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Sharp semiclassical spectral asymptotics for Schrödinger operators with non-smooth potentials

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Abstract. We consider semiclassical Schrödinger operators acting in $L^2(\mathbb{R}^d)$ with $d \geq 3$. For these operators, we establish sharp spectral asymptotics without full regularity. For the counting function, we assume the potential is locally integrable and that the negative part of the potential minus a constant is once differentiable, with its derivative being Hölder continuous with parameter $\mu \geq 1/2$. Moreover, we also consider sharp Riesz means of order γ with $\gamma \in (0, 1]$. Here, we assume the potential is locally integrable and that the negative part of the potential minus a constant is twice differentiable, with its second derivative being Hölder continuous with parameter μ that depends on γ.

1. Introduction

Consider a semiclassical Schrödinger operator $H_\hbar = -\hbar^2 \Delta + V$ acting in $L^2(\mathbb{R}^d)$, where $-\Delta$ is the positive Laplacian and V is the potential. For the Schrödinger operator $H_\hbar$, the Weyl law states that

$$\operatorname{Tr}[\mathbf{1}_{(-\infty,0]}(H_\hbar)] = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^{2d}} \mathbf{1}_{(-\infty,0]}(p^2 + V(x))\, dp dx + o(\hbar^{-d}), \tag{1.1}$$

where $\mathbf{1}_\Omega(t)$ is the characteristic function of the set Ω. Recently, Frank [6] proved that (1.1) holds under the conditions $d \geq 3$, $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, and $V_- \in L^{d/2}(\mathbb{R}^d)$, where $V_- = \max(0, -V)$. These conditions are the minimal requirements ensuring that both sides of the equality are well-defined and finite. For a brief historical overview of the development of (1.1) under minimal assumptions, see the introduction of [6].

Under additional assumptions on the potential V, Helffer and Robert established in [7] that

$$\operatorname{Tr}[\mathbf{1}_{(-\infty,0]}(H_\hbar)] = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^{2d}} \mathbf{1}_{(-\infty,0]}(p^2 + V(x))\, dp dx + \mathcal{O}(\hbar^{1-d}), \tag{1.2}$$

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for all $\hbar \in (0, \hbar_0]$, where $\hbar_0$ is sufficiently small. They proved this result under the conditions that $V \in C^\infty(\mathbb{R}^d)$, satisfies certain regularity conditions at infinity, and $V(x) \geq c > 0$ for all $x \in \Omega^c$, where $\Omega \subset \mathbb{R}^d$ is some open bounded set. Moreover, they assumed a non-critical condition on the energy surface $\{(x, p) \in \mathbb{R}^{2d} \mid p^2 + V(x) = 0\}$, which was later removed; see [22]. The error estimate in (1.2) is the best generic error bound one can obtain. As an example, consider the operator $H_\hbar = -\hbar^2\Delta + x^2 - \lambda$, for some $\lambda > 0$. For this operator, all eigenvalues can be explicitly computed, and one can verify by direct calculation that (1.2) holds with an explicit error term of order $\hbar^{1-d}$.

When comparing the two results in dimensions $d \geq 3$, it does raise the question: is formula (1.2) valid under less smoothness? Could it even be valid for all V satisfying the assumptions of the result by Frank? The last part of the question currently seems beyond reach for a positive answer, and to the author's knowledge, no counterexample exists yet. However, for the first part of the question we will give positive answers.

We will in fact not just consider the Weyl law but also Riesz means. That is, for $\gamma \in [0, 1]$, we will consider traces of the form

$$\mathrm{Tr}[g_\gamma(H_\hbar)], \tag{1.3}$$

where the function g_γ is given by

$$g_\gamma(t) = \begin{cases} \mathbf{1}_{(-\infty,0]}(t), & \gamma = 0, \\ (t)_-^\gamma, & \gamma \in (0, 1]. \end{cases}$$

Frank also considered traces of the form (1.3) in [6]. Helffer and Robert only considered Weyl asymptotics in [7], but proved the sharp estimate for Riesz means in [8]. For future reference and comparison, we recall the exact statement of the results obtained by Frank in [6].

Theorem 1.1. *Let $\gamma \geq 1/2$ if $d = 1$, $\gamma > 0$ if $d = 2$, and $\gamma \geq 0$ if $d \geq 3$. Let $\Omega \subset \mathbb{R}^d$ be an open set and let $V \in L^1_{\mathrm{loc}}(\Omega)$ with $V_- \in L^{\gamma+d/2}(\Omega)$. Then,*

$$\mathrm{Tr}[g_\gamma(H_\hbar)] = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^{2d}} g_\gamma(p^2 + V(x))\, dx\, dp + o(\hbar^{-d})$$

as $\hbar \to 0$, where $H_\hbar = -\hbar^2\Delta + V(x)$ is considered in $L^2(\Omega)$ with Dirichlet boundary conditions.

One thing to observe here is that this theorem is also valid when we are on bounded domains. We will only discuss the case where the domain is the whole space $\mathbb{R}^d$, $d \geq 3$. For results on sharp Weyl laws without full regularity and on bounded domains, we refer the reader to the works by Ivrii [11–16].

1.1. Sharp asymptotics

We will set up some notation and recall a definition before we give the assumptions for our main theorem and state it.

Definition 1.2. Let $f:\mathbb{R}^d \mapsto \mathbb{R}$ be a measurable function. For each $\nu \in \mathbb{R}$, we define the set

$$\Omega_{\nu,f} := \{x \in \mathbb{R}^d \mid f(x) < \nu\}.$$

Definition 1.3. For k in $\mathbb{N}$ and μ in $[0,1]$ and $\Omega \subset \mathbb{R}^d$ open, we denote by $C^{k,\mu}(\Omega)$ the subspace of $C^k(\Omega)$ defined by

$$\begin{aligned} C^{k,\mu}(\Omega) = \{f \in C^k(\Omega) \mid {} & \text{there exists } C > 0 \text{ such that} \\ & |\partial_x^\alpha f(x) - \partial_x^\alpha f(y)| \le C|x-y|^\mu \\ & \text{for all } \alpha \in \mathbb{N}^d \text{ with } |\alpha| = k \text{ and for all } x, y \in \Omega\}. \end{aligned}$$

These definitions are here to clarify the notation. We are now ready to state our assumptions on the potential V.

Assumption 1.4. Let $V \in L^1_{\text{loc}}(\mathbb{R}^d)$ be a real function. Suppose that there exist $\nu > 0$, $k \in \mathbb{N}_0$, and $\mu \in [0,1]$ such that the set $\Omega_{4\nu,V}$ is open and bounded and $V \in C^{k,\mu}(\Omega_{4\nu,V})$.

With our assumptions on the potential V in place, we can now state the main theorem.

Theorem 1.5. *Let $H_\hbar = -\hbar^2\Delta + V$ be a Schrödinger operator acting in $L^2(\mathbb{R}^d)$ and let $\gamma \in [0,1]$. If $\gamma = 0$, we assume $d \ge 3$, and if $\gamma \in (0,1]$, we assume $d \ge 4$. Suppose that V satisfies Assumption 1.4 with $\nu > 0$ and $k = 1$, $\mu \ge 1/2$ if $\gamma = 0$, and $k = 2$, $\mu \ge \max(3\gamma/2 - 1/2, 0)$ if $\gamma > 0$. Then, it holds that*

$$\left| \operatorname{Tr}[g_\gamma(H_\hbar)] - \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^{2d}} g_\gamma(p^2 + V(x))\, dx\, dp \right| \le C\hbar^{1+\gamma-d} \tag{1.4}$$

for all $\hbar$ sufficiently small. The constant C depends on ν and the potential V.

When comparing the assumptions for our main theorem and Theorem 1.1, we have that in both we assume the potential to be in $L^1_{\text{loc}}(\mathbb{R}^d)$. But in Theorem 1.1 the additional assumptions on the potential are on the negative part of V, whereas we need to assume regularity for the negative part of $V - 4\nu$ for some $\nu > 0$. One could have hoped to only have an assumption on the negative part of V. However, this does not seem obtainable with the methods we use here. Firstly, the way we prove the theorem requires us to have control of the potential just outside the classical allowed region

$(\{x \in \mathbb{R}^d \mid V(x) \leq 0\})$. Secondly, we have that the constant in (1.4) will diverge to infinity as ν tends to zero. Hence, we cannot hope to do an approximation argument.

The assumptions on dimensions are needed to ensure the integrability of some integrals. In the case of Theorem 1.1, there are counter examples to the Weyl asymptotics for $V \in L^{d/2}(\mathbb{R}^d)$ for $d = 1, 2$; for details see [1, 19].

This is not the first work considering sharp Weyl laws without full regularity. The first results in a semiclassical setting were obtained by Ivrii in [10], where he also considered higher order differential operators acting in $L^2(M)$, where M is a compact manifold without boundary. In this work, the coefficients are assumed to be differentiable and with a Hölder continuous first derivative. This was a generalization of works by Zielinski who previously had obtained sharp Weyl laws in high energy asymptotics in [23–26]. The results by Ivrii were generalized by Bronstein and Ivrii in [2], where they reduced the assumptions further by assuming the first derivative to have a modulus of continuity $\mathcal{O}(|\log(x-y)|^{-1})$, and then again by Ivrii in [11] to also include boundaries and removing the non-critical condition. The non-critical condition, used in cases without full regularity, for a semiclassical pseudo-differential operator $\operatorname{Op}_\hbar^{\mathrm{w}}(a)$ is

$$|\nabla_p a(x, p)| \geq c > 0 \quad \text{for all } (x, p) \in a^{-1}(\{0\}). \tag{1.5}$$

In [27], Zielinski considers the semiclassical setting with differential operators acting in $L^2(\mathbb{R}^d)$ and proves an optimal Weyl Law under the assumption that all coefficients are one-time differentiable with a Hölder continuous derivative. Moreover, it is assumed that the coefficients and the derivatives are bounded. In [27], it is remarked that it should be possible to consider unbounded coefficients in a framework of tempered variation models. This was generalized by the author in [17] to allow for the coefficients to be unbounded. Moreover, more general operators were also considered in [17]. Both of these works assumed a non-critical condition (1.5). This assumption makes the results of [17, 27] not valid for Schrödinger operators. Since the assumption is equivalent to assuming that

$$|V(x)| \geq c > 0 \quad \text{for all } x \in \mathbb{R}^d,$$

Zielinski managed to establish sharp Weyl laws without assuming a non-critical condition in [28] for Schrödinger operators with a bounded potential of class $C^{2,\mu}(\mathbb{R}^d)$ for $\mu > 0$ and $d \geq 3$. This result does not use the multiscale approach as we do here, but requires an additional geometric condition. Zielinski's approach seems to be favorable if higher-order differential operators are considered.

The author recently established sharp spectral asymptotics for operators that locally behave like a magnetic Schrödinger operators in [18]. The techniques used to establish those will be crucial for the results obtained here. It should be remarked that the assumptions we make on regularity here are "lower" than the regularity assumptions made in [18].

The results obtained by Bronstein and Ivrii [2] and Ivrii [10, 11] do assume less regularity than we do in the present work. However, the techniques used in these works do not seem to translate well to a non-compact setting.

