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Phase transition in the integrated density of states of the Anderson model arising from a supersymmetric sigma model

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Abstract. We study the integrated density of states (IDS) of the random Schrödinger operator appearing in the study of certain reinforced random processes in connection with a supersymmetric sigma-model. We rely on previous results on the supersymmetric sigma-model to obtain lower and upper bounds on the asymptotic behavior of the IDS near the bottom of the spectrum in all dimension. We show a phase transition for the IDS between weak and strong disorder regime in dimension larger or equal to three, that follows from a phase transition in the corresponding random process and supersymmetric sigma-model. In particular, we show that the IDS does not exhibit Lifshitz tails in the strong disorder regime, confirming a recent conjecture. This is in stark contrast with other disordered systems, like the Anderson model. A Wegner-type estimate is also derived, giving an upper bound on the IDS and showing the regularity of the function.

1. Introduction and main results

Transport phenomena in disordered materials can be described at the quantum mechanical level via random Schödinger operators. On the lattice $\mathbb{Z}^d$, $d \geq 1$, they generally take the form of an infinite random matrix $H_\omega = -\Delta + \lambda V_\omega \in \mathbb{R}^{\mathbb{Z}^d \times \mathbb{Z}^d}_{\mathrm{sym}}$ where $-\Delta$ is the negative discrete Laplacian and V_ω is a diagonal matrix with random entries.

In this paper, we consider a random Schrödinger operator H_β (defined in (1.1) below) arising from the supersymmetric hyperbolic sigma model $H^{2|2}$ introduced by Zirnbauer in the context of quantum diffusion [13, 35]. This can be seen as a statistical mechanics spin model, where the spins take values on a supersymmetric extension of the hyperbolic plane H^2. This model is expected to qualitatively reflect

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the phenomenon of Anderson localization and delocalization for real symmetric band matrices (see [12, Section 3]) and exhibits a dimension-dependent phase transition between a disordered phase [11] and an ordered phase with spontaneous symmetry breaking [12].

In recent years, $H^{2|2}$ attracted a growing interest from the mathematics community due to the discovery in [32–34] of surprising connections with two linearly reinforced random processes: the edge reinforced random walk introduced by Diaconis in 1986, and the vertex reinforced jump process conceived by Werner around 2000. A first spectacular application of this connection was the proof of a phase transition in the reinforced processes between a recurrent and a transient phase that follows from the disorder/order transition in the $H^{2|2}$ model [4, 10, 32].

In [33, 34], Sabot, Tarrès, and Zeng show that a key ingredient in proving many properties of the reinforced processes is the connection with the random Schrödinger operator H_β (1.1), which is the main object of study in the present paper. The spectrum of this random operator is deterministic (see [1, Theorem 3.10] and [30, Chapter 4]) and the existence/non-existence of an eigenvalue in 0 is related to transience/recurrence properties of the stochastic processes [33, 34]. At large disorder the spectrum is pure point [7].

In this paper, we pursue the study of spectral properties of this operator. Our aim is to study the asymptotics of the so-called *integrated density of states* (IDS) of the operator H_β for energies near the bottom of the spectrum. The IDS is a function on the spectrum of the operator that computes the average number of eigenvalues per unit volume. In disordered systems like the Anderson model with independent random variables, the IDS exhibits an exponential decay near the spectral edges at arbitrary dimension, known as Lifshitz tails. This is in stark contrast with the behavior of the IDS in periodic systems. The Lifshitz behavior of the IDS is a key ingredient to prove localization for random operators, although it is not a necessary condition (see, e.g., Delone–Anderson models for which the IDS might not even exist but localization still holds [14, 31]). The connection between the IDS behavior at the bottom of the spectrum and localization explains the important role played by the IDS in the spectral and dynamical study of random Schrödinger operators.

In [34], the authors conjecture that the asymptotic behavior of the IDS of the random Schrödinger operator H_β appearing in connection to reinforced random processes does not exhibit Lifshitz tails. This is due to dependencies in the random variables, that imply that the bottom of the spectrum is not attained by extreme values of the random variables, but can be attained by several configurations of the potential.

In this article, we show that the IDS $N(E, H_\beta)$ of the operator H_β does not exhibit Lifshitz tails, and undergoes a phase transition in its behavior as a function of E, depending on the dimension and the strength of the disorder. This follows from a phase transition in the associated reinforced random process and supersymmetric sigma-model. Namely, we prove that in dimension one, for any value of the disorder strength, the IDS behaves roughly as $\sqrt{E}$ as $E \downarrow 0$, while in dimension two and above, this behavior holds for large disorder. On the contrary, in dimension three and above the decay rate is bounded above by E at weak disorder.

To the best of our knowledge, the operator H_β is the first Anderson-type model for which the IDS is known to undergo a phase transition, whose dependence on the disorder strength and dimension is similar to the one in the metal-insulator transition conjectured for the Anderson model. Note that the transitions appearing in the literature for the IDS of Anderson-type models (the so-called classical-quantum transitions) are transitions in the exponents of the Lifshitz tails depending on the decay of the single site potential [16, 23]. A phase transition which does not involve Lifshitz tails has been observed in the IDS for certain random spin models [15]. As far as we know, the operator H_β provides a first physically motivated example where Lifshitz tails break down, even in presence of pure point spectrum. The latter contributes to the family of very specific models for which the violation of Lifshitz tails is known [6, 20, 21, 26, 28].

We proceed to define the random Schrödinger operator H_β on $\mathbb{Z}^d$ appearing in connection with the hyperbolic $H^{2|2}$ sigma-model and reinforced random processes. Let $\mathbb{Z}^d$ be the undirected square lattice, with vertex set $V(\mathbb{Z}^d)$ and edge set $E(\mathbb{Z}^d)$. By abuse of notation, we will often identify the set in $\mathbb{Z}^d$ with its vertex set and, in particular, write $\mathbb{Z}^d$ instead of $V(\mathbb{Z}^d)$. The operator H_β is defined as follows. Let $W_e = W_{i,j} > 0$ be the edge weight of $e = \{i, j\}$ on $\mathbb{Z}^d$, and P^W be the associated adjacency operator of $\mathbb{Z}^d$, or equivalently, P^W is the operator on $\ell^2(\mathbb{Z}^d)$ defined by

$$P^W f(i) = \sum_{j:j\sim i} W_{i,j} f(j) \quad \text{for all } f \in \ell^2(\mathbb{Z}^d),$$

where $j \sim i$ means that $\{i, j\}$ is an edge of the lattice $\mathbb{Z}^d$. We consider $\mathcal{H}_\beta \in \mathbb{R}^{\mathbb{Z}^d \times \mathbb{Z}^d}$, the infinite symmetric matrix defined by

$$\mathcal{H}_\beta := 2\beta - P^W, \tag{1.1}$$

where β is a diagonal matrix whose diagonal entries $(\beta_i)_{i\in\mathbb{Z}^d}$ form a family of positive random variables defined as follows (cf. [33, Theorem 1] and [34, Proposition 1]): for all $i \in \mathbb{Z}^d$, $\beta_i > 0$ a.s., and for all sub lattice $\Lambda \subset \mathbb{Z}^d$ finite, the Laplace transform of $(\beta_i)_{i\in\Lambda}$ equals

$$\mathbb{E}^W(e^{-\langle\lambda,\beta\rangle_\Lambda}) = e^{-\sum_{i,j\in\Lambda,i\sim j} W_{i,j}(\sqrt{(1+\lambda_i)(1+\lambda_j)}-1)-\sum_{i\in\Lambda,j\notin\Lambda,i\sim j} W_{i,j}(\sqrt{1+\lambda_i}-1)} \prod_{i\in\Lambda} \frac{1}{\sqrt{1+\lambda_i}}. \tag{1.2}$$

The law of this random field β is characterized by the above Laplace transform. This Laplace transform is a particular case of the general version, given in equation (2.1), with $\theta \equiv 1$. By means of the Laplace transform (1.2), one can see that if i and j are not related by an edge in $\mathbb{Z}^d$, then β_i and β_j are independent. We say that the field β is 1-*dependent.* The infinite-volume distribution of $(\beta_i)_{i\in\mathbb{Z}^d}$ will be denoted by ν^W and the associated expectation is denoted by $\mathbb{E}^W$. The operator $\mathcal{H}_\beta$ defined by

$$\mathcal{H}_\beta f(i) = 2\beta_i f(i) - \sum_{j:j\sim i} W_{i,j} f(j) \quad \text{for all } i \in \mathbb{Z}^d,$$

maps $\mathcal{D} \to \ell^2(\mathbb{Z}^d)$ almost surely, where $\mathcal{D} \subset \ell^2(\mathbb{Z}^d)$ is the set of sequences with finite support, which is dense.

In this paper, we will set all $W_{i,j}$ equal, and, in an abuse of notation, denote this common value W too (as it will not cause any ambiguity in the sequel). This condition ensures that the operator $\mathcal{H}_\beta$ is ergodic with respect to the translations in $\mathbb{Z}^d$. Ergodicity is a key ingredient to prove the existence of the IDS and the deterministic nature of the spectrum of $\mathcal{H}_\beta$ using standard arguments. In this case, the spectrum is given by $\sigma(\mathcal{H}_\beta) = [0,+\infty)$ (see [34, Theorem 2.(i)] and [30]). At times, we still write $W_{i,j}$ to specify the vertex i that we are considering or to emphasize the generality of the probability measures.

By [33, Proposition 1] or [34, Lemma 4], any finite marginal $(\beta_i)_{i\in\Lambda}$ (i.e., $\Lambda \subset \mathbb{Z}^d$ is a finite subset) has the following explicit probability density with respect to the product Lebesgue measure $d\beta = \prod_{i\in\Lambda} d\beta_i$:

$$\begin{aligned}\nu_\Lambda^{W,\eta^w}&(d\beta)\\ &= \mathbf{1}_{\mathcal{H}_{\beta,\Lambda}>0} e^{-\frac{1}{2}(\langle 1,\mathcal{H}_{\beta,\Lambda}1\rangle + \langle \eta^w, \mathcal{H}_{\beta,\Lambda}^{-1}\eta^w\rangle - 2\langle 1,\eta^w\rangle)} \frac{1}{\sqrt{\det \mathcal{H}_{\beta,\Lambda}}} \left(\frac{2}{\pi}\right)^{\frac{|\Lambda|}{2}} d\beta. \quad (1.3)\end{aligned}$$

The corresponding average will be denoted by $\mathbb{E}_\Lambda^{W,\eta^w}$. Here, η^w is a vector denoting a wired boundary condition on Λ, defined by

$$\eta^w(i) := \eta^w_\Lambda(i) = \sum_{j\notin\Lambda, j\sim i} W_{i,j}, \quad \text{for } i \in \Lambda, \tag{1.4}$$

and

$$\mathcal{H}_{\beta,\Lambda} := (2\beta - P^W)|_\Lambda = 1_\Lambda \mathcal{H}_\beta 1_\Lambda$$

is the operator $\mathcal{H}_\beta$ restricted on the set Λ with simple boundary condition, i.e., a finite matrix defined by

$$\mathcal{H}_{\beta,\Lambda} f(i) := 2\beta_i f(i) - \sum_{j\in\Lambda:j\sim i} W_{i,j} f(j), \quad \text{for all } f \in \mathbb{R}^\Lambda.$$

Here, 1_Λ is the projection operator on Λ. Note that, even if we replace η^w by an arbitrary $\eta \in \mathbb{R}^{\Lambda}_{\geq 0}$, (1.3) is still a probability density. In this case, the probability will be denoted by $\nu_\Lambda^{W,\eta}$ and the expectation by $\mathbb{E}_\Lambda^{W,\eta}$, to stress the η dependence. A more general finite volume density is given in Theorem 4 below.

Sometimes, we will write $\mathcal{H}^S_{\beta,\Lambda} := \mathcal{H}_{\beta,\Lambda}$ to insist on the type of boundary conditions considered, that we call *simple boundary condition*. We will also consider the operator with Dirichlet boundary condition, which will be denoted by $\mathcal{H}^D_{\beta,\Lambda}$ and is defined by

$$\mathcal{H}^D_{\beta,\Lambda} := (2\beta - P^W)_\Lambda + WM_{2d-n} = \mathcal{H}_{\beta,\Lambda} + WM_{2d-n},$$

where M_{2d-n} is the multiplicative operator by $2d - n$ acting on $\ell^2(\mathbb{Z}^d)$, where for every $i \in \Lambda$, $n_i := \deg(i)$ in Λ, i.e., $n_i = \sum_{j\in\Lambda,\ j\sim i} 1$.

In the usual Anderson model, the random Schrödinger operator $H = -\Delta + \lambda V$ with a bounded potential, the edge weight equals 1 (in the discrete Laplacian Δ, entries are 0 or 1), and the disorder parameter $\lambda > 0$ modulating the intensity of the random potential allows for two well-defined regimes, that of strong disorder ($\lambda \gg 1$) and that of weak disorder (small λ). In $\mathcal{H}_\beta$, however, the edge weight equals W, and the law of the random potential depends also on W, hence the disorder parameter does not appear as a coupling constant but is encoded in the law of β. To have an expression that resembles the Anderson model, we consider the rescaled operator H_β defined by

$$H_\beta := \frac{1}{W}\mathcal{H}_\beta = \frac{2\beta}{W} - P = (-\Delta) + \Big(\frac{2\beta}{W} - 2d\Big), \quad P_{ij} := \mathbf{1}_{i\sim j}.$$

The corresponding finite volume operator with Dirichlet boundary condition is then

$$H^D_{\beta,\Lambda} := \frac{1}{W}\mathcal{H}^D_{\beta,\Lambda} = \Big(\frac{2\beta}{W} - P\Big)_\Lambda + M_{2d-n} = H_{\beta,\Lambda} + M_{2d-n}, \tag{1.5}$$

where $H_{\beta,\Lambda} = H^S_{\beta,\Lambda}$ is the operator with simple boundary condition.

Note that, by the explicit Laplace transform (1.2) (cf. [7, Theorem C]), we have for every $j \in \mathbb{Z}^d$ and for every $\lambda > 0$,

$$\mathbb{E}^W[e^{-\lambda\beta_j}] = \frac{e^{-2dW(\sqrt{1+\lambda}-1)}}{\sqrt{1+\lambda}}.$$

Therefore, the one point marginal of the random potential is known to be a reciprocal inverse Gaussian distribution.

It follows that the mean of $2\beta_i$ is $2dW + 1$, and its variance is $2dW + 2$. The corresponding rescaled potential $V_i := 2\beta_i/W - 2d$ has mean $\mathbb{E}^W[V_i] = 1/W$ and variance $\mathrm{Var}[V_i] = 2d/W + 2/W^2$. Therefore, analogously to the case of the Anderson model, for H_β we can identify two regimes: W small corresponds to a strong disorder regime and W large, to a weak disorder regime. Indeed, for large W we have $\mathbb{E}^W[V_i] = 1/W \simeq 0$, and $\mathrm{Var}[V_i] = O(1/W) \simeq 0$, hence H_β is a small perturbation of $2d - P = -\Delta$. On the contrary, for small W both mean and variance are large, $\mathbb{E}^W[V_i] = 1/W$, and $\mathrm{Var}[V_i] \simeq 1/W^2$, hence H_β is dominated by the diagonal disorder.

Our main object of study is the integrated density of states (IDS)

$$N(E) = N(E, H_\beta)$$

for H_β at an energy $E \in \mathbb{R}$, defined by

$$N(E, H_\beta) = \lim_{L\to\infty} \mathbb{E}^W_{\Lambda_L}[N_{\Lambda_L}(E, H^\#_{\beta,\Lambda_L})], \tag{1.6}$$

where Λ_L is a box in $\mathbb{Z}^d$ of side $2L+1$ centered at zero and $N_{\Lambda_L}(E, H^\#_{\beta,\Lambda_L})$ is the finite volume IDS on Λ_L defined by

$$\begin{aligned} N_{\Lambda_L}(E, H^\#_{\beta,\Lambda_L}) &:= \frac{1}{|\Lambda_L|} \sum_{\lambda\in\sigma(H^\#_{\beta,\Lambda_L})\cap(-\infty,E]} 1 \\ &= \frac{1}{|\Lambda_L|} \mathrm{tr}(\mathbf{1}_{(-\infty,E]}(H^\#_{\beta,\Lambda_L})). \end{aligned} \tag{1.7}$$

Here, $\# \in \{D, S\}$ indicates if we have Dirichlet (see (1.5)) or simple boundary conditions. Note that the usual definition of the IDS (see e.g., [1, Corollary 3.16]) does not contain the expectation in the right-hand side of the equation (1.6). However, the equivalence between these two definitions follows from, e.g., [1, Lemma 4.12]. Also, the limiting function N does not depend on the boundary conditions in the finite-volume restriction of H_β to the box, so we can replace the simple boundary conditions with Dirichlet boundary conditions and the result still holds [1, Lemma 4.12].

We are interested in the asymptotics of $N(E)$ for $E \searrow 0$, that is, at the bottom of the spectrum of H_β. For a random Schrödinger operator with i.i.d. random potential, the Lifshitz tails estimate (e.g., [22]) claims that, near the bottom of the spectrum (assuming it is 0), the integrated density of states behaves like

$$N(E) = ce^{-E^{-\frac{1}{2}d+o(1)}} \tag{1.8}$$

in d dimensions. This is in stark contrast with the case of the free laplacian which exhibits Van Hove asymptotics, that is, $N(E) \asymp E^{d/2}$ (see [5, Theorem 3]). The exponential decay in (1.8) appears since, by the i.i.d. nature of the potential, configurations near the bottom of the spectrum are highly unlikely. Lifshitz tails also appear in models exhibiting correlations in the potential, for example in potentials given by a linear combination of i.i.d. random variables [19]. This behavior may be violated for example when the operator is not monotonous in the random variables [6,20,21,28], or the lattice is replaced by a random graph. In particular, the random Laplacian of the percolation subgraph of bond-percolation with parameter p on $\mathbb{Z}^d$, exhibits a transition between Van Hove and Lifshitz behavior depending of the percolation parameter p, see [26, 27].

For our H_β operator, the 1-dependence of the β variables entails that the number of realizations of the potential favoring low energy states is large and hence Lifshitz tails do not occur. Precisely, we will prove the following three results.

