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Fluid Mechanics. – *Remarks on a comparison principle for a doubly singular quasilinear anisotropic problem*, by Luigi Montoro and Berardino Sciunzi, communicated on 14 February 2025.

Abstract. – In these notes, using some arguments of Montoro, Sciunzi, and Trombetta (2025), we prove a new general version of a comparison principle for sub-supersolutions to a singular quasilinear problem driven by the anisotropic operator. As a consequence, we deduce a uniqueness result for weak solutions to the problem

$$(\mathcal{P}) \qquad \begin{cases} -\Delta_p^H u = \theta \frac{u^{p-1}}{H^0(x)^p} + \frac{1}{u^\gamma} + f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and then we analyze, in the anisotropic setting, the question of the existence of solutions to a subdiffusive problem in the whole $\mathbb{R}^N$.

Keywords. – comparison principle, Finsler anisotropic operator, Picone identity.

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1. Introduction and main results

In this paper, we consider the doubly singular quasilinear anisotropic problem

$$(\mathcal{P}) \qquad \begin{cases} -\Delta_p^H u = \theta \frac{u^{p-1}}{H^0(x)^p} + \frac{1}{u^\gamma} + f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded C^2 domain, $0 \in \Omega$, $1 < p < N$, $\theta \geq 0$, $\gamma > 0$, H°, f are suitable functions defined here below and $-\Delta_p^H u$ is the anisotropic p-Laplace operator, which for suitable smooth functions is given by

$$-\Delta_p^H u := -\operatorname{div}(H^{p-1}(\nabla u)\nabla H(\nabla u)). \tag{1.1}$$

The anisotropic function H in (1.1) is a Finsler norm that satisfies the following set of assumptions:

(h_H) (i) $H \in C^{2,\beta}_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ and such that $H(\xi) > 0$ $\forall \xi \in \mathbb{R}^N \setminus \{0\}$;

(ii) $H(s\xi) = |s| H(\xi)$ $\forall \xi \in \mathbb{R}^N \setminus \{0\}$, $\forall s \in \mathbb{R}$;

(iii) H is *uniformly elliptic*, which means set $B_1^H := \{\xi \in \mathbb{R}^N : H(\xi) < 1\}$ is uniformly convex, i.e.,

$$\text{(1.2)}\quad \exists \Lambda > 0 : \quad \big\langle D^2 H(\xi) v, v\big\rangle \geq \Lambda |v|^2 \quad \forall \xi \in \partial B_1^H,\ \forall v \in \nabla H(\xi)^{\perp}.$$

The function $H^\circ : \mathbb{R}^N \to [0, +\infty)$ in $(\mathcal{P})$ is the dual norm of H defined as

$$H^\circ(x) = \sup_{H(\xi) \leq 1} \langle \xi, x\rangle.$$

In all the paper, we assume that the nonlinearity f satisfies the following hypothesis (denoted by (hp_f) in the sequel):

(hp_f) $f : \Omega \times (0, \infty) \to \mathbb{R}_0^+$ is a measurable function such that $f(x,t) \leq a(x) + b(x) t^{p^*}$ for some nonnegative functions $a, b \in L^\infty(\Omega)$.

Note that hypothesis (hp_f) is required when considering $W^{1,p}_{\mathrm{loc}}$-solutions to state problem $(\mathcal{P})$; see Definition 1.1 below. In the case $\theta = 0$, if the solution u is a priori bounded, this assumption can be removed, e.g., in the case of locally Lipschitz continuous nonlinearities.

Definition 1.1. We say that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a weak supersolution (subsolution) to

$$-\Delta_p^H u = \theta \frac{u^{p-1}}{H^0(x)^p} + \frac{1}{u^\gamma} + f(x,u), \tag{1.3}$$

if

(i) $\forall \omega \Subset \Omega\ \exists c_\omega : u \geq c_\omega > 0$ in ω and

(ii) $\int_\Omega H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla \varphi\rangle\, dx \underset{(\leq)}{\geq} \int_\Omega \big(\theta \frac{u^{p-1}}{H^0(x)^p} + \frac{1}{u^\gamma} + f(x,u)\big)\varphi\, dx$, for all $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$.

Finally, we say that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a weak solution to (1.3) if u is both a supersolution and a subsolution to (1.3).

Remark 1.2. We observe that since (hp_f) holds and since $1 < p < N$ (and by Hardy inequality, e.g., see [9, Proposition 7.5]), the right-hand side of (1.3) is well defined.

Because the solutions to $(\mathcal{P})$ generally are not in $W_0^{1,p}(\Omega)$, the Dirichlet datum has to be understood in a generalized meaning.