1.2. Non-sharp asymptotics

The methods we use to establish Theorem 1.5 can also be used in cases where we have less regularity than we assume in the statement of the theorem. However, if we assume less regularity, we cannot obtain sharp remainder estimates. The results we can obtain are in the following two theorems.

Theorem 1.6. *Let $H_\hbar = -\hbar^2\Delta + V$ be a Schrödinger operator acting in $L^2(\mathbb{R}^d)$ with $d \geq 3$. Suppose that V satisfies Assumption 1.4 with $\nu > 0$, $k = 1$, and $0 \leq \mu \leq 1$. Then, it holds that*

$$\Big|\mathrm{Tr}[g_0(H_\hbar)] - \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^{2d}} g_0(p^2 + V(x))\, dx\, dp\Big| \leq C\hbar^{\kappa-d}$$

for all $\hbar$ sufficiently small, where $\kappa = \min[2(1+\mu)/3, 1]$. The constant C depends on ν and the potential V.

One can see that, for $\mu \geq 1/2$, we are in the setting of Theorem 1.5 and recover the sharp estimate. For the cases $\mu < 1/2$, we cannot currently get the optimal error. However, the "worst" error we can obtain is $\hbar^{2/3-d}$. This is still a significant improvement of the estimate $\hbar^{-d}$. Moreover, since a global Lipschitz function is almost everywhere differentiable, with these methods we obtain the error $\hbar^{2/3-d}$ when the potential V satisfies Assumption 1.4 with $\nu > 0$, $k = 0$, and $\mu = 1$. The author believes that this case should also have sharp estimates.

Theorem 1.7. *Let $H_\hbar = -\hbar^2\Delta + V$ be a Schrödinger operator acting in $L^2(\mathbb{R}^d)$ with $d \geq 4$ and let $\gamma \in (0, 1]$. Suppose that V satisfies Assumption 1.4 with $\nu > 0$, $k = 2$, and $0 \leq \mu \leq 1$. Then, it holds that*

$$\Big|\mathrm{Tr}[g_\gamma(H_\hbar)] - \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^{2d}} g_\gamma(p^2 + V(x))\, dx\, dp\Big| \leq C\hbar^{\kappa-d}$$

for all $\hbar$ sufficiently small where $\kappa = \min[2(2+\mu)/3, 1+\gamma]$. The constant C depends on ν and the potential V.

Again, we have that for $\mu \geq \min(3\gamma/2 - 1/2, 0)$ we again recover the sharp estimates from Theorem 1.5. Considering the result obtained here, we find that for the case $\gamma = 1$ and $\mu = 1$, the error terms are sharp. Even under a C^2 assumption, we obtain an error of the form $\hbar^{4/3-d}$.

We can also obtain results for $d = 2$ for the counting function and $d = 2$ and $d = 3$ for the Riesz means. These will also not be sharp. The following two theorems establish them.

Theorem 1.8. *Let $H_\hbar = -\hbar^2\Delta + V$ be a Schrödinger operator acting in $L^2(\mathbb{R}^2)$. Suppose that V satisfies Assumption 1.4 with $\nu > 0$, $k = 1$, and $0 \le \mu \le 1$. Then, it holds that*

$$\Big|\mathrm{Tr}[g_0(H_\hbar)] - \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^{2d}} g_0(p^2 + V(x))\,dx\,dp\Big| \le C\hbar^{\kappa-2}$$

for all $\hbar$ sufficiently small, where $\kappa = \min[(1+2\mu)/3, 2/3]$. The constant C depends on ν and the potential V.

Theorem 1.9. *Let $H_\hbar = -\hbar^2\Delta + V$ be a Schrödinger operator acting in $L^2(\mathbb{R}^d)$ with $d = 2$ or $d = 3$ and let $\gamma \in (0, 1]$. Suppose that V satisfies Assumption 1.4 with $\nu > 0$, $k = 2$, and $0 \le \mu \le 1$. Then, it holds that*

$$\Big|\mathrm{Tr}[g_\gamma(H_\hbar)] - \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^{2d}} g_\gamma(p^2 + V(x))\,dx\,dp\Big| \le C\hbar^{\kappa-d}$$

for all $\hbar$ sufficiently small, where $\kappa = \min[(1+2\mu+d-\gamma)/3, (d+2\gamma)/3]$. The constant C depends on ν and the potential V.

1.3. Organisation of the paper

This paper is structured as follows. In Section 2, we specify our notation and construct approximating and framing operators. Inspired by these framing operators, we define operators that locally behave like rough Schrödinger operators in Section 3. For these operators, we establish a sharp Weyl law at the end of the section. This result relies heavily on the findings of [18]. In Section 4, we first establish a result concerning trace localizations and a comparison of phase-space integrals. We conclude the section with a proof of the main theorems.

2. Preliminaries

For an operator A acting in a Hilbert space $\mathcal{H}$, we denote the operator norm by $\|A\|_{\mathrm{op}}$ and the trace norm by $\|A\|_1$. Moreover, in the following, we will use the convention that $\mathbb{N}$ is the set of the strictly positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Next, we will describe the operators we are working with. Under Assumption 1.4, we can define the operator

$$H_\hbar = -\hbar^2\Delta + V$$

as the Friedrichs extension of the quadratic form is given by

$$\mathfrak{h}[f,g] = \int_{\mathbb{R}^d} \hbar^2 \sum_{i=1}^{d} \partial_{x_i} f(x)\overline{\partial_{x_i} g(x)} + V(x) f(x)\overline{g(x)}\, dx, \qquad f,g \in \mathcal{D}(\mathfrak{h}),$$

where

$$\mathcal{D}(\mathfrak{h}) = \Big\{ f \in L^2(\mathbb{R}^d) \,\Big|\, \int_{\mathbb{R}^d} |p|^2 |\hat{f}(p)|^2 \, dp < \infty \text{ and } \int_{\mathbb{R}^d} |V(x)||f(x)|^2 \, dx < \infty \Big\}.$$

In this set up, the Friedrichs extension will be unique and self-adjoint; see e.g., [20]. In our analysis, we will use the Helffer–Sjöstrand formula. Before we state it, we will recall a definition of an almost analytic extension.

Definition 2.1 (Almost analytic exstension). For $f \in C_0^\infty(\mathbb{R})$, we call a function $\tilde{f} \in C_0^\infty(\mathbb{C})$ an *almost analytic extension* if it has the properties

$$|\bar{\partial}\tilde{f}(z)| \le C_n |\operatorname{Im}(z)|^n \quad \text{for all } n \in \mathbb{N}_0,$$
$$\tilde{f}(t) = f(t) \quad \text{for all } t \in \mathbb{R},$$

where $\bar{\partial} = (\partial_x + i\partial_y)/2$.

For a construct of the almost analytic extension of a given $f \in C_0^\infty(\mathbb{R})$, see, e.g., [4, 29]. The following theorem is a simplified version of a theorem in [3].

Theorem 2.2 (Helffer–Sjöstrand formula). *Let H be a self-adjoint operator acting on a Hilbert space $\mathcal{H}$ and f be a function from $C_0^\infty(\mathbb{R})$. Then, the bounded operator $f(H)$ is given by the equation*

$$f(H) = -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)(z-H)^{-1}\, L(dz),$$

where $L(dz) = dx\,dy$ is the Lebesgue measure on $\mathbb{C}$ and $\tilde{f}$ is an almost analytic extension of f.

2.1. Construction of framing operators and auxiliary asymptotics

The crucial part in this construction is Proposition 2.3, for which a proof can be found in either [2, Proposition 1.1] or [12, Proposition 4.A.2].

Proposition 2.3. *Let f be in $C^{k,\mu}(\mathbb{R}^d)$ for a μ in $[0,1]$. Then, for every $\varepsilon > 0$, there exists a function f_ε in $C^\infty(\mathbb{R}^d)$ such that*

$$\begin{aligned} |\partial_x^\alpha f_\varepsilon(x) - \partial_x^\alpha f(x)| &\le C_\alpha \varepsilon^{k+\mu-|\alpha|} \quad |\alpha| \le k, \\ |\partial_x^\alpha f_\varepsilon(x)| &\le C_\alpha \varepsilon^{k+\mu-|\alpha|} \quad |\alpha| \ge k+1, \end{aligned} \tag{2.1}$$

where C_α is independent of ε, but depends on f for all $\alpha \in \mathbb{N}_0^d$.

Lemma 2.4. *Let $H_\hbar = -\hbar^2\Delta + V$ be a Schrödinger operator acting in $L^2(\mathbb{R}^d)$ and suppose that V satisfies Assumption 1.4 with (ν, k, μ). Then, for all $\varepsilon > 0$, there exist two framing operators $H_{\hbar,\varepsilon}^\pm$ such that*

$$H_{\hbar,\varepsilon}^- \le H_\hbar \le H_{\hbar,\varepsilon}^+ \tag{2.2}$$

in the sense of quadratic forms. The operators $H_{\hbar,\varepsilon}^\pm$ are explicitly given by

$$H_{\hbar,\varepsilon}^\pm = -\hbar^2\Delta + V_\varepsilon^\pm,$$

where

$$V_\varepsilon^\pm(x) = V_\varepsilon^1(x) + V^2(x) \pm C\varepsilon^{k+\mu},$$

where the function $V_\varepsilon^1(x)$ is the smooth function from Proposition 2.3 associated to $V^1 = V\varphi$ and $V^2 = V(1-\varphi)$. The function φ is chosen such that $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\varphi(x) = 1$ for all $x \in \Omega_{3\nu,V}$ and $\operatorname{supp}(\varphi) \subset \Omega_{4\nu,V}$. Moreover, for all $\varepsilon > 0$ sufficiently small, there exists a $\tilde{\nu} > 0$ such that

$$\Omega_{4\tilde{\nu},V_\varepsilon^+} \cap \operatorname{supp}(V^2) = \emptyset \quad \textit{and} \quad \Omega_{4\tilde{\nu},V_\varepsilon^-} \cap \operatorname{supp}(V^2) = \emptyset. \tag{2.3}$$

Proof. Let φ be as in the statement of the lemma, and set

$$V^1 = V\varphi \quad \text{and} \quad V^2 = V(1-\varphi).$$

By assumption, we have that $V^1 \in C_0^{k,\mu}(\mathbb{R}^d)$. Hence, for all $\varepsilon > 0$, we get from Proposition 2.3 the existence of $V_\varepsilon^1(x)$ such that (2.1) is satisfied with f replaced by V^1. We now let

$$H_{\hbar,\varepsilon} = -\hbar^2\Delta + V_\varepsilon^1 + V^2.$$

This operator is well defined and self-adjoint since both potentials are in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Moreover, we have that $H_{\hbar,\varepsilon}$ and $H_\hbar$ will have the same domains. Let $f \in \mathcal{D}[H_\hbar]$; we then have that

$$\begin{aligned} |\langle H_\hbar f, f\rangle - \langle H_{\hbar,\varepsilon} f, f\rangle| &= |\langle (V^1 - V_\varepsilon^1) f, f\rangle| \\ &\le \|V^1 - V_\varepsilon^1\|_{L^\infty(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)}^2 \\ &\le c\varepsilon^{k+\mu} \|f\|_{L^2(\mathbb{R}^d)}^2. \end{aligned} \tag{2.4}$$

By choosing a sufficiently large constant C, we obtain from (2.4) that, by defining $H^{\pm}_{\hbar,\varepsilon} = -\hbar^2\Delta + V^{\pm}_{\varepsilon}$ with $V^{\pm}_{\varepsilon}(x) = V^1_{\varepsilon}(x) + V^2(x) \pm C\varepsilon^{k+\mu}$, equation (2.2) is satisfied with this choice of operators.

