lambda_0}(x,y) = \frac{G_{\lambda_0}(x,y)}{G_{\lambda_0}(o,y)}.$$

The Green kernel G of $\frac{\Delta^0}{2} = \Phi_0^{-1}(\Delta_{\lambda_0} \circ \Phi_0)$ with respect to m is (see Lemma 2.3)

$$G(x,y) = \int_0^{+\infty} \frac{e^{\lambda_0 t} p_t(x,y)\Phi_0(y)}{\Phi_0(x)}\, dt = G_{\lambda_0}(x,y)\frac{\Phi_0(y)}{\Phi_0(x)},$$

thus its Martin kernel $K(x,y)=\frac{G(x,y)}{G(o,y)}$ is given by

$$K(x,y)=\frac{K_{\lambda_0}(x,y)}{\Phi_0(x)}.$$

For any $b\in K/M$, let h_b be the Δ_{λ_0}-harmonic function on $\mathbb{M}$ defined by

$$h_b(x)=e^{-\langle\rho,H[g^{-1}k]\rangle},$$

if $x=g.o\in\mathbb{M}$ and $b=kM\in K/M$. The set $\{h_b,b\in K/M\}$ is a compact subset of the Martin boundary of $(\mathbb{M},\Delta_{\lambda_0})$ and it is shown in Guivarc'h, Taylor & Ji [27] that

$$\lim_{n\to+\infty}K_{\lambda_0}(x,y_n)=h_b(x)$$

if and only if $\dot{k}(y_n)$ converges to b in K/M and $\langle\alpha,C(y_n)\rangle\to+\infty$ for all $\alpha\in\Sigma^+$ (in order to give a precise reference for this claim, let us use the notations of [27] and set $h_{\{\emptyset\}}(x)=\exp-\langle\rho,H[g^{-1}]\rangle$ when $x=gK$. Then, for $b=kM\in K/M$, $h_b=S_kh_{\{\emptyset\}}$, where $S_kf(x)=f(k^{-1}.x)$ for any function f on $\mathbb{M}$. Thus, in the Martin topology for Δ_{λ_0}, a sequence y_n converges to h_b if it is $C_{\{\emptyset\}}$-fundamental in the sense of [27], that is if and only if $\dot{k}(y_n)\to b$ and $\langle\alpha,C(y_n)\rangle\to+\infty$ for all $\alpha\in\Sigma^+$, see [27], 7.27–7.33). The functions $h_b^0=h_b/\Phi_0$ are Δ^0-harmonic. The sequence y_n converges to the functions h_b^0 for the Martin topology of Δ^0, i.e. $K(x,y_n)\to h_b^0(x)$, if and only if $\dot{k}(y_n)\to b$ and $\langle\alpha,C(y_n)\rangle\to+\infty$ for any $\alpha\in\Sigma^+$. Since $\Phi_0(x)=\int_{K/M}h_b(x)\,d\nu(b)$, the Δ^0-harmonic function 1 has the representation

$$1=\int_{K/M}h_b^0(x)\,d\nu(b).$$

It follows from the Martin boundary theory that B_t^0, starting from o, converges almost surely in the Martin topology to a random point carried by $\{h_b^0,b\in K/M\}$ with distribution ν (see, e.g., Kunita & Watanabe [34], Pinsky [49], Theorem 7.2.2). This proves the proposition.

Remark 5.2 The proof of the proposition shows that $\langle\alpha,B_t^0\rangle\to+\infty$ a.s. for each $\alpha\in\Sigma^+$, when $t\to+\infty$.

5.2. Asymptotic behavior of the radial component of the I.B.L.

We have seen in Corollary 4.3 that the radial part $X=C\left(B^0\right)$ on $\mathfrak{a}_+\cup\{0\}$ of the infinite Brownian loop B^0 around o is a $\mathfrak{a}_+\cup\{0\}$-valued continuous Markov process starting from 0, with generator

$$\frac{1}{2}\;\mathrm{Rad}\left(\Delta^0\right)=\frac{1}{2}\,\Delta_{\mathfrak{a}}+\nabla_{\mathfrak{a}}\log(\delta^{\frac{1}{2}}\varphi_0)\cdot\nabla_{\mathfrak{a}}.$$

Thus $X_0 = 0$ and

$$\left\{ \begin{array}{l} \forall t > 0,\ X_t \in \mathfrak{a}_+, \\ X_t - X_0 - \displaystyle\int_0^t \nabla_{\mathfrak{a}} \log(\delta^{\frac{1}{2}}\varphi_0)\,(X_s)\ ds \text{ is a Brownian motion on } \mathfrak{a}. \end{array} \right. \tag{5.1}$$

We will consider the behavior of this Markov process starting from any point $x \in \mathfrak{a}_+ \cup \{0\}$. The case where $x \in \mathfrak{a}_+$ will be needed to establish the asymptotic independence of the radial and of the K/M-component of the I.B.L. around o.

5.2.1. Almost sure behavior

Let us recall that the Bessel process of dimension n is the norm of an n-dimensional Brownian motion.

Proposition 5.3 *Let X be a continuous process satisfying (5.1) starting from $x \in \mathfrak{a}_+ \cup \{0\}$. For some $\kappa \in \mathbb{N}$, there exists two Bessel processes $R^{(d)}$ and $R^{(p)}$, of dimension d and p, such that $R_0^{(d)} = \|x\| = R_0^{(p)}$ and such that, for all $t \geq 0$*

$$R_t^{(d)} \leq \|X_t\| \leq R_t^{(p)}.$$

Proof. By assumption $B_t = X_t - X_0 - \int_0^t \nabla_{\mathfrak{a}} \log(\delta^{\frac{1}{2}}\varphi_0)(X_s)\, ds$ is a Brownian motion. By Ito's formula,

$$\begin{aligned} \|X_t\|^2 &= \|X_0\|^2 + 2\int_0^t \langle X_s, dB_s\rangle + t\,d + 2\int_0^t \langle \nabla_{\mathfrak{a}} \log(\delta^{\frac{1}{2}}\varphi_0) X_s, X_s\rangle\, ds \\ &= \|X_0\|^2 + 2\int_0^t \langle X_s, dB_s\rangle + t\,d + 2\int_0^t \mathcal{E} \log(\delta^{\frac{1}{2}}\varphi_0)(X_s)\, ds \end{aligned}$$

where $d = \dim(\mathfrak{a})$ and $\mathcal{E} = \sum_{1 \leq j \leq d} x_j \dfrac{\partial}{\partial x_j}$ is the Euler operator on $\mathfrak{a}$. As shown in the appendix (Theorem 8.3), there is some integer q such that on $\mathfrak{a}_+$,

$$0 \leq \mathcal{E} \log(\delta^{\frac{1}{2}}\varphi_0) \leq q.$$

Let $H^{(n)}$ be the solution of the equation

$$H_t^{(n)} = \|X_0\|^2 + 2\int_0^t \sqrt{H_s^{(n)}}\, d\beta_s + nt$$

where β is the real Brownian motion defined by $\beta_t = \displaystyle\int_0^t \|X_s\|^{-1}\langle X_s, dB_s\rangle$. It is well known (see Yamada [56], Revuz & Yor [50], XI.1) that $H^{(n)}$ is the square of a Bessel process of dimension n. Let $p = d + 2q$. It follows from the comparison theorem ([50], Theorem IX.3.7) that almost surely, for all $t \geq 0$,

$$H_t^{(d)} \leq \|X_t\|^2 \leq H_t^{(p)}. \tag{5.2}$$

One obtains the proposition by setting $R_t^{(n)} = \sqrt{H_t^{(n)}}$ for $n = d, p$. ∎

Corollary 5.4 *Let $B^{(0)}$ be the I.B.L. around o. Then, almost surely,*

$$\lim_{t \to +\infty} \frac{d(o, B_t^{(0)})}{t} = 0.$$

More precisely (law of iterated logarithm), a.s.

$$\limsup_{t \to +\infty} \frac{d(o, B_t^{(0)})}{\sqrt{2t \log \log t}} = 1,$$

and for all $0 < \varepsilon < 1$, there is $C_\varepsilon > 0$ such that, for all $t, \eta \geq 1$,

$$\mathbb{P}\Big(\sup_{\varepsilon t \leq s \leq t} \frac{d(o, B_s^{(0)})}{s} \geq \eta \Big) \leq C_\varepsilon \, e^{-\frac{\eta^2 \varepsilon^2 t}{2p}}.$$

Proof. Since $d(o, B_t^{(0)}) = \|X_t\|$, where X is the solution of (5.1) starting from 0, the result follows easily from the proposition and from classical properties of the Brownian motion. ∎

5.2.2. Distributional behavior

In this part, we will prove the following theorem.

Theorem 5.5 *Let X be a continuous process satisfying (5.1) starting from some $x \in \mathfrak{a}_+ \cup \{0\}$ and let, for $T > 0$, $X^{(T)}$ be the rescaled process defined by*

$$X_t^{(T)} = \frac{1}{\sqrt{T}} X_{tT}.$$

As $T \to +\infty$, $X^{(T)}$ converges in distribution in $C(\mathbb{R}_+, \bar{\mathfrak{a}}_+)$ to the intrinsic Brownian motion of the Weyl Chamber $\mathfrak{a}_+$.

It can be useful to the reader to have a very sketchy and informal presentation of the strategy of our proof. Let $G_T(x) = (\delta^{\frac{1}{2}} \varphi_0)(x\sqrt{T})$. We will see that $X_t^{(T)}$ is a solution of the equation

$$X_t = X_0 + \beta_t + \int_0^t \nabla_{\mathfrak{a}} \left(\log G_T\right)(X_s) \, ds$$

where β is a Brownian motion on $\mathfrak{a}_+$. On the other hand, the intrinsic Brownian motion Z_t of $\mathfrak{a}_+$ is a solution of

$$Z_t = B_t + \int_0^t \nabla_{\mathfrak{a}} (\log \boldsymbol{\pi}) (Z_s) \, ds. \tag{5.3}$$

Imagine now that G_T and $\boldsymbol{\pi}$ are bounded away from 0 and with bounded derivatives on $\bar{\mathfrak{a}}_+$ (this is actually obviously false). Then by using the fact that for every smooth function h on $\mathfrak{a}$,

$$\Delta_{\mathfrak{a}} (\log h) - \frac{\Delta_{\mathfrak{a}} h}{h} + \|\nabla_{\mathfrak{a}} (\log h)\|^2 = 0 \tag{5.4}$$

one sees that

$$N_t^T = \frac{\boldsymbol{\pi}}{G_T}(X_t) \exp\left(-\frac{1}{2} \int_0^t \left(\frac{\Delta_{\mathfrak{a}} \boldsymbol{\pi}}{\boldsymbol{\pi}} (X_s) - \frac{\Delta_{\mathfrak{a}} G_T}{G_T} (X_s) \right) ds \right)$$

is a martingale and that, for all $b > 0$, when $0 \le t \le b$,

$$X_t^{(T)} = X_0^{(T)} + \overline{\beta}_t + \int_0^t \nabla_{\mathfrak{a}} (\log \boldsymbol{\pi}) (X_s^{(T)}) \, ds$$

where $\overline{\beta}$ is a Brownian motion on $\mathfrak{a}_+$ under the probability $N_b^T (N_0^T)^{-1} \cdot \mathbb{P}$ (using Girsanov's theorem). If, when $T \to +\infty$, N_b^T tends to 1 in an appropriate way, we will conclude that $X_t^{(T)}$ converges in distribution to the solution of (5.3).

Actually the behavior of the coefficients are singular near the walls of the Weyl chamber and in particular near 0 which is the starting point (at least of the limit) and thus cannot be avoided. The plan of the proof is now the following. In Lemma 5.6 we show that the processes $X^{(T)}$ are well behaved in a short time $t \le a$. At a fixed time $t = a$, we will use the convergence of the densities at time t (see Lemma 5.7). Then one localizes the processes in compact subsets of the open cone $\mathfrak{a}_+$. The convergence of (a localized variant of) N^T is dealt with in Lemma 5.8. The precise version of Girsanov's type argument alluded to above is given in Lemma 5.9. After these preliminaries the proof is easy and presented at the end of this section.

We will suppose without loss of generality that the process X satisfying (5.1) is the coordinate process on $\Omega = C(\mathbb{R}_+, \bar{\mathfrak{a}}_+)$ and we let $\mathcal{F}_t = \sigma\{X_s \mid 0 \le s \le t\}$. When $X_0 = x$, we let $\mathbb{P}_x$ be the distribution of X and $\mathbb{P}_x^{(T)}$ be the distribution of $X^{(T)}$. The distribution of the intrinsic Brownian motion of the Weyl chamber is denoted by $\mathbb{Q}$.

