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The wave kernel on asymptotically complex hyperbolic manifolds

Hadrian Quan

Abstract. We study the behavior of the wave kernel of the Laplacian on asymptotically complex hyperbolic manifolds for finite times. We show that the wave kernel on such manifolds belongs to an appropriate class of Fourier integral operators and analyze its trace. This construction proves that the singularities of its trace are contained in the set of lengths of closed geodesics and we obtain an asymptotic expansion for the trace at time zero.

1. Introduction

The focus of this work is an analysis of the behavior of the solutions of the wave equation for finite times in the setting of *asymptotically complex hyperbolic manifolds*. These spaces were first introduced by Epstein, Mendoza, and Melrose [8], and more recently have been investigated extensively by [10, 16, 16, 18, 19, 24, 25]. This class of manifolds includes certain quotients of complex hyperbolic space by discrete groups, as well as strictly pseudoconvex domains in Stein manifolds equipped with Kähler metrics of Bergman type. In this work, we extend major results which study the wave kernel of asymptotically real hyperbolic manifolds to this complex setting. Joshi and Sá Barreto [23] studied the wave kernel of such manifolds by exhibiting this operator as an element of a certain algebra of Fourier integral operators which have been adapted to the geometry at infinity of this class of real asymptotically hyperbolic manifolds.

We say a non-compact Riemannian manifold (X, g), of *complex* dimension $(n+1)$, is an *asymptotically complex hyperbolic manifold* (hereafter ACH manifold) if the following holds. We assume X compactifies to a $\mathcal{C}^\infty$ manifold $\overline{X}$, compact with boundary, equipped with a choice of *boundary defining function* r (hereafter, a bdf). This is a smooth nonnegative function on $\overline{X}$ which such that

$$\overline{X} = \{r = 0\}, \quad dr|_{\partial \overline{X}} \neq 0.$$

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We further assume the boundary admits: (1) a contact form $\theta \in \Omega^1(\partial\overline{X})$ defined as satisfying $\theta \wedge (d\theta)^n \neq 0$; (2) an almost complex structure $J\colon \operatorname{Ker}\theta \to \operatorname{Ker}\theta$; such that $d\theta(\cdot, J\cdot)$ is a symmetric, positive-definite bilinear form on $\operatorname{Ker}\theta$. Then, we say that (X, g) is an *ACH manifold* if there is a tubular neighborhood $\Phi\colon U \to \partial\overline{X} \times [0, \varepsilon)_r$ of the boundary $\partial\overline{X}$ such that

$$g \sim \Phi^* g_\theta \quad \text{as } r \to 0, \qquad g_\theta = \Big(\frac{4dr^2}{r^2} + \frac{d\theta(\cdot, J\cdot)}{r^2} + \frac{\theta^2}{r^4}\Big) = \frac{4dr^2 + g_0(r)}{r^2}. \tag{1.1}$$

In particular, for another choice of boundary defining function, $\tilde{r}$, we observe that $r^4 g|_{\partial\overline{X}} = e^{4f}\theta$, for some $f \in \mathcal{C}^\infty(\overline{X})$. Denoting the conformal class of our contact structure by $[\theta]$, we can consider the boundary as being endowed with the structure of a *conformal pseudohermitian manifold* $(\partial\overline{X}, [\theta], J)$.

Before continuing, we require an additional hypothesis, which is that g is an *even metric*; i.e., the dual metric g^{-1} defined on $T^*\overline{X}$ has only even powers of r in a Taylor expansion at $r = 0$. This is automatic in the case of $\mathbb{H}^{n+1}_{\mathbb{C}}$, and necessary for the existence of a meromorphic continuation of the resolvent of Δ_g to all of $\mathbb{C}$ (in fact, the failure of this hypothesis implies the existence of at least one essential singularity in the continuation of the resolvent, see [15, 16]).

In the case that the metric of (X, g) is even in the above sense, we can replace the smooth structure on this manifold with its *even smooth structure*, denoted X_{even}. In this case, the smooth structure on X has been modified by declaring that only functions which are even in r are smooth with respect to X_{even}. This change of the smooth structure permits us to define a square root of our original defining function, and guarantee that it is an element of $\mathcal{C}^\infty(X_{\text{even}})$. Equivalently, the even smooth structure can be defined by declaring X_{even} is a smooth manifold with boundary, with bdf r^2. Throughout, we shall denote the square root of our bdf $\rho = r^2$.

Now, we state our main results on the behavior of solutions to the wave equation for small times. This question can be approached by a study of the fundamental solution to the wave equation, as in the work of Joshi and Sá Barreto [23] who studied the wave operator $\cos(t\sqrt{\Delta_g - (n+1)^2/4})$ in the setting of real asymptotically hyperbolic manifolds. This operator has Schwartz kernel $U(t, p, p')$ satisfying

$$\begin{cases} \Big(\partial_t^2 + \Delta_g - \dfrac{(n+1)^2}{4}\Big) U(t, p, p') = 0, \\ U(0, p, p') = \delta(p, p'), \quad \partial_t U(0, p, p') = 0, \end{cases}$$

and they prove that $\cos(t\sqrt{\Delta_g - (n+1)^2/4})$ resides in an algebra of Fourier integral operators. Having shown this, they use the results of [7, 20, 21] to study its (regularized) trace.

This construction of the wave group $U(t, p, p')$ as a Fourier integral operator was motivated by the analysis of the resolvent of a real asymptotically hyperbolic manifold initiated in [26]. Mazzeo and Melrose obtained their results by exhibiting the resolvent as an element of the "large" calculus of zero pseudodifferential operators $\Psi_0^*(M)$; i.e., those pseudodifferential operators with Schwartz kernels constructed as distributions on the blown-up space $\bar{M} \times_0 \bar{M}$, obtained by blowing up the intersection of the corner $\partial\bar{M} \times \partial\bar{M}$ with the diagonal $\bar{M}_{\text{diag}} \hookrightarrow \bar{M} \times \bar{M}$ in $\bar{M} \times \bar{M}$. The new boundary hypersurface resulting from this blow up is called the *front face*. (For an extended treatment on such blow ups, see [26, Section 3] and [14, 27].)

Along such lines, [23] construct a class of zero Fourier integral operators as those operators whose Schwartz kernels, when lifted to $\bar{M} \times_0 \bar{M}$, have support away from the *left* and *right* boundary faces (i.e., the lifts of $\partial\bar{M} \times \bar{M}$ and $\bar{M} \times \partial\bar{M}$ respectively). This greatly simplifies the construction of this class of operators, as typically the corners formed by the intersections of the left face (resp. right) with the front face would need to be incorporated into the definition of the operators; requiring the support of the Schwartz kernels avoid such corners allows their contributions to be neglected. In particular, due to the finite speed of propagation for the wave equation, a distribution which is initially supported only on the front face (such as $U(t, p, p')$) will remain supported in the interior of the front face for all finite time. Thus, [23] can construct a small time parametrix for the wave group while remaining entirely in this restricted calculus of zero Fourier integral operators.

Following this strategy, we begin with the notion of the Θ-stretched product, $\bar{X} \times_\Theta \bar{X}$, which is the analogous blow up of the double space $\bar{X} \times \bar{X}$ defining the class of Θ-pseudodifferential operators $\Psi_\Theta^*(X)$ used in the study of the resolvent initiated by [8]. With the appropriate definition of Θ-Fourier integral operators, we can quickly conclude as follows.

Theorem 1.1. *Let G be the length functional on T^*X (i.e., the dual metric). For each $t \in \mathbb{R}$, the graph of the time-t flow-out of the diagonal in $T^*X \times T^*X$ by the Hamilton vector field H_G is a canonical relation, denoted C. Furthermore, the wave group $U(t)$ is a Θ-FIO with respect to this canonical relation.*

Once we know the wave group is a Θ-Fourier integral operator, it is straightforward to use the results of [7, 21] to analyze the trace of $U(t, p, p')$. One subtlety is that the trace needs to replaced with a regularized trace, defined using a Hadamard regularization procedure using our choice of bdf ρ. Defining the cut-off wave trace,

$$T_\varepsilon(t) = \int_{\{\rho>\varepsilon\}} U(t, p, p),$$

we obtain the following.

Proposition 1.2. *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, the singular support of T_ε is contained in the set of periods of closed geodesics of X.*

With this result in hand, after choosing a smooth cutoff $\chi(t) \in \mathcal{C}_0^\infty(\mathbb{R})$ supported away from the lengths of all non-zero periods of closed geodesics, and, using the results of [20], we obtain a Duistermaat–Guillemin-type result for the inverse Fourier transform cutoff wave trace.

Theorem 1.3. *There exists $\{\omega_k\}_{k\in\mathbb{N}_0} \subset \mathbb{R}$ such that the renormalized trace ${}^R\operatorname{Tr} U(t)$ satisfies,*

$$\int_{\mathbb{R}} {}^R\operatorname{Tr} U(t)\chi(t)e^{it\mu}dt \sim \frac{1}{(2\pi)^{2n+2}} \sum_{k=0}^{\infty} \omega_k \mu^{2n+2-2k},$$

as $\mu \to 0$ and is rapidly decaying as $\mu \to -\infty$. The leading term, $\omega_0 = {}^R\operatorname{Vol}_g(X)$, is called the renormalized volume, *and can be computed as*

$$ {}^R\operatorname{Vol}_g(X) = \lim_{\varepsilon\to 0}\left[\int_{\{\rho>\varepsilon\}} d\operatorname{Vol}_g - \sum_{j=-2n-2}^{-1} C_j\varepsilon^j - C_0\log\Big(\frac{1}{\varepsilon}\Big)\right], \tag{1.2}$$

where C_j are the unique real numbers such that this limit exists.

Finally, we remark on the appearance of the renormalized volume in Theorem 1.3. In the real hyperbolic setting, it is known that the renormalized volume is, in certain dimensions, independent on the choice of representative of the conformal infinity. Namely, for (M^{n+1}, g) a real asymptotically hyperbolic manifold, one can similarly define the renormalized volume as the finite part of the in the expansion of $\operatorname{Vol}_g(\{x \geq \varepsilon\})$ as $\varepsilon \to 0$, given a choice of bdf x. For n odd, the real hyperbolic renormalized volume is independent of h_0, the choice of conformal representative. On the other hand, for n even, we suddenly have the dependence of the renormalized volume on this choice of representative of $[h]$. This is result is the so-called holographic anomaly (see [17]) and motivates much of the interest of asymptotically hyperbolic manifolds in mathematical physics, for their connection with the anti-de Sitter/conformal field theory (AdS/CFT) correspondence.

More concretely, the volume expansion of (M^{n+1}, g) of an *Einstein* asymptotically hyperbolic metric, for n even, is given by

$$\begin{aligned}\operatorname{Vol}_g(\{x \geq \varepsilon\}) = V_{-n}\varepsilon^{-n} + V_{-n+2}\varepsilon^{-n+2} + \cdots + V_{-2}\varepsilon^{-2} \\ + V_0\log(1/\varepsilon) + {}^R\operatorname{Vol}_g(M) + o(1),\end{aligned}$$

and [13] first made the connection of V_0 to Branson's Q-curvature [4],

$$V_0 = 2c_{n/2}\int_{\partial M} Q,$$

for $c_{n/2}$ a dimensional constant. In the ACH setting, the renormalized volume was first studied at this level of generality by Matsumoto in [24]. Our construction of the renormalized wave trace thus provides an alternate proof of Matsumoto's result, via formula (1.2). For a general ACH metric, [24] generalizes this result for an analogue of Bransons Q-curvature. From his result, we obtain as a corollary that the constant d_0 in our Theorem 1.3 is given by

$$d_0 = \frac{2(-1)^{n+1}}{n!^2(n+1)!} \int_{\partial X} Q^g_\theta \theta \wedge (d\theta)^n.$$

This quantity is a global CR invariant of the boundary, thus leading to a pseudo-conformal analogue of the holographic anomaly. Given these results, there is strong connection between the renormalized volume of an ACH manifold and its spectrum. On the mathematical physics side, there seems to be relatively scarce work on this complex analogue of the AdS/CFT correspondence.