What remains is to establish (2.3). By construction,

$$\|V - V^{\pm}_{\varepsilon}\|_{L^\infty(\mathbb{R}^d)} \leq C\varepsilon^{k+\mu}.$$

Hence, if we choose $\tilde{\nu} \leq \nu/2$ and ε is sufficiently small, we can ensure that $\Omega_{4\tilde{\nu},V^+_{\varepsilon}} \subset \Omega_{3\nu,V}$ and $\Omega_{4\tilde{\nu},V^-_{\varepsilon}} \subset \Omega_{3\nu,V}$. Since, by construction $\operatorname{supp}(V^2) \subset \Omega^{c}_{3\nu,V}$, it follows that, with such a choice of $\tilde{\nu}$ and for ε sufficiently small, equation (2.3) holds true. This concludes the proof. ∎

Definition 2.5 (Rough Schrödinger operator). In what follows, for $\varepsilon > 0$, we will call a potential $V_\varepsilon \in C_0^\infty(\mathbb{R}^d)$ a *rough potential of regularity* $\tau \geq 0$ if

$$\sup_{x\in\mathbb{R}^d} |\partial_x^\alpha V_\varepsilon(x)| \leq C_\alpha \varepsilon^{\min(0,\tau-|\alpha|)} \quad \text{for all } \alpha \in \mathbb{N}_0^d,$$

where the constants C_α are independent of ε. Moreover, we call an operator $H_{\hbar,\varepsilon}$ a *rough Schrödinger operator of regularity* $\tau \geq 0$ if it is an operator of the form

$$H_{\hbar,\varepsilon} = -\hbar^2\Delta + V + V_\varepsilon,$$

where $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and V_ε is a rough potential of regularity τ.

Remark 2.6. Assume we are in the setting of Lemma 2.2. It follows from Theorem 1.1 that there exists a constant $C > 0$ such that

$$\operatorname{Tr}[g_\gamma(H^+_{\hbar,\varepsilon})] \leq \operatorname{Tr}[g_\gamma(H_\hbar)] \leq \operatorname{Tr}[g_\gamma(H^-_{\hbar,\varepsilon})] \leq C\hbar^{-d}$$

for $\hbar > 0$, $\varepsilon > 0$ sufficiently small. The constant C depends only on the dimension, the set $\Omega_{4\nu,V}$, and $\min(V)$. The first two inequalities follow from the min-max principle. For the third inequality, we can choose a potential $V^{\min}$ such that

$$V^{\min}(x) = \begin{cases} \min(V) - 1 & \text{if } x \in \Omega_{4\nu,V}. \\ 0 & \text{if } x \notin \Omega_{4\nu,V}. \end{cases}$$

Then, when we consider the operator

$$H^{\min}_\hbar = -\hbar^2\Delta + V^{\min},$$

defined as a Friedrichs extension of the associated form, we have that

$$H^{\min}_\hbar \leq H^-_{\hbar,\varepsilon}$$

in the sense of quadratic forms. Hence, using the min-max principle and Theorem 1.1, we obtain that

$$\begin{aligned}\operatorname{Tr}[g_\gamma(H^-_{\hbar,\varepsilon})] &\le \operatorname{Tr}[g_\gamma(H^{\min}_\hbar)]\\ &\le \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^{2d}} g_\gamma(p^2+V^{\min}(x))\,dx\,dp + \widetilde{C}\hbar^{-d} \le C\hbar^{-d},\end{aligned}$$

where the constant C only depends on the dimension, the set $\Omega_{4\nu,V}$, and $\min(V)$.

3. Auxiliary results and model problem

Inspired by the form of the framing operators, we make the following assumption, which is essentially the same as the one in [22], but with a rough potential and no magnetic field.

Assumption 3.1. Let $\mathcal{H}_{\hbar,\varepsilon}$ be an operator acting in $L^2(\mathbb{R}^d)$, where $\hbar,\varepsilon>0$. Suppose that

(i) $\mathcal{H}_{\hbar,\varepsilon}$ is self-adjoint and lower semi-bounded;

(ii) there exists an open set $\Omega\subset\mathbb{R}^d$ and a rough potential $V_\varepsilon\in C_0^\infty(\mathbb{R}^d)$ of regularity $\tau\ge 0$ such that $C_0^\infty(\Omega)\subset\mathcal{D}(\mathcal{H})$ and

$$\mathcal{H}_{\hbar,\varepsilon}\varphi = H_{\hbar,\varepsilon}\varphi \quad \text{for all } \varphi\in C_0^\infty(\Omega),$$

where $H_{\hbar,\varepsilon}=-\hbar^2\Delta+V_\varepsilon$.

For these operators, we will establish our model problem. The first auxiliary result we will need was established in [18, Lemma 4.6]. It is almost the full model problem, except that we consider only the operator $H_{\hbar,\varepsilon}$ and not the general operator $\mathcal{H}_{\hbar,\varepsilon}$.

Lemma 3.2. *Let $\gamma\in[0,1]$ and $H_{\hbar,\varepsilon}=-\hbar^2\Delta+V_\varepsilon$ be a rough Schrödinger operator acting in $L^2(\mathbb{R}^d)$ of regularity $\tau\ge 1$ if $\gamma=0$, and regularity $\tau\ge 2$ if $\gamma>0$, with $\hbar\in(0,\hbar_0]$, $\hbar_0$ sufficiently small. Assume that $V_\varepsilon\in C_0^\infty(\mathbb{R}^d)$ and there exists $\delta\in(0,1]$ such that $\varepsilon\ge\hbar^{1-\delta}$. Suppose there is an open set $\Omega\subset\operatorname{supp}(V_\varepsilon)$ and a $c>0$ such that*

$$|V_\varepsilon(x)|+\hbar^{2/3}\ge c \quad \text{for all } x\in\Omega.$$

Then, for $\varphi\in C_0^\infty(\Omega)$, it holds that

$$\Big|\operatorname{Tr}[\varphi g_\gamma(H_{\hbar,\varepsilon})]-\frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^{2d}} g_\gamma(p^2+V_\varepsilon(x))\varphi(x)\,dx\,dp\Big| \le C\hbar^{1+\gamma-d},$$

where C is a constants that depends only on the dimension and γ, $\|\partial^\alpha\varphi\|_{L^\infty(\mathbb{R}^d)}$, and $\varepsilon^{-\min(0,\tau-|\alpha|)}\|\partial^\alpha V_\varepsilon\|_{L^\infty(\Omega)}$, for all $\alpha\in N_0^d$.

In order to prove our model problem, it remains to establish that $\mathrm{Tr}[\varphi g_\gamma(H_{\hbar,\varepsilon})]$ and $\mathrm{Tr}[\varphi g_\gamma(\mathcal{H}_{\hbar,\varepsilon})]$ are close. To do this, we will need some additional notation and results.

Remark 3.3. In order to prove Lemma 3.2 as done in [18], one needs to understand the Schrödinger propagator $e^{i\hbar^{-1}tH_{\hbar,\varepsilon}}$ associated to $H_{\hbar,\varepsilon}$. Under the assumptions of the lemma, we can find an operator with an explicit kernel that locally approximates $e^{i\hbar^{-1}tH_{\hbar,\varepsilon}}$ in a suitable sense. This local construction is only valid for times of order $\hbar^{1-\delta/2}$. But if we locally have a non-critical condition, the approximation can be extended to a small time interval $[-T_0, T_0]$, where T_0 is independent of $\hbar$. For further details, see [17]. In the following, we will reference back to this remark and T_0.

Remark 3.4. Let T_0 be defined as in Remark 3.3 and T_1 as in Lemma 3.9. Let $T \in (0, \min(T_0, T_1)]$ and $\hat{\chi} \in C_0^\infty((-T, T))$ be a real valued function such that $\hat{\chi}(s) = \hat{\chi}(-s)$ and $\hat{\chi}(s) = 1$ for all $t \in (-T/2, T/2)$. Define

$$\chi_1(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\chi}(s) e^{ist}\, ds.$$

We assume that $\chi_1(t) \geq 0$ for all $t \in \mathbb{R}$ and there exist $T_2 \in (0, T)$ and $c > 0$ such that $\chi_1(t) \geq c$ for all $t \in [-T_2, T_2]$. We can guarantee these assumptions by replacing $\hat{\chi}$ with $\hat{\chi} * \hat{\chi}$ if necessary. We will denote by $\chi_\hbar(t)$ the function

$$\chi_\hbar(t) = \frac{1}{\hbar}\chi_1\Big(\frac{t}{\hbar}\Big).$$

Moreover, for any function $g \in L^1_{\mathrm{loc}}(\mathbb{R})$, we will use the notation

$$g^{(\hbar)}(t) = g * \chi_\hbar(t) = \int_{\mathbb{R}} g(s)\chi_\hbar(t-s).$$

Before we proceed, we recall the following classes of functions. These were first introduced in [21].

Definition 3.5. A function $g \in C^\infty(\mathbb{R} \setminus \{0\})$ is said to *belong to the class* $C^{\infty,\gamma}(\mathbb{R})$, $\gamma \in [0, 1]$, if $g \in C(\mathbb{R})$ for $\gamma > 0$ and if there exist constants $C > 0$ and $r > 0$ such that the following holds:

$$\begin{aligned}
&g(t) = 0 && \text{for all } t \geq C,\\
&|\partial_t^m g(t)| \leq C_m |t|^r && \text{for all } m \in \mathbb{N}_0 \text{ and } t \leq -C,\\
&|\partial_t^m g(t)| \leq \begin{cases} C_m & \text{if } \gamma = 0, 1,\\ C_m |t|^{\gamma - m} & \text{if } \gamma \in (0,1), \end{cases} && \text{for all } m \in \mathbb{N} \text{ and } t \in [-C, C] \setminus \{0\}.
\end{aligned}$$

A function g is said to *belong to* $C_0^{\infty,\gamma}(\mathbb{R})$ if $g \in C^{\infty,\gamma}(\mathbb{R})$ and g has compact support.

With this notation established, we recall the following Tauberian-type result. This result can be found in [22, Proposition 2.8].

Proposition 3.6. *Let A be a self-adjoint operator acting in a Hilbert space $\mathcal{H}$ and $g \in C_0^{\infty,\gamma}(\mathbb{R})$. Let χ_1 be defined as in Remark 3.4. If for a Hilbert–Schmidt operator B,*

$$\sup_{t \in \mathcal{D}(\delta)} \|B^* \chi_\hbar(A-t)B\|_1 \leq Z(\hbar),$$

(where $\mathcal{D}(\delta) = \{t \in \mathbb{R} \mid \operatorname{dist}(\operatorname{supp}(g)), t) \leq \delta\}$, $Z(\beta)$ is some positive function, and δ is strictly positive), then it holds that

$$\|B^*(g(A) - g^{(\hbar)}(A))B\|_1 \leq C\hbar^{1+\gamma} Z(\hbar) + C_N' \hbar^N \|B^* B\|_1 \quad \textit{for all } N \in \mathbb{N},$$

where the constants C and C' only depend on δ and the functions g and χ_1.