Lemma 5.6 *There exists $\kappa > 0$ such that for all $T > 0, r > 0, t > 0$,*

$$\mathbb{P}_x \left[\sup_{s\le t} \|X_s^{(T)}\| \ge r\right] \le \frac{2}{r^2}\left(\frac{\|x\|^2}{T} + \kappa t\right).$$

Proof. It follows from Proposition 5.3 that, for some $\kappa > 0$,

$$\mathbb{E}_x \left[\sup_{s\le t} \|X_s^{(T)}\|^2\right] \le \frac{1}{T}\mathbb{E}\left[\sup_{s\le tT} \left(R_s^{(\kappa)}\right)^2\right] \le 2(\frac{\|x\|^2}{T} + \kappa t).$$

This gives the lemma using Markov's inequality. ■

Lemma 5.7 *For any starting point x, the density $\theta_t^{(T)}$ of $X_t^{(T)}$ converges, as $T \to +\infty$, to the density $\theta_t = h_t(0,\cdot)$ at time t of the intrinsic Brownian motion of the Weyl chamber.*

Proof. It follows from Corollary 4.3 that $\theta_t^{(T)}(x,y) = T^{\frac{d}{2}} q_{tT}^0(x, \sqrt{T}y)$. We must show that, for each fixed $x \in \bar{\mathfrak{a}}_+, y \in \mathfrak{a}_+$ and $t > 0$,

$$\lim_{T\to+\infty} T^{\frac{d}{2}}\, q_{tT}^0(x, \sqrt{T}y) = h_t(0,y).$$

We may assume that $t = 1$ by scaling. Recall that

$$q_T^0(x, \sqrt{T}y) = e^{\frac{\|\rho\|^2}{2}T}\, \tilde{\delta}(\sqrt{T}y)\, \frac{\varphi_0(\sqrt{T}y)}{\varphi_0(x)} \int_K p_T\big(e^x.o, ke^{\sqrt{T}y}.o\big)\, dk$$

where $\tilde{\delta} = \mathrm{Vol}(K/M)\delta$. We have

$$p_T(e^x.o, ke^{\sqrt{T}y}.o) = p_T(e^z.o, o)$$

if $z = z(T,x,y,k) \in \bar{\mathfrak{a}}_+$ is the radial component of $e^{-\sqrt{T}y}k^{-1}e^x$. Notice that z remains at bounded distance from $\sqrt{T}(-w.y) \in \mathfrak{a}_+$, as $T \to +\infty$, where w is the element in the Weyl group W, which interchanges $\mathfrak{a}_+$ with $-\mathfrak{a}_+$. We have indeed

$$\|z + \sqrt{T}w.y\| \le d(e^{-\sqrt{T}y}k^{-1}e^x.o, e^{-\sqrt{T}y}.o) = d(e^x.o, o)$$

(see for instance Lemma 2.1.2 in [2]). Since $y \in \mathfrak{a}_+$, this implies in particular that for every $\alpha \in \Sigma^+$, $\langle \alpha, z\rangle/\sqrt{T}$ stays within two positive constants as $T \to +\infty$. The heat kernel analysis in Anker & Ji [2] (see Section 3, Step 6) yields the following asymptotics

$$p_T(e^z.o, o) \;=\; c_1\, e^{-\frac{\|y\|^2}{2}}\, T^{-\frac{D}{2}}\, e^{-\frac{\|\rho\|^2}{2}T}\, \varphi_0(z) \;+\; \mathrm{O}\left(T^{-\frac{D}{2}-\frac{1}{4}}\, e^{-\frac{\|\rho\|^2}{2}T}\, \varphi_0(z)\right)$$

as $T \to +\infty$, where $c_1 = 2^{n-\frac{3d}{2}} \pi^{-\frac{d}{2}} \operatorname{Vol}(\mathrm{K/M})^{-1} \boldsymbol{\pi}(\rho^0)\, \mathbf{b}(0)^{-2}$ and $n = \dim \mathbb{M}$. Notice that this estimate is uniform in the variable $k \in K$ involved in z. By integrating $\varphi_0(z)$ over K and by using the functional relation

$$\int_K \Phi_0\big(e^{-\sqrt{T}y} k^{-1} e^x\big)\, dk = \Phi_0\big(e^{\pm\sqrt{T}y}\big)\, \Phi_0\big(e^x\big)\,,$$

we obtain

$$\begin{aligned} \int_K p_T\big(e^x.0, ke^{\sqrt{T}y}.0\big)\, dk \;&=\; c_1\, \varphi_0(x)\, e^{-\frac{\|y\|^2}{2}}\, T^{-\frac{D}{2}}\, e^{-\frac{\|\rho\|^2}{2}T}\, \varphi_0(\sqrt{T}y) \\ &\qquad + \mathrm{O}\left(T^{-\frac{D}{2}-\frac{1}{4}}\, e^{-\frac{\|\rho\|^2}{2}T}\, \varphi_0(\sqrt{T}y) \right) \\ &\sim\; c_1\, \varphi_0(x)\, e^{-\frac{\|y\|^2}{2}}\, T^{-\frac{D}{2}}\, e^{-\frac{\|\rho\|^2}{2}T}\, \varphi_0(\sqrt{T}y) \end{aligned}$$

as $T \to +\infty$. By using the asymptotics

$$\tilde{\delta}(\sqrt{T}y) \;\sim\; \operatorname{Vol}(\mathrm{K/M}) 2^{d-n}\, e^{2\sqrt{T}\langle \rho, y\rangle}$$

and

$$\varphi_0(\sqrt{T}y) \;\sim\; \tfrac{\mathbf{b}(0)}{\boldsymbol{\pi}(\rho^0)}\, T^{\frac{|\Sigma_0^+|}{2}}\, \boldsymbol{\pi}(y)\, e^{-\sqrt{T}\langle \rho, y\rangle}$$

(see for instance [2], Proposition 2.2.12.ii), we further obtain

$$q_T^0(x, \sqrt{T}y) \;\sim\; (2\pi)^{-\frac{d}{2}}\, \boldsymbol{\pi}(\rho^0)^{-1}\, \boldsymbol{\pi}(y)^2\, e^{-\frac{\|y\|^2}{2}}\, T^{-\frac{d}{2}}\,, \tag{5.5}$$

We now remark that the limit depends only on the Weyl chamber $\mathfrak{a}_+$. Let us consider the symmetric space $\tilde{G}/\tilde{K}$ where $\tilde{G}$ is a complex group, which has this Weyl chamber. We have seen in Proposition 3.3 that in this case the radial part of the I.B.L. around o is equal to the intrinsic Brownian motion on $\mathfrak{a}_+$. Thus, in this case $q_t^0 = h_t$ where h_t is the density of the intrinsic Brownian motion. This process has the scaling property:

$$h_T(0, y\sqrt{T}) = h_1(0, y).$$

Thus, if we apply (5.5) to h with $x = 0$ we obtain that

$$h_1(0, y) = \lim_{t \to +\infty} h_T(0, y\sqrt{T}) = (2\pi)^{-\frac{d}{2}}\, \boldsymbol{\pi}(\rho^0)^{-1}\, \boldsymbol{\pi}(y)^2\, e^{-\frac{\|y\|^2}{2}}\, T^{-\frac{d}{2}}.$$

Thus (5.5) proves the lemma. ■

Let $a > 0, \eta > 0,\ R > 0$, we set

$$\begin{aligned} \mathfrak{a}_+^\eta &= \{x \in \mathfrak{a}_+;\ \langle \alpha, x\rangle \geq \eta,\, \forall \alpha \in \Sigma^+\}\,, \\ \sigma &= \inf \big\{ t \geq a;\ X_t \notin \mathfrak{a}_+^\eta \text{ or } \|X_t\| > R \big\}\,. \end{aligned} \tag{5.6}$$

Lemma 5.8 *Let* $G_T(x) = (\delta^{\frac{1}{2}}\varphi_0)(x\sqrt{T})$ *and, for* $b \geq a > 0$,

$$M_b^{(T)} = \frac{\boldsymbol{\pi}(X_{b\wedge\sigma})\, G_T(X_a)}{\boldsymbol{\pi}(X_a)\, G_T(X_{b\wedge\sigma})} \exp\left(\int_a^{b\wedge\sigma} \frac{\Delta_{\mathfrak{a}} G_T}{2G_T}(X_s)\, ds\right). \tag{5.7}$$

Then, almost surely, $M_b^{(T)} \to 1$ *as* $T \to \infty$.

Proof. Recall that $G = \delta^{\frac{1}{2}}\varphi_0$ and $G_T = G(\sqrt{T}\,\cdot)$. Obviously, for all $x \in \mathfrak{a}_+$,

$$\frac{\Delta_{\mathfrak{a}} G_T}{G_T}(x) = T\,\frac{\Delta_{\mathfrak{a}} G}{G}(\sqrt{T}x).$$

On one hand, $\Delta\Phi_0 = -\|\rho\|^2\Phi_0$ hence $\mathrm{Rad}(\Delta)\varphi_0 = -\|\rho\|^2\varphi_0$. On the other hand, $\varphi_0^{-1}\,\mathrm{Rad}(\Delta)\varphi_0 = G^{-1}\Delta_{\mathfrak{a}} G - \delta^{-\frac{1}{2}}\Delta_a\delta^{\frac{1}{2}}$. As a consequence, $G^{-1}\Delta_{\mathfrak{a}} G = \delta^{-\frac{1}{2}}\Delta_{\mathfrak{a}}\delta^{\frac{1}{2}} - \|\rho\|^2$. The latter expression occurs in the analysis of the Harish–Chandra expansion (8.3), as performed by Gangolli. It follows immediately from its expansion in [32], proof of Lemma IV.5.6, or from the following explicit formula noticed by Wallach [55] (see also Heckman ([29], Theorem 2.1.1):

$$\delta(x)^{-\frac{1}{2}}\Delta_{\mathfrak{a}}\delta^{\frac{1}{2}}(x) - \|\rho\|^2 = \frac{1}{4}\sum_{\alpha\in\Sigma^+}(m_\alpha(m_\alpha-2) + 2m_\alpha m_{2\alpha})\,\|\alpha\|^2\sinh^{-2}\langle\alpha, x\rangle$$

(where $m_{2\alpha} = 0$ when $\alpha \notin \Sigma_0^+$) that $\frac{\Delta_{\mathfrak{a}} G_T}{G_T}$ converges uniformly to 0 on $\mathfrak{a}_+^\eta$. Therefore

$$\exp\Big(\int_a^{b\wedge\sigma} \tfrac{\Delta_a G_T}{2G_T}(X_s)\, ds\Big) \to 1 \quad \text{as } T \to +\infty\,.$$

Besides, for every $x \in \mathfrak{a}_+$, if $n = \dim \mathbb{M}$,

$$G_T(x) = \delta(\sqrt{T}x)^{\frac{1}{2}}\varphi_0(\sqrt{T}x) \sim 2^{\frac{d-n}{2}} e^{\langle\rho,\sqrt{T}x\rangle}\varphi_0(\sqrt{T}x) \sim 2^{\frac{d-n}{2}}\gamma\, T^{|\Sigma_0^+|/2}\boldsymbol{\pi}(x)$$

as $T \to +\infty$ (see for instance Theorem 8.1.ii). Thus

$$\frac{G_T(x)}{G_T(y)} \to \frac{\boldsymbol{\pi}(x)}{\boldsymbol{\pi}(y)} \quad \text{as } T \to +\infty\,,$$

for all $x, y \in \mathfrak{a}_+$. This concludes the proof of the Lemma. ■

Recall that $\mathbb{Q}$ is the distribution of the intrinsic Brownian motion of the Weyl chamber, θ_t is its density at time t and $\theta_t^{(T)}$ is the density of X_t under $\mathbb{P}_x^{(T)}$.