Background and related work. There is a long-standing research program investigating the spectral and scattering theory of *real* asymptotically hyperbolic manifolds, see e.g., [1–3, 5, 9, 12, 13, 22, 28] and references contained therein, for a small sample of the surrounding work. A non-compact Riemannian manifold (M, g) of real dimension $(n+1)$ is called *asymptotically hyperbolic* if it compactifies to a $\mathcal{C}^\infty$ manifold $\bar{M}$ with compact boundary $\partial\bar{M}$, equipped with a boundary defining function ρ, and such that $\rho^2 g$ is a $\mathcal{C}^\infty$ metric which is non-degenerate up to the boundary, and moreover that $|d\rho|^2_{\rho^2 g} \equiv 1$ at $\partial\bar{M}$.

As proven in [22], these geometric hypotheses are equivalent to the existence of a product-type decomposition (cf. with (1.1)) of the metric at infinity $M \sim [0, \varepsilon)_\rho \times \partial M$, such that

$$g = \frac{d\rho^2 + g_0(\rho)}{\rho^2},$$

where $g_0(\rho)$ is a $\mathcal{C}^\infty$ 1-parameter family of $\mathcal{C}^\infty$ metrics on $\partial\bar{M}$. In this model, the boundary $\partial\bar{M}$ represents the geometric infinity of $\bar{M}$, analogous to the role played by the $\mathbb{S}^n$ at infinity in $\mathbb{H}^{n+1}_{\mathbb{R}}$. In particular, the metric $\rho^2 g|_{\partial M}$ fixes a conformal representative of a metric on $\partial\bar{M}$. This should be contrasted with the conformal pseudohermitian structure induced by an ACH metric.

The spectrum of the Laplacian of such manifolds was first studied by [26]; they determined that it is comprised of finitely many L^2-eigenvalues $\sigma_{\mathrm{pp}}(\Delta_g) \subset (0, (n+1)^2/4)$ and the absolutely continuous spectrum $\sigma_{\mathrm{ac}}(\Delta_g) = [(n+1)^2/4, \infty)$. These results followed by proving that the resolvent of $\Delta_g - \zeta(n+1-\zeta)$ could be constructed within an exotic pseudodifferential operator calculus. The results of [23] built on this argument by extending this class of operators to a calculus of certain Fourier integral operators, which they called 0-*Fourier integral operators*. By exhibiting the wave kernel as an element in this class of 0-FIOs, this construction allowed them to argue as in [7] to conclude versions of our main results in the asymptotically hyperbolic setting.

2. The geometry of asymptotically complex hyperbolic manifolds

Because the construction of our adapted FIO-calculus entails a finer understanding of the geometry of an asymptotically complex hyperbolic manifold, we briefly recall the geometry of the Bergman-type metric our manifold is endowed with.

Let $(X, \partial X)$ be a non-compact manifold with closed boundary. We assume the boundary admits a contact form θ and an almost complex structure $J\colon \mathrm{Ker}(\theta) \to \mathrm{Ker}(\theta)$ (i.e., an endomorphism satisfying $J \circ J = -\mathrm{Id}_{\mathrm{Ker}(\theta)}$) such that $d\theta(\cdot, J\cdot)$ is symmetric positive definite on $\mathrm{Ker}(\theta)$. We consider a metric g_{ACH} of the following form: there is a boundary defining function ρ,

$$\partial X = \{\rho = 0\}, \quad d\rho|_{\partial X} \neq 0,$$

such that in a collar neighborhood $\varphi\colon [0,1)_\rho \times \partial X_{\omega,z} \to U$ it takes the form

$$\varphi^* g_{\mathrm{ACH}} = \frac{d\rho^2}{\rho^2} + \frac{d\theta(\cdot, J\cdot)}{\rho^2} + \frac{\theta \otimes \theta}{\rho^4} + \rho Q_\rho = \frac{d\rho^2 + h(\rho, \omega, z, d\omega, dz)}{\rho^2}, \quad (2.1)$$

where $(\mathcal{D}_\rho^{\mathbb{H}})^* Q_\rho$ is a smooth section of $S^2(T^*X) \cap \mathrm{Ker}(\iota_{\partial_\rho})$. Here, $\mathcal{D}_\rho^{\mathbb{H}}$ denotes the anisotropic dilation map

$$T_q \partial X = \mathcal{H}_q \oplus \mathcal{V}_q \ni (v_H, v_V) \xmapsto{\mathcal{D}_\rho^{\mathbb{H}}} (\rho v_H, \rho^2 v_V) \in \mathcal{H}_q \oplus \mathcal{V}_q = T_q \partial X,$$

with splitting induced by the choice of contact structure $(\partial X, \theta)$ (i.e., $\mathcal{H} = \mathrm{Ker}\,\theta$).

We observe that for any other choice of defining function $\tilde{\rho}$ we have

$$\tilde{\rho}^4 g|_{\partial X} = e^{4\omega_0} \theta \otimes \theta, \text{ for some } \omega_0 \in \mathcal{C}^\infty(X),$$

thus it is more natural to associate to g_{ACH} a conformal class of 1-forms $[\Theta]$. The boundary manifold equipped with the data of $(\partial X, \theta, J)$ is a *closed pseudohermitian manifold*. The corresponding *conformal pseudohermitian structure* $([\Theta], J)$ was called a Θ-*structure* in [8].

This Riemannian metric structure describes a non-compact incomplete manifold whose metric is asymptotic to complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^{n+1}$. A useful model of complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^{n+1}$ is given by

$$\mathbb{H}_{\mathbb{C}}^{n+1} = \{\zeta \in \mathbb{C}^{n+1} : Q(\zeta, \zeta) > 0\}, \quad \text{where } Q(\zeta, \zeta') = -\frac{i}{2}(\zeta_1 - \zeta_1') - \frac{1}{2}\sum_{j>1} \zeta_j \bar{\zeta}_j{}',$$

with boundary sphere equal to a compactification of the $(2n+1)$-dimensional Heisenberg group,

$$H_n := \{\zeta \in \mathbb{C}^{n+1} : Q(\zeta, \zeta) = 0\} = \left\{(\zeta_1, w) \in \mathbb{C}^{n+1} : \frac{1}{2}|w|^2 = \Im(\zeta_1)\right\} \simeq \mathbb{C}^n \times \mathbb{R}.$$

This model of complex hyperbolic space realizes $\mathbb{H}_{\mathbb{C}}^{n+1} \simeq \mathbb{R}^+ \times H_n$ with the coordinates

$$\rho(\zeta) = Q(\zeta, \zeta)^{1/2}, \quad w \in \mathbb{C}^n, \quad z = \Re(\zeta_1),$$

foliating $\mathbb{H}_{\mathbb{C}}^{n+1}$ by a family of H_n-hypersurfaces. Writing

$$w = x + iy$$

in these coordinates, we can also write the contact form at the boundary as

$$\theta_0 = dz + \sum_{j=1}^{n} y_j dx^j - x_j dy^j,$$

and the metric on complex hyperbolic is the Bergman metric,

$$g_{\mathrm{Berg}} = \frac{4d\rho^2 + 2|dw|^2}{\rho^2} + \frac{\theta_0^2}{\rho^4}.$$

Finally, we discuss the complexified hyperbolic space as a semi-direct product: there is parabolic dilation on H_n (consistent with the bracket relations of the Lie algebra $\mathfrak{h}$) given by $\mathcal{D}_\delta(x, y, z) = (\delta x, \delta y, \delta^2 z)$. The group law on the semidirect product $\mathbb{H}_{\mathbb{C}}^{n+1} \simeq \mathbb{R}^+ \rtimes_{\mathcal{D}_\delta} H_n$ is expressed as

$$(\rho, W) \cdot_{\mathbb{H}_{\mathbb{C}}} (\rho', W') = (\rho\rho', W \cdot_H \mathcal{D}_\rho(W')). \tag{2.2}$$

The geometric picture described above of complex hyperbolic space being foliated by a family of Heisenberg groups as level-set hypersurfaces of ρ is compatible with this group law: an open set in $\{\rho = c\} \simeq H_n$ is related to the corresponding set in $\{\rho = c + \varepsilon\}$ by pullback along M_ε. We highlight also that this dilation structure is crucial for the definition of resolution of the product space $X \times X$ (see (3.2)).

3. The wave kernel on asymptotically complex hyperbolic manifolds

In this section, we begin the construction of a Fourier integral operator calculus, which is adapted to the asymptotic geometry of the metric (2.1). Such a calculus will be comprised of operators whose Schwartz kernels have prescribed asymptotics on a manifold with corners, the Θ-stretched product $X \times_\Theta X$ of [8].

Analogously to the 0-blow up, Epstein, Mendoza, and Melrose defined the Θ-blow up of an ACH manifold; this will be very similar to the 0-blow up of an AH manifold. The biggest distinction being the blow up at the front face is non-isotropic, reflecting the different asymptotics in ρ of boundary vector fields (namely those vector fields whose g_{ACH}-duals span $d\theta(\cdot, J\cdot)$ vs $\theta \otimes \theta$).

Following [8], we next explain how we will modify the product $X \times X$ to construct our algebra of Fourier integral operators. We begin with the notion of the Θ-vector fields $\mathcal{V}_\Theta$:

$$V \in \mathcal{V}_\Theta \iff V \in \rho \cdot \mathcal{C}^\infty(X; TX), \quad \tilde{\theta}(V) \in \rho^2 \cdot \mathcal{C}^\infty(X; TX),$$

where $\tilde{\theta} \in \mathcal{C}^\infty(X; TX)$ is any smooth extension of θ to all of X. It is shown in [8, Section 1] that this definition is dependent only on the choice of conformal class of $[\theta]$. This is partly because a representative of $[\theta]$ determines a local frame by requiring

$$\{X_1, \dots, X_n, Y_1, \dots, Y_n\} \quad \text{is an orthonormal frame of } d\theta(\cdot, J\cdot), \tag{3.1a}$$

$$\theta(Z) = 1, \tag{3.1b}$$

$$\theta(\partial_\rho) = 0, \tag{3.1c}$$

in which we can express

$$\mathcal{V}_\Theta = \operatorname{span}_{\mathcal{C}^\infty}\{\rho\partial_\rho, \rho X_1, \dots, \rho X_n, \rho Y_1, \dots, \rho Y_n, \rho^2 Z\},$$

and a different choice of bdf ρ' produces a frame as in (3.1) associated to a contact form θ' conformal to θ.