Theorem 1 (Lower bound on the IDS). *We define*

$$W_{\mathrm{cr}} = W_{\mathrm{cr}}(d) := \max\{W_c, W_c'\}, \tag{1.9}$$

where $W_c > 0$ (resp., $W_c' > 0$) is the (dimensional dependent) parameter introduced in Theorem 6 (resp., Theorem 8). In particular, $W_{\mathrm{cr}} = \infty$ for $d = 1$. Then, for each $0 < W < W_{\mathrm{cr}}$ there exist constants $c = c(W, d) > 0$ and $E_0 = E_0(W, d, c) > 0$ such that

$$N(E, H_\beta) \geq c(-\log E)^{-d}\sqrt{E}, \quad \textit{for all } 0 < E < E_0.$$

Actually, we will show in Lemma 10 below that $W_c'(d) > W_c(d)$ for all $d \geq 2$. Equation (2.12) yields $W_c'(d) \leq 0.1$.

The next result concerns the regularity of the finite volume IDS with simple/Dirichlet boundary condition defined in (1.7) above.

Theorem 2 (Wegner-type estimate). *For all $W > 0$, we have that the finite volume IDS $N_{\Lambda_L}(E, H^{\#}_{\beta,\Lambda_L})$, with $\# \in \{D, S\}$, satisfies the bound*

$$\mathbb{E}^{W,\eta^w}_{\Lambda_L}[N_{\Lambda_L}(E+\varepsilon, H^{\#}_{\beta,\Lambda_L}) - N_{\Lambda_L}(E-\varepsilon, H^{\#}_{\beta,\Lambda_L})] \leq 4\sqrt{\frac{W}{2\pi}}\sqrt{\varepsilon} \tag{1.10}$$

uniformly in Λ_L, $E \in \mathbb{R}$ and $\varepsilon > 0$.

Moreover, for $d \geq 3$, we define

$$W_0 = W_0(d) := \max\{W_0', 4^8\},$$

where $W_0' = W_0'(d) \geq 1$ is the parameter introduced in Theorem 27. Then, for all $W \geq W_0$, the following improved estimate holds:

$$\mathbb{E}^{W,\eta^w}_{\Lambda_L}[N_{\Lambda_L}(E+\varepsilon, H^{\#}_{\beta,\Lambda_L}) - N_{\Lambda_L}(E-\varepsilon, H^{\#}_{\beta,\Lambda_L})] \leq C\sqrt{W}\,\varepsilon \tag{1.11}$$

uniformly in Λ_L, $E \in \mathbb{R}$ and $\varepsilon > 0$, where $C > 0$ is some constant.

Theorem 3 (Upper bound and regularity for the IDS). *For all $W > 0$, the function $E \mapsto N(E, H_\beta)$ is Hölder continuous with exponent $1/2$ and Hölder seminorm*

$[N]_{C^{0,\frac{1}{2}}} \leq 2\sqrt{W/\pi}$. *In particular, it satisfies the bound*

$$N(E, H_\beta) \leq 2\sqrt{\frac{W}{\pi}}\sqrt{E} \quad \text{for all } E > 0.$$

Moreover, for $d \geq 3$ *and* $W \geq W_0$*, the function* $E \mapsto N(E)$ *is Lipschitz continuous with Lipschitz constant* $\mathrm{Lip}(N) \leq C\sqrt{W}/2$*, where* C *is the constant introduced in Theorem 2. In addition, it satisfies the bound*

$$N(E, H_\beta) \leq C'E \quad \text{for all } E > 0, \tag{1.12}$$

for some constant $C' > 0$ *independent of* W.

Discussion on the results and open questions

Behavior of the IDS near E = 0. Theorems 1 and 3 imply, in the strong disorder regime $W < W_{\mathrm{cr}}$ and for all dimension d,

$$c\frac{1}{|\ln E|^d}\sqrt{E} \leq N(E) \leq 2\sqrt{\frac{W}{\pi}}\sqrt{E}$$

as $E \searrow 0$. In particular, our results indicate that, for such a model, that is ergodic and features a 1-dependent random potential, Lifshitz tails do not emerge. Note that it is conjectured in [34] that, in the case of constant weights, the asymptotic behavior of the IDS of the operator H_β is $\sqrt{E}$. Therefore, we prove this conjecture in the strong disorder regime up to a logarithmic correction. We anticipate that this result will also apply to non-constant weights, provided they do not vary excessively and the relevant quantities can be defined.

Note that, for strong disorder $W \ll 1$, the random variables β are approximately iid with Gamma distribution (cf. the explicit Laplace transform in (1.2)). Therefore, one expects that

$$N(E, H_\beta) = N(EW, \mathcal{H}_\beta) \simeq \mathbb{P}(2\beta_0 < EW) \propto \sqrt{EW}$$

for $EW < 1$, which is indeed what we obtained.

For weak disorder $W \gg 1$, the random variables approach the constant value $2d$, hence one expects convergence to the IDS of $2d - P = -\Delta$ as $W \to \infty$ which is proportional to $E^{d/2}$. Note that the improved bound (1.12) is compatible with this expectation and moreover shows the IDS undergoes a phase transition at $d \geq 3$. A more precise comparison with $-\Delta$ would require also a lower bound for $N(E, H_\beta)$ at weak disorder, which is still missing. This leads to the following open question.

Open question 1. *What is the exact asymptotics of the IDS at the bottom of the spectrum at weak disorder* $W \gg 1$*, depending on the dimension? Does it approach the IDS of the Laplacian?*

Critical value of the disorder strength W. For dimension $d \geq 3$, we have proved a phase transition for the IDS in the following sense: for E near zero $N(E) \approx \sqrt{E}$ when $W \leq W_c'(d) \leq 0.1$ and $N(E) \leq C'E \ll \sqrt{E}$ when $W \geq W_0 > 1$. A natural question is whether this phase transition is unique. This would require to know the behaviour of $N(E)$ for intermediate values of W, which is at the moment out of reach for our techniques. This unicity is known in the case of the vertex reinforced jump process: in [29], Poudevigne-Auboiron proved that there is a unique transition point $W^*(d)$ between recurrence and transience for the vertex reinforced jump process with constant weights W on $\mathbb{Z}^d$ when $d \geq 3$. The proof uses a monotonicity property concerning $\mathcal{H}_\beta^{-1}$ that follows from a clever coupling (see also Theorem 24 in the appendix) and the 0–1 law in [34, Proposition 3]. As the operator $\mathcal{H}_\beta$ is crucial to define the random environment of the vertex reinforced jump process, the following question arises.

Open question 2. *Is the phase transition for the density of states of $H_\beta = \mathcal{H}_\beta / W$ unique and does it occur at the same value $W^*(d)$?*

The strong disorder behavior $N(E) \approx \sqrt{E}$ corresponds to the recurrence region in the vertex reinforced jump process. A sufficient condition to obtain this behavior is

$$\mathbb{E}_\Lambda^{W,\eta^w}[|\mathcal{H}_{\beta,\Lambda}^{-1}(0,i)|^s] \leq e^{-c|i|}$$

for some $s \in (0,1)$ uniformly in Λ. At the moment, we only have this bound for $W < W_c'(d) \leq 0.1$.

On the other hand, a sufficient condition to obtain the weak disorder behavior $N(E) \leq C'E$, which corresponds to the transience region in the vertex reinforced jump process, is to prove the L^2-integrability of the martingale $(\psi_L)_{L \in \mathbb{N}}$, defined in Section 6. For $W > W_0(d)$ and $d \geq 3$, this bound follows from Lemma 28. Note that, if L^2-integrability was equivalent to transience on $\mathbb{Z}^d$, then the bound $N(E) \leq C_W E$, for some constant C_W, would hold for all $W > W^*(d)$. In the case of trees, this equivalence was proved in [30]. However, on $\mathbb{Z}^d$ only uniform integrability of $(\psi_L)_{L \in \mathbb{N}}$ for $W > W^*(d)$ has been proven so far (see [30]). This result suggests another possible scenario, with an intermediate phase where the vertex reinforced jump process is transient but $(\psi_L)_{L \in \mathbb{N}}$ is not bounded in L^2 and may correspond to an intermediate phase for the behaviour of the density of states of H_β too. This could also be consistent with the presence of a phase where H_β exhibits singular continuous spectrum. The existence of singular continuous spectrum for certain random Schrödinger operators is conjectured to be true in high dimension by a growing community of physicists (see for example [2]).

The Anderson transition. Sabot and Zeng conjecture in [34] that the phase transition in linearly reinforced random processes between recurrent and transient regimes is related to the dynamical localization and delocalization transition of H_β. This provides an additional motivation for the study of this random operator, since the localization-delocalization transition for random Schrödinger operators is a long-standing open problem in the theory of disordered systems, going back to the seminal work of P. W. Anderson [3]. As in the standard Anderson model, the operator H_β exhibits dynamical localization for strong disorder $W \leq W'(d)$, see [7]. This motivates the following question.

Open question 3. *Does dynamical localization for H_β hold at spectral band edges for any fixed disorder parameter $W > 0$, in particular at weak disorder?*

This is so far out of reach, since the stardard proof relies on the Liftschitz tails of the IDS, which are absent in our case.

Theorem 1: Strategy of the proof. We argue in three steps.

Step 1. By standard arguments (see Section 3.1), we have, for all $L \geq 1$,

$$N(E, H_\beta) \geq \mathbb{E}_{\Lambda_L}^{W,\eta^w}[N_{\Lambda_L}(E, H_{\beta,\Lambda_L}^D)] \geq \frac{1}{|\Lambda_L|}\nu_{\Lambda_L}^{W,\eta^w}\Big((H_{\beta,\Lambda_L}^D)^{-1}(0,0) \geq \frac{1}{E}\Big),$$

where H_{β,Λ_L}^D was defined in (1.5) and $N_{\Lambda_L}(E, H_{\beta,\Lambda_L}^D)$ is the finite volume IDS with Dirichlet boundary condition defined in (1.7).

Step 2. We have no direct information on the probability density of $(H_{\beta,\Lambda_L}^D)^{-1}(0,0)$, but we do have detailed information on the distribution of

$$(H_{\beta,\Lambda_L}^S)^{-1}(0,0) = H_{\beta,\Lambda_L}^{-1}(0,0).$$

In Section 3.2, we show that, for $E \leq 1/2$, $W < W_{\rm cr}$, and $L > 1$ large enough, we have

$$\Big\{(H_{\beta,\Lambda_L}^S)^{-1}(0,0) > \frac{1}{E}\Big\} \implies \Big\{(H_{\beta,\Lambda_L}^D)^{-1}(0,0) > \frac{1}{2E}\Big\},$$

on a configuration set $\Omega_{\rm loc}$ of probability close to one, precisely $1 - e^{-\kappa L}$, for some positive constant $\kappa > 0$. Hence,

$$N(E, H_\beta) \geq \frac{1}{|\Lambda_L|}\nu_{\Lambda_L}^{W,\eta^w}\Big(\Omega_{\rm loc} \cap \Big\{H_{\beta,\Lambda_L}^{-1}(0,0) \geq \frac{1}{2E}\Big\}\Big).$$

Step 3. The conditional density of $y := 1/\mathcal{H}_{\beta,\Lambda_L}^{-1}(0,0) = W/H_{\beta,\Lambda_L}^{-1}(0,0)$, knowing $\beta_{0^c} = (\beta_j)_{j\in\Lambda_L\setminus\{0\}}$, is denoted by $d\rho_{a_0}$ and explicitly given in (4.2). All dependence on β_{0^c} is contained in the parameter a_0, defined in (4.3), which contains the two-point Green's function of the ground state of $H_{\beta,\Lambda\setminus\{0\}}$. Therefore,

$$\mathbb{E}_{\Lambda_L}^{W,\eta^w}[\mathbf{1}_{\Omega_{\rm loc}}\mathbf{1}_{\{H_{\beta,\Lambda_L}^{-1}(0,0)\geq\frac{1}{2E}\}}] = \mathbb{E}_{\Lambda_L}^{W,\eta^w}\Big[\int \mathbf{1}_{\Omega_{\rm loc}}\mathbf{1}_{\{y\leq 2EW\}}d\rho_{a_0(\beta_{0^c})}(y)\Big].$$

A subtle point is to show that we can choose Ω_{loc} as intersection of two events $\Omega_{\mathrm{loc}} = \Omega_{loc,0} \cap \Omega_{loc,1}$, where $\Omega_{loc,0}$ is measurable with respect to β_{0^c} and $\Omega_{loc,1} = \{y \geq e^{-\kappa L} W\}$ is measurable with respect to y. As a result, we can write

$$\mathbb{E}_{\Lambda_L}^{W,\eta^w}\left[\int \mathbf{1}_{\Omega_{\mathrm{loc}}} \mathbf{1}_{\{y \leq 2WE\}} d\rho_{a_0(\beta_{0^c})}(y)\right] = \mathbb{E}_{\Lambda_L}^{W,\eta^w}\left[\mathbf{1}_{\Omega_{loc,0}} \int_{e^{-\kappa L} W}^{2WE} d\rho_{a_0(\beta_{0^c})}(y)\right].$$

The set $\Omega_{loc,0}$ ensures that $a_0(\beta_{0^c}) \leq e^{-\kappa L/2}$ for all $\beta_{0^c} \in \Omega_{loc,0}$. Then, using the explicit form of ρ_a and the fact that $\Omega_{loc,0}$ has probability close to one, we get

$$\mathbb{E}_{\Lambda_L}^{W,\eta^w}\left[\mathbf{1}_{\Omega_{loc,0}} \int_{e^{-\kappa L} W}^{2WE} d\rho_{a_0(\beta_{0^c})}(y)\right] \geq (1 - ce^{-\kappa L}) \rho_{e^{-\kappa L/2}}(e^{-\kappa L} W \leq y \leq 2WE).$$

The result now follows from a direct analysis of the one-dimensional measure ρ_a. The details are explained in Section 4.

Theorem 2: Strategy of the proof. Note that $H_{\beta,\Lambda_L} \pm 2\varepsilon = H_{\beta \pm \varepsilon,\Lambda_L}$. Following a standard argument, we construct a sequence of potentials interpolating between $\beta + \varepsilon$ and $\beta - \varepsilon$, by switching ε one site at the time. As a result, we obtain the following estimate (cf. Lemma 17):

$$\begin{aligned} &\mathbb{E}_{\Lambda_L}^{W,\eta^w}[N_{\Lambda_L}(E + \varepsilon, H_{\beta,\Lambda_L}^{\#}) - N_{\Lambda_L}(E - \varepsilon, H_{\beta,\Lambda_L}^{\#})] \\ &\leq \frac{4}{|\Lambda_L|} \sum_{j \in \Lambda} \mathbb{E}_{\Lambda_L}^{W,\eta^w}[\mathcal{L}_{\rho_{a_j(\beta_{j^c})}}(4W\varepsilon)] \quad \text{for all } L > 1, \end{aligned}$$

where $\mathcal{L}_{\rho_{a_j}}$ denotes the Lévy concentration (defined in (5.2)) of the conditional measure ρ_{a_j}. Using the explicit formula for ρ_a, we then show that $\mathcal{L}_{\rho_a}(\varepsilon) \leq c\sqrt{\varepsilon}$ for some constant $c > 0$ independent of a (cf. (5.3)), which gives the first result. For $d \geq 3$, we bound the conditional density pointwise by

$$\rho_a(y) \leq \frac{1}{\sqrt{2\pi}}\left(\frac{1}{a} + \frac{1}{\sqrt{a}}\right) \quad \text{for all } y > 0.$$

The result now follows from the bound (cf. Lemma (19))

$$\mathbb{E}_{\Lambda_L}^{W,\eta^w}\left[\frac{1}{a}\right] = \mathbb{E}_{\Lambda_L}^{W,\eta^w}[e^{-u_0} \mathcal{H}_{\beta,\Lambda_L}^{-1}(0,0)] \leq \frac{C_d}{W},$$

where $C_d > 0$ is a constant depending only on the dimension.

Theorem 3: Strategy of the proof. The regularity bounds follow directly from the Wegner estimate and (1.6) by replacing E and ε by $E/2$. On the contrary, the improved upper bound (1.12) is proved in Proposition 21 using properties of the infinite volume distribution of β.

Organization of this paper. In Section 2, we review some definitions and known results, and derive the modifications of these results that will be needed in the rest of the paper. A few additional technical results that we will also use in this section are summarized in Appendix A. Section 3 covers the first two steps in the proof of Theorem 1. The final step is worked out in Section 4. Section 5 contains the proof of Theorem 2. Some estimates on the field u associated to the $H^{2|2}$-model (see (2.4) below) which are necessary for the proof, but are also interesting in their own, are collected in Appendix B. Note that all these proofs involve only properties of the *finite volume* marginal distribution but some of the above results can be recovered by exploiting properties of the *infinite volume* distribution. The main ideas are sketched in Section 6, while the detailed construction can be found in [30]. This alternative approach also provides the improved bound (1.12) in Theorem 3. All corresponding details are given in Section 6.

2. Some previous results on the $\mathcal{H}_\beta$ operator

As mentioned in the introduction, the operator $\mathcal{H}_\beta$ has been studied in the literature in connection with linearly reinforced random processes and the $H^{2|2}$ sigma-model. In this section, we collect some tools and results that we will use in the next sections. The following theorem can be found in [24, 33, 34].