Lemma 5.9 *Let* $b > a > 0$ *and let* ${}^aX$ *be the process defined by* ${}^aX_t = X_{\sup(a,t)}$. *Then, for any* $\mathcal{F}_b$*-measurable function* Ψ *on* Ω *and* $\varepsilon > \eta$,

$$\mathbb{E}_x^{(T)}\left[\Psi({}^aX)\, 1_{\{X_a\in\mathfrak{a}_+^\varepsilon,\sigma>b\}}\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{\theta_a^{(T)}(X_a)}{\theta_a(X_a)}\frac{1}{M_b^{(T)}}\Psi({}^aX)\, 1_{\{X_a\in\mathfrak{a}_+^\varepsilon,\sigma>b\}}\right] \tag{5.8}$$

Proof. The process X satisfies (5.1), thus we know that, for $t \geq a$,

$$X_t - X_a - \int_a^t \nabla_{\mathfrak{a}} \log G^{(1)}(X_s)\, ds$$

is a Brownian motion on $\mathfrak{a}_+$ under $\mathbb{P}_x$ starting from 0 at time a. Hence, under the probability measure $\mathbb{P}_x^{(T)}$,

$$\beta_t^{(T)} = X_t - X_a - \int_a^t \nabla_{\mathfrak{a}} \log G_T\,(X_s)\; ds$$

is also a Brownian motion for $t \geq a$. Since (see (5.4))

$$\|\nabla_{\mathfrak{a}} \log G_T\|^2 + \Delta_{\mathfrak{a}} \log G_T = \frac{\Delta_{\mathfrak{a}} G_T}{G_T} \text{ and } \|\nabla_{\mathfrak{a}} \log \boldsymbol{\pi}\|^2 + \Delta_{\mathfrak{a}} \log \boldsymbol{\pi} = 0,$$

it follows from Ito's formula that $M_b^{(T)}$ defined by (5.7) satisfies

$$M_b^{(T)} = \exp\left(\int_a^{b\wedge\sigma} k_T(X_s)\; d\beta_s^{(T)} - \frac{1}{2}\int_a^{b\wedge\sigma} k_T(X_s)^2\, ds\right)$$

where $k_T = \nabla_{\mathfrak{a}} \log(\boldsymbol{\pi}/G_T)$. This shows that $M_b^{(T)}$ is a local martingale under $\mathbb{P}_x^{(T)}$, for $b \geq a$. It is clear on (5.7) that it is bounded, hence

$$\mathbb{E}_x^{(T)}\left[M_b^{(T)} \mid \mathcal{F}_a\right] = 1.$$

From now on we work conditionally on $\{X_a \in \mathfrak{a}_+^{\varepsilon}\}$. By Girsanov's theorem, under the probability $M_b^{(T)} \cdot \mathbb{P}_x^{(T)}$, when $a \leq t \leq b$,

$$X_{t\wedge\sigma} - X_a - \int_a^{t\wedge\sigma} \nabla_{\mathfrak{a}} \log \boldsymbol{\pi}\,(X_s)\; ds$$

is a continuous local martingale with increasing process $1_{\{t\geq a\}}\,(t\wedge\sigma - a)\, I_d$. This implies that there is a Brownian motion $\tilde{\beta}$ such that, for $a \leq t \leq b$,

$$\tilde{\beta}_{t\wedge\sigma} - \tilde{\beta}_a = X_{t\wedge\sigma} - X_a - \int_a^{t\wedge\sigma} \nabla_{\mathfrak{a}} \log \boldsymbol{\pi}\,(X_s)\; ds. \tag{5.9}$$

Since $\log \boldsymbol{\pi}$ is C^∞ inside $\mathfrak{a}_+$, $(X_{t\wedge\sigma})_{t\geq a}$ is the unique solution of the stochastic integral equation (5.9) starting at X_a and stopped at its first exit time from $\mathfrak{a}_+^{\eta} \cap \{x \in \mathfrak{a}_+;\ \|x\| \leq R\}$. The intrinsic Brownian motion satisfies the same stochastic equation. Thus $X_{t\wedge\sigma}, a \leq t \leq b$, has the same distribution under the probability measure $M_b^{(T)}.\mathbb{P}_x^{(T)}$ and under $\mathbb{Q}$, conditionally on X_a. This implies that

$$\mathbb{E}_x^{(T)}\left[\Psi\left({}^aX\right)1_{\left\{X_a\in\mathfrak{a}_+^\varepsilon,\sigma>b\right\}}\mid\sigma(X_a)\right]=\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{M_b^{(T)}}\Psi\left({}^aX\right)1_{\left\{X_a\in\mathfrak{a}_+^\varepsilon,\sigma>b\right\}}\mid\sigma(X_a)\right],$$

which yields (5.8), using the Markov property. ■

We can now prove Theorem 5.5. Let $0<a<b, 0<\eta<\varepsilon$ and ${}^aX_t=X_{\sup(a,t)}$. We remark that

$$\sup_{t\le b}\left\|{}^aX_t-X_t\right\|=\sup_{t\le a}\left\|X_a-X_t\right\|\le 2\sup_{t\le a}\left\|X_t\right\|.$$

Let Ψ be a nonnegative function on $C\left([0,b],\bar{\mathfrak{a}}_+\right)$, uniformly continuous and bounded by 1. Let $\delta>0$ and $\beta>0$ such that $\left|\Psi\left(X\right)-\Psi\left(X'\right)\right|\le\delta$ when $\sup_{t\le b}\left\|X'_t-X_t\right\|<\beta$. We have:

$$\left|\mathbb{E}_x^{(T)}[\Psi\left(X\right)]-\mathbb{E}_x^{(T)}\left[\Psi\left({}^aX\right)\right]\right|\le\delta+2\,\mathbb{P}_x^{(T)}\left[\sup_{t\le a}\left\|X_t\right\|>\frac{1}{2}\beta\right]$$

and, applying Lemma 5.6,

$$\mathbb{E}_x^{(T)}\left[\Psi\left(X\right)\right]\ge\mathbb{E}_x^{(T)}\left[\Psi\left({}^aX\right)\right]-\tau,$$

if $\tau=\delta+\frac{16}{\beta^2}(\frac{\|x\|^2}{T}+\kappa\,a)$. Therefore, it follows from Lemma 5.9 that

$$\begin{aligned}\mathbb{E}_x^{(T)}\left[\Psi\left(X\right)\right]&\ge\mathbb{E}_x^{(T)}\left[\Psi\left({}^aX\right)1_{\left\{X_a\in\mathfrak{a}_+^\varepsilon,\sigma>b\right\}}\right]-\tau\\&\ge\mathbb{E}_{\mathbb{Q}}\left[\frac{\theta_a^{(T)}\left(X_a\right)}{\theta_a\left(X_a\right)}\frac{1}{M_b^{(T)}}\Psi\left({}^aX\right)1_{\left\{X_a\in\mathfrak{a}_+^\varepsilon,\sigma>b\right\}}\right]-\tau.\end{aligned}$$

Using the lemmas 5.7, 5.8 and Fatou's lemma, we have:

$$\begin{aligned}\liminf_{T\to+\infty}\mathbb{E}_x^{(T)}\left[\Psi\left(X\right)\right]&\ge\mathbb{E}_{\mathbb{Q}}\left[\Psi\left({}^aX\right)1_{\left\{X_a\in\mathfrak{a}_+^\varepsilon,\sigma>b\right\}}\right]-\delta-\frac{16}{\beta^2}\kappa\,a\\&\ge\mathbb{E}_{\mathbb{Q}}\left[\Psi\left(X\right)\right]-2\delta-\frac{16}{\beta^2}\kappa\,a-\mathbb{Q}\left[X_a\notin\mathfrak{a}_+^\varepsilon\right]\\&\qquad-\mathbb{Q}\left[\sigma\le b\right]-2\mathbb{Q}\left[\sup_{s\le a}\left\|X_s\right\|>\frac{1}{2}\beta\right].\end{aligned}$$

Recall that $\sigma=\inf\left(t\ge a\mid X_t\notin\mathfrak{a}_+^\eta\text{ or }\left\|X_t\right\|>R\right)$. As $\mathbb{Q}$-a.s., $X_t\in\mathfrak{a}_+$ for all $t>0$, obviously $\mathbb{Q}\left[\sigma\le b\right]\to0$ when $\eta\to0$ and $R\to+\infty$. It suffices to let successively (η,R) goes to $(0,+\infty)$, ε to 0, a to 0 and δ to 0 to conclude that

$$\liminf_{T\to+\infty}\ \mathbb{E}^{(T)}\left[\Psi\left(X\right)\right]\ge\mathbb{E}_{\mathbb{Q}}\left[\Psi\left(X\right)\right].$$

Replacing Ψ with $1-\Psi$ gives immediately that

$$\lim_{T\to+\infty}\mathbb{E}^{(T)}\left[\Psi\left(X\right)\right]=\mathbb{E}_{\mathbb{Q}}\left[\Psi\left(X\right)\right].$$

■

5.3. Asymptotic independence

We will now (define and) prove the asymptotic independence of the radial and of the K/M components of the I.B.L. This will be a consequence of the following proposition.

Proposition 5.10 *Let X be a continuous process satisfying (5.1), defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\widetilde{\mathbb{P}}$ be a probability on $\mathcal{F}$, absolutely continuous with respect to $\mathbb{P}$. Under $\widetilde{\mathbb{P}}$, as $T \to +\infty$, $X^{(T)}$ converges in distribution to the intrinsic Brownian motion of the Weyl Chamber $\mathfrak{a}_+$.*

Proof. Let $\mathcal{F}_\infty = \sigma\{X_r;\, r \geq 0\}$ and $\mathcal{F}_a = \sigma\{X_r;\, 0 \leq r \leq a\}$. We have to show that for any $Z \in \mathbf{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, for any $s > 0$ and any bounded continuous function Ψ on $C(\mathbb{R}_+, \bar{\mathfrak{a}}_+)$, $\mathcal{F}_s$-measurable,

$$\lim_{T\to+\infty} \mathbb{E}[Z\,\Psi(X^{(T)})] = \mathbb{E}[Z]\,\mathbb{Q}[\Psi], \tag{5.10}$$

where $\mathbb{Q}$ is the distribution of the intrinsic Brownian motion of $\mathfrak{a}_+$. Since

$$\mathbb{E}[Z\,\Psi(X^{(T)})] = \mathbb{E}[Z_\infty\,\Psi(X^{(T)})]$$

where $Z_\infty = \mathbb{E}[Z \mid \mathcal{F}_\infty]$ and since

$$\left|\mathbb{E}[U\,\Psi(X^{(T)})] - \mathbb{E}[V\,\Psi(X^{(T)})]\right| \leq \|\Psi\|_\infty\,\mathbb{E}[|U-V|],$$

it suffices by density in $\mathbf{L}^1(\boldsymbol{\Omega}, \mathcal{F}_\infty, \mathbb{P})$ to consider the case where Z is bounded and $\mathcal{F}_a$-measurable for some $a \geq 0$. Let

$$X^{(a,T)} : t \mapsto X_t^{(a,T)} = \frac{1}{\sqrt{T}} X_{\sup(Tt,a)}.$$

Since $\sup_t \|X_t^{(a,T)} - X_t^{(T)}\| \leq \frac{1}{\sqrt{T}} \sup_{t\leq a} \|X_t\|$, we may also replace $X^{(T)}$ by $X^{(a,T)}$. Using the Markov property at time a, we get

$$\mathbb{E}[Z\,\Psi(X^{(a,T)})] = \mathbb{E}\left[Z\,\mathbb{E}[\Psi(X^{(a,T)}) \mid \mathcal{F}_a]\right] = \mathbb{E}\left[Z\,\mathbb{E}_{X_a}[\Psi(\widetilde{X}^{(a,T)})]\right],$$

where $\widetilde{X_t}^{(a,T)} = \frac{1}{\sqrt{T}} X_{(Tt-a)_+}$ and where $\mathbb{P}_x$ is the law of the solution of (5.1) starting at $x \in \mathfrak{a}_+ \cup \{0\}$. By the dominated convergence theorem, it suffices finally to show that, for all $x \in \mathfrak{a}_+ \cup \{0\}$,

$$\lim_{T\to+\infty} \mathbb{E}_x[\Psi(\widetilde{X}^{(a,T)})] = \mathbb{Q}[\Psi]. \tag{5.11}$$

By Theorem 5.5 and Prokhorov's criterion, the family of the distributions of $X^{(T)}$, $T \geq 1$, under $\mathbb{P}_x$ is tight: for all $\varepsilon > 0$, there exists a compact set $\mathcal{K}$ in $C(\mathbb{R}_+, \bar{\mathfrak{a}}_+)$ such that for all $T \geq 1$

$$\mathbb{P}_x[X^{(T)} \notin \mathcal{K}] < \varepsilon.$$

Using the Ascoli's criterion of compactness, we know that, uniformly in $T \geq 1$, when $X^{(T)} \in \mathcal{K}$, the modulus of continuity

$$\sup_{0 \leq u < v \leq s, |u-v| \leq \eta} \|X_u^{(T)} - X_v^{(T)}\|$$

goes to 0 with η in $\mathbb{P}_x$-probability. Since

$$\begin{aligned} \sup_{t \leq s} \|\widetilde{X}_t^{(a,T)} - X_t^{(T)}\| &= \tfrac{1}{\sqrt{T}} \sup_{t \leq s} \|X_{(tT-a)_+} - X_{tT}\| \\ &\leq \sup_{0 \leq u < v \leq s, |u-v| \leq \frac{a}{T}} \|X_u^{(T)} - X_v^{(T)}\| \end{aligned}$$

and since Ψ is $\mathcal{F}_s$-measurable, this implies that

$$\lim_{T \to +\infty} \mathbb{E}_x[\Psi(\widetilde{X}^{(a,T)})] = \lim_{T \to +\infty} \mathbb{E}_x[\Psi(X^{(T)})]$$

thus (5.11) follows from Theorem 5.5. ■

We have seen that $\dot{k}(B_t^{(0)})$ converges almost surely to some random limit $b_\infty \in K/M$. For $b \in K/M$ let $\bar{b}$ be the constant path $\bar{b}_t = b$ for all $t > 0$. The process $\{\dot{k}(B_{tT}^{(0)}), t > 0\}$ converges in distribution in $C((0, \infty), K/M)$ to the process $\bar{b}_\infty$, where $C((0, \infty), K/M)$ is equipped with the uniform convergence on compact subsets of $(0, \infty)$. Notice that 0 has to be excluded.

