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On the ground state of lattice Schrödinger operators

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Abstract. We prove necessary and sufficient conditions for lattice Schrödinger operators to have a zero-energy bound state in arbitrary dimension. The two criteria are sharp, complementary, and depend crucially on both the dimension and asymptotic behaviour of the potential. The method relies on a discrete variant of Agmon's comparison principle which is also proven. Our results represent a discrete variant of the recent criteria obtained in the continuous setting by D. Hundertmark, M. Jex, and M. Lange [Forum Math. Sigma 11 (2023), artile no. e61].

1. Introduction

The bound states serves a crucial role in the stability of quantum systems. The special importance has a ground state as a most stable state of a given system which corresponds to the lowest eigenvalue of the Hamiltonian describing the system. We consider the Schrödinger operator

$$H_V = -\Delta + V$$

acting in $\ell^2(\mathbb{Z}^d)$, where

$$(-\Delta\psi)_n := \sum_{\substack{m\in\mathbb{Z}^d\\|m-n|=1}} (\psi_n - \psi_m)$$

is the discrete Laplacian on the lattice $\mathbb{Z}^d$ and V is a real valued potential such that H_V is a self-adjoint operator on its maximal domain (more details given in Section 2.1 below).

We focus on the situation when the ground state eigenvalue approaches the threshold of the essential spectrum. In particular, we are interested in the situation when the ground state eigenvalue is precisely at the threshold, thus becoming embedded in the essential spectrum. In general, the problem of embedded eigenvalues is of

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great interest. However, most of the results are focused on the eigenvalues embedded in the interior of the essential spectrum. It is known that the potential needs to satisfy specific behaviour to allow the existence of embedded eigenvalues, see e.g., [6,7,26,27,31,33].

The conditions for the existence and absence of threshold states were studied in the continuous setting for a long time due to its importance to the time-decay of solutions of the time-dependent Schrödinger equation [19,34]. The typical approach to this problem is to study the behaviour of the operator resolvent. This is much more complicated than the approach used recently in [16], where the conditions were derived from subharmonic estimates on the eigenfunctions.

The threshold states in the continuous settings were also studied in atomic systems for hydrogen anions with scaled Coulomb repulsion [13, 14], where authors have shown the existence of threshold states for singlet states and absence for triplet states. This was generalized for general repulsive Coulomb interaction in [4], where it is shown that the long range repulsion stabilizes the threshold states. The effect of Coulomb repulsion was revisited for atomic systems also recently in [2,9,15]. The general condition on the absence result for spherically symmetric potentials satisfying

$$V(x) < \frac{3}{4|x|^2} + \frac{1}{|x|^2 \log |x|}$$

in dimension 3 was proved in [3]. The existence condition on the critical potential was derived in the form

$$V(x) \geq \frac{3+\varepsilon}{4|x|^2}$$

in [11] using careful resolvent estimates. The most general condition for arbitrary dimension was done in [16] and its leading order terms can be written as

$$\begin{aligned} V(x) &\leq \frac{d(4-d)}{4|x|^2} + \frac{1}{|x|^2 \log |x|} \quad \text{for the absence,} \\ V(x) &\geq \frac{d(4-d)}{4|x|^2} + \frac{1+\varepsilon}{|x|^2 \log |x|} \quad \text{for the existence,} \end{aligned}$$

where $d \in \mathbb{N}$ denotes the dimension, $\varepsilon > 0$, and x is sufficiently large. It turns out that the properties of the zero-energy eigenstates are dictated by a long range behaviour of the system also in the discrete case.

Before we state our main results, we need to introduce a notation. The iterated logarithm $\log_k |n|$ is defined inductively as $\log_0 x := x$ for $x > 0$ and $\log_{k+1} x := \log(\log_k x)$ for $k \in \mathbb{N}_0$ and x greater than the k-th tetration of e. By $\sigma(H)$, $\sigma_{\mathrm{p}}(H)$, and $\sigma_{\mathrm{ess}}(H)$ we denote the spectrum, the point spectrum, and the essential spectrum of a self-adjoint operator H, respectively. Our first main result is the following *absence* condition.

Theorem 1. *If there exists $s \in \mathbb{N}_0$ such that the potential V of the discrete Schrödinger operator H_V on $\mathbb{Z}^d$ fulfils*

$$V_n \leq \frac{d(4-d)}{4|n|^2} + \frac{1}{|n|^2} \sum_{j=1}^{s} \prod_{k=1}^{j} \frac{1}{\log_k |n|} \tag{1.1}$$

for all $n \in \mathbb{Z}^d$ with $|n|$ sufficiently large, then H_V does not have a zero-energy ground state, i.e., $0 \notin \sigma_{\mathrm{p}}(H_V)$ or $0 \neq \inf \sigma(H_V)$.

To give the complementary existence condition, we require one additional condition on the potential. We say that an operator is *critical at* 0 whenever an arbitrary compact negative perturbation of the operator creates discrete (negative) eigenvalues below the threshold of the essential spectrum; see Definition 18 below for the exact definition. The *existence* condition formulated for critical operators in the next theorem is our second main result.

Theorem 2. *Let $\inf \sigma(H_V) = \inf \sigma_{\mathrm{ess}}(H_V) = 0$ and H_V be critical at 0. If there exist $s \in \mathbb{N}_0$ and $\varepsilon > 0$ such that*

$$V_n \geq \frac{d(4-d)}{4|n|^2} + \frac{1}{|n|^2} \sum_{j=1}^{s} \prod_{k=1}^{j} \frac{1}{\log_k |n|} + \frac{\varepsilon}{|n|^2} \prod_{k=1}^{s} \frac{1}{\log_k |n|} \tag{1.2}$$

for all $n \in \mathbb{Z}^d$ with $|n|$ sufficiently large, then $0 \in \sigma_{\mathrm{p}}(H_V)$.

An inspection of the expressions in (1.1) and (1.2) shows that the conditions are sharp due to the fact that s can be chosen arbitrary large and $\varepsilon > 0$ arbitrary small. It is worth noting that the existence condition in principle works also for higher eigenvalues at the critical coupling. It is important that even though the potential estimates (1.1) and (1.2) are discrete analogies to the ones in the continuous setting [16], their proofs are more involved. In fact, conditions (1.1) and (1.2) follow from more general Theorems 15 and 20 as by no means obvious particular cases. Since the discrete Laplacian is a bounded operator, H_V can have both the lower as well as the upper threshold of the essential spectrum. We prove analogous conditions to those in (1.1) and (1.2) for upper threshold eigenstates in Theorems 16 and 23 below.

From the expressions (1.1) and (1.2), we see that the critical decay of the potential is of order $1/|n|^2$ for $|n|$ large. The inverse square decay is present also in other situations as a borderline case. One is Hardy potentials, where the inverse square decay exhibit asymptotically optimal Hardy weights on $\mathbb{Z}^d$ for $d \geq 3$ or $d = 1$ with the Dirichlet condition at the origin. The case $d = 2$ is special and reminiscent to the case $d = 4$ in (1.1) and (1.2) for an extra logarithmic term is present in the respective Hardy weight; see [20] for details. Optimal discrete Hardy weights, which can be viewed as potentials V which, when subtracted from the Laplacian, the corresponding

Schrödinger operator becomes critical, are not completely analogical to their well-known continuous counterparts. Nevertheless, they still exhibit the inverse square decay for $d = 1$ with the Dirichlet condition at the origin and $d \geq 3$, see [22, Theorems 7.2 and 7.3].

Another example of the borderline case concerns the discrete spectrum of discrete Schrödinger operators which is known to be finite if $V_n \gtrsim -|n|^{-2-\varepsilon}$ for $\varepsilon > 0$; see [5, 28, 29] for more details. Let us remark that these results, which are well known in the continuous case, seem harder to prove in the discrete setting.

In the continuous setting. the spectral phase transition was observed for $d = 4$, see [16]. It manifests in the different behaviour of virtual levels for small dimensions and large dimensions, e.g., absence of zero-energy resonances for short range potentials in dimension $d > 4$, see [17–19]. Our results reveal the same behaviour in the discrete setting. It follows from the dependence of the sign of the leading term in (1.1) and (1.2) on the dimension d. The short range potentials satisfy condition (1.2) for $d > 4$ because the leading order term becomes negative.

Proofs of our main results rely on subharmonic comparison estimates in the spirit of Agmon deduced in Theorem 8 below. Agmon's comparison theorem [1, Theorem 2.7] proved itself to be a very useful tool in the continuous setting. We were not able to find its discrete variant in the literature and we are convinced that it is a result of independent interest in analysis of properties of eigenvectors in the discrete setting, too. Using the standard definition of subsolutions and supersolutions given precisely in Definition 5 below, the discrete Agmon's comparison principle proves existence of a constant $C > 0$ such that

$$u_n \leq C w_n$$

for all n in a set, where w_n is a strictly positive supersolution of the Schrödinger equation and u_n a subsolution of the same Schrödinger equation satisfying an additional mild summability condition, see (2.2) below. The summability condition is fulfilled when u_n decays sufficiently fast as $|n| \to \infty$, see Remark 11 below. In our proofs of the absence and existence conditions, one of the functions u and w is always an eigenvector of a Schrödinger operator while the second is chosen as a suitable comparison function.

1.1. Organization of the paper

Preliminary results are deduces in four subsections of Section 2. Basic definitions of lattice Schrödinger operators are recalled in Section 2.1. General ergodic theory is used to demonstrate simplicity and positivity of the ground state of a lattice Schrödinger operator in Section 2.2. The discrete variant of Agmon's comparison principle is proven in Section 2.3. Two essential inequalities are derived in Section 2.4.

Main results are proven in Section 3. Conditions for the absence of the zero-energy ground state are deduced in Section 3.1, while complementary conditions guaranteeing the existence of the ground state are given in Section 3.2. Similar conditions for the opposite edge point of the essential spectrum are also formulated in Sections 3.1 and 3.2. Finally, an example of a lattice Schrödinger operator demonstrating a transition between the existence and non-existence of the ground state is presented in last Section 3.3.

1.2. Notation

For readers convenience, we summarize the notation used throughout this paper. As it is common, $\mathbb{Z}$ denotes the set of integers; $\mathbb{N}_0$ and $\mathbb{N}$ are the sets of non-negative and positive integers, respectively.

