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Mathematical Physics. – *The Euler relativistic gasdynamic system. Chaplygin–Kármán–Tsien laws and substitution principle*, by Colin Rogers and Tommaso Ruggeri, communicated on 14 February 2025.

Abstract. – A seed class of exact solutions of the Euler system of two-dimensional relativistic gasdynamics is derived with an underlying Chaplygin–Kármán–Tsien type constitutive law. The invariance of the nonlinear relativistic system under multi-parameter reciprocal Bäcklund transformations is applied to generate a wide class of novel associated solutions with a barotropic relation. The latter reduces to the standard Chaplygin–Kármán–Tsien gas law in the non-relativistic limit. An additional invariance of the general relativistic gasdynamics system is established under a substitution principle. The latter represents an extension of a classical result in non-relativistic gasdynamics.

Keywords. – relativistic gasdynamics, reciprocal transformations, Chaplygin–Kármán–Tsien gas law.

Mathematics Subject Classification 2020. – 76Y05 (primary); 76N99 (secondary).

1. Introduction

The relativistic Euler system describes the dynamics of fluid when relativistic effects become significant. This system consists of partial differential equations derived from the conservation laws of particle number, momentum, and energy within the framework of special or general relativity.

To close the system, a constitutive equation of state is required, relating energy to pressure and particle number. However, unlike in the classical case, determining the equation of state in relativity is challenging. As a result, the equations of state used are often only valid in the classical regime where simplified relations apply. As observed in [34], this is one of the weak points of the theory.

In particular, relativistic gasdynamics systems with classical Chaplygin-type constitutive laws and their generalizations have been the subject of an extensive literature, in particular, related to Riemann problems (see [7,9,40,41] and works cited therein). Here, a novel relativistic version of the Chaplygin laws is introduced in the context of a classical two-dimensional relativistic system due to Taub [37]. In [27], this nonlinear system of conservation laws has been shown to be invariant under a multi-parameter class of reciprocal transformations. The invariance of the 1+1-dimensional Taub relativistic gas dynamics system under analogous reciprocal transformations had previously been established in [26]. This represented an extension of invariance under multi-parameter reciprocal transformations originally derived in 1+1-dimensional non-relativistic gas-dynamics in [20, 21]. In terms of application, this kind of invariant transformations has been used in [6] in the analysis of the motion of a gas between a driven piston and a non-uniform shock.

Reciprocal transformations were introduced in [2] in connection with lift and drag phenomena in two-dimensional isentropic gasdynamics. They were subsequently shown by Bateman in [3] to constitute a particular class of Bäcklund transformations [28, 32]. In two-dimensional subsonic gasdynamics, these have a key application in the reduction of the governing hodograph system with a Chaplygin–Kármán–Tsien constitutive law to the tractable Cauchy–Riemann system of classical hydrodynamics [38]. Loewner [13, 14] subsequently undertook the systematic construction via Bäcklund transformations of model constitutive laws in gasdynamics which allow the reduction of the hodograph system to viable canonical forms in subsonic, transonic, and supersonic régimes, respectively. In [11, 12], a re-interpretation and extension of the class of infinitesimal Bäcklund transformations applied in a gasdynamics context in [14] proved key to the construction of a novel master 2+1-dimensional solitonic system.

Reciprocal transformations associated with admitted conservation laws were introduced in soliton theory in [10] and conjugated with a nonlinear superposition principle associated with the classical Bianchi permutability theorem which allows the iterative generation of multi-soliton solutions. Such reciprocal transformations were subsequently applied in [33] in the linkage of the canonical AKNS and WKI inverse scattering schemes of [1] and [39], respectively. They likewise connect certain classes of 1+1-dimensional solitonic hierarchies [5, 23, 25]. Reciprocal transformations in 2+1-dimensions were introduced in [22]. Later, the triad of 2+1-dimensional integrable hierarchies of Kadomtsev–Petviashvilli, modified Kadomtsev–Petviashvilli, and extended 2+1-dimensional Dym type were shown to be linked by conjugation of gauge and reciprocal transformations.