Theorem 4 (Multivariate inverse Gaussian distribution). *Let $\mathcal{G} = (V, E)$ be a finite graph. For any $W \in \mathbb{R}^E_{>0}$, $\theta \in \mathbb{R}^V_{>0}$ and $\eta \in \mathbb{R}^V_{\geq 0}$, the following holds:*

$$\int_{\mathcal{H}_{\beta,V}>0} e^{-\frac{1}{2}(\langle \theta, \mathcal{H}_{\beta,V}\theta\rangle + \langle \eta, \mathcal{H}_{\beta,V}^{-1}\eta\rangle - 2\langle \theta,\eta\rangle)} \frac{\prod_i \theta_i}{\sqrt{\det \mathcal{H}_{\beta,V}}} \left(\frac{2}{\pi}\right)^{\frac{1}{2}|V|} d\beta = 1,$$

where $\mathcal{H}_{\beta,V} := 2\beta - P^W \in \mathbb{R}^{V\times V}$. We denote by $\nu_{\mathcal{G}}^{W,\theta,\eta}$ the probability defined by the above integral, in particular, $\nu_{\mathcal{G}}^{W,\eta} = \nu_{\mathcal{G}}^{W,\theta,\eta}$ with $\theta \equiv 1$. The associated expectations are denoted by $\mathbb{E}_{\mathcal{G}}^{W,\theta,\eta}$ and the Laplace transform is given by

$$\begin{aligned} &\mathbb{E}_{\mathcal{G}}^{W,\theta,\eta}(e^{-\langle\lambda,\beta\rangle}) \\ &\quad = e^{-\sum_{i,j\in V, i\sim j} W_{i,j}(\sqrt{(\theta_i^2+\lambda_i)(\theta_j^2+\lambda_j)}-\theta_i\theta_j) - \sum_{i\in V}\eta_i(\sqrt{\theta_i^2+\lambda_i}-\theta_i)} \prod_{i\in V} \frac{\theta_i}{\sqrt{\theta_i^2+\lambda_i}}. \end{aligned} \tag{2.1}$$

Moreover, if $(\beta_i)_{i\in V}$ is distributed according to $\nu_{\mathcal{G}}^{W,\theta,0}$, and $\mathcal{G}' = (V', E')$ is the subgraph obtained by taking $V' \subset V$ and $E' := \{\{i,j\} \in E \mid i,j \in V'\}$, then the marginal law of $(\beta_i)_{i\in V'}$ is $\nu_{\mathcal{G}'}^{W',\theta',\eta}$, where W', θ' equal W, θ restricted on Λ', and η, is defined by $\eta_i = \sum_{j\in V\setminus V'} W_{i,j}\theta_j$. Note that η is a generalization of $\eta_{V'}^w$, defined in (1.4).

In the sequel of the paper, we will always assume that $\theta \equiv 1$ and we will use the notation $\nu_{\mathcal{G}}^{W,\eta}$ instead of $\nu_{\mathcal{G}}^{W,1,\eta}$.

Remarks on various marginals. Note that (see, e.g., [7, Remark 3.5]), in the case $\eta \equiv 0$, for any $i \in V$,

$$\gamma := \frac{1}{2\mathcal{H}_{\beta,V}^{-1}(i,i)} \text{ has density } \mathbf{1}_{\gamma>0}\frac{1}{\sqrt{\pi\gamma}}e^{-\gamma}, \tag{2.2}$$

i.e., it is a Gamma random variable of parameter $1/2$. This holds for any W and any finite graph $\mathcal{G}$.

If we consider a box Λ_L of side $2L+1$ in $\mathbb{Z}^d$, a Borel function $f\colon \mathbb{R}^{\Lambda_L} \to \mathbb{R}$, and $\Lambda_L \subset \Lambda' \subset\subset \mathbb{Z}^d$, then

$$\mathbb{E}_{\Lambda_L}^{W,\eta^w_{\Lambda_L}}[f(\beta_{\Lambda_L})] = \mathbb{E}_{\Lambda_{L+1}}^{W,0}[f(\beta_{\Lambda_L})] = \mathbb{E}_{\Lambda_L\cup\{\delta\}}^{W,0}[f(\beta_{\Lambda_L})] = \mathbb{E}_{\Lambda'}^{W,\eta^w_{\Lambda'}}[f(\beta_{\Lambda_L})], \tag{2.3}$$

where $\eta^w_{\Lambda_L}, \eta^w_{\Lambda'}$ are given in (1.4), the graph $\Lambda_L \cup \delta$ has vertex set $\Lambda_L \cup \{\delta\}$ and edge set $E(\Lambda_L) \cup \{\{i,\delta\} \mid i \in \Lambda_L\}$, and we defined $W_{i\delta} = \eta^w_{\Lambda_L}(i)$, for all $i \in \Lambda_L$. Note that η can be seen as the boundary condition of the law of the random potential. The case $\eta \equiv 0$ is called *zero boundary condition.*

Connection with the $H^{2|2}$ model. Let $\mathcal{G}$ be the graph associated to a box Λ of $\mathbb{Z}^d$ and $\eta \in [0,\infty)^\Lambda$ with at least one strictly positive component. The following expression defines a probability measure for $u \in \mathbb{R}^\Lambda$ (cf. [11]):

$$\mu_\Lambda^{W,\eta}(u) = e^{-\sum_{i\sim j, i,j\in\Lambda} W_{ij}(\cosh(u_i-u_j)-1)} e^{-\sum_{j\in\Lambda}\eta_j(\cosh u_j - 1)} \sqrt{\det \mathcal{H}_{\beta(u),\Lambda}}\, \frac{1}{\sqrt{2\pi}^{|\Lambda|}} du_\Lambda, \tag{2.4}$$

where we defined

$$2\beta_i(u) = \sum_{j\in\Lambda} W_{ij} e^{u_j-u_i} + \eta_i e^{-u_i} \quad \text{for all } i \in \Lambda. \tag{2.5}$$

The corresponding average is denoted by $\mathbb{E}_{u,\Lambda}^{W,\eta}$. Note that the measure $\mu_\Lambda^{W,\eta}(u)$ is also the effective bosonic field measure in [12, ySection 2.3], which is obtained as a marginal of the $H^{2|2}$ measure after inserting horospherical coordinates.

The next lemma connects $\mu_\Lambda^{W,\eta}(u)$ with $\nu_\Lambda^{W,\eta}(\beta)$ and can be found in [33, Proposition 2 and Theorem 3].

Lemma 5 (Connection to $H^{2|2}$). *Let $\mathcal{G}$ be the graph associated to a box Λ of $\mathbb{Z}^d$ and $\eta \in [0,\infty)^\Lambda$ with at least one strictly positive component. It holds*

$$\mathbb{E}_\Lambda^{W,\eta}[f(\beta_\Lambda)] = \mathbb{E}_{u,\Lambda}^{W,\eta}[f(\beta_\Lambda(u))] \tag{2.6}$$

for any function f integrable with respect to the measure $\nu_\Lambda^{W,\eta}$.

Moreover, remembering that $\nu_{\Lambda}^{W,\eta}$ corresponds to the marginal of $\nu_{\Lambda\cup\delta}^{W,0}$ with $W_{j,\delta} = \eta_j$ for all $j \in \Lambda$ (cf. equation (2.3)), it holds

$$e^{u_i} = \frac{\mathcal{H}_{\beta,\Lambda\cup\delta}^{-1}(i,\delta)}{\mathcal{H}_{\beta,\Lambda\cup\delta}^{-1}(\delta,\delta)} \quad \textit{for all } i \in \Lambda,$$

where the above fraction is independent of β_δ. In particular, we have

$$\mathbb{E}_{u,\Lambda}^{W,\eta}[f(u)] = \mathbb{E}_{\Lambda\cup\delta}^{W,0}[f(u(\beta))] = \mathbb{E}_{\Lambda}^{W,\eta}[f(u(\beta_\Lambda))].$$

Note that, by the resolvent identity, we have, for all $j \in \Lambda$,

$$\begin{aligned}\mathcal{H}_{\beta,\Lambda\cup\delta}^{-1}(j,\delta) &= 0 + \sum_{k\in\Lambda} \mathcal{H}_{\beta,\Lambda}^{-1}(j,k) W_{k,\delta} \mathcal{H}_{\beta,\Lambda\cup\delta}^{-1}(\delta,\delta) \\ &= \mathcal{H}_{\beta,\Lambda\cup\delta}^{-1}(\delta,\delta) \sum_{k\in\Lambda} \mathcal{H}_{\beta,\Lambda}^{-1}(j,k)\eta_k = \mathcal{H}_{\beta,\Lambda\cup\delta}^{-1}(\delta,\delta)\ (\mathcal{H}_{\beta,\Lambda}^{-1}\eta)(j),\end{aligned}$$

and hence

$$e^{u_j} = \frac{\mathcal{H}_{\beta,\Lambda\cup\delta}^{-1}(j,\delta)}{\mathcal{H}_{\beta,\Lambda\cup\delta}^{-1}(\delta,\delta)} = (\mathcal{H}_{\beta,\Lambda}^{-1}\eta)(j). \tag{2.7}$$

It follows

$$\mathbb{E}_{u,\Lambda}^{W,\eta}[f(e^{u_j - u_{j'}})] = \mathbb{E}_{\Lambda}^{W,\eta}\Big[f\Big(\frac{(\mathcal{H}_{\beta,\Lambda}^{-1}\eta)(j)}{(\mathcal{H}_{\beta,\Lambda}^{-1}\eta)(j')}\Big)\Big].$$

for any function f, as long as the left and right-hand side are well defined. In the special case $\eta_j = \eta\delta_{jj_0}$ (pinning at one point), the formula simplifies to

$$e^{u_j - u_{j_0}} = \frac{\mathcal{H}_{\beta,\Lambda}^{-1}(j,j_0)}{\mathcal{H}_{\beta,\Lambda}^{-1}(j_0,j_0)} = \frac{\mathcal{H}_{\beta,\Lambda}^{-1}(j_0,j)}{\mathcal{H}_{\beta,\Lambda}^{-1}(j_0,j_0)}. \tag{2.8}$$

With these notations, we can translate [11, Theorems 1 and 2] into the following.

Theorem 6 (Decay of the ground state Green's function (1)). *Let Λ_L be a finite box of side $2L+1$ in $\mathbb{Z}^d$, and $\eta \in [0,\infty)^\Lambda$ with at least one strictly positive component. We define*

$$I_W := \sqrt{W} \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} e^{-W(\cosh t - 1)}$$

and $W_c > 0$ as the unique solution of $I_{W_c} e^{W_c(2d-2)}(2d-1) = 1$. Then, for all $0 < W < W_c$, $I_W e^{W(2d-2)}(2d-1) < 1$, and for $d = 1$, we have $W_c = +\infty$. Finally, set $C_0 = C_0(W) := 2e^{2W}(1 - I_W e^{W(2d-2)}(2d-1))^{-1}$.

The following holds.

(i) *For all $i, j \in \Lambda$ such that $\eta_i > 0$, $\eta_j > 0$, we have*

$$\mathbb{E}^{W,\eta}_{\Lambda_L}[\mathcal{H}^{-1}_{\beta,\Lambda_L}(i,j)] \\ \leq C_0 e^{\sum_{k\in\Lambda_L}\eta_k}(\eta_i^{-1}+\eta_j^{-1})[I_W e^{W(2d-2)}(2d-1)]^{|i-j|}, \tag{2.9}$$

where $|i-j|$ is the graph distance between i, j on $\mathbb{Z}^d$.

(ii) *Assume there is only one pinning at $j_0 \in \Lambda_L$, i.e. $\eta_j = \eta\delta_{jj_0}$. Then, for all $j \in \Lambda_L$,*

$$\mathbb{E}^{W,\eta}_{\Lambda_L}\left[\sqrt{\frac{\mathcal{H}^{-1}_{\beta,\Lambda_L}(j_0,j)}{\mathcal{H}^{-1}_{\beta,\Lambda_L}(j_0,j_0)}}\right] \leq C_0(I_W e^{W(2d-2)}(2d-1))^{|j-j_0|}. \tag{2.10}$$

How the results above follow from [11]. Note that in [11] η is called ε and it is assumed that $\sum_{k\in\Lambda}\varepsilon_k \leq 1$. This implies that the term $e^{\sum_{k\in\Lambda}\varepsilon_k}$ in [11, equation (2.24)] is bounded by e^1, which is absorbed by the global constant C_0 in [11, equation (1.18)]. Also, although in [11] the global constant C_0 is not given explicitely, it follows directly from the second line of [11, equation (2.20)], so we have given its precise value here.

Note that statement (ii) above is slightly different from the one in [11, Theorem 2]. Namely, in [11] the pinning point is set to $j_0 = 0$ and the observable is $e^{(1/2)u_j}$, while, by (2.8), the observable in (2.10) is $e^{(1/2)(u_j-u_{j_0})}$. Since the two observables differ only by a function of u_{j_0} (the variable at the pinning point), the proof for this modified observable is identical to the one in [11] up to the final line in [11, equation (3.5)], where the integral I_{ε_0} is replaced by

$$\sqrt{\varepsilon_0/(2\pi)}\int_{\mathbb{R}} dt e^{-(1/2)t} e^{-\varepsilon_{j_0}(\cosh t-1)} = 1.$$

As a consequence, the bound in (2.10) is written in terms of the same constant C_0 used in (2.9) and is independent from the pinning stregth ε_{j_0}, while the bound in [11] does depend on ε_{j_0} via the function $I_{\varepsilon_{j_0}}$.

Remark. The function $W \mapsto I_W$ is monotone increasing (cf. [11, Remark 1 after Theorem 1]). Therefore, the function $W \mapsto F_d(W) := I_W e^{W(2d-2)}(2d-1)$ is also monotone increasing and W_c is well defined.

In this article, we will use the following extension of Theorem 6.

Theorem 7 (Decay with wired bc (1)). *Let Λ_L be a finite box of side $2L+1$ in $\mathbb{Z}^d$. We consider $(\beta_i)_{i\in\Lambda_L} \sim \nu^{W,\eta^w}_{\Lambda_L}$, where $\eta^w_{\Lambda_L}$ is the wired boundary condition introduced in* (1.4)*. Let W_c be as in Theorem 6 above.*

For all $0 < W < W_c$, $j, j_0 \in \Lambda$, *we have*

$$\mathbb{E}_{\Lambda_L}^{W,\eta^w}\left[\sqrt{\frac{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j)}{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j_0)}}\right] \le C_0 e^{-\kappa|j-j_0|}. \tag{2.11}$$

where

$$\kappa = \kappa(W,d) := -\log(I_W e^{W(2d-2)}(2d-1)) > 0,$$

and $C_0 = C_0(W)$ *is the constant introduced in Theorem 6. In particular, in* $d = 1$, *we have* $W_c = \infty$, *hence the bound holds for all* $W > 0$.

Proof. By (2.3), we have

$$\mathbb{E}_{\Lambda_L}^{W,\eta^w}\left[\sqrt{\frac{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j)}{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j_0)}}\right] = \mathbb{E}_{\Lambda_{L+1}}^{W,0}\left[\sqrt{\frac{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j)}{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j_0)}}\right].$$

By a random walk representation (cf. [34, Proposition 6 and notations therein]),

$$\frac{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j)}{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j_0)} = \sum_{\sigma\in\overline{\mathcal{P}}_{jj_0}^{\Lambda_L}} \frac{W_\sigma}{(2\beta)_\sigma^-}$$

where $\overline{\mathcal{P}}_{j_0 j}^{\Lambda_L}$ is the set of nearest neighbor paths from j_0 to j in Λ_L that visit j_0 only once. Moreover, for every path σ,

$$W_\sigma = \prod_{k=0}^{|\sigma|-1} W_{\sigma_k,\sigma_{k+1}} \text{ and } (2\beta)_\sigma^- = \prod_{k=0}^{|\sigma|-1} (2\beta_{\sigma_k}).$$

It follows, for all $j, j_0 \in \Lambda_L$,

$$\frac{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j)}{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j_0)} \le \frac{\mathcal{H}_{\beta,\Lambda_{L+1}}^{-1}(j_0,j)}{\mathcal{H}_{\beta,\Lambda_{L+1}}^{-1}(j_0,j_0)},$$

since in the term in the right-hand side contains more paths. Hence,

$$\begin{aligned}\mathbb{E}_{\Lambda_L}^{W,\eta^w}\left[\sqrt{\frac{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j)}{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j_0)}}\right] &= \mathbb{E}_{\Lambda_{L+1}}^{W,0}\left[\sqrt{\frac{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j)}{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j_0)}}\right] \\ &\le \mathbb{E}_{\Lambda_{L+1}}^{W,0}\left[\sqrt{\frac{\mathcal{H}_{\beta,\Lambda_{L+1}}^{-1}(j_0,j)}{\mathcal{H}_{\beta,\Lambda_{L+1}}^{-1}(j_0,j_0)}}\right].\end{aligned}$$

By the monotonicity result [29, Theorem 6] (cf. Corollary 25 in Appendix A), we have, setting $\eta_j = W\delta_{jj_0}$ for all $j \in \Lambda_{L+1}$,

$$\mathbb{E}^{W,0}_{\Lambda_{L+1}}\left[\sqrt{\frac{\mathcal{H}^{-1}_{\beta,\Lambda_{L+1}}(j_0,j)}{\mathcal{H}^{-1}_{\beta,\Lambda_{L+1}}(j_0,j_0)}}\right] \le \mathbb{E}^{W,\eta}_{\Lambda_{L+1}}\left[\sqrt{\frac{\mathcal{H}^{-1}_{\beta,\Lambda_{L+1}}(j_0,j)}{\mathcal{H}^{-1}_{\beta,\Lambda_{L+1}}(j_0,j_0)}}\right]$$
$$= \mathbb{E}^{W,\eta}_{u,\Lambda_{L+1}}[\sqrt{e^{u_j-u_{j_0}}}] \le C_0 e^{-\kappa|j-j_0|},$$

with $\kappa = -\log(I_W e^{W(2d-2)}(2d-1)) > 0$, and $C_0 = 2e^{2W}$. In the last step, we used Lemma 5 and Theorem 6 (ii). ■

Another useful result on the decay of the ground state Green's function is [7, equation (5.4) in Theorem 2.1]. We state this result for our applications.

Theorem 8 (Decay of the ground state Green's function (2)). *Let Λ_L be a finite box of side $2L+1$ in $\mathbb{Z}^d$, and define*

$$W'_c = W'_c(d) := \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)2^{\frac{3}{4}}d}. \tag{2.12}$$

Let $(\beta_i)_{i\in\Lambda_L} \sim \nu^{W,0}_{\Lambda_L}$. Then, for all $0 < W < W'_c$, there are constants $\kappa' = \kappa'(d,W)$ and $C'_0(d,W)$ such that, for any $i,j \in \Lambda_L$,

$$\mathbb{E}^{W,0}_{\Lambda_L}[\mathcal{H}^{-1}_{\beta,\Lambda_L}(i,j)^{\frac{1}{4}}] \le C'_0 e^{-\kappa'|i-j|}.$$

In this paper, we will use the following corollary of the above result.