Corollary 5.11 *The two processes* $\{\dot{k}(B_{tT}^{(0)}), t > 0\}$ *and* $\{\frac{C(B_{tT}^{(0)})}{\sqrt{T}}, t \geq 0\}$ *are asymptotically independent in the sense that their joint distribution on*

$$C((0, \infty), K/M) \times C([0, \infty), \bar{\mathfrak{a}}_+)$$

converges to the distribution of two independent processes, as $T \to +\infty$.

Proof. Let us actually prove that, if F is a bounded continuous function on $C((0, \infty), K/M)$ and G is a bounded continuous function on $C(\mathbb{R}_+, \mathfrak{a}_+ \cup \{0\})$, then

$$\lim_{T \to +\infty} \mathbb{E}\left[F(\dot{k}(B_{tT}^{(0)}), t > 0) G(\frac{C(B_{tT}^{(0)})}{\sqrt{T}}, t \geq 0)\right] = \mathbb{E}\left[F(\bar{b}_\infty)\right] \mathbb{E}_{\mathbb{Q}}[G]$$

By density, one may suppose that, for $\omega \in C((0,\infty), K/M)$, $F(\omega)$ does not depends on ω_t for $t \leq \varepsilon$. Then

$$\lim_{T\to+\infty} F(\dot{k}(B^{(0)}_{tT}), t > 0) = F(\bar{b}_\infty).$$

Thus

$$\left| \mathbb{E}\left[F(\dot{k}(B^{(0)}_{\cdot T}))G(\frac{C(B^{(0)}_{\cdot T})}{\sqrt{T}})\right] - \mathbb{E}\left[F(\bar{b}_\infty)G(\frac{C(B^{(0)}_{\cdot T})}{\sqrt{T}})\right]\right| \\ \leq \|G\|_\infty \mathbb{E}\left[\left| F(\dot{k}(B^{(0)}_{\cdot T})) - F(\bar{b}_\infty)\right|\right]$$

converges to 0 as $T \to +\infty$ and one concludes with Proposition 5.9. ■

6. The other ground state processes

Motivated by the infinite Brownian loop on the symmetric space $\mathbb{M}$, we have so far studied the relativized Φ_0-process of the Brownian motion. The function Φ_0 is the unique K-invariant ground state, but there are many others. Recall that we call a ground state any positive solution φ of $(\Delta + 2\lambda_0)\varphi = 0$. To every ground state φ corresponds a unique probability measure λ on K/M such that

$$\varphi(x) = \varphi(o) \int_{K/M} h_b(x)\, d\lambda(b)$$

(see, e.g., Guivarc'h, Ji & Taylor [27], 7.1).

Theorem 6.1 *Let B^φ be the relativized φ-process of the Brownian motion on a noncompact symmetric space $\mathbb{M}$, starting from o. Then, as $T \to +\infty$,*

- *(i) Almost surely, $\frac{1}{t} d(o, B^\varphi_T) \to 0$ and $\dot{k}(B^\varphi_T)$ converges to a random variable with distribution λ on K/M.*
- *(ii) $\{\frac{1}{\sqrt{T}} C(B^\varphi_{tT}), t \geq 0\}$ converges in distribution to the intrinsic Brownian motion on the Weyl chamber.*
- *(iii) $\{\dot{k}(B^\varphi_{tT}), t > 0\}$ and $\{\frac{1}{\sqrt{T}} C(B^\varphi_{tT}), t \geq 0\}$ are asymptotically independent.*

Proof. The proof of the almost sure convergence of $\dot{k}(B^\varphi_T)$ is the same as the proof of Proposition 5.1. Let us consider the radial part. Let $F : C(\mathbb{R}_+, \bar{\mathfrak{a}}_+) \to \mathbb{R}_+$ be a $\mathcal{F}_T$-measurable function for the canonical filtration. Then, by definition of the φ-process,

$$\mathbb{E}[F(C(B^\varphi))] = \frac{e^{\lambda_0 T}}{\varphi(o)} \mathbb{E}[F(C(B))\varphi(B_T)]$$

where B is the Brownian motion on $\mathbb{M}$ starting from o. Notice that, if $b = \dot{k}_0$ and $x = g.o$

$$\int_K h_b(k.x)\,dk = \int_K e^{-\langle \rho, H[g^{-1}k^{-1}k_0]\rangle}\,dk = \Phi_0(x)$$

therefore

$$\int_K \varphi(k.x)dk = \varphi(o)\int_K \int_{K/M} h_b(k.x)\,d\lambda(b)dk = \varphi(o)\Phi_0(x).$$

We use this equality, the invariance invariance of the Brownian motion under K and the relation $C(k.x) = C(x)$ for $k \in K$ and $x \in \mathbb{M}$ to write

$$\begin{aligned} \mathbb{E}[F(C(B^\varphi))] &= \frac{e^{\lambda_0 T}}{\varphi(o)}\int_K \mathbb{E}[F(C(B))\varphi(k.B_T)]\,dk \\ &= e^{\lambda_0 T}\mathbb{E}[F(C(B))\Phi_0(B_T)] \\ &= \mathbb{E}[F(C(B^0))]. \end{aligned}$$

This proves that the process $C(B^\varphi)$ has the same distribution as the radial part of the I.B.L. Thus (ii) follows from Theorem 1.4 and (iii) is proved exactly in the same way as Proposition 5.11 was.

Remark 6.2 Let us consider the case where λ is the Dirac measure on the class M in K/M, i.e. the case where

$$\varphi(x) = e^{-\langle \rho, H[g^{-1}]\rangle},\ \text{if } x = g.o.$$

In this case the generator of the φ-process is the so-called distinguished Laplacian on the solvable group AN, (see [8], Cowling, Giulini, Hulanicki & Mauceri [13]). This generator is

$$\frac{1}{2}\sum_i H_i^2 + \sum_k N_k^2$$

where (H_i) is an orthonormal basis of $\mathfrak{a}$ and (N_k) is an orthonormal basis of $\mathfrak{n}$, compatible with the root space decomposition. It is left invariant under AN and the relativized φ-process is a symmetric continuous-time random walk on AN (the distributions of B_t^φ form a symmetric convolution semigroup on the group AN). The above proposition thus gives a precise description of this process (notice that $\dot{k}(B_T^\varphi)$ converges to the class M in this case).

7. Asymptotic behavior of the double-ended I.B.L. on a symmetric space

We now consider the double-ended infinite Brownian loop $\{(B_t^0, \tilde{B}_t^0), t \geq 0\}$ from $q \in \mathbb{M}$ to $p \in \mathbb{M}$. Without loss of generality we will suppose that $q = o$. Let us define $\Psi : \mathbb{M} \times \mathbb{M} \to \mathbb{R}$ by

$$\Psi(g.o, h.o) = \Phi_0(h^{-1}g.o)$$

where Φ_0 is the Harish–Chandra function. It follows immediately from Proposition 2.6 and Proposition 4.2 that this double-ended I.B.L. $(B^0, \tilde{B}^0)$ is the relativized Ψ-process of the Brownian motion $(B, \tilde{B})$ on $\mathbb{M} \times \mathbb{M}$, starting from (o, p). Observe that B and $\tilde{B}$ are two independent Brownian motions on $\mathbb{M}$. Since $\Psi(g.o, h.o) \neq \Phi_0(g.o)\Phi_0(h.o)$, the two components B^0 and $\tilde{B}^0$ are not independent. Actually, loosely speaking, these two components remember that the K/M-component of the Brownian motion on $\mathbb{M}$ converges almost surely. This is the intuitive explanation of the assertion (i) of the following theorem. Observe also that it follows from this theorem that the asymptotic behavior of the first component B^0 itself does also depend on the end p.

For each $p \in \mathbb{M}$, let ν_p be the distribution on K/M defined by

$$\nu_p(V) = \int_V \frac{h_b(p)}{\Phi_0(p)}\, d\nu(b)$$

for all Borel set V in K/M, where ν is the K-invariant probability measure on K/M and $h_b(g.o) = e^{-\langle \rho, H[g^{-1}k]\rangle}$ if $b = kM, k \in K$. We choose some $\gamma \in G$ such that $\gamma.p = o$. Notice that when $p = o$, the following theorem (except (ii)) also follows from Theorem 6.1, since the double ended $I.B.L$ is a ground state process of the Brownian motion on $\mathbb{M}^2$.

Theorem 7.1 *Let $\{(B_t^0, \tilde{B}_t^0), t \geq 0\}$ be the double-ended infinite Brownian loop from o to p on the symmetric space $\mathbb{M}$. Then*

(i) The "angular parts" $\dot{k}(B_t^0)$ and $\dot{k}(\tilde{B}_t^0)$ converge almost surely to the same limit b_∞, which has the distribution ν_p on K/M.

(ii) The processes B^0 and $C(\gamma.\tilde{B}^0)$ are independent.

(iii) The normalized radial parts $\{\frac{1}{\sqrt{T}}(C(B_{tT}^0), C(\tilde{B}_{tT}^0)), t \geq 0\}$ converge in distribution to two independent copies of the Brownian motion in the Weyl chamber, as $T \to +\infty$.

(iv) The three processes $\frac{1}{\sqrt{T}}C(B_{tT}^0)$, $\frac{1}{\sqrt{T}}C(\tilde{B}_{tT}^0)$, and $(\dot{k}(B_t^0), \dot{k}(\tilde{B}_t^0))$ are asymptotically independent (in the sense similar to that of Corollary 5.11).