Lemma 3.7. *Let $\mathcal{H}_{\hbar,\varepsilon}$ be an operator acting in $L^2(\mathbb{R}^d)$ which satisfies Assumption 3.1 with the open set Ω and let $H_{\hbar,\varepsilon} = -\hbar^2\Delta + V_\varepsilon$ be the associated rough Schrödinger operator of regularity $\tau \geq 1$. Assume that $\hbar \in (0, \hbar_0]$, with $\hbar_0$ sufficiently small. Then, for $f \in C_0^\infty(\mathbb{R})$ and $\varphi \in C_0^\infty(\Omega)$, we have for any $N \in \mathbb{N}_0$ that*

$$\|\varphi[f(\mathcal{H}_{\hbar,\varepsilon}) - f(H_{\hbar,\varepsilon})]\|_1 \leq C_N \hbar^N, \tag{3.1}$$

$$\|\varphi[(z - \mathcal{H}_{\hbar,\varepsilon})^{-1} - (z - H_{\hbar,\varepsilon})^{-1}]\|_1 \leq C_N \frac{\langle z \rangle^{N+\frac{d+1}{2}} \hbar^{2N-d}}{|\operatorname{Im}(z)|^{2N+2}}, \quad z \in \mathbb{C} \setminus \mathbb{R} \tag{3.2}$$

and

$$\|\varphi f(\mathcal{H}_{\hbar,\varepsilon})\|_1 \leq C\hbar^{-d}, \tag{3.3}$$

The constant C_N depends on $\operatorname{supp}(f)$, N, and $\|\partial^\alpha f\|_{L^\infty(\mathbb{R})}$ and $\|\partial^\alpha \varphi\|_{L^\infty(\mathbb{R}^d)}$, for all $\alpha \in \mathbb{N}_0^d$.

Proof. Estimates (3.1) and (3.2) follow as in the proof of [18, Lemma 4.3], while estimate (3.3) follows similarly to [18, (4.8)]. ■

Lemma 3.8. *Let $H_{\hbar,\varepsilon} = -\hbar^2\Delta + V_\varepsilon$ be a rough Schrödinger operator acting in $L^2(\mathbb{R}^d)$ of regularity $\tau \geq 1$ with $\hbar \in (0, \hbar_0]$, $\hbar_0$ sufficiently small. Assume that $V_\varepsilon \in C_0^\infty(\mathbb{R}^d)$ and there exists $\delta \in (0, 1]$ such that $\varepsilon \geq \hbar^{1-\delta}$. Suppose there is an open set $\Omega \subset \operatorname{supp}(V_\varepsilon)$ and $c > 0$ such that*

$$|V_\varepsilon(x)| + \hbar^{2/3} \geq c \quad \textit{for all } x \in \Omega.$$

Let $\chi_\hbar(t)$ be the function from Remark 3.4, $f \in C_0^\infty(\mathbb{R})$ and $\varphi \in C_0^\infty(\Omega)$, then it holds for $s \in \mathbb{R}$ that

$$\|\varphi f(H_{\hbar,\varepsilon}) \chi_\hbar(H_{\hbar,\varepsilon} - s) f(H_{\hbar,\varepsilon})\varphi\|_1 \leq C\hbar^{-d}.$$

The constant C depends only on the dimension, $\mathrm{supp}(f)$, $\|f\|_{L^\infty(\mathbb{R})}$, $\|\partial^\alpha \varphi\|_{L^\infty(\mathbb{R})}$ *for all* $\alpha \in \mathbb{N}_0^d$*, and* $\varepsilon^{-\min(0,\tau-|\alpha|)}\|\partial^\alpha V_\varepsilon\|_{L^\infty(\Omega)}$ *for all* $\alpha \in \mathbb{N}_0^d$*.*

Proof. The proof is analogous to that of [18, Lemma 4.5]. Note that in the proof of [18, Lemma 4.5] it is assumed the operator is of regularity $\tau \geq 2$. However, it suffices to assume that the operator is of regularity $\tau \geq 1$, without changing the proof. ■

The next lemma is a result on the propagation of singularities, which we will need to obtain certain estimates later. We will not give the full proof of the lemma here as it is almost identical to the proof of [18, Lemma 4.7]. There is also a version of this result in [21, Lemma 5.1]. This result is stated and proven for semiclassical pseudo-differential operators.

Lemma 3.9. *Let* $H_{\hbar,\varepsilon} = -\hbar^2\Delta + V_\varepsilon$ *be a rough Schrödinger operator of regularity* $\tau \geq 1$ *acting in* $L^2(\mathbb{R}^d)$*. Let* $\theta_1 \in C_0^\infty(\mathbb{R}^{2d})$ *and* $\theta_2 \in \mathcal{B}^\infty(\mathbb{R}^{2d})$ *such that*

$$\mathrm{dist}\{\mathrm{supp}(\theta_1), \mathrm{supp}(\theta_2)\} \geq c > 0,$$

and let $f \in C_0^\infty(\mathbb{R})$*. Then, there exists* $T_1 > 0$ *sufficiently small such that, for any* $N \in \mathbb{N}$*, it holds that*

$$\|\mathrm{Op}_\hbar^{\mathrm{w}}(\theta_2)e^{it\hbar^{-1}H_{\hbar,\mu}} f(H_{\hbar,\mu})\,\mathrm{Op}_\hbar^{\mathrm{w}}(\theta_1)\|_{\mathrm{op}} \leq C\hbar^N,$$

uniformly for $t \in [-T_1, T_1]$*. The constant C depends on* $\mathrm{supp}(f)$*, the dimension, N,* $\|\partial^\alpha f\|_{L^\infty(\mathbb{R}^d)}$, $\|\partial^\alpha \theta_1\|_{L^\infty(\mathbb{R}^d)}$, $\|\partial^\alpha \theta_2\|_{L^\infty(\mathbb{R}^d)}$ *for all* $\alpha \in \mathbb{N}_0^d$*, and the constant c.*

Proof. The proof follows the argument in [18, Lemma 4.7]. The main difference between the two cases is that here we consider an operator that is already a rough pseudo-differential operator. Hence, the step where we exchange operators in the proof of [18, Lemma 4.7] can be omitted. This omission applies from [18, equation (4.27)] up to the first full stop after [18, equation (4.28)]. Once this step is removed, the remaining changes involve choosing the constants large enough to ensure that an error of order $\hbar^N$ is obtained, rather than just $\hbar^{3+\gamma}$. ■

Lemma 3.10. *Let* $\mathcal{H}_{\hbar,\varepsilon}$ *be an operator acting in* $L^2(\mathbb{R}^d)$ *satisfying Assumption 3.1 with the open set* Ω *and let* $H_{\hbar,\varepsilon} = -\hbar^2\Delta + V_\varepsilon$ *be the associated rough Schrödinger operator of regularity* $\tau \geq 1$*. Assume that* $\hbar \in (0, \hbar_0]$*, with* $\hbar_0$ *sufficiently small, and that there exists* $\delta \in (0, 1]$ *such that* $\varepsilon \geq \hbar^{1-\delta}$ *Let* $\chi_\hbar(t)$ *be the function from Remark 3.4,* $f \in C_0^\infty(\mathbb{R})$*, and* $\varphi \in C_0^\infty(\Omega)$*. For* $s \in \mathbb{R}$ *and* $N \in \mathbb{N}$*, it holds that*

$$\|\varphi f(\mathcal{H}_{\hbar,\varepsilon})\chi_\hbar(\mathcal{H}_{\hbar,\varepsilon} - s) f(\mathcal{H}_{\hbar,\varepsilon})\varphi - \varphi f(H_{\hbar,\varepsilon})\chi_\hbar(H_{\hbar,\varepsilon} - s) f(H_{\hbar,\varepsilon})\varphi\|_1 \leq C_N\hbar^N. \tag{3.4}$$

Moreover, suppose there exists $c > 0$ such that

$$|V_\varepsilon(x)| + \hbar^{2/3} \geq c \quad \text{for all } x \in \Omega.$$

Then,

$$\|\varphi f(\mathcal{H}_{\hbar,\varepsilon})\chi_\hbar(\mathcal{H}_{\hbar,\varepsilon} - s) f(\mathcal{H}_{\hbar,\varepsilon})\varphi\|_1 \leq C\hbar^{-d}. \tag{3.5}$$

The constants C_N and C depend on the dimension and $\|\partial^\alpha f\|_{L^\infty(\mathbb{R})}$, $\|\partial^\alpha \varphi\|_{L^\infty(\mathbb{R})}$, and $\varepsilon^{-\min(0,\tau-|\alpha|)}\|\partial^\alpha V_\varepsilon\|_{L^\infty(\Omega)}$ for all $\alpha \in \mathbb{N}_0^d$.

Proof. We start by considering the left-hand side of (3.4). We have that

$$\begin{aligned}
&\|\varphi f(\mathcal{H}_{\hbar,\varepsilon})\chi_\hbar(\mathcal{H}_{\hbar,\varepsilon} - s) f(\mathcal{H}_{\hbar,\varepsilon})\varphi - \varphi f(H_{\hbar,\varepsilon})\chi_\hbar(H_{\hbar,\varepsilon} - s) f(H_{\hbar,\varepsilon})\varphi\|_1 \\
&\quad \leq \|\varphi f(\mathcal{H}_{\hbar,\varepsilon})\chi_\hbar(\mathcal{H}_{\hbar,\varepsilon} - s) f(H_{\hbar,\varepsilon})\varphi - \varphi f(\mathcal{H}_{\hbar,\varepsilon})\chi_\hbar(H_{\hbar,\varepsilon} - s) f(H_{\hbar,\varepsilon})\varphi\|_1 \\
&\qquad + C\hbar^{-1}\|\varphi f(\mathcal{H}_{\hbar,\varepsilon}) - \varphi f(H_{\hbar,\varepsilon})\|_1 + C\hbar^{-1}\|f(\mathcal{H}_{\hbar,\varepsilon})\varphi - f(H_{\hbar,\varepsilon})\varphi\|_1 \\
&\quad \leq \|\varphi f(\mathcal{H}_{\hbar,\varepsilon})\chi_\hbar(\mathcal{H}_{\hbar,\varepsilon} - s) f(H_{\hbar,\varepsilon})\varphi - \varphi f(\mathcal{H}_{\hbar,\varepsilon})\chi_\hbar(H_{\hbar,\varepsilon} - s) f(H_{\hbar,\varepsilon})\varphi\|_1 \\
&\qquad + C\hbar^N,
\end{aligned} \tag{3.6}$$

where in the first inequality we have added and subtracted the two terms

$$\varphi f(\mathcal{H}_{\hbar,\varepsilon})\chi_\hbar(H_{\hbar,\varepsilon} - s) f(H_{\hbar,\varepsilon})\varphi \quad \text{and} \quad \varphi f(\mathcal{H}_{\hbar,\varepsilon})\chi_\hbar(\mathcal{H}_{\hbar,\varepsilon} - s) f(H_{\hbar,\varepsilon})\varphi,$$

used the triangle inequality and that the function $\chi_\hbar(t)$ is bounded by $C\hbar^{-1}$ uniformly in t. In the second inequality, we have used Lemma 3.7. We observe that, using how we defined the function $\chi_\hbar(z - s)$, we have that

$$\chi_\hbar(z - s) = \mathcal{F}_\hbar^{-1}[\chi](z - s) = \frac{1}{2\pi\hbar}\int_{\mathbb{R}} e^{i\hbar^{-1}t(z-s)}\chi(t)\,dt.$$

