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Algebraic Geometry. – *Tangentially degenerate monomial curves*, by Satoru Fukasawa, communicated on 8 November 2024.

Abstract. – This paper shows that a certain space monomial curve is a tangentially degenerate curve such that all nonzero orders of hyperplanes at any point are not divisible by the characteristic of the ground field. This indicates that a conjecture presented by Kaji in 2014 does not hold true.

Keywords. – tangentially degenerate curves, positive characteristic, order sequences, monomial curves.

Mathematics Subject Classification 2020. – 14N05 (primary); 14H37 (secondary).

1. Introduction

An irreducible projective curve $X \subset \mathbb{P}^N$ over an algebraically closed field k of characteristic $p \geq 0$ is said to be *tangentially degenerate* if for a general point $P \in X$, the projective tangent line $T_P X \subset \mathbb{P}^N$ to X at P meets X again (see [5]). Throughout this paper, we assume that $N \geq 3$, X is not contained in any hyperplane, and $\varphi : C \to \mathbb{P}^N$ is a morphism induced by the normalisation $C \to X \subset \mathbb{P}^N$. It would be natural to ask the existence of tangentially degenerate curves. When $p = 0$, this problem was posed by Terracini in 1932 [9, p. 143]. For the case where $p = 0$ and φ is unramified, the nonexistence of such curves was proved by Kaji [5] in 1986 (see also [1,6]). The problem of Terracini is still open in general if $p = 0$.

In the case of positive characteristic, many examples of tangentially degenerate curves have been given (see [4,5,7]). In most known cases, the Gauss map $\gamma : X \dashrightarrow \mathbb{G}(1, \mathbb{P}^N)$; $P \mapsto T_P X$ is *not* separable, where $\mathbb{G}(1, \mathbb{P}^N)$ is the Grassmann variety parameterising lines of $\mathbb{P}^N$. In 1994, Esteves and Homma presented the first example of a tangentially degenerate curve in $\mathbb{P}^3$ with $p \geq 5$ whose order-sequence (at a general point) is classical; in particular, the Gauss map is birational onto its image [2, p. 39]. Recently, the present author described a method of constructing tangentially degenerate curves admitting a birational Gauss map, focusing on the non-classicality of automorphisms of the curves [3].

Automorphisms $\sigma \in \mathrm{Aut}(C)$ appearing in Esteves–Homma's example and being considered mainly in the previous paper [3] are *additive*; that is, there exist a point

$P \in C$ and a local parameter x of P such that $\sigma^* x - x \in k \setminus \{0\}$. This paper considers *multiplicative* automorphisms and proves the following proposition.

PROPOSITION 1. *Let $p \geq 5$, and let $\sigma : \mathbb{P}^1 \to \mathbb{P}^1$ be an automorphism defined by $(1 : t) \mapsto (1 : (-2)t)$. We consider a morphism*

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^3; \quad (1 : t) \mapsto (1 : t : t^3 : t^{3p-1}),$$

which is birational onto its image. Then, the following hold:

(a) *For any point $P \in \mathbb{P}^1 \setminus \{(0 : 1)\}$,*

$$\varphi\big(\sigma(P)\big) \in T_{\varphi(P)}\varphi(\mathbb{P}^1);$$

namely, $\varphi(\mathbb{P}^1)$ is tangentially degenerate.

(b) *It follows that*

$$\begin{aligned} &\{\mathrm{ord}_P \varphi^* H \mid H \subset \mathbb{P}^3 : \textit{hyperplane}\} \\ &= \begin{cases} \{0, 1, 2, 3\}, & \textit{if } P \neq (1 : 0), (0 : 1); \\ \{0, 1, 3, 3p-1\}, & \textit{if } P = (1 : 0); \\ \{0, 3p-4, 3p-2, 3p-1\}, & \textit{if } P = (0 : 1). \end{cases} \end{aligned}$$

In particular, for any point $P \in \mathbb{P}^1$ and any hyperplane $H \subset \mathbb{P}^3$ with $H \ni \varphi(P)$, $\mathrm{ord}_P \varphi^* H$ is not divisible by p.

This proposition indicates that the following conjecture presented by Kaji [6, Conjecture 4.1] in 2014 does *not* hold true.

CONJECTURE 1. *For a non-degenerate projective curve $X \subset \mathbb{P}^N$ in arbitrary characteristic p, if $N \geq 3$ and for any point P of the normalisation of X, there exist distinct $i, j, k > 0$ such that none of the orders $b_i(P)$, $b_j(P)$, nor $b_k(P)$ of X at P is divisible by p; then X is not tangentially degenerate.*

Some people might expect that the tangential degeneration is caused by the divisibility of orders of curves by the characteristic p; however, such an expectation is negated by virtue of Proposition 1. It turns out that whether a space curve (in the case $p > 0$) is tangentially degenerate cannot be understood only from the order sequences.

