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Spectral estimates of dynamically-defined and amenable operator families

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Abstract. We consider kernel operators defined by a dynamical system. The Hausdorff distance of spectra is estimated by the Hausdorff distance of subsystems. We prove that the spectrum map is $1/2$-Hölder continuous provided the group action and kernel are Lipschitz continuous and the group has strict polynomial growth. Also, we prove that the continuity can be improved resulting in the spectrum map being Lipschitz continuous provided the kernel is instead locally-constant. This complements a result by J. Avron, P. van Mouche, and B. Simon (1990) establishing that one-dimensional discrete quasiperiodic Schrödinger operators with Lipschitz continuous potentials, e.g., the almost-Mathieu operator, exhibit spectral $1/2$-Hölder continuity. Also, this complements a result by S. Beckus, J. Bellissard, and H. Cornean (2019) establishing that d-dimensional discrete subshift Schrödinger operators with locally-constant potentials, e.g., the Fibonacci Hamiltonian, exhibit spectral Lipschitz continuity. Our work exposes the connection between the past two results, and the group, e.g., the Heisenberg group, needs not be the integer lattice nor abelian.

1. Introduction

The *almost-Mathieu operator* (AMO) is the linear operator $H_{\lambda,\alpha,\omega}: \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by

$$(H_{\lambda,\alpha,\omega}\psi)(n) = \psi(n+1) + \psi(n-1) + 2\lambda\cos(2\pi(n\alpha+\omega))\psi(n)$$

for every $\psi \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$. Here, λ, α, ω are parameters in $\mathbb{R}$; without loss of generality,

$$\omega \in \mathbb{T} := \mathbb{R}/\mathbb{Z} = [0,1]/\sim.$$

The spectrum is a nonempty compact subset of $\mathbb{R}$ since the AMO is both bounded and self-adjoint. The AMO has its origins in solid-state physics and the study of electrons; the name was introduced by B. Simon in 1982 [58], but the operator dates back

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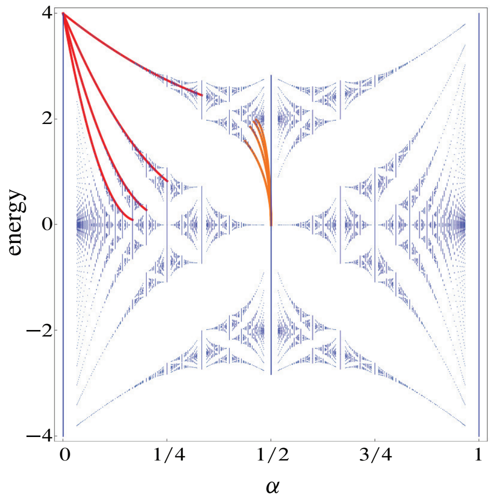

Figure 1. Hofstadter's butterfly [37].

to 1955 when P. G. Harper [35, 36] under the tutelage of R. E. Peierls published one of the first descriptions of the spectrum with $\lambda = 1$ and $\omega = 0$ by utilizing a tight-binding approximation: electrons in a crystal and in a magnetic field have nondiscrete nonevenly-spaced broadened energy values. Physicists sought to better understand the spectrum; M. Azbel in 1964 [8] conjectured and D. Hofstadter in 1976 [40] computationally supported (see Figure 1) that the spectrum has the characteristic of being either band-like for rational α or fractal-like for irrational α. Decades later, A. Avila and S. Jitomirskaya in 2009 [2] made the final step towards the complete solution of the Ten Martini Problem which sought to confirm the conjectured topological structure of the spectrum: if $\alpha = p/q$ is rational, then the spectrum is a disjoint union of at most q-many compact intervals; if α is irrational, then the spectrum is a Cantor set, i.e., a nonempty compact subset of $\mathbb{R}$ not having isolated points nor interior points.

Our motivation. J. Avron, P. van Mouche, and B. Simon [7] considered a general operator family (one-dimensional discrete quasiperiodic Schrödinger operators) containing the AMO and established that the Hausdorff distance $\mathrm{dist}_{\mathrm{H}}$ of spectra has a square root behavior with respect to the distance of the associated frequencies or, equivalently, the spectrum map is $1/2$-Hölder continuous.

To be more precise, let the phase $\omega \in \mathbb{T}$ and let the frequency $\alpha \in \mathbb{R}$. Then, the rotation by α on $\mathbb{T}$ is defined by $T_\alpha\colon \mathbb{T} \to \mathbb{T}, \omega \mapsto -\alpha + \omega$. A potential is a continuous function $v\colon \mathbb{T} \to \mathbb{R}$ defining the following operator family:

$$H_{v,\alpha,\omega}\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \quad (H_{v,\alpha,\omega}\psi)(n) = \psi(n+1) + \psi(n-1) + v(T_\alpha^{-n}\omega)\psi(n)$$

for every $\psi \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$. Observe that if $v(\omega) = 2\lambda\cos(2\pi\omega)$, then we recover the AMO. The 1990 result [7] is the following theorem: if $v\colon \mathbb{T} \to \mathbb{R}$ is Lipschitz continuous, then there exist $\delta > 0$ and $C > 0$ such that for each $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ with $|\alpha_1 - \alpha_2| < \delta$,

$$\operatorname{dist}_{\mathsf{H}}\Big(\overline{\bigcup_{\omega\in\mathbb{T}} \operatorname{spec}(H_{v,\alpha_1,\omega})}, \overline{\bigcup_{\omega\in\mathbb{T}} \operatorname{spec}(H_{v,\alpha_2,\omega})}\Big) \le C|\alpha_1 - \alpha_2|^{1/2}.$$

The rotation T_α by α defines a dynamical system on the torus and then $H_{v,\alpha,\omega}$ is a dynamically-defined operator. We recover this spectral estimate in Section 2.1 as a consequence of our main Theorem 1.3 dealing with a large class of dynamically-defined operators. Here the distance between the different frequencies can be seen as the Hausdorff distance d_{H} of the associated dynamical subsystems $\mathcal{S}$ defined below. This approach is motivated by the 2018 result [13] and the 2019 result [12]. In [13], the continuity of the spectrum of dynamically-defined operators (in the more general realm of groupoids) with respect to d_{H} on $\mathcal{S}$ was proven. In [12], the spectra has a quantitative Lipschitz behavior with respect to d_{H} on $\mathcal{S}$ for symbolic dynamical systems over the group $\mathbb{Z}^d$.

In this work, we establish that the Hausdorff distance $\operatorname{dist}_{\mathsf{H}}$ of spectra has a quantitative (1/2-Hölder or Lipschitz) behavior with respect to the Hausdorff distance d_{H} of subsystems. In contrast to [12, 13], our proof is based on approximate eigenvectors. Moreover, the interplay of the amenability assumption and regularity of the spectral map is explicitly observed. We will demonstrate that our unified theory applies for the following special cases recovering previously discussed results:

- one-dimensional discrete quasiperiodic Schrödinger operators (known [7]);
- d-dimensional discrete subshift Schrödinger operators (known [12]);
- one-dimensional discrete skew-shift Schrödinger operators (new);
- one-dimensional discrete limit-periodic Schrödinger operators (new).

These special cases are defined by a group action of $\mathbb{Z}^d$. Our main Theorem 1.3 applies also to non-abelian group actions such as the discrete Heisenberg group. Our result was recently applied in [10] to construct periodic approximations for an aperiodic subshift over the discrete Heisenberg group. Our main constraint is that the underlying group is amenable and unimodular.

We emphasize that the estimates provided in [7] and also in our main Theorem 1.3 cannot be improved in general since, for the AMO model, the square root behavior is sharp due to $\alpha = 1/2$ in the middle of Figure 1; see [37, 57]. Specifically, for fixed n such as $n = 0, 1, 2, 3$ in Figure 1,

$$\alpha = \frac{p}{q} \to \frac{1}{2} \implies \sup_{n,0} \operatorname{spec}(H^{\mathrm{AMO}}_{1,\alpha,0}) \asymp \pm\sqrt{16\pi n\Big|\alpha - \frac{1}{2}\Big| - 16\pi^2 n^2\Big|\alpha - \frac{1}{2}\Big|^2},$$

where $\operatorname{spec}_{n,E}$ is the n-th band nearby energy E, but the behavior can be linear such as at $\alpha = 0^+$

$$\alpha = \frac{p}{q} \to 0^+ \implies \sup \operatorname*{spec}_{n,4}(H^{\mathrm{AMO}}_{1,\alpha,0}) \asymp 4 - 2\pi(2n+1)\alpha + \frac{1}{2}\pi^2(2n^2+2n+1)\alpha^2.$$

Our work: Amenability and continuity. Let G be a second-countable locally-compact Hausdorff group and ℓ be a left-invariant proper metric on G generating its topology. A dynamical system $\langle Z, (G,\tau)\rangle$ is a metric space (Z,d) endowed with a left continuous action $\tau\colon G \times Z \to Z$, namely $\tau(e,x) = x$ for all $x \in Z$ and $\tau(g,\tau(h,x)) = \tau(gh,x)$ for every $g,h \in G$, $x \in Z$. Throughout this work, we use the notation $g \cdot_\tau x = \tau(g,x)$. In Section 3 onward, we suppress the notation further and write $gx = g \cdot_\tau x$. Let $k\colon G \times Z \to \mathbb{C}$, where k is measurable and

$$\sup_{x\in Z} \int \|k(h, (\cdot)^{-1} \cdot_\tau x)\|_\infty \, dh < +\infty.$$

Here, we integrate with respect to the left Haar measure λ on G. For $x \in Z$, the dynamically-defined operator $A_x\colon L^2(G) \to L^2(G)$ is defined by

$$(A_x\psi)(g) = \int k(g^{-1}h, g^{-1} \cdot_\tau x)\psi(h)\, dh$$

for every $\psi \in L^2(G)$, $g \in G$.

Let $\mathcal{S}$ be the collection of all left-invariant nonempty closed subsets of $\langle Z, (G,\tau)\rangle$ equipped with the Hausdorff metric

$$d_{\mathsf{H}}\colon \mathcal{S} \times \mathcal{S} \to [0,+\infty], \quad (X,Y) \mapsto \max\{\sup_{z\in X} d(z,Y), \sup_{z\in Y} d(z,X)\},$$

and define, for $X \in \mathcal{S}$, $\operatorname{spec}(A_X) := \overline{\bigcup_{x\in X} \operatorname{spec}(A_x)}$. The dynamically-defined operators A_x and their spectra are the main objects of study in this paper. Our attention is on the regularity of the spectrum map $X \mapsto \operatorname{spec}(A_X)$. Using standard techniques such as working with approximate eigenvectors and utilizing strong continuity, we immediately obtain that if G is unimodular and A_x is normal for every $x \in Z$ and k satisfies both a continuity condition and a decay condition, then $\operatorname{spec}(A_x) = \operatorname{spec}(A_{X(x)})$, where $X(x) := \overline{\{g \cdot_\tau x : g \in G\}}$ (orbit-closure); see, e.g., Lemma 3.2 (c). We forgo describing the conditions on k in detail, but they can be found within Section 5. It suffices to say that stricter conditions on k are sufficient to prove the following proposition (proven in Proposition 4.1) regarding the spectrum map.

Proposition 1.1. *Assume G is unimodular, A_x is normal for every $x \in Z$, and the following hold.*

(i) *τ satisfies a uniform continuity condition: for each $\varepsilon > 0$ and for each nonempty compact subset K of G, there exists $\delta > 0$ such that*

$$d(y,x) < \delta \implies \sup_{g \in K} d(g \cdot_\tau y, g \cdot_\tau x) < \varepsilon.$$

(ii) *k satisfies a uniform continuity condition: for each $\varepsilon > 0$ and for each nonempty compact subset F of G, there exists $\delta > 0$ such that*

$$d(y,x) < \delta \implies \int |k(h,y) - k(h,x)| \mathbb{1}_F(h)\, dh < \varepsilon.$$

(iii) *k satisfies a uniform decay condition: for each $\varepsilon > 0$, there exists a nonempty compact subset F of G such that*

$$\sup_{x \in Z} \int \| k(h, (\cdot)^{-1} \cdot_\tau x) \mathbb{1}_{G \setminus F}(h) \|_\infty \, dh < \varepsilon.$$

Fix $X \in \mathcal{S}$. Then,

$$\lim_{Y \underset{\mathcal{S}}{\to} X} \sup_{E \in \mathrm{spec}(A_X)} \mathrm{dist}(E, \mathrm{spec}(A_Y)) = 0.$$

The previous proposition asserts that the spectrum $\mathrm{spec}(A_X)$ is approximated but note that the limit (if it exists) of $\mathrm{spec}(A_Y)$ may contain more points, namely

$$\mathrm{spec}(A_X) \subseteq \lim_{Y \underset{\mathcal{S}}{\to} X} \mathrm{spec}(A_Y)$$

(if the limit exists). More can be obtained provided the group G is amenable; see Section 3.1.2 for two definitions. This is the context of the following theorem (proven in Theorem 4.4) which establishes that the spectrum map is continuous.

Theorem 1.2. *Assume G is unimodular, A_x is normal for every $x \in Z$, and the following hold.*

(i) *τ satisfies a uniform continuity condition: for each $\varepsilon > 0$ and for each nonempty compact subset K of G, there exists $\delta > 0$ such that*

$$d(y,x) < \delta \implies \sup_{g \in K} d(g \cdot_\tau y, g \cdot_\tau x) < \varepsilon.$$

(ii) *k satisfies a uniform continuity condition: for each $\varepsilon > 0$ and for each nonempty compact subset F of G, there exists $\delta > 0$ such that*

$$d(y,x) < \delta \implies \int |k(h,y) - k(h,x)| \mathbb{1}_F(h)\, dh < \varepsilon.$$

(iii) *k satisfies a uniform decay condition: for each $\varepsilon > 0$, there exists a nonempty compact subset F of G such that*

$$\sup_{x \in Z} \int \|k(h, (\cdot)^{-1} \cdot_\tau x)\mathbb{1}_{G \setminus F}(h)\|_\infty \, dh < \varepsilon.$$

(iv) *G is amenable.*

Then,

$$\lim_{\delta \to 0^+} \sup_{X, Y \in \mathcal{S}, d_{\mathsf{H}}(X,Y) < \delta} \mathrm{dist}_{\mathsf{H}}(\mathrm{spec}(A_X), \mathrm{spec}(A_Y)) = 0.$$

Remark 1.2.1. (a) In the AMO-like case, this result is known; publications of this prior result in the 1980s can be found on [29, p. 258], on [5, p. 382], and on [6, p. 2205].

(b) Theorem 1.2 intersects a 2018 result by S. Beckus, J. Bellissard, and G. De Nittis [13] establishing that the spectrum map is continuous. Indeed, the étale groupoid within [13] is assumed to satisfy an amenability condition. We remark that this prior result implies Theorem 1.2; our proof is different and significant since Theorem 1.3 – γ-Hölder continuity – is obtainable.

(c) If the reader is comfortable with the phase space Z being compact, then (i) immediately holds. If the reader is comfortable with the potential v being discontinuous in the AMO-like case, then (ii) needs not hold thus $x \mapsto H_{v,x}\psi$ needs not be continuous. We remark that models of quasicrystals have potentials defined on $\mathbb{T}$ being indicator functions $\mathbb{1}_{[1-\alpha,1)}$ in the AMO-like case thus discontinuous in the euclidean topology. The spectrum map is discontinuous at rational frequencies α, but in 1991 [19] it was established (and plotted) that the spectrum map is continuous at irrational frequencies α, where – in a different perspective – the potential defined on $\{0, 1\}^{\mathbb{Z}}$ is "evaluation at zero," thus continuous in the product topology; see Section 2.2. A new dynamical perspective on this is elaborated in [14]. If the reader is comfortable with the kernel k having compact range, e.g., the self-adjoint Jacobi operator case [63], then (iii) immediately holds. If the reader is comfortable with the group G being $\mathbb{Z}^d$ or $\mathbb{R}^d$ or compact or abelian or the Heisenberg group or the lamplighter group, then $\langle G, \lambda \rangle$ is both unimodular and amenable (iv).

More can be obtained provided stricter conditions replace (i)–(iv). This is the context of the following main theorem (proven in Theorem 4.6 and Theorem 4.7) which establishes that the spectrum map is γ-Hölder continuous ($\gamma = 1/2$ or 1) provided the group is strictly-polynomial-growing. To define this, $B(r) := \{g \in G : |g| < r\}$ denotes the open ball with radius $r \geq 0$, where for $g \in G$, $|g| := \ell(g, e)$.