Using this expression and the fundamental theorem of calculus, we get that

$$\begin{aligned}
&\chi_\hbar(\mathcal{H}_{\hbar,\varepsilon} - s) - \chi_\hbar(H_{\hbar,\varepsilon} - s) \\
&\quad = \frac{1}{2\pi\hbar}\int_{\mathbb{R}} (e^{i\hbar^{-1}t(\mathcal{H}_{\hbar,\varepsilon}-s)} - e^{i\hbar^{-1}t(H_{\hbar,\varepsilon}-s)})\chi(t)\,dt \\
&\quad = \frac{i}{2\pi\hbar^2}\int_{\mathbb{R}} e^{-i\hbar^{-1}ts}\chi(t)\int_0^t e^{i\hbar^{-1}\tau\mathcal{H}_{\hbar,\varepsilon}}(\mathcal{H}_{\hbar,\varepsilon} - H_{\hbar,\varepsilon})e^{i\hbar^{-1}(t-\tau)H_{\hbar,\varepsilon}}\,d\tau dt.
\end{aligned} \tag{3.7}$$

Letting $\tilde{f} \in C_0^\infty(\mathbb{R})$ such that

$$\tilde{f}(t) f(t) = f(t) \quad \text{for all } t \in \mathbb{R}$$

and using the identity obtained in (3.7), we get that

$$\begin{aligned}
&\|\varphi f(\mathcal{H}_{\hbar,\varepsilon})\chi_\hbar(\mathcal{H}_{\hbar,\varepsilon}-s)f(H_{\hbar,\varepsilon})\varphi-\varphi f(\mathcal{H}_{\hbar,\varepsilon})\chi_\hbar(H_{\hbar,\varepsilon}-s)f(H_{\hbar,\varepsilon})\varphi\|_1\\
&\quad\leq\frac{1}{2\pi\hbar^2}\int_{\mathbb{R}}\chi(t)\int_0^t\|\varphi f(\mathcal{H}_{\hbar,\varepsilon})e^{i\hbar^{-1}t\mathcal{H}_{\hbar,\varepsilon}}\tilde{f}_1(\mathcal{H}_{\hbar,\varepsilon})(\mathcal{H}_{\hbar,\varepsilon}-H_{\hbar,\varepsilon})\\
&\qquad\qquad\times\tilde{f}(H_{\hbar,\varepsilon})e^{i\hbar^{-1}(t-\tau)H_{\hbar,\varepsilon}}f(H_{\hbar,\varepsilon})\varphi\|_1\,d\tau dt. \qquad (3.8)
\end{aligned}$$

We let $\theta,\tilde{\theta}\in C_0^\infty(\Omega\times B(0,K+1))$ such that

$$\operatorname{supp}(\varphi)\cap\operatorname{supp}(1-\theta)\cap\operatorname{supp}(f(a^f_{\varepsilon,0}))=\emptyset$$

and

$$\operatorname{dist}(\operatorname{supp}(\theta),\operatorname{supp}(1-\tilde{\theta}))\geq c,$$

where $c>0$ is some positive constant. With these functions and by using standard pseudo-differential techniques, we have for any $N\in\mathbb{N}$ that

$$\begin{aligned}
&\|\varphi f(\mathcal{H}_{\hbar,\varepsilon})e^{i\hbar^{-1}t\mathcal{H}_{\hbar,\varepsilon}}\tilde{f}_1(\mathcal{H}_{\hbar,\varepsilon})(\mathcal{H}_{\hbar,\varepsilon}-H_{\hbar,\varepsilon})\tilde{f}(H_{\hbar,\varepsilon})e^{i\hbar^{-1}(t-\tau)H_{\hbar,\varepsilon}}f(H_{\hbar,\varepsilon})\varphi\|_1\\
&\quad\leq C\hbar^{-d}[\|\tilde{f}_1(\mathcal{H}_{\hbar,\varepsilon})(\mathcal{H}_{\hbar,\varepsilon}-H_{\hbar,\varepsilon})\tilde{f}(H_{\hbar,\varepsilon})\operatorname{Op}_\hbar^{\mathrm{w}}(\tilde{\theta})\|_{\mathrm{op}}\\
&\qquad+\|\operatorname{Op}_\hbar^{\mathrm{w}}(1-\tilde{\theta})e^{i\hbar^{-1}(t-\tau)H_{\hbar,\varepsilon}}f(H_{\hbar,\varepsilon})\operatorname{Op}_\hbar^{\mathrm{w}}(\theta)\varphi\|_{\mathrm{op}}]+C_N\hbar^N, \qquad (3.9)
\end{aligned}$$

where the constants C and C_N depend on the variables as stated in the Lemma. Using the assumptions on our operators and standard pseudo-differential techniques, we have for any $N\in\mathbb{N}$ that

$$\|\tilde{f}_1(\mathcal{H}_{\hbar,\varepsilon})(\mathcal{H}_{\hbar,\varepsilon}-H_{\hbar,\varepsilon})\tilde{f}(H_{\hbar,\varepsilon})\operatorname{Op}_\hbar^{\mathrm{w}}(\tilde{\theta})\|_{\mathrm{op}}\leq C_N\hbar^N,$$

where again the constant C_N depends on the variables as stated in the Lemma. We can also bound the last term on the right-hand side of (3.9). This follows from Lemma 3.9. Hence, we have for any $N\in\mathbb{N}$ that

$$\begin{aligned}
&\|\varphi f(\mathcal{H}_{\hbar,\varepsilon})e^{i\hbar^{-1}t\mathcal{H}_{\hbar,\varepsilon}}\tilde{f}_1(\mathcal{H}_{\hbar,\varepsilon})(\mathcal{H}_{\hbar,\varepsilon}-H_{\hbar,\varepsilon})\tilde{f}(H_{\hbar,\varepsilon})e^{i\hbar^{-1}(t-\tau)H_{\hbar,\varepsilon}}f(H_{\hbar,\varepsilon})\varphi\|_1\\
&\quad\leq C_N\hbar^N, \qquad (3.10)
\end{aligned}$$

where again the constant C_N depends on the variables as stated in the lemma. Combining estimates (3.6), (3.8), and (3.10), we have established estimate (3.4). Combining this estimate with Lemma 3.8 gives us (3.5). This concludes the proof. ■

Lemma 3.11. *Let $\gamma\in[0,1]$ and $\mathcal{H}_{\hbar,\varepsilon}$ be an operator acting in $L^2(\mathbb{R}^d)$. Suppose $\mathcal{H}_{\hbar,\varepsilon}$ satisfies Assumption 3.1 with the open set Ω and let $H_{\hbar,\varepsilon}=-\hbar^2\Delta+V_\varepsilon$ be the associated rough Schrödinger operator of regularity $\tau\geq1$ if $\gamma=0$ and $\tau\geq2$ if $\gamma>0$.*

Assume that $\hbar \in (0, \hbar_0]$*, with* $\hbar_0$ *sufficiently small, and that there exists* $\delta \in (0, 1]$ *such that* $\varepsilon \geq \hbar^{1-\delta}$*. Suppose that there exists* $c > 0$ *such that*

$$|V_\varepsilon(x)| + \hbar^{2/3} \geq c \quad \textit{for all } x \in \Omega$$

Then, for $\varphi \in C_0^\infty(\Omega)$*, it holds that*

$$|\operatorname{Tr}[\varphi g_\gamma(\mathcal{H}_{\hbar,\varepsilon})] - \operatorname{Tr}[\varphi g_\gamma(H_{\hbar,\varepsilon})]| \leq C\hbar^{1+\gamma-d} + C_N'\hbar^N.$$

The constants C *and* C_N' *depend on the dimension, as well as on the parameters* γ*,* $\|\partial^\alpha \varphi\|_{L^\infty(\mathbb{R})}$*, and* $\varepsilon^{-\min(0,\tau-|\alpha|)}\|\partial^\alpha V_\varepsilon\|_{L^\infty(\Omega)}$ *for all* $\alpha \in \mathbb{N}_0^d$*.*

Proof. Since both operators are lower semi-bounded, we may assume that g is compactly supported. Let $f \in C_0^\infty(\mathbb{R})$ such that $f(t)g_\gamma(t) = g_\gamma(t)$ for all $t \in \mathbb{R}$ and let $\varphi_1 \in C_0^\infty(\Omega)$ such that $\varphi(x)\varphi_1(x) = \varphi(x)$ for all $x \in \mathbb{R}^d$. Moreover, let $\chi_\hbar(t)$ be the function from Remark 3.4 and set $g_\gamma^{(\hbar)}(t) = g_\gamma * \chi_\hbar(t)$. With this notation, we have that

$$\begin{aligned}
&|\operatorname{Tr}[\varphi g_\gamma(\mathcal{H}_{\hbar,\varepsilon})] - \operatorname{Tr}[\varphi g_\gamma(H_{\hbar,\varepsilon})]| \\
&\quad \leq \|\varphi\varphi_1 f(\mathcal{H}_{\hbar,\varepsilon})(g_\gamma(\mathcal{H}_{\hbar,\varepsilon}) - g_\gamma^{(\hbar)}(\mathcal{H}_{\hbar,\varepsilon}))f(\mathcal{H}_{\hbar,\varepsilon})\varphi_1\|_1 \\
&\qquad + \|\varphi\varphi_1 f(H_{\hbar,\varepsilon})(g_\gamma(H_{\hbar,\varepsilon}) - g_\gamma^{(\hbar)}(H_{\hbar,\varepsilon}))f(H_{\hbar,\varepsilon})\varphi_1\|_1 \\
&\qquad + \|\varphi\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}} g_\gamma(s)\, ds \sup_{s\in\mathbb{R}} \|\varphi\varphi_1 f(\mathcal{H}_{\hbar,\varepsilon})\chi_\hbar(\mathcal{H}_{\hbar,\varepsilon} - s)f(\mathcal{H}_{\hbar,\varepsilon})\varphi_1 \\
&\qquad\qquad - \varphi_1 f(H_{\hbar,\varepsilon})\chi_\hbar(H_{\hbar,\varepsilon} - s)f(H_{\hbar,\varepsilon})\varphi_1\|_1.
\end{aligned} \tag{3.11}$$