Given this $\mathcal{C}^\infty(X)$-module, we can define the Θ-tangent bundle ${}^\Theta TX$. This is a vector bundle over X, with a bundle map $\iota_\Theta \colon {}^\Theta TX \to TX$, which is an isomorphism over $X \setminus \partial X$ such that

$$\mathcal{C}^\infty(X; {}^\Theta TX) = \iota_\Theta^*(\mathcal{V}_\Theta).$$

Next, we construct the Θ-stretched product of [8, Section 8]. Notice first that in the product $X \times X$, the boundary of the diagonal $\partial X_{\mathrm{diag}} \simeq \partial X$ is an embedded submanifold,

$$\partial X_{\mathrm{diag}} \hookrightarrow \partial X \times \partial X \hookrightarrow X \times X$$

and is a *clean submanifold* in the sense of [7], since it is an embedded submanifold of the corner, and thus all differentials of bdfs vanish at $\partial X_{\mathrm{diag}}$. The 1-form θ on X defines a line subbundle

$$\mathcal{H}^* \subset N^*_{X \times X}(\partial X_{\mathrm{diag}})$$

spanned by

$$\pi_L^*\theta - \pi_R^*\theta,$$

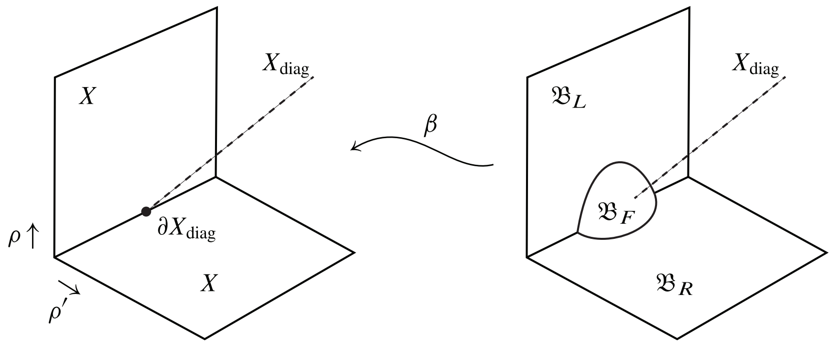

Figure 1. The blow-down map β of the Θ-stretched product space $X \times_\Theta X$.

with $\pi_{(\cdot)}: X \times X \to X$ denoting the projection onto the left and right factors respectively. With this trivialization of the conormal bundle, we define the Θ-blow up of the corner as the $\mathcal{H}^*$-parabolic blow up (defined using the dilation structure on fibers given in (2.2)) of the boundary diagonal:

$$X \times_\Theta X = [X \times X; \partial X_{\mathrm{diag}}, \mathcal{H}^*] := (X \times X \setminus \partial X_{\mathrm{diag}}) \sqcup \mathbb{S}N_{\mathcal{H},+}(\partial X_{\mathrm{diag}})$$
$$\mathbb{S}N_{\mathcal{H},+}(\partial X_{\mathrm{diag}}) = (N_{X\times X}\partial X_{\mathrm{diag}})/\mathbb{R}^+_{\sim_{\mathcal{D}^{\mathcal{H}}}}$$

(see Figure 1), where the equivalence on fibers $\mathcal{D}^{\mathcal{H}}$ is defined using the decomposition $N(\partial X_{\mathrm{diag}})_+ = \mathcal{H} \oplus \mathcal{H}^\perp$, with $\mathcal{H} = \mathrm{Ann}(\mathcal{H}^*)$,

$$(W, Z) \sim_{\mathcal{D}^{\mathcal{H}}_\delta} (W', Z') \iff \exists \delta > 0, (W, Z) = (\delta W', \delta^2 Z').$$

This real unoriented blow up replaces the submanifold $\partial X_{\mathrm{diag}}$ with its inward-pointing *parabolic*-sphere bundle. This blow-up procedure furnishes a blow-down map

$$\beta: X \times_\Theta X \to X \times X,$$

which is the identity on $X \times X \setminus \partial X_{\mathrm{diag}}$, and given by the bundle projection map of the parabolic-sphere bundle on $\mathbb{S}N_{\mathcal{H},+}(\partial X_{\mathrm{diag}})$. This is a manifold with corners, and has three new boundary faces:

$$\begin{aligned}
\mathfrak{B}_F &= \beta^{-1}(\partial X_{\mathrm{diag}}) = \mathbb{S}N_{\mathcal{H},+}(\partial X_{\mathrm{diag}}),\\
\mathfrak{B}_L &= \overline{\beta^{-1}\{(X \times \partial X) \setminus \partial X_{\mathrm{diag}}\}},\\
\mathfrak{B}_R &= \overline{\beta^{-1}\{(\partial X \times X) \setminus \partial X_{\mathrm{diag}}\}}.
\end{aligned}$$

By construction, the front face $\mathfrak{B}_F$ is a fiber bundle over $\partial X_{\mathrm{diag}}$ with fiber a projective quotient of the inward pointing normal bundle $N_{X\times X}(\partial X_{\mathrm{diag}})_+$; the front face has fiber over $p \in \partial X$ given by

$$\mathfrak{B}_F|_p = [N_{X\times X}(\partial X_{\mathrm{diag}})_+ \setminus \partial X_{\mathrm{diag}}]/ \sim_{\mathcal{D}_\delta^{\mathcal{H}}} . \tag{3.2}$$

For more details and the proof of diffeomorphism invariance of this construction, see [8, Sections 5–7].

3.1. The Θ-symplectic structure on ${}^{\Theta}T^*X$

Similarly to [23], associated to the Lie algebra $\mathcal{V}_\Theta$, we can define the notion of a Θ-*Fourier integral operator,* which will be operators whose Schwartz kernels have prescribed asymptotics on a resolution of the product $X \times X$, the Θ-stretched product $X \times_\Theta X$. A *standard Fourier integral operator* is characterized by its Schwartz kernel having singular support conormal to a Lagrangian inside $(T^*X \setminus o) \times (T^*X \setminus o)$; to generalize this notion we must first understand the symplectic structure of ${}^{\Theta}T^*X$.

In a neighborhood of the boundary U, if we use coordinates

$$(x,\zeta) = ((\rho, w, z); (\xi, \eta_H, \eta_V)) \in {}^{\Theta}T^*X$$

which satisfy both that ρ is a boundary defining function and that

$$\theta|_{U\cap\{\rho=0\}} = dz - \frac{1}{2}\sum_{j=1}^{n} w_{x_j}\, dw_{y_j} - w_{y_j}\, dw_{x_j},$$

then the induced map on dual bundles can be expressed in these coordinates by

$$\begin{aligned} &\bar{\iota}_\Theta \colon T^*X \to {}^{\Theta}T^*X, \\ &((\rho, w, z); (\xi, \eta_H, \eta_V)) \mapsto ((\rho, w, z); (\rho\xi, \rho\eta_H, \rho^2\eta_V)) =: ((\rho, w, z); (\mu, u, t)). \end{aligned}$$

In these coordinates, the canonical 1-form

$$\alpha = \xi d\rho + \eta_H \cdot dw + \eta_V dz$$

pulls back to the 1-form

$$\bar{\iota}_\Theta\alpha = {}^{\Theta}\alpha = \frac{\mu}{\rho}d\rho + \frac{u}{\rho}dw + \frac{t}{\rho^2}dz,$$

and hence we have a symplectic form

$$\begin{aligned} {}^{\Theta}\omega = d({}^{\Theta}\alpha) &= \frac{1}{\rho}d\mu \wedge d\rho + \frac{1}{\rho}du \wedge d\omega + \frac{1}{\rho^2}(dt \wedge dz - d\rho \wedge (u d\omega)) \\ &\quad - \frac{2}{\rho^3}d\rho \wedge (t dz). \end{aligned} \tag{3.3}$$

With this symplectic structure on ${}^{\Theta}T^*X$, we can define as usual the familiar notions of symplectic geometry such as Hamiltonian flows, Lagrangians, etc.

The next subtlety is that because we want our putative Θ-Fourier integral operators to generalize the existing notion of Θ-pseudodifferential operators, we should define our Θ-FIOs as those operators whose Schwartz kernels *are Lagrangian distributions of* $T^*(X \times_\Theta X)$, *rather than of* ${}^{\Theta}T^*X \times {}^{\Theta}T^*X$. This is because Θ-pseudodifferential operators are defined as having Schwartz kernels conormal to the diagonal in $X \times_\Theta X$. So, we seek also to define Lagrangians of $T^*(X \times_\Theta X)$. Given the canonical identification of $T^*M \times T^*M \simeq T^*(M \times M)$, for M closed, this distinction is only an issue in the category of manifolds with corners; in our case, such an identification will hold only over the interior of $X \times X$ (cf. (3.5)).

Unlike ${}^{\Theta}T^*X$, the symplectic structure on $T^*(X \times_\Theta X)$ will be the usual one, arising from the canonical 1-form of the (non-rescaled) cotangent bundle of $X \times_\Theta X$. In this case, Lagrangians are defined as usual, so we investigate those Lagrangians of $T^*(X \times_\Theta X)$ which could be induced by relations on ${}^{\Theta}T^*X \times {}^{\Theta}T^*X$. Following [23], we can define extendible Lagrangians. Set

$$(X \times_\Theta X)_d = X \times_\Theta X \bigsqcup_{\mathfrak{B}_F} X \times_\Theta X,$$

the double of the Θ-stretched product across the front face. We say that a smooth conic closed Lagrangian submanifold $\Lambda \subset T^*(X \times_\Theta X)$ is *extendible* if it intersects $T^*(X \times_\Theta X)|_{\mathfrak{B}_F}$ transversely. This implies there exists a smooth conic Lagrangian $\Lambda_{ext} \subset T^*(X \times_\Theta X)_d$ such that

$$\Lambda = \Lambda_{ext} \cap T^*(X \times_\Theta X), \quad \Lambda_\Theta := \Lambda \pitchfork T^*(X \times_\Theta X)|_{\mathfrak{B}_F}.$$

One reason for the interest in extendible Lagrangians is that their intersection with the cotangent bundle over the front face $\mathfrak{B}_F$ is again a Lagrangian submanifold.

Lemma 3.1. *If $\Lambda \subset T^*(X \times_\Theta X)$ is extendible, then $\Lambda_\Theta = \Lambda \cap T^*(X \times_\Theta X)|_{\mathfrak{B}_F}$ is a Lagrangian submanifold of $T^*\mathfrak{B}_F$*

Proof. Fix coordinates $(\rho, w_1, \ldots, w_{2n}, z)$ of $X \times_\Theta X$ valid near $\mathfrak{B}_F = \{\rho = 0\}$, and with dual variables $(\xi, \eta_H^1, \ldots, \eta_H^{2n}, \eta_V)$. Then, $(\rho, w, z; \xi, \eta_H, \eta_V)$ give local coordinates for $T^*(X \times_\Theta X)$ near $\mathfrak{B}_F$. By transversality, $d\rho|_\Lambda \neq 0$, thus ρ and some subset of $(w, z; \xi, \eta_H, \eta_V)$ must give local coordinates for Λ. Since Λ is Lagrangian, the canonical 2-form

$$\omega_{T^*(X \times_\Theta X)} = d\rho \wedge d\xi + \sum_{j=1}^{2n} dw^j \wedge d\eta_H^j + dz \wedge d\eta_V$$

must vanish on Λ; hence, it vanishes on Λ_Θ as well. From the overall vanishing of this symplectic form, and the non-vanishing of $d\rho$ on Λ, we must have that $d\xi$ restricted to $T\Lambda|_{\Lambda_\Theta}$ is a multiple of $d\rho$. This implies existence of a function $\phi(\rho, w, z; \eta_H^j, \eta_V)$ satisfying

$$\Lambda \subset \{\xi = \rho\phi(\rho, w, z; \eta_H^j, \eta_V)\}.$$

In particular, $\xi|_{\Lambda_\Theta} = 0$ and $\sum dw_j \wedge d\eta_H^j + dz \wedge d\eta_V = 0$ on $T\Lambda_\Theta$. ∎

Having introduced extendible Lagrangians, we immediately explain their relation to our the class of distributions we will ultimately be concerned with. We define a *Lagrangian distribution associated to an extendible Lagrangian* (either $\Lambda \subset T^*(X \times_\Theta X)$ or $\Lambda \subset T^*\mathbb{R} \times T^*(X \times_\Theta X)$) to be the restriction to $X \times_\Theta X$ of a distribution which is Lagrangian with respect to an extension Λ_{ext} of Λ across $\mathfrak{B}_F$. As usual, we denote the set of order m distributions which are Lagrangian with respect to Λ by $I^m(X \times_\Theta X; \Lambda, {}^\Theta\Omega^{1/2})$ (resp. $I^m(\mathbb{R} \times X \times_\Theta X; \Lambda, {}^\Theta\Omega^{1/2})$).