Theorem 1.3. *Assume G is unimodular, A_x is normal for every $x \in Z$, and the following hold.*

(i) *τ satisfies a Lipschitz continuity condition: there exists $c_\tau > 0$ such that for each $g \in G$, $d(g \cdot_\tau x, g \cdot_\tau y) \le (c_\tau |g| + 1) d(x, y)$.*

(ii) *k satisfies one of two quantitative continuity conditions considered; see part (a) and (b) below.*

(iii) *k satisfies a linear decay condition: there exists $c_s > 0$ such that for each $r \ge 0$, $\sup_{x \in Z} \int \|k(h, (\cdot)^{-1} \cdot_\tau x) \min\{|h|, r\}\|_\infty dh \le c_s$.*

(iv) *$\langle (G, \ell), \lambda \rangle$ is strictly-polynomial-growing: there exist $b \ge 1$ and $c_1 \ge c_0 > 0$ such that for each $r \ge 1$, $c_0 r^b \le \lambda(B(r)) \le c_1 r^b$.*

Then,

(a) *if k satisfies a Lipschitz continuity condition, there exists $c_k \in L^1_+(G)$ such that for λ-a.e. $h \in G$, $|k(h,x) - k(h,y)| \le c_k(h) d(x,y)$, then for all $X, Y \in \mathcal{S}$,*

$$d_{\mathsf{H}}(X,Y) \le \min\{\delta, 1\} \implies \operatorname{dist}_{\mathsf{H}}(\operatorname{spec}(A_X), \operatorname{spec}(A_Y)) \le C d_{\mathsf{H}}(X,Y)^{1/2},$$

where

$$\delta := \frac{c_s}{\|c_k\|_1 c_\tau} \Big(\frac{(2+b)^{2+b} c_1}{2 b^b c_0} \Big)^{1/2}$$

and

$$C := 2(\|c_k\|_1 c_\tau c_s \Big(\frac{(2+b)^{2+b} c_1}{2 b^b c_0} \Big)^{1/2})^{1/2} + \|c_k\|_1;$$

(b) *if k satisfies a locally-constant condition, there exists $c_k \in L^\infty_+(G)$ such that for λ-a.e. $h \in G$, $d(x,y) < 1/c_k(h) \implies k(h,x) = k(h,y)$, then for all $X, Y \in \mathcal{S}$,*

$$d_{\mathsf{H}}(X,Y) \le \delta \implies \operatorname{dist}_{\mathsf{H}}(\operatorname{spec}(A_X), \operatorname{spec}(A_Y)) \le C d_{\mathsf{H}}(X,Y),$$

where

$$\delta := \frac{1}{(\|c_k\|_\infty + 1)(c_\tau + 1)}$$

and

$$C := (\|c_k\|_\infty + 1)(c_\tau + 1) c_s \Big(\frac{(2+b)^{2+b} c_1}{2 b^b c_0} \Big)^{1/2}.$$

Remark 1.3.1. (a) We remark that Theorem 1.3 (a) generalizes the 1990 result by J. Avron, P. van Mouche, and B. Simon [7], and Theorem 1.3 (b) generalizes the 2019 result by S. Beckus, J. Bellissard, and H. Cornean [12]. Our work exposes the connection between the past two results and reveals that quantitative continuity of the spectrum map follows from quite general operator and dynamical properties.

(b) Lipschitz continuity of the action (i) and strict polynomial growth of the group (iv) are specific conditions covering a large class of models. We remark that our proof allows one to weaken or strengthen both conditions. For example, we will demonstrate that our unified theory applies in Section 2.4 after condition (i) is weakened so that the group action is instead assumed to be exponential Lipschitz leading to a log-Hölder continuity result. Also, one can generalize Lemma 4.5 (where the strict polynomial growth-estimates of the group are utilized) so that yet another quantitative continuity result can be obtained for groups with (sub)exponential growth-estimates; see a discussion below. Specifically, Lemma 4.5 is a machine that has as an input the polynomial growth-estimates and that has as an output a suitable family of almost invariant vectors to be utilized later as cutoff functions. The generalization reduces to replacing growth-estimates of the group with estimates from the suitable family of almost invariant vectors. We remark that the lamplighter group has exponential growth but is amenable.

(c) Theorem 1.3 (a) is sharp due to the 1990 works [39, 57] discussed above; see also [17]. We remark that the regularity can be improved to be Lipschitz provided the kernel is instead locally-constant; see Theorem 1.3 (b). Theorem 1.3 (b) is also sharp due to the recent work [14] for Sturmian Hamiltonians. Also, in 1994 [17] it was established that the regularity at the boundary of a spectral gap can be improved to be Lipschitz for a general operator family containing the AMO provided the gap width is locally-positive. Also, in 1998 [45] it was established that the regularity can be improved to be Lipschitz-log for the AMO provided the coupling λ is in the open set $(0, 2/29) \cup (29/2, +\infty)$; see also [44, 68]. Also, in 2014 [42] it was established that the regularity can be generalized to be $\gamma/(\gamma + 1)$-Hölder for one-dimensional discrete quasiperiodic Schrödinger operators with γ-Hölder continuous potentials; our proofs utilize almost invariant vectors as cutoff functions which is inspired by the 1990 works [7, 24] which itself inspired the generalization within [42], and we remark that our regularity can too be generalized to be $\gamma/(\gamma + 1)$-Hölder in the setting for Theorem 1.3 (a) provided the kernel is instead γ-Hölder continuous.

(d) As discussed before, our spectral estimates are sharp for Lipschitz continuous kernels as well as for locally-constant kernels. In both cases, sharpness is proven by studying cases where spectral gaps close in the limit. In contrast to that, we discussed before that these estimates can be improved if spectral gaps do not close. This is a general phenomena discussed thoroughly in [11]. The decrease of regularity is reflected in our proof by the interplay between amenability (implemented through Lemma 4.5) and the regularity of the group action τ and the kernel k as described below in the strategy of the proof.

Strategy of the proof. We briefly explain the main strategy to prove Theorem 1.3. For the sake of simplicity, let us assume that

$$X = \overline{\{g \cdot_\tau x : g \in G\}} \quad \text{and} \quad Y = \overline{\{g \cdot_\tau y : g \in G\}}$$

for some $x, y \in Z$. Since the operators are normal, the first main ingredient of our proof is that for $z \in Z$ and $E \in \mathbb{R}$, the estimate $\operatorname{dist}(E, \operatorname{spec}(A_z)) < \delta$ holds if and only if there is a $\varphi \in C_c(G)$ satisfying $\|(A_z - E)\varphi\| \le \varepsilon \|\varphi\|$. Here, $C_c(G)$ is the space of continuous functions with compact support on G and φ is also called an *approximate eigenvector* of A_z for E. This correlation is used to obtain $\operatorname{dist}(E, \operatorname{spec}(A_y)) < \delta = C d_{\mathsf{H}}(X, Y)^\gamma$ for $E \in \operatorname{spec}(A_x)$ and a suitable $\gamma \in [0, 1]$.

Let $E \in \operatorname{spec}(A_x)$, $\varepsilon > 0$ and $\chi \in C_c(G)$ be a cutoff function. Then, there is an approximate eigenvector φ satisfying $\|(A_x - E)\varphi\| \le \varepsilon \|\varphi\|$. Next, we construct an approximate eigenvector of A_z for E and a suitable $z \in Y$ using φ. The first step is to localize the support of φ onto the shift of the support of χ. We prove that there is a $j \in G$ such that for $\varphi_j \in C_c(G)$ (defined by $\varphi_j(g) = \chi(j^{-1}g)\varphi(g)$ for $g \in G$) the formula

$$\|(A_x - E)\varphi_j\| < \delta_1(\varepsilon, \chi)\|\varphi_j\|$$

is satisfied for a suitable choice of $\delta_1(\varepsilon, \chi)$, confer Lemma 4.2 and the first part of the proof in Lemma 4.3. By definition of the Hausdorff metric on $\mathcal{S}$, there is a $z \in Y$ satisfying $d(j^{-1} \cdot_\tau x, j^{-1} \cdot_\tau z) \le d_{\mathsf{H}}(X, Y) + \varepsilon$. We already note here that $\operatorname{spec}(A_z) \subseteq \operatorname{spec}(A_y)$ since $z \in Y = \overline{\{g \cdot_\tau y : g \in G\}}$.

Using $A_z = A_z - A_x + A_x$, we prove in Lemma 4.3 that

$$\|(A_z - E)\varphi_j\| < (\delta_1(\varepsilon, \chi) + \delta_2(\varepsilon, \chi))\|\varphi_j\|$$

where

$$\delta_2(\varepsilon, \chi) = \left\| (A_z - A_x) \frac{\varphi_j}{\|\varphi_j\|} \right\|.$$

Hence, we conclude, for $\delta(\varepsilon, \chi) = \delta_1(\varepsilon, \chi) + \delta_2(\varepsilon, \chi)$,

$$\emptyset \neq B_{\delta(\varepsilon,\chi)}(E) \cap \operatorname{spec}(A_z) \subseteq B_{\delta(\varepsilon,\chi)}(E) \cap \operatorname{spec}(A_y).$$

Thus, $\operatorname{dist}(E, \operatorname{spec}(A_y)) < \delta(\varepsilon, \chi)$ is concluded. In order to get $\delta(\varepsilon, \chi) = C d_{\mathsf{H}}(X, Y)^\gamma$, we need to choose ε and the cutoff χ accordingly where we use

- for $\delta_1(\varepsilon, \chi)$ assumptions (iii) and (iv), and
- for $\delta_2(\varepsilon, \chi)$ assumptions (i) and (ii).

We observe that $\delta_1(\varepsilon, \chi)$ is small provided we choose the support of χ such that its boundary is small compared to the size of the support. This means that the support needs to be large. Here, the concept of amenability enters via our choice of cutoff functions in Lemma 4.5. On the other hand, $\delta_2(\varepsilon, \chi)$ is estimated by using the regularity of the group action and the kernel. Here, the size of the support enters via the Lipschitz constant of the group action in (i). Thus, the larger the support, the larger is $\delta_2(\varepsilon, \chi)$. This interplay between a large and a small support leads to the square root behavior in Theorem 1.3 (a).

The correct choice of the cutoff function χ may influence the exponent γ. On the one hand, it is chosen to be supported on Følner sequences of the underlying group G – here balls since the group is strictly-polynomial-growing. In [24], the authors choose χ to be the characteristic function on balls leading to an exponent $\gamma = 1/3$ for the AMO. Instead, [7] choose a tent-like function on the balls leading to an exponent $\gamma = 1/2$, which cannot be improved as discussed above and in Remark 4.5.1.

As demonstrated in Section 2.3 and Section 2.4, one can follow the same strategy if a different regularity of the group action τ or the kernel k is chosen. Then, the optimization of the size of the support of χ (to minimize $\delta_1(\varepsilon, \chi)$ and $\delta_2(\varepsilon, \chi)$ simultaneously) leads to a different exponent γ. The estimate improves in Theorem 1.3 (b) since for a suitable choice of ε and χ, the term $\delta_2(\varepsilon, \chi)$ vanishes using that the kernel is locally constant.

The strict polynomial growth enters only when estimating $\delta_1(\varepsilon, \chi)$ via Lemma 4.5, where an estimate of a certain cutoff for such groups is proven. The existence of a cutoff function χ is guaranteed by assuming that the group is amenable. If, however, the group is not strictly polynomial growing, then the analog estimate of Lemma 4.5 for the cutoff function may vary in the size of the support of χ. This affects the quantitative behavior of the spectrum accordingly.

Our work: final remarks. To explain why the continuous behavior of the spectra has been studied for various models, there are cases where one has a sequence of (periodic) approximations and thus a sequence of convergent spectra. This has been useful to analyze the spectral theory of operators. For example, rational approximations provide a deeper understanding of the almost-Mathieu operator and related quasiperiodic models. Similarly, periodic approximations provide a deeper understanding of the Fibonacci Hamiltonian and related Sturmian models. For more papers about "almost periodic" operators, we recommend the 1990 paper by A. Sütő [61] (also [16]) and the references therein. Also, we recommend the 2016 paper by D. Damanik, A. Gorodetski, and W. Yessen [27], the recent solution to the Dry Ten Martini Problem for Sturmian systems [9] that uses periodic approximations, and the book by D. Damanik and J. Fillman [26]. Approximations are a practical tool to compute the spectrum. Indeed, D. Hofstadter in 1976 [40] plotted one such spectral approximation for the AMO (see Figure 1), but what is more important is that it can be utilized to derive spectral properties of the limit-operator A_x from the periodic approximants A_{y_j}.

Indeed, physicists sought to better understand electrons in a crystal and in a magnetic field; E. J. Austin and M. Wilkinson in 1990 [64] computationally studied a general operator family containing the AMO and plotted a Hofstadter-like butterfly. Also, physicists sought to better understand the fractal dimension of $\mathrm{spec}(H^{\mathrm{AMO}}_{1,\alpha,0})$; M. Kohmoto and C. Tang in 1986 [62] computationally studied $\alpha = (\sqrt{5}-1)/2$, $\sqrt{2}-1$, $(\pi-3)\sqrt{3}$ and conjectured that the dimension is $1/2$ for irrational α (the dimension is 1 for rational α); see also [33, 59]. Years later Y. Last in 1994 [48] proved that the Hausdorff dimension does not exceed $1/2$ for topologically-generic irrational $\alpha^{(\mathcal{T})}$ – the proof utilizes $1/2$-Hölder regularity of the spectrum map. But, E. J. Austin and M. Wilkinson in 1994 [65] proposed that the box-counting dimension is nonconstant for irrational α and approaches zero for specific α_n, and T. Geisel, R. Ketzmerick, K. Kruse, and F. Steinbach in 1998 [32] proposed that, given an assumption, the Hausdorff dimension does not exceed $1/2$ for all irrational α. A decade later, S. Jitomirskaya and I. Krasovsky in 2019 [41] made steps toward the conjectured fractal dimension of the spectrum – the proof utilizes Lipschitz-log regularity of the spectrum map. We remark that Y. Last and M. Shamis in 2016 [49] proved that the Hausdorff dimension is 0 for topologically-generic irrational $\alpha^{(\mathcal{T}')}$ – the proof utilizes $1/2$-Hölder regularity of the spectrum map – and S. Jitomirskaya and S. Zhang in 2022 [43] proved that the box-counting dimension is 1 for the same topologically-generic irrational $\alpha^{(\mathcal{T}')}$ and $\mathcal{T}' \subseteq \mathcal{T}$. Also, B. Helffer, Q. Liu, Y. Qu, and Q. Zhou in 2019 [38] proved that the collection $\mathcal{T}''$ of all irrational α such that the Hausdorff dimension is positive (thus $\mathcal{T}'' \cap \mathcal{T}' = \varnothing$) is topologically-dense and itself has positive Hausdorff dimension.

Within [3], periodic Jacobi matrices on trees are studied, and our theorem does not apply since we consider groups but they consider trees. We notice that the athors of [3] remark that what makes their objects fascinating is that the fundamental group is nonabelian. Within [23], Schrödinger operators on Lie groups are studied, and our theorem does not apply since we consider dynamically-defined operators but they do not have a dynamical system. Also, their group needs not be unimodular thus needs not have polynomial growth but their group satisfies a weak polynomial growth condition. We notice that the authors of [23] remark that what makes their theorem fascinating is that their work applies to the Heisenberg group (a nonabelian group). In recent elaborations [10], it was shown that aperiodic subshifts over the discrete Heisenberg admit periodic approximations and our main Theorem 1.3 was there applied to obtain corresponding spectral estimates. Within [50], random Schrödinger operators on manifolds are studied. We notice that the authors of [50] remark that their work continues the [20] approach (see also [15]), and what makes their theorem fascinating is that they use techniques from Connes' noncommutative integration theory and von Neumann algebras. For more related papers, see [25, 34].

Within [20], our operator families are studied, and we consider spectral continuity results but they consider the integrated density of states and spectral gap-labeling results. We notice that the authors of [20] remark that their work generalizes results by both R. Johnson and J. Moser (1982) [46] and F. Delyon and B. Souillard (1983) [28] and what makes their theorem fascinating is that they can deal with an arbitrary dimension both in the discrete and in the continuous case.

In Section 2 we demonstrate example applications. In Section 3 we provide the preliminaries. In Section 4 we prove the main results. In Section 5 we prove a standard lemma.

2. Examples of applications

2.1. One-dimensional discrete quasiperiodic Schrödinger operators

We demonstrate that our unified theory applies for this titular setting.

Let $\langle \mathbb{Z}, \# \rangle$ be the one-dimensional lattice endowed with the counting measure #. Let ℓ be the metric on $\mathbb{Z}$ defined by

$$\ell(n, m) = |-n + m|$$

for every $n, m \in \mathbb{Z}$. Define the phase space $Z := \mathbb{R} \times \mathbb{T}$. Let d be the metric on Z defined by

$$d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

for every $x, y \in Z$. Let $\langle Z, (\mathbb{Z}, \tau) \rangle$ be the (quasiperiodic) dynamical system defined by

$$-n \cdot_\tau x = \begin{bmatrix} x_1 \\ n x_1 + x_2 \end{bmatrix}$$

for every $n \in \mathbb{Z}$, $x \in Z$. Using the transformation $T_\alpha \colon \mathbb{T} \to \mathbb{T}, \omega \mapsto -\alpha + \omega$ defined in the introduction, observe $-n \cdot_\tau \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_1 \\ T_{x_1}^{-n} x_2 \end{smallmatrix}\right]$ for $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in Z$. Define, for $v \in C(\mathbb{T})$, the kernel $k_v \colon \mathbb{Z} \times Z \to \mathbb{C}$ by

$$k_v\left(m, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{cases} 1 & \text{if } m \in \{-1, 1\}, \\ v(x_2) & \text{if } m = 0, \\ 0 & \text{else}, \end{cases}$$

for every $m \in \mathbb{Z}$, $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in Z$. A short computation gives that the kernel k_v defines the dynamically-defined operator family $\{A_x = A_{v,x}\}_{v \in C(\mathbb{T}), x \in Z} \subseteq \mathcal{L}(\ell^2(\mathbb{Z}))$ by

$$\begin{aligned}(A_x\psi)(n) &= \sum_{m\in\mathbb{Z}} k_v(-n+m, -n \cdot_\tau x)\psi(m) \\ &= \psi(n+1) + \psi(n-1) + v(nx_1 + x_2)\psi(n)\end{aligned}$$

for every $\psi \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$. Observe that if $v(\omega) = 2\lambda\cos(2\pi\omega)$, then we recover the AMO.