Definition 1.3. We say that $u \leq 0$ on $\partial\Omega$ if $(u - \delta)^+ \in W_0^{1,p}(\Omega)$ for every $\delta > 0$. Finally, $u = 0$ on $\partial\Omega$ if u is nonnegative and $u \leq 0$ on $\partial\Omega$.

We point out that in the study of quasilinear problems involving singular nonlinearities such as the case of $u^{-\gamma}$ in $(\mathcal{P})$, we have to face the loss of regularity at the boundary; that is, the problem is singular at the boundary. Moreover, due to the singularity introduced by the presence of the Hardy potential in the critical term $u^{p-1}/H^0(x)^p$, in all the paper, we assume the following natural assumption:

$$u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty\big(\bar{\Omega} \setminus \{0\}\big). \tag{1.4}$$

First of all, we recall some behavior at the boundary and at zero for sub-supersolutions to $(\mathcal{P})$ that we need in the proof of Theorem 1.5. These results follow mainly exploiting [8, Theorem 1.4] and [4, Proposition 3.4, Theorem 1.1]. Let $d : \mathbb{R}^N \to \mathbb{R}$ be the distance function for $\partial\Omega$. Moreover, in our case, d is C^2 in $I_\varepsilon(\partial\Omega)$, namely, a neighborhood of $\partial\Omega$ with the *unique nearest point* property (see [6]) (recall that by assumption $\partial\Omega$ is C^2).

Lemma 1.4. *Let us assume that* (hp_f) *holds, let* $\check{u} \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\bar{\Omega} \setminus \{0\})$ *be a subsolution to* $(\mathcal{P})$ *and let* $\hat{u} \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\bar{\Omega} \setminus \{0\})$ *be a supersolution to* (1.3). *Then, the following hold:*

(i) *If* $\gamma > 1$, *then there exist two positive constants* $\check{c}, \hat{c}$, *and there exists* ε *sufficiently small such that*

$$\hat{u} \geq \hat{c}\, d^{\frac{p}{\gamma+p-1}}, \quad \check{u}(x) \leq \check{c}\, d^{\frac{p}{\gamma+p-1}} \quad \forall x \in I_\varepsilon(\partial\Omega). \tag{1.5}$$

(ii) *If* $0 < \gamma \leq 1$, *then there exist two positive constants* $\check{c}, \hat{c}$, *and there exists* ε *sufficiently small such that*

$$\hat{u} \geq \hat{c}\, d, \quad \check{u}(x) \leq \check{c}\, d^{\frac{p-1}{\gamma+p-1}} \quad \forall x \in I_\varepsilon(\partial\Omega). \tag{1.6}$$

(iii) *There exist constants* $\hat{c}, \check{c}, R > 0$, $0 < \mu < (N-p)/p$ *such that*

$$\hat{u} \geq \hat{c}[H^0(x)]^{-\mu}, \quad \check{u}(x) \leq \check{c}[H^0(x)]^{-\mu} \quad \forall x \in B_R^{H^\circ}(0), \tag{1.7}$$

where $B_R^{H^\circ}(0) := \{x \in \mathbb{R}^N : H^\circ(x) < R\}$.

All the numerical constants depend on $\check{u}$ *and* $\hat{u}$.

The following theorem is a comparison principle for sub-supersolutions to $(\mathcal{P})$ with a singular-type right-hand side.

Theorem 1.5 (Comparison principle). *Let* $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\bar{\Omega} \setminus \{0\})$ *be a subsolution to* $(\mathcal{P})$, *and let* $v \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\bar{\Omega} \setminus \{0\})$ *be a supersolution to* (1.3). *Let us assume* $u \leq v$ *on* $\partial\Omega$, *that* (hp_f) *holds and that*

$$t \to \frac{f(x,t)}{t^{p-1}} \text{ is (strictly) decreasing for a.e. } x \in \Omega.$$

Then,

$$u \le v \quad \text{in } \Omega.$$

The proof relies on the one of Theorem 1.5 in [7], also correcting an inaccuracy in the choice of the test functions. An immediate consequence of Theorem 1.5 in a more regular context is the following uniqueness result.

COROLLARY 1.6. *Let* $0 < q < p-1$ *and* $0 \le h(x) \in L^\infty(\Omega)$. *The problem*

$$\begin{cases} -\Delta_p^H u = h(x)u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.8}$$

has at most one positive weak solution $u \in C^1(\overline{\Omega})$.