In magnetogasdynamics, invariance under multi-parameter reciprocal transformations has been established in [24]. In addition, in [30], a Bäcklund transformation was coupled with the action on seed vortex motions of a multi-parameter class of reciprocal transformations to construct periodic solutions of breather-type in super Alfénic magnetogasdynamics. The constitutive law adopted there was of generalized Chaplygin–Kármán–Tsien type.

Here, a novel seed class of exact solutions of the Taub system of steady relativistic gasdynamics is constructed. This seed class is then extended via the action upon it of multi-parameter reciprocal transformations. In addition, the invariance of the relativistic gas dynamic system under an extension of the classical Prim substitution principle is established.

2. A class of seed solutions in relativistic gasdynamics. Admittance of a generalized Chaplygin–Kármán–Tsien law

Here, a two-dimensional steady relativistic gasdynamic system with origin in the work of Taub is considered, which reduces to the system of conservation laws [27]

$$\partial_x(\rho\Gamma u) + \partial_y(\rho\Gamma v) = 0, \tag{2.1}$$

$$\partial_x\left(\frac{e+p}{c^2}\Gamma^2 u^2 + p\right) + \partial_y\left(\frac{e+p}{c^2}\Gamma^2 uv\right) = 0, \tag{2.2}$$

$$\partial_x\left(\frac{e+p}{c^2}\Gamma^2 uv\right) + \partial_y\left(\frac{e+p}{c^2}\Gamma^2 v^2 + p\right) = 0, \tag{2.3}$$

$$\partial_x\big[(e+p)\Gamma^2 u\big] + \partial_y\big[(e+p)\Gamma^2 v\big] = 0. \tag{2.4}$$

In the above, p denotes the pressure, $\rho = nm_0$ is the mass density, n is the particle number, and m_0 is the mass in the rest frame, while e is the energy density composed of the internal energy denoted by $\rho\varepsilon$ and the rest energy density according to

$$e = \rho(\varepsilon + c^2),$$

where c is the speed of light. In addition,

$$\Gamma = \frac{1}{\sqrt{1 - q^2/c^2}}$$

is the Lorentz factor, where $q^2 = u^2 + v^2$ is the square modulus of the relativistic gas velocity $\mathbf{q} = u\mathbf{i} + v\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the x, y direction, respectively.

In the sequel, new variables R and S are introduced according to

$$R = \Gamma\rho, \qquad S = \Gamma^2\frac{e+p}{c^2}, \tag{2.5}$$

whence the system of conservation laws (2.1)–(2.4) becomes

$$(Ru)_x + (Rv)_y = 0, \tag{2.6}$$

$$(p + Su^2)_x + (Suv)_y = 0, \tag{2.7}$$

$$(Suv)_x + (p + Sv^2)_y = 0, \tag{2.8}$$

$$(Su)_x + (Sv)_y = 0. \tag{2.9}$$

The seed ansätze

$$u = \frac{\alpha}{S} + \beta, \tag{2.10a}$$

$$v = \frac{\gamma}{S} + \delta, \tag{2.10b}$$

$$p = \frac{\zeta}{S} + \epsilon, \tag{2.10c}$$

$$\alpha, \beta, \gamma, \delta, \epsilon, \zeta \in \mathbb{R}$$

are now introduced wherein the insertion of (2.10a) and (2.10b) into (2.9) shows that

$$S = S(\eta)$$

with $\eta = \delta x - \beta y$. The relation (2.10c) then yields

$$p_x = -\zeta\delta S'/S^2,$$

whence the conservation law (2.7) requires that

$$(p + Su^2)_x + (Suv)_y = (S'/S^2)[-\zeta\delta - \alpha^2\delta + \alpha\beta\gamma] = 0.$$