Theorem 2.1 (1/2-Hölder regularity). *Assume the potential $v\colon \mathbb{T} \to \mathbb{R}$ is Lipschitz continuous, i.e., there exists $\mathrm{c}_v > 0$ such that for all $\omega_1, \omega_2 \in \mathbb{T}$, $|v(\omega_1) - v(\omega_2)| \leq \mathrm{c}_v|\omega_1 - \omega_2|$. Then,*

(a) *τ satisfies a Lipschitz continuity condition: for each $n \in \mathbb{Z}$,*

$$d(n \cdot_\tau x, n \cdot_\tau y) \leq (|n| + 1)d(x, y);$$

(b) *k_v satisfies a Lipschitz continuity condition: for each $m \in \mathbb{Z}$,*

$$|k_v(m, x) - k_v(m, y)| \leq \mathrm{c}_v\delta_0(m)d(x, y);$$

(c) *k_v satisfies a linear decay condition: for each $r \geq 0$,*

$$\sup_{x\in Z} \sum_{m\in\mathbb{Z}} \|k_v(m, -(\cdot) \cdot_\tau x)\min\{|m|, r\}\|_\infty \leq 2d;$$

(d) *$\langle(\mathbb{Z}, \ell), \#\rangle$ is strictly-polynomial-growing: for each $r \geq 1$,*

$$r \leq \#B(r) \leq 2r;$$

(e) *for all $X, Y \in \mathcal{S}$,*

$$\begin{aligned}&d_{\mathsf{H}}(X, Y) \leq \min\{\delta, 1\} \\ &\quad\Longrightarrow\ \operatorname{dist}_{\mathsf{H}}(\operatorname{spec}(A_X), \operatorname{spec}(A_Y)) \leq C d_{\mathsf{H}}(X, Y)^{1/2},\end{aligned}$$

where

$$\delta := \frac{2}{\mathrm{c}_v}\frac{3^{3/2}}{1} \quad \text{and} \quad C := 2(\mathrm{c}_v 2(3^{3/2}))^{1/2} + \mathrm{c}_v.$$

Proof. The claims (a)–(d) are easy to see. Then, (e) follows from a combination of (a)–(d) and Theorem 1.3 (a). ■

Now, we can directly conclude the following.

Corollary 2.1.1. *Consider the almost-Mathieu operator family*

$$H_{\lambda,\alpha,\omega}\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$$

for $\lambda, \alpha \in \mathbb{R}$ *and* $\omega \in \mathbb{T}$. *If* $|\alpha_1 - \alpha_2| < 3^{3/2}/(2\pi\lambda)$, *then*

$$\mathrm{dist}_{\mathsf{H}}\Big(\overline{\bigcup_{\omega\in\mathbb{T}} \mathrm{spec}(H_{\lambda,\alpha_1,\omega})}, \overline{\bigcup_{\omega\in\mathbb{T}} \mathrm{spec}(H_{\lambda,\alpha_2,\omega})}\Big) \le (2\sqrt{8\pi\lambda(3^{3/2})} + 4\pi\lambda)|\alpha_1 - \alpha_2|^{1/2}.$$

Assume α_1 *is irrational and* $\alpha_2 = p/q$ *with* p, q *coprime and* $|\alpha_1 - \alpha_2| < 1/q^2 < 1$*; then, for* $\omega \in \mathbb{T}$, $\overline{\bigcup_{\omega'\in\mathbb{T}} \mathrm{spec}(H_{\lambda,\alpha_1,\omega'})} = \mathrm{spec}(H_{\lambda,\alpha_1,\omega})$ *and*

$$\mathrm{dist}_{\mathsf{H}}(\mathrm{spec}(H_{\lambda,\alpha_1,\omega}), \mathrm{spec}(H_{\lambda,\alpha_2,\omega})) \le (2\sqrt{8\pi\lambda(3^{3/2})} + 4\pi\lambda)\frac{1}{\sqrt{2q}}.$$

Proof. Consider the kernel k_v defined by $v(\omega) = 2\lambda\cos(2\pi\omega)$. Then, the Lipschitz constant is $c_v = 4\pi\lambda$ and

$$\delta = \frac{2}{\mathrm{c}_v}\frac{3^{3/2}}{1} = \frac{3^{3/2}}{2\pi\lambda}$$

and

$$C = 2(\mathrm{c}_v 2(3^{3/2}))^{1/2} + \mathrm{c}_v = 2\sqrt{8\pi\lambda(3^{3/2})} + 4\pi\lambda.$$

For the first estimate, observe that $X = \{\alpha_1\} \times \mathbb{T}$ and $Y = \{\alpha_2\} \times \mathbb{T}$ are elements of $\mathcal{S}$ and $d_{\mathsf{H}}(X, Y)^{1/2} = |\alpha_1 - \alpha_2|^{1/2}$. For the second estimate, let $\omega \in \mathbb{T}$. Recall the orbit-closure of $x \in Z$ is defined by $X(x) := \overline{\{n \cdot_\tau x : n \in \mathbb{Z}\}}$. Observe $X([\begin{smallmatrix}\alpha_2\\ \omega\end{smallmatrix}]) = \{\alpha_2\} \times \{np/q + \omega : n \in \mathbb{Z}\}$ and $X([\begin{smallmatrix}\alpha_1\\ \omega\end{smallmatrix}]) = \{\alpha_1\} \times \mathbb{T}$ since α_1 is irrational. Then the estimate follows from

$$d_{\mathsf{H}}\left(X\left(\begin{bmatrix}\alpha_1\\ \omega\end{bmatrix}\right), X\left(\begin{bmatrix}\alpha_2\\ \omega\end{bmatrix}\right)\right)^{1/2} = \max\left\{|\alpha_1 - \alpha_2|, \frac{1}{2q}\right\}^{1/2} = \frac{1}{\sqrt{2q}}.$$

The equality $\overline{\bigcup_{\omega'\in\mathbb{T}} \mathrm{spec}(H_{\lambda,\alpha_1,\omega'})} = \mathrm{spec}(H_{\lambda,\alpha_1,\omega})$ follows from standard arguments that the irrational rotation is minimal. It follows also from Theorem 1.3 using that if α_1 is irrational, then $X\left([\begin{smallmatrix}\alpha_1\\ \omega_1\end{smallmatrix}]\right) = X\left([\begin{smallmatrix}\alpha_1\\ \omega_2\end{smallmatrix}]\right)$ for all $\omega_1, \omega_2 \in \mathbb{T}$, ∎

We remark that a 1990 result by both [24] and [7] established that the spectrum map is Hölder continuous in this specific setting; the regularity was established to be 1/3-Hölder [24] but was improved to be 1/2-Hölder [7]. Our estimate is different and significant since

$$\delta = \frac{2}{\mathrm{c}_v}\frac{3^{3/2}}{1}$$

is explicitly written; another C constant is explicitly written within [7].

2.2. Multidimensional discrete subshift Schrödinger operators

We demonstrate that our unified theory applies for this titular setting.

Let $\langle \mathbb{Z}^d, \# \rangle$ be the multidimensional lattice endowed with the counting measure #. Let ℓ be the metric on $\mathbb{Z}^d$ defined by

$$\ell(n,m) = \max\{|-n_1 + m_1|, \ldots, |-n_d + m_d|\}$$

for every $n, m \in \mathbb{Z}^d$. Define the phase space $Z := \mathcal{A}^{\mathbb{Z}^d} = \{x : \mathbb{Z}^d \to \mathcal{A}\}$ for a finite set of symbols $\mathcal{A}$ "a finite alphabet." For instance, $\mathcal{A} = \{0, 1\}$. Let d be the metric on Z defined by

$$d(x,y) = \inf\Big\{\frac{1}{r+1} : r \in [0, +\infty), x|_{B(r)} = y|_{B(r)}\Big\}$$

for every $x, y \in Z$. Let $\langle Z, (\mathbb{Z}^d, \tau)\rangle$ be the (subshift) dynamical system defined by

$$-n \cdot_\tau x = (m \mapsto x(n+m)) =: T^{-n} x$$

for every $n \in \mathbb{Z}^d, x \in Z$. Define, for $v \in C(Z)$, the kernel $k_v : \mathbb{Z}^d \times Z \to \mathbb{C}$ by

$$k_v(m,x) = \begin{cases} 1 & \text{if } \|m\|_1 = 1, \\ v(x) & \text{if } m = \vec{0}, \\ 0 & \text{else}, \end{cases}$$

for every $m \in \mathbb{Z}^d, x \in Z$. Define the operator family

$$\{A_x = A_{v,x}\}_{v \in C(Z), x \in Z} \subseteq \mathcal{L}(\ell^2(\mathbb{Z}^d))$$

by

$$\begin{aligned}(A_x\psi)(n) &= \sum_{m \in \mathbb{Z}^d} k_v(-n+m, -n \cdot_\tau x)\psi(m) \\ &= \Big(\sum_{\|m\|_1 = 1} \psi(n+m)\Big) + v(T^{-n}x)\psi(n)\end{aligned}$$

for every $\psi \in \ell^2(\mathbb{Z}^d)$, $n \in \mathbb{Z}^d$. Observe if $\mathcal{A} = \{0,1\}$, $v(x) = x(\vec{0})$ and $x \in Z$ is a Fibonacci sequence, then we recover the Fibonacci Hamiltonian [27].

Theorem 2.2 (Lipschitz regularity). *Assume the potential $v : Z \to \mathbb{R}$ is locally-constant, i.e., there exist $\mathrm{r}_v \geq 0$ and $\mathrm{w}_v : \{0,1\}^{B(\mathrm{r}_v)} \to \mathbb{R}$ such that for each $x \in Z$, $v(x) = \mathrm{w}_v(x|_{B(\mathrm{r}_v)})$. Then,*

(a) *τ satisfies a Lipschitz continuity condition: for each $n \in \mathbb{Z}^d$,*

$$d(n \cdot_\tau x, n \cdot_\tau y) \leq (|n| + 1) d(x,y);$$

(b) *k_v satisfies a locally-constant condition: for each $m \in \mathbb{Z}^d$,*

$$d(x,y) < \frac{1}{(\mathrm{r}_v + 1)\delta_{\vec{0}}(m)} \implies k_v(m,x) = k_v(m,y);$$

(c) *k_v satisfies a linear decay condition: for each $r \geq 0$,*

$$\sup_{x \in Z} \sum_{m \in \mathbb{Z}^d} \|k_v(m, -(\cdot) \cdot_\tau x) \min\{|m|, r\}\|_\infty \leq 2d;$$

(d) *$\langle(\mathbb{Z}^d, \ell), \#\rangle$ is strictly-polynomial-growing: for each $r \geq 1$,*

$$r^d \leq \#B(r) \leq 2^d r^d;$$

(e) *for all $X, Y \in \mathcal{S}$,*

$$d_{\mathsf{H}}(X,Y) \leq \delta \implies \operatorname{dist}_{\mathsf{H}}(\operatorname{spec}(A_X), \operatorname{spec}(A_Y)) \leq C d_{\mathsf{H}}(X,Y),$$

where

$$\delta := \frac{1}{2(\mathrm{r}_v + 2)} \quad \textit{and} \quad C := 4(\mathrm{r}_v + 2)d\left(\frac{(2+d)^{2+d}2^d}{2d^d}\right)^{1/2}.$$

Proof. The claims (a)–(d) are easy to see. Then, (e) follows from a combination of (a)–(d) and Theorem 1.3 (b). ∎

We remark that a 2019 result by [12] established that the spectrum map is Lipschitz continuous in this specific setting. Our estimate is different and significant since $\delta = 1/(2(\mathrm{r}_v + 2))$ is explicitly written; another C constant is explicitly written within [12].

2.3. One-dimensional discrete skew-shift Schrödinger operators

We demonstrate that our unified theory applies for this titular setting provided a lenient condition replaces condition (i) within Theorem 1.3; see Theorem 2.3 (a).

Let $\langle\mathbb{Z}, \#\rangle$ be the one-dimensional lattice endowed with the counting measure #. Let ℓ be the metric on $\mathbb{Z}$ defined by

$$\ell(n,m) = |-n+m|$$

for every $n, m \in \mathbb{Z}$. Define the phase space $Z := \mathbb{R} \times \mathbb{T} \times \mathbb{T}$. Let d be the metric on Z defined by

$$d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|\}$$

for every $x, y \in Z$. Let $\langle Z, (\mathbb{Z}, \tau)\rangle$ be the (skew-shift) dynamical system defined by

$$-n \cdot_\tau x = \begin{bmatrix} x_1 \\ nx_1 + x_2 \\ \frac{n(n-1)}{2}x_1 + nx_2 + x_3 \end{bmatrix}$$

for every $n \in \mathbb{Z}, x \in Z$. Define, for $v \in C(\mathbb{T})$, the kernel $k_v : \mathbb{Z} \times Z \to \mathbb{C}$ by

$$k_v(m, x) = \begin{cases} 1 & \text{if } m \in \{-1, 1\}, \\ v(x_3) & \text{if } m = 0, \\ 0 & \text{else,} \end{cases}$$

for every $m \in \mathbb{Z}, x \in Z$. Define the operator family

$$\{A_x = A_{v,x}\}_{v \in C(\mathbb{T}), x \in Z} \subseteq \mathcal{L}(\ell^2(\mathbb{Z}))$$

by

$$\begin{aligned} (A_x \psi)(n) &= \sum_{m \in \mathbb{Z}} k_v(-n + m, -n \cdot_\tau x) \psi(m) \\ &= \psi(n+1) + \psi(n-1) + v\Big(\frac{n(n-1)}{2} x_1 + n x_2 + x_3\Big) \psi(n) \end{aligned}$$

for every $\psi \in \ell^2(\mathbb{Z}), n \in \mathbb{Z}$. We remark that this operator family can be found within [21, 22, 47] and the following regularity result is new.

Theorem 2.3 (1/3-Hölder regularity). *Assume the potential $v : \mathbb{T} \to \mathbb{R}$ is Lipschitz continuous, i.e., there exists $c_v > 0$ such that for all $\omega_1, \omega_2 \in \mathbb{T}$, $|v(\omega_1) - v(\omega_2)| \leq c_v |\omega_1 - \omega_2|$. Then,*

(a) *τ satisfies a power Lipschitz continuity condition: for each $n \in \mathbb{Z}$,*

$$d(n \cdot_\tau x, n \cdot_\tau y) \leq \Big(\frac{n(n-1)}{2} + |n| + 1\Big) d(x, y);$$

(b) *k_v satisfies a Lipschitz continuity condition: for each $m \in \mathbb{Z}$,*

$$|k_v(m, x) - k_v(m, y)| \leq c_v \delta_0(m) d(x, y);$$

(c) *k_v satisfies a linear decay condition: for each $r \geq 0$,*

$$\sup_{x \in Z} \sum_{m \in \mathbb{Z}} \| k_v(m, -(\cdot) \cdot_\tau x) \min\{|m|, r\} \|_\infty \leq 2;$$

(d) *$\langle (\mathbb{Z}, \ell), \# \rangle$ is strictly-polynomial-growing: for each $r \geq 1$,*

$$r \leq \# B(r) \leq 2r;$$

(e) *for all $X, Y \in \mathcal{S}$,*

$$\begin{aligned} r \geq 1 \implies \; & \mathrm{dist}_{\mathsf{H}}(\mathrm{spec}(A_X), \mathrm{spec}(A_Y)) \\ & \leq c_v \Big(\frac{r(r-1)}{2} + r + 1\Big) d_{\mathsf{H}}(X, Y) + \frac{2}{r}(3^{3/2}) \end{aligned}$$

and there exists $\delta(\mathrm{c}_v) > 0$ *(independent of* X *and* Y*) such that*

$$\begin{aligned} & d_{\mathsf{H}}(X,Y) \le \min\{\delta(\mathrm{c}_v), 1\} \\ & \qquad \Longrightarrow \ \operatorname{dist}_{\mathsf{H}}(\operatorname{spec}(A_X), \operatorname{spec}(A_Y)) \le C d_{\mathsf{H}}(X,Y)^{1/3}, \end{aligned}$$

where

$$C := \min\Big\{\mathrm{c}_v\Big(\frac{\delta^{2/3}}{2} + \frac{\delta^{1/3}}{2} + 1\Big) + \frac{2}{\delta^{1/3}}(3^{3/2}) : \delta > 0\Big\}.$$