Corollary 9 (Decay with wired b.c. (2)). *Let Λ_L be a finite box of side $2L+1$ in $\mathbb{Z}^d$, and let $(\beta_i)_{i\in\Lambda_L} \sim \nu^{W,\eta^w}_{\Lambda_L}$ where $\eta^w_{\Lambda_L}$ is the wired boundary condition introduced in (1.4). Remember the definition of W'_c in (2.12).*

For all $0 < W < W'_c$, there are constants $\kappa' = \kappa'(W,d) > 0$ and $C'_0(W,d) > 0$ such that, for any $i,j \in \Lambda_L$,

$$\mathbb{E}^{W,\eta^w}_{\Lambda_L}[\mathcal{H}^{-1}_{\beta,\Lambda_L}(i,j)^{\frac{1}{4}}] \le C'_0 e^{-\kappa'|i-j|}. \tag{2.13}$$

In an abuse of notation, in the rest of the paper we will write C_0 for the constant in both decay results in Theorem 6(ii) and Theorem 8.

Proof of Corollary 9. By (2.3), we have

$$\mathbb{E}^{W,\eta^w}_{\Lambda_L}[\mathcal{H}^{-1}_{\beta,\Lambda_L}(i,j)^{\frac{1}{4}}] = \mathbb{E}^{W,0}_{\Lambda_{L+1}}[\mathcal{H}^{-1}_{\beta,\Lambda_L}(i,j)^{\frac{1}{4}}].$$

By random walk representation (cf. [34, Proposition 6 and notations therein]),

$$\mathcal{H}^{-1}_{\beta,\Lambda_L}(i,j) = \sum_{\sigma\in\mathcal{P}^{\Lambda_L}_{ij}} \frac{W_\sigma}{(2\beta)_\sigma},$$

where $\mathcal{P}_{ij}^{\Lambda_L}$ is the set of nearest neighbor paths from j_0 to j in Λ_L. It follows, for all $i, j \in \Lambda_L$,

$$\mathcal{H}_{\beta,\Lambda_L}^{-1}(i,j) \le \mathcal{H}_{\beta,\Lambda_{L+1}}^{-1}(i,j) \tag{2.14}$$

since in the second term we have more paths. Therefore,

$$\begin{aligned}\mathbb{E}_{\Lambda_L}^{W,\eta^w}[\mathcal{H}_{\beta,\Lambda_L}^{-1}(i,j)^{\frac{1}{4}}] &= \mathbb{E}_{\Lambda_{L+1}}^{W,0}[\mathcal{H}_{\beta,\Lambda_L}^{-1}(i,j)^{\frac{1}{4}}] \\ &\le \mathbb{E}_{\Lambda_{L+1}}^{W,0}[\mathcal{H}_{\beta,\Lambda_{L+1}}^{-1}(i,j)^{\frac{1}{4}}] \le C_0' e^{-\kappa'|i-j|},\end{aligned}$$

where in the last step we applied Theorem 8. ∎

Comparing W_c and W_c'. In Section 3.2, we will construct two sets of measure close to one Ω_1 (resp Ω_2) using Theorem 7 (resp., Corollary 9). Both sets can be used to construct the same lower bound on the IDS (cf. Section 4), which will be valid for all $W < W_c = W_c(d)$ (resp., for all $W < W_c' = W_c'(d)$), if we use Ω_1 (resp., Ω_2).

It is then reasonable to ask which result works for a larger set of parameters W, i.e., which of the two critical values is larger. While we have an explicit numeric expression for $W_c'(d)$, $W_c(d)$ is only indirectly determined as the unique solution of $F_d(W) = I_W e^{W(2d-2)}(2d-1) = 1$ (cf. the remark before Theorem 7). The next lemma shows that $W_c(d) < W_c'(d)$ for all $d \ge 2$.

Lemma 10. *For $d = 1$, $W_c(1) = \infty > W_c'(1)$. For $d \ge 2$, $W_c(d) < W_c'(d)$.*

Proof. Recalling the definition of modified Bessel function of the second kind

$$K_\alpha(x) := \int_0^\infty \cosh(\alpha t) e^{-x\cosh t}\, dt,$$

we have

$$I_W = 2e^W \sqrt{\frac{W}{2\pi}} K_0(W)$$

and

$$F_d(W) = \sqrt{\frac{2W}{\pi}} K_0(W) e^{W(2d-1)}(2d-1).$$

Note that, from (2.12), $W_c'(d) = C/d$, where C is a constant independent of d, and hence $f(d) := F_d(W_c'(d)) = F_d(C/d)$. Moreover,

$$\partial_W F_d(W) = \Big(\frac{1}{2W} + (2d-1) - \frac{K_1(W)}{K_0(W)}\Big) F_d(W),$$

$$\partial_d F_d(W) = \Big(2W + \frac{2}{2d-1}\Big) F_d(W).$$

It follows

$$f'(d) = -\frac{C}{d^2}\partial_W F_d\Big(\frac{C}{d}\Big) + \partial_d F_d\Big(\frac{C}{d}\Big)$$
$$= F_d\Big(\frac{C}{d}\Big)\Big[\frac{2d+1}{2d(2d-1)} + \frac{C}{d^2} + \frac{C}{d^2}\frac{K_1(\frac{C}{d})}{K_0(\frac{C}{d})}\Big] > 0,$$

which proves that f is monotone increasing.

We compute numerically $F_2(W_c'(2)) \approx 2.908$, hence $W_c(2) < W_c'(2)$. As f is increasing, we have $F_d(W_c'(d)) \geq F_2(W_c'(2)) \geq 2.9$, which implies $W_c'(d) > W_c(d)$ for all $d \geq 2$. ∎

Before going to the proof of the lower bound, we list an additional useful corollary on the probability distribution of $\mathcal{H}^{-1}_{\beta,\Lambda_L}(0,0)$.

Corollary 11. *Let Λ_L be a finite box of side $2L+1$ in $\mathbb{Z}^d$, and let $(\beta_i)_{i\in\Lambda_L} \sim \nu^{W,\eta^w}_{\Lambda_L}$, where $\eta^w_{\Lambda_L}$ is the wired boundary condition introduced in (1.4).*

Then, for any $\delta > 0$, we have

$$\nu^{W,\eta^w}_{\Lambda_L}(\mathcal{H}^{-1}_{\beta,\Lambda_L}(0,0) > \delta) \leq \int_0^{1/(2\delta)} \frac{1}{\sqrt{\pi\gamma}} e^{-\gamma}\, d\gamma.$$

Proof. We argue

$$\nu^{W,\eta^w}_{\Lambda_L}(\mathcal{H}^{-1}_{\beta,\Lambda_L}(0,0) > \delta) = \nu^{W,0}_{\Lambda_{L+1}}(\mathcal{H}^{-1}_{\beta,\Lambda_L}(0,0) > \delta).$$

By (2.14), $\mathcal{H}^{-1}_{\beta,\Lambda_{L+1}}(0,0) \geq \mathcal{H}^{-1}_{\beta,\Lambda_L}(0,0)$, and hence

$$\mathcal{H}^{-1}_{\beta,\Lambda_L}(0,0) > \delta \implies \mathcal{H}^{-1}_{\beta,\Lambda_{L+1}}(0,0) > \mathcal{H}^{-1}_{\beta,\Lambda_L}(0,0) > \delta.$$

It follows

$$\nu^{W,0}_{\Lambda_{L+1}}(\mathcal{H}^{-1}_{\beta,\Lambda_L}(0,0) > \delta) \leq \nu^{W,0}_{\Lambda_{L+1}}(\mathcal{H}^{-1}_{\beta,\Lambda_{L+1}}(0,0) > \delta)$$
$$= \nu^{W,0}_{\Lambda_{L+1}}\Big(\frac{1}{2\mathcal{H}^{-1}_{\beta,\Lambda_{L+1}}(0,0)} < \frac{1}{2\delta}\Big) = \int_0^{1/(2\delta)} \frac{1}{\sqrt{\pi\gamma}} e^{-\gamma}\, d\gamma,$$

where in the last step we used that $1/[2\mathcal{H}^{-1}_{\beta,\Lambda_{L+1}}(0,0)]$ is Gamma distributed (cf. equation (2.2)). ∎

3. Preliminary results

3.1. Connection between $N(E)$ and the Green's function with Dirichlet b.c.

To obtain a lower bound on $N(E, H_\beta)$, we use the following classical argument (see, e.g., [18])

Lemma 12. *For any finite box Λ_L of side $2L+1$, we have*

$$N(E, H_\beta) \geq \mathbb{E}_{\Lambda_L}^{W,\eta^w}[N_{\Lambda_L}(E, H_{\beta,\Lambda_L}^D)],$$

where $N_{\Lambda_L}(E, H_{\beta,\Lambda_L}^D)$ is the finite volume IDS with Dirichlet boundary condition defined in (1.7), $\mathbb{E}_{\Lambda_L}^{W,\eta^w}$ denotes the expectation with respect to the finite marginal $\nu_{\Lambda_L}^{W,\eta^w}$ of $(\beta_i)_{i\in\Lambda_L}$ given in (1.3), and $\eta^w = \eta_{\Lambda_L}^w$ are the wired boundary condition given in (1.4).

Proof. Recalling the definition of the integrated density of states N, by (1.6) we have

$$N(E) = \lim_{K\to\infty} \frac{1}{|\Lambda_K|} \mathbb{E}_{\Lambda_K}^{W,\eta_{\Lambda_K}^w}[\mathrm{tr}(\mathbf{1}_{(-\infty,E]}(H_{\beta,\Lambda_K}))],$$

where Λ_K is a finite box of side $2K+1$. We split the large box Λ_K into a tiling of smaller boxes of side $2L+1$ with $L<K$, $\Lambda_K = \bigcup_{j=1}^{N_K} \Lambda_{L,j}$. Using $(v_i - v_j)^2 \leq 2(v_i^2 + v_j^2)$ (Dirichlet–Neumann bracketing), we obtain

$$H_{\beta,\Lambda_K} \leq \bigoplus_{j=1}^{N_K} H_{\beta,\Lambda_{L,j}}^D,$$

as a quadratic form. Note that, by the min-max principle, if $A > B$, then $\lambda_{A,j} > \lambda_{B,j}$, where $\lambda_{A,j}$ are ordered eigenvalues of A. This, together with translation invariance, the relation $|\Lambda_L| N_K = |\Lambda_K|$, and (2.3) yields

$$\begin{aligned}
\frac{1}{|\Lambda_K|} \mathbb{E}_{\Lambda_K}^{W,\eta_{\Lambda_K}^w}[\mathrm{tr}(\mathbf{1}_{(-\infty,E]}(H_{\beta,\Lambda_K}))] &\geq \sum_{j=1}^{N_K} \frac{1}{|\Lambda_K|} \mathbb{E}_{\Lambda_K}^{W,\eta_{\Lambda_K}^w}[\mathrm{tr}(\mathbf{1}_{(-\infty,E]}(H_{\beta,\Lambda_{L,j}}^D))] \\
&= \sum_{j=1}^{N_K} \frac{1}{|\Lambda_K|} \mathbb{E}_{\Lambda_{L,j}}^{W,\eta_{\Lambda_{L,j}}^w}[\mathrm{tr}(\mathbf{1}_{(-\infty,E]}(H_{\beta,\Lambda_{L,j}}^D))] \\
&= \frac{N_K}{|\Lambda_K|} \mathbb{E}_{\Lambda_L}^{W,\eta_{\Lambda_L}^w}[\mathrm{tr}(\mathbf{1}_{(-\infty,E]}(H_{\beta,\Lambda_L}^D))] \\
&= \mathbb{E}_{\Lambda_L}^{W,\eta_{\Lambda_L}^w}[N_{\Lambda_L}(E, H_{\beta,\Lambda_L}^D)],
\end{aligned}$$

for any finite box Λ_L in the family $\{\Lambda_{L,j}\}_{j=1}^{N_K}$. Taking the limit $K\to\infty$ keeping L fixed gives the desired result. ∎

As we are looking for a lower bound of $N(E, H_\beta)$, we can consider any finite box Λ_L (usually a larger L gives a better bound). We will fix $\Lambda_L = [-L, L]^d \cap \mathbb{Z}^d$ in the sequel. At the end, we will choose L depending on the energy E.

Lemma 13. *Let $\Lambda_L = [-L, L]^d \cap \mathbb{Z}^d$. It holds*

$$\mathbb{E}_{\Lambda_L}^{W,\eta^w}[N_{\Lambda_L}(E, H_{\beta,\Lambda_L}^D)] \geq \frac{1}{|\Lambda_L|}\nu_{\Lambda_L}^{W,\eta^w}\Big((H_{\beta,\Lambda_L}^D)^{-1}(0,0) \geq \frac{1}{E}\Big)$$

Proof. H_{β,Λ_L}^D is a self adjoint finite random matrix, and by definition it is a.s. positive definite. As a consequence, its smallest eigenvalue λ_1 satisfies

$$\lambda_1 > 0 \quad \text{and} \quad \frac{1}{\lambda_1} = \|(H_{\beta,\Lambda_L}^D)^{-1}\|_{\text{op}},$$

where $\|\cdot\|_{\text{op}}$ stands for the operator norm. It follows that

$$\begin{aligned}
\mathbb{E}_{\Lambda_L}^{W,\eta^w}[N_{\Lambda_L}(E, H_{\beta,\Lambda_L}^D)] &= \frac{1}{|\Lambda_L|}\mathbb{E}_{\Lambda_L}^{W,\eta^w}[\text{tr}(\mathbf{1}_{(-\infty,E]}(H_{\beta,\Lambda_L}^D))] \\
&\geq \frac{1}{|\Lambda_L|}\nu_{\Lambda_L}^{W,\eta^w}(\text{tr}(\mathbf{1}_{(-\infty,E]}(H_{\beta,\Lambda_L}^D) \geq 1)) \\
&= \frac{1}{|\Lambda_L|}\nu_{\Lambda_L}^{W,\eta^w}(\lambda_1 \leq E) \\
&= \frac{1}{|\Lambda_L|}\nu_{\Lambda_L}^{W,\eta^w}\Big(\|(H_{\beta,\Lambda_L}^D)^{-1}\|_{\text{op}} \geq \frac{1}{E}\Big).
\end{aligned}$$

Note that

$$\begin{aligned}
\|(H_{\beta,\Lambda_L}^D)^{-1}\|_{\text{op}} &= \sup_{\psi:\|\psi\|=1} \|(H_{\beta,\Lambda_L}^D)^{-1}\psi\| \\
&\geq \|(H_{\beta,\Lambda_L}^D)^{-1}e_0\| \geq |(H_{\beta,\Lambda_L}^D)^{-1}(0,0)| = (H_{\beta,\Lambda_L}^D)^{-1}(0,0),
\end{aligned}$$

where $e_0 = (\delta_{j0})_{j\in\mathbb{Z}^d}$. In the last step, we used that, since the matrix is a.s. an M-matrix, the entries of its Green's function are all positive. Therefore,

$$\begin{aligned}
\mathbb{E}_{\Lambda_L}^{W,\eta^w}[N_{\Lambda_L}(E, H_{\beta,\Lambda_L}^D)] &\geq \frac{1}{|\Lambda_L|}\nu_{\Lambda_L}^{W,\eta^w}\Big(\|(H_{\beta,\Lambda_L}^D)^{-1}\|_{\text{op}} \geq \frac{1}{E}\Big) \\
&\geq \frac{1}{|\Lambda_L|}\nu_{\Lambda_L}^{W,\eta^w}\Big((H_{\beta,\Lambda_L}^D)^{-1}(0,0) \geq \frac{1}{E}\Big).
\end{aligned}$$

This concludes the proof of the lemma. ∎

3.2. From Dirichlet to simple boundary conditions

Lemma 14 (Dirichlet versus simple bc (1)). *Let Λ_L be the finite box in $\mathbb{Z}^d$ of side $2L+1$ centered at* 0. *We consider $(\beta_i)_{i\in\Lambda_L} \sim \nu_{\Lambda_L}^{W,\eta^w}$, where η^w is the wired boundary* *condition introduced in* (1.4). *Define* $\Omega_1 = \Omega_{1,0} \cap \Omega_{1,1}$, *with*

$$\Omega_{1,0} := \left\{ \sqrt{\frac{\mathcal{H}^{-1}_{\beta,\Lambda_L}(0,i)}{\mathcal{H}^{-1}_{\beta,\Lambda_L}(0,0)}} \le e^{-\frac{1}{2}\kappa|i|} \text{ for all } i \in \partial\Lambda_L \right\}, \tag{3.1a}$$

$$\Omega_{1,1} := \{H^{-1}_{\beta,\Lambda_L}(0,0) \le e^{\kappa L}\}, \tag{3.1b}$$

where κ *is the constant introduced in Theorem* 7, *and remember that*

$$\mathcal{H}_{\beta,\Lambda_L} = W H_{\beta,\Lambda_L}.$$

Let W_c *be as in Theorem* 6.

There are constants $L_0 = L_0(W,d) > 1$ *and* $C_1 = C_1(d, W_c)$ *such that, for all* $L \ge L_0$ *and* $0 < W < W_c$, *we have*

$$\nu^{W,\eta^w}_{\Lambda_L}(\Omega_{1,j}) \ge 1 - C_1\, e^{-\kappa L/4} \quad \textit{for } j = 0, 1, \tag{3.2}$$

and hence $\nu^{W,\eta^w}_{\Lambda_L}(\Omega_1) \ge 1 - 2C_1\, e^{-\kappa L/4}$. *Moreover, on the set* Ω_1, *for any* $E > 0$, *it holds*

$$\left\{(H^S_{\beta,\Lambda_L})^{-1}(0,0) > \frac{1}{E}\right\} \implies \left\{(H^D_{\beta,\Lambda_L})^{-1}(0,0) > \frac{1}{2E}\right\}.$$

In particular, when $d = 1$, *this result holds for all* $W > 0$, *since* $W_c = \infty$.

Note that the set $\Omega_{1,0}$ is measurable with respect to $\{\beta_j\}_{j\in\Lambda\setminus\{0\}}$, while $\Omega_{1,1}$ is measurable with respect to $(H_{\beta,\Lambda_L})^{-1}(0,0)$. This fact will be important in the proof of the lower bound for the IDS.