Thus, with $S' \neq 0$,

$$\zeta = \alpha(\beta\gamma - \alpha\delta)/\delta, \tag{2.11}$$

where the parameters α, β and $\beta\gamma - \alpha\delta$ are taken as non-zero. In a similar manner,

$$p_y = \zeta\beta S'/S^2$$

and the conservation law (2.8), in turn, yields

$$(Suv)_x + (p + Sv^2)_y = (S'/S^2)[\zeta\beta + \beta\gamma^2 - \alpha\gamma\delta] = 0,$$

so that

$$\zeta = \gamma(\alpha\delta - \beta\gamma)/\beta, \tag{2.12}$$

where β, γ and $\alpha\delta - \beta\gamma$ are assumed non-zero. The compatibility of the relations (2.11) and (2.12) now requires that

$$\alpha(\beta\gamma - \alpha\delta)/\delta = \gamma(\alpha\delta - \beta\gamma)/\beta,$$

whence

$$(\alpha\beta + \gamma\delta)(\beta\gamma - \alpha\delta) = 0, \quad \beta\gamma - \alpha\delta \neq 0,$$

so that

$$\alpha\beta + \gamma\delta = 0. \tag{2.13}$$

Companion relations to (2.10a) and (2.10b) are now introduced in terms of R according to

$$u = \frac{\lambda}{R} + \mu,$$
$$v = \frac{\sigma}{R} + \tau,$$

so that

$$R = \lambda S/\big[\alpha + (\beta - \mu)S\big] = \sigma S/\big[\gamma + (\delta - \tau)S\big].$$

Thus,

$$\lambda\gamma = \alpha\sigma,$$
$$\lambda(\delta - \tau) = \sigma(\beta - \mu),$$

while the conservation law (2.6) requires that

$$\mu R_x + \tau R_y = 0,$$

where $R = R(\eta)$ so that

$$\mu\delta = \beta\tau. \tag{2.14}$$

The relation (2.5) on the use of (2.13) now yields

$$\lambda/\big[\alpha + (\beta - \mu)S\big] = \rho/\sqrt{S^2 - \big(\alpha^2 + \gamma^2 + (\beta^2 + \delta^2)S^2\big)/c^2} \tag{2.15}$$

wherein

$$S = \zeta/(p - \epsilon).$$

The latter pair of relations together show then the seed class of exact solutions (2.10) of the relativistic gas dynamic system (2.1)–(2.4) is compatible with a barotropic $p(\rho)$-relation.

Therein, the parametric constraints (2.11) and (2.13) combine to show that $\zeta = -(\alpha^2 + \gamma^2) < 0$, whence

$$p = \epsilon - \frac{\alpha^2 + \gamma^2}{S}, \tag{2.16}$$

where $S \equiv S(\rho, c)$ is determined by the relation (2.15).

Consider now the particular case $\beta = \mu$. In this case, (2.14) require $\delta = \tau$. The relation (2.15) then yields (taking the positive value for S)

$$S = \sqrt{\frac{\frac{\alpha^2+\gamma^2}{c^2} + \left(\frac{\alpha}{\lambda}\right)^2 \rho^2}{1 - \frac{\beta^2+\delta^2}{c^2}}}. \tag{2.17}$$

Thus, (2.16) with (2.17) is a generalized Chaplygin–Kármán–Tsien type law result as when c become large, (2.16) reduce the classical Chaplygin–Kármán–Tsien equation:

$$p = b - \frac{a^2}{\rho}, \quad b = \epsilon > 0,\ a^2 = (\alpha^2 + \gamma^2)\left|\frac{\lambda}{\alpha}\right| > 0.$$

We remark that from (2.16) and (2.17), the derivative of p with respect to ρ is positive according to the hyperbolicity condition of the Euler system.