- The Euclidean norm of $n \in \mathbb{Z}^d$ is denoted by
$$|n| := \sqrt{\sum_{j=1}^{d} |n_j|^2}.$$
- The Hilbert space of square-summable functions on $\mathbb{Z}^d$ is denoted by
$$\ell^2(\mathbb{Z}^d) := \{\psi\colon \mathbb{Z}^d \to \mathbb{C} \mid \|\psi\| < \infty\},$$
where $\|\cdot\|$ is the ℓ^2-norm induced by the inner product
$$\langle \phi, \psi \rangle := \sum_{n\in\mathbb{Z}^d} \overline{\phi_n}\psi_n.$$
We use the notation $\psi_n := \psi(n)$ for values of a function $\psi\colon \mathbb{Z}^d \to \mathbb{C}$.
- For $\Omega \subset \mathbb{Z}^d$, $C_c(\Omega)$ denotes the space of functions $\psi\colon \mathbb{Z}^d \to \mathbb{C}$ compactly supported in Ω.
- e_n for $n \in \mathbb{Z}^d$ and δ_j for $j = 1, \dots, d$ denote vectors of standard bases of spaces $\ell^2(\mathbb{Z}^d)$ and $\mathbb{C}^d$, respectively.
- The positive part of a function $\psi\colon \mathbb{Z}^d \to \mathbb{C}$ is denoted by $\psi_+ := \max(0, \psi)$.
- By $\sigma(H)$, $\sigma_{\mathrm{p}}(H)$, and $\sigma_{\mathrm{ess}}(H)$ we denote the spectrum, the point spectrum, and the essential spectrum of a self-adjoint operator H, respectively.
- For $N > 0$, we define
$$\mathbb{Z}_{\geq N} := \{n \in \mathbb{Z}^d \mid |n| \geq N\}.$$
- The empty product is set to 1 and the empty sum equals 0.

2. Preliminaries

2.1. Discrete Schrödinger operators

Recall that the *discrete Laplacian* on the d-dimensional lattice $\mathbb{Z}^d$ is defined by the formula

$$(-\Delta\psi)_n := \sum_{\substack{m\in\mathbb{Z}^d\\|m-n|=1}}(\psi_n-\psi_m) = 2d\psi_n - \sum_{\substack{m\in\mathbb{Z}^d\\|m-n|=1}}\psi_m, \quad n\in\mathbb{Z}^d,$$

for any $\psi\colon\mathbb{Z}^d\to\mathbb{C}$. When regarded as an operator on $\ell^2(\mathbb{Z}^d)$, $-\Delta$ is bounded, self-adjoint, and diagonalized by the d-dimensional discrete Fourier transform

$$F : \ell^2(\mathbb{Z}^d) \to L^2([-\pi,\pi]^d) : \psi \mapsto \frac{1}{(2\pi)^{d/2}}\sum_{n\in\mathbb{Z}^d} e^{-\mathrm{i}n\cdot\xi}\,\psi_n.$$

Concretely, for any $f\in L^2([-\pi,\pi]^d)$, one has the dispersion relation

$$F(-\Delta)F^{-1}f(\xi) = h(\xi)f(\xi),$$

where

$$h(\xi) := 2\sum_{j=1}^{d}(1-\cos\xi_j).$$

Consequently, the spectrum of $-\Delta$ is absolutely continuous and $\sigma(-\Delta)=[0,4d]$.

Given $V\colon\mathbb{Z}^d\to\mathbb{R}$, we define an operator by the multiplication of V, denoted again by the letter V with some abuse of the notation, by the formula

$$(V\psi)_n := V_n\psi_n, \quad n\in\mathbb{Z}^d,$$

on the maximal domain

$$\operatorname{Dom} V := \{\psi\in\ell^2(\mathbb{Z}^d) \mid V\psi\in\ell^2(\mathbb{Z}^d)\}.$$

As it is standard, both the function and the operator V are referred to as the *potential*. Then the *discrete Schrödinger operator* on the lattice $\mathbb{Z}^d$ is the operator sum $H_V := -\Delta+V$. Since $-\Delta$ is bounded $\operatorname{Dom} H_V = \operatorname{Dom} V$ and H_V is self-adjoint.

2.2. Simplicity and positivity of the ground state of H_V

It is well known that under certain assumptions, if a continuous Schrödinger operator has an eigenvalue as the lowest spectral point, this eigenvalue is simple and the corresponding eigenvector can be chosen strictly positive, see [30, Section XIII.12]. We will use the theory of [30, Section XIII.12] to show that the same holds in the discrete setting, too.

First, we recall a terminology. A vector $\psi \in \ell^2(\mathbb{Z}^d)$ is called *positive* if $\psi \neq 0$ and $\psi_n \geq 0$ for all $n \in \mathbb{Z}^d$. A vector $\psi \in \ell^2(\mathbb{Z}^d)$ is called *strictly positive* if $\psi_n > 0$ for all $n \in \mathbb{Z}^d$. A bounded operator A acting on $\ell^2(\mathbb{Z}^d)$ is called *positivity preserving* or *positivity improving* if $A\psi$ is positive whenever ψ is positive or $A\psi$ is strictly positive whenever ψ is positive, respectively. Lastly, a bounded operator A is called *ergodic* if A is positivity preserving and for any $\phi, \psi \in \ell^2(\mathbb{Z}^d)$ both positive there exists $k \in \mathbb{N}$ for which $\langle \phi, A^k \psi \rangle \neq 0$.

A combination of results from [30, Section XIII.12] particularly yields the following useful proposition.

Proposition 3. *Let H and H_0 be self-adjoint operators bounded from below on a Hilbert space and $E := \inf \sigma(H)$ be an eigenvalue of H. Suppose that there exists a sequence of operators of multiplication by bounded functions W_N so that $H_0 + W_N \to H$ and $H - W_N \to H_0$ in the strong resolvent sense as $N \to \infty$. Suppose, in addition, that $H_0 + W_N$ and $H - W_N$ are uniformly bounded from below. If the semi-group $\exp(-tH_0)$ is positivity improving for all $t > 0$, then E is a simple eigenvalue of H and the corresponding eigenvector can be chosen strictly positive.*

Proof. If $\exp(-tH_0)$ is positivity improving, it is ergodic by definition. The proof of the implication (b) $\implies$ (c) in [30, Theorem XIII.43] shows that the set of bounded multiplication operators and $\exp(-tH_0)$ act irreducibly (here, we do not need the assumption that $\|\exp(-tH_0)\|$ is an eigenvalue). Then, [30, Theorem XIII.45] tells us that also the set of bounded diagonal operators and $\exp(-tH)$ act irreducibly and $\exp(-tH)$ is positivity preserving for all $t > 0$. It means, by [30, Theorem XIII.43], that $\exp(-tH)$ is ergodic for all $t > 0$, and [30, Theorem XIII.44] implies the statement. ■

Theorem 4. *Let H_V be the discrete Schrödinger operator and $E := \inf \sigma(H_V) > -\infty$ be an eigenvalue of H_V. Then, E is simple and the corresponding eigenvector can be chosen strictly positive.*

Proof. For $N \in \mathbb{N}_0$, we denote by P_N the orthogonal projection onto $\operatorname{span}\{e_n \mid |n| \leq N\}$, where $\{e_n \mid n \in \mathbb{Z}^d\}$ is the standard basis of $\ell^2(\mathbb{Z}^d)$, i.e., $(e_n)_m = 1$, if $m = n$, and $(e_n)_m = 0$, if $m \neq n$. Define $V_N := VP_N = P_N V$. Clearly, V_N is bounded for all $N \in \mathbb{N}_0$. Notice that H_V is bounded from below if and only if the potential V is a bounded function from below. Hence, both V_N and $V - V_N$ are uniformly bounded from below by the lower bound of V. Next, it is easy to see that $V_N \to V$ in the strong resolvent sense.

We intend to apply Proposition 3 with $H := H_V$, $H_0 := -\Delta - 2d - 1$, and $W_N := V_N + 2d + 1$. If we verify that $\exp(-tH_0)$ is positivity improving for all $t > 0$, then all assumptions of Proposition 3 are fulfilled and the claim follows. To this end, we use the fact that if a bounded operator A on $\ell^2(\mathbb{Z}^d)$ satisfies

$$\langle e_n, Ae_m \rangle > 0,$$

for all $m, n \in \mathbb{Z}^d$, then A is positivity improving. We will show that this is the case for $\exp(-tH_0)$.

Using the definition of $-\Delta$, we can express the action of $\Delta + 2d + 1$ to a function ψ as

$$((\Delta + 2d + 1)\psi)_n = \psi_n + \sum_{\substack{s \in \mathbb{Z}^d \\ |s-n|=1}} \psi_s = \sum_{\substack{s \in \mathbb{Z}^d \\ |s-n| \leq 1}} \psi_s.$$

Then one readily checks by induction in $k \in \mathbb{N}$ that, for all $\psi \geq 0$ and $n \in \mathbb{Z}^d$, one has

$$[(\Delta + 2d + 1)^k \psi]_n \geq \sum_{\substack{s \in \mathbb{Z}^d \\ |s-n| \leq k}} \psi_s.$$

By the boundedness of $\Delta + 2d + 1$, we have

$$\exp(-tH_0) = \sum_{k=0}^{\infty} \frac{t^k}{k!} (\Delta + 2d + 1)^k, \quad t > 0,$$

where the series converges in the operator norm. Hence, for all $t > 0$, $\psi \geq 0$, and $n \in \mathbb{Z}^d$, we get the inequality

$$[\exp(-tH_0)\psi]_n \geq \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{\substack{s \in \mathbb{Z}^d \\ |s-n| \leq k}} \psi_s.$$

Consequently, for any $m, n \in \mathbb{Z}^d$ and $t > 0$, we find that

$$\langle e_n, \exp(-tH_0) e_m \rangle \geq \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{\substack{s \in \mathbb{Z}^d \\ |s-n| \leq k}} \delta_{m,s} \geq \frac{t^\ell}{\ell!} > 0,$$

where $\ell \in \mathbb{N}_0$ is large enough so that $|m - n| \leq \ell$. The proof is complete. ■

2.3. Discrete Agmon's comparison principle

We deduce a discrete variant of Agmon's comparison principle given in [1, Theorem 2.7]. Although here it is derived as an essential tool for proofs of our main results, it is definitely a claim of independent interest, too.

First, we define discrete analogues to terms *subsolution* and *supersolution* of the equation $(H_V - \lambda)\psi = 0$ in a subset of $\mathbb{Z}^d$, where $\lambda \in \mathbb{R}$.

Definition 5. Let $\Omega \subset \mathbb{Z}^d$ and $\lambda \in \mathbb{R}$. Functions $u, w : \mathbb{Z}^d \to \mathbb{R}$ are called *subsolution* and *supersolution* of the equation $(H_V - \lambda)\psi = 0$ in Ω, if

$$[(H_V - \lambda)u]_n \leq 0 \quad \text{and} \quad [(H_V - \lambda)w]_n \geq 0,$$

for all $n \in \Omega$, respectively.