Now that we have introduced Lagrangians in this setting, we can see some ways they arise naturally. If X, Y are two ACH manifolds, a Θ-*canonical relation* between them is a $\mathcal{C}^\infty$-map

$$\chi\colon \Gamma \subset {}^\Theta T^*X \to {}^\Theta T^*Y$$

defined on an open conic subset $\Gamma \subset {}^\Theta T^*X$ such that $\chi^*({}^\Theta\alpha_Y) = {}^\Theta\alpha_X$. Certain Θ-canonical relations will define Lagrangian submanifolds in $T^*(X \times_\Theta X)$, by associating to χ its graph relation

$$\chi\colon {}^\Theta T^*X \to {}^\Theta T^*X \leftrightsquigarrow \mathrm{Gr}(\chi) \subset {}^\Theta T^*X \times {}^\Theta T^*X,$$

and we denote such Lagrangians by Λ_χ. Particularly relevant Lagrangians will arise from *liftable canonical transformations;* these are homogeneous canonical transformations $\chi\colon {}^\Theta T^*X \to {}^\Theta T^*X$, whose projections to the base is the identity over ∂X.

Using the left and right projections, we can define a symplectic form on ${}^\Theta T^*X \times {}^\Theta T^*X$ by

$$\omega = \pi_1^*\omega_\Theta - \pi_2^*\omega_\Theta. \tag{3.4}$$

Further, the dual to the differential of the blow-down map $\beta\colon X \times_\Theta X \to X \times X$ induces a smooth map

$$T^*X \times T^*X \simeq T^*(X \times X) \to T^*(X \times_\Theta X) \tag{3.5}$$

which is an isomorphism over $\mathrm{Int}(X \times X)$ between ω and the standard symplectic form on $T^*X \times T^*X$.

Lemma 3.2 (Liftable canonical transformations inducing extendible Lagrangians). *Let $\chi\colon {}^{\Theta}T^*X \to {}^{\Theta}T^*X$ be a liftable canonical transformation. The map (3.5), combined with the identification (over* $\mathrm{Int}(X \times X)$*) $T^*X \times T^*X \sim {}^{\Theta}T^*X \times {}^{\Theta}T^*X$, gives a smooth map*

$$\varphi_{\Theta}\colon {}^{\Theta}T^*X \times {}^{\Theta}T^*X \xrightarrow{\simeq} T^*(X \times_{\Theta} X) \quad \textit{over } \mathrm{Int}(X \times X) \tag{3.6}$$

which, restricted to the graph of χ, extends by continuity to the boundary and embeds into it as a smooth Lagrangian of $T^*(X \times_{\Theta} X)$, denoted Λ_χ. Further, Λ_χ intersects the boundary of $T^*(X \times_{\Theta} X)$ only over $T^*_{\mathfrak{B}_F}(X \times_{\Theta} X)$, it is extendible across the front face, and this intersection

$$\Lambda_{\chi\Theta} := \Lambda_\chi \cap T^*_{\mathfrak{B}_F}(X \times_{\Theta} X)$$

defines a Lagrangian submanifold of $T^*\mathfrak{B}_F$.

Proof. On the two copies of X in the product $X \times X$, we consider respectively coordinates (ρ, w, z) and (ρ', w', z') valid near the boundary. These induce corresponding local coordinates on the cotangent bundles, which we denote by

$$(\rho, w, z; \xi, \eta_H, \eta_V) \quad \text{and} \quad (\rho', w', z'; \xi', \eta'_H, \eta'_V) \quad \text{corresponding to } T^*X,$$

and

$$(\rho, w, z; \mu, u, t) \quad \text{and} \quad (\rho', w', z'; \mu', u', t') \qquad \text{corresponding to } {}^{\Theta}T^*X, \tag{3.7}$$

on the left and right copies of the respective cotangent bundles. We fix

$$V = \frac{\rho}{\rho'} \quad W = \frac{w - w'}{\rho'}, \quad Z = \frac{z - z'}{(\rho')^2}$$

as coordinates valid near the front face $\mathfrak{B}_F$, away from $\beta^{\#}(\{\rho' = 0\})$. The map (3.6) gives an identification between the 1-forms

$$\frac{\mu}{\rho} d\rho - \frac{\mu'}{\rho'} d\rho' + \frac{u}{\rho} dw - \frac{u'}{\rho'} dw' + \frac{t}{\rho^2} dz - \frac{t'}{(\rho')^2} dz',$$

and

$$\alpha dV + \tilde{\xi} d\rho' + \beta dW + \tilde{\kappa} dw' + \gamma dZ + \tilde{\eta} dz',$$

defined on ${}^{\Theta}T^*X \times {}^{\Theta}T^*X$ and $T^*(X \times_{\Theta} X)$, respectively. We will first determine how the coefficients of these 1-forms are related under the map (3.6), in this neighborhood of $\mathfrak{B}_F$. Since $\rho = V\rho'$, $w = w' + \rho' W$, and $z = z' + (\rho')^2 Z$, we have

$$\begin{aligned} d\rho &= V d\rho' + \rho' dV, \\ dw &= dw' + \rho' dW + W d\rho', \\ dz &= dz' + 2\rho' Z d\rho' + (\rho')^2 dZ, \end{aligned}$$

and so the canonical 1-form in $T^*(X \times_\Theta X)$ is given by

$$\begin{aligned}
&\Big(\frac{\mu}{\rho}V - \frac{\mu'}{\rho'} + \frac{u}{\rho}W + \frac{2t\rho'}{\rho^2}Z\Big)d\rho' + \frac{\mu\rho'}{\rho}dV \\
&+ \Big(\frac{u}{\rho} - \frac{u'}{\rho'}\Big)dw' + \frac{u\rho'}{\rho}dW + \Big(\frac{t}{\rho^2} - \frac{t'}{(\rho')^2}\Big)dz' + \frac{t(\rho')^2}{\rho^2}dZ \\
&\quad = \alpha dV + \tilde{\xi} d\rho' + \beta dW + \tilde{\kappa} dw' + \gamma dZ + \tilde{\eta} dz',
\end{aligned}$$

where

$$\alpha = \mu\frac{\rho'}{\rho}, \quad \beta = u\frac{\rho'}{\rho}, \quad \gamma = t\Big(\frac{\rho'}{\rho}\Big)^2,$$

$$\tilde{\xi} = \frac{\mu}{\rho'} - \frac{\mu'}{\rho'} + \frac{u}{\rho'}\frac{w - w'}{\rho} + \frac{2t}{\rho'}\frac{z - z'}{\rho^2}, \quad \tilde{\kappa} = \frac{u}{\rho} - \frac{u'}{\rho'}, \quad \tilde{\eta} = \frac{t}{\rho^2} - \frac{t'}{(\rho')^2}.$$

Next, we shall leverage the fact that χ is a Θ-canonical relation (and thus that ${}^\Theta\alpha_X - \chi^*({}^\Theta\alpha_X) = 0$), and the fact that χ restricts to the identity over ∂X, in order to determine $\mathrm{Gr}(\chi)$ in the coordinates (3.7). First, observe that ρ and ρ' are both bdfs on X and thus conformal: $\rho' = f\rho$. Further, we have that $\pi_X \colon {}^\Theta T^*X \to X$ and

$$(0, w', z') := (\pi_X \circ \chi)|_{\partial X}(x, \zeta) = (0, w, z),$$

hence $w' = w + \rho A$, $z' = z + \rho^2 B$ for some smooth functions A, B on ${}^\Theta T^*X$. Finally, we use the relation between the fundamental 1-forms to observe that

$$\begin{aligned}
&\mu\frac{d\rho}{\rho} + u\frac{dw}{\rho} + t\frac{dz}{\rho^2} \\
&\quad = \chi^*\Big(\mu'\frac{d\rho'}{\rho'} + u'\frac{dw'}{\rho'} + t'\frac{dz'}{(\rho')^2}\Big) \\
&\quad = \mu'\Big(\frac{d\rho}{\rho} + \frac{df}{f}\Big) + \frac{u'}{a}\Big(\frac{dw}{\rho} + dA + A\frac{d\rho}{\rho}\Big) + \frac{t'}{a^2}\Big(\frac{dz}{\rho^2} + 2B\frac{d\rho}{\rho} + dB\Big) \\
&\quad = \Big(\mu' + \frac{u'}{f}A + \frac{2t'}{f^2}\Big)\frac{d\rho}{\rho} + \frac{u'}{f}\frac{dw}{\rho} + \frac{2t'}{f^2}\frac{dz}{\rho^2} + \Big(\frac{\mu'}{f}df + \frac{u'}{f}dA + \frac{t'}{f^2}dB\Big).
\end{aligned}$$

The final bracketed term will only contribute terms which are $\mathcal{O}(\rho)$ or $\mathcal{O}(\rho^2)$ after computing their Θ-differential (e.g., $df = \rho\partial_\rho f d\rho/\rho + \rho\partial_w f dw/\rho + \rho^2\partial_z f dz/\rho^2$); thus, after grouping such terms, we obtain

$$\begin{aligned}
&\mu\frac{d\rho}{\rho} + u\frac{dw}{\rho} + t\frac{dz}{\rho^2} \\
&\quad = \Big(\mu' + \frac{u'}{f}A + \frac{2t'}{f^2} + \rho C\Big)\frac{d\rho}{\rho} + \Big(\frac{u'}{f} + \rho D\Big)\frac{dw}{\rho} + \Big(\frac{2t'}{f^2} + \rho^2 E\Big)\frac{dz}{\rho^2},
\end{aligned}$$

where $f > 0, A, B, C, D$ are smooth functions of (ρ, w, z, μ, u, t). Taken together, these computations imply that its graph is of the form

$$\begin{aligned}\mathrm{Gr}(\chi) = \{((\rho, w, z, \mu, u, t), (\rho', w', z', \mu', u', t')) \mid \\ \rho' = f\rho,\ w' = w + \rho A,\ z' = z + \rho^2 B,\ \mu' = \mu - uA - 2tB + \rho C, \\ u' = fu + \rho D,\ t' = f^2 t + \rho^2 E\}.\end{aligned}$$

From this, we can see that

$$\alpha = e^f \mu, \quad \beta = e^f u, \quad \gamma = e^{2f} t, \quad \tilde{\xi} = -e^{-f} C, \quad \tilde{\kappa} = -e^{-f} D, \quad \tilde{\eta} = -e^{-2f} E.$$

Since $e^f = \rho'/\rho$ is smooth and positive on Λ_χ, the map (3.6) (defined over the interior $\mathrm{Int}(X \times X)$), extends by continuity to the boundary when restricted to $\mathrm{Gr}(\chi)$, thus identifying Λ_χ and $\mathrm{Gr}(\chi)$,

$$\mathrm{Gr}(\chi) \simeq \Lambda_\chi \hookrightarrow T^*(X \times_\Theta X)$$

as the image of $\mathrm{Gr}(\chi)$ under the map (3.6). Further, this shows that Λ_χ intersects the boundary of $T^*(X \times_\Theta X)$ only over $\mathfrak{B}_F = \{\rho' = 0\}$ and does so transversely. Thus, it is an extendible Lagrangian, and we have by the previous lemma that this intersection $\Lambda_{\chi\Theta}$ is a Lagrangian submanifold of $T^*\mathfrak{B}_F$. ∎

This lemma elucidates the name liftable canonical transformation as they provide examples of canonical transformation with "good" lifts to $T^*(X \times_\Theta X)$ as the associated Lagrangian meets the diagonal only in the front face $\mathfrak{B}_F$.