Finally, we shall use Theorem 1.5 for the study of some subdiffusive problems in $\mathbb{R}^N$. In particular, let us consider the following problem:

$$\begin{cases} -\Delta_p^H u = h(x) \text{ in } \mathcal{D}'(\mathbb{R}^N), \quad u > 0 \text{ in } \mathbb{R}^N, \\ u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N), \end{cases} \tag{$\mathcal{P}_1$}$$

with $h \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, $h \ge 0$. Let us define also the following second problem:

$$\begin{cases} -\Delta_p^H u = h(x)u^q \text{ in } \mathcal{D}'(\mathbb{R}^N), \quad q < p-1, \ u > 0 \text{ in } \mathbb{R}^N, \\ u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N). \end{cases} \tag{$\mathcal{P}_2$}$$

Our result in this context is the Brezis–Kamin result [2, Theorem 1] in the anisotropic framework.

THEOREM 1.7. *Problem* ($\mathcal{P}_1$) *has a bounded solution if and only if* ($\mathcal{P}_2$) *has a bounded solution.*

In the next section, we prove Theorems 1.5 and 1.7.

2. PROOF OF THEOREMS 1.5 AND 1.7

We start with the proof of Theorem 1.5.

PROOF OF THEOREM 1.5. For $\delta > 0$, let us define $v_\delta = (v + \delta)$ and

$$w_\delta = (u^p - v_\delta^p).$$

Since $v > 0$ a.e. in Ω, then, by continuity,

$$\operatorname{supp}(u^p - v_\delta^p)^+ \subset \Omega.$$

Therefore, recalling Definition 1.1, using (iii) of Lemma 1.4, we deduce that

$$\frac{w_\delta^+}{u^{p-1}} \quad \text{and} \quad \frac{w_\delta^+}{v_\delta^{p-1}}$$

are good test functions for ($\mathscr{P}$) and (1.3). Therefore,

$$\begin{aligned}(2.1)\quad &\int_\Omega H^{p-1}(\nabla u)\Big\langle \nabla H(\nabla u), \nabla\Big(\frac{w_\delta^+}{u^{p-1}}\Big)\Big\rangle\,dx \\ &\quad - \int_\Omega H^{p-1}(\nabla v)\Big\langle \nabla H(\nabla v), \nabla\Big(\frac{w_\delta^+}{v_\delta^{p-1}}\Big)\Big\rangle\,dx \\ &\le \int_\Omega \Big(\frac{u^{p-1}}{H^0(x)^p u^{p-1}} - \frac{v^{p-1}}{H^0(x)^p v_\delta^{p-1}}\Big) w_\delta^+\,dx \\ &\quad + \int_\Omega \Big(\frac{1}{u^\gamma u^{p-1}} - \frac{1}{v^\gamma v_\delta^{p-1}}\Big) w_\delta^+\,dx + \int_\Omega \Big(\frac{f(x,u)}{u^{p-1}} - \frac{f(x,v)}{v_\delta^{p-1}}\Big) w_\delta^+\,dx.\end{aligned}$$

We start evaluating the left-hand side of (2.1). We observe that

$$\nabla w_\delta^+ = p(u^{p-1}\nabla u - v_\delta^{p-1}\nabla v)\chi_{\{u \ge v_\delta\}},$$

where $\chi_{\{u\ge v\}}$ denotes the characteristic function of the set $\{x \in \Omega : u \ge v\}$:

$$\begin{aligned}(2.2)\quad &\int_\Omega H^{p-1}(\nabla u)\Big\langle \nabla H(\nabla u), \nabla\Big(\frac{w_\delta^+}{u^{p-1}}\Big)\Big\rangle\,dx \\ &\quad - \int_\Omega H^{p-1}(\nabla v)\Big\langle \nabla H(\nabla v), \nabla\Big(\frac{w_\delta^+}{v_\delta^{p-1}}\Big)\Big\rangle\,dx \\ &= \int_\Omega H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \frac{\nabla w_\delta^+ u^{p-1} - (p-1)u^{p-2}\nabla u w_\delta^+}{u^{2(p-1)}}\rangle\,dx \\ &\quad - \int_\Omega H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \frac{\nabla w_\delta^+ v_\delta^{p-1} - (p-1)v_\delta^{p-2}\nabla v w_\delta^+}{v_\delta^{2(p-1)}}\rangle\,dx \\ &= \int_\Omega H^p(\nabla u) - pH^{p-1}(\nabla u)\frac{v_\delta^{p-1}}{u^{p-1}}\langle \nabla H(\nabla u), \nabla v_\delta\rangle + (p-1)H^p(\nabla u)\frac{v_\delta^p}{u^p}\,dx \\ &\quad + \int_\Omega H^p(\nabla v_\delta) - pH^{p-1}(\nabla v_\delta)\frac{u^{p-1}}{v_\delta^{p-1}}\langle \nabla H(\nabla v_\delta), \nabla u\rangle + (p-1)H^p(\nabla v_\delta)\frac{u^p}{v_\delta^p}\,dx\end{aligned}$$