3. Reciprocal invariance

In [27], it was established that the relativistic gasdynamic system (2.6)–(2.9) is invariant under the 4-parameter class of reciprocal transformations:

$$\begin{aligned}
u' &= -\frac{a_1 u}{p + a_2}, \quad v' = -\frac{a_1 v}{p + a_2}, \quad p' = a_4 - \frac{a_1^2 a_3}{p + a_2}, \\
R' &= \frac{a_3(p + a_2)R}{p + Sq^2 + a_2}, \quad S' = \frac{a_3(p + a_2)S}{p + Sq^2 + a_2}, \\
dx' &= -(p + Sv^2 + a_2)\,dx + Suv\,dy, \\
dy' &= Suv\,dx - (p + Sv^2 + a_2)\,dy,
\end{aligned} \tag{3.1}$$

with

$$0 < \left|J(x', y'; x, y)\right| < \infty.$$

The reciprocal relation R' in (3.1) yields

$$\frac{\rho'}{\sqrt{1 - q'^2/c^2}} = \frac{a_3(p + a_2)}{p + Sq^2 + a_2}\left(\frac{\rho}{\sqrt{1 - q^2/c^2}}\right),$$

whence

$$\rho' = \frac{a_3 \rho (p + a_2)}{(p + Sq^2 + a_2)} \sqrt{\frac{1 - a_1^2 q^2/\left[c^2(p + a_2)^2\right]}{1 - q^2/c^2}}.$$

The reciprocal S' relation, in turn, gives

$$e' = \frac{a_3(p + a_2)(e + p)}{p + Sq^2 + a_2}\left(\frac{1 - q^2/c^2}{1 - a_1^2 q^2/\left[c^2(p + a_2)^2\right]}\right) - p',$$

where p' is the reciprocal pressure.

The action of the above reciprocal transformations on the seed class of solutions (2.10) of the Taub relativistic gasdynamic system, in view of the relations

$$S = \zeta/(\epsilon - p), \quad q^2 = \frac{\alpha^2 + \gamma^2}{S^2} + \beta^2 + \delta^2,$$

results in a barotropic relation $p' = p'(\rho')$ for the associated multi-parameter class.

In the non-relativistic reduction $c \to \infty$, the multi-parameter class of reciprocal transformations (3.1) reduces to that originally obtained by Bateman in [2]. These have applications in the linkage of subsonic two-dimensional motions of a Chaplygin–Kármán–Tsien gas to associated incompressible motions determined by the classical Cauchy–Riemann system. In that context, the reciprocal variables x and y are related to the lift and drag functions [2].

In the case of the present relativistic seed class (2.10) integrals of the reciprocal coordinates dx' and dy' in (3.1) in turn yield

$$x' = -\frac{1}{\beta}\int\left[\frac{\alpha\gamma}{S(\eta)} + \beta\delta S(\eta)\right]d\eta - [\epsilon + a_2 + 2\gamma\delta]x + [\beta\gamma + \alpha\delta]y,$$

$$y' = \frac{1}{\delta}\int\left[\frac{\alpha\gamma}{S(\eta)} + \beta\delta S(\eta)\right]d\eta + [\beta\gamma + \alpha\delta]x - [\epsilon + a_2 + 2\gamma\delta]y.$$

4. A Substitution principle in relativistic gasdynamics

The invariance of non-relativistic gasdynamics under what are termed substitution principles was originally established in the work by Munk and Prim [15, 19]. In steady magnetogasdynamics in which the Maxwellian and constant total pressure surfaces coincide, invariance under a substitution principle was subsequently established in [29]. This correspondence to an invariance was admitted by a constrained solitonic Pohlmeyer–Lund–Regge system. The invariance under a substitution principle in non-steady magnetogasdynamics was established in [18]. Lie group connections with invariance under substitution principles in both non-relativistic gasdynamics and magnetogasdynamics have an extensive literature (see e.g. [16, 17] and works cited therein).