Given $p \in \mathcal{C}^\infty({}^\Theta T^* X)$, we define its Θ-Hamiltonian vector field by the relation ${}^\Theta\omega(-, {}^\Theta H_p) = dp$. In local coordinates in which ${}^\Theta\omega$ is given by (3.3), ${}^\Theta H_p$ is given by

$$\begin{aligned}{}^\Theta H_p = \rho \frac{\partial p}{\partial \mu} \partial_\rho - \Big(\rho \frac{\partial p}{\partial \rho} + w_j \frac{\partial p}{\partial w_j} + 2t \frac{\partial p}{\partial t}\Big) \partial_\mu \\ + \sum_{j=1}^{2n} \Big(\rho \frac{\partial p}{\partial u_j}\Big) \partial_{w_j} - \Big(\rho \frac{\partial p}{\partial w_j} - u_j \frac{\partial p}{\partial \mu}\Big) \partial_{u_j} + \rho^2 \frac{\partial p}{\partial t} \partial z \\ - \Big(\rho^2 \frac{\partial p}{\partial z} - 2t \frac{\partial p}{\partial \mu}\Big) \partial_t.\end{aligned}$$

Observe that this vector field has the special property that the projection of the vector field to the base vanishes when restricted to ∂X.

Because our focus is the wave equation, we are most interested in the Hamiltonian associated to our ACH metric. Since our metric satisfies

$$g_{\mathrm{ACH}} = \frac{d\rho^2 + h_\Theta(w, z, dw, dz) + \rho Q(\rho, w, z, dw, dz)}{\rho^2},$$

we can conclude its dual metric on T^*X has the form

$$G = (\rho\xi)^2 + \rho^2 h_\Theta(w, z, \eta_H, \eta_V) + \rho^3 Q(\rho, w, z, \eta_H, \eta_V)$$

or, in the coordinates (ρ, w, z, μ, u, t) on ${}^\Theta T^*X$, our dual metric is given by

$$G = \mu^2 + h_\Theta(w, z, u, t) + \rho Q(\rho, w, z, u, t). \tag{3.8}$$

This function on ${}^\Theta T^*X$ will be the Hamiltonian of interest in our study of the wave equation.

Lemma 3.3 (Θ-canonical flowouts). *Let $G \in \mathcal{C}^\infty({}^\Theta T^*X)$ be the dual metric associated to the metric g_{ACH} and let ${}^\Theta H_G$ be its Θ-Hamilton vector field. For all $s > 0$, the canonical transformation $\chi_s\colon {}^\Theta T^*X \to {}^\Theta T^*X$, given as the flow-out of the Hamiltonian*

$$\chi_s(q) := \exp(s\,{}^\Theta H_G)(q)$$

is a liftable canonical transformation. Thus, we have that the graph of χ_s defines a smooth extendible Lagrangian submanifold of $T^*(X \times_\Theta X)$. Further, the intersection

$$\Lambda_{\mathfrak{B}_F}(s) := \Lambda_s \cap T^*(X \times_\Theta X)|_{\mathfrak{B}_F}$$

is a smooth Lagrangian submanifold of $T^*\mathfrak{B}_F$ given by

$$\exp(sH_{G_\Theta})(T^*\mathfrak{B}_F|_{D_\Theta \cap \mathfrak{B}_F}) = \Lambda_{\mathfrak{B}_F}(s),$$

where $G_\Theta = \widetilde{G}|_{\mathfrak{B}_F}$, the restriction to the front face of the lift of G to $T^*(X \times_\Theta X)$.

Proof. Since the flow-out of a Hamilton vector field is always a canonical transformation, the first claim follows from the fact that only the projection onto the base vanishes. Thus, we only to check the claim regarding $\Lambda_{\mathfrak{B}_F}(s)$. We can study the graph of χ_s after viewing $G \in \mathcal{C}^\infty({}^\Theta T^*X \times {}^\Theta T^*X)$ as a function depending only on the second copy of ${}^\Theta T^*X$.

On this space, we can write our canonical 1-form in the coordinates

$$\pi_1^*({}^\Theta\alpha) - \pi_2^*({}^\Theta\alpha) := \frac{\mu}{\rho}d\rho - \frac{\mu'}{\rho'}d\rho' + \frac{u}{\rho}dw - \frac{u'}{\rho'}dw' + \frac{t}{\rho^2}dz - \frac{t'}{(\rho')^2}dz'; \tag{3.9}$$

thus, we can write the Hamilton vector field of a function on this space with respect to this 1-form, with the same formula as we calculated above. In this case, χ_s is the flow-out of the diagonal in ${}^\Theta T^*X \times {}^\Theta T^*X$ along the vector field ${}^\Theta H_G$. In these coordinates, our length function is given by

$$G = (\mu')^2 + h_\Theta(w', z', u', t') + \rho' Q(\rho', w', z', u', t')$$

and we can consider local coordinates near the front face, projective with respect to the left face,

$$V = \frac{\rho'}{\rho}, \quad W = \frac{w' - w}{\rho}, \quad Z = \frac{z' - z}{\rho^2},$$

with blow-down map

$$\begin{aligned}
&\beta \colon X \times_\Theta X \to X \times X, \\
&(\rho, w, z, V, W, Z) \mapsto (\rho, w, z, \rho', w', z') = (\rho, w, z, \rho V, w + \rho W, z + \rho^2 Z).
\end{aligned}$$

The pullback of (3.9) by β is

$$a d\rho + a_f dV + b dw + b_f dW + c dz + c_f dZ,$$

and in these coordinates we have that $\mathfrak{B}_F = \{\rho = 0\}$ and the interior lift of the diagonal D_Θ is given by $D_\Theta = \{V = 1, W = Z = 0\}$. The lift of p to $T^*(X \times_\Theta X)$ is given by

$$\begin{aligned}
\tilde{p} &= (a_f V)^2 + h_0(w + \rho W, z + \rho^2 Z, -V b_f, -V^2 c_f) \\
&\quad + \rho V Q(\rho V, w + \rho W, z + \rho^2 Z, -V b_f, -V^2 c_f) \\
&= (a_f V)^2 + V^2 h_0(w + \rho W, z + \rho^2 W, b_f, c_f) \\
&\quad + \rho V^3 Q(\rho V, w + \rho W, z + \rho^2 Z, b_f, c_f),
\end{aligned}$$

where the functions h_0, Q are $\mathcal{D}_\rho^{\mathbb{H}}$-homogeneous of order 2 in the fiber variables.

Now, we lift our symplectic form (3.4) to $T^*(X \times_\Theta X)$, and denote it by $\tilde{\omega}$; ${}^\Theta H_G$ lifts to $H_{\widetilde{G}}$. In the coordinates

$$[(\rho, w, z, V, W, Z); (a, b, c, a_f, b_f, c_f)] \in T^*(X \times_\Theta X),$$

our lifted Hamilton vector field has the form

$$\begin{aligned}
H_{\widetilde{G}} &= \Big(\frac{\partial \widetilde{G}}{\partial a}\partial_\rho - \frac{\partial \widetilde{G}}{\partial \rho}\partial_a\Big) + \Big(\frac{\partial \widetilde{G}}{\partial a_f}\partial_V - \frac{\partial \widetilde{G}}{\partial V}\partial_{a_f}\Big) \\
&\quad + \sum_{j=1}^{2n}\Big(\frac{\partial \widetilde{G}}{\partial b^j}\partial_{w_j} - \frac{\partial \widetilde{G}}{\partial w^j}\partial_{b_j}\Big) + \sum_{j=1}^{2n}\Big(\frac{\partial \widetilde{G}}{\partial b_f^j}\partial_{W_j} - \frac{\partial \widetilde{G}}{\partial W^j}\partial_{(b_f)_j}\Big) \\
&\quad + \Big(\frac{\partial \widetilde{G}}{\partial c}\partial_z - \frac{\partial \widetilde{G}}{\partial z}\partial_c\Big) + \Big(\frac{\partial \widetilde{G}}{\partial c_f}\partial_Z - \frac{\partial \widetilde{G}}{\partial Z}\partial_{c_f}\Big) \\
&= 2a_f V^2 \partial_V - 2V[a_f^2 + h_\Theta]\partial_{a_f} - V^2 \sum_j W_j \frac{\partial \widetilde{G}}{\partial w^j}\partial_a - V^2 \sum_j \frac{\partial h_\Theta}{\partial w_j}\partial_{b_j} \\
&\quad + V^2 \sum_j \frac{\partial h_\Theta}{\partial b_f^j}\partial_{W_j} + V^2 \frac{\partial h_\Theta}{\partial c_f}\partial_Z + \mathcal{O}(\rho),
\end{aligned}$$

thus $H_{\tilde{p}}$ is smooth all the way down to $\mathfrak{B}_F = \{\rho = 0\}$. Further, with respect to our coordinate transformation on $T^*(X \times_\Theta X)$ induced by the blow-down map β, we have that the diagonal

$$\{\rho = \rho', w = w', z = z', \mu = \mu', u = u', t = t'\} = ({}^\Theta T^* X)_{\mathrm{diag}} \subset {}^\Theta T^* X \times {}^\Theta T^* X$$

lifts to

$$\widetilde{\mathfrak{D}}_\Theta = \{V = 1, W = Z = a = b = c = 0\} \subset T^*(X \times_\Theta X).$$

Thus, $\widetilde{\mathfrak{D}}_\Theta$ transversely intersects $T^*(X \times_\Theta X)|_{\mathfrak{B}_F}$ at $\{\rho = 0, V = 1, W = Z = a = b = c = 0\}$. Finally, we see that $H_{\tilde{p}}$ projects down to $T^*\mathfrak{B}_F$ as

$$\begin{aligned} &2a_f V^2 \partial_V - 2V(a_f^2 + h_\Theta(w, z, b_f, c_f))\partial_{a_f} \\ &\qquad + 2V^2 \sum_{i,j} h_\Theta^{ij}(w, z) \cdot b_f^{(j)} \frac{\partial}{\partial b_f^{(j)}} + 2V^2 h_\Theta^{0,0}(w, z) \cdot c_f \frac{\partial}{\partial c_f} \end{aligned}$$

which is precisely the Hamilton vector field of $a_f^2 V^2 + V^2 h_\Theta(w, z, b_f, c_f)$. ■

Remark 3.4. Notice that, because

$$\widetilde{G}|_{\mathfrak{B}_F} = V^2(a_f^2 + h_\Theta(w, z, b_f, c_f)) =: G_\Theta,$$

the projection of the Hamilton vector field of G to $T^*\mathfrak{B}_F$ is precisely the Hamilton vector field of the restriction of G to $\mathfrak{B}_F$, with respect to the induced symplectic form on $T^*\mathfrak{B}_F$.