Remark 6. To compare the above definition with its continuous traditional form, one needs to realize that u is a subsolution of $(H_V - \lambda)\psi = 0$ in Ω if and only if

$$\langle \phi, (H_V - \lambda)u \rangle \leq 0,$$

for all *non-negative* $\phi \in C_c(\Omega)$, where $C_c(\Omega)$ is the space of compactly supported functions in Ω. A similar claim with the opposite inequality holds for a supersolution w.

Further, we will need two auxiliary observations. First, notice that

$$-\Delta = \sum_{j=1}^{d} D_j^* D_j \tag{2.1}$$

where D_j is the first order partial difference operator defined by

$$(D_j \psi)_n := \psi_n - \psi_{n-\delta_j}$$

and D_j^* its adjoint acting as

$$(D_j^* \psi)_n := \psi_n - \psi_{n+\delta_j}$$

for all $n \in \mathbb{Z}^d$ and $\psi : \mathbb{Z}^d \to \mathbb{C}$, where δ_j denotes the j-th vector of the standard basis of $\mathbb{C}^d$. The second observation is formulated in the following lemma. The result is known and holds even in a more abstract setting, see [21, Lemma 1.9]. We provide its short proof for the reader's convenience.

Lemma 7. *If u is a subsolution of $(H_V - \lambda)\psi = 0$ in $\Omega \subset \mathbb{Z}^d$, then $u_+ := \max(0, u)$ is also a subsolution of $(H_V - \lambda)\psi = 0$ in Ω.*

Proof. Let $n \in \Omega$ is fixed. If $u_n \leq 0$, then $(u_+)_n = 0$ and therefore the inequality $[(H_V - \lambda)u_+]_n \leq 0$ is equivalent to the inequality

$$-\sum_{\substack{m \in \mathbb{Z}^d \\ |m-n|=1}} (u_+)_m \leq 0,$$

which is true since $u_+ \geq 0$.

If $u_n > 0$, then $u_n = (u_+)_n$. Taking also into account that $u \le u_+$ and the assumption, we estimate $[(H_V - \lambda)u_+]_n$ by

$$(2d + V_n - \lambda)(u_+)_n - \sum_{\substack{m\in\mathbb{Z}^d\\|m-n|=1}} (u_+)_m \le (2d + V_n - \lambda)u_n - \sum_{\substack{m\in\mathbb{Z}^d\\|m-n|=1}} u_m = [(H_V - \lambda)u]_n \le 0,$$

which concludes the proof. ∎

Next, we prove a discrete variant of Agmon's comparison theorem. Recall the notation $\mathbb{Z}^d_{\ge N} := \{n \in \mathbb{Z}^d \mid |n| \ge N\}$ for $N > 0$.

Theorem 8. *Let $N \in \mathbb{N}$, $\lambda \in \mathbb{R}$, and let w be a strictly positive supersolution of $(H_V - \lambda)\psi = 0$ in $\mathbb{Z}^d_{\ge N}$. Suppose further that u is a subsolution of $(H_V - \lambda)\psi = 0$ in $\mathbb{Z}^d_{\ge N}$ satisfying*

$$\liminf_{M\to\infty} \frac{1}{M^2} \sum_{M\le|n|\le\alpha M} \sum_{j=1}^{d} |u_n u_{n-\delta_j}| = 0 \tag{2.2}$$

for some $\alpha > 1$. Then, for all $n \in \mathbb{Z}^d_{\ge N}$, one has

$$u_n \le C w_n, \tag{2.3}$$

where C is any positive constant such that

$$C \ge \max\left\{ \frac{u_n}{w_n} \;\middle|\; N - 1 \le |n| < N + 1 \right\}. \tag{2.4}$$

Remark 9. Notice that a constant $C > 0$ satisfying (2.4) always exists because the set of indices $n \in \mathbb{Z}^d$ with $N - 1 \le |n| < N + 1$ is finite and $w_n > 0$ for all $n \in \mathbb{Z}^d$.

Proof of Theorem 8. Pick any $C > 0$ satisfying (2.4). Notice that then inequality (2.3) holds for all $n \in \mathbb{Z}^d$ with $N - 1 \le |n| < N + 1$. Define the auxiliary function

$$t := (u - Cw)_+. \tag{2.5}$$

Then $t_n = 0$ for $n \in \mathbb{Z}^d$ with $N - 1 \le |n| < N + 1$. We show that then t must vanish identically in $\mathbb{Z}^d_{\ge N}$, which completes the proof. The proof proceeds in several steps. Throughout the proof, summation indices are elements of $\mathbb{Z}^d$ restricted further by displayed inequalities.

Step 0. *A summation by parts identity.* Let $\phi, \psi \in C_c(\mathbb{Z}^d)$. We show that, if $\phi_n = 0$ for all $n \in \mathbb{Z}^d$ such that $N - 1 \le |n| < N + 1$, then the identity

$$\sum_{|n|\ge N} \phi_n (D_j^* \psi)_n = \sum_{|n|\ge N} (D_j\phi)_n \psi_n \tag{2.6}$$

holds for all $j \in \{1, \dots, d\}$. Indeed, to verify (2.6), we expand the left-hand side getting

$$\sum_{|n|\geq N} \phi_n (D_j^* \psi)_n = \sum_{|n|\geq N} \phi_n \psi_n - \sum_{|n|\geq N} \phi_n \psi_{n+\delta_j} = \sum_{|n|\geq N} \phi_n \psi_n - \sum_{|n-\delta_j|\geq N} \phi_{n-\delta_j} \psi_n. \tag{2.7}$$

Notice that if a multi-index $n \in \mathbb{Z}^d$ satisfies $|n-\delta_j| \geq N$ and $|n| < N$, then $N \leq |n-\delta_j| < N+1$ and $\phi_{n-\delta_j} = 0$ by the assumption. Similarly, if $|n-\delta_j| < N$ and $|n| \geq N$, then $N-1 \leq |n-\delta_j| < N$ and so again $\phi_{n-\delta_j} = 0$ by the assumption. It means that the last sum in (2.7) remains unchanged when the restriction $|n-\delta_j| \geq N$ is replaced by $|n| \geq N$. Thus, we arrive at the equality

$$\sum_{|n|\geq N} \phi_n (D_j^* \psi)_n = \sum_{|n|\geq N} \phi_n \psi_n - \sum_{|n|\geq N} \phi_{n-\delta_j} \psi_n$$

which amounts to (2.6).

Step 1. *An inequality from the subsolution.* We establish the identity

$$\sum_{|n|\geq N} \xi_n^2 t_n (-\Delta t)_n = \sum_{|n|\geq N} \sum_{j=1}^{d} [D_j(\xi t)]_n^2 - \sum_{|n|\geq N} \sum_{j=1}^{d} t_n t_{n-\delta_j} (D_j \xi)_n^2, \tag{2.8}$$

which holds for any $t\colon \mathbb{Z}^d \to \mathbb{R}$ and $\xi \in C_c(\mathbb{Z}^d)$ such that $\xi_n t_n = 0$ whenever $N-1 \leq |n| < N+1$. By applying (2.1) and formula (2.6) with $\phi = \xi^2 t$ and $\psi = D_j t$, we find

$$\sum_{|n|\geq N} \xi_n^2 t_n (-\Delta t)_n = \sum_{j=1}^{d} \sum_{|n|\geq N} \xi_n^2 t_n (D_j^* D_j t)_n = \sum_{j=1}^{d} \sum_{|n|\geq N} [D_j(\xi^2 t)]_n (D_j t)_n.$$

Next, by expanding the right-hand side and using the elementary identity

$$(\xi_n^2 t_n - \xi_{n-\delta_j}^2 t_{n-\delta_j})(t_n - t_{n-\delta_j}) = (\xi_n t_n - \xi_{n-\delta_j} t_{n-\delta_j})^2 - t_n t_{n-\delta_j} (\xi_n - \xi_{n-\delta_j})^2,$$

we arrive at (2.8).

Now, suppose additionally that t is a positive subsolution of $(H_V - \lambda)\psi = 0$ in $\mathbb{Z}^d_{\geq N}$. It follows that

$$0 \geq \sum_{|n|\geq N} \xi_n^2 t_n [(H_V - \lambda)t]_n = \sum_{|n|\geq N} \xi_n^2 t_n (-\Delta t)_n + \sum_{|n|\geq N} (V_n - \lambda) \xi_n^2 t_n^2.$$

An application of identity (2.8) yields the inequality

$$\sum_{|n|\geq N} \sum_{j=1}^{d} [D_j(\xi t)]_n^2 + \sum_{|n|\geq N} (V_n - \lambda) \xi_n^2 t_n^2 \leq \sum_{|n|\geq N} \sum_{j=1}^{d} t_n t_{n-\delta_j} (D_j \xi)_n^2 \tag{2.9}$$

for any $\xi \in C_c(\mathbb{Z}^d)$ and a positive subsolution t of $(H_V - \lambda)\psi = 0$ in $\mathbb{Z}^d_{\geq N}$ satisfying $t_n = 0$ if $N-1 \leq |n| < N+1$ (to guarantee that $t_n \xi_n = 0$).

Step 2. *An inequality from the supersolution.* Let $w\colon\mathbb{Z}^d \to \mathbb{R}$ and $\phi \in C_c(\mathbb{Z}^d)$ is such that $\phi_n = 0$ if $N-1 \le |n| < N+1$. Then, identity (2.8) with t replaced by w and ξ replaced by ϕ reads

$$\sum_{|n|\ge N} w_n\phi_n^2(-\Delta w)_n = \sum_{|n|\ge N}\sum_{j=1}^{d}[D_j(w\phi)]_n^2 - \sum_{|n|\ge N}\sum_{j=1}^{d} w_n w_{n-\delta_j}(D_j\phi)_n^2. \tag{2.10}$$

Suppose additionally that w is a positive supersolution of $(H_V-\lambda)\psi = 0$ in $\mathbb{Z}^d_{\ge N}$. Then,

$$0 \le \sum_{|n|\ge N} w_n\phi_n^2[(H_V-\lambda)w]_n = \sum_{|n|\ge N} w_n\phi_n^2(-\Delta w)_n + \sum_{|n|\ge N}(V_n-\lambda)w_n^2\phi_n^2.$$

An application of identity (2.10) yields the inequality

$$\sum_{|n|\ge N}\sum_{j=1}^{d}[D_j(w\phi)]_n^2 + \sum_{|n|\ge N}(V_n-\lambda)w_n^2\phi_n^2 \ge \sum_{|n|\ge N}\sum_{j=1}^{d} w_n w_{n-\delta_j}(D_j\phi)_n^2 \tag{2.11}$$

for any positive supersolution w of $(H_V-\lambda)\psi = 0$ in $\mathbb{Z}^d_{\ge N}$ and $\phi \in C_c(\mathbb{Z}^d)$ satisfying $\phi_n = 0$ if $N-1 \le |n| < N+1$.