2. PROOF AND REMARKS

The linear system Λ induced by the morphism $\varphi : C \to \mathbb{P}^N$ can be considered the set of all divisors given by the intersection $X \cap H$ of X and a hyperplane $H \subset \mathbb{P}^N$. If a divisor $\sum_{P \in C} n_P P \in \Lambda$ corresponds to a hyperplane-section $X \cap H$, then for any

$P \in C$, $n_P = \mathrm{ord}_P \varphi^* H$. For any point $P \in C$, the set

$$\left\{ n_P \;\middle|\; \sum_{Q \in C} n_Q Q \in \Lambda \right\}$$

consists of exactly $N+1$ values (see, for example, [8]). Such values are denoted by

$$b_0(P) < b_1(P) < \cdots < b_N(P).$$

Proof of Proposition 1. Since $(-2)^{3p-1} = ((-2)^{p-1})^3 \times (-2)^2 = 4$ in k, it follows that

$$\varphi \circ \sigma = (1 : -2t : -8t^3 : 4t^{3p-1}).$$

On the other hand, it follows that

$$(\sigma^* t - t)\frac{d\varphi}{dt} = (-3t)\big(0, 1, 3t^2, (3p-1)t^{3p-2}\big) = (0, -3t, -9t^3, 3t^{3p-1}).$$

Therefore,

$$\varphi + (\sigma^* t - t)\frac{d\varphi}{dt} = (1, -2t, -8t^3, 4t^{3p-1}) = \varphi \circ \sigma.$$

Assertion (a) follows.

We consider assertion (b). It follows that if $P = (1 : 0)$, then

$$\{\mathrm{ord}_P \varphi^* H \mid H \subset \mathbb{P}^3 : \text{hyperplane}\} = \{0, 1, 3, 3p-1\}.$$

For a point $P = (0 : 1)$, $\varphi = (s^{3p-1} : s^{3p-2} : s^{3p-4} : 1)$, where $s = 1/t$. This implies that

$$\{\mathrm{ord}_P \varphi^* H \mid H \subset \mathbb{P}^3 : \text{hyperplane}\} = \{0, 3p-4, 3p-2, 3p-1\}.$$

We consider the case where $P = (1 : \alpha)$, where $\alpha \in k \setminus \{0\}$. Let $u = t - \alpha$. Then, for a system $(x_0 : x_1 : x_2 : x_3)$ of coordinates of $\mathbb{P}^3$, the power series expansions are expressed as follows:

$$\begin{aligned}
\varphi^* x_0 &= 1, \\
\varphi^* x_1 &= \alpha + u, \\
\varphi^* x_2 &= \alpha^3 + 3\alpha^2 u + 3\alpha u^2 + u^3, \\
\varphi^* x_3 &= \alpha^{3p-1} + (3p-1)\alpha^{3p-2} u + \binom{3p-1}{2}\alpha^{3p-3}u^2 + \binom{3p-1}{3}\alpha^{3p-4}u^3 + \cdots.
\end{aligned}$$

Note that

$$\binom{3p-1}{2} = \frac{(3p-1)(3p-2)}{2} = 1,$$

$$\binom{3p-1}{3} = \frac{(3p-1)(3p-2)(3p-3)}{6} = -1$$

as elements in k. It follows that the rank of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ \alpha^3 & 3\alpha^2 & 3\alpha & 1 \\ \alpha^{3p-1} & -\alpha^{3p-2} & \alpha^{3p-3} & -\alpha^{3p-4} \end{pmatrix}$$

is equal to four, where the (i, j) entry is the coefficient of degree $j-1$ of the function $\varphi^* x_{i-1}$ for $i, j = 1, 2, 3, 4$. This implies that

$$\{\mathrm{ord}_P \varphi^* H \mid H \subset \mathbb{P}^3 : \text{hyperplane}\} = \{0, 1, 2, 3\}.$$

Assertion (b) follows. ■

REMARK 1. A counter-example of a smooth space curve exists, as follows. Let $p \geq 5$, $N \geq 5$, and $n = N - 4$. A morphism

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^N; \quad (1 : t) \mapsto (1 : t : t^3 : t^{3p-1} : t^{p(p-1)} : \cdots : t^{p^n(p^n-1)} : t^{p^n(p^n-1)+1})$$

is an embedding, $\varphi(\mathbb{P}^1)$ is tangentially degenerate, and orders $b_1(P), b_2(P), b_3(P)$ are not divisible by p for any point $P \in \mathbb{P}^1$.

REMARK 2. For any $N \geq 3$, there exists a counter-example. When $N = 3$ or $N \geq 5$, we constructed it in Proposition 1 and Remark 1. For $N = 4$, the birational embedding

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^4; \quad (1 : t) \mapsto (1 : t : t^3 : t^{3p-1} : t^{p(p-1)})$$

gives a counter-example.

REMARK 3. The method of constructing the example in Proposition 1 is related to that in the previous paper [3]. The automorphism σ with $\sigma(t) = -2t$ as in Proposition 1 is non-classical with respect to the morphism φ, in the sense of Levcovitz [7]. It follows that

$$t^3, t^{3p-1}, t^{p^n(p^n-1)}, t^{p^n(p^n-1)+1} \in V_{\sigma,t} := \left\{ g \in k(t) \mid \sigma^* g - g = (\sigma^* t - t)\frac{dg}{dt} \right\}$$

for any integer $n \geq 1$; the vector space $V_{\sigma,t}$ was introduced in [3]. For an automorphism σ of $\mathbb{P}^1$ defined by $\sigma(t) = at$ with $a \in k \setminus \{0, 1\}$ and a monomial t^m, the condition $t^m \in V_{\sigma,t}$ is satisfied if and only if

$$a^m - 1 = m(a-1)$$

holds. The condition $a^m - 1 = m(a-1)$ appeared in several papers [1, p. 962], [5, p. 439], [6, p. 749], [7, p. 146], and [9, p. 123].

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