Lemma 3.8 and Lemma 3.10 give us that the assumptions of Proposition 3.6 are fulfilled with B equal to $\varphi_1 f(H_\hbar)$ and $\varphi_1 f(H_{\hbar,\varepsilon})$ respectively. Hence, we have that

$$\|\varphi\varphi_1 f(\mathcal{H}_{\hbar,\varepsilon})(g_\gamma(\mathcal{H}_{\hbar,\varepsilon}) - g_\gamma^{(\hbar)}(\mathcal{H}_{\hbar,\varepsilon}))f(\mathcal{H}_{\hbar,\varepsilon})\varphi_1\|_1 \leq C\hbar^{1+\gamma-d} \tag{3.12}$$

and

$$\|\varphi\varphi_1 f(H_{\hbar,\varepsilon})(g_\gamma(H_{\hbar,\varepsilon}) - g_\gamma^{(\hbar)}(H_{\hbar,\varepsilon}))f(H_{\hbar,\varepsilon})\varphi_1\|_1 \leq C\hbar^{1+\gamma-d}. \tag{3.13}$$

By applying Lemma 3.10, we get for all $N \in \mathbb{N}$ that

$$\begin{aligned}
&\sup_{s\in\mathbb{R}} \|\varphi\varphi_1 f(\mathcal{H}_\hbar)\chi_\hbar(\mathcal{H}_\hbar - s)f(\mathcal{H}_\hbar)\varphi_1 - \varphi_1 f(H_{\hbar,\varepsilon})\chi_\hbar(H_{\hbar,\varepsilon} - s)f(H_{\hbar,\varepsilon})\varphi_1\|_1 \\
&\quad \leq C_N\hbar^N.
\end{aligned} \tag{3.14}$$

Finally, by combining estimates (3.11)–(3.14), we obtain the desired estimate. ■

For operators that satisfy Assumption 3.1, we can establish the following model theorem. The proof of the theorem is similar to the proof of [18, Theorem 5.2].

Theorem 3.12. *Let $\gamma \in [0,1]$ and $\mathcal{H}_{\hbar,\varepsilon}$ be an operator acting in $L^2(\mathbb{R}^d)$. Suppose that $\mathcal{H}_{\hbar,\varepsilon}$ satisfies Assumption 3.1 with the open set Ω and let $H_{\hbar,\varepsilon} = -\hbar^2\Delta + V_\varepsilon$ be the associated rough Schrödinger operator of regularity $\tau \geq 1$ if $\gamma = 0$, and $\tau \geq 2$ if $\gamma > 0$. Assume that $\hbar \in (0, \hbar_0]$, with $\hbar_0$ sufficiently small and there exists $\delta \in (0,1]$ such that $\varepsilon \geq \hbar^{1-\delta}$. Suppose there exists $c > 0$ such that*

$$|V_\varepsilon(x)| + \hbar^{2/3} \geq c \quad \text{for all } x \in \Omega. \tag{3.15}$$

Then, for any $\varphi \in C_0^\infty(\Omega)$, it holds that

$$\Big|\mathrm{Tr}[\varphi g_\gamma(\mathcal{H}_{\hbar,\varepsilon})] - \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^{2d}} g_\gamma(p^2 + V_\varepsilon(x))\varphi(x)\,dx\,dp\Big| \leq C\hbar^{1+\gamma-d},$$

where the constant C depends only on the dimension, as well as on $\|\partial^\alpha\varphi\|_{L^\infty(\mathbb{R}^d)}$ and $\varepsilon^{-\min(0,\tau-|\alpha|)}\|\partial^\alpha V_\varepsilon\|_{L^\infty(\Omega)}$ for all $\alpha \in \mathbb{N}_0^d$.

Proof. Firstly, observe that, under the assumptions of this theorem, $\mathcal{H}_{\hbar,\varepsilon}$ and $H_{\hbar,\varepsilon}$ satisfy the assumptions of Lemma 3.11. Furthermore, $H_{\hbar,\varepsilon}$ satisfies the assumptions of Lemma 3.2. Thus, applying Lemma 3.11 and Lemma 3.2, we conclude that

$$\begin{aligned}
&\Big|\mathrm{Tr}[\varphi g_\gamma(\mathcal{H}_{\hbar,\varepsilon})] - \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^{2d}} g_\gamma(p^2 + V_\varepsilon(x))\varphi(x)\,dx\,dp\Big| \\
&\quad\leq |\,\mathrm{Tr}[\varphi g_\gamma(\mathcal{H}_{\hbar,\varepsilon})] - \varphi g_\gamma(H_{\hbar,\varepsilon})| \\
&\qquad + \Big|\mathrm{Tr}[\varphi g_\gamma(H_{\hbar,\varepsilon})] - \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^{2d}} g_\gamma(p^2 + V_\varepsilon(x))\varphi(x)\,dx\,dp\Big| \\
&\quad\leq C\hbar^{1+\gamma-d}.
\end{aligned}$$

This concludes the proof. ∎

4. Towards a proof of the main theorem

At the end of this section, we will prove our main theorem. Before that, we introduce some lemmas needed for the proof. The first lemma allows us to localize the trace we consider, while the second provides a comparison of phase space integrals.

4.1. Localization of traces and comparison of phase-space integrals

Before we state the lemma on localization of the trace, we recall the following Agmon-type estimate from [5, Lemma A.1].

Lemma 4.1. *Let $H_\hbar = -\hbar^2\Delta + V$ be a Schrödinger operator acting in $L^2(\mathbb{R}^d)$, where V is in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and suppose that there exist $\nu > 0$ and an open bounded set U such that*

$$V(x) \geq \nu \quad \text{when } x \in U^{\mathrm{c}}. \tag{4.1}$$

Let $d(x) = \operatorname{dist}(x, U_a)$, where

$$U_a = \{x \in \mathbb{R}^d \mid \operatorname{dist}(x, U) < a\}$$

and let ψ be a normalized solution to the equation

$$H_\hbar \psi = E\psi,$$

with $E < \nu/4$. Then, there exists $C > 0$ such that

$$\|e^{\delta\hbar^{-1}d}\psi\|_{L^2(\mathbb{R}^d)} \leq C,$$

for $\delta = \sqrt{\nu}/8$. The constant C depends on a and is uniform in V, ν, and U satisfying (4.1).

In the formulation of the lemma presented here, we consider U_a for $a > 0$, rather than just U_1 as in [5]. There are no differences in the proof. However, one point to remark is that the constant C diverges to infinity as a tends to 0. Moreover, in the statement, we highlight the uniformity of the constant with respect to the potential V, ν, and the set U. That this constant is indeed uniform in these parameters follows directly from the proof given in [5].

Lemma 4.2. *Let $\gamma \in [0, 1]$ and $H_\hbar = -\hbar^2\Delta + V$ be a Schrödinger operator acting in $L^2(\mathbb{R}^d)$, where V is in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and suppose that there exist an $\nu > 0$ and a open bounded sets U such that $V(x)\mathbf{1}_U(x) \in L^{\gamma+d/2}(\mathbb{R}^d)$ and*

$$V(x) \geq \nu \quad \text{when } x \in U^{\mathrm{c}}. \tag{4.2}$$

Fix $a > 0$ and let $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $\varphi(x) = 1$ for all $x \in U_a$, where

$$U_a = \{x \in \mathbb{R}^d \mid \operatorname{dist}(x, U) < a\}.$$

Then for every $N \in \mathbb{N}$, it holds that

$$\operatorname{Tr}[g_\gamma(H_\hbar)] = \operatorname{Tr}[g_\gamma(H_\hbar)\varphi] + C_N\hbar^N,$$

where the constant is C_N depends on a and is uniform in V, ν, and U satisfying (4.2).

Proof. Using the linearity of the trace, we have that

$$\operatorname{Tr}[g_\gamma(H_\hbar)] = \operatorname{Tr}[g_\gamma(H_\hbar)\varphi] + \operatorname{Tr}[g_\gamma(H_\hbar)(1-\varphi)]. \tag{4.3}$$

For the second term on the right-hand side of (4.3), we calculate the trace in a normalized basis of eigenfunctions for $H_\hbar$, denoted ψ_n, with eigenvalue E_n:

$$\begin{aligned}\operatorname{Tr}[g_\gamma(H_\hbar)(1-\varphi)] &= \sum_{E_n\le 0} \langle g_\gamma(H_\hbar)(1-\varphi)\psi_n, \psi_n\rangle \\ &= \sum_{E_n\le 0} g_\gamma(E_n)\|\sqrt{1-\varphi}\psi_n\|^2_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{4.4}$$

To estimate the L^2-norms, we let $d(x) = \operatorname{dist}(x, U_{a/2})$. For all $x \in \operatorname{supp}(1-\varphi)$, we have that $d(x) > 0$, since $\varphi(x) = 1$ for all $x \in U_a$. By Lemma 4.1, there exists a constant C, depending on a, such that, for all normalised eigenfunctions ψ_n with eigenvalue less than $\nu/4$, we have the estimate

$$\|e^{\tilde{\delta}\hbar^{-1}d}\psi_n\|_{L^2(\mathbb{R}^d)} \le C,$$

where $\tilde{\delta} = \sqrt{\nu}/8$ and C is uniform in V, ν, and U satisfying (4.2). Using this estimate and the observations made for $d(x)$, we get, for all norms in (4.4) and all $N \in \mathbb{N}$, the estimate

$$\begin{aligned}\|\sqrt{1-\varphi}\psi_n\|^2_{L^2(\mathbb{R}^d)} &\le \|\sqrt{1-\varphi}e^{-\tilde{\delta}\hbar^{-1}d}\|^2_{L^\infty(\mathbb{R}^d)}\|e^{\tilde{\delta}\hbar^{-1}d}\psi_n\|^2_{L^2(\mathbb{R}^d)} \\ &\le C\Big\|\sqrt{1-\varphi}\Big(\frac{\hbar}{\tilde{\delta}d}\Big)^N\Big(\frac{\tilde{\delta}d}{\hbar}\Big)^N e^{-\tilde{\delta}\hbar^{-1}d}\Big\|^2_{L^\infty(\mathbb{R}^d)} \\ &\le C_N\hbar^{2N}. \end{aligned} \tag{4.5}$$

Combining (4.4) with the estimate obtained in (4.5), we get, for all $N \in \mathbb{N}$, that

$$\begin{aligned}\operatorname{Tr}[g_\gamma(H_\hbar)(1-\varphi)] &\le C_N\hbar^{2N}\sum_{E_n\le 0} g_\gamma(E_n) = C_N\hbar^{2N}\operatorname{Tr}[g_\gamma(H_\hbar)] \\ &\le \tilde{C}_N\hbar^{2N-d}, \end{aligned} \tag{4.6}$$

where, in the last estimate, we have used Theorem 1.1. Combining (4.3) and (4.6), we obtain the desired estimate. ∎

Remark 4.3. When applying the above lemma, we must ensure that the constant remains the same for both cases under consideration. To achieve this, we use Theorem 1.1, as explained in Remark 2.6 at the end of the proof.

The next lemma is a result on comparing phase-space integrals. Similar estimates are obtained with different methods in [17]. These are parts of larger proofs and not an independent lemma. The following lemma is taken from [18, Lemma 5.1].