Step 3. *A combined inequality.* Assuming that w is a *strictly* positive supersolution of $(H_V-\lambda)\psi = 0$ in $\mathbb{Z}^d_{\ge N}$, we may plug $\phi = \xi t/w$ into (2.11). Then, its left-hand side coincides with the left-hand side of (2.9) and we deduce the inequality

$$\sum_{|n|\ge N}\sum_{j=1}^{d} w_n w_{n-\delta_j}\left[D_j\left(\frac{\xi t}{w}\right)\right]_n^2 \le \sum_{|n|\ge N}\sum_{j=1}^{d} t_n t_{n-\delta_j}(D_j\xi)_n^2 \tag{2.12}$$

for any $\xi \in C_c(\mathbb{Z}^d)$, t is a positive subsolution of $(H_V-\lambda)\psi = 0$ in $\mathbb{Z}^d_{\ge N}$ satisfying $t_n = 0$ if $N-1 \le |n| < N+1$, and w a strictly positive supersolution $(H_V-\lambda)\psi = 0$ in $\mathbb{Z}^d_{\ge N}$.

Step 4. *Proof of the statement.* Let the assumptions of the statement be fulfilled and define t by (2.5). Then, by Lemma 7, t is a subsolution of $(H_V-\lambda)\psi = 0$ in $\mathbb{Z}^d_{\ge N}$ and $t_n = 0$ for all $n \in \mathbb{Z}^d$ such that $N-1 \le |n| < N+1$.

For $M > N$ sufficiently large and $\alpha > 1$, we substitute $\xi_n = g^{(M)}(|n|)$ into (2.12) with the piece-wise linear function

$$g^{(M)}(r) := \begin{cases} 1 & \text{if } r \le M+1, \\ \dfrac{\alpha M - 1 - r}{(\alpha-1)M-2} & \text{if } M+1 < r < \alpha M - 1, \\ 0 & \text{if } \alpha M - 1 \le r. \end{cases}$$

With this choice, one finds that

$$(D_j g^{(M)})_n^2 \leq \frac{1}{[(\alpha-1)M-2]^2}\, \chi_{\{M\leq |n|\leq \alpha M\}}(n) \leq \frac{A}{M^2}\, \chi_{\{M\leq |n|\leq \alpha M\}}(n)$$

for a sufficiently large constant $A>0$, where χ stands for the indicator function. Taking also into account that $t \leq u_+ \leq |u|$, we infer from (2.12) that

$$\sum_{|n|\geq N} \sum_{j=1}^{d} w_n w_{n-\delta_j} \left[D_j \left(\frac{g^{(M)} t}{w} \right) \right]_n^2 \leq \frac{A}{M^2} \sum_{M\leq |n|\leq \alpha M} \sum_{j=1}^{d} |u_n u_{n-\delta_j}|$$

for all M sufficiently large. Applying Fatou's lemma and assumption (2.2), we get

$$\sum_{|n|\geq N} \sum_{j=1}^{d} w_n w_{n-\delta_j} \left[D_j \left(\frac{t}{w} \right) \right]_n^2 \leq \liminf_{M\to\infty} \frac{A}{M^2} \sum_{M\leq |n|\leq \alpha M} \sum_{j=1}^{d} |u_n u_{n-\delta_j}| = 0.$$

Since w is strictly positive, all terms of the sum on the left-hand side have to vanish. It implies that

$$\left[D_j \left(\frac{t}{w} \right) \right]_n = \frac{t_n}{w_n} - \frac{t_{n-\delta_j}}{w_{n-\delta_j}} = 0$$

for all $n \in \mathbb{Z}^d_{\geq N}$ and $j \in \{1,\dots,d\}$. Recalling the assumption $t_n = 0$ in $N-1 \leq |n| < N+1$, we conclude that $t_n = 0$ in all of $\mathbb{Z}^d_{\geq N}$ and the proof is completed. ■

Remark 10. Following exactly the definitions, the strictly positive supersolution w of $(H_V - \lambda)\psi = 0$ in $\mathbb{Z}^d_{\geq N}$ is a function $w\colon \mathbb{Z}^d \to (0,\infty)$ such that the inequality $[(H_V-\lambda))w]_n \geq 0$ holds for every index $n \in \mathbb{Z}^d_{\geq N}$. However, the proof of Theorem 8 takes only the values of w_n with $|n| \geq N-1$ into account. For $|n| < N-1$, values of w_n can be taken arbitrary positive numbers. Therefore, in proofs of Theorems 15 and 20 below, where Theorem 8 is applied with N large, we need not care when w (and similarly u) are chosen as functions not defined on a ball of finite radius.

Remark 11. If the subsolution u in Theorem 8 satisfies $u_n = O(|n|^{-\gamma})$ for $|n|\to\infty$, with $\gamma > (d-2)/2$, then (2.2) holds. Indeed, since the number of lattice points in $\mathbb{R}^d$ inside a ball of radius R equals $O(R^d)$, as $R\to\infty$, we deduce that

$$\sum_{M\leq |n|\leq \alpha M} 1 = O(M^d), \qquad \text{as } M\to\infty,$$

for any $\alpha > 1$. Hence, with a sufficiently large constant $C>0$, we have

$$\frac{1}{M^2} \sum_{M\leq |n|\leq \alpha M} \sum_{j=1}^{d} |u_n u_{n-\delta_j}| \leq \frac{C}{M^{2+2\gamma}} \sum_{M\leq |n|\leq \alpha M} 1 = O(M^{d-2\gamma-2}), \qquad \text{as } M\to\infty.$$

2.4. Expansions and inequalities

We prove two, in a sense complementary, inequalities which will be essential in the forthcoming derivation of the absence and existence conditions for potentials. To this end, we recall that by $\log_k$ we denote the composition of k natural logarithms with the convention that $\log_0$ stands for the identity, i.e.,

$$\log_0 x = x \quad \text{and} \quad \log_k x = \log(\log_{k-1} x) \ \text{ for } k \in \mathbb{N}.$$

Clearly, $\log_k x > 0$ if $x > \mathrm{e}_{k-1}$, where

$$\mathrm{e}_{-1} = 0 \quad \text{and} \quad \mathrm{e}_k = \exp \mathrm{e}_{k-1} \ \text{ for } k \in \mathbb{N}_0.$$

Further, for $s \in \mathbb{N}_0$ and $\varepsilon \geq 0$, we define positive functions

$$b_n^s(\varepsilon) := |n|^{-d/2}\Big(\prod_{i=1}^{s} \log_i^{-1/2}|n|\Big)\log_s^{-\varepsilon/4}|n|, \tag{2.13}$$

where $n \in \mathbb{Z}^d$ with $|n| > \mathrm{e}_{s-1}$. For $\varepsilon = 0$, we briefly write

$$b_n^s := b_n^s(0) = |n|^{-d/2}\prod_{i=1}^{s} \log_i^{-1/2}|n|. \tag{2.14}$$

The two key inequalities will be direct consequences of the following asymptotic expansions.

Proposition 12. *Let $s \in \mathbb{N}$ and $\varepsilon > 0$. For $|n| \to \infty$, we have*

$$\begin{aligned}\frac{(\Delta b^s)_n}{b_n^s} = {}& \frac{d(4-d)}{4|n|^2} + \frac{1}{|n|^2}\sum_{j=1}^{s}\prod_{k=1}^{j}\frac{1}{\log_k|n|} + \frac{3}{4|n|^2\log^2|n|} \\ &+ O\Big(\frac{1}{|n|^2\log^2|n|\log_2|n|}\Big)\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}\frac{(\Delta b^s(\varepsilon))_n}{b_n^s(\varepsilon)} = {}& \frac{d(4-d)}{4|n|^2} + \frac{1}{|n|^2}\sum_{j=1}^{s}\prod_{k=1}^{j}\frac{1}{\log_k|n|} + \frac{\varepsilon}{2|n|^2}\prod_{k=1}^{s}\frac{1}{\log_k|n|} \\ &+ O_\varepsilon\Big(\frac{1}{|n|^2\log^2|n|}\Big),\end{aligned} \tag{2.16}$$

where the index ε in the second Landau symbol indicates the dependence of the reminder on ε.

Proof. Notation. It turns out to be advantageous to start with expansions in terms of the quantity

$$N_j := \frac{2n_j + 1}{|n|^2},$$

where $j \in \{1, \dots, d\}$. Notice that $N_j = O(1/|n|)$ as $|n| \to \infty$. Further, for $k \in \mathbb{N}_0$, we define

$$x_k := \frac{\log_k |n + \delta_j|}{\log_k |n|},$$

where the dependence on $n \in \mathbb{Z}^d$ and $j \in \{1, \dots, d\}$ is suppressed in the notation. Whenever needed, the norm of $n \in \mathbb{Z}^d$ is assumed to be sufficiently large so the iterated logarithms are well defined and no division by zero occurs. Finally, we will abbreviate

$$\ell_k := \log_k |n|.$$

Step 1. *Asymptotic expansion of* x_k. Notice that, for $k \in \mathbb{N}$, we have the recurrence

$$x_k = 1 + \frac{\log x_{k-1}}{\ell_k}. \tag{2.17}$$

As the first step, we deduce auxiliary expansions of x_k for $|n|$ large. Specifically, we prove that, for $k \in \mathbb{N}$ and $j \in \{1, \dots, d\}$, we have

$$x_k = 1 + \frac{N_j}{2\ell_1 \dots \ell_k} - \frac{N_j^2}{4\ell_1 \dots \ell_k}\Big(1 + \frac{1}{2}\sum_{r=1}^{k-1} \frac{1}{\ell_1 \dots \ell_r}\Big) + O(N_j^3), \tag{2.18}$$

as $|n| \to \infty$.

The proof of (2.18) proceeds by induction in k. If $k = 1$, then

$$x_1 = 1 + \frac{\log(1 + N_j)}{2\ell_1}.$$

Using the elementary expansion

$$\log(1 + X) = X - \frac{X^2}{2} + O(X^3), \quad X \to 0, \tag{2.19}$$

we find that

$$x_1 = 1 + \frac{1}{2\ell_1}\Big(N_j - \frac{N_j^2}{2} + O(N_j^3)\Big) = 1 + \frac{N_j}{2\ell_1} - \frac{N_j^2}{4\ell_1} + (N_j^3), \quad |n| \to \infty,$$

as claimed.