Here, the relativistic gasdynamic system (2.6)–(2.9) is seen to be under a substitution principle wherein

$$u^\dagger = u/\Lambda^{1/2}, \quad v^\dagger = v/\Lambda^{1/2}, \quad p^\dagger = p,$$

together with

$$R^\dagger = \Lambda R, \quad S^\dagger = \Lambda S, \tag{4.1}$$

and the constraint

$$\mathbf{q}\cdot\nabla\Lambda = 0.$$

The relations (4.1) in turn determine the density $\rho^\dagger$ and internal energy according to

$$\rho^\dagger = \rho\Lambda\sqrt{\frac{1 - q^2/(\Lambda c^2)}{1 - q^2/c^2}}, \quad e^\dagger = \Lambda\left(\frac{1 - q^2/(\Lambda c^2)}{1 - q^2/c^2}\right)(e + p) - p.$$

The invariance of the Taub relativistic gasdynamic system (2.1)–(2.4) as derived under the multi-parameter reciprocal transformations may accordingly be extended via conjugation with the above substitution principle to boost known classes of seed solutions.

The connection between the Lie group invariance of the relativistic gasdynamic Taub system (2.1)–(2.4) and the above novel substitution principle remains to be investigated in analogy with the procedures adopted in [16] and [17] in gasdynamics and magnetogasdynamics, respectively. Likewise, a study of the Lie group invariance of the Taub system under appropriate constitutive laws has yet to be undertaken. In the latter regard, properties of generalized Chaplygin gas laws such as introduced by Boillat in [4] in connection with the propagation of relativistic gasdynamic exceptional waves, namely,

$$p = b - \frac{a^2}{\rho + bc^{-2}}, \quad a, b \in \mathbb{R},$$

would be of potential research interest.

5. Conclusion

There has been extensive application of admitted invariance principles to the generalization of solutions of Einstein's equations (see e.g. [36] and literature cited therein). Here, the concern has been with the two-dimensional relativistic gasdynamic system (2.1)–(2.4) and its admitted invariance under multi-parameter reciprocal relations augmented by a novel extension of the Prim substitution principle of classical gasdynamics. The application in relativity of analytic methods of the modern soliton theory such as invariance under Laplace–Darboux and Bäcklund transformations along with concomitant nonlinear superposition principles for the iterative generation of exact solutions is well established. This has notably involved the celebrated Ernst equation [8] and its integrable variants [35]. In the most recent work [31], binary Darboux-type transformations with origin in soliton theory have been applied to seed solutions of the relativistic gasdynamic system to generate wide classes of novel solutions.

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References

[1] M. J. Ablowitz – D. J. Kaup – A. C. Newell – H. Segur, Nonlinear-evolution equations of physical significance. *Phys. Rev. Lett.* **31** (1973), 125–127. Zbl 1243.35143 MR 0406176

[2] H. Bateman, The lift and drag functions for an elastic fluid in two dimensional irrotational flow. *Proc. Natl. Acad. Sci. USA* **24** (1938), 246–251. Zbl 0019.41001

[3] H. Bateman, The transformation of partial differential equations. *Quart. Appl. Math.* **1** (1944), 281–296. Zbl 0061.20706 MR 0009686

[4] G. Boillat, Covariant disturbances and exceptional waves. *J. Math. Phys.* **14** (1973), 973–976. Zbl 0262.76041

[5] S. Carillo – B. Fuchssteiner, The abundant symmetry structure of hierarchies of nonlinear equations obtained by reciprocal links. *J. Math. Phys.* **30** (1989), no. 7, 1606–1613. Zbl 0693.35137 MR 1002268

[6] S. P. Castell – C. Rogers, Applications of invariant transformations in one-dimensional non-steady gasdynamics. *Quart. Appl. Math.* **32** (1974), 241–251. Zbl 0289.76034

[7] H. Cheng – H. Yang, Riemann problem for the relativistic Chaplygin Euler equations. *J. Math. Anal. Appl.* **381** (2011), no. 1, 17–26. Zbl 1220.35126 MR 2796189

[8] F. J. Ernst, New formulation of the axially symmetric gravitational field problem. *Phys. Rev.* **167** (1968), no. 5, 1175–1178.