Lemma 4.4. *Suppose that $\Omega \subset \mathbb{R}^d$ is an open set and let $\varphi \in C_0^\infty(\Omega)$. Let $\varepsilon > 0$, $\hbar \in (0, \hbar_0]$ and $V, V_\varepsilon \in L^1_{\text{loc}}(\mathbb{R}^d) \cap C(\Omega)$. Suppose that*

$$\|V - V_\varepsilon\|_{L^\infty(\Omega)} \le c\varepsilon^{k+\mu}. \tag{4.7}$$

Then, for $\gamma \in [0,1]$ *and* ε *sufficiently small, it holds that*

$$\left| \int_{\mathbb{R}^{2d}} [g_\gamma(p^2 + V_\varepsilon(x)) - g_\gamma(p^2 + V(x))]\varphi(x)\, dx\, dp \right| \leq C\varepsilon^{k+\mu},$$

where the constant C *depends on the dimension and* γ *and* c *in* (4.7).

4.2. Proof of main theorem

The proof of the main theorem relies on a multi-scale argument. Before using this technique to establish the theorem, we first recall the following crucial lemma.

Lemma 4.5. *Let* $\Omega \subset \mathbb{R}^d$ *be an open set and let* l *be a function in* $C^1(\overline{\Omega})$ *such that* $l > 0$ *on* $\overline{\Omega}$*, and assume that there exists* ρ *in* $(0,1)$ *such that*

$$|\nabla_x l(x)| \leq \rho \quad \textit{for all } x \textit{ in } \Omega.$$

The following statements hold true.

(i) *There exists a sequence* $\{x_k\}_{k=0}^{\infty}$ *in* Ω *such that the open balls* $B(x_k, l(x_k))$ *form a covering of* Ω*. Furthermore, there exists a constant* N_ρ*, depending only on the constant* ρ*, such that the intersection of more than* N_ρ *balls are empty.*

(ii) *One can choose a sequence* $\{\varphi_k\}_{k=0}^{\infty}$ *such that* $\varphi_k \in C_0^\infty(B(x_k, l(x_k)))$ *for all* k *in* $\mathbb{N}$*. Moreover, for all multiindices* α *and all* k *in* $\mathbb{N}$*,*

$$|\partial_x^\alpha \varphi_k(x)| \leq C_\alpha l(x_k)^{-|\alpha|},$$

and

$$\sum_{k=1}^{\infty} \varphi_k(x) = 1,$$

for all x *in* Ω*.*

This lemma is taken from [22, Lemma 5.4]. The proof is analogous to the proof of [9, Theorem 1.4.10]. We are now ready to prove the main theorem.

Proof of Theorem 1.5. Let $H_{\hbar,\varepsilon}^{-}$ and $H_{\hbar,\varepsilon}^{+}$ be the two framing operators constructed in Lemma 2.4, where we choose $\varepsilon = \hbar^{1-\delta}$. For $\gamma = 0$, we choose $\delta = \mu/(1+\mu)$ and if $\gamma > 0$ we choose $\delta = (1+\mu-\gamma)/(2+\mu)$. Note that our assumptions on μ will in all cases ensure that $\delta \geq 1/3$. Moreover, we get that

$$\begin{aligned} \varepsilon^{1+\mu} &= \hbar, \qquad &\gamma = 0, \\ \varepsilon^{2+\mu} &= \hbar^{1+\gamma}, \quad &\gamma > 0. \end{aligned} \tag{4.8}$$

Since we have that $H_{\hbar,\varepsilon}^{-} \leq H_{\hbar} \leq H_{\hbar,\varepsilon}^{+}$ in the sense of quadratic forms, it follows from the min-max theorem that

$$\mathrm{Tr}[g_\gamma(H_{\hbar,\varepsilon}^{+})] \leq \mathrm{Tr}[g_\gamma(H_{\hbar})] \leq \mathrm{Tr}[g_\gamma(H_{\hbar,\varepsilon}^{-})]. \tag{4.9}$$

Our aim is now to obtain spectral asymptotics for $\mathrm{Tr}[g_\gamma(H_{\hbar,\varepsilon}^{+})]$ and $\mathrm{Tr}[g_\gamma(H_{\hbar,\varepsilon}^{-})]$. Since the arguments will be analogous, we drop the superscript $\pm$ for the operator $H_{\hbar,\varepsilon}^{\pm}$ in what follows. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\varphi(x) = 1$ for all $x \in \Omega_{\tilde{\nu},V_\varepsilon}$ and $\mathrm{supp}(\varphi) \subset \Omega_{2\tilde{\nu},V_\varepsilon}$. Then, applying Lemma 4.2, we obtain for all $N \in \mathbb{N}$ that

$$\mathrm{Tr}[g_\gamma(H_{\hbar,\varepsilon})] = \mathrm{Tr}[g_\gamma(H_{\hbar,\varepsilon})\varphi] + C_N \hbar^N. \tag{4.10}$$

For the terms $\mathrm{Tr}[g_\gamma(H_{\hbar,\varepsilon})\varphi]$, we use a multiscale argument such that we can locally apply Theorem 3.12. Recall that, by Lemma 2.4, we have

$$V_\varepsilon(x) = V_\varepsilon^1(x) + V^2(x) \pm C\varepsilon^{\tau+\mu},$$

where $\mathrm{supp}(V^2) \cap \Omega_{4\tilde{\nu},V_\varepsilon}^{\mathrm{c}} = \emptyset$ and $V_\varepsilon^1 \in C_0^\infty(\mathbb{R}^d)$. We define $\varphi_1 \in C_0^\infty(\mathbb{R}^d)$ such that $\varphi_1(x) = 1$ for all $x \in \Omega_{2\tilde{\nu},V_\varepsilon}$ and $\mathrm{supp}(\varphi_1) \subset \Omega_{4\tilde{\nu},V_\varepsilon}$. With this function, we obtain

$$\varphi_1(x)V_\varepsilon^{\pm}(x) = \varphi_1(x)(V_\varepsilon^1(x) \pm C\varepsilon^{\tau+\mu}).$$

Note that, under these assumptions on $\varphi_1(x)$, we have $\varphi_1(x)\varphi(x) = \varphi(x)$ for all $x \in \mathbb{R}^d$. This observation ensures that when we define our localization function $l(x)$ below, it remains positive on the set $\mathrm{supp}(\varphi)$.

Before defining our localization functions, we remark that due to the continuity of V_ε on $\Omega_{4\tilde{\nu},V_\varepsilon}$, there exists $\epsilon > 0$ such that

$$\mathrm{dist}(\mathrm{supp}(\varphi), \Omega_{2\tilde{\nu},V_\varepsilon}^{c}) > \epsilon.$$

The parameter ϵ is important for our localization functions. As we need to ensure the supports are contained in the set $\Omega_{2\tilde{\nu},V_\varepsilon}$, we let

$$l(x) = A^{-1}\sqrt{|\varphi_1(x)V_\varepsilon(x)|^2 + \hbar^{4/3}} \quad \text{and} \quad f(x) = \sqrt{l(x)},$$

Where we choose $A > 0$ sufficiently large that

$$l(x) \leq \frac{\epsilon}{9} \quad \text{and} \quad |\nabla l(x)| \leq \rho < \frac{1}{8} \tag{4.11}$$

for all $x \in \overline{\mathrm{supp}(\varphi)}$. Note that, due to our assumptions on V_ε, we can choose A independent of $\hbar$ and uniformly for $\hbar \in (0, \hbar_0]$. Moreover, we have that

$$|\varphi_1(x)V_\varepsilon(x)| \leq A l(x), \quad \text{for all } x \in \mathbb{R}^d. \tag{4.12}$$

By Lemma 4.5, with the set $\operatorname{supp}(\varphi)$ and the function $l(x)$ there exists a sequence $\{x_k\}_{k=1}^{\infty}$ in $\operatorname{supp}(\varphi)$ such that $\operatorname{supp}(\varphi) \subset \bigcup_{k\in\mathbb{N}} B(x_k, l(x_k))$ and there exists a constant $N_{1/8}$ such that at most $N_{1/8}$ of the sets $B(x_k, l(x_k))$ can have a non-empty intersection. Moreover, there exists a sequence $\{\varphi_k\}_{k=1}^{\infty}$ such that $\varphi_k \in C_0^\infty(B(x_k, l(x_k)))$,

$$|\partial_x^\alpha \varphi_k(x)| \leq C_\alpha l(x_k)^{-|\alpha|} \quad \text{for all } \alpha \in \mathbb{N}_0, \tag{4.13}$$

and

$$\sum_{k=1}^{\infty} \varphi_k(x) = 1 \quad \text{for all } \operatorname{supp}(\varphi).$$

We have that $\bigcup_{k\in\mathbb{N}} B(x_k, l(x_k))$ is an open covering of $\operatorname{supp}(\varphi)$ and since this set is compact, there exists a finite subset $\mathcal{I}' \subset \mathbb{N}$ such that

$$\operatorname{supp}(\varphi) \subset \bigcup_{k\in\mathcal{I}'} B(x_k, l(x_k)).$$

In order to ensure that we have a finite partition of unity over the set $\operatorname{supp}(\varphi)$, we define the set

$$\mathcal{I} = \bigcup_{j\in\mathcal{I}'} \{k \in \mathbb{N} \mid B(x_k, l(x_k)) \cap B(x_j, l(x_j)) \neq \emptyset\}.$$

We observe that $\mathcal{I}$ is still finite since at most $N_{1/8}$ balls can have a non-empty intersection. Moreover, we have that

$$\sum_{k\in\mathcal{I}} \varphi_k(x) = 1 \quad \text{for all } \operatorname{supp}(\varphi).$$

From this, we get the identity

$$\operatorname{Tr}[\varphi \mathbf{1}_{(-\infty,0]}(H_{\hbar,\varepsilon})] = \sum_{k\in\mathcal{I}} \operatorname{Tr}[\varphi_k \varphi \mathbf{1}_{(-\infty,0]}(H_{\hbar,\varepsilon})],$$

where we have used the linearity of the trace. For the remaining part of the proof, we will use the following notation:

$$l_k = l(x_k), \quad f_k = f(x_k), \quad h_k = \frac{\hbar}{l_k f_k}, \quad \varepsilon_k = \hbar_k^{1-\delta}.$$

We notice that h_k is uniformly bounded from above, since

$$l(x)f(x) = A^{-3/2}(|\varphi_1(x)V_\varepsilon(x)|^2 + \hbar^{4/3})^{3/4} \geq A^{-3/2}\hbar \quad \text{for all } x.$$

By assumption, $\delta \geq 1/3$ and $l_k = f_k^2$; therefore we obtain that

$$l_k \varepsilon^{-1} \leq \varepsilon_k^{-1}. \tag{4.14}$$

We define the two unitary operators U_l and T_z by

$$U_l f(x) = l^{d/2} f(lx) \quad \text{and} \quad T_z f(x) = f(x+z) \quad \text{for } f \in L^2(\mathbb{R}^d).$$