Proof of Lemma 14. The decay estimate (2.11), together with the Markov inequality, entails that, for all $0 < W < W_c$,

$$\begin{aligned} \nu^{W,\eta^w}_{\Lambda_L}(\Omega^c_{1,0}) &\le \sum_{i\in\partial\Lambda_L} \nu^{W,\eta^w}_{\Lambda_L}\left(\left(\sqrt{\frac{\mathcal{H}^{-1}_{\beta,\Lambda_L}(0,i)}{\mathcal{H}^{-1}_{\beta,\Lambda_L}(0,0)}}\right) > e^{-\frac{\kappa|i|}{2}}\right) \\ &\le C_0|\partial\Lambda_L| e^{-\frac{1}{2}\kappa L} \le C_1 e^{-\frac{1}{4}\kappa L} \end{aligned}$$

for some constants C_1, and for L large enough depending on W and d. Corollary 11 with $\delta = e^{\kappa L/2}/W$ gives

$$\begin{aligned} \nu^{W,\eta^w}_{\Lambda_L}(\Omega^c_{1,1}) &= \nu^{W,\eta^w}_{\Lambda_L}(H^{-1}_{\beta,\Lambda_L}(0,0) > e^{\kappa L}) \\ &= \nu^{W,\eta^w}_{\Lambda_L}\left(\mathcal{H}^{-1}_{\beta,\Lambda_L}(0,0) > \frac{e^{\kappa L}}{W}\right) \le C_1 e^{-\frac{1}{4}\kappa L}. \end{aligned}$$

Therefore, (3.2) holds.

Assume now we are in Ω_1. By the resolvent identity, we have

$$\begin{aligned}(H^S_{\beta,\Lambda_L})^{-1}(0,0) - (H^D_{\beta,\Lambda_L})^{-1}(0,0)\\ = \sum_{j\in\partial\Lambda_L}(H^S_{\beta,\Lambda_L})^{-1}(0,j)(2d-n_j)(H^D_{\beta,\Lambda_L})^{-1}(j,0).\end{aligned} \tag{3.3}$$

By random walk representation, we have, setting $\mathcal{P}^{\Lambda_L}_{j_0,j}$ = the set of nearest neighbor paths from j_0 to j in Λ_L (cf. [34, Proposition 6] and notations therein),

$$\mathcal{H}^{-1}_{\beta,\Lambda_L}(0,i) = \sum_{\sigma\in\mathcal{P}^{\Lambda_L}_{0,i}}\frac{W_\sigma}{(2\beta)_\sigma} \geq \sum_{\sigma\in\mathcal{P}^{\Lambda_L}_{0,i}}\frac{W_\sigma}{(2\beta+W(2d-n))_\sigma} = (\mathcal{H}^D_{\beta,\Lambda_L})^{-1}(0,i). \tag{3.4}$$

Therefore, on the set Ω_1, we have

$$\begin{aligned}\left|\frac{(H^D_{\beta,\Lambda_L})^{-1}(0,0)}{(H^S_{\beta,\Lambda_L})^{-1}(0,0)}-1\right| &= 1-\frac{(H^D_{\beta,\Lambda_L})^{-1}(0,0)}{(H^S_{\beta,\Lambda_L})^{-1}(0,0)}\\ &= \sum_{j\in\partial\Lambda_L}(H^S_{\beta,\Lambda_L})^{-1}(0,j)(2d-n_j)\frac{(H^D_{\beta,\Lambda_L})^{-1}(j,0)}{(H^S_{\beta,\Lambda_L})^{-1}(0,0)}\\ &\leq (H^S_{\beta,\Lambda_L})^{-1}(0,0)\sum_{j\in\partial\Lambda_L}\frac{(\mathcal{H}^S_{\beta,\Lambda_L})^{-1}(0,j)}{(\mathcal{H}^S_{\beta,\Lambda_L})^{-1}(0,0)}(2d-n_j)\frac{(\mathcal{H}^S_{\beta,\Lambda_L})^{-1}(j,0)}{(\mathcal{H}^S_{\beta,\Lambda_L})^{-1}(0,0)}\\ &\leq e^{\kappa L}2d\sum_{j\in\partial\Lambda_L}e^{-2\kappa|j|} \leq 2d|\partial\Lambda_L|e^{+\kappa L}e^{-2\kappa L} \leq e^{-\frac{1}{4}\kappa L} \leq \frac{1}{2},\end{aligned}$$

for $L\geq L_0$, where L_0 depends on W,d. Thus, on Ω_1, it holds

$$\begin{aligned}(H^D_{\beta,\Lambda})^{-1}(0,0) &= (H^S_{\beta,\Lambda})^{-1}(0,0)\Big[1-\Big(1-\frac{(H^D_{\beta,\Lambda_L})^{-1}(0,0)}{(H^S_{\beta,\Lambda_L})^{-1}(0,0)}\Big)\Big]\\ &\geq \frac{1}{2}(H^S_{\beta,\Lambda})^{-1}(0,0),\end{aligned}$$

and hence

$$(H^S_{\beta,\Lambda_L})^{-1}(0,0) > \frac{1}{E} \implies (H^D_{\beta,\Lambda})^{-1}(0,0) \geq \frac{1}{2E}. \qquad \blacksquare$$

Lemma 15 (Dirichlet versus simple bc (2)). *Let Λ_L be the finite box in $\mathbb{Z}^d$ of side $2L+1$ centered at 0. We consider $(\beta_i)_{i\in\Lambda_L}\sim\nu^{W,\eta^w}_{\Lambda_L}$, where η^w is the wired boundary condition introduced in (1.4).*

Define $\Omega_2=\Omega_{2,0}\cap\Omega_{2,1}$, with

$$\Omega_{2,0} := \{\max_{j\in\partial\Lambda_L, i\sim 0}(\mathcal{H}^{-1}_{\beta,\Lambda_L\setminus\{0\}}(i,j)) \leq e^{-\frac{3}{2}\kappa' L}\}, \tag{3.5a}$$

$$\Omega_{2,1} := \{H^{-1}_{\beta,\Lambda_L}(0,0) \leq e^{\kappa' L}\}, \tag{3.5b}$$

where κ' is the constant introduced in Theorem 8, and remember that

$$\mathcal{H}_{\beta,\Lambda_L} = WH_{\beta,\Lambda_L}.$$

Let W_c' be as in Theorem 8.

There is $L_0(W,d) > 1$ such that, for all $L \geq L_0$ and $0 < W < W_c'$, we have $\kappa' L > 1$, and there is a constant $C_1' = C_1'(W,d) > 0$ such that

$$\nu_{\Lambda_L}^{W,\eta^w}(\Omega_{2,j}) \geq 1 - C_1' e^{-\frac{1}{4}\kappa' L} \quad \text{for } j = 0, 1, \tag{3.6}$$

and hence $\nu_{\Lambda_L}^{W,\eta^w}(\Omega_2) \geq 1 - 2C_1' e^{-\kappa' L/4}$. Moreover, on the set Ω_2 it holds

$$\left\{ (H_{\beta,\Lambda_L}^S)^{-1}(0,0) > \frac{1}{E} \right\} \implies \left\{ (H_{\beta,\Lambda_L}^D)^{-1}(0,0) > \frac{1}{2E} \right\},$$

for all energy $0 < E < 1/2$.

Note that also here the set $\Omega_{2,0}$ is measurable with respect to $\{\beta_j\}_{j\in\Lambda\setminus\{0\}}$, while $\Omega_{2,1}$ is measurable with respect to $(H_{\beta,\Lambda_L})^{-1}(0,0)$. This fact will be important in the proof of the lower bound on the IDS.

Proof of Lemma 15. Using the random path representation, as in (2.14), we obtain $\mathcal{H}^{-1}_{\beta,\Lambda_L\setminus\{0\}}(i,j) \leq \mathcal{H}^{-1}_{\beta,\Lambda_L}(i,j)$. Then, the decay estimate (2.13) together with the Markov inequality entails, for all $0 < W < W_c'$,

$$\begin{aligned}
\nu_{\Lambda_L}^{W,\eta^w}(\Omega_{2,0}^c) &\leq \sum_{j\in\partial\Lambda_L, i\sim 0} \nu_{\Lambda_L}^{W,\eta^w}\big(\mathcal{H}^{-1}_{\beta,\Lambda_L\setminus\{0\}}(i,j) > e^{-\frac{3}{2}\kappa' L}\big) \\
&\leq \sum_{j\in\partial\Lambda_L, i\sim 0} e^{\frac{3}{8}\kappa' L} \mathbb{E}_{\Lambda_L}^{W,\eta^w}\big[(\mathcal{H}^{-1}_{\beta,\Lambda_L\setminus\{0\}}(i,j))^{\frac{1}{4}}\big] \\
&\leq \sum_{j\in\partial\Lambda_L, i\sim 0} e^{\frac{3}{8}\kappa' L} \mathbb{E}_{\Lambda_L}^{W,\eta^w}\big[(\mathcal{H}^{-1}_{\beta,\Lambda_L}(i,j))^{\frac{1}{4}}\big] \\
&\leq C_0 \sum_{j\in\partial\Lambda_L, i\sim 0} e^{\frac{3}{8}\kappa' L} e^{-\kappa'|i-j|} \leq C_0 |\partial\Lambda_L| e^{-\frac{5}{8}\kappa' L} \leq C_1' e^{-\frac{1}{4}\kappa' L}
\end{aligned}$$

for some constant $C_1' = C_1'(W,d)$. The bound for $\Omega_{2,1}$ works exactly as the one for $\Omega_{1,1}$ in Lemma 14. Therefore, (3.6) holds.

Assume now we are on Ω_2. We have, for all $j \in \partial\Lambda_L$,

$$\mathcal{H}^{-1}_{\beta,\Lambda_L}(0,j) = \sum_{i\sim 0} \mathcal{H}^{-1}_{\beta,\Lambda_L}(0,0) W \mathcal{H}^{-1}_{\beta,\Lambda_L\setminus\{0\}}(i,j).$$

Therefore,

$$\mathcal{H}^{-1}_{\beta,\Lambda_L}(0,j) \leq W 2d e^{\kappa' L} e^{-\frac{3}{2}\kappa' L} \leq W e^{-\frac{1}{4}\kappa' L}.$$

By (3.3) and (3.4),

$$\begin{aligned}
&|(H^D_{\beta,\Lambda_L})^{-1}(0,0) - (H^S_{\beta,\Lambda_L})^{-1}(0,0)| \\
&\quad = \sum_{j\in\partial\Lambda_L} (H^S_{\beta,\Lambda_L})^{-1}(0,j)(2d-n_j)(H^D_{\beta,\Lambda_L})^{-1}(j,0) \\
&\quad \le \sum_{j\in\partial\Lambda_L} (H^S_{\beta,\Lambda_L})^{-1}(0,j)(2d-n_j)(H^S_{\beta,\Lambda_L})^{-1}(j,0) \\
&\quad \le W^2|\partial\Lambda_L|2de^{-\frac{1}{2}\kappa' L} \le e^{-\frac{1}{4}\kappa' L},
\end{aligned}$$

for L large enough, depending on W and d. It follows that if $E \le 1/2$, then

$$\begin{aligned}
&(H^S_{\beta,\Lambda_L})^{-1}(0,0) > \frac{1}{E} \\
&\quad \Longrightarrow\ (H^D_{\beta,\Lambda_L})^{-1}(0,0) \ge (H^S_{\beta,\Lambda_L})^{-1}(0,0) - e^{-\frac{1}{4}\kappa' L} \ge \frac{1}{E} - e^{-\frac{1}{4}\kappa' L} \ge \frac{1}{2E}
\end{aligned}$$

for $L \ge L_0 = L_0(W,d)$. ∎

4. Lower bound on the IDS

We are now ready to prove Theorem 1. By Lemma 12 and Lemma 13 in Section 3.1, we have

$$N(E,H_\beta) \ge \frac{1}{|\Lambda_L|}\nu^{W,\eta^w}_{\Lambda_L}\Big((H^D_{\beta,\Lambda_L})^{-1}(0,0) \ge \frac{1}{E}\Big).$$

Remember the definition of $W_{\rm cr}$ in (1.9) and the configuration sets $\Omega_{1,0}, \Omega_{1,1}, \Omega_{2,0}, \Omega_{2,1}$ introduced in (3.1) and (3.5). We define, for $j = 0, 1$,

$$\Omega_{\mathrm{loc},j} := \begin{cases} \Omega_{1,j} & \text{if } W_{\rm cr} = W_c, \\ \Omega_{2,j} & \text{if } W_{\rm cr} = W_c'. \end{cases}$$

and $\Omega_{\rm loc} := \Omega_{\mathrm{loc},0} \cap \Omega_{\mathrm{loc},1}$. Note that the constants $\kappa, C_0, C_1, \kappa', C_0', C_1'$ appearing in the definition of the sets $\Omega_{1,j}, \Omega_{2,j}$ and the results of Lemma 14 and 15 play the same role. To alleviate the notation, in the following we will not distinguish them.

By Lemma 14 and 15, we have

$$N(E,H_\beta) \ge \frac{1}{|\Lambda_L|}\mathbb{E}^{W,\eta^w}_{\Lambda_L}[\mathbf{1}_{\Omega_{\rm loc}}\mathbf{1}_{\{H^{-1}_{\beta,\Lambda_L}(0,0)\ge\frac{1}{2E}\}}] \tag{4.1}$$

for all $0 < W < W_{\rm cr}$, $E < 1/2$ and L large. Remember that $H_{\beta,\Lambda_L} = \mathcal{H}_{\beta,\Lambda_L}/W$, and set $0^c = \Lambda_L \setminus \{0\}$. By Schur decomposition,

$$\frac{1}{W}H^{-1}_{\beta,\Lambda_L}(0,0) = \mathcal{H}^{-1}_{\beta,\Lambda_L}(0,0) = \frac{1}{2\beta_0 - P^W_{0,0^c}\mathcal{H}^{-1}_{\beta,\Lambda_L\setminus\{0\}}P^W_{0^c,0}} =: \frac{1}{y}.$$

Lemma 16. *The conditional density of the variable* $y := 2\beta_0 - P^W_{0,0^c}\mathcal{H}^{-1}_{\beta,\Lambda_L\setminus\{0\}}P^W_{0^c,0}$, *given* $\beta_{0^c} = (\beta_j)_{i\in 0^c}$, *is*

$$d\rho_{a_0}(y) = \rho_{a_0}(y)dy = \frac{e^{a_0}}{\sqrt{2\pi}}e^{-\frac{1}{2}(y+\frac{a_0^2}{y})}\frac{1}{\sqrt{y}}\mathbf{1}_{y>0}dy, \tag{4.2}$$

where

$$a_0 = a_0(\beta_{0^c}) = \sum_{j\in\partial\Lambda_L}\frac{\mathcal{H}^{-1}_{\beta,\Lambda_L}(0,j)}{\mathcal{H}^{-1}_{\beta,\Lambda_L}(0,0)}\eta^w_j = W\sum_{i\sim 0, j\in\partial\Lambda_L}\mathcal{H}^{-1}_{\beta,\Lambda_L\setminus\{0\}}(i,j)\,\eta^w_j. \tag{4.3}$$

The corresponding average will be denoted by $\mathbb{E}^{a_0}$.

Proof. The result follows from the factorization in [34, equation (5.14)] with $U = \Lambda\setminus\{0\}$ and $U^c = \{0\}$.

Alternatively, one may insert the following relations in (1.3):

$$\begin{aligned}
\langle 1, \mathcal{H}_{\beta,\Lambda_L}1\rangle &= 2\beta_0 + \langle 1_{0^c}, \mathcal{H}_{\beta,\Lambda_L\setminus\{0\}}1_{0^c}\rangle - 2\sum_{i\sim 0}W = y + F(\beta_{0^c}),\\
\langle \eta^w, \mathcal{H}^{-1}_{\beta,\Lambda_L}\eta^w\rangle &= \langle \eta^w, \mathcal{H}^{-1}_{\beta,\Lambda_L\setminus\{0\}}\eta^w\rangle + a_0^2\mathcal{H}^{-1}_{\beta,\Lambda_L}(0,0) = G(\beta_{0^c}) + \frac{a_0^2}{y},\\
\det\mathcal{H}_{\beta,\Lambda_L} &= y\det\mathcal{H}_{\beta,\Lambda_L\setminus\{0\}},
\end{aligned}$$

where F, G are functions of β_{0^c} and in second line we combined $\eta^w_0 = 0$ and the resolvent identity

$$A^{-1} = B^{-1} + B^{-1}(B-A)B^{-1} + B^{-1}(B-A)A^{-1}(B-A)B^{-1}$$

with

$$A = \mathcal{H}_{\beta,\Lambda_L} \quad\text{and}\quad B = 2\beta_0 \oplus \mathcal{H}_{\beta,\Lambda_L\setminus\{0\}}.$$

This yields

$$\rho_{a_0}(y) = c_{a_0}e^{-\frac{1}{2}(y+\frac{a_0^2}{y})}\frac{1}{\sqrt{y}}\mathbf{1}_{y>0}.$$

To determine the normalizing constant c_{a_0}, note that

$$\int_0^\infty e^{-\frac{1}{2}(y+\frac{a_0^2}{y})}\frac{1}{\sqrt{y}}dy = \frac{\sqrt{2\pi}}{a_0}e^{-a_0}\,\mathbb{E}_{IG(a_0,a_0^2)}[y] = \sqrt{2\pi}e^{-a_0},$$

where $\mathbb{E}_{IG(a_0,a_0^2)}$ denotes the expectation with respect to the probability distribution of the inverse Gaussian $IG(a_0, a_0^2)$. ∎

Using these results, the average in (4.1) can be reformulated as follows:

$$\begin{aligned}
\mathbb{E}_{\Lambda_L}^{W,\eta^w}[\mathbf{1}_{\Omega_{\mathrm{loc}}}\mathbf{1}_{\{H^{-1}_{\beta,\Lambda_L}(0,0)\geq\frac{1}{2E}\}}] &= \mathbb{E}_{\Lambda_L}^{W,\eta^w}[\mathbf{1}_{\Omega_{\mathrm{loc}}}\mathbf{1}_{\{\frac{1}{W}H^{-1}_{\beta,\Lambda_L}(0,0)\geq\frac{1}{2EW}\}}] \\
&= \mathbb{E}_{\Lambda_L}^{W,\eta^w}\left[\mathbf{1}_{\Omega_{\mathrm{loc},0}}\int \mathbf{1}_{\Omega_{\mathrm{loc},1}}\mathbf{1}_{y\leq 2EW}\frac{e^{a_0}}{\sqrt{2\pi}}e^{-\frac{1}{2}(y+\frac{a_0^2}{y})}\frac{1}{\sqrt{y}}\mathbf{1}_{y>0}dy\right] \\
&= \mathbb{E}_{\Lambda_L}^{W,\eta^w}[\mathbf{1}_{\Omega_{\mathrm{loc},0}}\mathbb{E}^{a_0(\beta_{0^c})}[\mathbf{1}_{\Omega_{\mathrm{loc},1}}\mathbf{1}_{y\leq 2EW}]] \\
&= \mathbb{E}_{\Lambda_L}^{W,\eta^w}[\mathbf{1}_{\Omega_{\mathrm{loc},0}}\,\mathbb{E}^{a_0(\beta_{0^c})}[\mathbf{1}_{We^{-\kappa L}\leq y\leq 2EW}]\,],
\end{aligned}$$

where we used that $\Omega_{\mathrm{loc},0}$ is measurable with respect to β_{0^c} and

$$\Omega_{\mathrm{loc},1} = \{H^{-1}_{\beta,\Lambda_L}(0,0) < e^{\kappa L}\} = \left\{\frac{y}{W} > e^{-\kappa L}\right\}.$$