Also, $\delta(\mathrm{c}_v)$ *minimizes* C*.*

Proof. The claims (a)–(d) are easy to see. Then, (e) follows from a combination of (a)–(d) and Lemma 4.3 (where amenability twinkles) and Lemma 4.5 as described next. Fix $r \ge 1$. Define $\chi_r = \chi\colon \mathbb{Z} \to [0, +\infty), n \mapsto ((r - |n|)/r)\mathbb{1}_{B(r)}(n)$. Write $\|\chi\| := \|\chi\|_2$. Let $\rho > 0$. Observe

$$\begin{aligned} & \operatorname{dist}_{\mathsf{H}}(\operatorname{spec}(A_X), \operatorname{spec}(A_Y)) \\ & \quad \le \sup_{\substack{\check{x}\in X, \check{y}\in Y, \check{z}\in Z \\ d(\check{x},\check{y}) < d_{\mathsf{H}}(X,Y)+\rho}} \sum_{m\in\mathbb{Z}} \|(k_v(m, -(\cdot)\cdot_\tau \check{y}) - k_v(m, -(\cdot)\cdot_\tau \check{x}))\mathbb{1}_{\operatorname{supp}(\chi)}(\cdot)\|_\infty \\ & \qquad + \|k_v(m, -(\cdot)\cdot_\tau \check{z})\|_\infty \frac{\|\chi^m - \chi\|}{\|\chi\|} && \text{(Lemma 4.3)} \\ & \quad \le \sup_{\substack{\check{x}\in X, \check{y}\in Y, \check{z}\in Z \\ d(\check{x},\check{y}) < d_{\mathsf{H}}(X,Y)+\rho}} \sum_{m\in\mathbb{Z}} \|\mathrm{c}_v\delta_0(m) d(-(\cdot)\cdot_\tau \check{y}, -(\cdot)\cdot_\tau \check{x})\mathbb{1}_{\operatorname{supp}(\chi)}(\cdot)\|_\infty \\ & \qquad + \|k_v(m, -(\cdot)\cdot_\tau \check{z})\|_\infty \frac{\|\chi^m - \chi\|}{\|\chi\|} && \text{(property (b))} \\ & \quad \le \sup_{\substack{\check{x}\in X, \check{y}\in Y, \check{z}\in Z \\ d(\check{x},\check{y}) < d_{\mathsf{H}}(X,Y)+\rho}} \sum_{m\in\mathbb{Z}} \mathrm{c}_v\delta_0(m)\Big(\frac{r(r-1)}{2} + r + 1\Big) d(\check{y}, \check{x}) \\ & \qquad + \|k_v(m, -(\cdot)\cdot_\tau \check{z})\|_\infty \frac{\|\chi^m - \chi\|}{\|\chi\|} && \text{(property (a))} \\ & \quad \le \mathrm{c}_v\Big(\frac{r(r-1)}{2} + r + 1\Big)(d_{\mathsf{H}}(X,Y) + \rho) \\ & \qquad + \sup_{\check{z}\in Z} \sum_{m} \|k_v(m, -(\cdot)\cdot_\tau \check{z})\|_\infty \min\Big\{\frac{|m|}{r}(3^{3/2}), 2^{1/2}\Big\} && \text{(Lemma 4.5)} \\ & \quad \le \mathrm{c}_v\Big(\frac{r(r-1)}{2} + r + 1\Big)(d_{\mathsf{H}}(X,Y) + \rho) + \frac{2}{r}(3^{3/2}). && \text{(property (c))} \end{aligned}$$

Here, $\chi^m\colon \mathbb{Z} \to [0, +\infty), n \mapsto \chi(n+m)$. Also, if

$$d_{\mathsf{H}}(X,Y) \le \min\{\delta, 1\},$$

then $1 \leq (d_{\mathsf{H}}(X,Y)^{-1}\delta)^{1/3}$ and $d_{\mathsf{H}}(X,Y) \leq d_{\mathsf{H}}(X,Y)^{1/3}$ and

$$\begin{aligned}
&\mathrm{dist}_{\mathsf{H}}(\mathrm{spec}(A_X), \mathrm{spec}(A_Y)) \\
&\quad \leq \mathrm{c}_v\Big(\frac{r(r-1)}{2} + r + 1\Big) d_{\mathsf{H}}(X,Y) + \frac{2}{r}(3^{3/2})\Big|_{r=(d_{\mathsf{H}}(X,Y)^{-1}\delta)^{1/3}} \\
&\quad \leq C d_{\mathsf{H}}(X,Y)^{1/3}.
\end{aligned}$$

■

2.4. One-dimensional discrete limit-periodic Schrödinger operators

We demonstrate that our unified theory applies for this titular setting provided a lenient condition replaces condition (i) within Theorem 1.3; see Theorem 2.4 (a).

Let $\langle \mathbb{Z}, \# \rangle$ be the one-dimensional lattice endowed with the counting measure #. Let ℓ be the metric on $\mathbb{Z}$ defined by

$$\ell(n,m) = |-n + m|$$

for every $n, m \in \mathbb{Z}$. Define the space $\bar{Z} := [0,1]^{\mathbb{Z}}$. Let d be the metric on $\bar{Z}$ defined by

$$d(x,y) = \sum_{m \geq 0} \frac{1}{2^m} \max\{|x_m - y_m|, |x_{-m} - y_{-m}|\}$$

for every $x, y \in \bar{Z}$. Let $\langle \bar{Z}, (\mathbb{Z}, \tau) \rangle$ be the dynamical system defined by

$$-n \cdot_\tau x = (m \mapsto x(n+m)) =: T^{-n} x$$

for every $n \in \mathbb{Z}, x \in \bar{Z}$. Define the phase (sub)space $Z := \bar{Z} \cap \{\text{limit-periodic}\}$, i.e., for each $x \in \bar{Z}$, $x \in Z$ if and only if there exists a sequence $\{y_j\} \subseteq \bar{Z}$ such that $\#\{n \cdot_\tau y_j : n \in \mathbb{Z}\} < +\infty$ for every j and $\|x - y_j\|_\infty \to 0$ as $j \to +\infty$. Define, for $v \in C(Z)$, the kernel $k_v \colon \mathbb{Z} \times Z \to \mathbb{C}$ by

$$k_v(m,x) = \begin{cases} 1 & \text{if } m \in \{-1, 1\}, \\ v(x) & \text{if } m = 0, \\ 0 & \text{else,} \end{cases}$$

for every $m \in \mathbb{Z}, x \in Z$. Define the operator family

$$\{A_x = A_{v,x}\}_{v \in C(Z), x \in Z} \subseteq \mathcal{L}(\ell^2(\mathbb{Z}))$$

by

$$\begin{aligned}
(A_x\psi)(n) &= \sum_{m \in \mathbb{Z}} k_v(-n+m, -n \cdot_\tau x)\psi(m) \\
&= \psi(n+1) + \psi(n-1) + v(T^{-n}x)\psi(n)
\end{aligned}$$

for every $\psi \in \ell^2(\mathbb{Z}), n \in \mathbb{Z}$. We remark that this operator family can be found within [1, 4, 31] and the following regularity result is new.

Theorem 2.4 (log-Hölder regularity). *Assume the potential $v\colon Z \to \mathbb{R}$ is Lipschitz continuous, i.e., there exists $\mathrm{c}_v > 0$ such that for all $x, y \in Z$, $|v(x) - v(y)| \leq \mathrm{c}_v d(x, y)$. Then,*

(a) *τ satisfies an exponential Lipschitz continuity condition: for each $n \in \mathbb{Z}$,*

$$d(n \cdot_\tau x, n \cdot_\tau y) \leq 2^{|n|} d(x, y);$$

(b) *k_v satisfies a Lipschitz continuity condition: for each $m \in \mathbb{Z}$,*

$$|k_v(m, x) - k_v(m, y)| \leq \mathrm{c}_v \delta_0(m) d(x, y);$$

(c) *k_v satisfies a linear decay condition: for each $r \geq 0$,*

$$\sup_{x \in Z} \sum_{m \in \mathbb{Z}} \|k_v(m, -(\cdot) \cdot_\tau x) \min\{|m|, r\}\|_\infty \leq 2;$$

(d) *$\langle(\mathbb{Z}, \ell), \#\rangle$ is strictly-polynomial-growing: for each $r \geq 1$,*

$$r \leq \# B(r) \leq 2r;$$

(e) *for all $X, Y \in \mathcal{S}$,*

$$r \geq 1 \implies \operatorname{dist}_{\mathsf{H}}(\operatorname{spec}(A_X), \operatorname{spec}(A_Y)) \leq \mathrm{c}_v 2^r d_{\mathsf{H}}(X, Y) + \frac{2}{r}(3^{3/2})$$

and for

$$C := 2\Big(\mathrm{c}_v + \frac{2}{1}(3^{3/2})\Big),$$

$$d_{\mathsf{H}}(X, Y) \leq \frac{1}{4}$$
$$\implies \operatorname{dist}_{\mathsf{H}}(\operatorname{spec}(A_X), \operatorname{spec}(A_Y)) \leq C(\log_2(d_{\mathsf{H}}(X, Y)^{-1}))^{-1}.$$

Proof. The claims (a)–(d) are easy to see. Then, (e) follows from a combination of (a)–(d) and Lemma 4.3 (where amenability twinkles) and Lemma 4.5 as described next. Fix $r \geq 1$. Define

$$\chi_r = \chi\colon \mathbb{Z} \to [0, +\infty), \quad n \mapsto \Big(\frac{r - |n|}{r}\Big) \mathbb{1}_{B(r)}(n).$$

Write

$$\|\chi\| := \|\chi\|_2.$$

Let $\rho > 0$. Observe

$$
\begin{aligned}
&\operatorname{dist}_{\mathrm{H}}(\operatorname{spec}(A_X), \operatorname{spec}(A_Y)) \\
&\quad \leq \sup_{\substack{\check{x}\in X, \check{y}\in Y, \check{z}\in Z \\ d(\check{x},\check{y})<d_{\mathrm{H}}(X,Y)+\rho}} \sum_{m\in\mathbb{Z}} \|(k_v(m, -(\cdot)\cdot_\tau \check{y}) - k_v(m, -(\cdot)\cdot_\tau \check{x}))\mathbb{1}_{\operatorname{supp}(\chi)}(\cdot)\|_\infty \\
&\qquad + \|k_v(m, -(\cdot)\cdot_\tau \check{z})\|_\infty \frac{\|\chi^m - \chi\|}{\|\chi\|} && \text{(Lemma 4.3)} \\
&\quad \leq \sup_{\substack{\check{x}\in X, \check{y}\in Y, \check{z}\in Z \\ d(\check{x},\check{y})<d_{\mathrm{H}}(X,Y)+\rho}} \sum_{m\in\mathbb{Z}} \|\mathrm{c}_v \delta_0(m) d(-(\cdot)\cdot_\tau \check{y}, -(\cdot)\cdot_\tau \check{x})\mathbb{1}_{\operatorname{supp}(\chi)}(\cdot)\|_\infty \\
&\qquad + \|k_v(m, -(\cdot)\cdot_\tau \check{z})\|_\infty \frac{\|\chi^m - \chi\|}{\|\chi\|} && \text{(property (b))} \\
&\quad \leq \sup_{\substack{\check{x}\in X, \check{y}\in Y, \check{z}\in Z \\ d(\check{x},\check{y})<d_{\mathrm{H}}(X,Y)+\rho}} \sum_{m\in\mathbb{Z}} \mathrm{c}_v \delta_0(m)(2^r) d(\check{y}, \check{x}) \\
&\qquad + \|k_v(m, -(\cdot)\cdot_\tau \check{z})\|_\infty \frac{\|\chi^m - \chi\|}{\|\chi\|} && \text{(property (a))} \\
&\quad \leq \mathrm{c}_v(2^r)(d_{\mathrm{H}}(X,Y) + \rho) \\
&\qquad + \sup_{\check{z}\in Z} \sum_{m\in\mathbb{Z}} \|k_v(m, -(\cdot)\cdot_\tau \check{z})\|_\infty \min\Big\{\frac{|m|}{r}(3^{3/2}), 2^{1/2}\Big\} && \text{(Lemma 4.5)} \\
&\quad \leq \mathrm{c}_v\Big(\frac{2^{2r}}{r}\Big)(d_{\mathrm{H}}(X,Y) + \rho) + \frac{2}{r}(3^{3/2}). && \text{(property (c))}
\end{aligned}
$$

Here, $\chi^m : \mathbb{Z} \to [0, +\infty), n \mapsto \chi(n + m)$. Also, if

$$d_{\mathrm{H}}(X,Y) \leq 1/4,$$

then $1 \leq \log_2(d_{\mathrm{H}}(X,Y)^{-1})/2$ and

$$
\begin{aligned}
&\operatorname{dist}_{\mathrm{H}}(\operatorname{spec}(A_X), \operatorname{spec}(A_Y)) \\
&\quad \leq \mathrm{c}_v\Big(\frac{2^{2r}}{r}\Big) d_{\mathrm{H}}(X,Y) + \frac{2}{r}(3^{3/2})\Big|_{r=\log_2(d_{\mathrm{H}}(X,Y)^{-1})/2} \\
&\quad \leq C(\log_2(d_{\mathrm{H}}(X,Y)^{-1}))^{-1}.
\end{aligned}
$$

■

3. Preliminaries

3.1. Locally-compact Hausdorff groups

We provide a terse exposition of topological groups. The focus is on the existence of a translation-invariant sigma-finite measure and the existence of a translation-invariant proper metric. The topics on unimodularity and amenability are principal axes.

Say G is a *topological group* if G is a group with a topology $\mathcal{T}$, where the group product and the group inverse are continuous. Let G be a topological group. Assume G is both Hausdorff and locally-compact. Say λ is a *Radon measure* on G if λ is a nonzero Borel measure on G, where λ is outer regular for Borel sets and inner regular for open sets and finite for compact sets. Say λ is a *left Haar measure* on G if λ is a left-invariant Radon measure on G.

Theorem 3.1. *Let G be a locally-compact Hausdorff group.*

(a) *There exists a left Haar measure λ on G. Also, λ is unique modulo scaling.*

(b) *If G is sigma-compact, then λ is sigma-finite and λ is inner regular for Borel sets.*

(c) *If G is first-countable, then there exists a left-invariant $\mathcal{T}$-generating metric on G.*

(d) *If G is second-countable, then there exists a left-invariant proper $\mathcal{T}$-generating metric on G.*

(e) *G is sigma-compact and G is first-countable if and only if G is second-countable.*

The proof of Theorem 3.1 (a)–(d) can be found within the works [30, 52, 60, 67]. Theorem 3.1 (e) follows from the observation that every sigma-compact space is a Lindelöf space and the observation that, for metric spaces, Lindelöfness and separability and second-countability are equivalent; see [66].

3.1.1. Unimodularity. Let $\langle G, \lambda \rangle$ be a second-countable locally-compact Hausdorff group endowed with a left Haar measure. Let $\mathcal{B}$ be the collection of all Borel subsets of G. Define, for $g \in G$ and $B \in \mathcal{B}$, $\lambda^g(B) := \lambda(Bg)$ and $\lambda^*(B) := \lambda(B^{-1})$. Observe λ^g is a left Haar measure on G and there exists $c(g) > 0$ such that $\lambda^g = c(g)\lambda$. It can be shown that $c\colon G \to (0, +\infty)$, called the *modular function,* is a continuous homomorphism. Say $\langle G, \lambda \rangle$ is *unimodular* if c is identically 1 or, equivalently, $\lambda^g = \lambda$ for every $g \in G$. It can be shown that if $\langle G, \lambda \rangle$ is unimodular, then $\lambda^* = \lambda$.

Observe if G is discrete or G is abelian or G is compact, then $\langle G, \lambda \rangle$ is unimodular. To see how unimodularity follows from $G/[G, G]$ being compact, the image of $[G, G] := \overline{\{[g, h] : g, h \in G\}}$ under c is trivial since c is a homomorphism and $c(G) = c(G/[G, G])$ is compact since c is continuous and the only compact subgroup of $(0, +\infty)$ is $\{1\}$. The details and more examples can be found within [30] such as connected Lie groups that are nilpotent or semisimple.

3.1.2. Amenability. Let $\langle G, \lambda \rangle$ be a second-countable locally-compact Hausdorff group endowed with a left Haar measure. Say $\mathfrak{m}$ is a *mean* on G if $\mathfrak{m}$ is a nonzero bounded linear functional on $L^\infty(G)$, where $\mathfrak{m}(\varphi) \geq 0$ for every $\varphi \in L^\infty_+(G)$ and

$\mathfrak{m}(\mathbb{1}_G) = 1$. Say $\langle G, \lambda \rangle$ is *amenable* if a left-invariant mean on G exists or, equivalently, for each $\varepsilon > 0$ and for each nonempty compact subset F of G, there exists a compactly-supported continuous nonzero $\chi \in L^2_+(G)$ such that $\sup_{h \in F} \|\chi^h - \chi\| < \varepsilon \|\chi\|$. Here $\chi^h : G \to [0, +\infty), g \mapsto \chi(gh)$.

It can be shown that if G is abelian or G is compact, then $\langle G, \lambda \rangle$ is amenable. Also, if G is solvable, then $\langle G, \lambda \rangle$ is amenable since G is an extension of an amenable group by an amenable closed normal subgroup. We remark that if G has a closed subgroup isomorphic to the free group with two generators F_2, then $\langle G, \lambda \rangle$ is not amenable. There is an extent to which nonamenability is related to the presence of F_2; see [54, Chapter 3]: if G is almost-connected, then $\langle G, \lambda \rangle$ is amenable if and only if G does not have a closed subgroup isomorphic to F_2; if G is a connected Lie group, then G is solvable (thus amenable) if and only if G does not have a closed subgroup isomorphic to F_2. The details and more examples can be found within [54, 56] such as groups that are uniformly-distributed or polynomial-growing.