$$
\begin{aligned}
&\geq \int_\Omega H^p(\nabla v_\delta) - pH^{p-1}(\nabla u)\Big(\frac{v_\delta}{u}\Big)^{p-1} \langle \nabla H(\nabla u), \nabla v_\delta\rangle + (p-1)H^p(\nabla u)\frac{v_\delta^p}{u^p}\,dx \\
&\quad + \int_\Omega H^p(\nabla u) - pH^{p-1}(\nabla v_\delta)\Big(\frac{u}{v_\delta}\Big)^{p-1} \langle \nabla H(\nabla v_\delta), \nabla u\rangle + (p-1)H^p(\nabla v_\delta)\frac{u^p}{v_\delta^p}\,dx \\
&:= \int_\Omega A_1(x)\,dx + \int_\Omega A_2(x)\,dx \geq 0,
\end{aligned}
$$

where we used the fact that $A_1(x), A_2(x) \geq 0$ a.e. in Ω. This follows, using a density argument, by the fact that

$$
\begin{aligned}
H^p(\nabla v_\delta) &\geq pH^{p-1}(\nabla u)\Big(\frac{v_\delta}{u}\Big)^{p-1} \langle \nabla H(\nabla u), \nabla v_\delta\rangle + (p-1)H^p(\nabla u)\frac{v_\delta^p}{u^p}, \\
H^p(\nabla u) &\geq pH^{p-1}(\nabla v_\delta)\Big(\frac{u}{v_\delta}\Big)^{p-1} \langle \nabla H(\nabla v_\delta), \nabla u\rangle + (p-1)H^p(\nabla v_\delta)\frac{u^p}{v_\delta^p},
\end{aligned}
$$

in Ω; see Proposition [8, Proposition 3.1]. Therefore, using (2.1), we get

$$
\begin{aligned}
&\int_\Omega \Big(\frac{u^{p-1}}{H^0(x)^p u^{p-1}} - \frac{v^{p-1}}{H^0(x)^p v_\delta^{p-1}}\Big) w_\delta^+\,dx \\
&\quad + \int_\Omega \Big(\frac{1}{u^{\gamma+(p-1)}} - \frac{1}{v^\gamma v_\delta^{(p-1)}}\Big) w_\delta^+\,dx + \int_\Omega \Big(\frac{f(x,u)}{u^{p-1}} - \frac{f(x,v)}{v_\delta^{p-1}}\Big) w_\delta^+\,dx \geq 0
\end{aligned}
$$

and then, by the monotonicity of $t \to 1/t^\alpha$ and that $v < v_\delta$, it follows that

$$
\begin{aligned}
&\int_\Omega \Big(\frac{u^{p-1}}{H^0(x)^p u^{p-1}} - \frac{v^{p-1}}{H^0(x)^p v_\delta^{p-1}}\Big) w_\delta^+\,dx \\
&\quad + \int_\Omega \Big(\frac{f(x,u)}{u^{p-1}} - \frac{f(x,v)}{v_\delta^{p-1}}\Big) w_\delta^+\,dx \geq 0.
\end{aligned} \tag{2.3}
$$

We use dominated convergence in both terms of (2.3). Indeed, for the first term, we have that

$$
\Big|\frac{u^{p-1}}{H^0(x)^p u^{p-1}} - \frac{v^{p-1}}{H^0(x)^p v_\delta^{p-1}}\Big| w_\delta^+ \leq 2\frac{u^p}{H^0(x)^p} \in L^1(\Omega).
$$

For the second term, in the set $\{x \in \Omega : u \geq v\}$, we deduce

$$
\begin{aligned}
\Big|\frac{f(x,u)}{u^{p-1}} - \frac{f(x,v)}{v_\delta^{p-1}}\Big| w_\delta^+ &\leq \frac{f(x,u)}{u^{p-1}}u^p + \frac{f(x,v)}{v^{p-1}}u^p \\
&= \frac{f(x,u)}{u^{p-1}}u^p\chi_{\{u\leq 1\}} + \frac{f(x,v)}{v^{p-1}}u^p\chi_{\{u\leq 1\}} \\
&\quad + \frac{f(x,u)}{u^{p-1}}u^p\chi_{\{u> 1\}} + \frac{f(x,v)}{v^{p-1}}u^p\chi_{\{u> 1\}} \\
&\leq C(1 + u^p\chi_{\{u>1\}}) \in L^1(\Omega).
\end{aligned} \tag{2.4}
$$