We have, for L large enough, for all $\beta_{0^c}\in\Omega_{1,0}$,

$$a_0(\beta_{0^c}) = \sum_{j\in\partial\Lambda_L}\frac{\mathcal{H}^{-1}_{\beta,\Lambda_L}(0,j)}{\mathcal{H}^{-1}_{\beta,\Lambda_L}(0,0)}\eta_j^w \leq 2dW|\partial\Lambda_L|e^{-\kappa L}\leq We^{-\frac{1}{2}\kappa L}$$

and, for all $\beta_{0^c}\in\Omega_{2,0}$,

$$a_0(\beta_{0^c}) = W\sum_{i\sim 0,j\in\partial\Lambda_L}\mathcal{H}^{-1}_{\beta,\Lambda_L\setminus\{0\}}(i,j)\eta_j \leq 2dW|\partial\Lambda_L|e^{-\frac{3}{2}\kappa L}\leq We^{-\frac{1}{2}\kappa L}.$$

Setting $\bar{a}_0 := We^{-\kappa L/2}$, and remarking that $a_0\geq 0$, we argue

$$e^{a_0(\beta_{0^c})}e^{-\frac{(a_0(\beta_{0^c}))^2}{2y}}\geq e^{-\frac{(a_0(\beta_{0^c}))^2}{2y}}\geq e^{-\frac{\bar{a}_0^2}{2y}} = e^{\bar{a}_0}e^{-\frac{\bar{a}_0^2}{2y}}e^{-\bar{a}_0},$$

hence

$$\rho_{a_0(\beta_{0^c})}(y)\geq\rho_{\bar{a}_0}(y)e^{-\bar{a}_0}\quad\text{for all }\beta_{0^c}\in\Omega_{\mathrm{loc},1}.$$

Therefore, we obtain

$$\begin{aligned}
&\mathbb{E}_{\Lambda_L}^{W,\eta^w}[\mathbf{1}_{\Omega_{\mathrm{loc},0}}\,\mathbb{E}^{a_0(\beta_{0^c})}[\mathbf{1}_{We^{-\kappa L}\leq y\leq 2WE}]\,] \\
&\quad\geq\; e^{-\bar{a}_0}\,\nu_{\Lambda_L}^{W,\eta^w}(\Omega_{\mathrm{loc},0})\,\rho_{\bar{a}_0}(We^{-\kappa L}\leq y\leq 2WE) \\
&\quad\geq (1-C_2e^{-\frac{1}{4}\kappa L})\,\rho_{\bar{a}_0}(We^{-\kappa L}\leq y\leq 2WE)
\end{aligned}$$

for some constant C_2, where we used (3.2), (3.6), and the bound $e^{-\bar{a}_0} = e^{-We^{-\kappa L/2}}\geq (1-ce^{-\kappa L/4})$ for some constant $c>0$.

It remains to extract a lower bound on $\rho_{\bar{a}_0}(We^{-\kappa L}\leq y\leq 2WE)$. Set

$$L = L_E := \frac{1}{\kappa}\ln\frac{1}{E}.$$

For E small, L_E is large enough for all our results to hold. Then $2WE = 2We^{-\kappa L}$ and hence $We^{-\kappa L} < 2WE < \bar{a}_0$. Moreover,

$$\frac{(\bar{a}_0 - y)^2}{y} \leq W \quad \text{for all } y \text{ such that } We^{-\kappa L} \leq y \leq 2We^{-\kappa L}.$$

It follows

$$\begin{aligned}\rho_{\bar{a}_0}(We^{-\kappa L} \leq y \leq 2WE) &= \frac{1}{\sqrt{2\pi}} \int_{We^{-\kappa L}}^{2WE} e^{-\frac{(\bar{a}_0 - y)^2}{2y}} \frac{1}{\sqrt{y}} dy \\ &\geq \frac{e^{-\frac{1}{2}W}}{\sqrt{2\pi}} \int_{We^{-\kappa L}}^{2WE} \frac{1}{\sqrt{y}} dy = ce^{-\frac{1}{2}W}\sqrt{WE}\end{aligned}$$

for some constant $c > 0$ independent of E, κ, L, W. Putting all these results together, we obtain

$$\begin{aligned}N(E, H_\beta) &\geq \frac{1}{|\Lambda_{L_E}|} \nu_{\Lambda_{L_E}}^{W,\eta^w} \Big((H^D_{\beta,\Lambda_{L_E}})^{-1}(0,0) \geq \frac{1}{E} \Big) \\ &\geq \frac{1}{|\Lambda_{L_E}|} (1 - C_2 e^{-\frac{1}{4}\kappa L_E})\, ce^{-\frac{1}{2}W} \sqrt{W}\sqrt{E} \geq c' \frac{1}{|\log E|^d} \sqrt{E}\end{aligned}$$

for some constant $c' > 0$ depending on W, d, κ. This concludes the proof of the lower bound.

5. Wegner estimate

In this section, we prove Theorem 2. We work out in detail the proof only for the case of simple boundary conditions. The proof in the case of Dirichlet boundary conditions works exactly in the same way.

Proof of Theorem 2. Note that, since $H = \mathcal{H}/W$, we have

$$N_{\Lambda_L}(E, H_{\beta,\Lambda_L}) = N_{\Lambda_L}(WE, \mathcal{H}_{\beta,\Lambda_L}) = \frac{1}{|\Lambda_L|} \operatorname{tr} \mathbf{1}_{(-\infty,0]}(\mathcal{H}_{\beta,\Lambda_L} - WE).$$

We start with a regularity bound on

$$N_{\Lambda_L}(E+\varepsilon, H_{\beta,\Lambda_L}) - N_{\Lambda_L}(E-\varepsilon, H_{\beta,\Lambda_L}) = \frac{1}{|\Lambda_L|} \operatorname{tr} \mathbf{1}_{[-\varepsilon,\varepsilon]}(\mathcal{H}_{\beta,\Lambda_L} - WE).$$

We smooth out the discontinuous function $\mathbf{1}_{[-\varepsilon,\varepsilon]}(x)$ as follows. Let ρ be a smooth non-decreasing function ρ satisfying $\rho = 0$ on $(-\infty, -\varepsilon)$ and $\rho = 1$ on (ε, ∞). Then,

$$\mathbf{1}_{[-\varepsilon,\varepsilon]}(x) \leq \rho(x + 2\varepsilon) - \rho(x - 2\varepsilon) \quad \text{for all } x \in \mathbb{R}.$$

Setting

$$\delta_L(E,\varepsilon) := \mathbb{E}_{\Lambda_L}^{W,\eta^w}[\mathrm{tr}(\rho(\mathcal{H}_{\beta,\Lambda_L} - E + 2\varepsilon) - \rho(\mathcal{H}_{\beta,\Lambda_L} - E - 2\varepsilon))],$$

we have

$$0 \le \mathbb{E}_{\Lambda_L}^{W,\eta^w}[N_{\Lambda_L}(E+\varepsilon, H_{\beta,\Lambda_L}) - N_{\Lambda_L}(E-\varepsilon, H_{\beta,\Lambda_L})] \le \frac{1}{|\Lambda_L|}\delta_L(WE, W\varepsilon).$$

Remember that the conditional measure for $2\beta_j - P^W_{j,j^c}\mathcal{H}^{-1}_{\beta,\Lambda_L\setminus\{j\}}P^W_{j^c,j}$, given $\beta_{j^c} = (\beta_j)_{i\in\Lambda_L\setminus\{j\}}$, is ρ_{a_j} defined in (4.2), where, for general $j \in \Lambda_L$,

$$a_j = a_j(\beta_{j^c}) := \eta_j^w + W\sum_{i\sim j, k\in\Lambda_L\setminus\{j\}} \mathcal{H}^{-1}_{\beta,\Lambda_L\setminus\{j\}}(i,k)\,\eta_k^w = \frac{\sum_{k\in\partial\Lambda_L}\mathcal{H}^{-1}_{\beta,\Lambda_L}(j,k)\eta_k^w}{\mathcal{H}^{-1}_{\beta,\Lambda_L}(j,j)} \tag{5.1}$$

(cf. [34, equation (5.14)]). We also recall that the Lévy concentration of a measure μ on $\mathbb{R}$ is defined by

$$\mathcal{L}_\mu(\varepsilon) = \sup_x \mu([x, x+\varepsilon)). \tag{5.2}$$

By Lemma 17 below, we have

$$\begin{aligned}
&\mathbb{E}_{\Lambda_L}^{W,\eta^w}[N_{\Lambda_L}(E+\varepsilon, H_{\beta,\Lambda_L}) - N_{\Lambda_L}(E-\varepsilon, H_{\beta,\Lambda_L})]\\
&\quad\le \frac{1}{|\Lambda_L|}\delta_L(WE, W\varepsilon)\\
&\quad\le \frac{1}{|\Lambda_L|}\sum_{j\in\Lambda_L}\mathbb{E}_{\Lambda_L}^{W,\eta^w}[\mathcal{L}_{\rho_{a_j(\beta_{j^c})}}(4W\varepsilon)].
\end{aligned}$$

Now, (1.10) and (1.11) follow by inserting the bounds (5.3) and (5.4), stated below. This concludes the proof of Theorem 2. ∎

Lemma 17. *For all $E > 0$ and $0 < \varepsilon < E$, it holds*

$$\delta_L(E,\varepsilon) \le \sum_{j\in\Lambda_L}\mathbb{E}_{\Lambda_L}^{W,\eta^w}[\mathcal{L}_{\rho_{a_j(\beta_{j^c})}}(4\varepsilon)].$$

Proof. Note that

$$\mathcal{H}_{\beta,\Lambda_L} \pm 2\varepsilon = 2(\beta \pm \varepsilon) - P^W = \mathcal{H}_{\beta\pm\varepsilon,\Lambda_L}.$$

Order the vertices in Λ_L as $\{1, 2, \ldots, |\Lambda_L|\}$. For each $1 \le k \le |\Lambda_L|$, we define

$$\beta'_k = \beta'_k(\varepsilon) := (\beta_1 + \varepsilon, \ldots, \beta_{k-1} + \varepsilon, \beta_k - \varepsilon, \ldots, \beta_{|\Lambda_L|} - \varepsilon)$$

and

$$\beta'_{|\Lambda_L|+1} = \beta'_{|\Lambda_L|+1}(\varepsilon) := (\beta_1 + \varepsilon, \ldots, \beta_{|\Lambda_L|} + \varepsilon).$$

With this convention, we have

$$\mathcal{H}_{\beta'_{k+1},\Lambda_L}(i,j) = \mathcal{H}_{\beta'_k,\Lambda_L}(i,j) + \mathbf{1}_{i=j=k}4\varepsilon.$$

Expanding in a telescopic sum, we get

$$\delta_L(E,\varepsilon) = \sum_{k=1}^{|\Lambda_L|} \mathbb{E}_{\Lambda_L}^{W,\eta^w}[\mathrm{tr}(\rho(\mathcal{H}_{\beta'_{k+1},\Lambda_L} - E) - \rho(\mathcal{H}_{\beta'_k,\Lambda_L} - E))].$$

We concentrate now on the k-th term in the sum. For a fixed configuration β_{k^c}, we define $y_k := 2\beta_k - P^W_{k,k^c}\mathcal{H}^{-1}_{\beta,\Lambda_L\setminus\{k\}}P^W_{k^c,k}$. Note that $\beta'_k = \beta'_k(y_k,\beta_{k^c})$ is a function of y_k and β_{k^c}. We consider the function $y_k \mapsto F_k(y_k) := \mathrm{tr}\,\rho(\mathcal{H}_{\beta'_k(y_k,\beta_{k^c}),\Lambda_L} - E)$. We can write

$$\begin{aligned}
&\mathbb{E}_{\Lambda_L}^{W,\eta^w}[\mathrm{tr}(\rho(\mathcal{H}_{\beta'_{k+1},\Lambda_L} - E) - \rho(\mathcal{H}_{\beta'_k,\Lambda_L} - E))] \\
&\quad = \mathbb{E}_{\Lambda_L}^{W,\eta^w}\left[\int (F_k(y_k + 4\varepsilon) - F_k(y_k))\rho_{a_k}(y_k)dy_k\right].
\end{aligned}$$

We take the following primitive of ρ_{a_k}:

$$G_k(y) := \int_0^y \rho_{a_k}(t)dt.$$

This function is differentiable and satisfies $G(\infty) = 1$ and $G(0) = 0$. Moreover, we can write

$$\begin{aligned}
&\int_0^\infty \rho_{a_k}(y)[F_k(y + 4\varepsilon) - F_k(y)]dy \\
&\quad = \lim_{M\to\infty}\int_0^M \rho_{a_k}(y)[F_k(y + 4\varepsilon) - F_k(y_k)]dy =: \lim_{M\to\infty} I^k_M.
\end{aligned}$$

Performing integration by parts, we argue, using $G_k(0) = 0$,

$$\begin{aligned}
I^k_M &= G_k(M)(F_k(M + 4\varepsilon) - F_k(M)) - \int_0^M G_k(y)(F'_k(y + 4\varepsilon) - F'_k(y))dy \\
&= \int_M^{M+4\varepsilon} G_k(M)F'_k(y)dy - \int_0^M G_k(y)F'_k(y + 4\varepsilon)dy + \int_0^M G_k(y)F'_k(y)dy.
\end{aligned}$$

We write the second integral as

$$\int_0^M G_k(y)F_k'(y+4\varepsilon) = \int_{4\varepsilon}^M G_k(y-4\varepsilon)F_k'(y)dy + \int_M^{M+4\varepsilon} G_k(y-4\varepsilon)F_k'(y)dy$$

and the third integral as

$$\int_0^M G_k(y)F_k'(y)dy = \int_{4\varepsilon}^M G_k(y)F_k'(y)dy + \int_0^{4\varepsilon} G_k(y)F_k'(y)dy.$$

Putting everything together, we get

$$\begin{aligned} I_M^k &= \int_M^{M+4\varepsilon} (G_k(M) - G_k(y-4\varepsilon))F_k'(y)dy \\ &\quad + \int_{4\varepsilon}^M (G_k(y) - G_k(y-4\varepsilon))F_k'(y)dy + \int_0^{4\varepsilon} G_k(y)F_k'(y)dy. \end{aligned}$$

Now, we argue

$$G_k(y) - G_k(y-4\varepsilon) = \int_{y-4\varepsilon}^{y} \rho_{a_k}(t)dt \leq \mathcal{L}_{\rho_{a_k}}(4\varepsilon) \quad \text{for all } y \in [4\varepsilon, M].$$

The same bound holds for $G_k(M) - G_k(y-4\varepsilon)$ for $y \in [M, M+4\varepsilon]$ and $G_k(y) = G_k(y) - G_k(0)$ for $y \in [0, 4\varepsilon]$. Therefore,

$$I_M^k \leq \mathcal{L}_{\rho_{a_k}}(4\varepsilon) \int_0^{M+4\varepsilon} F_k'(y)dy = \mathcal{L}_{\rho_{a_k}}(4\varepsilon)(F_k(M+4\varepsilon) - F_k(0)).$$

Finally, using a standard argument of rank-one perturbation (see, e.g., [18, Lemma 5.25]), we get $(F_k(M+4\varepsilon) - F_k(0)) \leq 1$ uniformly in M. The result follows. ∎

Lemma 18. *It holds, for all $\varepsilon > 0$ and $a > 0$,*

$$\mathcal{L}_{\rho_a}(\varepsilon) \leq \sqrt{\frac{2\varepsilon}{\pi}}. \tag{5.3}$$

Moreover, for $d \geq 3$ and $W \geq W_0$ (as defined in Theorem 2), the following improved estimate holds for all $\varepsilon > 0$:

$$\mathbb{E}_{\Lambda_L}^{W,\eta^w}[\mathcal{L}_{\rho_{a_j(\beta_j c)}}(\varepsilon)] \leq \frac{C_1}{\sqrt{W}}\varepsilon \quad \textit{for all } j \in \Lambda_L, \tag{5.4}$$

where $C_1 > 0$ is a constant depending only on the dimension.