Moreover, we set

$$\widetilde{H}_{\varepsilon,h_k} = f_k^{-2}(T_{x_k}U_{l_k})H_\hbar(T_{x_k}U_{l_k})^* = -h_k^2\Delta + \widetilde{V}_\varepsilon(x),$$

where $\widetilde{V}_\varepsilon(x) = f_k^{-2}V_\varepsilon(l_k x + x_k)$. We need to check that this rescaled operator satisfies the assumptions of Theorem 3.12 with $\hbar_k$, ε_k, and the set $B(0,8)$. To establish this, we first observe that, by (4.11), we have

$$(1-8\rho)l_k \le l(x) \le (1+8\rho)l_k \quad \text{for all } x \in B(x_k, 8l_k). \tag{4.15}$$

We start by verifying that the operator $\widetilde{H}_{\varepsilon,h_k}$ satisfies Assumption 3.1. It follows from Lemma 2.4 that $\widetilde{H}_{\varepsilon,h_k}$ is lower semi-bounded and self-adjoint. By our choice of φ_1, we have that $\widetilde{H}_{\varepsilon,h_k}$ satisfies Assumption 3.1 (ii) with the set $B(0,8)$ and the potential

$$\widetilde{\varphi_1 V}_\varepsilon(x) = \varphi_1(l_k x + x_k) f_k^{-2} V_\varepsilon(l_k x + x_k), \tag{4.16}$$

where, by (4.16), we have that $\widetilde{\varphi_1 V}_\varepsilon(x) \in C_0^\infty(\mathbb{R}^d)$. What remains to verify is that we have obtained a non-critical condition (3.15). Using (4.12), for $x \in B(0,8)$, we have that

$$\begin{aligned}
|\widetilde{\varphi_1 V}_\varepsilon(x)| + h_k^{2/3} &= f_k^{-2}|\varphi_1 V_\varepsilon(l_k x + x_k)| + \Big(\frac{\hbar}{f_k l_k}\Big)^{2/3} \\
&= l_k^{-1}(|\varphi_1 V_\varepsilon(l_k x + x_k)| + \hbar^{2/3}) \\
&\ge l_k^{-1} A l(l_k x + x_k) \ge (1-8\rho)A.
\end{aligned}$$

Hence, we have obtained the non-critical condition on $B(0,8)$. So all assumptions of Theorem 3.12 are fulfilled. But before applying it, we verify that the constant from Theorem 3.12 are independent of k and $\hbar$. Firstly, we have the norm estimate for the potential

$$\|\widetilde{\varphi_1 V}_\varepsilon\|_{L^\infty(B(0,8))} = \sup_{x\in B(0,8)} |\varphi_1(l_k x + x_k) f_k^{-2} V_\varepsilon(l_k x + x_k)| \le (1+8\rho)A,$$

where we have used (4.12) and (4.15). When we consider the derivatives, for $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \ge 1$, we have that

$$\begin{aligned}
&\varepsilon_k^{-\min(0,\tau-|\alpha|)}\|\partial^\alpha \widetilde{\varphi_1 V}_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \\
&\quad\le \varepsilon_k^{-\min(0,\tau-|\alpha|)} f_k^{-2} l_k^{|\alpha|} \varepsilon^{\min(0,\tau-|\alpha|)} \sup_{x\in\mathbb{R}^d} \sum_{\beta\le\alpha} \binom{\alpha}{\beta} |(\partial^{\alpha-\beta}\varphi_1)(\partial^\beta V_\varepsilon)(l_k x + x_k)| \\
&\quad\le C_\alpha,
\end{aligned}$$

where C_α is independent of k and $\hbar$. In the estimate, we have used the definition of ε_k, f_k, (4.14) and Proposition 2.3. Hence, all these estimates are independent of $\hbar$ and k. The last quantities we check are $\|\partial_x^\alpha \widetilde{\varphi_k\varphi}\|_{L^\infty(\mathbb{R}^d)}$ for all $\alpha \in \mathbb{N}_0^d$, where $\widetilde{\varphi_k\varphi} = (T_{x_k}U_{l_k})\varphi_k\varphi(T_{x_k}U_{l_k})^*$. Here, by construction of φ_k (4.13), for all $\alpha \in \mathbb{N}_0^d$, we have

$$\begin{aligned}\|\partial_x^\alpha \widetilde{\varphi_k\varphi}\|_{L^\infty(\mathbb{R}^d)} &= \sup_{x\in\mathbb{R}^d}\Big| l_k^{|\alpha|}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(\partial_x^\beta\varphi_k)(l_kx+x_k)(\partial_x^{\alpha-\beta}\varphi)(l_kx+x_k)\Big| \\ &\leq C_\alpha \sup_{x\in\mathbb{R}^d}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta} l_k^{|\alpha-\beta|}|(\partial_x^{\alpha-\beta}\varphi)(l_kx+x_k)| \leq \widetilde{C}_\alpha.\end{aligned}$$

With this, we have established that all the constant from Theorem 3.12 are independent of $\hbar$ and k. By applying Theorem 3.12, we get that

$$\begin{aligned}&\Big|\mathrm{Tr}[\varphi g_\gamma(H_{\varepsilon,\hbar})] - \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^{2d}} g_\gamma(p^2+V(x))\varphi(x)\,dx\,dp\Big| \\ &\leq \sum_{k\in\mathcal{I}}\Big|\mathrm{Tr}[\varphi_k\varphi g_\gamma(H_{\varepsilon,\hbar})] - \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^{2d}} g_\gamma(p^2+V(x))\varphi_k\varphi(x)\,dx\,dp\Big| \\ &\leq \sum_{k\in\mathcal{I}} f_k^{2\gamma}\Big|\mathrm{Tr}[g_\gamma(\widetilde{H}_{\varepsilon,h_k})\widetilde{\varphi_k\varphi}] - \frac{1}{(2\pi h_k)^d}\int_{\mathbb{R}^{2d}} g_\gamma(p^2+\widetilde{V}(x))\widetilde{\varphi_k\varphi}(x)\,dx\,dp\Big| \\ &\leq \sum_{k\in\mathcal{I}} f_k^{2\gamma}\Big|\mathrm{Tr}[g_\gamma(\widetilde{H}_{\varepsilon,h_k})\widetilde{\varphi_k\varphi}] \\ &\qquad - \frac{1}{(2\pi h_k)^d}\int_{\mathbb{R}^{2d}} g_\gamma(p^2+\widetilde{\varphi_1 V_\varepsilon}(x))\widetilde{\varphi_k\varphi}(x)\,dx\,dp\Big| \\ &\quad + \sum_{k\in\mathcal{I}}\frac{f_k^{2\gamma}}{(2\pi h_k)^d}\Big|\int_{\mathbb{R}^{2d}}[g_\gamma(p^2+\widetilde{\varphi_1 V_\varepsilon}(x)) - g_\gamma(p^2+\widetilde{V}(x))]\widetilde{\varphi_k\varphi}(x)\,dx\,dp\Big| \\ &\leq C\sum_{k\in\mathcal{I}}\frac{f_k^{2\gamma}}{h_k^d}\Big[h_k^{1+\gamma} + \Big|\int_{\mathbb{R}^{2d}}[g_\gamma(p^2+\widetilde{\varphi_1 V_\varepsilon}(x)) - g_\gamma(p^2+\widetilde{V}(x))] \\ &\qquad\qquad \times\widetilde{\varphi_k\varphi}(x)\,dx\,dp\Big|\Big].\end{aligned} \tag{4.17}$$

To estimate the remaining integrals, we use Lemma 4.4. Combining this lemma with (4.8), we obtain that

$$\begin{aligned}&\Big|\int_{\mathbb{R}^{2d}}[g_\gamma(p^2+\widetilde{\varphi_1 V_\varepsilon}(x))\widetilde{\varphi_k\varphi}(x) - g_\gamma(p^2+\widetilde{V}(x))]\widetilde{\varphi_k\varphi}(x)\,dx\,dp\Big| \\ &\quad\leq C\hbar^{1+\gamma} \leq C h_k^{1+\gamma}.\end{aligned} \tag{4.18}$$

Hence, by combining (4.17) and (4.18), we obtain that

$$\Big|\mathrm{Tr}[\varphi g_\gamma(H_{\varepsilon,\hbar})] - \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^{2d}} g_\gamma(p^2+V(x))\varphi(x)\,dx\,dp\Big| \le C\sum_{k\in\mathcal{I}} f_k^{2\gamma} h_k^{1+\gamma-d}. \tag{4.19}$$

By considering the sum over k on the right-hand side of (4.19) and by using (4.15), we get

$$\begin{aligned}\sum_{k\in\mathcal{I}} C h_k^{1+\gamma-d} f_k^{2\gamma} &= \sum_{k\in\mathcal{I}} \tilde{C}\hbar^{1+\gamma-d}\int_{B(x_k,l_k)} l_k^{-d} f_k^{2\gamma}(l_k f_k)^{d-1-\gamma}\,dx \\ &= \sum_{k\in\mathcal{I}} \tilde{C}\hbar^{1+\gamma-d}\int_{B(x_k,l_k)} l_k^{\gamma-d} l_k^{(3d-3-3\gamma)/2}\,dx \\ &\le \sum_{k\in\mathcal{I}} \hat{C}\hbar^{1+\gamma-d}\int_{B(x_k,l_k)} l(x)^{(d-3-\gamma)/2}\,dx \le C\hbar^{1+\gamma-d},\end{aligned} \tag{4.20}$$

where in the last inequality we have used that $\mathrm{supp}(\varphi)\subset\Omega_{2\tilde{\nu},V_\varepsilon}$ and that $\Omega_{2\tilde{\nu},V_\varepsilon}$ is assumed to be compact. This ensures that the constant obtained in the last inequality is finite. By combining the estimates and identities in (4.9), (4.10), (4.19), and (4.20), we obtain that

$$\Big|\mathrm{Tr}[\mathbf{1}_{(-\infty,0]}(H_{\hbar,\varepsilon})] - \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^{2d}} \mathbf{1}_{(-\infty,0]}(p^2+V_\varepsilon(x))\,dx\,dp\Big| \le C\hbar^{1-d}$$

for all $\hbar\in(0,\hbar_0]$. This concludes the proof. ∎

Proof of Theorem 1.6 *and Theorem* 1.7. The proofs are almost analogous to the one just given for Theorem 1.5. The key difference here is that δ is always chosen to be $1/3$ when determining the scaling of the framing operators $H^{\pm}_{\hbar,\varepsilon}$ with $\varepsilon=\hbar^{1-\delta}$. After this choice, the remainder of the proof remains identical.

∎

Proof of Theorem 1.8 *and Theorem* 1.9. The proofs are again almost analogous to the one just given for Theorem 1.5, with the same differences as before. Specifically, we always choose $\delta=1/3$ when determining the scaling of the framing operators $H^{\pm}_{\hbar,\varepsilon}$ with $\varepsilon=\hbar^{1-\delta}$. After this choice, the remainder of the proof is identical up to (4.20).

For the cases considered here, we have $d-3-\gamma<0$. Consequently, we obtain a negative power of $l(x)$ and must use the lower bound $l(x)\ge C\hbar^{2/3}$, rather than relying on an upper bound for $l(x)$. Using this bound and calculating the resulting power of the semiclassical parameter, we derive the errors stated in the theorems. This concludes the proof. ■

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