In other words, for the Hamiltonian given by our length functional (3.8), we have that, for all $s > 0$, the twisted graph of $\exp(s{}^\Theta H_G)\colon {}^\Theta T^* X \to {}^\Theta T^* X$ defines a Lagrangian submanifold $\Lambda(s) \subset T^*(X \times_\Theta X)$. Further, this Lagrangian intersects the boundary only over $\mathfrak{B}_F$, and it does so transversely. The transversal intersection is itself a Lagrangian flow-out

$$\Lambda_F(s) := \exp(sH_{G_\Theta})(T^*\mathfrak{B}_F|_{D_\Theta \cap \mathfrak{B}_F}) \subset T^*\mathfrak{B}_F,$$

which is the flow-out by the Hamilton vector field of $G_\Theta = \sigma_2(N_{\mathfrak{B}_F}(\Delta))$, the principal symbol of the normal operator at the front face.

3.2. Θ-FIOs and the wave kernel

Here, we construct the calculus of operators that our wave group $\cos(t\sqrt{\Delta - n^2/4})$ will lie in. These shall be restricted to the subclass of Lagrangian distributions whose support does not meet the left or right faces, $\beta^*(\partial X \times X)$ and $\beta^*(X \times \partial X)$, respectively. Due to the finite speed of propagation, initial data $U(t, p, p')$ supported in the interior of $\mathfrak{B}_F$ which evolves according to the wave equation

$$\begin{cases} \left(D_t^2 + \Delta_g - \frac{n^2}{4}\right)U(t,p,p') = 0 \\ U(0,p,p') = \delta(p,p'), \quad \partial_t U(0,p,p') = 0, \end{cases}$$

remain supported away from the left and right faces, $\mathfrak{B}_L, \mathfrak{B}_R$. In particular, when considering our calculus of FIOs, we can ignore the complement of the front face in the corner, and restrict ourselves to Lagrangians which meet the boundary only at $\mathfrak{B}_F$.

Since the canonical relation C of the wave group will be a Lagrangian in $T^*\mathbb{R} \times T^*(X \times_\Theta X)$, we mildly extend our class of Lagrangians from the last section. The canonical 1-form on $T^*\mathbb{R} \times {}^\Theta T^*X \times {}^\Theta T^*X$ is given by

$$\alpha = t\,d\tau + \frac{\mu}{\rho}d\rho - \frac{\mu'}{\rho'}d\rho' + \frac{u}{\rho}dw - \frac{u'}{\rho'}dw' + \frac{s}{\rho^2}dz - \frac{s'}{(\rho')^2}dz'.$$

With this 1-form, we can define a canonical relation

$$\begin{aligned} C &= \{(t,\tau,\zeta_1,\zeta_2) \mid \tau + \sqrt{G(\zeta_1,\zeta_1)} = 0, \zeta_2 = \exp(t\,{}^\Theta H_G)(\zeta_1)\} \\ &\subset T^*\mathbb{R} \times {}^\Theta T^*X \times {}^\Theta T^*X, \end{aligned}$$

and this canonical relation in turn defines a Lagrangian of $T^*\mathbb{R} \times T^*(X \times_\Theta X)$ given by

$$\Lambda_C = \{(t,\tau,\zeta_1,\zeta_2) \mid \tau + \sqrt{\widetilde{G}(\zeta_1,\zeta_2)} = 0, (\zeta_1,\zeta_2) \in \Lambda_t\} \subset T^*\mathbb{R} \times T^*(X \times_\Theta X),$$

where Λ_t is an extendible Lagrangian associated to the graph of the liftable canonical transformation

$$\chi_t := \exp(t\,{}^\Theta H_G)(q) \colon {}^\Theta T^*X \to {}^\Theta T^*X,$$

and $\widetilde{G}$ is the lift of G from the second copy of ${}^\Theta T^*X$. In particular, this Lagrangian intersects the boundary only over the front face $\mathfrak{B}_F$, and

$$\Lambda_C^\Theta = \{(t,\tau,\bar{\zeta}_1,\bar{\zeta}_2) \mid \tau + \sqrt{G_\Theta(\bar{\zeta}_1,\bar{\zeta}_2)} = 0, (\bar{\zeta}_1,\bar{\zeta}_2) \in \Lambda_{\mathfrak{B}_F}(t)\} \subset T^*\mathbb{R} \times T^*\mathfrak{B}_F,$$

where G_Θ is the restriction of $\widetilde{G}$ to $\mathfrak{B}_F$.

Now, finally, we can give the definition of a Θ-FIO. Given a (liftable) canonical transformation $\chi\colon {}^\Theta T^*X \to {}^\Theta T^*X$, we know by Lemma 3.2, that its graph induces an (extendible) Lagrangian Λ_χ inside of $T^*(X \times_\Theta X)$. It is only with respect to such Lagrangians that we shall define Θ-FIOs, namely as those operators whose Schwartz kernels are conormal to Lagrangians obtained in this manner. More precisely, given a liftable canonical transformation $\chi\colon {}^{\Theta}T^*X \to {}^{\Theta}T^*X$, we define our Θ-*Fourier integral operators* associated to χ to be the linear operators $L\colon \mathcal{E}'(X) \to \mathcal{D}'(X)$ whose Schwartz kernels lie in the space of distributions

$$I_{\Theta}^{m,s}(X;\chi,{}^{\Theta}\Omega^{1/2}) := \{\rho_F^s \mathcal{K}_L \mid \mathcal{K}_L \in I^m(X \times_{\Theta} X;\Lambda_\chi,{}^{\Theta}\Omega^{1/2}),\ \rho_F^s\mathcal{K}_L \text{ vanishes in a neighborhood of } \partial(X\times_\Theta X)\setminus \mathfrak{B}_F\},$$

where Λ_χ is the extendible Lagrangian submanifold of $T^*(X\times_\Theta X)$ associated to χ by Lemma 3.2. Similarly, for the canonical relation C defined above, we say that Θ-*Fourier integral operators* associated to C are the linear operators $B\colon \mathcal{E}'(\mathbb{R}\times X) \to \mathcal{D}'(X)$ whose Schwartz kernels lie in the space of distributions

$$\begin{aligned} &I_{\Theta}^{m,s}(\mathbb{R}\times X, X; C,{}^{\Theta}\Omega^{1/2}) \\ &\quad := \{\rho_F^s\mathcal{K}_B \mid \mathcal{K}_B \in I^m(\mathbb{R}\times X\times_\Theta X;\Lambda_C,{}^{\Theta}\Omega^{1/2}),\ \rho_F^s\mathcal{K}_B \text{ vanishes} \\ &\qquad\qquad \text{in a neighborhood of } \partial(\mathbb{R}\times X\times_\Theta X)\setminus(\mathbb{R}\times\mathfrak{B}_F)\}. \end{aligned}$$

In both cases, such operators are those whose Schwartz kernels are Lagrangian distributions with respect to Λ_χ (Λ_C resp.) and vanish to order s at the front face $\mathfrak{B}_F$. Such operators carry two different principal symbol mappings: one is the usual symbol of a Lagrangian distribution, in the interior; the second operator is obtained by the principal symbol of the normal operator $\mathcal{K}_L|_{\mathfrak{B}_F}$ (resp. $\mathcal{K}_B|_{\mathbb{R}\times\mathfrak{B}_F}$) associated to the Lagrangian in $T^*\mathfrak{B}_F$ (resp. $T^*\mathbb{R}\times T^*\mathfrak{B}_F$).

This second symbol is again the symbol of a Lagrangian distribution from the fact that our Lagrangian Λ_χ (resp. Λ_C) has transversal intersection with $T^*(X\times_\Theta X)|_{\mathfrak{B}_F}$ (resp. $T^*\mathbb{R}\times T^*(X\times_\Theta X)|_{\mathfrak{B}_F}$), thus the restriction of Lagrangian distribution to $\mathfrak{B}_F$ is again a Lagrangian distribution with respect to Λ_χ (resp. Λ_C).

We now take a moment again to highlight the normal operator. If $\mathcal{K}_A \in I_\Theta^{m,s}$, then $N_p(A) = (\rho_F^{-s}\mathcal{K}_A)|_{\mathfrak{B}_F}$, and $N_p(A)$ is a Lagrangian distribution with respect to Λ_χ (resp. Λ_C). Further, the normal operator satifies an analogue of the short exact sequence for principal symbols of operators.

Proposition 3.5. *The normal operator participates in a short exact sequence*

$$\begin{aligned} 0 \to I_\Theta^{m,1}(\mathbb{R}\times X, X; C,{}^{\Theta}\Omega^{1/2}) &\hookrightarrow I_\Theta^{m,0}(\mathbb{R}\times X, X; C,{}^{\Theta}\Omega^{1/2}) \\ &\xrightarrow{N_p(-)} I^m(\mathbb{R}\times\mathfrak{B}_F;\Lambda_C^\Theta,\Omega^{1/2}) \to 0 \end{aligned}$$

such that for any Θ-differential operator $P \in \mathrm{Diff}_\Theta^m(X)$ and any Θ-Fourier integral operator $B \in I_\Theta^{m,s}(\mathbb{R}\times X, X; C,{}^{\Theta}\Omega^{1/2})$ *we have*

$$N_p((D_t^2 - P)\circ B) = (D_t^2 - N_p(P)) * N_p(B).$$

Proof. This is an analogue of [26, Proposition 5.19] and [23, Proposition 3.1].

The injectivity portion of the statement of exactness is immediate from the definition, since we have that $N_{\hat{p}}(-)$ is $\mathcal{C}^\infty$ in $\hat{p} \in \partial X$ and defines an operator on $\mathcal{C}^\infty(\mathfrak{B}_{F_{\hat{p}}})$ for each $\hat{p}$ fixed. In particular, since the kernels of these operators are smooth up to the front face, it makes sense to consider their Taylor series on $\mathfrak{B}_{F_{\hat{p}}}$. The surjectivity of $N_{\hat{p}}(-)$ thus arises from a version of Borel's lemma for the Taylor series of $N_{\hat{p}}(-)$ in local coordinates for $\mathfrak{B}_{F_{\hat{p}}}$.

To prove the composition formula, we can use the structure of the Normal operator at $\mathfrak{B}_F$, and the fact that we are not blowing up in the t variable, so it commutes with the normal operator.