Suppose $k \geq 2$ is fixed and formula (2.18) holds for x_{k-1}. Then, recurrence (2.17) and the induction hypothesis imply

$$x_k = 1 + \frac{1}{\ell_k}\log\Big(1 + \frac{N_j}{2\ell_1 \dots \ell_{k-1}} - \frac{N_j^2}{4\ell_1 \dots \ell_{k-1}}\Big(1 + \frac{1}{2}\sum_{r=1}^{k-2} \frac{1}{\ell_1 \dots \ell_r}\Big) + O(N_j^3)\Big),$$

as $|n| \to \infty$. Using again (2.19), we find

$$x_k = 1 + \frac{1}{\ell_k}\Big[\frac{N_j}{2\ell_1 \dots \ell_{k-1}} - \frac{N_j^2}{4\ell_1 \dots \ell_{k-1}}\Big(1 + \frac{1}{2}\sum_{r=1}^{k-2}\frac{1}{\ell_1 \dots \ell_r}\Big) - \frac{N_j^2}{8\ell_1^2 \dots \ell_{k-1}^2}\Big] + O(N_j^3)$$
$$= 1 + \frac{N_j}{2\ell_1 \dots \ell_k} - \frac{N_j^2}{4\ell_1 \dots \ell_k}\Big(1 + \frac{1}{2}\sum_{r=1}^{k-1}\frac{1}{\ell_1 \dots \ell_r}\Big) + O(N_j^3),$$

for $|n| \to \infty$, which concludes the proof of (2.18).

Step 2. *Proof of expansion* (2.15). Notice that the expression from the left side of (2.15) can be expressed in terms of x_k as follows:

$$\frac{(\Delta b^s)_n}{b_n^s} = -2d + \sum_{j=1}^{d}\frac{b_{n+\delta_j}^s}{b_n^s} + \frac{b_{n-\delta_j}^s}{b_n^s} = -2d + \sum_{j=1}^{d}\Big(x_0^{-d/2}\prod_{k=1}^{s}x_k^{-1/2} + \text{s.c.}\Big). \tag{2.20}$$

Above and hereafter, we use the abbreviation s.c. for the same term as the one displayed, where each occurrence of n_j is replaced by $-n_j$, for brevity.

With the aid of (2.18) and the expansion

$$(1+X)^{-1/2} = 1 - \frac{X}{2} + \frac{3X^2}{8} + O(X^3), \quad X \to 0,$$

we deduce, for $k \in \mathbb{N}$, that

$$x_k^{-1/2} = 1 - \alpha_k N_j + \beta_k N_j^2 + O(N_j^3), \quad |n| \to \infty,$$

where

$$\alpha_k = \alpha_k(\ell_1, \dots, \ell_k) := \frac{1}{4\ell_1 \dots \ell_k} \tag{2.21}$$

and

$$\beta_k = \beta_k(\ell_1, \dots, \ell_k) := \frac{1}{8\ell_1 \dots \ell_k}\Big(1 + \frac{1}{2}\sum_{r=1}^{k-1}\frac{1}{\ell_1 \dots \ell_r} + \frac{3}{4\ell_1 \dots \ell_k}\Big). \tag{2.22}$$

Taking the product over $k = 1, \dots, s$ yields the formula

$$\prod_{k=1}^{s}x_k^{-1/2} = 1 - \Big(\sum_{k=1}^{s}\alpha_k\Big)N_j + \Big(\sum_{k=1}^{s}\beta_k + \sum_{k=1}^{s}\sum_{l=k+1}^{s}\alpha_k\alpha_l\Big)N_j^2 + O(N_j^3) \tag{2.23}$$

for $|n| \to \infty$.

Similarly, since

$$(1+X)^{-d/4} = 1 - \frac{dX}{4} + \frac{d(d+4)X^2}{32} + O(X^3), \quad X \to 0, \tag{2.24}$$

we readily get

$$x_0^{-d/2} = (1+N_j)^{-d/4} = 1 - \frac{dN_j}{4} + \frac{d(d+4)N_j^2}{32} + O(N_j^3), \quad |n| \to \infty.$$

Multiplying the above formula with (2.23) yields

$$\begin{aligned} x_0^{-d/2} \prod_{k=1}^{s} x_k^{-1/2} = 1 &- \Big(\frac{d}{4} + \sum_{k=1}^{s} \alpha_k\Big) N_j \\ &+ \Big(\frac{d(d+4)}{32} + \frac{d}{4}\sum_{k=1}^{s} \alpha_k + \sum_{k=1}^{s} \beta_k + \sum_{k=1}^{s} \sum_{l=k+1}^{s} \alpha_k \alpha_l\Big) N_j^2 \\ &+ O(N_j^3) \end{aligned} \tag{2.25}$$

for $|n| \to \infty$. Taking also into account that

$$N_j + \text{s.c.} = \frac{2}{|n|^2} \quad \text{and} \quad N_j^2 + \text{s.c.} = \frac{8n_j^2 + 2}{|n|^4}, \tag{2.26}$$

we deduce, for $|n| \to \infty$, the expansion

$$\begin{aligned} &\sum_{j=1}^{d} \Big(x_0^{-d/2} \prod_{k=1}^{s} x_k^{-1/2} + \text{s.c.} \Big) \\ &= 2d - \Big(\frac{d}{4} + \sum_{k=1}^{s} \alpha_k\Big) \frac{2d}{|n|^2} \\ &\quad + \Big(\frac{d(d+4)}{32} + \frac{d}{4}\sum_{k=1}^{s} \alpha_k + \sum_{k=1}^{s} \beta_k + \sum_{k=1}^{s} \sum_{l=k+1}^{s} \alpha_k \alpha_l\Big) \frac{8}{|n|^2} + O\Big(\frac{1}{|n|^3}\Big), \end{aligned}$$

where we have also used that $N_j = O(1/|n|)$ as $|n| \to \infty$.

Recalling (2.20), a slight simplification of the last formula provides us with the expansion

$$\frac{(\Delta b^s)_n}{b_n^s} = \Big(\frac{d(4-d)}{4} + 8\sum_{k=1}^{s} \beta_k + 8\sum_{k=1}^{s} \sum_{l=k+1}^{s} \alpha_k \alpha_l\Big) \frac{1}{|n|^2} + O\Big(\frac{1}{|n|^3}\Big)$$

for $|n| \to \infty$. Finally, substituting for α_k and β_k from (2.21) and (2.22), one finds that the coefficient in front of $1/|n|^2$ equals

$$\frac{d(4-d)}{4} + \sum_{k=1}^{s} \frac{1}{\ell_1 \dots \ell_k} + \frac{3}{4\ell_1^2} + O\Big(\frac{1}{\ell_1^2 \ell_2}\Big)$$

for $|n| \to \infty$. The proof of formula (2.15) follows.

Step 3. *Proof of expansion* (2.16). An analogous expression to (2.20) for the left-hand side of (2.16) reads

$$\frac{(\Delta b^s(\varepsilon))_n}{b_n^s(\varepsilon)} = -2d + \sum_{j=1}^{d}\Big[x_0^{-d/2}\Big(\prod_{k=1}^{s} x_k^{-1/2}\Big)x_s^{-\varepsilon/4} + \text{c.s.}\Big].$$

When compared to (2.20), one sees that the proof proceeds similarly to previous step taking into account only the presence of the additional term $x_s^{-\varepsilon/4}$.

Using (2.24) with d replaced by ε together with (2.18), we get

$$x_s^{-\varepsilon/4} = 1 - \frac{\varepsilon N_j}{8\ell_1\dots\ell_s} + \frac{\varepsilon N_j^2}{16\ell_1\dots\ell_s}\Big(1 + \frac{1}{2}\sum_{r=1}^{s-1}\frac{1}{\ell_1\dots\ell_r} + \frac{\varepsilon+4}{8\ell_1\dots\ell_s}\Big) + O_\varepsilon(N_j^3),$$

which, when rewritten in terms of (2.21) and (2.22), reads

$$x_s^{-\varepsilon/4} = 1 - \frac{\varepsilon\alpha_s}{2}N_j + \frac{4\varepsilon\beta_s + \varepsilon(\varepsilon-2)\alpha_s^2}{8}N_j^2 + O_\varepsilon(N_j^3)$$

for $|n| \to \infty$. Multiplying the above expansion with (2.25) yields

$$x_0^{-d/2}\Big(\prod_{k=1}^{s} x_k^{-1/2}\Big)x_s^{-\varepsilon/4} = 1 - A_s N_j + B_s N_j^2 + O_\varepsilon(N_j^3)$$

for $|n| \to \infty$, where

$$A_s := \frac{d}{4} + \sum_{k=1}^{s}\alpha_k + \frac{\varepsilon\alpha_s}{2}$$

and

$$B_s := \frac{d(d+4)}{32} + \frac{d}{4}\sum_{k=1}^{s}\alpha_k + \sum_{k=1}^{s}\beta_k + \sum_{k=1}^{s}\sum_{l=k+1}^{s}\alpha_k\alpha_l + \frac{\varepsilon\alpha_s}{2}\Big(\frac{d}{4} + \sum_{k=1}^{s}\alpha_k + \frac{(\varepsilon-2)\alpha_s}{4}\Big) + \frac{\varepsilon\beta_s}{2}.$$

By summing up with respect to $j = 1, \dots, d$, recalling (2.26), and using that $N_j = O(1/|n|)$ for $|n| \to \infty$, we arrive at the expansion

$$\sum_{j=1}^{d}\Big[x_0^{-d/2}\Big(\prod_{k=1}^{s} x_k^{-1/2}\Big)x_s^{-\varepsilon/4} + \text{c.s.}\Big] = 2d - \frac{2dA_s - 8B_s}{|n|^2} + O_\varepsilon\Big(\frac{1}{|n|^3}\Big).$$

Noticing further that

$$B_s = \frac{d(d+4)}{32} + \Big(\frac{d}{4} + \frac{1}{2}\Big)\sum_{k=1}^{s}\alpha_k + \Big(\frac{d}{4} + \frac{1}{2}\Big)\frac{\varepsilon\alpha_s}{2} + O_\varepsilon\Big(\frac{1}{\ell_1^2}\Big),$$

we compute

$$2dA_s - 8B_s = \frac{d(d-4)}{4} - 4\sum_{k=1}^{s}\alpha_k - 2\varepsilon\alpha_s + O_\varepsilon\Big(\frac{1}{\ell_1^2}\Big)$$

for $|n| \to \infty$. In total, we obtain the expansion

$$\frac{(\Delta b^s(\varepsilon))_n}{b_n^s(\varepsilon)} = \Big(\frac{d(4-d)}{4} + 4\sum_{k=1}^{s}\alpha_k + 2\varepsilon\alpha_s\Big)\frac{1}{|n|^2} + O_\varepsilon\Big(\frac{1}{|n|^2\ell_1^2}\Big)$$

for $|n| \to \infty$. Finally, substituting from (2.21), we infer (2.16). ∎

As an immediate corollary, we obtain from expansions (2.15) and (2.16) the following inequalities.