3.2. Dynamically-defined operator families

We provide a terse exposition of operator families induced by elements of a C^*-algebra itself induced by a topological dynamical system. The focus is on their covariant representation and properties, e.g., uniform boundedness and strong continuity.

Let $\langle G, \lambda \rangle$ be a second-countable locally-compact Hausdorff group endowed with a left Haar measure. Define, for $\varphi \in L^1(G)$, $\int \varphi(h)\, dh := \int \varphi\, d\lambda$. Let $\langle Z, (G, \tau) \rangle$ be a compact Hausdorff space endowed with a left continuous action. Specifically, τ is a left continuous action if and only if $\tau : G \times Z \to Z$ is continuous and $e \cdot_\tau x = x$ for every $x \in Z$ and $g \cdot_\tau (h \cdot_\tau x) = (gh) \cdot_\tau x$ for every $g, h \in G, x \in Z$. Define, for $g \in G$ and $x \in Z$, $gx := g \cdot_\tau x$. Define, for $g \in G$ and $f \in C(Z)$ and $x \in Z$, $(gf)(x) := f(g^{-1}x)$. Here, $C(Z)$ (or $C_b(Z)$) is the unital C^*-algebra of all (bounded) continuous complex-valued functions on Z. Define, for $x \in Z$, $[\,]_x : C_c(G, C(Z)) \to \mathcal{L}(L^2(G))$ by

$$[f_{(\cdot)}]_x = \int dh [f_h((\cdot)^{-1}x)] R_h = (f_{g^{-1}h}(g^{-1}x))_{g,h \in G}$$

for every $f_{(\cdot)} \in C_c(G, C(Z))$. Here, $C_c(G, C(Z))$ is the linear space of all compactly-supported continuous $C(Z)$-valued functions on G. Also, $[f_h((\cdot)^{-1}x)] : L^2(G) \to L^2(G)$, a multiplication operator, is defined by $([f_h((\cdot)^{-1}x)]\psi)(g) = f_h(g^{-1}x)\psi(g)$ for every $\psi \in L^2(G), g \in G$. Also, $R_h : L^2(G) \to L^2(G)$, is defined by $(R_h\psi)(g) = \psi(gh)$ for every $\psi \in L^2(G), g \in G$. Also, $(f_{g^{-1}h}(g^{-1}x))$ is a matrix whose (g, h)-th component is $f_{g^{-1}h}(g^{-1}x)$. More conventionally,

$$([f_{(\cdot)}]_x\psi)(g) = \int f_h(g^{-1}x)\psi(gh)\,dh = \int f_{g^{-1}h}(g^{-1}x)\psi(h)\,dh$$

for every $x \in Z$, $f_{(\cdot)} \in C_c(G, C(Z))$, $\psi \in L^2(G)$, $g \in G$. Observe $[\,]_x$ is linear and $[f_{(\cdot)}]_x$ is linear. Also, $U_{j^{-1}}[f_h((\cdot)^{-1}x)]U_j = [f_h((\cdot)^{-1}j^{-1}x)]$ and $[U_j, R_h] = 0$. Here, $U_j\colon L^2(G) \to L^2(G)$, called the *left-regular representation,* is defined by $(U_j\psi)(g) = \psi(j^{-1}g)$ for every $\psi \in L^2(G)$, $g \in G$.

We remark that $C_c(G, C(Z))$ is a *-algebra and $\|f_{(\cdot)} * \tilde{f}_{(\cdot)}\| \le \|f_{(\cdot)}\|\|\tilde{f}_{(\cdot)}\|$ and $\|f_{(\cdot)}{}^* * f_{(\cdot)}\| = \|f_{(\cdot)}\|^2$, where

$$\begin{aligned}
(f_{(\cdot)} * \tilde{f}_{(\cdot)})_h(x) &:= \int f_j(x)\tilde{f}_{j^{-1}h}(j^{-1}x)\,dj,\\
(f_{(\cdot)}{}^*)_h(x) &:= \overline{f_{h^{-1}}(h^{-1}x)},\\
\|f_{(\cdot)}\| &:= \sup_{x\in Z}\|[f_{(\cdot)}]_x\|.
\end{aligned}$$

Also, $C_c(G, C(Z))$ embeds into a C^*-algebra, denoted $C(Z) \rtimes_\tau G$, where

$$\operatorname{spec}(f_{(\cdot)}) = \overline{\bigcup_{x\in Z}\operatorname{spec}([f_{(\cdot)}]_x)}$$

for normal elements; see [13,16,20,55]. The C^*-algebra $C(Z) \rtimes_\tau G$, called a *reduced crossed product,* and its K-theory were utilized to prove the gap labeling theorems; see [15, 16, 18, 20, 51]: if $\langle G, \lambda\rangle$ is both unimodular and amenable and Z is second-countable, then for each ergodic left-invariant Borel probability measure μ on Z, there exists a countable subgroup of $\mathbb{R}$ containing all gap labels. The 1985 result by J. Bellissard, R. Lima, and D. Testard [20] and the 1986 result by J. Bellissard [15] considered the case where $G = \mathbb{Z}^d$ or $G = \mathbb{R}^d$ or G is both discrete and amenable and utilized that G sometimes has a regular Følner sequence so that the pointwise ergodic theorem holds for the "time-average" approximations of the "space-average" integrated density of states – whose image over the gaps in the spectra of self-adjoint elements of $C(Z) \rtimes_\tau G$ are precisely the gap labels. The existence of regular Følner sequences is not known for amenable groups with exponential growth [53], but the 2001 result by E. Lindenstrauss [51] establishes that amenable groups always have tempered Følner sequences. We remark that tempered Følner sequences are suitable.

This subsection is the only place where C^*-algebras are mentioned, and it is done so to expose the spectrum map

$$X \mapsto \overline{\bigcup_{x\in X}\operatorname{spec}([f_{(\cdot)}]_x)},$$

where X is a subsystem of $\langle Z, (G, \tau)\rangle$. Specifically, X is a subsystem of $\langle Z, (G, \tau)\rangle$ if and only if X is a left-invariant nonempty closed subset of Z. Here, X is left invariant if $gX := \{gx : x \in X\} \subseteq X$ for all $g \in G$.

We remark that in the below expression we suppress the notations involving λ and τ. For example, $\int \varphi(h)\, dh := \int \varphi\, d\lambda$ and $gx := g \cdot_\tau x$. Define $k(h, x) := f_h(x)$. Let $\mathcal{S}$ be the collection of all subsystems of $\langle Z, (G, \tau)\rangle$. Define, for $X \in \mathcal{S}$, $\operatorname{spec}(A_X) := \overline{\bigcup_{x \in X} \operatorname{spec}(A_x)}$.

Lemma 3.2. *Let $\langle G, \lambda\rangle$ be a second-countable locally-compact Hausdorff group endowed with a left Haar measure. Let $\langle Z, (G, \tau)\rangle$ be a topological space endowed with a left continuous action. Let $f_{(\cdot)} \in C_c(G, C_b(Z))$. Define, for $x \in Z$,*

$$A_x \colon L^2(G) \to L^2(G)$$

by

$$(A_x\psi)(g) = \int k(h, g^{-1}x)\psi(gh)\, dh = \int k(g^{-1}h, g^{-1}x)\psi(h)\, dh$$

for every $\psi \in L^2(G)$, $g \in G$ where $f_h(x) = k(h, x)$. Then,

(a) *$k \colon G \times Z \to \mathbb{C}$ is measurable and $\sup_{x \in Z} \int \|k(h, (\cdot)^{-1}x)\|_\infty\, dh < +\infty$;*

(b) *if $\langle G, \lambda\rangle$ is unimodular, then for all $x, y \in Z$ and for each $\psi \in L^2(G)$ and for each $j \in G$,*

$$\|A_x\psi\| \le \left(\int \|k(h, (\cdot)^{-1}x)\mathbb{1}_{\operatorname{supp}(\psi)h^{-1}}(\cdot)\|_\infty\, dh\right)\|\psi\|,$$

$$\|A_x^*\psi\| \le \left(\int \|k(h, (\cdot)^{-1}x)\mathbb{1}_{\operatorname{supp}(\psi)}(\cdot)\|_\infty\, dh\right)\|\psi\|,$$

$$\|(A_y - A_x)\psi\| \le \left(\int \|(k(h, (\cdot)^{-1}y) - k(h, (\cdot)^{-1}x))\mathbb{1}_{\operatorname{supp}(\psi)h^{-1}}(\cdot)\|_\infty\, dh\right)\|\psi\|,$$

$$\|(A_y - A_x)^*\psi\| \le \left(\int \|(k(h, (\cdot)^{-1}y) - k(h, (\cdot)^{-1}x))\mathbb{1}_{\operatorname{supp}(\psi)}(\cdot)\|_\infty\, dh\right)\|\psi\|,$$

$$U_{j^{-1}}A_xU_j = A_{j^{-1}x}.$$

(c) *If $\langle G, \lambda\rangle$ is unimodular and A_x is normal for every $x \in Z$, then for each $x \in Z$,*

$$\lim_{y \xrightarrow[Z]{} x} \sup_{E \in \operatorname{spec}(A_x)} \operatorname{dist}(E, \operatorname{spec}(A_y)) = 0$$

and

$$\operatorname{spec}(A_x) = \operatorname{spec}(A_{X(x)}),$$

where $X(x) := \overline{\{gx : g \in G\}}$, the "orbit-closure."

The proof of – a suitable generalization of – Lemma 3.2 (b)–(c) can be found within Section 5.

3.3. Hausdorff distance

Let Z be a metrizable space. Let d be a $\mathcal{T}$-generating metric on Z. Specifically, for each metric d on Z, d is $\mathcal{T}$-generating if and only if the balls $\{y \in Z : d(y, x) < r\}$ are open and generate the topology on Z. Also, the balls are precompact if and only if d is proper. Let S_1 and S_2 be subsets of Z. The *Hausdorff distance* between S_1 and S_2 with respect to d is

$$\max\{\sup_{s \in S_1} d(s, S_2), \sup_{s \in S_2} d(s, S_1)\} =: d_{\mathsf{H}}(S_1, S_2).$$

Proposition 3.3. *Let G be a second-countable locally-compact Hausdorff group, and let $\langle Z, (G, \tau)\rangle$ be a compact metrizable (or second-countable compact Hausdorff) space endowed with a left continuous action, and let $\mathcal{S}$ be the collection of all subsystems of $\langle Z, (G, \tau)\rangle$. Let d be a $\mathcal{T}$-generating metric on Z. Then $(\mathcal{S}, d_{\mathsf{H}})$ is a compact metric space.*

The proof of Proposition 3.3 can be found within [13, Proposition 4].

3.4. Strict polynomial growth

Let $\langle G, \lambda\rangle$ be a second-countable locally-compact Hausdorff group endowed with a left Haar measure. Let ℓ be a left-invariant proper $\mathcal{T}$-generating metric on G. Define, for $g \in G$, $|g| := \ell(g, e)$. Define, for $r \geq 0$, $B(r) := \{g \in G : |g| < r\}$. Say $\langle (G, \ell), \lambda\rangle$ is *strictly-polynomial-growing* if there exist $b \geq 1$ and $c_1 \geq c_0 > 0$ such that for each $r \geq 1$,

$$c_0 r^b \leq \lambda(B(r)) \leq c_1 r^b.$$

Proposition 3.4. *Let $\langle G, \lambda\rangle$ be a second-countable locally-compact Hausdorff group endowed with a left Haar measure. Let ℓ be a left-invariant proper $\mathcal{T}$-generating metric on G. Assume $\langle (G, \ell), \lambda\rangle$ is strictly-polynomial-growing. Then $\langle G, \lambda\rangle$ is both unimodular and amenable.*

The proof of Proposition 3.4 can be found within [53], see also Lemma 4.5.

4. Proof of main results

Let $\langle G, \lambda\rangle$ be a second-countable locally-compact Hausdorff group endowed with a left Haar measure. Let $\langle Z, (G, \tau)\rangle$ be a metrizable space endowed with a left continuous action. Let $k\colon G \times Z \to \mathbb{C}$, where k is measurable and

$$\sup_{x\in Z} \int \|k(h, (\cdot)^{-1}x)\|_\infty \, dh < +\infty.$$

Define, for $x \in Z$, $A_x\colon L^2(G) \to L^2(G)$ by

$$(A_x\psi)(g) = \int k(h, g^{-1}x)\psi(gh)\, dh = \int k(g^{-1}h, g^{-1}x)\psi(h)\, dh$$

for every $\psi \in L^2(G), g \in G$. We remark that in the above expression we suppress the notations involving λ and τ. For example, $\int \varphi(h)\, dh := \int \varphi \, d\lambda$ and $gx := g \cdot_\tau x$. Let $\mathcal{S}$ be the collection of all subsystems of $\langle Z, (G, \tau)\rangle$. Define, for $X \in \mathcal{S}$, $\operatorname{spec}(A_X) := \overline{\bigcup_{x\in X} \operatorname{spec}(A_x)}$. Let ℓ be a left-invariant proper $\mathcal{T}$-generating metric on G. Let d be a $\mathcal{T}$-generating metric on Z.

Proposition 4.1. *Assume $\langle G, \lambda\rangle$ is unimodular, A_x is normal for every $x \in Z$, and the following.*

(i) *τ satisfies a uniform continuity condition: for each $\varepsilon > 0$ and for each nonempty compact subset K of G, there exists $\delta > 0$ such that*

$$d(y, x) < \delta \implies \sup_{g\in K} d(g^{-1}y, g^{-1}x) < \varepsilon.$$

(ii) *k satisfies a uniform continuity condition: for each $\varepsilon > 0$ and for each nonempty compact subset F of G, there exists $\delta > 0$ such that*

$$d(y, x) < \delta \implies \int |k(h, y) - k(h, x)| \mathbb{1}_F(h)\, dh < \varepsilon.$$

(iii) *k satisfies a uniform decay condition: for each $\varepsilon > 0$, there exists a nonempty compact subset F of G such that*

$$\sup_{x\in Z} \int \|k(h, (\cdot)^{-1}x)\mathbb{1}_{G\setminus F}(h)\|_\infty \, dh < \varepsilon.$$

Fix $X \in \mathcal{S}$. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that for each $Y \in \mathcal{S}$, if $d_{\mathsf{H}}(X, Y) < \delta$, then $\sup_{E\in\operatorname{spec}(A_X)} \operatorname{dist}(E, \operatorname{spec}(A_Y)) < \varepsilon$, i.e.,

$$\lim_{Y \xrightarrow[\mathcal{S}]{} X} \sup_{E\in\operatorname{spec}(A_X)} \operatorname{dist}(E, \operatorname{spec}(A_Y)) = 0.$$

Proof. It suffices to show that for each $E \in \operatorname{spec}(A_X)$ and for each $\varepsilon > 0$, there exists $\delta > 0$ such that for each $Y \in \mathcal{S}$, if $d_{\mathsf{H}}(X, Y) < \delta$, then $\operatorname{dist}(E, \operatorname{spec}(A_Y)) < \varepsilon$ since spec(A_X) is compact. Fix $E \in \operatorname{spec}(A_X)$ and $\varepsilon > 0$. Observe there exists a nonempty compact subset F of G such that

$$\sup_{x \in Z} \int \|k(h, (\cdot)^{-1}x)\mathbb{1}_{G \setminus F}(h)\|_\infty \, dh < \frac{1}{5}\varepsilon$$

since k satisfies a uniform decay condition. Observe there exist $x \in X$ and $E' \in \operatorname{spec}(A_x)$ such that $|E - E'| < \varepsilon/5$. Without loss of generality, assume $E' \notin \operatorname{spec}(A_Y)$. Observe A_x is normal and, by Weyl's criterion, there exists a nonzero vector φ such that $\|(A_x - E')\varphi\| < \varepsilon\|\varphi\|/5$. Without loss of generality, assume φ is both continuous and compactly-supported. Define $K := \operatorname{supp}(\varphi)$. Observe

$$d(y, x) < 2\delta \implies \int \|(k(h, (\cdot)^{-1}y) - k(h, (\cdot)^{-1}x))\mathbb{1}_K(\cdot)\|_\infty \mathbb{1}_F(h) \, dh < \frac{1}{5}\varepsilon$$

for some $\delta > 0$ since the group action and k satisfy a uniform continuity condition. Fix $Y \in \mathcal{S}$. Assume $d_{\mathrm{H}}(X, Y) < \delta$. Observe there exists $y \in Y$ such that $d(x, y) < d(x, Y) + \delta < 2\delta$. Also, A_y is normal and

$$\begin{aligned}
\|(A_y - E')^{-1}\|^{-1}\|\varphi\| &\le \|(A_y - E')\varphi\| = \|(A_y - E')^*\varphi\| \\
&< \left(\left\|(A_y - A_x)^* \frac{\varphi}{\|\varphi\|}\right\| + \frac{1}{5}\varepsilon\right)\|\varphi\|
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{dist}(E, \operatorname{spec}(A_Y)) \\
&\quad < \operatorname{dist}(E', \operatorname{spec}(A_y)) + \frac{1}{5}\varepsilon = \|(A_y - E')^{-1}\|^{-1} + \frac{1}{5}\varepsilon \\
&\quad < \left\|(A_y - A_x)^* \frac{\varphi}{\|\varphi\|}\right\| + \frac{2}{5}\varepsilon \\
&\quad \le \int \|(k(h, (\cdot)^{-1}y) - k(h, (\cdot)^{-1}x))\mathbb{1}_K(\cdot)\|_\infty \, dh + \frac{2}{5}\varepsilon \qquad \text{(Proposition 5.1)} \\
&\quad < \int \|(k(h, (\cdot)^{-1}y) - k(h, (\cdot)^{-1}x))\mathbb{1}_K(\cdot)\|_\infty \mathbb{1}_F(h) \, dh + \frac{4}{5}\varepsilon < \varepsilon. \qquad \blacksquare
\end{aligned}$$

Remark 4.1.1. Proposition 4.1 will be improved provided the group is amenable.