We observe first that such a $P \in \mathrm{Diff}^m_\Theta(X)$ can be written with respect to our frame $\{\rho\partial_\rho, \rho V_{w_i}, \rho^2\partial_z\}$ for ${}^\Theta TM$:

$$\begin{aligned} P &= \sum_{j+|\alpha|+k\le m} a_{j\alpha k}(\rho, w, z)(\rho\partial_\rho)^j(\rho V_w)^\alpha(\rho^2\partial_z)^k \\ &\implies N_{\hat{p}}(P) = \sum_{j+|\alpha|+k\le m} a_{j\alpha k}(0, w, z)(\rho\partial_\rho)^j(\rho V_w)^\alpha(\rho^2\partial_z)^k. \end{aligned}$$

As usual, we choose to identify this as acting on $1/2$-densities: if we choose coordinates (ρ, w, z), these induce a trivialization of the square root of the Θ-density bundle $\Omega^{1/2}_\Theta = \Omega^{1/2}$

$$\gamma = (\rho)^{-(2n+3)}|d\rho\, dw\, dz|^{1/2}$$

and P acts on $f \in \mathcal{C}^\infty(X; \Omega^{1/2}_\Theta)$ by $Pf = P(f\gamma^{-1})\gamma$. Of course, this is simply for Θ-differential operators. More generally, Θ-FIOs will act on $1/2$-densities via their normal operator: $N_{\hat{p}}(A) = (\rho_F^{-s}\mathcal{K}_A)|_{\mathfrak{B}_{F_{\hat{p}}}}$,

$$\begin{aligned} &(Bf)(\rho, w, z)\cdot\gamma \\ &\quad = \int_{\mathfrak{B}_{Fp}} \mathcal{K}_B(0, w, z; V, W, Z) f\Big(\frac{\rho}{V}, w - \frac{\rho}{V}W, z - \Big(\frac{\rho}{V}\Big)^2 Z\Big)\frac{dVdWdZ}{V}\cdot\gamma. \end{aligned}$$

In particular, this implies that the normal operator of $P \circ B$ satisfies

$$\begin{aligned} &N_{\hat{p}}(P \circ B) \\ &\quad = \Big(\sum_{j+|\alpha|+k\le m} a_{j\alpha k}(0, w, z)(\rho\partial_\rho)^j(\rho V_w)^\alpha(\rho^2\partial_z)^k \circ (\mathcal{K}_B(0, w, z; V, W, Z))\Big)\cdot\gamma. \end{aligned}$$

∎

Having proven this lemma, we arrive at a short time parametrix for the wave group.

Proposition 3.6. *For each $t \in \mathbb{R}$, for the canonical relation*

$$\begin{aligned} C = \{&[(t,\tau),(\rho,w,z;\mu,u,s),(\rho',w',z';\mu',u',s')] : \\ &\tau + \sqrt{G(\rho,w,z;\mu,u,s)} = 0, \\ &(\rho',w',z';\mu',u',s') = \exp(t\,{}^{\Theta}H_G)(\rho,w,z;\mu,u,s)\} \end{aligned}$$

the wave group $U(t)$ is Θ-Fourier integral operator of the class

$$U(t) = \cos(t\sqrt{\Delta_g - n^2/4}) \in I_{\Theta}^{-1/4,0}(\mathbb{R}\times X, X; C, {}^{\Theta}\Omega^{1/2}).$$

Proof. Given the normal sequence, the argument reduces to a purely local one: using Proposition 3.5, and the fact that

$$N_{\hat{p}}(\mathrm{Id}) = \delta(V-1)\delta(W)\delta(Z)\gamma = \delta(0_p)\gamma,$$

we can take as ansatz $U_0(t,\hat{p}) = N_{\hat{p}}(U(t))$ the wave group in this fiber $\mathfrak{B}_{F_{\hat{p}}}$:

$$\begin{cases} \Big(D_t^2 + N_{\hat{p}}(\Delta_g) - \dfrac{n^2}{4}\Big)U_0(t,\hat{p}) = 0, \\ U(0,\hat{p}) = \delta(0_{\hat{p}}), \quad \partial_t U(0,\hat{p}) = 0; \end{cases}$$

here $0_{\hat{p}} \in \mathfrak{B}_{F_{\hat{p}}} \simeq \mathbb{X}_{\hat{p}}$ corresponds to the identity element in the group. Note also that the specific form of the model Laplacian

$$N_{\hat{p}}(\Delta_g) = -\frac{1}{4}(\rho\partial_\rho)^2 + \frac{n+1}{2}\rho\partial_\rho + \rho^2\Delta_H(\hat{p}) - \rho^4 Z^2(\hat{p})$$

means we can also construct the model wave group, and study its asymptotics via analyzing those of the wave group in $\mathbb{H}_{\mathbb{C}}^{n+1}$.

Since $0_{\hat{p}} \in \mathrm{Int}(\mathfrak{B}_{F_{\hat{p}}})$, it does not meet the corners of $\mathfrak{B}_{F_{\hat{p}}}$. Similarly, Λ_C does not meet the corners in finite time, so we can follow the argument of Duistermaat and Guillemin [7, Proposition 1.1] to conclude $U_0(t) \in I^{-1/4}(\mathbb{R}\times\mathfrak{B}_F; \Lambda_C^{\Theta}, \Omega)$.

Now, we iterate. Choose a $u_0 \in I_{\Theta}^{-1/4,0}(X\times\mathbb{R}, X; C, {}^{\Theta}\Omega^{1/2})$ such that $N_{\hat{p}}(u_0) = U_0(t)$. Then,

$$\beta_L^*(D_t^2 + \Delta_g - n^2/4)(U(t) - u_0) = r_0 \in I_{\Theta}^{-1/4,1}(X\times\mathbb{R}, X; C, {}^{\Theta}\Omega^{1/2})$$

and $\rho^{-1}r_0 \in I^{-1/4,0}$, where ρ is a defining function for the left face. (This is well defined, since r_0 is supported away from the left face, as u_0 was, and the wave operator preserves this support due to the condition on wave front of U_0, via [21, Theorem 2.5.15].) Now, we solve the inhomogeneous wave equation to find a $u_1 \in I_{\Theta}^{-1/4,0}$ solving

$$\begin{cases} \Big(D_t^2 + N_{\hat{p}}(\Delta_g) - \dfrac{n^2}{4}\Big)N_{\hat{p}}(u_1) = N_{\hat{p}}(\rho^{-1}r_0), \\ N_{\hat{p}}(u_1)|_{t=0} = N_{\hat{p}}(\rho^{-1}r_0), \quad \partial_t N_{\hat{p}}(u_1)|_{t=0} = \partial_t N_{\hat{p}}(\rho^{-1}r_0)|_{t=0}, \end{cases}$$

as before, we obtain such a u_1. We now have

$$\beta_L^*\Big(D_t^2 + \Delta_g - \frac{n^2}{4}\Big)(U(t) - u_0 - \rho u_1) = r_1 \in I_\Theta^{-1/4,-2}.$$

Proceeding iteratively, we obtain $U_\infty \sim \sum_{j\geq 0} \rho^j u_j$ such that $\beta_L^*(D_t^2 + \Delta_g - n^2/4)U_\infty$ vanishes to infinite order at $\mathfrak{B}_F$. The error term also has infinite order vanishing at $\mathfrak{B}_F$ in the Cauchy data from the construction. Finally, after extending this error term to be identically zero across the front face, we can use Hörmander's transverse intersection calculus to remove this error term (see e.g., [21, Theorem 2.5.15]). ■

Unfortunately, this is a short time parametrix, as this construction is only valid for finite t. If we allow $t \to \infty$, our Lagrangian flow-out $\Lambda(t)$ will meet the corners of $\mathfrak{B}_F$, which would require a more sophisticated composition formula.

4. Wave trace asymptotics

Now that we know the wave group is a Θ-Fourier integral operator, we can ask whether its trace can be studied, as in the case of the wave trace on a compact manifold without boundary. This presents some technical difficulties, since the operator $\cos(t\sqrt{\Delta_g - (n+1)^2/4})$ is not trace class, so we need to introduce a regularization of its trace.

Heuristically, our goal is to study the trace,

$$\operatorname{Tr} U(t) = \int_{\bar{X}_{\mathrm{diag}}} U(t,x,x)\, d\mathrm{Vol}_g = \Pi_* \iota_{\mathrm{diag}}^* U \tag{4.1}$$

using appropriate maps ι_{diag} and Π to define this integral via pullback and pushforward. An analysis of the wavefront sets of these maps will permit an analysis of their associated operators, and prove that the resulting object is a well-defined distribution on $\mathbb{R}$, with wavefront set to be determined.

First, notice that for all $p, p' \not\in \partial\bar{X}$, the restriction of $U(t,p,p')$ to the diagonal $\bar{X}_{\mathrm{diag}}$ is well defined. To see this, we proceed as in [7, Section 1] by introducing the map

$$\iota_{\mathrm{diag}}\colon \mathbb{R} \times \bar{X}_{\mathrm{diag}} \to \mathbb{R} \times \bar{X} \times \bar{X}, \quad (t,p) \mapsto (t,p,p)$$

of the inclusion of the diagonal. The pullback along this map is a Fourier integral operator of order $(n+1)/2$, defined by the canonical relation

$$\begin{aligned}\mathrm{WF}'(\iota_{\mathrm{diag}}^*) &= \{(((t,\tau),(p,\zeta+\zeta')),((t,\tau),(p,\zeta),(p,\zeta')))\}\\ &= N^*\{\iota_{\mathrm{diag}}(t,p) = (t,p,p')\}.\end{aligned}$$

Now, using the fact that $\mathrm{WF}(U) = C$ (as defined in Proposition 3.6), assuming $p, p' \notin \partial \bar{X}$, then, whenever $((t,\tau),(p,\zeta),(p,\zeta')) \in \mathrm{WF}'(U)$, we have $\tau \neq 0$; thus, we get $\mathrm{WF}(U) \cap N_{\iota_{\mathrm{diag}}} = \emptyset$ at such points where

$$N_{\iota_{\mathrm{diag}}} = \{(\iota(t,p),\tau,\zeta,\zeta') \in T^*(\mathbb{R} \times \bar{X} \times \bar{X}) : D\iota_{\mathrm{diag}}^{\mathsf{T}}(\tau,\zeta,\zeta') = 0\}$$

is the set of normals of the map. Thus, we can apply [21, Theorem 2.5.11′] to conclude that $\iota_{\mathrm{diag}}^* U$ is a well-defined distribution on $\mathbb{R} \times (\bar{X} \setminus \partial \bar{X})$ with wavefront set

$$\mathrm{WF}'(\iota_{\mathrm{diag}}^* U) = \{((t,\tau),(p,\zeta-\zeta')) : \tau + \sqrt{G(p,\zeta)} = 0,\ (p,\zeta) = \exp(t^{\Theta} H_G)(p,\zeta')\}.$$

Duistermaat and Guillemin next study the wavefront set of the projection $\Pi : \mathbb{R} \times \bar{X} \to \mathbb{R}$. In our case, we now introduce the regularization procedure. For $\varepsilon > 0$, define $\bar{X}_\varepsilon = \{\rho > \varepsilon\}$ for our bdf ρ. Consider the cutoff projection

$$\Pi_\varepsilon : \mathbb{R} \times \bar{X}_\varepsilon \to \mathbb{R}, \quad (t,p) \mapsto t,$$

for which integration over the range p is equal to the pushforward along Π_ε (the transpose of the operator Π^*). This map thus defines a Fourier integral operator of order $1/2 - (n+1)/2$ given by the canonical relation

$$\mathrm{WF}'(\Pi_*) = \{((t,\tau),\ ((t,\tau),(x,0)))\}.$$

Again, applying Hörmander's theorem [21, Theorem 2.5.11′], we can conclude that the cutoff wave trace

$$T_\varepsilon(t) = \int\limits_{\rho > \varepsilon} U(t,p,p) = (\Pi_\varepsilon)_*(\iota_{\mathrm{diag}}^* U(t))$$

is a well-defined distribution on $\mathbb{R}$ satisfying

$$\begin{aligned}\mathrm{WF}(T_\varepsilon(t)) = \{(t,\tau) : \tau < 0 \text{ and} \\ (p,\zeta) = \exp(t^{\Theta} H_G)(p,\zeta') \text{ for some } (p,\zeta),\ \rho(p) > \varepsilon\}.\end{aligned}$$

We obtain, as a corollary, the following result.