Corollary 13. *Let $s \in \mathbb{N}$ and $\varepsilon > 0$. For all $n \in \mathbb{Z}^d$ with $|n|$ sufficiently large, we have inequalities*

$$\frac{(\Delta b^s)_n}{b_n^s} \geq \frac{d(4-d)}{4|n|^2} + \frac{1}{|n|^2}\sum_{j=1}^{s}\prod_{k=1}^{j}\frac{1}{\log_k |n|} \tag{2.27}$$

and

$$\frac{(\Delta b^s(\varepsilon))_n}{b_n^s(\varepsilon)} \leq \frac{d(4-d)}{4|n|^2} + \frac{1}{|n|^2}\sum_{j=1}^{s}\prod_{k=1}^{j}\frac{1}{\log_k |n|} + \frac{\varepsilon}{|n|^2}\prod_{k=1}^{s}\frac{1}{\log_k |n|}, \tag{2.28}$$

where sequences $b^s(\varepsilon)$ and b^s are defined in (2.13) *and* (2.14).

Remark 14. If $s = 0$, inequality (2.27) does not hold.

3. The absence and existence conditions

The main goal of this section is to derive conditions for potential V of the discrete Schrödinger operator H_V implying either absence or existence of the zero-energy ground state of H_V. The absence condition is a sufficient condition restricting only the entries of the potential from above for indices with sufficiently large Euclidean norm. The existence condition requires a complementary restriction on the entries of the potential from below and, in addition, a criticality of the Schrödinger operator in question. Analogous results concerning the right-most spectral point will be also derived.

The strategy of proofs relies on the discrete Agmon's comparison principle (Theorem 8). As comparison sequences, $b^s(\varepsilon)$ and $b^s \equiv b^s(0)$ defined by (2.13) and (2.14) will be used. It is important to notice that, for any $\varepsilon \geq 0$, $s \in \mathbb{N}_0$, and $N \geq \mathrm{e}_{s-1}$, $b_n^s(\varepsilon) > 0$ for all $n \in \mathbb{Z}^d_{\geq N}$, and

$$b^s(\varepsilon) \in \ell^2(\mathbb{Z}^d_{\geq N}) \text{ if } \varepsilon > 0, \quad \text{but} \quad b^s \notin \ell^2(\mathbb{Z}^d_{\geq N}).$$

3.1. The absence condition

Recall that the discrete Schrödinger operator H_V is said to have a *zero-energy ground state* if and only if $0 = \inf \sigma(H_V) \in \sigma_{\mathrm{p}}(H_V)$.

Theorem 15. *If there exists $s \in \mathbb{N}_0$ such that potential V of the discrete Schrödinger operator H_V on $\mathbb{Z}^d$ fulfills*

$$V_n \leq -2d + \sum_{j=1}^{d} \Bigg[\left(\frac{|n|}{|n+\delta_j|} \right)^{d/2} \prod_{i=1}^{s} \sqrt{\frac{\log_i |n|}{\log_i |n+\delta_j|}} + \left(\frac{|n|}{|n-\delta_j|} \right)^{d/2} \prod_{i=1}^{s} \sqrt{\frac{\log_i |n|}{\log_i |n-\delta_j|}} \Bigg] \tag{3.1}$$

for all $n \in \mathbb{Z}^d$ with $|n|$ sufficiently large, then H_V does not have a zero-energy ground state, i.e., $0 \notin \sigma_{\mathrm{p}}(H_V)$ or $0 \neq \inf \sigma(H_V)$. In particular, it is the case if

$$V_n \leq \frac{d(4-d)}{4|n|^2} + \frac{1}{|n|^2} \sum_{j=1}^{s} \prod_{k=1}^{j} \frac{1}{\log_k |n|} \tag{3.2}$$

for all $n \in \mathbb{Z}^d$ with $|n|$ sufficiently large.

Proof. Suppose V satisfies (3.1) for some $s \in \mathbb{N}_0$ and all $n \in \mathbb{Z}^d_{\geq N}$, where N is a sufficiently large positive integer. Notice that (3.1) is equivalent to the inequality

$$V_n \leq \frac{(\Delta b^s)_n}{b_n^s},$$

where the sequence b^s is defined by (2.14). Therefore,

$$(H_V b^s)_n = (-\Delta b^s)_n + V_n b_n^s \leq 0$$

for all $n \in \mathbb{Z}^d_{\geq N}$, and so b^s is a subsolution of the equation $H_V \psi = 0$ in $\mathbb{Z}^d_{\geq N}$.

Assume, for contradiction, that $0 = \inf \sigma(H_V) \in \sigma_{\mathrm{p}}(H_V)$. Then, the eigenvector $\phi \in \ell^2(\mathbb{Z}^d)$ of H_V corresponding to the eigenvalue 0 can be chosen strictly positive by Theorem 4. We apply Theorem 8 to vectors $w := \phi$ and $u := b^s$. Using Remark 11, one sees that all assumptions of Theorem 8 are indeed satisfied. Theorem 8 implies the existence of a constant $C > 0$ such that

$$b_n^s \leq C \phi_n$$

for all $n \in \mathbb{Z}^d_{\geq N}$. It follows, however, that $b^s \in \ell^2(\mathbb{Z}^d_{\geq N})$, which is a contradiction.

The second claim follows readily from inequality (2.27) for $s \in \mathbb{N}$. The case $s = 0$ also follows as the right-hand side of (3.2) is an increasing function of s. ∎

Recall that the spectrum of $H_0 = -\Delta$ equals $[0, 4d]$. Since the spectrum of the unperturbed operator is a compact interval and conditions of Theorem 15 concern its left end-point 0, it is relevant to ask whether there is a similar condition also for the right end-point $4d$. Of course, this question has no continuous counterpart since the classical Laplacian regarded as an operator acting in $L^2(\mathbb{R}^d)$ is not bounded from above. In the discrete setting, such a complementary condition exists and is analogous to (3.1).

Theorem 16. *If there exists $s \in \mathbb{N}_0$ such that potential V of the discrete Schrödinger operator H_V on $\mathbb{Z}^d$ fulfills*

$$V_n \geq 2d - \sum_{j=1}^{d}\Bigg[\left(\frac{|n|}{|n+\delta_j|}\right)^{d/2}\prod_{i=1}^{s}\sqrt{\frac{\log_i |n|}{\log_i |n+\delta_j|}} + \left(\frac{|n|}{|n-\delta_j|}\right)^{d/2}\prod_{i=1}^{s}\sqrt{\frac{\log_i |n|}{\log_i |n-\delta_j|}}\Bigg] \tag{3.3}$$

for all $n \in \mathbb{Z}^d$ with $|n|$ sufficiently large, then $4d \notin \sigma_{\mathrm{p}}(H_V)$ or $4d \neq \sup \sigma(H_V)$. In particular, it is the case if

$$V_n \geq -\frac{d(4-d)}{4|n|^2} - \frac{1}{|n|^2}\sum_{j=1}^{s}\prod_{k=1}^{j}\frac{1}{\log_k |n|}$$

for all $n \in \mathbb{Z}^d$ with $|n|$ sufficiently large.

Proof. Consider the unitary involution U on $\ell^2(\mathbb{Z}^d)$ defined by the equation

$$(U\psi)_n := (-1)^{n_1+\cdots+n_d}\,\psi_n$$

for any $n \in \mathbb{Z}^d$. Then, by using the respective definitions, one readily verifies that

$$U(-\Delta)U = 4d + \Delta,$$

which implies the relation

$$H_{-V} = 4d - UH_VU \tag{3.4}$$

for any potential V.

It follows from (3.4) that

$$\inf \sigma(H_{-V}) = 4d + \inf -\sigma(H_V) = 4d - \sup \sigma(H_V)$$

and

$$0 \in \sigma_{\mathrm{p}}(H_{-V}) \iff 4d \in \sigma_{\mathrm{p}}(H_V).$$

Now, it suffices to note that, if V fulfils (3.3), then $-V$ satisfies (3.1), and apply Theorem 15 to H_{-V}. ∎

Remark 17. Spectral properties of H_V are particularly understood if $d = 1$, in which case H_V is a Jacobi operator. We complement our conditions with other results. When imposed jointly, the simplest form of conditions (3.1) and (3.3) for $s = 0$ requires

$$|V_n| \leq 2 - \sqrt{\frac{n}{n+1}} - \sqrt{\frac{n}{n-1}} \tag{3.5}$$

for all $n \in \mathbb{Z}$ of sufficiently large modulus. It follows that $V_n = O(1/n^2)$, as $|n| \to \infty$. So V is a trace class operator which implies that there are no embedded eigenvalues of H_V in $(0, 4)$. This is a well-known discrete analogue to a result of the scattering theory about an asymptotic behavior of the Jost solutions; see for instance [10, eq. (1.17)]. Hence, the potential V can produce only eigenvalues in $(-\infty, 0] \cup [4, \infty)$. If we restrict the class of potentials even more to non-trivial potentials satisfying

$$\sum_{n\in\mathbb{Z}} n^{1+\varepsilon}|V_n| < \infty \quad \text{and} \quad \sum_{n\in\mathbb{Z}} V_n \leq 0 \tag{3.6}$$

for some $\varepsilon > 0$, then the weak coupling analysis tells us that there is always a negative eigenvalue in the spectrum of H_V, see [24, Theorem 1.4] or [12, Theorem A.19]. Similarly, if we alter the second inequality in (3.6) to ≥ 0, there is always an eigenvalue of H_V greater than 4. Hence, in these particular cases, Theorems 15 and 16 do not imply anything new. However, condition (3.6) is stronger than (3.5). For a potential that fulfills (3.5) (or its logarithmic refinements) but not (3.6), Theorems 15 and 16 yield new results even for the simplest $d = 1$ case. If, in addition to (3.5), V does not produce weakly coupled bound states, i.e., H_V enjoys the spectral stability $\sigma(H_V) = \sigma_{\mathrm{ess}}(H_V)$, our results imply that the spectrum of H_V is purely continuous and fills the interval $[0, 4]$. Conditions for spectral stability of discrete Schrödinger operators on $\mathbb{N}$ allowing even complex-valued potentials have been studied recently in [25, Section 5].

3.2. The existence condition

First, we define the notion of criticality adapted to the discrete setting. As the discrete Schrödinger operators H_V are bounded if and only if V is bounded, in contrast to the continuous case, one may define two kinds of criticality of H_V reflecting the fact that the spectrum of H_V can have two finite end points. Below, letters V and W are generically used for multiplication operators equipped with their maximal domains.