[9] M. Huang – Z. Shao, Riemann problem with delta initial data for the relativistic Chaplygin Euler equations. *J. Appl. Anal. Comput.* **6** (2016), no. 2, 376–395. Zbl 1463.35345 MR 3423922

[10] J. G. Kingston – C. Rogers, Reciprocal Bäcklund transformations of conservation laws. *Phys. Lett. A* **92** (1982), no. 6, 261–264. MR 0683534

[11] B. Konopel'chenko – C. Rogers, On $(2 + 1)$-dimensional nonlinear systems of Loewner-type. *Phys. Lett. A* **158** (1991), no. 8, 391–397. MR 1130347

[12] B. G. Konopel'chenko – C. Rogers, On generalized Loewner systems: novel integrable equations in $2 + 1$ dimensions. *J. Math. Phys.* **34** (1993), no. 1, 214–242. Zbl 0767.35087 MR 1198634

[13] C. Loewner, A transformation theory of the partial differential equations of gas dynamics. *Tech. Notes Nat. Adv. Comm. Aeronaut.* **1950** (1950), no. 2065, 1–56. MR 0044712

[14] C. Loewner, Generation of solutions of systems of partial differential equations by composition of infinitesimal Baecklund transformations. *J. Analyse Math.* **2** (1953), 219–242. Zbl 0053.38801 MR 0057426

[15] M. Munk – R. Prim, On the multiplicity of steady gas flows having the same streamline pattern. *Proc. Natl. Acad. Sci. USA* **33** (1947), 137–141.

[16] F. Oliveri – M. P. Speciale, Exact solutions to the equations of perfect gases through Lie group analysis and substitution principles. *Internat. J. Non-Linear Mech.* **34** (1999), no. 6, 1077–1087. Zbl 1006.76083 MR 1698004

[17] F. Oliveri – M. P. Speciale, Exact solutions to the ideal magneto-gas-dynamics equations through Lie group analysis and substitution principles. *J. Phys. A* **38** (2005), no. 40, 8803–8820. Zbl 1075.76061 MR 2185853

[18] G. Powers – C. Rogers, Substitution principles in nonsteady magneto-gas-dynamics. *Appl. Sci. Res.* **21** (1969), 176–184. MR 0270628

[19] R. C. Prim III, A note on the substitution principle for steady gas flow. *J. Appl. Phys.* **20** (1949), 448–450. Zbl 0033.41702 MR 0030377

[20] C. Rogers, Reciprocal relations in non-steady one-dimensional gasdynamics. *Z. Angew. Math. Phys.* **19** (1968), 58–63. Zbl 0159.57402

[21] C. Rogers, Invariant transformations in non-steady gasdynamics and magneto-gasdynamics. *Z. Angew. Math. Phys.* **20** (1969), 370–382. Zbl 0187.24304

[22] C. Rogers, Reciprocal transformations in $(2+1)$ dimensions. *J. Phys. A* **19** (1986), no. 9, L491–L496. Zbl 0614.45019 MR 0847602

[23] C. Rogers – S. Carillo, On reciprocal properties of the Caudrey-Dodd-Gibbon and Kaup-Kupershmidt hierarchies. *Phys. Scripta* **36** (1987), no. 6, 865–869. Zbl 1063.37551 MR 0921595

[24] C. Rogers – J. G. Kingston – W. F. Shadwick, Reciprocal-type transformations in magnetogasdynamics. *J. Math. Phys.* **21** (1980), 395–397. Zbl 0439.76095

[25] C. Rogers – M. C. Nucci, On reciprocal Bäcklund transformations and the Korteweg-de Vries hierarchy. *Phys. Scripta* **33** (1986), no. 4, 289–292. Zbl 1063.37565 MR 0870139