Proof. The proof of (i) will use the same approach as Proposition 5.1 and the following formula (see Helgason [32], Lemma IV.4.4), for all $g, h \in G$,

$$\Phi_0(h^{-1}g.o) = \int_{K/M} e^{-\langle\rho, H[h^{-1}k]\rangle} e^{-\langle\rho, H[g^{-1}k]\rangle}\, d\nu(\dot{k}),$$

which can be written: for all $x_1, x_2 \in \mathbb{M}$,

$$\Psi(x_1, x_2) = \int_{K/M} h_b(x_1) h_b(x_2)\, d\nu(b). \tag{7.1}$$

Let $\mathbf{D} = \Delta_{x_1} + \Delta_{x_2}$ be the Laplace Beltrami operator on $\mathbb{M}^2$. The bottom of the spectrum of $-\mathbf{D}/2$ is $2\lambda_0$. Let $p_t^{(2)}$ be the heat kernel on $\mathbb{M}^2$ and $G_{2\lambda_0}^{(2)} = \int_0^{+\infty} e^{2\lambda_0} p_t^{(2)}\, dt$ be the Green kernel of $\frac{1}{2}\mathbf{D} + 2\lambda_0$. Consider

$$\mathbf{D}^0 = \Psi^{-1}\,(\frac{1}{2}\mathbf{D} + 2\lambda_0) \circ \Psi,$$

and its Green kernel G. We introduce the two following Martin kernels

$$K^{(2)}(x, y) = \frac{G_{2\lambda_0}^{(2)}(x, y)}{G_{2\lambda_0}^{(2)}((o, o), y)}, \qquad K_p(x, y) = \frac{G(x, y)}{G((o, p), y)}, \quad x, y \in \mathbb{M}^2.$$

For all $(b_1, b_2) \in K/M \times K/M$, the functions

$$\bar{h}_{(b_1,b_2)}(x) = \frac{h_{b_1}(x_1) h_{b_2}(x_2) \Psi(o, p)}{h_{b_2}(p) \Psi(x_1, x_2)}, \qquad x = (x_1, x_2) \in \mathbb{M}^2$$

are $\mathbf{D}^0$-harmonic and equal to 1 at (o, p). The formula (7.1) can be written:

$$1 = \int_{K/M} \bar{h}_{(b,b)}(x) \frac{h_b(p)}{\Psi(o, p)}\, d\nu(b) = \int_{K/M} \bar{h}_{(b,b)}(x)\, d\nu_p(b)$$

for all $x \in \mathbb{M}^2$, or

$$1 = \int_{K/M \times K/M} \bar{h}_{(b_1,b_2)}(x)\, d\bar{\nu}_p(b_1, b_2)$$

where $\bar{\nu}_p$ is the image on the diagonal of $K/M \times K/M$ of ν_p by the map $b \mapsto (b, b)$. It follows from Martin boundary theory and from this representation formula of the harmonic function 1 that the diffusion $(B^0, \tilde{B}^0)$ associated with $\mathbf{D^0}$ starting from (o, p) converges almost surely in the Martin topology of $(\mathbf{D}^0, K_p)$ to a random limit (b_∞, b_∞) where b_∞ has the distribution ν_p (see [34]). By definition, this means that

$$\lim_{t \to +\infty} K_p(x, (B_t^0, \tilde{B}_t^0)) = \bar{h}_{b_\infty}(x).$$

Now, since $G(x,y) = G^{(2)}_{2\lambda_0}(x,y)\Psi(y)/\Psi(x)$, it is easy to see that

$$K^{(2)}(x,y) = \frac{K_p(x,y)\Psi(x)}{K_p(0,y)}.$$

Therefore, for all $x_1, x_2 \in \mathbb{M}$,

$$\lim_{t\to+\infty} K^{(2)}((x_1,x_2),(B^0_t,\tilde{B}^0_t)) = h_{b_\infty}(x_1)h_{b_\infty}(x_2). \tag{7.2}$$

As in the proof of Proposition 5.1, we see, using the description of the Martin boundary of $\frac{1}{2}\mathbf{D} + 2\lambda_0$ given by Guivarc'h, Ji & Taylor [27], that if

$$\lim_{n\to+\infty} K^{(2)}((x_1,x_2),y^{(n)}) = h_b(x_1)h_b(x_2), \qquad \forall (x_1,x_2) \in \mathbb{M}^2,$$

then the two K/M-components $\dot{k}(y_1^{(n)})$ and $\dot{k}(y_2^{(n)})$ of $y_n = (y_1^{(n)}, y_2^{(n)}) \in \mathbb{M}^2$ both converge to $b \in K/M$. Thus (i) follows from (7.2).

Let us prove (ii). Let $f : \mathbb{M}^2 \mapsto \mathbb{R}$ and $g : \mathfrak{a}_+ \times \mathfrak{a}_+ \mapsto \mathbb{R}$ be measurable bounded functions. One has, for all $t, s \geq 0$,

$$\begin{aligned}
&\mathbb{E}[f(B^0_s, B^0_{s+t})g(C(\gamma.\tilde{B}^0_s), C(\gamma.\tilde{B}^0_{s+t}))] = \\
&= \mathbb{E}[f(B_s, B_{s+t})g(C(\gamma.\tilde{B}_s), C(\gamma.\tilde{B}_{s+t}))e^{-(s+t)2\lambda_0}\Psi(B_{s+t}, \tilde{B}_{s+t})] \\
&= e^{-(s+t)2\lambda_0}\int f(x_1,x_2)g(C(\gamma.x_3), C(\gamma.x_4))p_s(o,x_1)p_t(x_1,x_2) \\
&\qquad \times p_s(\gamma^{-1}.o, x_3)p_t(x_3,x_4)\Psi(x_2,x_4)dm(x_1)dm(x_2)dm(x_3)dm(x_4)
\end{aligned}$$

where m is the Riemannian measure on $\mathbb{M}$. Since both m and p_t are invariant under the action of G, one has

$$\begin{aligned}
&\int g(C(\gamma.x_3), C(\gamma.x_4))p_s(\gamma^{-1}.o, x_3)p_t(x_3,x_4)\Psi(x_2,x_4)dm(x_3)dm(x_4) \\
&= \int g(C(x_3), C(x_4))p_s(o,x_3)p_t(\gamma^{-1}.x_3, \gamma^{-1}.x_4)\Psi(x_2, \gamma^{-1}.x_4)dm(x_3)dm(x_4) \\
&= \int g(C(x_3), C(x_4))p_s(o,x_3)p_t(x_3,x_4)\Psi(x_2, \gamma^{-1}.x_4)dm(x_3)dm(x_4) \\
&= \int g(C(x_3), r_4)p_s(o,x_3)p_t(x_3, k_4e^{r_4}.o)\Psi(x_2, \gamma^{-1}k_4e^{r_4}.o)dm(x_3)dk_4\tilde{\delta}(r_4)dr_4 \\
&= \int g(C(k_4.x_3), r_4)p_s(k_4.o, k_4.x_3)p_t(k_4.x_3, k_4e^{r_4}.o) \\
&\qquad\qquad \times \Psi(x_2, \gamma^{-1}k_4e^{r_4}.o)dm(x_3)dk_4\tilde{\delta}(r_4)dr_4 \\
&= \int g(C(x_3), r_4)p_s(o,x_3)p_t(x_3, e^{r_4}.o)\Psi(x_2, \gamma^{-1}k_4e^{r_4}.o)dm(x_3)dk_4\tilde{\delta}(r_4)dr_4
\end{aligned}$$

where we have used the expression of m in polar coordinates (if $x = ke^r.o$, $dm(x) = dk\tilde{\delta}(r)dr$).

On the other hand, since Φ_0 as a function on G is spherical,

$$\int_K \Phi_0(gkh)\, dk = \Psi_0(g)\Phi_0(h)\,,$$

and symmetric ($\Phi_0(g) = \Phi_0(g^{-1})$) one has

$$\int \Psi(x_2, \gamma^{-1}k_4 e^{r_4}.o)\, dk_4 = \int \Phi_0(e^{-r_4}k_4^{-1}\gamma.x_2)\, dk_4 = \Phi_0(\gamma.x_2)\Phi_0(e^{r_4}.o)$$

hence,

$$\begin{aligned}
&\mathbb{E}[f(B^0_s, B^0_{s+t})g(C(\gamma.\tilde{B}^0_s), C(\gamma.\tilde{B}^0_{s+t}))] = e^{-2\lambda_0(s+t)}\times \\
&\quad \times \int f(x_1,x_2)p_s(o,x_1)p_t(x_1,x_2)\Phi_0(\gamma.x_2)dm(x_1)dm(x_2) \\
&\quad \times \int g(r_3,r_4)p_s(o,e^{r_3}.o)p_t(k_3e^{r_3}.o, e^{r_4}.o)\Phi_0(e^{r_4}.o)\, \tilde{\delta}(r_3)\tilde{\delta}(r_4)\, dr_3dr_4dk_3. \\
&= e^{-\lambda_0(s+t)} \int f(x_1,x_2)p_s(o,x_1)p_t(x_1,x_2)\Phi_0(\gamma.x_2)dm(x_1)dm(x_2) \\
&\quad \times e^{-\lambda_0(s+t)} \int g(C(x_3),C(x_4))p_s(o,x_3)p_t(x_3,x_4)\Phi_0(x_4)dm(x_3)dm(x_4).
\end{aligned}$$

The same proof applies for f and g depending on an arbitrary finite number of coordinates of the processes $B^0, \tilde{B}^0$. We first deduce from this formula that the process $C(\gamma.\tilde{B}^0)$ has the same distribution as the radial part of the I.B.L. around o, and then that B^0 and $(C(\tilde{B}^0)$ are independent. This proves (ii).

To prove (iii) one first observes that $C(B^0)$ and $C(\gamma.\tilde{B}^0)$ are independent and have the same distribution as the radial component of the I.B.L. around o. Hence, the processes $\{\frac{1}{\sqrt{T}}(C(B^0_{tT}), C(\gamma.\tilde{B}^0_{tT})), t \geq 0\}$ converge in distribution to two independent copies of the Brownian motion in the Weyl chamber, as $T \to +\infty$. It follows from the next lemma that $C(\gamma.\tilde{B}^0_t) - C(\tilde{B}^0_t)$ is bounded when $t \to +\infty$. This implies (iii).

In order to show (iv), let us prove that if F is a bounded continuous function on $C((0,\infty),(K/M)^2)$ and G_1, G_2 are bounded continuous function on $C(\mathbb{R}_+, \mathfrak{a}_+ \cup \{0\})$, then, as $T \to +\infty$,

$$\mathbb{E}\left[F(\dot{k}(B^{(0)}_{\cdot T}, \dot{k}(\tilde{B}^{(0)}_{\cdot T}))\, G_1(\frac{C(B^{(0)}_{\cdot T})}{\sqrt{T}})\, G_2(\frac{C(\tilde{B}^{(0)}_{\cdot T})}{\sqrt{T}})\right] \to \mathbb{E}[F(\bar{b}_\infty)]\mathbb{E}_{\mathbb{Q}}(G_1)\mathbb{E}_{\mathbb{Q}}(G_2)$$

where $\bar{b}$ is the constant function $\bar{b}(t) = b$. One may suppose that, for some $\varepsilon > 0$, $F(\omega)$ does not depend on $\omega_s, s \leq \varepsilon$. It follows from (i) that

$$\lim_{T\to+\infty} F(\dot{k}(B^{(0)}_{tT}), \dot{k}(\tilde{B}^{(0)}_{tT}), t > 0)) = F(\bar{b}_\infty).$$

This implies that

$$\begin{aligned}
&\mathbb{E}\left[F(\dot{k}(B^{(0)}_{\cdot T}, \dot{k}(\tilde{B}^{(0)}_{\cdot T}))\, G_1(\frac{C(B^{(0)}_{\cdot T})}{\sqrt{T}})\, G_2(\frac{C(\tilde{B}^{(0)}_{\cdot T})}{\sqrt{T}})\right] \\
&\quad = \lim_{T\to+\infty} \mathbb{E}\left[F(\bar{b}_\infty)\, G_1(\frac{C(B^{(0)}_{\cdot T})}{\sqrt{T}})\, G_2(\frac{C(\tilde{B}^{(0)}_{\cdot T})}{\sqrt{T}})\right] \\
&\quad = \lim_{T\to+\infty} \mathbb{E}\left[F(\bar{b}_\infty)\, G_1(\frac{C(B^{(0)}_{\cdot T})}{\sqrt{T}})\, G_2(\frac{C(\gamma.\tilde{B}^{(0)}_{\cdot T})}{\sqrt{T}})\right]
\end{aligned}$$

where we use Lemma 7.2 to replace $C(\tilde{B}^{(0)})$ by $C(\gamma.\tilde{B}^{(0)})$. Since b_∞ is $\sigma(B^0_t, t \geq 0)$-measurable, this is equal to

$$\lim_{T\to+\infty} \mathbb{E}\left[F(\bar{b}_\infty)\, G_1(\frac{C(B^{(0)}_{\cdot T})}{\sqrt{T}})\right] \mathbb{E}\left[G_2(\frac{C(\gamma.\tilde{B}^{(0)}_{\cdot T})}{\sqrt{T}})\right]$$

by (ii) and one concludes with Corollary 5.11. ■

Lemma 7.2 *Let x_j be a sequence in $\bar{\mathfrak{a}}_+$, and let $g, h \in G$. Denote by $y_j \in \bar{\mathfrak{a}}_+$ the radial component of $g\, e^{x_j} h$ in the Cartan decomposition $G = K(\exp \bar{\mathfrak{a}}_+)K$. Then y_j remains at bounded distance from x_j, as $j \to +\infty$.*