Definition 18. Suppose H_V is the discrete Schrödinger operator on $\mathbb{Z}^d$ which is bounded from below and denote $s_- := \inf \sigma(H_V)$. Then, we call H_V

(i) *critical at s_- (from below)* if

$$(\forall W \geq 0 \text{ compact})(\inf \sigma(H_V - W) = s_- \implies W = 0),$$

(ii) *subcritical at s_- (from below)* if

$$(\exists W \geq 0 \text{ compact})(\inf \sigma(H_V - W) = s_- \text{ and } W \neq 0).$$

Similarly, assuming H_V to be bounded from above and $s_+ := \sup \sigma(H_V)$, then H_V is called

(i) *critical at s_+ (from above)* if

$$(\forall W \geq 0 \text{ compact})(\sup \sigma(H_V + W) = s_+ \implies W = 0),$$

(ii) *subcritical at s_+ (from above)* if

$$(\exists W \geq 0 \text{ compact})(\sup \sigma(H_V + W) = s_+ \text{ and } W \neq 0).$$

Remark 19. Analogously to the continuous case, the discrete Laplacian H_0 on the lattice $\mathbb{Z}^d$ is critical at 0 for $d = 1, 2$, which demonstrates the existence of weakly coupled bound states [24], and subcritical at 0 for $d \geq 3$, which follows from the existence of Hardy inequalities [20,32]. A theory on critical Schrödinger operators on lattices and more general graph structures is discussed in [22,23].

Now, we are ready to state the condition for the existence of the zero-energy ground state of H_V on $\mathbb{Z}^d$. We provide a proof which relies only on results of this paper and general principles of functional analysis. However, by employing a more advanced results from recent developments of the criticality theory in the discrete setting, the proof can be streamlined which is indicated in Remark 21 below the proof.

Theorem 20. *Let* $\inf \sigma(H_V) = \inf \sigma_{\text{ess}}(H_V) = 0$ *and* H_V *be critical at* 0. *If there exist* $s \in \mathbb{N}_0$ *and* $\varepsilon > 0$ *such that*

$$\begin{aligned} V_n \geq -2d &+ \sum_{j=1}^{d} \left(\frac{|n|}{|n+\delta_j|} \right)^{d/2} \left(\prod_{i=1}^{s} \sqrt{\frac{\log_i |n|}{\log_i |n+\delta_j|}} \right) \left(\frac{\log_s |n|}{\log_s |n+\delta_j|} \right)^{\varepsilon} \\ &+ \sum_{j=1}^{d} \left(\frac{|n|}{|n-\delta_j|} \right)^{d/2} \left(\prod_{i=1}^{s} \sqrt{\frac{\log_i |n|}{\log_i |n-\delta_j|}} \right) \left(\frac{\log_s |n|}{\log_s |n-\delta_j|} \right)^{\varepsilon} \end{aligned} \tag{3.7}$$

for all $n \in \mathbb{Z}^d$ *with* $|n|$ *sufficiently large, then* $0 \in \sigma_{\rm p}(H_V)$. *In particular, if there exist* $s \in \mathbb{N}_0$ *and* $\varepsilon > 0$ *such that*

$$V_n \geq \frac{d(4-d)}{4|n|^2} + \frac{1}{|n|^2}\sum_{j=1}^{s}\prod_{k=1}^{j}\frac{1}{\log_k |n|} + \frac{\varepsilon}{|n|^2}\prod_{k=1}^{s}\frac{1}{\log_k |n|} \tag{3.8}$$

for all $n \in \mathbb{Z}^d$ *with* $|n|$ *sufficiently large, then* $0 \in \sigma_{\rm p}(H_V)$.

Proof. Let us denote by δ_0 the Dirac delta potential which acts on $\ell^2(\mathbb{Z}^d)$ as

$$(\delta_0\psi)_n = \begin{cases} \psi_0 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Suppose H_V satisfies the assumptions. We define the auxiliary sequence of potentials

$$V^k := V - \frac{1}{k}\delta_0, \quad k \in \mathbb{N}.$$

Then, $H_{V^k} = H_V - \delta_0/k$ and $\operatorname{Dom} H_{V^k} = \operatorname{Dom} H_V = \operatorname{Dom} V$. Since H_V is critical at 0, $\inf\sigma(H_{V^k}) < 0$ for all $k \in \mathbb{N}$. Moreover, $\sigma_{\rm ess}(H_{V^k}) = \sigma_{\rm ess}(H_V) \subset [0,\infty)$ because δ_0 is a rank one operator and hence $H_{V^k} - H_V$ is compact. It follows that

$$E_k := \inf\sigma(H_{V^k})$$

is a discrete eigenvalue of H_{V^k}. By the min-max principle, $-1/k \leq E_k < 0$, therefore $E_k \to 0$ as $k \to \infty$. Let us denote by ϕ^k the *normalized* eigenvector of H_{V^k} corresponding to the eigenvalue E_k. By Theorem 4, we may assume ϕ^k to be strictly positive.

As the unit ball in any reflexive Banach space is weakly precompact, $\{\phi^k\}_{k=1}^\infty$ contains a weakly convergent subsequence which we again denote by $\{\phi^k\}_{k=1}^\infty$ with some abuse of notation. Notice that weak convergence in $\ell^2(\mathbb{Z}^d)$ means nothing but point-wise convergence. In the course of the proof, we will show that, under the assumption (3.7), $\{\phi^k\}_{k=1}^\infty$ converges even strongly, i.e., in the norm of $\ell^2(\mathbb{Z}^d)$. To this end, it suffices to verify that $\|\phi\| = 1$, where ϕ is the weak limit of $\{\phi^k\}_{k=1}^\infty$.

Condition (3.7) is equivalent to the inequality

$$V_n \geq \frac{(\Delta b^s(\delta))_n}{b^s(\delta)_n},$$

where $\delta := 4\varepsilon$ and sequence $b^s(\delta)$ is defined by (2.13). In other words, $[H_V b^s(\delta)]_n \geq 0$ for all $n \in \mathbb{Z}^d_{\geq N}$, where N is a positive integer. Taking also into account that $[\delta_0 b^s(\delta)]_n = 0$ for $n \neq 0$, we get

$$[(H_{V^k} - E_k)b^s(\delta)]_n = [H_V b^s(\delta)]_n - E_k b_n^s(\delta) \geq -E_k b_n^s(\delta) > 0,$$

for all $n \in \mathbb{Z}^d_{\geq N}$. Hence, $b^s(\delta)$ is a strictly positive supersolution of the equation $(H_{V^k} - E_k)\psi = 0$ in $\mathbb{Z}^d_{\geq N}$.

Fix $N \in \mathbb{N}$ such that assumption (3.7) holds for all $n \in \mathbb{Z}^d_{\geq N}$. For any $k \in \mathbb{N}$, we may apply Theorem 8 to $w := b^s(\delta)$ and $u := \phi^k$, which implies

$$\phi^k_n \leq C_k b^s_n(\delta).$$

for all $n \in \mathbb{Z}^d_{\geq N}$, with the constant $C_k > 0$ chosen as

$$C_k := \max_{N-1 \leq |n| < N+1} \frac{\phi^k_n}{b^s_n(\delta)}.$$

By the normalization and positivity of all ϕ^k, we have $0 \leq \phi^k_n \leq 1$ for all $k \in \mathbb{N}$ and $n \in \mathbb{Z}^d$. Then

$$C_k \leq C := \max_{N-1 \leq |n| < N+1} \frac{1}{b^s_n(\delta)}$$

for all $k \in \mathbb{N}$ and we obtain the estimate

$$\phi^k_n \leq C b^s_n(\delta)$$

valid for all $n \in \mathbb{Z}^d_{\geq N}$ with the k-independent constant C. Since $b^s(\delta) \in \ell^2(\mathbb{Z}^d_{\geq M})$, it follows

$$\|\phi\|^2 = \sum_{n \in \mathbb{Z}^d} |\phi_n|^2 = \lim_{k \to \infty} \sum_{n \in \mathbb{Z}^d} |\phi^k_n|^2 = 1$$

by the Lebesgue dominated convergence. Thus, $\phi^k \to \phi$ in the norm of $\ell^2(\mathbb{Z}^d)$.

To finish the proof of the first claim, we show that vector ϕ is an eigenvector of H_V corresponding to the eigenvalue 0. Clearly, $0 \neq \phi \in \ell^2(\mathbb{Z}^d)$ since $\|\phi\| = 1$. So, we are done once we show that $\phi \in \operatorname{Dom} V$ and, as $k \to \infty$,

$$(H_{V^k} - E_k)\phi^k \to H_V \phi$$

in $\ell^2(\mathbb{Z}^d)$. Since $E_k \to 0$, δ_0 is bounded, and ϕ^k are uniformly bounded, both $k^{-1}\delta_0\phi^k$ and $E_k\phi^k$ tend to 0 in $\ell^2(\mathbb{Z}^d)$. Further, Δ is bounded and so $\Delta\phi^k \to \Delta\phi$ in $\ell^2(\mathbb{Z})$. Thus, it suffices to show that $\phi \in \operatorname{Dom} V$ and $V\phi^k \to V\phi$ in $\ell^2(\mathbb{Z}^d)$. This is true if $V\phi^k$ converges in $\ell^2(\mathbb{Z}^d)$ because V is a closed operator on its maximal domain. Finally, the last assertion is true indeed, because

$$V\phi^k = \Delta\phi^k + \frac{1}{k}\delta_0\phi^k + E_k\phi^k \to \Delta\phi$$

in $\ell^2(\mathbb{Z}^d)$.

To verify the second claim for $s \in \mathbb{N}$, it suffices to note that, if V satisfies (3.8), then V fulfils (3.7) with ε replaced by $\varepsilon/4$ by inequality (2.28). The factor $1/4$ is of course inessential and we may apply the already proven first claim. Finally, the case $s = 0$ is also covered since, if condition (3.8) is true for $s = 0$, then it is true also for $s = 1$ and all $n \in \mathbb{Z}^d$ with $|n|$ sufficiently large. ∎

Remark 21. If the technique of recent preprint [8] is employed, the previous proof admits a simplification which is as follows. In our proof, we show that $b^s(\delta)$ is a positive supersolution of $H_V\psi = 0$ in $\mathbb{Z}^d_{\geq N}$ for N large enough. Since H_V is assumed to be critical at 0 it has the so-called Agmon ground state ϕ, which is in particular a positive solution of minimal growth at infinity of equation $H_V\psi = 0$ in $\mathbb{Z}^d$, see [8, Theorem 2.6]. By the definition of the positive solution of minimal growth, see [8, Definition 2.4], there exists a constant $C > 0$ such that $\phi_n \leq C b^s_n(\delta)$ for all $n \in \mathbb{Z}^d_{\geq N}$. Since $b^s(\delta)$ is square summable $\phi \in \ell^2(\mathbb{Z}^d)$, and hence ϕ is an eigenvector of H_V corresponding to the eigenvalue 0.