Let χ be a compactly-supported continuous nonzero vector in $L^2(G)$. The *cutoff operator* with respect to χ is

$$[\chi] \colon L^2(G) \to L^2(G)$$

defined by

$$([\chi]\psi)(g) = \chi(g)\psi(g)$$

for every $\psi \in L^2(G)$, $g \in G$. Let A and B be operators on $L^2(G)$. The *commutator operator* with respect to A and B is

$$[A, B]\colon L^2(G) \to L^2(G)$$

defined by

$$([A, B]\psi)(g) = ((AB - BA)\psi)(g)$$

for every $\psi \in L^2(G)$, $g \in G$.

Lemma 4.2. *Assume $\langle G, \lambda\rangle$ is unimodular. Let χ be a compactly-supported continuous nonzero vector. Define, for $j, h \in G$,*

$$\chi_j\colon G \to \mathbb{C}, \quad g \mapsto \chi(j^{-1}g), \qquad \chi^h\colon G \to \mathbb{C}, \quad g \mapsto \chi(gh).$$

Fix $x \in Z$ and $\psi \in L^2(G)$. Then

$$\int \|[A_x, [\chi_j]]\psi\|^2\, dj \le \Big(\int \|k(h, (\cdot)^{-1}x)\mathbb{1}_{\mathrm{supp}(\psi)h^{-1}}(\cdot)\|_\infty \|\chi^h - \chi\|\, dh\Big)^2 \|\psi\|^2,$$

$$\int \|[A_x{}^*, [\chi_j]]\psi\|^2\, dj \le \Big(\int \|k(h^{-1}, (\cdot)^{-1}x)\mathbb{1}_{\mathrm{supp}(\psi)}(\cdot)\|_\infty \|\chi^h - \chi\|\, dh\Big)^2 \|\psi\|^2,$$

$$\int \|[\chi_j]\psi\|^2\, dj = \|\chi\|^2\|\psi\|^2.$$

Proof. Observe

$$\begin{aligned}
&\int \|[A_x, [\chi_j]]\psi\|^2\, dj \\
&\quad= \iint \Big|\int k(h, g^{-1}x)(\chi_j(gh) - \chi_j(g))\psi(gh)\, dh\Big|^2 dg\, dj \\
&\quad\le \iint \Big(\int |k(h, g^{-1}x)||\chi_j(gh) - \chi_j(g)||\psi(gh)|\, dh\Big)^2 dg\, dj \\
&\quad\le \iint \Big(\int |k(h, g^{-1}x)|\mathbb{1}_{\mathrm{supp}(\psi)}(gh)\|\chi^h - \chi\|\, dh\Big) \\
&\qquad\quad \times \Big(\int |k(h, g^{-1}x)||\chi_j(gh) - \chi_j(g)|^2|\psi(gh)|^2\|\chi^h - \chi\|^{-1}\, dh\Big)\, dg\, dj \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(Hölder's inequality)} \\
&\quad\le \iint \Big(\int \|k(h, (\cdot)^{-1}x)\mathbb{1}_{\mathrm{supp}(\psi)h^{-1}}(\cdot)\|_\infty \|\chi^h - \chi\|\, dh\Big) \\
&\qquad\quad \times \Big(\int \|k(h, (\cdot)^{-1}x)\mathbb{1}_{\mathrm{supp}(\psi)h^{-1}}(\cdot)\|_\infty \\
&\qquad\qquad \times |\chi_j(gh) - \chi_j(g)|^2|\psi(gh)|^2\|\chi^h - \chi\|^{-1}\, dh\Big)\, dj\, dg \\
&\quad= \Big(\int \|k(h, (\cdot)^{-1}x)\mathbb{1}_{\mathrm{supp}(\psi)h^{-1}}(\cdot)\|_\infty \|\chi^h - \chi\|\, dh\Big)^2 \|\psi\|^2
\end{aligned}$$

and

$$
\begin{aligned}
&\int \|[A_x{}^*, [\chi_j]]\psi\|^2 dj \\
&\quad = \iint \left| \int \overline{k(h^{-1}, h^{-1}g^{-1}x)}(\chi_j(gh) - \chi_j(g))\psi(gh)\, dh \right|^2 dg\, dj \\
&\quad \le \iint \left(\int |\overline{k(h^{-1}, h^{-1}g^{-1}x)}||\chi_j(gh) - \chi_j(g)||\psi(gh)|\, dh \right)^2 dg\, dj \\
&\quad \le \iint \left(\int |k(h^{-1}, (gh)^{-1}x)| \mathbb{1}_{\mathrm{supp}(\psi)}(gh) \|\chi^h - \chi\|\, dh \right) \\
&\qquad \times \left(\int |k(h^{-1}, (gh)^{-1}x)||\chi_j(gh) - \chi_j(g)|^2 |\psi(gh)|^2 \|\chi^h - \chi\|^{-1}\, dh \right) dg\, dj \\
&\quad \le \iint \left(\int \|k(h^{-1}, (\cdot)^{-1}x) \mathbb{1}_{\mathrm{supp}(\psi)}(\cdot)\|_\infty \|\chi^h - \chi\|\, dh \right) \\
&\qquad \times \left(\int \|k(h^{-1}, (\cdot)^{-1}x) \mathbb{1}_{\mathrm{supp}(\psi)}(\cdot)\|_\infty \right. \\
&\qquad\qquad \left. \times |\chi_j(gh) - \chi_j(g)|^2 |\psi(gh)|^2 \|\chi^h - \chi\|^{-1}\, dh \right) dj\, dg \\
&\quad = \left(\int \|k(h^{-1}, (\cdot)^{-1}x) \mathbb{1}_{\mathrm{supp}(\psi)}(\cdot)\|_\infty \|\chi^h - \chi\|\, dh \right)^2 \|\psi\|^2.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\int \|[\chi_j]\psi\|^2\, dj &= \iint |\chi_j(g)|^2 |\psi(g)|^2\, dg\, dj \\
&= \iint |\chi_j(g)|^2 |\psi(g)|^2\, dj\, dg = \|\chi\|^2 \|\psi\|^2. \qquad \blacksquare
\end{aligned}
$$

Lemma 4.3. *Assume $\langle G, \lambda \rangle$ is unimodular and A_x is normal for every $x \in Z$. Let χ be a compactly-supported continuous nonzero vector. Fix $X, Y \in \mathcal{S}$. Then, for each $\rho > 0$,*

$$
\begin{aligned}
\Delta_{\mathsf{H}} \le \sup_{\substack{\check{x}\in X, \check{y}\in Y, \check{z}\in Z \\ d(\check{x},\check{y}) < \delta_{\mathsf{H}} + \rho}} \int &\|(k(h, (\cdot)^{-1}\check{y}) - k(h, (\cdot)^{-1}\check{x})) \mathbb{1}_{\mathrm{supp}(\chi)}(\cdot)\|_\infty \\
&+ \|k(h, (\cdot)^{-1}\check{z})\|_\infty \frac{\|\chi^h - \chi\|}{\|\chi\|}\, dh,
\end{aligned}
$$

where

$$\Delta_{\mathsf{H}} := \mathrm{dist}_{\mathsf{H}}(\mathrm{spec}(A_X), \mathrm{spec}(A_Y)) \quad \textit{and} \quad \delta_{\mathsf{H}} := d_{\mathsf{H}}(X, Y)$$

and

$$\chi^h : G \to \mathbb{C}, \quad g \mapsto \chi(gh).$$

Proof. Fix $\rho > 0$. Let $E \in \operatorname{spec}(A_X)$. Let $\varepsilon > 0$. Observe there exist $x \in X$ and $E' \in \operatorname{spec}(A_x)$ such that $|E - E'| < \varepsilon$. Without loss of generality, assume $E' \notin \operatorname{spec}(A_Y)$. Observe A_x is normal and, by Weyl's criterion, there exists a nonzero vector φ such that $\|(A_x - E')\varphi\| < \varepsilon\|\varphi\|$. Without loss of generality, assume φ is both continuous and compactly-supported. Define, for $j \in G$,

$$\chi_j : G \to \mathbb{C}, \quad g \mapsto \chi(j^{-1}g).$$

Define

$$r := \frac{1}{\varepsilon} \int \|k(h, (\cdot)^{-1}x)\|_\infty \frac{\|\chi^h - \chi\|}{\|\chi\|}\, dh.$$

Observe

$$\begin{aligned}
&\int \|(A_x - E')[\chi_j]\varphi\|^2\, dj \\
&\quad = \int \|[(A_x - E'), [\chi_j]]\varphi + [\chi_j](A_x - E')\varphi\|^2\, dj \\
&\quad \le \int \Big(1 + \frac{1}{r}\Big)\|[(A_x - E'), [\chi_j]]\varphi\|^2 + (1 + r)\|[\chi_j](A_x - E')\varphi\|^2\, dj \\
&\qquad\qquad \text{(Peter–Paul's inequality)} \\
&\quad \le \Big(1 + \frac{1}{r}\Big)\Big(\int \|k(h, (\cdot)^{-1}x)\|_\infty \|\chi^h - \chi\|\, dh\Big)^2 \|\varphi\|^2 \\
&\qquad + (1 + r)\|\chi\|^2\|(A_x - E')\varphi\|^2 \qquad \text{(Lemma 4.2)} \\
&\quad < \Big(1 + \frac{1}{r}\Big)\Big(\int \|k(h, (\cdot)^{-1}x)\|_\infty \|\chi^h - \chi\|\, dh\Big)^2 \|\varphi\|^2 + (1 + r)\varepsilon^2\|\chi\|^2\|\varphi\|^2 \\
&\quad = \int \Big(1 + \frac{1}{r}\Big)\Big(\int \|k(h, (\cdot)^{-1}x)\|_\infty \|\chi^h - \chi\|\, dh\Big)^2 \|\chi\|^{-2}\|[\chi_j]\varphi\|^2 \\
&\qquad + (1 + r)\varepsilon^2\|[\chi_j]\varphi\|^2\, dj \\
&\quad = \int \bigg(\Big(1 + \frac{1}{r}\Big)\Big(\int \|k(h, (\cdot)^{-1}x)\|_\infty \frac{\|\chi^h - \chi\|}{\|\chi\|}\, dh\Big)^2 + (1 + r)\varepsilon^2\bigg)\|[\chi_j]\varphi\|^2\, dj \\
&\quad = \int \Big(\int \|k(h, (\cdot)^{-1}x)\|_\infty \frac{\|\chi^h - \chi\|}{\|\chi\|}\, dh + \varepsilon\Big)^2 \|[\chi_j]\varphi\|^2\, dj.
\end{aligned}$$

Observe there exists $j \in G$ such that

$$\|(A_x - E')[\chi_j]\varphi\| < \Big(\int \|k(h, (\cdot)^{-1}x)\|_\infty \frac{\|\chi^h - \chi\|}{\|\chi\|}\, dh + \varepsilon\Big)\|[\chi_j]\varphi\|.$$

Define $\varphi_j := [\chi_j]\varphi$. Observe

$$\operatorname{supp}(\varphi_j) \subseteq \operatorname{supp}(\chi_j).$$

Observe that there exists $y \in Y$ such that

$$d(j^{-1}x, j^{-1}y) < d(j^{-1}x, Y) + \rho \leq \delta_{\mathsf{H}} + \rho,$$

since Y is left invariant. Also, A_y is normal and

$$\begin{aligned}
&\|(A_y - E')\varphi_j\| \\
&\quad = \|(A_y - E')^*\varphi_j\| \\
&\quad < \left(\left\|(A_y - A_x)^*\frac{\varphi_j}{\|\varphi_j\|}\right\| + \int \|k(h, (\cdot)^{-1}x)\|_\infty \frac{\|\chi^h - \chi\|}{\|\chi\|}\,dh + \varepsilon\right)\|\varphi_j\|
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{dist}(E, \operatorname{spec}(A_Y)) \\
&\quad < \operatorname{dist}(E', \operatorname{spec}(A_y)) + \varepsilon = \|(A_y - E')^{-1}\|^{-1} + \varepsilon \leq \left\|(A_y - E')\frac{\varphi_j}{\|\varphi_j\|}\right\| + \varepsilon \\
&\quad < \left\|(A_y - A_x)^*\frac{\varphi_j}{\|\varphi_j\|}\right\| + \int \|k(h, (\cdot)^{-1}x)\|_\infty \frac{\|\chi^h - \chi\|}{\|\chi\|}\,dh + 2\varepsilon \\
&\quad \leq \int \|(k(h, (\cdot)^{-1}y) - k(h, (\cdot)^{-1}x))\mathbb{1}_{\operatorname{supp}(\varphi_j)}(\cdot)\|_\infty \\
&\qquad\qquad\qquad + \|k(h, (\cdot)^{-1}x)\|_\infty \frac{\|\chi^h - \chi\|}{\|\chi\|}\,dh + 2\varepsilon \\
&\quad \leq \int \|(k(h, (\cdot)^{-1}y) - k(h, (\cdot)^{-1}x))\mathbb{1}_{\operatorname{supp}(\chi_j)}(\cdot)\|_\infty \\
&\qquad\qquad\qquad + \|k(h, (\cdot)^{-1}x)\|_\infty \frac{\|\chi^h - \chi\|}{\|\chi\|}\,dh + 2\varepsilon \\
&\quad = \int \|(k(h, (\cdot)^{-1}j^{-1}y) - k(h, (\cdot)^{-1}j^{-1}x))\mathbb{1}_{\operatorname{supp}(\chi)}(\cdot)\|_\infty \\
&\qquad\qquad\qquad + \|k(h, (\cdot)^{-1}x)\|_\infty \frac{\|\chi^h - \chi\|}{\|\chi\|}\,dh + 2\varepsilon \\
&\quad \leq \sup_{\substack{\check{x}\in X, \check{y}\in Y, \check{z}\in Z \\ d(\check{x},\check{y})<\delta_{\mathsf{H}}+\rho}} \int \|(k(h, (\cdot)^{-1}\check{y}) - k(h, (\cdot)^{-1}\check{x}))\mathbb{1}_{\operatorname{supp}(\chi)}(\cdot)\|_\infty \\
&\qquad\qquad\qquad + \|k(h, (\cdot)^{-1}\check{z})\|_\infty \frac{\|\chi^h - \chi\|}{\|\chi\|}\,dh + 2\varepsilon.
\end{aligned}$$

As a result, $\sup_{E\in\operatorname{spec}(A_X)} \operatorname{dist}(E, \operatorname{spec}(A_Y))$ and Δ_{H} are bounded from above by

$$\begin{aligned}
\sup_{\substack{\check{x}\in X, \check{y}\in Y, \check{z}\in Z \\ d(\check{x},\check{y})<\delta_{\mathsf{H}}+\rho}} \int &\|(k(h, (\cdot)^{-1}\check{y}) - k(h, (\cdot)^{-1}\check{x}))\mathbb{1}_{\operatorname{supp}(\chi)}(\cdot)\|_\infty \\
&\qquad + \|k(h, (\cdot)^{-1}\check{z})\|_\infty \frac{\|\chi^h - \chi\|}{\|\chi\|}\,dh
\end{aligned}$$

as desired. ∎

Theorem 4.4. *Assume $\langle G, \lambda\rangle$ is unimodular, A_x is normal for every $x \in Z$, and the following.*

(i) *τ satisfies a uniform continuity condition: for each $\varepsilon > 0$ and for each nonempty compact subset K of G, there exists $\delta > 0$ such that*

$$d(y,x) < \delta \implies \sup_{g \in K} d(g^{-1}y, g^{-1}x) < \varepsilon.$$

(ii) *k satisfies a uniform continuity condition: for each $\varepsilon > 0$ and for each nonempty compact subset F of G, there exists $\delta > 0$ such that*

$$d(y,x) < \delta \implies \int |k(h,y) - k(h,x)| \mathbb{1}_F(h)\, dh < \varepsilon.$$

(iii) *k satisfies a uniform decay condition: for each $\varepsilon > 0$, there exists a nonempty compact subset F of G such that*

$$\sup_{x \in Z} \int \|k(h,(\cdot)^{-1}x)\mathbb{1}_{G\setminus F}(h)\|_\infty\, dh < \varepsilon.$$