Proof. Let $k_j, k'_j \in K$ such that $g\, e^{x_j} h = k_j\, e^{y_j} k'_j$ and let us decompose $g = k\, e^z n$ in the Iwasawa decomposition $G = K(\exp \mathfrak{a})N$. Since

$$\|y_j - x_j\| = d(Ke^{y_j}.o, Ke^{x_j}.o)$$

(see for instance Lemma 2.1.2 in [2]), we have

$$\begin{aligned}
\|y_j - x_j\| &\leq d(k^{-1}k_j e^{y_j}.o, e^{x_j}.o) \;=\; d(e^z n\, e^{x_j} h.o, e^{x_j}.o) \\
&\leq d(e^z n\, e^{x_j} h.o, e^z n\, e^{x_j}.o) + d(e^z n\, e^{x_j}.o, e^{x_j}.o) \\
&= d(h.o, o) + d(e^{-x_j} n\, e^{x_j}.o, e^{-z}.o) \\
&\leq d(h.o, o) + d(e^{-x_j} n\, e^{x_j}.o, o) + \|z\|\,,
\end{aligned}$$

which remains bounded, since $\mathrm{Ad}(e^{-x_j}) = e^{-\operatorname{ad} x_j}$ acts by contractions on $\mathfrak{n}$. ■

8. Appendix. Some estimates of φ_0

We consider a symmetric space $\mathbb{M} = G/K$ of the noncompact type. The ground spherical function

$$\Phi_0(g) = \int_K e^{-\langle \rho, H[g^{\pm 1}k]\rangle}\, dk = \int_{K/M} e^{-\langle \rho, H[g^{-1}k]\rangle}\, d\nu(\dot{k})$$

plays a fundamental role in harmonic analysis on semisimple Lie groups. Let us recall its behavior, which was fully determined in Anker [1] and in Anker & Ji [2] (see Proposition 2.2.12), by resuming carefully Harish–Chandra's analysis (see Gangolli & Varadarajan [22], Section 4.6 & Theorem 5.9.5). We use the notation of these references with a few modifications: Σ_0^+ is the set of indivisible positive roots, S is the set of simple positive roots, $\rho^0 = \frac{1}{2}\sum_{\alpha\in\Sigma_0^+}\alpha$ and $d = \mathrm{rank}(\mathbb{M})$. Recall that $\varphi_0(x) = \Phi_0(e^x), x \in \mathfrak{a}$.

Theorem 8.1 (i) Global estimate[1] :

$$\varphi_0(x) \asymp \Big\{ \prod_{\alpha\in\Sigma_0^+} (1 + \langle \alpha, x\rangle) \Big\}\, e^{-\langle \rho, x\rangle} \qquad \forall\, x \in \bar{\mathfrak{a}}_+ .$$

(ii) Asymptotics away from the walls :

$$\varphi_0(x) \sim \gamma\, \boldsymbol{\pi}(x)\, e^{-\langle \rho, x\rangle}$$

when $\langle \alpha, x\rangle \to +\infty \ \ \forall\, \alpha \in \Sigma^+$. *The positive constant* γ *is equal to* $\frac{\mathbf{b}(0)}{\boldsymbol{\pi}(\rho^0)}$, *where* $\mathbf{b}(\lambda) = \boldsymbol{\pi}(i\lambda)\, \mathbf{c}(\lambda), \boldsymbol{\pi}(x) = \prod_{\alpha\in\Sigma_0^+}\langle \alpha, \lambda\rangle$, *and* $\mathbf{c}$ *is the Harish–Chandra function.*

(iii) Asymptotics along a face :

$$\frac{\varphi_0(x)}{\gamma\, \boldsymbol{\pi}(x)\, e^{-\langle \rho, x\rangle}} \sim \frac{\varphi_{F,0}(x)}{\gamma_F\, \boldsymbol{\pi}_F(x)\, e^{-\langle \rho_F, x\rangle}} \quad \textit{i.e.} \quad \varphi_0(x) \sim \gamma^F \pi^F(x)\, \varphi_{F,0}(x)\, e^{-\langle \rho^F, x\rangle}$$

when $x \in \bar{\mathfrak{a}}_+$, $\omega^F(x) = \min_{\alpha\in S\smallsetminus F}\langle \alpha, x\rangle \to +\infty$, *while* $\omega_F(x) = \max_{\alpha\in F}\langle \alpha, x\rangle$ *remains* $\mathrm{o}\big(\omega^F(x)\big)$.

Recall that subsets F of S are in one–to–one correspondence with faces

$$\mathfrak{a}_+^F = \{\, x \in \mathfrak{a} \mid \langle \alpha, x\rangle = 0 \ \ \forall\, \alpha \in F \ \text{ and } \ \langle \alpha, x\rangle > 0 \ \ \forall\, \alpha \in S \smallsetminus F\,\}$$

of $\bar{\mathfrak{a}}_+$ and with standard parabolic subgroups $P^F = G_F A^F N^F$ of G. We use F as a subscript for quantities attached to the reductive component G_F,

[1] The symbol $\asymp$ between two positive expressions means that there ratio is bounded above and below

and as a superscript for complementary quantities, which are most generally attached to the split component A^F. For instance

$$\rho_F = \frac{1}{2} \sum_{\alpha \in \Sigma_F^+} m_\alpha \, \alpha \in \mathfrak{a}_F \,, \quad \gamma_F = \frac{\mathbf{b}_F(0)}{\boldsymbol{\pi}_F(\rho_F^0)} \,, \quad \dots$$

while

$$\rho^F = \rho - \rho_F = \frac{1}{2} \sum_{\alpha \in \Sigma^+ \smallsetminus \Sigma_F^+} m_\alpha \, \alpha \in \mathfrak{a}^F, \quad \gamma^F = \frac{\gamma}{\gamma_F} = \frac{\mathbf{b}^F(0)}{\boldsymbol{\pi}^F(\rho^0)} \,, \quad \dots$$

In this appendix, we analyze some logarithmic derivatives of φ_0, which are used in Proposition 5.3. Consider the Euler operator

$$\mathcal{E} f(x) = \partial_x f(x) = \frac{\partial}{\partial t}\Big|_{t=1} f(tx)$$

on $\mathfrak{a}$, which writes

$$\mathcal{E} f(x) = \sum_{j=1}^{d} x_j \, \frac{\partial}{\partial x_j} f(x)$$

in Euclidean coordinates or, after a short calculation,

$$\mathcal{E} f(x) = \sum_{\alpha \in \Sigma} m_\alpha \, \langle \alpha, x \rangle \, \partial_\alpha f(x) = \, 2 \sum_{\alpha \in \Sigma^+} m_\alpha \, \langle \alpha, x \rangle \, \partial_\alpha f(x)$$

with respect to Σ (we will not use this one). The expression we are interested in is

$$\chi = \mathcal{E} \log \big(\delta^{\frac{1}{2}} \varphi_0 \big) \,. \tag{8.1}$$

It is well–known that spherical analysis is elementary when G is complex. In this case, Σ is reduced, all roots have multiplicity 2, and $\delta^{\frac{1}{2}} \varphi_0 = \boldsymbol{\pi}$ is a homogeneous polynomial of degree $|\Sigma^+|$. Consequently

$$\chi(x) \equiv |\Sigma^+| \,. \tag{8.2}$$

In general, since, if $n = \dim \mathbb{M}$ and $d = \dim \mathfrak{a}$,

$$\delta(x) \sim 2^{d-n} e^{\,2\langle \rho, x \rangle} \quad \text{and} \quad \varphi_0(x) \sim \gamma \, \boldsymbol{\pi}(x) \, e^{-\langle \rho, x \rangle}$$

when $x \in \mathfrak{a}_+$ tends to infinity away from the walls, it is conceivable that (8.2) holds asymptotically. This will be established next.

Proposition 8.2 *The expression* $\chi(x)$ *tends to* $|\Sigma_0^+|$ *when* $\langle\alpha,x\rangle\to+\infty$ *for all* $\alpha\in\Sigma^+$.

Proof. Since we are working away from the walls, we can expand the spherical functions

$$\Phi_\lambda(g)=\int_K e^{\langle i\lambda-\rho,H[gk]\rangle}\,dk$$

according to Harish–Chandra, actually in the following modified way, due to Gangolli: if $\varphi_\lambda(x)=\Phi_\lambda(e^x), x\in\mathfrak{a}_+$,

$$\delta(x)^{\frac12}\boldsymbol{\pi}(i\lambda)\,\varphi_\lambda(x)=\sum_{w\in W}\sum_{q\in 2Q}(\det w)\,\mathbf{b}(w.\lambda)\,a_q(w.\lambda)\,e^{\langle i\,w.\lambda-q,\,x\rangle} \tag{8.3}$$

(see for instance Gangolli & Varadarajan [22], Section 4.5). Here W is the Weyl group, Q is the positive lattice generated by the (simple) positive roots, $\mathbf{b}(\lambda)=\boldsymbol{\pi}(i\lambda)\,\mathbf{c}(\lambda)$ is an analytic function on $\mathfrak{a}$ with polynomial growth, $a_0(\lambda)\equiv1$ and the other $a_q(\lambda)$ are rational functions with no singularities on $\mathfrak{a}$, which can be estimated as follows, together with their derivatives:

$$\left|\left(\tfrac{\partial}{\partial\lambda}\right)^j a_q(\lambda)\right|\le C_j\,(1+\|q\|)^N\qquad\forall\,\lambda\in\mathfrak{a}\,.$$

As a consequence, the series in (8.3) converges for $\lambda\in\mathfrak{a}$ and $x\in\mathfrak{a}_+$, and it can be differentiated term by term in both variables. By applying successively $\boldsymbol{\pi}\left(-i\frac{\partial}{\partial\lambda}\right)\big|_{\lambda=0}$ and the Euler operator, we deduce from (8.3) the expansions

$$\begin{aligned}\delta(x)^{\frac12}\,\varphi_0(x)&=\tfrac{1}{c}\,\boldsymbol{\pi}\left(-i\tfrac{\partial}{\partial\lambda}\right)\big|_{\lambda=0}\left\{\,\delta(x)^{\frac12}\,\boldsymbol{\pi}(i\lambda)\,\varphi_\lambda(x)\,\right\}\\&=\sum_{R\subset\Sigma_0^+}\sum_{q\in2Q}c_{R,q}\left\{\prod_{\alpha\in R}\langle\alpha,x\rangle\right\}e^{-\langle q,\,x\rangle}\end{aligned} \tag{8.4}$$

where $c=\partial(\boldsymbol{\pi})\,\boldsymbol{\pi}=|W|\,\boldsymbol{\pi}(\rho^0)$ (for the last equality see for instance [2], Proof of Proposition 2.2.12.ii.), and

$$\mathcal{E}(\delta^{\frac12}\varphi_0)(x)=\sum_{R\subset\Sigma_0^+}\sum_{q\in2Q}c_{R,q}\,(|R|-\langle q,x\rangle)\left\{\prod_{\alpha\in R}\langle\alpha,x\rangle\right\}e^{-\langle q,\,x\rangle}\,. \tag{8.5}$$

The constant c is positive, as well as the leading coefficient $c_{\Sigma_0^+,0}=\frac{\mathbf{b}(0)}{\boldsymbol{\pi}(\rho^0)}$. The other $c_{R,q}$ are $\mathrm{O}\left(1+\|q\|^N\right)$. Thus we deduce from (8.4) and (8.5) that

$$\delta(x)^{\frac12}\,\varphi_0(x)=\left\{c_{\Sigma_0^+,0}+\mathrm{o}\,(1)\right\}\boldsymbol{\pi}(x) \tag{8.6}$$

and

$$\mathcal{E}(\delta^{\frac12}\varphi_0)(x)=\left\{c_{\Sigma_0^+,0}\,|\Sigma_0^+|+\mathrm{o}\,(1)\right\}\boldsymbol{\pi}(x) \tag{8.7}$$

when $\langle\alpha,x\rangle\to+\infty \ \ \forall\,\alpha\in\Sigma^+$, hence

$$\chi(x)=\frac{\mathcal{E}(\delta^{\frac12}\varphi_0)(x)}{\delta(x)^{\frac12}\varphi_0(x)}\;\longrightarrow\;|\Sigma_0^+|\,.$$

■

Remarks:

(a) Notice that (8.6) reproves Theorem 8.1.ii.

(b) By combining (8.7) with Theorem 8.1.i, we see that χ is bounded in every $\mathfrak{a}_+^\eta$.