We point out that to get (2.4), we used the fact that by our assumptions (see (hp_f)) $f(x,t) \leq C$ if $t \leq 1$, $f(x,t)/t^{p-1}$ is decreasing (together with the fact that $f(x,1) \in L^\infty(\Omega)$) and that $u/v \leq C$ in some neighborhood of the boundary $\partial\Omega$, thanks to (i)–(ii) of Lemma 1.4. Passing to the limit in (2.3), we have

$$\int_\Omega \left(\frac{f(x,u)}{u^{p-1}} - \frac{f(x,v)}{v^{p-1}} \right) (u^p - v^p)^+ \, dx \geq 0.$$

This actually implies ($f(x,t)/t^{p-1}$ is strictly decreasing) $(u^p - v^p)^+ = 0$ a.e. Hence, $u \leq v$ in Ω. ∎

Proof of Theorem 1.7. We start proving that

$$\text{Existence for } (\mathcal{P}_1) \implies \text{Existence for } (\mathcal{P}_2).$$

Let us consider the solution u_R of the problem

$$\begin{cases} -\Delta_p^H u_n = h(x) u_n^q & \text{in } B_n(0), \\ u_n > 0 & \text{in } B_n(0), \\ u_n = 0 & \text{on } \partial B_n(0). \end{cases} \tag{2.5}$$

Such a solution exists by minimization and belongs to $W_0^{1,p}(B_n(0)) \cap C^1(\overline{B}_n(0))$; see [1,3,5]. Moreover, u_n is unique by Corollary 1.6. The sequence u_n is increasing in n: indeed, if $n' > n$, u'_n is a supersolution to (2.5). By using Theorem 1.5 in this more regular context, we deduce that $u_n \leq u'_n$. Let C be a positive constant such that $C^{p-1-q} \geq \|u\|^q_{L^\infty(\mathbb{R}^N)}$ and u a solution to $(\mathcal{P}_1)$. Then, $v = Cu$ is a supersolution to $(\mathcal{P}_2)$. In fact,

$$-\Delta_p^H v = C^{p-1} h(x) \geq h(x) v^q, \quad \text{in } \mathbb{R}^N.$$

Using the same comparison argument, we have that $u_n \leq v$. Therefore,

$$u^* := \lim_{n \to +\infty} u_n$$

since u_n is increasing and consequently $u^* \leq v$. Using the $C^{1,\alpha}$ regularity results, exploiting the Arzelà–Ascoli theorem, we have

$$\int_{\mathbb{R}^N} H^{p-1}(\nabla u^*) \langle \nabla H(\nabla u^*), \nabla\varphi \rangle \, dx = \int_{\mathbb{R}^N} h(x) {u^*}^q \varphi \, dx,$$

for every $\varphi \in C_c^\infty(\mathbb{R}^N)$; namely, u^* is a solution to $(\mathcal{P}_2)$.

Finally, we show that

$$\text{Existence for } (\mathcal{P}_2) \implies \text{Existence for } (\mathcal{P}_1).$$

Assuming u a bounded solution to ($\mathcal{P}_2$), by the classical regularity result, we deduce that $u \in C^1(\mathbb{R}^N)$. Let us define

$$v = \frac{p-1}{p-q-1} u^{\frac{p-q-1}{p-1}}.$$

Testing with $\varphi \in C_c^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} (2.6) \quad & \int_{\mathbb{R}^N} H(\nabla v)^{p-1} \big(\nabla H(\nabla v), \nabla\varphi\big)\, dx \\ &= \int_{\mathbb{R}^N} H(\nabla u)^{p-1} \big(\nabla H(\nabla u), \nabla(u^{-q}\varphi)\big)\, dx + q \int_{\mathbb{R}^N} u^{-q-1} H(\nabla u)^p \varphi\, dx \\ &\geq \int_{\mathbb{R}^N} h(x)\varphi\, dx. \end{aligned}$$

Let $u_n \in W^{1,p}(B_n(0)) \cap C^{1,\alpha}(\overline{B}_n(0))$ the solution to

$$(2.7) \qquad \begin{cases} -\Delta_p^H u_n = h(x) & \text{in } B_n(0), \\ u_n > 0 & \text{in } B_n(0), \\ u_n = 0 & \text{on } \partial B_n(0). \end{cases}$$

Using (2.6) and (2.7), by the comparison principle, we deduce $u_n \leq v$. Moreover, u_n increase as $n \to +\infty$, again by the comparison principle. Passing to the limit, we get that $u := \lim_{n\to+\infty} u_n$ is a bounded solution to ($\mathcal{P}_1$). ∎

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