(iv) *$\langle G, \lambda\rangle$ is amenable.*

Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that for all $X, Y \in \mathcal{S}$, if $d_{\mathsf{H}}(X,Y) < \delta$, then $\mathrm{dist}_{\mathsf{H}}(\mathrm{spec}(A_X), \mathrm{spec}(A_Y)) < \varepsilon$, *i.e.,*

$$\lim_{\delta \to 0^+} \sup_{X,Y \in \mathcal{S}, d_{\mathsf{H}}(X,Y) < \delta} \mathrm{dist}_{\mathsf{H}}(\mathrm{spec}(A_X), \mathrm{spec}(A_Y)) = 0.$$

Proof. Fix $\varepsilon > 0$. Observe there exists a nonempty compact subset F of G such that

$$\sup_{x \in Z} \int \|k(h,(\cdot)^{-1}x)\mathbb{1}_{G\setminus F}(h)\|_\infty\, dh < \frac{1}{6}\varepsilon$$

since k satisfies a uniform decay condition. Define

$$\|k\| := \sup_{x \in Z} \int \|k(h,(\cdot)^{-1}x)\|_\infty\, dh.$$

Observe there exists a compactly-supported continuous nonzero vector $\chi \in L^2_+(G)$ such that

$$\sup_{h \in F} \frac{\|\chi^h - \chi\|}{\|\chi\|} < \frac{1}{6(\|k\| + 1)}\varepsilon$$

since $\langle G, \lambda\rangle$ is amenable; see Section 3.1.2. Here

$$\chi^h \colon G \to [0, +\infty), \quad g \mapsto \chi(gh).$$

Define

$$K := \mathrm{supp}(\chi).$$

Observe there exists $\delta > 0$ such that

$$d(y,x) < 2\delta \implies \int \|(k(h,(\cdot)^{-1}y) - k(h,(\cdot)^{-1}x))\mathbb{1}_K(\cdot)\|_\infty \mathbb{1}_F(h)\, dh < \frac{1}{6}\varepsilon$$

since the group action and k satisfy a uniform continuity condition. Fix $X, Y \in \mathcal{S}$. Assume $d_{\mathsf{H}}(X,Y) < \delta$. Observe

$$\begin{aligned}
&\operatorname{dist}_{\mathsf{H}}(\operatorname{spec}(A_X), \operatorname{spec}(A_Y)) \\
&\quad \le \sup_{\substack{\check{x}\in X, \check{y}\in Y, \check{z}\in Z\\ d(\check{x},\check{y})<d_{\mathsf{H}}(X,Y)+\delta}} \int \|(k(h,(\cdot)^{-1}\check{y}) - k(h,(\cdot)^{-1}\check{x}))\mathbb{1}_K(\cdot)\|_\infty \\
&\qquad\qquad + \|k(h,(\cdot)^{-1}\check{z})\|_\infty \frac{\|\chi^h - \chi\|}{\|\chi\|}\, dh \qquad \text{(Lemma 4.3)} \\
&\quad \le \sup_{\substack{\check{x}\in X, \check{y}\in Y, \check{z}\in Z\\ d(\check{x},\check{y})<2\delta}} \Bigg(\int \Big(\|(k(h,(\cdot)^{-1}\check{y}) - k(h,(\cdot)^{-1}\check{x}))\mathbb{1}_K(\cdot)\|_\infty \\
&\qquad\qquad + \|k(h,(\cdot)^{-1}\check{z})\|_\infty \frac{\|\chi^h - \chi\|}{\|\chi\|}\Big)\mathbb{1}_F(h)\, dh \\
&\qquad + \int \Big(\|(k(h,(\cdot)^{-1}\check{y}) - k(h,(\cdot)^{-1}\check{x}))\mathbb{1}_K(\cdot)\|_\infty \\
&\qquad\qquad + \|k(h,(\cdot)^{-1}\check{z})\|_\infty \frac{\|\chi^h - \chi\|}{\|\chi\|}\Big)\mathbb{1}_{G\setminus F}(h) dh\Bigg) \\
&\quad < \frac{2}{6}\varepsilon + \frac{4}{6}\varepsilon.
\end{aligned}$$

■

Remark 4.4.1. Theorem 4.4 will be improved provided the group action and k satisfy a Lipschitz continuity condition, k satisfies a linear decay condition, and the group has strict polynomial growth.

Lemma 4.5. *Assume $\langle G, \lambda\rangle$ is unimodular. Let $r \ge 1$. Define*

$$\chi_r = \chi \colon G \to [0,+\infty), \quad g \mapsto \Big(\frac{r - |g|}{r}\Big)\mathbb{1}_{B(r)}(g).$$

Assume $\langle (G,\ell), \lambda\rangle$ is strictly-polynomial-growing: there exist $b \ge 1$ and $c_1 \ge c_0 > 0$ such that for each $r \ge 1$, $c_0 r^b \le \lambda(B(r)) \le c_1 r^b$. Fix $h \in G$. Then,

$$\|\chi^h - \chi\| \le \min\Big\{\frac{|h|}{r}\Big(\frac{(2+b)^{2+b}c_1}{2b^b c_0}\Big)^{1/2}, 2^{1/2}\Big\}\|\chi\|,$$

where

$$\chi^h \colon G \to [0,+\infty), \quad g \mapsto \chi(gh).$$

Proof. Define

$$r' := \Big(\frac{2}{2+b}\Big)r.$$

Observe

$$
\begin{aligned}
&\|\chi^h - \chi\|^2 \\
&= \int\limits_{B(r)h^{-1}\setminus B(r)} \Big(\frac{r-|gh|}{r}\Big)^2 dg + \int\limits_{B(r)\setminus B(r)h^{-1}} \Big(\frac{r-|g|}{r}\Big)^2 dg + \int\limits_{B(r)\cap B(r)h^{-1}} \Big(\frac{|g|-|gh|}{r}\Big)^2 dg \\
&\le \int\limits_{B(r)h^{-1}\setminus B(r)} \Big(\frac{|g|-|gh|}{r}\Big)^2 dg + \int\limits_{B(r)\setminus B(r)h^{-1}} \Big(\frac{|gh|-|g|}{r}\Big)^2 dg + \int\limits_{B(r)\cap B(r)h^{-1}} \Big(\frac{|g|-|gh|}{r}\Big)^2 dg \\
&\le \int\limits_{B(r)h^{-1}\setminus B(r)} r^{-2}|h|^2\, dg + \int\limits_{B(r)\setminus B(r)h^{-1}} r^{-2}|h|^2\, dg + \int\limits_{B(r)\cap B(r)h^{-1}} r^{-2}|h|^2\, dg \le r^{-2}|h|^2 2(c_1 r^b)
\end{aligned}
$$

and

$$
\begin{aligned}
\|\chi\|^2 &= \int\limits_{B(r)\setminus B(r-r')} \Big(\frac{r-|g|}{r}\Big)^2 dg + \int\limits_{B(r-r')} \Big(\frac{r-|g|}{r}\Big)^2 dg \ge \int\limits_{B(r-r')} \Big(\frac{r-|g|}{r}\Big)^2 dg \\
&\ge \int\limits_{B(r-r')} r^{-2}(r')^2\, dg \ge r^{-2}(r')^2(c_0(r-r')^b).
\end{aligned}
$$

Also,

$$
\|\chi^h - \chi\|^2 \le \|\chi^h\|^2 + \|\chi\|^2 = 2\|\chi\|^2. \qquad \blacksquare
$$

Remark 4.5.1. (a) For each nonempty compact subset F of G,

$$
\sup_{h\in F} \|\chi^h - \chi\|/\|\chi\| \to 0 \quad \text{as } r \to +\infty.
$$

(b) If G is discrete, then

$$
\begin{aligned}
\|\mathbb{1}_{B(r)}\|^2 &= \#B(r), \\
\|\mathbb{1}_{B(r)h^{-1}} - \mathbb{1}_{B(r)}\|^2 &= \#(B(r)h^{-1} \triangle B(r)).
\end{aligned}
$$

(c) Define

$$
\acute{r} := \max\{n \in \mathbb{Z} : 0 \le n < r\}.
$$

If $G = \mathbb{Z}$, then

$$
\begin{aligned}
\|\mathbb{1}_{B(r)}\|^2 &= 2\acute{r} + 1, \\
\|\mathbb{1}_{B(r)-m} - \mathbb{1}_{B(r)}\|^2 &= \min\{2|m|, 2(2\acute{r}+1)\}, \\
\frac{\|\mathbb{1}_{B(r)-m} - \mathbb{1}_{B(r)}\|}{\|\mathbb{1}_{B(r)}\|} &= O(r^{-1/2})\big|_{m\in\mathbb{Z}, r\to+\infty}
\end{aligned}
$$

but

$$\|\chi\|^2 = \frac{2}{3}\Big(\frac{(2\acute{r}+1)(\acute{r}+1)}{2r^2} + \frac{3(r-\acute{r}-1)}{r}\Big)\acute{r} + 1,$$

$$\|\chi^m - \chi\|^2 = \begin{cases} 0 & \text{if } m = 0,\\ \frac{m^2}{r}\Big(2 - \frac{m}{r} + \frac{2(r-\acute{r})^2 - 2(r-\acute{r}) + 1}{mr}\Big) & \text{if } 1 \le m \le \acute{r},\\ \frac{m}{3}\Big(\Big(\frac{m}{r}\Big)^2 - 6\Big(\frac{m}{r}\Big) + 12\Big(\frac{\acute{r}}{r}\Big)\Big) - \frac{\acute{r}}{3}\Big(4\Big(\frac{\acute{r}}{r}\Big)\Big) & \\ +\frac{m}{3}\Big(6\Big(1 - \Big(\frac{\acute{r}}{r}\Big)^2\Big) + 6\Big(1 - \frac{\acute{r}}{r}\Big)\Big(\frac{1}{r}\Big) - \Big(\frac{1}{r}\Big)^2\Big) & \\ -\frac{\acute{r}}{3}\Big(4(2\acute{r}+1)\Big(1 - \frac{\acute{r}}{r}\Big)\Big(\frac{1}{r}\Big) - \Big(\frac{1}{r}\Big)^2\Big) & \text{if } \acute{r}+1 \le m \le 2\acute{r},\\ 2\|\chi\|^2 & \text{if } 2\acute{r}+1 \le m, \end{cases}$$

$$\frac{\|\chi^m - \chi\|}{\|\chi\|} = O(r^{-1})|_{m\in\mathbb{Z}, r\to+\infty}.$$

(d) Lemma 4.5 establishes that strict polynomial growth implies amenability; see Proposition 3.4 and (a) above. In the case where G is discrete, one could consider indicator functions on a Følner sequence instead of tent functions – which requires a metric; see (b) above. An advantage of using tent functions instead of indicator functions on balls $B(r)$ is that the limiting behavior is of order r^{-1} which is sharp; see Lemma 4.5 and (c) above.

(e) When $G = \mathbb{Z}$, a 1990 result by M. Choi, G. A. Elliott, and N. Yui [24] establishes spectral $1/3$-Hölder continuity for one-dimensional discrete quasiperiodic Schrödinger operators with Lipschitz continuous potentials, e.g., the almost-Mathieu operator, by utilizing indicator functions. Within the same year, J. Avron, P. van Mouche, and B. Simon [7] improved the result and established spectral $1/2$-Hölder continuity by utilizing tent functions.

Theorem 4.6. *Assume $\langle G, \lambda\rangle$ is unimodular, A_x is normal for every $x \in Z$, and the following.*

(a) *τ satisfies a Lipschitz continuity condition: there exists $c_\tau > 0$ such that for each $g \in G$,*

$$d(gx, gy) \le (c_\tau|g| + 1)d(x, y).$$

(b) *k satisfies a Lipschitz continuity condition: there exists $c_k \in L^1_+(G)$ such that for λ-a.e. $h \in G$,*

$$|k(h, x) - k(h, y)| \le c_k(h)d(x, y).$$

(c) *k satisfies a linear decay condition: there exists $c_s > 0$ such that for each $r \geq 0$,*

$$\sup_{x \in Z} \int \|k(h, (\cdot)^{-1}x) \min\{|h|, r\}\|_\infty \, dh \leq c_s.$$

(d) *$\langle (G, \ell), \lambda \rangle$ is strictly-polynomial-growing: there exist $b \geq 1$ and $c_1 \geq c_0 > 0$ such that for each $r \geq 1$,*

$$c_0 r^b \leq \lambda(B(r)) \leq c_1 r^b.$$

Fix $X, Y \in \mathcal{S}$. Then, for each $r \geq 1$,

$$\begin{aligned}&\operatorname{dist}_{\mathsf{H}}(\operatorname{spec}(A_X), \operatorname{spec}(A_Y)) \\ &\quad \leq \|c_k\|_1 (c_\tau r + 1) d_{\mathsf{H}}(X, Y) + \frac{c_s}{r} \Big(\frac{(2+b)^{2+b} c_1}{2 b^b c_0} \Big)^{1/2}\end{aligned}$$

and

$$\begin{aligned}&d_{\mathsf{H}}(X, Y) \leq \min\{\delta, 1\} \\ &\quad \Longrightarrow \ \operatorname{dist}_{\mathsf{H}}(\operatorname{spec}(A_X), \operatorname{spec}(A_Y)) \leq C d_{\mathsf{H}}(X, Y)^{1/2},\end{aligned}$$

where

$$\delta := \frac{c_s}{\|c_k\|_1 c_\tau} \Big(\frac{(2+b)^{2+b} c_1}{2 b^b c_0} \Big)^{1/2}$$

and

$$C := 2 \Big(\|c_k\|_1 c_\tau c_s \Big(\frac{(2+b)^{2+b} c_1}{2 b^b c_0} \Big)^{1/2} \Big)^{1/2} + \|c_k\|_1.$$

Proof. Fix $r \geq 1$. Define

$$\chi_r = \chi \colon G \to [0, +\infty), \quad g \mapsto \Big(\frac{r - |g|}{r} \Big) \mathbb{1}_{B(r)}(g).$$

Let $\rho > 0$. Observe

$$\begin{aligned}&\operatorname{dist}_{\mathsf{H}}(\operatorname{spec}(A_X), \operatorname{spec}(A_Y)) \\ &\quad \leq \sup_{\substack{\check{x} \in X, \check{y} \in Y, \check{z} \in Z \\ d(\check{x}, \check{y}) < d_{\mathsf{H}}(X,Y) + \rho}} \int \|(k(h, (\cdot)^{-1}\check{y}) - k(h, (\cdot)^{-1}\check{x})) \mathbb{1}_{\operatorname{supp}(\chi)}(\cdot)\|_\infty \\ &\qquad\qquad + \|k(h, (\cdot)^{-1}\check{z})\|_\infty \frac{\|\chi^h - \chi\|}{\|\chi\|} \, dh \qquad \text{(Lemma 4.3)} \\ &\quad \leq \sup_{\substack{\check{x} \in X, \check{y} \in Y, \check{z} \in Z \\ d(\check{x}, \check{y}) < d_{\mathsf{H}}(X,Y) + \rho}} \int \|c_k(h) d((\cdot)^{-1}\check{y}, (\cdot)^{-1}\check{x}) \mathbb{1}_{\operatorname{supp}(\chi)}(\cdot)\|_\infty \\ &\qquad\qquad + \|k(h, (\cdot)^{-1}\check{z})\|_\infty \frac{\|\chi^h - \chi\|}{\|\chi\|} \, dh \qquad \text{(property (b))}\end{aligned}$$

$$\leq \sup_{\substack{\check{x}\in X,\check{y}\in Y,\check{z}\in Z\\ d(\check{x},\check{y})<d_{\mathsf{H}}(X,Y)+\rho}} \int c_k(h)(c_\tau r+1)d(\check{y},\check{x}) \\ + \|k(h,(\cdot)^{-1}\check{z})\|_\infty \frac{\|\chi^h-\chi\|}{\|\chi\|}\,dh \quad \text{(property (a))}$$

$$\leq \|c_k\|_1(c_\tau r+1)(d_{\mathsf{H}}(X,Y)+\rho) \\ + \sup_{\check{z}\in Z}\int \|k(h,(\cdot)^{-1}\check{z})\|_\infty \min\Big\{\frac{|h|}{r}\Big(\frac{(2+b)^{2+b}c_1}{2b^b c_0}\Big)^{1/2}, 2^{1/2}\Big\}\,dh \quad \text{(Lemma 4.5)}$$

$$\leq \|c_k\|_1(c_\tau r+1)(d_{\mathsf{H}}(X,Y)+\rho)+\frac{c_s}{r}\Big(\frac{(2+b)^{2+b}c_1}{2b^b c_0}\Big)^{1/2}. \quad \text{(property (c))}$$

Here

$$\chi^h\colon G\to[0,+\infty),\quad g\mapsto\chi(gh).$$

Also, if

$$d_{\mathsf{H}}(X,Y)\leq\min\{\delta,1\},$$

then

$$1\leq(d_{\mathsf{H}}(X,Y)^{-1}\delta)^{1/2}\quad\text{and}\quad d_{\mathsf{H}}(X,Y)\leq d_{\mathsf{H}}(X,Y)^{1/2}$$

and

$$\begin{aligned}&\mathrm{dist}_{\mathsf{H}}(\mathrm{spec}(A_X),\mathrm{spec}(A_Y))\\ &\quad\leq \|c_k\|_1(c_\tau r+1)d_{\mathsf{H}}(X,Y)+\frac{c_s}{r}\Big(\frac{(2+b)^{2+b}c_1}{2b^b c_0}\Big)^{1/2}\Big|_{r=(d_{\mathsf{H}}(X,Y)^{-1}\delta)^{1/2}}\\ &\quad\leq C d_{\mathsf{H}}(X,Y)^{1/2}.\end{aligned}$$