The rest of this appendix is devoted to the proof of the following result, which requires to analyze the expression $\chi(x)$ along the various faces $\mathfrak{a}_+^F$ of $\bar{\mathfrak{a}}_+$.

Theorem 8.3 *On $\bar{\mathfrak{a}}_+$, χ is positive and bounded, both above and below. Moreover*

$$\chi(x)\;\longrightarrow\;|\,\Sigma_0^+\smallsetminus\Sigma_{F,0}^+\,|\;+\sum_{\alpha\in\Sigma_F^+}\tfrac{m_\alpha}{2}$$

when $x\in\bar{\mathfrak{a}}_+$ tends to infinity tangentially to the face $\mathfrak{a}_+^F$ *i.e.*

(8.8)
$$\begin{cases}\langle\alpha,x\rangle\to0 & \forall\,\alpha\in F,\\ \langle\alpha,x\rangle\to+\infty & \forall\,\alpha\in S\smallsetminus F.\end{cases}$$

We shall first replace the factor $\delta(x)^{\frac12}$ by $e^{\langle\rho,x\rangle}$ in the definition (8.1) of χ.

Lemma 8.4 (i) *The expression $\chi_1=\mathcal{E}\log(\delta^{\frac12}e^{-\rho})$ is positive and bounded above on $\bar{\mathfrak{a}}_+$.*

(ii) *$\chi_1(x)$ tends to $\sum_{\alpha\in\Sigma_F^+}\frac{m_\alpha}{2}$ under the assumption* (8.8).

(iii) *$\chi_1(x)$ tends to 0 if and only if $\langle\alpha,x\rangle\to\infty \ \forall\,\alpha\in\Sigma^+$.*

Proof. Everything follows from the explicit formula

$$\chi_1(x)=\sum_{\alpha\in\Sigma^+}m_\alpha\,\tfrac{\langle\alpha,x\rangle}{e^{\,2\langle\alpha,x\rangle}-1}$$

and from the behavior of the function $x\mapsto\frac{x}{e^x-1}$ for $x\geq0$.

In order to prove Theorem 8.3, it remains for us to establish the following properties of the expression

(8.9)
$$\chi_2=\chi-\chi_1=\mathcal{E}\log(e^\rho\varphi_0)\,.$$

Proposition 8.5 *On $\bar{\mathfrak{a}}_+$, χ_2 is nonnegative and bounded above. Moreover $\chi_2(x)$ tends to $|\,\Sigma_0^+ \smallsetminus \Sigma_{F,0}^+\,|$ under the assumption* (8.8).

Apart maybe from the lower bound, Theorem 8.3 follows obviously from Lemma 8.4 and Proposition 8.5. If χ were not bounded below, there would be a sequence $x_n \in \bar{\mathfrak{a}}_+$ such that $\chi(x_n) \to 0$. Since χ_1 and χ_2 are nonnegative, according to Lemma 8.4 and Proposition 8.5, this implies successively $\chi_2(x_n) \to 0$, $\langle \alpha, x_n \rangle \to 0 \;\; \forall\, \alpha \in \Sigma^+$ and $\chi(x_n) \to |\Sigma_0^+|$, by Lemma 8.4 and Proposition 8.2. Hence a contradiction.

Proposition 8.5 will be proved in several steps. After the first step, which is independent of the rest, we shall follow Harish–Chandra's strategy, using his *constant term theory* for φ_0 along faces, which consists in first order asymptotics, and reducing this way to semisimple symmetric subspaces of lower rank. Thus, beginning with Step 2, we shall argue by induction over the semisimple split rank and assume that Proposition 8.5 holds for every proper symmetric subspace $\mathbb{M}_F = G_F/K_F$ of $\mathbb{M} = G/K$. Notice that the rank zero case $F = \emptyset$ is trivial and that the rank one case $|F| = 1$ is already covered by Proposition 8.2.

Step 1 : Let us first show that $\chi_2 \geq 0$.

When applying the Euler operator $\mathcal{E}$ to the expression

$$e^{\langle \rho, x \rangle}\, \varphi_0(x) = \int_K e^{\langle \rho,\, x - H[e^x k] \rangle}\, dk$$

one is essentially reduced to differentiating the Iwasawa map

$$x \mapsto H[e^x k] = H[e^{\operatorname{Ad} k^{-1}.x}]$$

Recall that the derivative at the origin of the Iwasawa projection $H \circ \exp : \mathfrak{p} \mapsto \mathfrak{a}$ is the orthogonal projection $\mathrm{pr}_{\mathfrak{a}} : \mathfrak{p} \mapsto \mathfrak{a}$. Setting $y = \operatorname{Ad} k^{-1}.x$ and $g = e^y = k^{-1} e^x k$, we have

$$\begin{aligned} \frac{d}{dt}\Big|_{t=1} H[e^{tx} k] &= \frac{d}{dt}\Big|_{t=1} H \circ \exp(ty) = \frac{d}{dt}\Big|_{t=0} H[e^{ty} g] \\ &= \frac{d}{dt}\Big|_{t=0} H\big[e^{ty} k[g] e^{H[g]} n[g]\big]\,, \end{aligned}$$

which is equal to

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} H \circ \exp(t \operatorname{Ad} k[g]^{-1}.y) &= \mathrm{pr}_{\mathfrak{a}}(\operatorname{Ad} k[g]^{-1}.y) \\ &= \mathrm{pr}_{\mathfrak{a}}(\operatorname{Ad} k[e^x k]^{-1}.x)\,, \end{aligned} \tag{8.10}$$

since $H\big[e^{ty}k[g]e^{H[g]}n[g]\big]=H\big[e^{ty}k[g]\big]+H[g]$. Thus

$$\mathcal{E}(e^{\rho}\varphi_0)(x)=\int_K\langle\rho,x-\mathrm{pr}_{\mathfrak{a}}(\mathrm{Ad}\,k[e^xk]^{-1}.x)\,\rangle\,e^{\,\langle\rho,\,x-H[e^xk]\rangle}\,dk\,. \tag{8.11}$$

According to Kostant's convexity theorem in the flat case (see for instance Helgason [32], Theorem IV.10.2), $\mathrm{pr}_{\mathfrak{a}}(\mathrm{Ad}\,K.x)$ coincides with the convex hull of $W.x$. In particular,

$$\langle\rho,\mathrm{pr}_{\mathfrak{a}}(\mathrm{Ad}\,k[e^xk]^{-1}.x)\rangle\le\langle\rho,x\rangle\qquad\forall\,k\in K\,,\ \forall\,x\in\bar{\mathfrak{a}}_+\,.$$

As a consequence, the expression (8.11) hence $\chi_2(x)=\frac{\mathcal{E}(e^{\rho}\varphi_0)(x)}{e^{\,\langle\rho,x\rangle}\varphi_0(x)}$ is nonnegative.

Step 2 : Analysis along a face

This is the actual induction step. Assume that x tends to infinity in $\bar{\mathfrak{a}}_+$ in the following way :

$$\begin{cases}\langle\alpha,x\rangle=\mathrm{o}\,(\|x\|) & \forall\,\alpha\in F,\\ \langle\alpha,x\rangle\asymp\|x\| & \forall\,\alpha\in S\smallsetminus F,\end{cases} \tag{8.12}$$

where F is a proper subset of S. Then

$$\varphi_0(x)=\psi_0^F(x)\,e^{-\langle\rho^F,x^F\rangle}+\mathrm{O}\left(e^{-\langle\rho,x\rangle-2\,\omega^F(x)}\right), \tag{8.13}$$

where ψ_0^F is the *constant term* of φ_0 at infinity along the face $\mathfrak{a}_+^F$ (see for instance Gangolli & Varadarajan [22], Theorem 5.9.3.a) and $\omega^F(x)=\min_{\alpha\in S\smallsetminus F}\langle\alpha,x\rangle\asymp\|x\|$. Specifically,

$$\psi_0^F(x)=p^F\big(\tfrac{\partial}{\partial\lambda}\big)\big|_{\lambda=0}\,\varphi_\lambda^F(x)=\int_{K_F}p^F(H_F[e^{x_F}k_F]+x^F)\,e^{-\langle\rho_F,H_F[e^{x_F}k_F]\rangle}\,dk,$$

where

$$p^F(x)=\sum_{\substack{R\subset\Sigma_0^+\\ |R|\le|\Sigma_0^+\smallsetminus\Sigma_{F,0}^+|}}\gamma_R^F\,\big\{\prod_{\alpha\in R}\langle\alpha,x\rangle\big\}$$

is a polynomial with leading coefficient $\gamma^F_{\Sigma_0^+\smallsetminus\Sigma_{F,0}^+}=\gamma^F=\frac{\mathbf{b}^F(0)}{\boldsymbol{\pi}^F(\rho^0)}>0$ (see Gangolli & Varadarajan [22], Corollary 5.8.12 and Anker & Ji [2], proof of Proposition 2.2.12.ii), and

$$\varphi_\lambda^F(x)=\int_{K_F}e^{\,\langle i\lambda-\rho_F,\,H[e^xk_F]\rangle}\,dk_F=\int_{K_F}e^{\,\langle i\lambda_F-\rho_F,\,H_F[e^{x_F}k_F]\rangle}e^{\,\langle i\lambda^F,x^F\rangle}\,dk_F$$

denotes the spherical function of index $\lambda=\lambda_F+\lambda^F$ on $G^F=G_F\,A^F$. Hence

$$(8.14)\quad e^{\langle\rho_F,\,x_F\rangle}\,\psi_0^F(x)\;=\sum_{\substack{R\subset\Sigma_0^+\smallsetminus\Sigma_{F,0}^+,\,R'\subset\Sigma_0^+\smallsetminus R\\ |R|+|R'|\,\le\,|\Sigma_0^+\smallsetminus\Sigma_{F,0}^+|}}\gamma_{R,R'}^F\,\Big\{\prod_{\alpha\in R}\langle\alpha,x^F\rangle\Big\}\;\times$$
$$\times\int_{K_F}\Big\{\prod_{\beta\in R'}\langle\beta,H_F[e^{x_F}k_F]\rangle\Big\}\,e^{\langle\rho_F,\,x_F-H_F[e^{x_F}k_F]\rangle}\,dk_F\,,$$

with $\gamma^F_{\Sigma_0^+\smallsetminus\Sigma_{F,0}^+,\emptyset}=\gamma^F_{\Sigma_0^+\smallsetminus\Sigma_{F,0}^+}=\gamma^F$. As shown in Anker & Ji [2] (see Proof of Proposition 2.2.12.ii), the leading term

$$(8.15)\quad \gamma^F\,\boldsymbol{\pi}^F(x^F)\int_{K_F}e^{\langle\rho_F,\,x_F-H_F[e^{x_F}k_F]\rangle}\,dk_F$$
$$=\gamma^F\,\boldsymbol{\pi}^F(x)\,e^{\langle\rho_F,\,x_F\rangle}\,\varphi_{F,0}(x_F)\;+\;\mathrm{o}\Big\{\prod_{\alpha\in\Sigma_0^+}(1+\langle\alpha,x\rangle)\Big\}$$
$$\asymp\prod_{\alpha\in\Sigma_0^+}(1+\langle\alpha,x\rangle)$$

in (8.14) is obtained by taking $R=\Sigma_0^+\smallsetminus\Sigma_{F,0}^+$ and $R'=\emptyset$, while the other terms are

$$\mathrm{o}\Big\{\prod_{\alpha\in\Sigma_0^+}(1+\langle\alpha,x\rangle)\Big\}.$$

This yields the behavior

$$\begin{aligned} e^{\langle\rho,x\rangle}\,\varphi_0(x)\;&=\;e^{\langle\rho_F,\,x_F\rangle}\,\psi_0^F(x)\;+\;\mathrm{O}\big(\,\|x\|^{-\infty}\big)\\ &=\gamma^F\,\boldsymbol{\pi}^F(x)\,e^{\langle\rho_F,\,x_F\rangle}\,\varphi_{F,0}(x_F)\;+\;\mathrm{o}\;\Big\{\prod_{\alpha\in\Sigma_0^+}(1+\langle\alpha,x\rangle)\Big\}\\ &\asymp\prod_{\alpha\in\Sigma_0^+}(1+\langle\alpha,x\rangle)\end{aligned}$$

stated in Theorem 8.1. Here $\|x\|^{-\infty}$ denotes an arbitrary negative power of $\|x\|$. The expression $\mathcal{E}(e^\rho\varphi_0)(x)$ can be analyzed in a similar way. Since the asymptotic (8.13) holds also for derivatives: for each polynomial P there is $N_P>0$ such that

$$P\big(\tfrac{\partial}{\partial x}\big)\big\{e^{\langle\rho,x\rangle}\varphi_0(x)\big\}=P\big(\tfrac{\partial}{\partial x}\big)\big\{e^{\langle\rho_F,\,x\rangle}\psi_0^F(x)\big\}+\mathrm{O}\big\{(1+\|x\|)^{N_P}e^{-2\,\omega^F(x)}\big\}$$