Proof. Note that, for all $y > 0$, we have

$$\rho_a(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2y}(y-a)^2} \frac{1}{\sqrt{y}} \le \frac{1}{\sqrt{2\pi y}},$$

and, therefore,

$$\mathcal{L}_{\rho_a}(\varepsilon) = \sup_{x\ge 0} \rho_a([x, x+\varepsilon)) \le \sup_{x\ge 0} \int_x^{x+\varepsilon} \frac{1}{\sqrt{2\pi y}}\, dy = \int_0^{\varepsilon} \frac{1}{\sqrt{2\pi y}}\, dy = \sqrt{\frac{2\varepsilon}{\pi}}.$$

This gives the first bound (5.3). To obtain the improved bound (5.4), note that, by (5.1), $a = a_j(\beta_{j^c}) > 0$ almost surely; hence, the function $y \mapsto \rho_a(y)$ takes its maximum value in

$$y_a := \frac{1}{2}(-1 + \sqrt{1 + 4a^2}).$$

Therefore, we have $\mathcal{L}_{\rho_a}(\varepsilon) \le \rho(y_a)\varepsilon$. Now, using

$$\frac{1}{2y_a} = \frac{1 + \sqrt{1+4a^2}}{4a^2} \le \frac{2+2a}{4a^2} = \frac{1}{2}\Big(\frac{1}{a^2} + \frac{1}{a}\Big),$$

we obtain

$$\rho(y_a) = \frac{1}{\sqrt{2\pi y_a}} e^{-\frac{1}{2}\frac{(y_a-a)^2}{y_a}} \le \frac{1}{\sqrt{2\pi y_a}} \le \frac{1}{\sqrt{2\pi}}\Big(\frac{1}{a^2} + \frac{1}{a}\Big)^{\frac{1}{2}} \le \frac{1}{\sqrt{2\pi}}\Big(\frac{1}{a} + \frac{1}{\sqrt{a}}\Big).$$

It follows

$$\begin{aligned} \mathbb{E}^{W,\eta^w}_{\Lambda_L}[\mathcal{L}_{\rho_{a_j(\beta_{j^c})}}(\varepsilon)] &\le \frac{\varepsilon}{\sqrt{2\pi}}\Big(\mathbb{E}^{W,\eta^w}_{\Lambda_L}\Big[\frac{1}{a_j}\Big] + \mathbb{E}^{W,\eta^w}_{\Lambda_L}\Big[\frac{1}{\sqrt{a_j}}\Big]\Big) \\ &\le \frac{\varepsilon}{\sqrt{2\pi}}\Big(\mathbb{E}^{W,\eta^w}_{\Lambda_L}\Big[\frac{1}{a_j}\Big] + \mathbb{E}^{W,\eta^w}_{\Lambda_L}\Big[\frac{1}{a_j}\Big]^{\frac{1}{2}}\Big). \end{aligned}$$

The result now follows from Lemma 19 below setting $C_1 := \sqrt{2C_2/\pi}$. ■

Lemma 19. *For $d \ge 3$ and $W \ge W_0$, (as defined in Theorem 2), it holds*

$$\mathbb{E}^{W,\eta^w}_{\Lambda_L}\Big[\frac{1}{a_j(\beta_{j^c})}\Big] \le \frac{C_2}{W} \quad \textit{for all } j \in \Lambda_L, \tag{5.5}$$

where $C_2 > 0$ is a constant depending only on the dimension.

Proof. In the case $j \in \partial\Lambda_L$, by (5.1), we have $a_j \ge \eta^w_j \ge W$ a.s., and, hence, (5.5) holds with $C_2 = 1$.

Assume now $j \in \Lambda_L \setminus \partial\Lambda_L$. Using (2.7) and (5.1), we have

$$a_j = \frac{\sum_{k\in\partial\Lambda_L} \mathcal{H}^{-1}_{\beta,\Lambda_L}(j,k)\, \eta^w_k}{\mathcal{H}^{-1}_{\beta,\Lambda_L}(j,j)} = \frac{e^{u_j}}{\mathcal{H}^{-1}_{\beta,\Lambda_L}(j,j)},$$

therefore

$$\mathbb{E}^{W,\eta^w}_{\Lambda_L}\left[\frac{1}{a_j}\right] = \mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[\mathcal{H}^{-1}_{\beta(u),\Lambda_L}(j,j)e^{-u_j}] = \frac{1}{W}\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[D^{-1}(j,j)e^{u_j}],$$

where we used (2.5) and (2.6). The matrix $D = D(u) := e^u H_{\beta(u),\Lambda_L} e^u$ can be characterized via the quadratic form

$$\langle v, D(u)v\rangle = \sum_{k\sim k'\in\Lambda_L} e^{u_j+u_k}(\nabla_{kk'}v)^2 + \sum_{k\in\Lambda_L}\tilde{\eta}^w_k e^{u_k} v_k^2, \tag{5.6}$$

where we defined $\tilde{\eta}^w_k := \eta^w_k/W$ and $\nabla_{kk'}v := v_k - v_{k'}$. To estimate the average of $D^{-1}(j,j)e^{u_j}$, we use the same strategy as in [12, proof of Theorem 3]. We can write $D^{-1}(j,j)e^{u_j} = \langle f, D^{-1}f\rangle$, where $f_k := \delta_{kj}e^{u_j/2} = e^{u_j/2}(\delta_j)(k)$. Setting $D_0 := D(0) = -\Delta + \tilde{\eta}^w$, we argue

$$\begin{aligned}
\langle f, D^{-1}f\rangle &= \langle D_0 D_0^{-1} f, D^{-1}f\rangle \\
&= \sum_{k\sim k'}(\nabla_{kk'}D_0^{-1}f)(\nabla_{kk'}D^{-1}f) + \sum_k \tilde{\eta}^w_k (D_0^{-1}f)_k (D^{-1}f)_k \\
&= \sum_{k\sim k'}\frac{(\nabla_{kk'}D_0^{-1}f)}{e^{\frac{1}{2}(u_k+u_{k'})}}\frac{(\nabla_{kk'}D^{-1}f)}{e^{-\frac{1}{2}(u_k+u_{k'})}} + \sum_k \tilde{\eta}^w_k \frac{(D_0^{-1}f)_k}{e^{\frac{1}{2}u_k}}\frac{(D^{-1}f)_k}{e^{-\frac{1}{2}u_k}} \\
&\le \Big(\sum_{k\sim k'}\frac{(\nabla_{kk'}D_0^{-1}f)^2}{e^{(u_k+u_{k'})}} + \sum_k \tilde{\eta}^w_k \frac{(D_0^{-1}f)_k^2}{e^{u_k}}\Big)^{\frac{1}{2}}\langle f, D^{-1}f\rangle^{\frac{1}{2}},
\end{aligned}$$

where in the last step we used the Cauchy–Schwarz inequality. It follows

$$\begin{aligned}
\langle f, D^{-1}f\rangle &\le \sum_{k\sim k'}\frac{(\nabla_{kk'}D_0^{-1}f)^2}{e^{(u_k+u_{k'})}} + \sum_k \tilde{\eta}^w_k \frac{(D_0^{-1}f)_k^2}{e^{u_k}} \\
&= \sum_{k\sim k'}(\nabla_{kk'}D_0^{-1}\delta_j)^2\, e^{u_j-(u_k+u_{k'})} + \sum_k \tilde{\eta}^w_k (D_0^{-1}\delta_j)_k^2\, e^{u_j-u_k}
\end{aligned}$$

where we used the explicit form of f. Therefore,

$$\begin{aligned}
\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[\langle f, D^{-1}f\rangle] \le &\sum_{k\sim k'}(\nabla_{kk'}D_0^{-1}\delta_j)^2\,\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[e^{u_j-(u_k+u_{k'})}] \\
&+ \sum_k \tilde{\eta}^w_k (D_0^{-1}\delta_j)_k^2\,\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[e^{u_j-u_k}].
\end{aligned}$$

Note that

$$\begin{aligned}
&\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[e^{u_j-(u_k+u_{k'})}] \le 4\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[(\cosh(u_j-u_k))^2]^{\frac{1}{2}}\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[(\cosh u_{k'})^2]^{\frac{1}{2}}, \\
&\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[e^{u_j-u_k}] \le\ 2\,\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[\cosh(u_j-u_k)].
\end{aligned}$$

The bounds (B.3) and (B.4) in Appendix B ensure $\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[(\cosh(u_j - u_k))^m] \le 2$ for all $j,k \in \Lambda_L$, and $\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[(\cosh u_k)^2] \le 8$ for all $j \in \Lambda_L$. Putting all these bounds together, we obtain

$$\begin{aligned}\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[\langle f, D^{-1} f\rangle] &\le 16\Big(\sum_{k\sim k'} (\nabla_{kk'} D_0^{-1}\delta_j)^2 + \sum_k \tilde{\eta}^w_k (D_0^{-1}\delta_j)^2_k\Big) \\ &= 16\langle \delta_j, D_0^{-1}\delta_j\rangle = 16(-\Delta_{\Lambda_L} + \tilde{\eta})^{-1}_{jj} \le C_2, \end{aligned} \tag{5.7}$$

for some constant C_2 independent of j and L, since we are in dimension $d \ge 3$. This concludes the proof of the lemma. ∎

6. An alternative approach

Some of the above results can also be obtained by using the properties of the *infinite volume* measure ν^W, defined in (1.2). This alternative approach also provides the improved bound (1.12) in Theorem 3. In this section, we higlight the main differences with the finite volume approach and give the proof of (1.12). For more details, see [30].

To explain the strategy, we need to introduce a few preliminary notions and results. Recall that $\Lambda_L = [-L, L]^d \cap \mathbb{Z}^d$. We define, for every $i \in \mathbb{Z}^d$ and $L \in \mathbb{N}_{\ge 1}$,

$$\psi_L(i) := \begin{cases} 1 & \text{if } i \notin \Lambda_L, \\ \sum_{k\in\partial\Lambda_L} \mathcal{H}^{-1}_{\beta,\Lambda_L}(i,k)\eta^w_{\Lambda_L}(k) = e^{u_i(\beta_{\Lambda_L})} & \text{if } i \in \Lambda_L. \end{cases}$$

The following result is an extract of [34, Theorem 1].

Proposition 20. *The following facts hold true.*

(1) *For every $(i,j) \in \mathbb{Z}^d$, $(\mathcal{H}^{-1}_{\beta,\Lambda_L}(i,j))_{L\in\mathbb{N}_{\ge 1}}$ is increasing ν^W-a.s. Moreover, it converges toward some almost surely finite random variable which is denoted by $\widehat{G}(i,j)$.*

(2) *For every $i \in \mathbb{Z}^d$, $(\psi_L(i))_{L\in\mathbb{N}_{\ge 1}}$ is a positive martingale with respect to the filtration $(\sigma(\beta_i, i \in \Lambda_L), L \in \mathbb{N}_{\ge 1})$.*

(3) *For every $i \in \mathbb{Z}^d$, the bracket of $(\psi_L(i))_{L\in\mathbb{N}_{\ge 1}}$ equals $(\mathcal{H}^{-1}_{\beta,\Lambda_L}(i,i))_{L\in\mathbb{N}_{\ge 1}}$. In particular, $(\psi_L(i)^2 - \mathcal{H}^{-1}_{\beta,\Lambda_L}(i,i))_{L\in\mathbb{N}_{\ge 1}}$ is a martingale for every $i \in \mathbb{Z}^d$.*

By [34, Theorem 2], $\widehat{G}$ is the inverse of the infinite volume operator $\mathcal{H}_\beta$ in the following sense: $\widehat{G}(i,j) := \lim_{\varepsilon\to 0}(\mathcal{H}_\beta + \varepsilon)^{-1}(i,j)$, ν^W-a.s. Moreover, for every i,j, $\varepsilon \mapsto (\mathcal{H}_\beta + \varepsilon)^{-1}(i,j)$ is increasing ν^W-a.s. These facts are the key for the construction of the infinite volume environment of the related vertex reinforced jump process. A first application is the improved bound (1.12).

Proposition 21 (Upper bound on the IDS for large W). *For $d \geq 3$, there exists $W_0 > 1$ such that, for all $W \geq W_0$, the function $E \mapsto N(E)$ satisfies the bound*

$$N(E, H_\beta) \leq C'E \quad \textit{for all } E > 0,$$

for some constant $C' > 0$ independent of W.

Proof. Note that $N(E, H_\beta) = N(WE, \mathcal{H}_\beta) =: \widetilde{N}(WE)$. In the rest of the proof, we will work with $\widetilde{N}$. By [1, Section 3.3], for every bounded continuous function f,

$$\int_0^{+\infty} f(u)d\widetilde{N}(u) = \mathbb{E}^W[f(\mathcal{H}_\beta)(0,0)]$$

where $f(\mathcal{H}_\beta)$ is an operator which is well defined because $\mathcal{H}_\beta$ is self adjoint. In particular, for every $\varepsilon > 0$, it holds that

$$\int_0^{+\infty} \frac{1}{u+\varepsilon} d\widetilde{N}(u) = \mathbb{E}^W[(\mathcal{H}_\beta + \varepsilon)^{-1}(0,0)]. \tag{6.1}$$

Furthermore, as remarked above,

$$(\mathcal{H}_\beta + \varepsilon)^{-1}(0,0) \xrightarrow[\varepsilon \to 0]{} \widehat{G}(0,0),$$

ν^W-a.s. and this convergence is increasing. Therefore, by monotone convergence theorem, for every $\varepsilon > 0$,

$$\mathbb{E}^W[(\mathcal{H}_\beta + \varepsilon)^{-1}(0,0)] \xrightarrow[\varepsilon \to 0]{} \mathbb{E}^W[\widehat{G}(0,0)]$$

and

$$\int_0^{+\infty} \frac{1}{u+\varepsilon} d\widetilde{N}(u) \xrightarrow[\varepsilon \to 0]{} \int_0^{+\infty} \frac{1}{u} d\widetilde{N}(u).$$

Thus, if we make ε go to 0 in (6.1), this implies that, ν^W-a.s.,

$$\int_0^{+\infty} \frac{1}{u} d\widetilde{N}(u) = \mathbb{E}^W[\widehat{G}(0,0)].$$

Using Fatou's lemma, it yields

$$\begin{aligned} \int_0^{+\infty} \frac{1}{u} d\widetilde{N}(u) = \mathbb{E}^W[\widehat{G}(0,0)] &\leq \liminf_{L \to +\infty} \mathbb{E}^W[\mathcal{H}_{\beta,\Lambda_L}^{-1}(0,0)] \\ &= \liminf_{L \to +\infty} \frac{1}{W} \mathbb{E}_{u,\Lambda_L}^{W,\eta^w}[D^{-1}(0,0)e^{2u_0}] \\ &= \liminf_{L \to +\infty} \frac{1}{W} \mathbb{E}_{u,\Lambda_L}^{W,\eta^w}[\langle f, D^{-1} f \rangle], \end{aligned}$$

where the matrix D was defined in (5.6) and $f_k := \delta_{k0} e^{u_0}$ (instead of $f_k := \delta_{kj} e^{u_j/2}$ in the proof of Lemma 19). Repeating the same arguments as in the proof of Lemma 19, we obtain

$$\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[\langle f, D^{-1} f\rangle] \le \sum_{k\sim k'} (\nabla_{kk'} D_0^{-1}\delta_0)^2\, \mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[e^{2u_0-(u_k+u_{k'})}] + \sum_k \tilde{\eta}^w_k (D_0^{-1}\delta_0)^2_k\, \mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[e^{2u_0-u_k}],$$

where remember that $D_0 := D(0) = -\Delta_{\Lambda_L} + \tilde{\eta}^w$ and $\tilde{\eta}^w_k := \eta^w_k / W$. Note that

$$\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[e^{2u_0-(u_k+u_{k'})}] \le\ 4\,\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[(\cosh(u_0-u_k))^2]^{\frac12}\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[(\cosh(u_0-u_{k'}))^2]^{\frac12},$$
$$\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[e^{2u_0-u_k}] \le\ 4\,\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[(\cosh(u_0-u_k))^2]^{\frac12}\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[(\cosh u_0)^2]^{\frac12}.$$

Using (B.3) and Lemma 28, we obtain (cf. (5.7))

$$\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[\langle f, D^{-1} f\rangle] \le 16\langle\delta_0, D_0^{-1}\delta_0\rangle = 16(-\Delta_{\Lambda_L} + \tilde{\eta}^w)^{-1}(0,0) \le C_2,$$

where $C_2 > 0$ is the same constant we obtained in (5.7) and we used that we are in dimension $d \ge 3$. Hence, $\int_0^{+\infty} \frac{1}{u} d\tilde{N}(u) \le C_2/W$. It follows

$$N(E, H_\beta) = \tilde{N}(WE) = \int_0^{WE} \frac{u}{u} d\tilde{N}(u) \le WE \int_0^{+\infty} \frac{1}{u} d\tilde{N}(u)\ \le\ C_2 E.$$

This concludes the proof setting $C' := C_2$. ■

Remark. Note that $(\mathcal{H}^{-1}_{\beta,\Lambda_L}(0,0))_{L\in\mathbb{N}_{\ge1}}$ is the quadratic variation of the martingale $(\psi_L(0))_{L\in\mathbb{N}_{\ge1}}$ (cf. Proposition 20). This observation gives the slightly weaker estimate

$$\int_0^{+\infty} \frac{1}{u} d\tilde{N}(u) = \mathbb{E}^W[\hat{G}(0,0)] \le \liminf_{L\to+\infty} \mathbb{E}^W[\mathcal{H}^{-1}_{\beta,\Lambda_L}(0,0)] = \liminf_{L\to+\infty} \mathbb{E}^W[\psi_L(0)^2] \le 16,$$

where in the last inequality we used Lemma 28 together with

$$\mathbb{E}^W[\psi_L(0)^2] = \mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[e^{2u_0}] \le 4\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[(\cosh u_0)^2] \le 16.$$

The infinite volume measure approach also gives an alternative proof of the lower bound for $N(E)$. For this, we need some more definitions. Setting for $i \in \mathbb{Z}^d$, we define

$$\tilde{\beta}_i := \beta_i - \delta_{i,0}\frac{1}{2\hat{G}(0,0)}. \tag{6.2}$$

We have the following result.

Proposition 22 ([17, Proposition 2.4]). *Recall the definition of $W_{\rm cr}$ in (1.9). Then, for all $W < W_{\rm cr}$, $1/(2\widehat{G}(0,0))$ has density $\mathbf{1}_{\gamma>0}\, e^{-\gamma}/\sqrt{\pi\gamma}$. Moreover, $\tilde{\beta}$ and $1/(2\widehat{G}(0,0))$ are independent random variables.*

Note that this proposition works for any W such that the corresponding reinforced jump process is recurrent. This is true in particular for $W < W_{\rm cr}$. The variable $\tilde{\beta}_i$ arises naturally as the jump rate of the vertex reinforced jump process at vertex i (see [34, Theorem 1.(iii)]). In the following, we will consider $\widetilde{\mathcal{H}}_\beta := 2\tilde{\beta} - P^W$ and its Dirichlet restriction on the finite box Λ_L $\widetilde{\mathcal{H}}^D_{\beta,\Lambda_L}$ (cf (1.5)).