Corollary 4.1. *For $\varepsilon > 0$, the singular support of $T_\varepsilon \in \mathcal{D}'(\mathbb{R})$ is contained in the set of periods of closed geodesics in $\bar{X}_\varepsilon$. Moreover, there exists $\varepsilon_0 > 0$ such that all closed geodesics of (X,g) with period greater than zero are contained in $\bar{X}_{\varepsilon_0}$.*

In particular, for all $\varepsilon < \varepsilon_0$, the singular support of T_ε is contained in the set of period of closed geodesics of X.

Proof. Only the claim regarding closed geodesics remaining in $\bar{X}_{\varepsilon_0}$ remains to be proven. This is a statement about strict convexity of the geodesic flow in a neighborhood of infinity (see e.g., [23, Proposition 4.1] or [6, Lemma 4.1]). We show that if ε is sufficiently small, any geodesic γ which intersects $\{\rho < \varepsilon\}$ cannot be closed.

Next, we introduce coordinates (ρ, w, z) with corresponding dual coordinates (ξ, η_H, η_V), such that ρ is a boundary defining function for $\partial\overline{X}$. In these coordinates, we write the metric in a collar neighborhood of the boundary as

$$g = \frac{4d\rho^2 + \tilde{g}_\rho}{\rho^2}, \quad \tilde{g}_\rho = h_{\mathcal{H}} + \rho^{-2}\theta^2$$

and we write

$$G_\rho(\eta, \eta) = h_{\mathcal{H}}(\eta_H, \eta_H) + \rho^2\theta^2(\eta_V, \eta_V)$$

for the bilinear form on T^*X induced by the dual metric of $\tilde{g}_\rho$. In these coordinates, the geodesic Hamiltonian is given by

$$|\zeta|_g^2 = \sigma^2 + \overline{G}(\mu, \mu) = \sigma^2 + h_{\mathcal{H}}(\mu_H, \mu_H) + \theta^2(\mu_V, \mu_V)$$

where $\sigma = \rho\xi$, $\mu_H = \rho\eta_H$, $\mu_V = \rho^2\eta_V$, and $\overline{G} = \rho^2 G_\rho$. The Hamilton vector field of this function is given by

$$\begin{aligned} H_{|\zeta|_g^2} &= \partial_\xi|\zeta|^2\partial_\rho - \partial_\rho|\zeta|^2\partial_\xi + (\partial_{\eta_H}|\zeta|^2)\cdot Y - (Y|\zeta|^2)\cdot\partial_{\eta_H} \\ &\quad + (\partial_{\eta_V}|\zeta|^2)\partial_z - (\partial_z|\zeta|^2)\partial_{\eta_V}, \end{aligned}$$

where $\{Y_j\}_{j=1}^{2n}$ is a local $h_{\mathcal{H}}$-orthonormal frame dual to $\{d\eta_H^j\}_{j=1}^{2n}$. Computing the change in these vector fields with respect to the change of coordinates

$$(\rho, w, z, \xi, \eta_H, \eta_V) \mapsto (\rho, w, z, \sigma, \mu_H, \mu_V)$$

gives

$$\begin{gathered} \partial_\xi = \rho\partial_\sigma, \quad \partial_{\eta_H} = \rho\partial_{\mu_H}, \quad \partial_{\eta_V} = \rho^2\partial_{\mu_V}, \\ \partial_\rho = \partial_\rho + \rho^{-1}\sigma\partial_\sigma + \rho^{-1}(\mu_H\cdot\partial_{\mu_H} + 2\mu_V\partial_{\mu_V}), \quad Y = Y, \quad \partial_z = \partial_z. \end{gathered}$$

Thus, the Hamilton vector field can be re-expressed as

$$\begin{aligned} H_{|\zeta|^2} &= (\rho\partial_\sigma|\zeta|^2)(\partial_\rho + \rho^{-1}\mathcal{R}_{\mathrm{CC}}) - (\rho\partial_\rho + \mathcal{R}_{\mathrm{CC}})(|\zeta|^2)\partial_\sigma \\ &\quad + \rho[(\partial_{\mu_H}|\zeta|^2)\cdot Y - (Y|\zeta|^2)\cdot\partial_{\mu_H}] \\ &\quad + \rho^2[(\partial_{\mu_V}|\zeta|^2)\partial_z - (\partial_z|\zeta|^2)\partial_{\mu_V}] \end{aligned}$$

where we have defined $\mathcal{R}_{\mathrm{CC}} = \mu_H\cdot\partial_{\mu_H} + 2\mu_V\partial_{\mu_V}$, the infinitesimal generator of the Heisenberg dilation action on $T^*\partial\overline{X}$. Using the facts that

$$\partial_\sigma|\zeta|^2 = 2\sigma, \quad \mathcal{R}_{\mathrm{CC}}|\zeta|^2 = 2\overline{G}(\mu, \mu),$$

and writing the vector field $H_{\tilde{g}_\rho} = [(\partial_{\mu_H}|\zeta|^2)\cdot Y - (Y|\zeta|^2)\cdot\partial_{\mu_H}] + \rho[(\partial_{\mu_V}|\zeta|^2)\partial_z - (\partial_z|\zeta|^2)\partial_{\mu_V}]$, we can re-express this formula as

$$H_{|\zeta|^2} = 2\sigma\rho\partial_\rho + 2\sigma\mathcal{R}_{\mathrm{CC}} - (2\overline{G}(\mu, \mu) + \rho\partial_\rho\overline{G})\partial_\sigma + \rho\cdot H_{\tilde{g}_\rho}.$$

Thus, along integral curves of the vector field $H_{|\xi|^2}$ we have $\dot{\rho} = 2\sigma\rho$, $\dot{\tau} = -(2\overline{G} + \rho\partial_\rho\overline{G})$. Thus, at a critical point of ρ along the flow which is an interior point of X, we have

$$\dot{\rho} = 0 \implies \sigma = 0,$$

hence, at such points, we have

$$\ddot{\rho} = 0 + 2\dot{\sigma}\rho = -2\rho(2\overline{G} + \rho\partial_\rho\overline{G}) = -4\rho\overline{G} - 2\rho^2\partial_\rho\overline{G}.$$

Now, we use the fact that $\overline{G}|_{\rho=0}$ is positive definite, thus for sufficiently small ρ this quantity is negative. Thus, we have shown that for all geodesic curves γ which intersect $\{\rho \leq \varepsilon\}$ satisfy

$$\dot{\rho} \circ \gamma = 0 \implies \ddot{\rho} \circ \gamma < 0.$$

Now, assume, for the sake of contradiction, that γ is closed. Then, there exists $\delta \in (0, \varepsilon)$ such that γ intersects $\{\rho = \delta\}$ in at least two points. Therefore, there exists a s_0 with $\rho \circ \gamma(s_0) > 0$ where $\rho \circ \gamma$ has a minimum. However, at such a minimum we have $\dot{\rho} \circ \gamma(s_0) = 0$ and $\ddot{\rho} \circ \gamma(s_0) > 0$, contradicting our convexity statement. ■

Using this corollary, we can now begin an analysis of the renormalized wave trace. We denote by $u_j \in I_\Theta^{-1/4,j}(\mathbb{R} \times X, X; C, {}^\Theta\Omega^{1/2})$ the operators defined in the proof of Proposition 3.6. The same arguments used above can be used to show that the distribution

$$I_j(t, \varepsilon) = \int_{\rho > \varepsilon} \rho^j u_j(t, p, p)$$

is well defined, with singular support satisfying the conclusions of corollary 4.1. Since $\mathfrak{B}_F$ and Λ_C intersect transversally, only the density factor implicit in this operator can obstruct the convergence of $I_j(t, \varepsilon)$ as $\varepsilon \to 0$. Since this density, a trivialization of the ${}^\Theta\Omega^{1/2}$-bundle, diverges at the rate $\rho^{-(2n+3)}$ at $\partial\overline{X}$, the integrals $I_j(t, \varepsilon)$ converges for any $j \geq 2n + 3$. Applying Taylor's theorem to $u_j(t, p, p)$ as $\rho \to 0$, we see that there exists constants C_j such that the limit

$${}^R\operatorname{Tr} U(t) = \lim_{\varepsilon \to 0}\left[\int_{\{\rho > \varepsilon\}} U(t, p, p) - \sum_{j=-2n-2}^{-1} C_j \varepsilon^j - C_0 \log\left(\frac{1}{\varepsilon}\right)\right]$$

exists. We call this integral the *renormalized wave trace*. By the construction of the wave trace [7], these coefficients C_ε arise as integrals over $\{\rho > \varepsilon\}$ of universal polynomials in the metric, curvature, and covariant derivative of the metric and curvature (see also [11]). Thus, these specific constants are obtained from the particular choice of metric for this ACH manifold. See also [11]. From Corollary 4.1, we immediately obtain the following result.

Proposition 4.2. *The singular support of* ${}^R\operatorname{Tr} U(t)$ *is contained in the set of periods of closed geodesics of* (X, g).

Finally, we can begin our analysis of the renormalized wave trace as $t \to 0$ (in fact its inverse Fourier transform). First, we choose a cutoff function $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$, with the appropriate support to study the transform of the cutoff wave trace. If we denote the first non-zero period of a closed geodesic on (X, g) as t_0, then we choose χ such that $\chi(t) = 1$ for $|t| > t_0/2$ and $\chi(t) \equiv 0$ for $|t| > 2t_0/3$.

Now, using the arguments of [20] (which are purely local, applying to any para-compact manifold), or alternatively the proof of [7, Proposition 2.1], we immediately obtain the following.

Proposition 4.3. *There exist coefficients* $\{w_k\}_{k\in\mathbb{N}_0} \subset \mathbb{R}$ *such that the cutoff wave trace* $T_\varepsilon(t)$ *satisfies*

$$\int_{\mathbb{R}} T_\varepsilon(t)\chi(t)e^{it\mu}dt \sim \frac{1}{(2\pi)^{2n+2}} \sum_{k=0}^{\infty} w_k \mu^{2n+2-2k},$$

as $\mu \to 0$ *and rapidly decaying as* $\mu \to -\infty$*. The leading term of this expansion is* $\omega_0 = \operatorname{Vol}_g(\bar{X}_\varepsilon)$

Given this result for the asymtotics of the cutoff wave trace $T_\varepsilon(t)$, we can then conclude similarly for the full wave trace (4.1) that the following statement holds true.

Theorem 4.4. *There exist coefficients* $\{\omega_k\}_{k\in\mathbb{N}_0} \subset \mathbb{R}$ *such that the renormalized trace* ${}^R\operatorname{Tr} U(t)$ *satisfies*

$$\int_{\mathbb{R}} {}^R\operatorname{Tr} U(t)\chi(t)e^{it\mu}dt \sim \frac{1}{(2\pi)^{2n+2}} \sum_{k=0}^{\infty} \omega_k \mu^{2n+2-2k},$$

as $\mu \to 0$ *and rapidly decaying as* $\mu \to -\infty$*. The leading term,* $\omega_0 = {}^R\operatorname{Vol}_g(X)$*, is called the* renormalized volume, *and can be computed as*

$${}^R\operatorname{Vol}_g(X) = \lim_{\varepsilon\to 0}\left[\int_{\{\rho>\varepsilon\}} d\operatorname{Vol}_g - \sum_{j=-2n-2}^{-1} d_j\varepsilon^j - d_0\log\left(\frac{1}{\varepsilon}\right)\right],$$

where the d_j *are the unique real numbers such that this limit exists.*

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Hadrian Quan
Department of Mathematics, University of California, Santa Cruz, 1156 High Street, Santa Cruz, CA 95064, USA; hquan1@ucsc.edu