[26] C. Rogers – T. Ruggeri, On invariance in 1+1-dimensional isentropic relativistic gasdynamics. *Wave Motion* **94** (2020), article no. 102527. Zbl 1524.76536 MR 4060163

[27] C. Rogers – T. Ruggeri – W. K. Schief, On relativistic gasdynamics: invariance under a class of reciprocal-type transformations and integrable Heisenberg spin connections. *Proc. A.* **476** (2020), no. 2243, article no. 20200487. Zbl 1472.76130 MR 4187938

[28] C. Rogers – W. K. Schief, *Bäcklund and Darboux transformations*. Cambridge Texts Appl. Math., Cambridge University Press, Cambridge, 2002. Zbl 1019.53002 MR 1908706

[29] C. Rogers – W. K. Schief, Novel integrable reductions in nonlinear continuum mechanics via geometric constraints. *J. Math. Phys.* **44** (2003), no. 8, 3341–3369. Zbl 1062.37079 MR 2006754

[30] C. Rogers – W. K. Schief, Vortex trains in super-Alfvénic magnetogasdynamics. Application of reciprocal-Bäcklund transformations. *J. Nonlinear Math. Phys.* **12** (2005), 548–564. Zbl 1362.76065 MR 2118886

[31] C. Rogers – W. K. Schief, Relativistic perfect fluids: integrability of the classical Taub system. *Proc. A.* **479** (2023), no. 2271, article no. 20220645. MR 4574811

[32] C. Rogers – W. F. Shadwick, *Bäcklund transformations and their applications*. Math. Sci. Eng. 161, Academic Press, New York, 1982. Zbl 0492.58002 MR 0658491

[33] C. Rogers – P. Wong, On reciprocal Bäcklund transformations of inverse scattering schemes. *Phys. Scripta* **30** (1984), no. 1, 10–14. Zbl 1063.37552 MR 0757711

[34] T. Ruggeri – Q. Xiao – H. Zhao, Nonlinear hyperbolic waves in relativistic gases of massive particles with Synge energy. *Arch. Ration. Mech. Anal.* **239** (2021), no. 2, 1061–1109. Zbl 1456.76164 MR 4201622

[35] W. K. Schief – C. Rogers, On a Laplace sequence of nonlinear integrable Ernst-type equations. In *Algebraic aspects of integrable systems*, pp. 315–321, Progr. Nonlinear Differential Equations Appl. 26, Birkhäuser, Boston, MA, 1997. Zbl 0869.35092 MR 1418882

[36] H. Stephani – D. Kramer – M. MacCallum – C. Hoenselaers – E. Herlt, *Exact solutions of Einstein's field equations*. 2nd edn., Cambridge Monogr. Math. Phys., Cambridge University Press, Cambridge, 2003. Zbl 1057.83004 MR 2003646

[37] A. H. Taub, Approximate solutions of the Einstein equations for isentropic motions of plane-symmetric distributions of perfect fluids. *Phys. Rev. (2)* **107** (1957), 884–900. Zbl 0080.18306 MR 0092647

[38] H.-S. Tsien, Two-dimensional subsonic flow of compressible fluids. *J. Aeronaut. Sci.* **6** (1939), 399–407. Zbl 0063.07865 MR 0003135

[39] M. Wadati – K. Konno – Y. H. Ichikawa, New integrable nonlinear evolution equations. *J. Phys. Soc. Japan* **47** (1979), no. 5, 1698–1700. Zbl 1334.35256 MR 0552572

[40] H. Yang – Y. Zhang, Pressure and flux-approximation to the isentropic relativistic Euler equations for modified Chaplygin gas. *J. Math. Phys.* **60** (2019), no. 7, article no. 071502. Zbl 1419.76727 MR 3976419

[41] G. Yin – K. Song, Vanishing pressure limits of Riemann solutions to the isentropic relativistic Euler system for Chaplygin gas. *J. Math. Anal. Appl.* **411** (2014), no. 2, 506–521. Zbl 1331.35029 MR 3128409

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