Finally, recall that the graph $\Lambda_L \cup \delta$ has vertex set $\Lambda_L \cup \{\delta\}$ and edge set $E(\Lambda_L) \cup \{\{i,\delta\}|\ i \in \Lambda_L\}$, and we defined $W_{i,\delta} = \eta_i^w = \sum_{j\sim i, j\notin\Lambda_L} W$ for all $i \in \Lambda_L$. Now, consider an electrical network on $\Lambda_L \cup \delta$ with conductances

$$
\begin{aligned}
c(i,j) &:= W\frac{\widehat{G}(0,i)\widehat{G}(0,j)}{\widehat{G}(0,0)^2} && \text{for all } i,j \in \Lambda_L,\\
c(i,\delta_L) &:= \sum_{\substack{j\sim i\\ j\notin\Lambda_L}} W\Big(\frac{\widehat{G}(0,i)\widehat{G}(0,j)}{\widehat{G}(0,0)^2} + \frac{\widehat{G}(0,i)^2}{\widehat{G}(0,0)^2}\Big) && \text{for all } i \in \Lambda_L,
\end{aligned}
$$

and let $\mathcal{R}_L(0 \leftrightarrow \delta)$ be the effective resistance of the random walk associated with this network. The following proposition is proved in [30].

Proposition 23. *Let $W < W_{\rm cr}$. Then, for every $L \in \mathbb{N}_{\geq 1}$,*

$$(\widetilde{\mathcal{H}}^D_{\beta,\Lambda_L})^{-1}(0,0) = \mathcal{R}_L(0 \leftrightarrow \delta).$$

Thanks to this result, we can use some tricks from the theory of electrical networks (e.g., [25, Chapter 2]) to construct an alternative proof of Theorem 1. We sketch below the argument.

Proof of Theorem 1 (II). As in the proof given in Section 4, we start from

$$N(E,\mathcal{H}_\beta) \geq \frac{1}{|\Lambda_L|}\nu^W\Big((\mathcal{H}^D_{\beta,\Lambda_L})^{-1}(0,0) \geq \frac{1}{E}\Big).$$

Note that

$$
\begin{aligned}
\nu^W\Big(\frac{1}{2\widehat{G}(0,0)} \leq \frac{E}{4}\Big) &= \nu^W\Big(\widehat{G}(0,0) \geq \frac{2}{E}\Big)\\
&\leq \nu^W\Big(\widehat{G}(0,0) - (\mathcal{H}^D_{\beta,\Lambda_L})^{-1}(0,0) \geq \frac{1}{E}\Big)\\
&\quad + \nu^W\Big((\mathcal{H}^D_{\beta,\Lambda_L})^{-1}(0,0) \geq \frac{1}{E}\Big).
\end{aligned}
$$

Consequently,

$$N(E, \mathcal{H}_\beta) \geq \frac{1}{|\Lambda_L|}\Big[\nu^W\Big(\frac{1}{2\widehat{G}(0,0)} \leq \frac{E}{4}\Big) - \nu^W\Big(\widehat{G}(0,0) - (\mathcal{H}^D_{\beta,\Lambda_L})^{-1}(0,0) \geq \frac{1}{E}\Big)\Big]$$
$$\geq \frac{1}{|\Lambda_L|}\Big[C\sqrt{E} - \nu^W\Big(\widehat{G}(0,0) - (\mathcal{H}^D_{\beta,\Lambda_L})^{-1}(0,0) \geq \frac{1}{E}\Big)\Big],$$

where $C > 0$ is some constant and we used that $1/(2\widehat{G}(0,0))$ is a $\Gamma(1/2, 1)$ random variable (cf. Proposition 22). We claim that, for E small and L large enough,

$$\nu^W\Big(\widehat{G}(0,0) - (\mathcal{H}^D_{\beta,\Lambda_L})^{-1}(0,0) \geq \frac{1}{E}\Big) \ll \sqrt{E},$$

which implies the result. To prove this asymptotic domination, note that $\widetilde{\mathcal{H}}^D_{\beta,\Lambda_L} - \mathcal{H}^D_{\beta,\Lambda_L} = -\delta_0 1/\widehat{G}(0,0)$ (cf. equation (6.2)). Therefore,

$$(\mathcal{H}^D_{\beta,\Lambda_L})^{-1}(0,0) = \frac{(\widetilde{\mathcal{H}}^D_{\beta,\Lambda_L})^{-1}(0,0)}{1 + \frac{(\widetilde{\mathcal{H}}^D_{\beta,\Lambda_L})^{-1}(0,0)}{\widehat{G}(0,0)}}$$

and thus, using Proposition 23,

$$\widehat{G}(0,0) - (\mathcal{H}^D_{\beta,\Lambda_L})^{-1}(0,0) \leq \frac{\widehat{G}(0,0)^2}{(\widetilde{\mathcal{H}}^D_{\beta,\Lambda_L})^{-1}(0,0)} = \frac{\widehat{G}(0,0)^2}{\mathcal{R}_L(0 \leftrightarrow \delta)}.$$

Therefore, we have

$$\begin{aligned}\nu^W\Big(\widehat{G}(0,0) - (\mathcal{H}^D_{\beta,\Lambda_L})^{-1}(0,0) \geq \frac{1}{E}\Big) &\leq \nu^W\Big(\frac{\widehat{G}(0,0)^2}{\mathcal{R}_L(0 \leftrightarrow \delta)} \geq \frac{1}{E}\Big)\\ &= \int_0^{+\infty} \frac{e^{-\gamma}}{\sqrt{\pi\gamma}}\nu^W\Big(\frac{1}{\mathcal{R}_L(0 \leftrightarrow \delta)} \geq \frac{4\gamma^2}{E}\Big)d\gamma\end{aligned} \tag{6.3}$$

where, in the last equality of (6.3), we used Proposition 22 and the measurability of $(\mathcal{H}^D_{\beta,\Lambda_L})^{-1}(0,0) = \mathcal{R}_L(0 \leftrightarrow \delta)$ with respect to $\tilde{\beta}$. Then, one can use classical a result in electrical networks (the Nash–Williams inequality) to control the inverse of $\mathcal{R}_L(0 \leftrightarrow \delta)$, using the local conductances on the boundary of Λ_L. Finally, we can control these local conductances thanks to Corollary 9 if we choose a "good" L as a function of E. ∎

A. Monotonicity

The following monotonicity result can be found in [29, Theorem 6].

Theorem 24. *Let V be a finite set, $W^+, W^- \in \mathbb{R}_{\geq 0}^{V\times V}$ two families of non-negative weights satisfying $W_{jj}^{\pm} = 0$ for all $j \in V$, and*

$$W_{ji}^- = W_{ij}^- \leq W_{ij}^+ = W_{ji}^+ \quad \text{for all } i \neq j.$$

Let E^+ (resp., E^-) be the set of pairs with positive weight $W_{ij}^+ > 0$ (resp $W_{ij}^- > 0$) and denote by $\mathcal{G}^{\pm} = (V, E^{\pm})$ the corresponding graphs.

If $i, j \in V$ are connected by $\mathcal{G}^-$, then it holds

$$\mathbb{E}_{\mathcal{G}^-}^{W^-,0}\Big[f\Big(\frac{\mathcal{H}_{\beta,V,W^-}^{-1}(j,k)}{\mathcal{H}_{\beta,V,W^-}^{-1}(j,j)}\Big)\Big] \leq \mathbb{E}_{\mathcal{G}^+}^{W^+,0}\Big[f\Big(\frac{\mathcal{H}_{\beta,V,W^+}^{-1}(j,k)}{\mathcal{H}_{\beta,V,W^+}^{-1}(j,j)}\Big)\Big]$$

for any concave function f. Here, we write $\mathcal{H}_{\beta,V,W^{\pm}}$ instead of $\mathcal{H}_{\beta,V}$ to emphasize the dependence of $W^{\pm}$.

In this paper, we use the following corollary.

Corollary 25. *Let $\mathcal{G} = (V, E)$ be a connected finite graph and $W \in \mathbb{R}_{>0}^{E}$ a given set of weights. Fix a vertex $j_0 \in V$ and set $\eta_j = \eta\delta_{jj_0}$ with $\eta > 0$ (one pinning at j_0). It holds, for all $j \in V$,*

$$\mathbb{E}_{\mathcal{G}}^{W,\eta}\left[\sqrt{\frac{\mathcal{H}_{\beta,V}^{-1}(j_0,j)}{\mathcal{H}_{\beta,V}^{-1}(j_0,j_0)}}\right] \geq \mathbb{E}_{\mathcal{G}}^{W,0}\left[\sqrt{\frac{\mathcal{H}_{\beta,V}^{-1}(j_0,j)}{\mathcal{H}_{\beta,V}^{-1}(j_0,j_0)}}\right].$$

Proof. The measure $\nu_{\mathcal{G}}^{W,\eta}$ is the marginal of $\nu_{\mathcal{G}^\delta}^{W,0}$, where the graph $\mathcal{G}^\delta$ has vertex set $V \cup \{\delta\}$ and edge set $E \cup \{j_0, \delta\}$, and we defined $W_{j_0,\delta} = \eta$. Moreover, by resolvent expansion, we have

$$\begin{aligned}
\mathcal{H}_{\beta,V\cup\{\delta\}}^{-1}(j_0,j) &= \mathcal{H}_{\beta,V}^{-1}(j_0,j)\,[1 + \eta^2\mathcal{H}_{\beta,V}^{-1}(j_0,j_0)\mathcal{H}_{\beta,V\cup\{\delta\}}^{-1}(\delta,\delta)],\\
\mathcal{H}_{\beta,V\cup\{\delta\}}^{-1}(j_0,j_0) &= \mathcal{H}_{\beta,V}^{-1}(j_0,j_0)\,[1 + \eta^2\mathcal{H}_{\beta,V}^{-1}(j_0,j_0)\mathcal{H}_{\beta,V\cup\{\delta\}}^{-1}(\delta,\delta)],
\end{aligned}$$

and hence

$$\begin{aligned}
\mathbb{E}_{\mathcal{G}}^{W,\eta}\left[\sqrt{\frac{\mathcal{H}_{\beta,V}^{-1}(j_0,j)}{\mathcal{H}_{\beta,V}^{-1}(j_0,j_0)}}\right] &= \mathbb{E}_{\mathcal{G}^\delta}^{W,0}\left[\sqrt{\frac{\mathcal{H}_{\beta,V}^{-1}(j_0,j)}{\mathcal{H}_{\beta,V}^{-1}(j_0,j_0)}}\right]\\
&= \mathbb{E}_{\mathcal{G}^\delta}^{W,0}\left[\sqrt{\frac{\mathcal{H}_{\beta,V\cup\{\delta\}}^{-1}(j_0,j)}{\mathcal{H}_{\beta,V\cup\{\delta\}}^{-1}(j_0,j_0)}}\right].
\end{aligned}$$

Define $\widetilde{W}_{ij} = W_{ij}$ for all $i \sim j \in V$ and $\widetilde{W}_{j_0\delta} = 0$. Then, $W_{ij} \geq \widetilde{W}_{ij}$ for all $i \sim j \in \mathcal{G}^\delta$ and the graph generated by $\widetilde{W}$ connects j_0 to j for all $j \in V$. Since $f(x) = \sqrt{x}$ is a concave function, by Theorem 24 we have

$$\mathbb{E}_{\mathcal{G}^\delta}^{W,0}\left[\sqrt{\frac{\mathcal{H}^{-1}_{\beta,V\cup\{\delta\},W}(j_0,j)}{\mathcal{H}^{-1}_{\beta,V\cup\{\delta\},W}(j_0,j_0)}}\right] \geq \mathbb{E}_{\mathcal{G}^\delta}^{\widetilde{W},0}\left[\sqrt{\frac{\mathcal{H}^{-1}_{\beta,V\cup\{\delta\},\widetilde{W}}(j_0,j)}{\mathcal{H}^{-1}_{\beta,V\cup\{\delta\},\widetilde{W}}(j_0,j_0)}}\right].$$

Since $\widetilde{W}_{j_0\delta} = 0$ and $\widetilde{W} = W$ on V, $2\beta_\delta$ is independent of the other random variables and we have $\mathcal{H}_{\beta,V\cup\{\delta\},\widetilde{W}} = 2\beta_\delta \oplus \mathcal{H}_{\beta,V}$, where we abbreviated $\mathcal{H}_{\beta,V} = \mathcal{H}_{\beta,V,W}$. Therefore,

$$\mathcal{H}^{-1}_{\beta,V\cup\{\delta\},\widetilde{W}}(j_0,j) = \mathcal{H}^{-1}_{\beta,V}(j_0,j), \quad \mathcal{H}^{-1}_{\beta,V\cup\{\delta\},\widetilde{W}}(j_0,j_0) = \mathcal{H}^{-1}_{\beta,V}(j_0,j_0).$$

It follows

$$\begin{aligned}\mathbb{E}_{\mathcal{G}^\delta}^{\widetilde{W},0}\left[\sqrt{\frac{\mathcal{H}^{-1}_{\beta,V\cup\{\delta\},\widetilde{W}}(j_0,j)}{\mathcal{H}^{-1}_{\beta,V\cup\{\delta\},\widetilde{W}}(j_0,j_0)}}\right] &= \mathbb{E}_{\mathcal{G}^\delta}^{\widetilde{W},0}\left[\sqrt{\frac{\mathcal{H}^{-1}_{\beta,V}(j_0,j)}{\mathcal{H}^{-1}_{\beta,V}(j_0,j_0)}}\right] \\ &= \mathbb{E}_{\mathcal{G}}^{W,0}\left[\sqrt{\frac{\mathcal{H}^{-1}_{\beta,V}(j_0,j)}{\mathcal{H}^{-1}_{\beta,V}(j_0,j_0)}}\right],\end{aligned}$$

where in the last step we used that β_δ is independent of the other variables. This concludes the proof. ∎

B. Long range order estimates on the field u associated to the $H^{2|2}$-model

Recall the definition of $\mu_\Lambda^{W,\eta}(u)$ and η^w in (2.4) and (1.4), respectively.

Lemma 26. *For any $W > 0$, $d \geq 1$, $j \in \Lambda$ such that $\eta_j > 0$ and $m \leq \eta_j/2$, we have*

$$\mathbb{E}_{u,\Lambda}^{W,\eta}[(\cosh u_j)^m] \leq \frac{1}{1-\frac{m}{\eta_j}} \leq 2. \tag{B.1}$$

In particular, in the case $\Lambda = \Lambda_L$ and $\eta = \eta^w$, we have

$$\mathbb{E}_{u,\Lambda_L}^{W,\eta^w}[(\cosh u_j)^m] \leq \frac{1}{1-\frac{m}{W}} \leq 2 \tag{B.2}$$

for all $j \in \partial\Lambda_L$ and $m \leq W/2$.

Proof. This inequality follows by a supersymmetric Ward identity analog to the one in [12, Section 5.1]. See also in [9, Lemma 4.2]. We sketch here the argument. We refer to the notation in [12], in particular,

$$B_j = \cosh u_j + \frac{1}{2}s_j^2 e^{u_j}$$

where s_j is a real variable. Denoting the supersymmetric mean by $\langle\cdot\rangle_{\text{susy}}$, we have

$$1 = \langle (B_j + \overline{\psi}_j \psi_j e^{u_j})^m \rangle_{\text{susy}} = \mathbb{E}^{W,\eta}_{u,\Lambda}\Big[B_j^m \Big(1 - \frac{m}{B_j} \mathcal{H}^{-1}_{\beta,\Lambda_L}(j,j) \Big) \Big],$$

where $\overline{\psi}_j, \psi_j$ are anti-commuting variables. The bound (B.1) now follows from $B_j \geq 1$ and $\mathcal{H}^{-1}_{\beta,\Lambda_L}(j,j) \leq 1/\eta_j$. The bound (B.2) is obtained observing that $\eta_j^w \geq W$ for all $j \in \partial\Lambda_L$. ∎

The following result has been proved in [12, Theorem 1].

Theorem 27. *Let $d \geq 3$. There exists $W_0' = W_0'(d) > 1$ such that, for all $W \geq W_0'$, we have*

$$\mathbb{E}^{W,\eta}_{u,\Lambda_L}[(\cosh(u_j - u_k))^m] \leq 2 \quad \textit{for all } j,k \in \Lambda_L,\ m \leq W^{\frac{1}{8}}. \tag{B.3}$$

This bound holds for all η.

Proof. Although the model considered in [12] has uniform pinning $\eta_j = \varepsilon > 0$ for all $j \in \Lambda_L$, the proof of Theorem 1 is completely independent from the pinning choice. Also, while in [12] only $d = 3$ is considered, the same proof works for any $d \geq 3$. Indeed, the key dimension-dependent result (Lemma 5) is proved for general dimension $d \geq 3$. The same strategy was used in [10] in the case when the edge weights W_e are independent Gamma distributed variables. For a related result on a one-dimensional chain with non homogeneous weights, see [8]. ∎

Lemma 28. *Let $d \geq 3$ and $W_0' = W_0'(d) > 1$ be the parameter introduced in Theorem 27 above. For all $W \geq W_0 := \max\{W_0', 4^8\}$, we have*

$$\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[(\cosh u_k)^2] \leq 8 \quad \textit{for all } k \in \Lambda_L. \tag{B.4}$$

Proof. By Lemma 26, for any $W \geq W_0$, $j \in \partial\Lambda$, and $m \leq W/2$, we have

$$\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[(\cosh u_j)^m] \leq \frac{1}{1 - \frac{m}{W}} \leq 2. \tag{B.5}$$

Fix now $k \in \Lambda_L \setminus \partial\Lambda_L$ and let j be some vertex on $\partial\Lambda_L$. We have the bound $\cosh u_k \leq 2\cosh u_j \cosh(u_k - u_j)$, and hence

$$\mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[(\cosh u_k)^2] \leq\ 4\, \mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[(\cosh u_j)^4]^{\frac{1}{2}} \mathbb{E}^{W,\eta^w}_{u,\Lambda_L}[(\cosh(u_j - u_k))^4]^{\frac{1}{2}}.$$

The constraint $W \geq \max\{W_0, 4^8\}$ ensures $4 \leq W^{1/8}$ and $4 \leq W/2$. The result now follows from (B.3) and (B.5). ∎

Remark. Note that in [12] the bound (B.4) is proved in Theorem 2 and requires quite some work due to the presence of a uniform *small* pinning $\varepsilon \sim 1/|\Lambda_L|^{1-s}$, $0 < s \ll 1$. Here, the same bound follows easily from (B.3) and the fact that we have *large* pinning at the boundary $\eta_j^w \geq W \gg 1$ for all $j \in \partial\Lambda_L$.

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