Remark 22. It is not obvious that potentials satisfying the assumptions of Theorem 20 really exist. This is demonstrated by an example in Section 3.3.

The variant of Theorem 20 for the right spectral end-point can be proven in a similar fashion as is Theorem 16 deduced from Theorem 15. Note that it follows from equation (3.4) that H_V is critical at $4d$ from above if and only if H_{-V} is critical at 0 from below.

Theorem 23. *Let* $\sup\sigma(H_V) = \sup\sigma_{\mathrm{ess}}(H_V) = 4d$ *and* H_V *be critical at* $4d$ *from above. If there exist* $s \in \mathbb{N}_0$ *and* $\varepsilon > 0$ *such that*

$$V_n \leq 2d - \sum_{j=1}^{d}\left(\frac{|n|}{|n+\delta_j|}\right)^{d/2}\left(\prod_{i=1}^{s}\sqrt{\frac{\log_i|n|}{\log_i|n+\delta_j|}}\right)\left(\frac{\log_s|n|}{\log_s|n+\delta_j|}\right)^{\varepsilon}$$
$$- \sum_{j=1}^{d}\left(\frac{|n|}{|n-\delta_j|}\right)^{d/2}\left(\prod_{i=1}^{s}\sqrt{\frac{\log_i|n|}{\log_i|n-\delta_j|}}\right)\left(\frac{\log_s|n|}{\log_s|n-\delta_j|}\right)^{\varepsilon}$$

for all $n \in \mathbb{Z}^d$ *with* $|n|$ *sufficiently large, then* $4d \in \sigma_{\mathrm{p}}(H_V)$. *In particular, if there exist* $s \in \mathbb{N}_0$ *and* $\varepsilon > 0$ *such that*

$$V_n \leq -\frac{d(4-d)}{4|n|^2} - \frac{1}{|n|^2}\sum_{j=1}^{s}\prod_{k=1}^{j}\frac{1}{\log_k|n|} - \frac{\varepsilon}{|n|^2}\prod_{k=1}^{s}\frac{1}{\log_k|n|}$$

for all $n \in \mathbb{Z}^d$ *with* $|n|$ *sufficiently large, then* $4d \in \sigma_{\mathrm{p}}(H_V)$.

3.3. An example

For a parameter $\gamma > 0$, we consider potential $V = V(\gamma)$ defined by

$$V_n := \frac{(\Delta a)_n}{a_n}, \quad n \in \mathbb{Z}^d, \tag{3.9}$$

where

$$a_n := \begin{cases} |n|^{-\gamma} & \text{if } n \neq 0, \\ 1 & \text{if } n = 0. \end{cases} \tag{3.10}$$

Explicitly, the entries of V read

$$V_n = \begin{cases} 0 & \text{if } n = 0, \\ 1 + 2^{-\gamma} + (d-1)2^{1-\gamma/2} - 2d & \text{if } n = \pm\delta_i,\ i \in \{1, \ldots, d\}, \\ -2d + \displaystyle\sum_{j=1}^{d} \frac{|n|^\gamma}{|n+\delta_j|^\gamma} + \frac{|n|^\gamma}{|n-\delta_j|^\gamma} & \text{otherwise.} \end{cases}$$

The existence or non-existence of the zero-energy ground state of the Schrödinger operator $H_V = -\Delta + V$ depends on the value of the parameter γ, which is shown in the following proposition.

Proposition 24. *Let $\gamma > 0$ and H_V be the discrete Schrödinger operator with potential defined by* (3.9) *and* (3.10)*. Then,*

(i) $\sigma_{\mathrm{ess}}(H_V) = [0, 4d]$,

(ii) *H_V is bounded and non-negative,*

(iii) *H_V is critical at 0 for $\gamma > (d-1)/2$,*

(iv) *$0 \in \sigma_{\mathrm{p}}(H_V)$ if and only if $\gamma > d/2$.*

Proof. (i) A straightforward computation shows that

$$V_n = \frac{\gamma(\gamma + 2 - d)}{|n|^2} + O\Big(\frac{1}{|n|^3}\Big), \quad |n| \to \infty. \tag{3.11}$$

It follows that V is a compact operator and therefore $\sigma_{\mathrm{ess}}(H_V) = \sigma_{\mathrm{ess}}(-\Delta) = [0, 4d]$ by the Weyl criterion.

(ii) The boundedness of H_V follows from boundedness of V which, in its turn, is obvious from (3.11).

Next, we prove that $H_V \geq 0$. Since H_V is bounded it suffices to verify that $\langle\psi, H_V\psi\rangle \geq 0$ for all compactly supported sequences $\psi \in \ell^2(\mathbb{Z}^d)$. In addition, we can assume that ψ is real without loss of generality. Define the auxiliary sequence ϕ by putting $\phi_n := \psi_n/a_n$ for all $n \in \mathbb{Z}^d$, where the positive sequence a is given by (3.10). Then, using (2.1) and (3.9), we find

$$\begin{aligned} \langle\psi, H_V\psi\rangle &= \sum_{j=1}^{d} \|D_j(a\phi)\|^2 + \langle a\phi, Va\phi\rangle \\ &= \sum_{j=1}^{d} \sum_{n\in\mathbb{Z}^d} a_n a_{n-\delta_j}(a_n\phi_n - a_{n-\delta_j}\phi_{n-\delta_j})^2 - 2d \sum_{n\in\mathbb{Z}^d} a_n^2\phi_n^2 \\ &\quad + \sum_{n\in\mathbb{Z}^d} a_n\phi_n^2 \sum_{j=1}^{d} (a_{n+\delta_j} + a_{n-\delta_j}) \end{aligned}$$

$$
\begin{aligned}
&= -2\sum_{j=1}^{d}\sum_{n\in\mathbb{Z}^d} a_n a_{n-\delta_j} a_n a_{n-\delta_j}\phi_n\phi_{n-\delta_j} + \sum_{j=1}^{d}\sum_{n\in\mathbb{Z}^d} a_n a_{n-\delta_j}\phi_{n-\delta_j}^2\\
&\quad + \sum_{j=1}^{d}\sum_{n\in\mathbb{Z}^d} a_n a_{n-\delta_j} a_n a_{n-\delta_j}\phi_n^2\\
&= \sum_{j=1}^{d}\sum_{n\in\mathbb{Z}^d} a_n a_{n-\delta_j}(\phi_n - \phi_{n-\delta_j})^2 \geq 0.
\end{aligned}
$$

(iii) Using the above computation with $\psi^N := a\phi^N$, where

$$
\phi_n^N := \begin{cases} 1 & \text{if } n \in [-N,N]^d,\\ 0 & \text{if } n \notin [-N,N]^d, \end{cases}
$$

one finds that

$$
\langle \psi^N, H_V \psi^N\rangle = \sum_{j=1}^{d}\Bigg(\sum_{\substack{n\in[-N,N]^d\\ n_j=N+1}} a_n a_{n-\delta_j} + \sum_{\substack{n\in[-N,N]^d\\ n_j=-N}} a_n a_{n-\delta_j}\Bigg)
$$

for any $N \in \mathbb{N}$. Taking definition (3.10) into account, the right-hand side can be further simplified getting

$$
\begin{aligned}
\langle \psi^N, H_V \psi^N\rangle &= 2d \sum_{n_2=-N}^{N}\cdots\sum_{n_d=-N}^{N} a_{(N+1,n_2,\dots,n_d)} a_{(N,n_2,\dots,n_d)}\\
&\leq 2d \sum_{n_2=-N}^{N}\cdots\sum_{n_d=-N}^{N} a_{(N,n_2,\dots,n_d)}^2\\
&= \frac{2d}{N^{2\gamma-d+1}} \sum_{n_2=-N}^{N}\cdots\sum_{n_d=-N}^{N} \frac{1}{N^{d-1}}\left[1+\left(\frac{n_2}{N}\right)^2+\cdots+\left(\frac{n_d}{N}\right)^2\right]^{-\gamma}.
\end{aligned}
$$

Notice the last multi-sum, which is to be interpreted as 1 if $d = 1$, is the Riemann sum for a multi-variable function. Consequently, as $N \to \infty$, we have the finite limit

$$
\begin{aligned}
&\sum_{n_2=-N}^{N}\cdots\sum_{n_d=-N}^{N} \frac{1}{N^{d-1}\left[1+\left(\frac{n_2}{N}\right)^2+\cdots+\left(\frac{n_d}{N}\right)^2\right]^{\gamma}}\\
&\quad \to \int_{-1}^{1}\cdots\int_{-1}^{1} \frac{\mathrm{d}x_2\cdots\mathrm{d}x_d}{(1+x_2^2+\cdots+x_d^2)^{\gamma}}.
\end{aligned}
$$

Therefore, we may conclude that there exists a constant $C > 0$ such that

$$\langle \psi^N, H_V \psi^N \rangle \leq \frac{C}{N^{2\gamma-d+1}}$$

for all $N \in \mathbb{N}$.

Now, suppose that $2\gamma > d-1$ and $W \geq 0$ is a compact potential such that $H_V \geq W$ in the sense of quadratic forms. We show that $W = 0$ which implies claim (iii). For all $N \in \mathbb{N}$, we have

$$\sum_{n\in[-N,N]^d} a_n^2 W_n = \langle \psi^N, W\psi^N \rangle \leq \langle \psi^N, H_V \psi^N \rangle \leq \frac{C}{N^{2\gamma-d+1}}.$$

Taking the limit $N \to \infty$, we find that

$$\sum_{n\in\mathbb{Z}^d} a_n^2 W_n \leq 0.$$

Since $W_n \geq 0$ and $a_n > 0$ for all $n \in \mathbb{Z}^d$, the last inequality implies that $W_n = 0$ for all $n \in \mathbb{Z}^d$, i.e., $W = 0$.

Alternatively, one can show that ψ^N is a null-sequence in the sense of [23, Definition 5.2]. Then the criticallity follows from [23, Theorem 5.3 (i) and (iv′)].

(iv) Suppose $\gamma > d/2$. Then $a \in \ell^2(\mathbb{Z}^d)$ and it follows readily from definition (3.9) that $H_V a = 0$. Alternatively, one deduces the same result from (3.11) and Theorem 20 with $s = 0$.

If $0 < \gamma < d/2$, then a comparison of (3.11) with the condition (3.2) of Theorem 15 for $s = 0$ yields that H_V does not have a zero-energy ground state. Recalling that $H_V \geq 0$, it follows that $0 \notin \sigma_{\mathrm{p}}(H_V)$. If $\gamma = d/2$, one proceeds similarly using the refined condition (3.2) of Theorem 15 with $s = 1$. ∎

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