■

Theorem 4.7. *Assume $\langle G,\lambda\rangle$ is unimodular, A_x is normal for every $x\in Z$, and the following.*

(a) *τ satisfies a Lipschitz continuity condition: there exists $c_\tau>0$ such that for each $g\in G$,*

$$d(gx,gy)\leq(c_\tau|g|+1)d(x,y).$$

(b) *k satisfies a locally-constant condition: there exists $c_k\in L^\infty_+(G)$ such that for λ-a.e. $h\in G$,*

$$d(x,y)<\frac{1}{c_k(h)}\implies k(h,x)=k(h,y).$$

(c) *k satisfies a linear decay condition: there exists $c_s>0$ such that for each $r\geq 0$,*

$$\sup_{x\in Z}\int\|k(h,(\cdot)^{-1}x)\min\{|h|,r\}\|_\infty\,dh\leq c_s.$$

(d) $\langle (G,\ell),\lambda\rangle$ *is strictly-polynomial-growing: there exist* $b \geq 1$ *and* $c_1 \geq c_0 > 0$ *such that for each* $r \geq 1$,

$$c_0 r^b \leq \lambda(B(r)) \leq c_1 r^b.$$

Fix $X, Y \in \mathcal{S}$. *Then,*

$$d_{\mathsf{H}}(X,Y) \leq \delta \implies \operatorname{dist}_{\mathsf{H}}(\operatorname{spec}(A_X), \operatorname{spec}(A_Y)) \leq C d_{\mathsf{H}}(X,Y),$$

where

$$\delta := \frac{1}{(\|c_k\|_\infty + 1)(c_\tau + 1)}$$

and

$$C := (\|c_k\|_\infty + 1)(c_\tau + 1)c_s\Big(\frac{(2+b)^{2+b}c_1}{2b^b c_0}\Big)^{1/2}.$$

Proof. Assume $d_{\mathsf{H}}(X,Y) \leq \delta$. Observe

$$\begin{aligned}\frac{1}{(\|c_k\|_\infty + 1)\delta c_\tau} - \frac{1}{c_\tau} &= 1,\\ \frac{c_\tau}{1 - (\|c_k\|_\infty + 1)d_{\mathsf{H}}(X,Y)} &\leq c_\tau + 1.\end{aligned}$$

Define

$$r := \frac{1}{(\|c_k\|_\infty + 1)d_{\mathsf{H}}(X,Y)c_\tau} - \frac{1}{c_\tau}.$$

Observe

$$r \geq 1, \quad \frac{1}{r} \leq (\|c_k\|_\infty + 1)(c_\tau + 1)d_{\mathsf{H}}(X,Y).$$

Define, for $h \in G$,

$$\begin{aligned}\chi_r = \chi\colon G &\to [0,+\infty), \quad g \mapsto \Big(\frac{r - |g|}{r}\Big)\mathbb{1}_{B(r)}(g),\\ \chi^h\colon G &\to [0,+\infty), \quad g \mapsto \chi(gh).\end{aligned}$$

Define

$$\rho := \frac{d_{\mathsf{H}}(X,Y)}{\|c_k\|_\infty}.$$

Observe, by property (a),

$$\begin{aligned}&d(x,y) < d_{\mathsf{H}}(X,Y) + \rho\\ &\implies \|d((\cdot)^{-1}y, (\cdot)^{-1}x)\mathbb{1}_{\operatorname{supp}(\chi)}(\cdot)\|_\infty \leq (c_\tau r + 1)d(x,y)\\ &\qquad\qquad < \frac{1}{\|c_k\|_\infty}\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{dist}_{\mathsf{H}}(\mathrm{spec}(A_X), \mathrm{spec}(A_Y)) \\
&\leq \sup_{\substack{\check{x}\in X, \check{y}\in Y, \check{z}\in Z \\ d(\check{x},\check{y})<d_{\mathsf{H}}(X,Y)+\rho}} \int \|(k(h,(\cdot)^{-1}\check{y}) - k(h,(\cdot)^{-1}\check{x}))\mathbb{1}_{\mathrm{supp}(\chi)}(\cdot)\|_\infty \\
&\qquad\qquad + \|k(h,(\cdot)^{-1}\check{z})\|_\infty \frac{\|\chi^h - \chi\|}{\|\chi\|} dh \quad \text{(Lemma 4.3)} \\
&\leq 0 + \sup_{\check{z}\in Z} \int \|k(h,(\cdot)^{-1}\check{z})\|_\infty \frac{\|\chi^h - \chi\|}{\|\chi\|}\, dh \quad \text{(property (b))} \\
&\leq \sup_{\check{z}\in Z} \int \|k(h,(\cdot)^{-1}\check{z})\|_\infty \min\Big\{\frac{|h|}{r}\Big(\frac{(2+b)^{2+b}c_1}{2b^b c_0}\Big)^{1/2}, 2^{1/2}\Big\}\, dh \quad \text{(Lemma 4.5)} \\
&\leq \frac{c_s}{r}\Big(\frac{(2+b)^{2+b}c_1}{2b^b c_0}\Big)^{1/2} \quad \text{(property (c))} \\
&\leq C d_{\mathsf{H}}(X,Y). \qquad \blacksquare
\end{aligned}$$

5. Proof of Lemma 3.2

Let $\langle G,\lambda\rangle$ be a second-countable locally-compact Hausdorff group endowed with a left Haar measure. Let $\langle Z,(G,\tau)\rangle$ be a topological space endowed with a left continuous action. Let $k\colon G\times Z\to\mathbb{C}$, where k is measurable and

$$\sup_{x\in Z}\int \|k(h,(\cdot)^{-1}x)\|_\infty\, dh < +\infty.$$

Define, for $x\in Z$, $A_x\colon L^2(G)\to L^2(G)$ by

$$(A_x\psi)(g) = \int k(h,g^{-1}x)\psi(gh)\, dh = \int k(g^{-1}h,g^{-1}x)\psi(h)\, dh$$

for every $\psi\in L^2(G), g\in G$. We remark that in the above expression we suppress the notations involving λ and τ. For example, $\int\varphi(h)\,dh := \int\varphi\, d\lambda$ and $gx := g\cdot_\tau x$. Let $\mathcal{S}$ be the collection of all subsystems of $\langle Z,(G,\tau)\rangle$. Define, for $X\in\mathcal{S}$, $\mathrm{spec}(A_X) := \overline{\bigcup_{x\in X}\mathrm{spec}(A_x)}$.

Proposition 5.1 (Lemma 3.2 (b)). *Assume $\langle G,\lambda\rangle$ is unimodular. Fix $x,y\in Z$, $\psi\in L^2(G)$, $j\in G$. Then,*

$$\|A_x\psi\| \leq \Big(\int \|k(h,(\cdot)^{-1}x)\mathbb{1}_{\mathrm{supp}(\psi)h^{-1}}(\cdot)\|_\infty\, dh\Big)\|\psi\|,$$

$$\|A_x{}^*\psi\| \leq \Big(\int \|k(h,(\cdot)^{-1}x)\mathbb{1}_{\mathrm{supp}(\psi)}(\cdot)\|_\infty\, dh\Big)\|\psi\|,$$

$$\begin{aligned}
\|(A_y - A_x)\psi\| &\le \left(\int \|(k(h,(\cdot)^{-1}y) - k(h,(\cdot)^{-1}x))\mathbb{1}_{\operatorname{supp}(\psi)h^{-1}}(\cdot)\|_\infty \, dh\right)\|\psi\|,\\
\|(A_y - A_x)^*\psi\| &\le \left(\int \|(k(h,(\cdot)^{-1}y) - k(h,(\cdot)^{-1}x))\mathbb{1}_{\operatorname{supp}(\psi)}(\cdot)\|_\infty \, dh\right)\|\psi\|,\\
U_{j^{-1}}A_xU_j &= A_{j^{-1}x}.
\end{aligned}$$

Proof. Observe

$$\begin{aligned}
&\|A_x\psi\|^2\\
&\quad= \int \left|\int k(h,g^{-1}x)\psi(gh)\,dh\right|^2 dg\\
&\quad\le \int \left(\int |k(h,g^{-1}x)||\psi(gh)|\,dh\right)^2 dg\\
&\quad\le \int \left(\int |k(h,g^{-1}x)|\mathbb{1}_{\operatorname{supp}(\psi)}(gh)\,dh\right)\left(\int |k(h,g^{-1}x)||\psi(gh)|^2\,dh\right) dg\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(Hölder's inequality)}\\
&\quad\le \int \left(\int \|k(h,(\cdot)^{-1}x)\mathbb{1}_{\operatorname{supp}(\psi)h^{-1}}(\cdot)\|_\infty\,dh\right)\\
&\qquad\quad\times \left(\int \|k(h,(\cdot)^{-1}x)\mathbb{1}_{\operatorname{supp}(\psi)h^{-1}}(\cdot)\|_\infty|\psi(gh)|^2\,dh\right) dg\\
&\quad= \left(\int \|k(h,(\cdot)^{-1}x)\mathbb{1}_{\operatorname{supp}(\psi)h^{-1}}(\cdot)\|_\infty\,dh\right)^2 \|\psi\|^2
\end{aligned}$$

and

$$\begin{aligned}
&\|{A_x}^*\psi\|^2\\
&\quad= \int \left|\int \overline{k(h^{-1},h^{-1}g^{-1}x)}\psi(gh)\,dh\right|^2 dg\\
&\quad\le \int \left(\int |\overline{k(h^{-1},h^{-1}g^{-1}x)}||\psi(gh)|\,dh\right)^2 dg\\
&\quad\le \int \left(\int |k(h^{-1},(gh)^{-1}x)|\mathbb{1}_{\operatorname{supp}(\psi)}(gh)\,dh\right)\\
&\qquad\quad\times \left(\int |k(h^{-1},(gh)^{-1}x)||\psi(gh)|^2\,dh\right) dg\\
&\quad\le \int \left(\int \|k(h^{-1},(\cdot)^{-1}x)\mathbb{1}_{\operatorname{supp}(\psi)}(\cdot)\|_\infty\,dh\right)\\
&\qquad\quad\times \left(\int \|k(h^{-1},(\cdot)^{-1}x)\mathbb{1}_{\operatorname{supp}(\psi)}(\cdot)\|_\infty|\psi(gh)|^2\,dh\right) dg\\
&\quad= \left(\int \|k(h^{-1},(\cdot)^{-1}x)\mathbb{1}_{\operatorname{supp}(\psi)}(\cdot)\|_\infty\,dh\right)^2 \|\psi\|^2
\end{aligned}$$

and

$$
\begin{aligned}
&\|(A_y - A_x)\psi\|^2 \\
&\quad = \int \left| \int (k(h, g^{-1}y) - k(h, g^{-1}x))\psi(gh)\, dh \right|^2 dg \\
&\quad \le \left(\int \|(k(h, (\cdot)^{-1}y) - k(h, (\cdot)^{-1}x))\mathbb{1}_{\mathrm{supp}(\psi)h^{-1}}(\cdot)\|_\infty \, dh \right)^2 \|\psi\|^2
\end{aligned}
$$

and

$$
\begin{aligned}
&\|(A_y - A_x)^*\psi\|^2 \\
&\quad = \int \left| \int (\overline{k(h^{-1}, h^{-1}g^{-1}y)} - \overline{k(h^{-1}, h^{-1}g^{-1}x)})\psi(gh)\, dh \right|^2 dg \\
&\quad \le \left(\int \|(k(h^{-1}, (\cdot)^{-1}y) - k(h^{-1}, (\cdot)^{-1}x))\mathbb{1}_{\mathrm{supp}(\psi)}(\cdot)\|_\infty \, dh \right)^2 \|\psi\|^2.
\end{aligned}
$$

Also,

$$
\begin{aligned}
U_{j^{-1}} A_x U_j &= \int dh U_{j^{-1}}[k(h, (\cdot)^{-1}x)]U_j R_h \\
&= \int dh[k(h, (\cdot)^{-1}j^{-1}x)]R_h = A_{j^{-1}x}.
\end{aligned}
$$

■

Proposition 5.2 (Lemma 3.2 (c)). *Assume $\langle G, \lambda \rangle$ is unimodular, A_x is normal for every $x \in Z$, and the following.*

(i) *k satisfies a continuity condition: for each $x \in Z$ and for each $\varepsilon > 0$ and for each nonempty compact subset F of G,*

$$\int |k(h, y) - k(h, x)|\mathbb{1}_F(h)\, dh < \varepsilon$$

for every y in some neighborhood of x.

(ii) *k satisfies a decay condition: for each $x \in Z$ and for each $\varepsilon > 0$, there exists a nonempty compact subset F of G such that*

$$\int \|k(h, (\cdot)^{-1}y)\mathbb{1}_{G \setminus F}(h)\|_\infty \, dh < \varepsilon$$

for every y in some neighborhood of x.

Fix $x \in Z$. Then for each $\varepsilon > 0$, $\sup_{E \in \mathrm{spec}(A_x)} \mathrm{dist}(E, \mathrm{spec}(A_y)) < \varepsilon$ for every y in some neighborhood of x, i.e.,

$$\lim_{y \xrightarrow[Z]{} x} \sup_{E \in \mathrm{spec}(A_x)} \mathrm{dist}(E, \mathrm{spec}(A_y)) = 0.$$

Proof. It suffices to show that for each $E \in \operatorname{spec}(A_x)$ and for each $\varepsilon > 0$,

$$\operatorname{dist}(E, \operatorname{spec}(A_y)) < \varepsilon$$

for every y in some neighborhood of x since $\operatorname{spec}(A_x)$ is compact. Fix $E \in \operatorname{spec}(A_x)$ and $\varepsilon > 0$. Without loss of generality, assume $E \notin \operatorname{spec}(A_y)$. Observe there exists a nonempty compact subset F of G such that

$$\int \|k(h, (\cdot)^{-1} y) \mathbb{1}_{G \setminus F}(h)\|_\infty \, dh < \frac{1}{4}\varepsilon$$

for every y in some neighborhood U of x since k satisfies a decay condition. Observe A_x is normal and, by Weyl's criterion, there exists a nonzero vector φ such that $\|(A_x - E)\varphi\| < \varepsilon\|\varphi\|/4$. Without loss of generality, assume φ is both continuous and compactly-supported. Define $K := \operatorname{supp}(\varphi)$. Observe

$$\int \|(k(h, (\cdot)^{-1} y) - k(h, (\cdot)^{-1} x)) \mathbb{1}_K(\cdot)\|_\infty \mathbb{1}_F(h) \, dh < \frac{1}{4}\varepsilon$$

for every y in some neighborhood U' of x since the group action is continuous and k satisfies a continuity condition. Since A_z is normal for $z \in X$, observe for each $y \in U \cap U'$,

$$\begin{aligned} \|(A_y - E)^{-1}\|^{-1} \|\varphi\| \le \|(A_y - E)\varphi\| &= \|(A_y - E)^* \varphi\| \\ &< \left(\left\{ (A_y - A_x)^* \frac{\varphi}{\|\varphi\|} \right\} + \frac{1}{4}\varepsilon \right) \|\varphi\| \end{aligned}$$

and

$$\begin{aligned} &\operatorname{dist}(E, \operatorname{spec}(A_y)) \\ &\quad = \|(A_y - E)^{-1}\|^{-1} < \left\| (A_y - A_x)^* \frac{\varphi}{\|\varphi\|} \right\| + \frac{1}{4}\varepsilon \\ &\quad \le \int \|(k(h, (\cdot)^{-1} y) - k(h, (\cdot)^{-1} x)) \mathbb{1}_K(\cdot)\|_\infty \, dh + \frac{1}{4}\varepsilon \qquad \text{(Proposition 5.1)} \\ &\quad < \int \|(k(h, (\cdot)^{-1} y) - k(h, (\cdot)^{-1} x)) \mathbb{1}_K(\cdot)\|_\infty \mathbb{1}_F(h) \, dh + \frac{3}{4}\varepsilon < \varepsilon. \qquad \blacksquare \end{aligned}$$

Remark 5.2.1. (a) To see how conditions (i) and (ii) follow from the setting for Lemma 3.2, recall the setting has $f_{(\cdot)} \in C_c(G, C_b(Z))$ and $f_h(x) = k(h, x)$.

(b) To see how $\operatorname{spec}(A_x) = \operatorname{spec}(A_{X(x)})$ follows from Proposition 5.1 and Proposition 5.2, if $y = j^{-1}x$ for some $j \in G$, then $\operatorname{spec}(A_y) = \operatorname{spec}(A_x)$ since $A_y = U_{j^{-1}} A_x U_j$ and if y is a limit-point of the orbit of x, then $\operatorname{spec}(A_y) \subseteq \operatorname{spec}(A_x)$ since

$$\inf_{j \in G} \sup_{E \in \operatorname{spec}(A_y)} \operatorname{dist}(E, \operatorname{spec}(A_{j^{-1}x})) = 0.$$

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