(see Gangolli & Varadarajan [22], Theorem 5.9.3.a), we have

$$\mathcal{E}(e^\rho\varphi_0)(x)=\mathcal{E}(e^{\rho_F}\psi_0^F)(x)+\mathrm{O}\,\big\{(1+\|x\|)^Ne^{-2\,\omega^F(x)}\big\}$$

and are thus reduced to studying $\mathcal{E}(e^{\rho_F}\psi_0^F)(x)$. When the Euler operator is applied to the right hand side of (8.14), one obtains three kinds of expressions, depending whether $\mathcal{E}$ acts on $\prod_{\alpha\in R}\langle\alpha,x^F\rangle$, on $\prod_{\beta\in R'}\langle\beta,H_F[e^{x_F}k_F]\rangle$ or on $e^{\langle\rho_F,\,x_F-H_F[e^{x_F}k_F]\rangle}$, namely

$$(8.16)\qquad \begin{aligned} I^F_{R,R'} \;&=\; \gamma^F_{R,R'}\,|R|\,\Big\{\prod_{\alpha\in R}\langle\alpha,x^F\rangle\Big\}\;\times\\ &\times\int_{K_F}\Big\{\prod_{\beta\in R'}\langle\beta,H_F[e^{x_F}k_F]\rangle\Big\}\,e^{\langle\rho_F,\,x_F-H_F[e^{x_F}k_F]\rangle}\,dk_F\,, \end{aligned}$$

$$(8.17)\qquad \begin{aligned} II^F_{R,R'} \;&=\; \gamma^F_{R,R'}\,\Big\{\prod_{\alpha\in R}\langle\alpha,x^F\rangle\Big\}\;\times\\ &\times\int_{K_F}\Big[\sum_{\beta\in R'}\langle\beta,\mathrm{pr}_{\mathfrak{a}_F}(\mathrm{Ad}\,k_F[e^{x_F}k_F]^{-1}.\,x_F)\rangle\;\times\\ &\times\Big\{\prod_{\beta'\in R'\smallsetminus\{\beta\}}\langle\beta',H_F[e^{x_F}k_F]\rangle\Big\}\,e^{\langle\rho_F,\,x_F-H_F[e^{x_F}k_F]\rangle}\Big]\,dk_F \end{aligned}$$

and

$$(8.18)\qquad \begin{aligned} III^F_{R,R'} = \gamma^F_{R,R'}\,\Big\{\prod_{\alpha\in R}\langle\alpha,x^F\rangle\Big\}\int_{K_F}\Big[\Big\{\prod_{\beta\in R'}\langle\beta,H_F[e^{x_F}k_F]\rangle\Big\}\times\\ \times\;\langle\rho_F,x_F-\mathrm{pr}_{\mathfrak{a}_F}(\mathrm{Ad}\,k_F[e^{x_F}k_F]^{-1}.\,x_F)\rangle\,e^{\langle\rho_F,\,x_F-H_F[e^{x_F}k_F]\rangle}\Big]\,dk_F \end{aligned}$$

using (8.10) in the last two cases. Let us analyze all these expressions. First of all, $I^F_{\Sigma_0^+\smallsetminus\Sigma_{F,0}^+,\emptyset}$ is equal to the left hand side of (8.15), multiplied by $|\,\Sigma_0^+\smallsetminus\Sigma_{F,0}^+\,|$. The other expressions (8.16) are smaller:

$$\begin{aligned} |I^F_{R,R'}| \;&\le\; C\int_{K_F}\|x^F\|^{|R|}\,\|x_F\|^{|R'|}\int_{K_F}e^{\langle\rho_F,\,x_F-H_F[e^{x_F}k_F]\rangle}\,dk_F\\ &=\;\mathrm{o}\,\Big\{\prod_{\alpha\in\Sigma_0^+}(1+\langle\alpha,x\rangle)\Big\}. \end{aligned}$$

since $\|x^F\|\asymp\|x\|$ and $\|x_F\|=\mathrm{o}\,(\|x\|)$. Same for the expressions (8.17). The expressions (8.18) are more delicate to handle and will require the induction hypothesis. Consider first

$$\begin{aligned} III^F_{\Sigma_0^+\smallsetminus\Sigma_{F,0}^+,\emptyset} \;&=\; \gamma^F\,\boldsymbol{\pi}^F(x^F)\times\\ \times\int_{K_F}&\langle\rho_F,x_F-\mathrm{pr}_{\mathfrak{a}_F}(\mathrm{Ad}\,k_F[e^{x_F}k_F]^{-1}.\,x_F)\rangle\,e^{\langle\rho_F,\,x_F-H_F[e^{x_F}k_F]\rangle}\,dk_F \end{aligned}$$

and observe that this integral coincides with $\mathcal{E}_F(e^{\rho_F}\varphi_{F,0})(x_F)$, which is the expression under investigation for the symmetric subspace $\mathbb{M}_F = G_F/K_F$. Thus, by induction,

$$\begin{aligned} III^F_{\Sigma_0^+\smallsetminus\Sigma_{F,0}^+,\emptyset} &= \gamma^F\,\boldsymbol{\pi}^F(x)\,\mathcal{E}_F(e^{\rho_F}\varphi_{F,0})(x_F) + \mathrm{o}\Big\{\prod_{\alpha\in\Sigma_0^+}(1+\langle\alpha,x\rangle)\Big\} \\ &\le C\,\boldsymbol{\pi}^F(x)\,e^{\langle\rho_F,x_F\rangle}\,\varphi_{F,0}(x_F) \\ &\asymp \prod_{\alpha\in\Sigma_0^+}(1+\langle\alpha,x\rangle)\,. \end{aligned}$$

For the other expressions (8.18), we use in addition the nonnegativity of

$$\langle\rho_F, x_F - \mathrm{pr}_{\mathfrak{a}_F}(\mathrm{Ad}\,k_F[e^{x_F}k_F]^{-1}.\,x_F)\rangle$$

(see Step 1) to estimate

$$\begin{aligned} &|III^F_{R,R'}| \le C\,\|x^F\|^{|R|}\,\|x_F\|^{|R'|}\,\times \\ &\times\int_{K_F}\langle\rho_F, x_F - \mathrm{pr}_{\mathfrak{a}_F}(\mathrm{Ad}\,k_F[e^{x_F}k_F]^{-1}.\,x_F)\rangle\,e^{\langle\rho_F,\,x_F - H_F[e^{x_F}k_F]\rangle}\,dk_F\,. \end{aligned}$$

Notice that the last integral is equal to $\mathcal{E}_F(e^{\rho_F}\varphi_{F,0})(x_F)$. Thus $|III^F_{R,R'}| = \mathrm{o}\big\{\prod_{\alpha\in\Sigma_0^+}(1+\langle\alpha,x\rangle)\big\}$, again by induction. By putting everything together, we obtain eventually that

$$\begin{aligned} \text{(8.19)}\quad \mathcal{E}(e^{\rho}\varphi_0)(x) &= I^F_{\Sigma_0^+\smallsetminus\Sigma_{F,0}^+,\emptyset} + III^F_{\Sigma_0^+\smallsetminus\Sigma_{F,0}^+,\emptyset} + \mathrm{o}\Big\{\prod_{\alpha\in\Sigma_0^+}(1+\langle\alpha,x\rangle)\Big\} \\ &= |\,\Sigma_0^+\smallsetminus\Sigma_{F,0}^+\,|\,\gamma^F\,\boldsymbol{\pi}^F(x)\,e^{\langle\rho_F,x_F\rangle}\,\varphi_{F,0}(x_F) + \\ &\quad + \gamma^F\,\boldsymbol{\pi}^F(x)\,\mathcal{E}_F(e^{\rho_F}\varphi_{F,0})(x_F) + \mathrm{o}\Big\{\prod_{\alpha\in\Sigma_0^+}(1+\langle\alpha,x\rangle)\Big\} \end{aligned}$$

is bounded above by

$$e^{\langle\rho,x\rangle}\,\varphi_0(x) \asymp \prod_{\alpha\in\Sigma_0^+}(1+\langle\alpha,x\rangle)\,,$$

see Theorem 8.1.ii.

Moreover, by induction, (8.19) behaves asymptotically like

$$|\,\Sigma_0^+\smallsetminus\Sigma_{F',0}^+\,|\,\gamma^F\,\boldsymbol{\pi}^F(x)\,e^{\langle\rho_F,x_F\rangle}\,\varphi_{F,0}(x_F) \sim |\,\Sigma_0^+\smallsetminus\Sigma_{F',0}^+\,|\,e^{\langle\rho,x\rangle}\,\varphi_0(x)\,,$$

if for some $F'\subset F$, $\langle\alpha,x\rangle\to 0$ for all $\alpha\in F'$ and $\langle\alpha,x\rangle\to+\infty$ for all $\alpha\in F\smallsetminus F'$.

Step 3 : Conclusion

We argue by contradiction.

(i) If the expression (8.9)

$$\chi_2(x) \;=\; \mathcal{E}\log(e^{\rho}\varphi_0)(x) \;=\; \tfrac{\mathcal{E}(e^{\rho}\varphi_0)(x)}{e^{\langle\rho,x\rangle}\varphi_0(x)}$$

were unbounded on $\bar{\mathfrak{a}}_+$, there would be a sequence $x_n \in \bar{\mathfrak{a}}_+$ such that $\chi_2(x_n) \to +\infty$. Since χ_2 is analytic on $\mathfrak{a}$, hence locally bounded, the sequence $\|x_n\|$ must tend to $+\infty$. By passing to a subsequence, we may assume that $\frac{x_n}{\|x_n\|}$ tends to a unit vector x_∞ in $\bar{\mathfrak{a}}_+$, which lies in some face $\mathfrak{a}_+^F$, with $F \subsetneq S$. Thus

$$\begin{cases} \langle\alpha,x_n\rangle = \mathrm{o}\,(\|x_n\|) & \forall\,\alpha\in F \\ \langle\alpha,x_n\rangle \asymp \|x_n\| & \forall\,\alpha\in S\smallsetminus F \end{cases}$$

as in (8.12). According to Step 2 (or Proposition 8.2 when $F=\varnothing$), the sequence $\chi_2(x_n)$ is bounded. Hence a contradiction.

(ii) Similarly, if $\chi_2(x)$ would not tend to $|\,\Sigma_0^+\smallsetminus\Sigma_{F,0}^+\,|$ under the assumption (8.8), there would be a sequence $x_n \in \bar{\mathfrak{a}}_+$ such that

$$\begin{cases} \langle\alpha,x_n\rangle \to 0 & \forall\,\alpha\in F\,, \\ \langle\alpha,x_n\rangle \to +\infty & \forall\,\alpha\in S\smallsetminus F\,, \\ \inf_n \big|\,\chi_2(x_n) - |\,\Sigma_0^+\smallsetminus\Sigma_{F,0}^+\,|\,\big| \;>\; 0\,. \end{cases}$$

We may assume again the existence of a limit direction $\frac{x_n}{\|x_n\|} \to x_\infty$ in some face $C^{F'}$, with $F\subset F'\subsetneq S$. Hence again

$$\begin{cases} \langle\alpha,x_n\rangle = \mathrm{o}\,(\|x_n\|) & \forall\,\alpha\in F' \\ \langle\alpha,x_n\rangle \asymp \|x_n\| & \forall\,\alpha\in S\smallsetminus F' \end{cases}$$

and a contradiction with the convergence $\chi_2(x_n)\to|\,\Sigma_0^+\smallsetminus\Sigma_{F,0}^+\,|$ established in Step 2 (or in Proposition 8.2 when $F=\varnothing$).

This concludes the proof of Proposition 8.5. ■

Remark : All these results extend trivially to reductive symmetric spaces in the Harish–Chandra class. Except of course the lower bound on $\mathcal{E}\log\delta^{1/2}\varphi_0$ in the purely Euclidean case, where all expressions φ_0, δ, e^{ρ} are identically equal to 1.

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Recibido: 3 de marzo de 2000

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