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On localisation of eigenfunctions of the Laplace operator

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Abstract. We prove (i) a simple sufficient geometric condition for localisation of a sequence of first Dirichlet eigenfunctions provided the corresponding Dirichlet Laplacians satisfy a uniform Hardy inequality and (ii) localisation of a sequence of first Dirichlet eigenfunctions for a wide class of elongating horn-shaped domains. We give examples of sequences of simply connected, planar, polygonal domains for which the corresponding sequence of first eigenfunctions with either Dirichlet or Neumann boundary conditions κ-localise in L^2.

Dedicated to the memory of our friend and colleague
Thomas Kappeler

1. Introduction

In this paper, we study the phenomenon of localisation for eigenfunctions of the Laplace operator for domains in Euclidean space. Let Ω be a non-empty open, bounded, and connected set in $\mathbb{R}^m$ with Lebesgue measure $|\Omega|$. The spectrum of the Dirichlet Laplacian acting in $L^2(\Omega)$ is discrete and consists of eigenvalues $\{\lambda_1(\Omega) \le \lambda_2(\Omega) \le \cdots\}$ accumulating at infinity only. We denote a corresponding orthonormal sequence of Dirichlet eigenfunctions by $\{u_{j,\Omega},\ j \in \mathbb{N}\}$. Throughout, we denote the L^p norm, $1 \le p \le \infty$, by $\|\cdot\|_p$. Since Ω is connected, the first eigenvalue is simple, and the corresponding eigenspace is one-dimensional. The corresponding eigenfunction is determined up to a sign and we choose $u_{1,\Omega} > 0$ and write $u_\Omega := u_{1,\Omega}$. The question of localisation is the following. Does there exist, given a small $\varepsilon \in (0,1)$, a measurable set $A_\varepsilon \subset \Omega$ with

$$\frac{|A_\varepsilon|}{|\Omega|} \le \varepsilon, \quad \int_{A_\varepsilon} u_\Omega^2 \ge 1 - \varepsilon. \tag{1.1}$$

If (1.1) holds, then

$$1 - \varepsilon \le \|u_\Omega\|_\infty^2 |A_\varepsilon| \le \varepsilon \|u_\Omega\|_\infty^2 |\Omega|. \tag{1.2}$$

We recall that (see [25, equation (26)])

$$\|u_\Omega\|_\infty \le \left(\frac{e}{2\pi m}\right)^{m/4} \lambda_1(\Omega)^{m/4}. \tag{1.3}$$

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By (1.2) and (1.3), we have that

$$\lambda_1(\Omega)|\Omega|^{2/m} \geq \frac{2\pi m}{e}\Big(\frac{1-\varepsilon}{\varepsilon}\Big)^{2/m}, \tag{1.4}$$

and the first eigenvalue is, for small ε, large compared with the Faber–Krahn lower bound. The latter states that

$$\lambda_1(\Omega)|\Omega|^{2/m} \geq \lambda_1(B_1)|B_1|^{2/m},$$

where B_1 is an open ball with radius 1.

The torsion function for an open set Ω, $0 < |\Omega| < \infty$ is the unique solution of

$$-\Delta v = 1, \quad v \in H_0^1(\Omega),$$

and is denoted by v_Ω. The torsion function is non-negative, bounded, and monotone under set inclusion. A much studied quantity is the torsional rigidity, defined by

$$T(\Omega) = \int_\Omega v_\Omega,$$

see, for example, [7] and some of the references therein. It turns out that the localisation question for the torsion function stated below in L^1 is closely related to the localisation question for the first Dirichlet eigenfunction in L^2 (see the paragraph above (1.14)). Does there exist, given a small $\varepsilon \in (0,1)$, a measurable set $A_\varepsilon \subset \Omega$ with

$$\frac{|A_\varepsilon|}{|\Omega|} \leq \varepsilon, \qquad \frac{\int_{A_\varepsilon} v_\Omega}{\int_\Omega v_\Omega} \geq 1-\varepsilon. \tag{1.5}$$

If there exists A_ε satisfying (1.5), then

$$1-\varepsilon \leq \int_{A_\varepsilon} \frac{v_\Omega}{T(\Omega)} \leq T(\Omega)^{-1}\|v_\Omega\|_\infty |A_\varepsilon| \leq T(\Omega)^{-1}\|v_\Omega\|_\infty |\Omega|\varepsilon. \tag{1.6}$$

By [28, Theorem 1], we have

$$\|v_\Omega\|_\infty \leq (4+3m\log 2)\lambda_1(\Omega)^{-1}, \tag{1.7}$$

and by the Kohler–Jobin inequality (see [21,22]), we have

$$T(\Omega)\lambda_1(\Omega)^{(m+2)/2} \geq T(B_1)\lambda_1(B_1)^{(m+2)/2}. \tag{1.8}$$

We find that

$$\lambda_1(\Omega)|\Omega|^{2/m} \geq K_m\Big(\frac{1-\varepsilon}{\varepsilon}\Big)^{2/m}, \tag{1.9}$$

where $K_m > 0$ can be read-off from (1.6), (1.7), and (1.8). Again, we see that if there exists A_ε satisfying (1.5), then the first eigenvalue is, for small ε, large compared with the Faber–Krahn lower bound.

To simplify the discussion, we define localisation for sequences. Let $p \in [1, \infty)$ be fixed, and let (Ω_n) be a sequence of open sets in $\mathbb{R}^m$ with $0 < |\Omega_n| < \infty, n \in \mathbb{N}$. For $n \in \mathbb{N}$, let $f_n \in L^p(\Omega_n), 0 < \| f_n\|_p < \infty$. Define the following collection of sequences:

$$\mathfrak{A}((\Omega_n)) = \left\{(A_n) : (\forall n \in \mathbb{N})(A_n \subset \Omega_n, A_n \text{ measurable}), \lim_{n\to\infty} \frac{|A_n|}{|\Omega_n|} = 0\right\},$$

and let

$$\kappa = \sup\left\{\limsup_{n\to\infty} \frac{\| f_n \mathbf{1}_{A_n}\|_p^p}{\| f_n\|_p^p} : (A_n) \in \mathfrak{A}((\Omega_n))\right\}, \tag{1.10}$$

where $\mathbf{1}_{A_n}$ is the indicator function. Note that $0 \le \kappa \le 1$.

We write (f_n) for the sequence of functions $f_n : \Omega_n \to \mathbb{R}, n \in \mathbb{N}$ in the following definition (see [32]).

Definition 1. We say that

(i) the sequence (f_n) κ-localises in L^p if $0 < \kappa < 1$,

(ii) the sequence (f_n) localises in L^p if $\kappa = 1$,

(iii) the sequence (f_n) does not localise in L^p if $\kappa = 0$.

We see that, using Cantor's diagonalisation procedure, the supremum in (1.10) is achieved by a maximising sequence. Let (A_n) be such a sequence. This sequence is not unique since modification by sets of measure 0 does not change κ.

For $p = 2$ and $f_n = u_{\Omega_n}$, Definition 1 (ii) is equivalent to the following. There exist sequences (ε_n) with $\lim_{n\to\infty} \varepsilon_n = 0$, and $(A_n) \in \mathfrak{A}((\Omega_n))$ such that

$$\frac{|A_n|}{|\Omega_n|} \le \varepsilon_n, \qquad \int_{A_n} u_{\Omega_n}^2 \ge 1 - \varepsilon_n. \tag{1.11}$$

Similarly, for $p = 1$ and $f_n = v_{\Omega_n}$, Definition 1 (ii) is equivalent to the following. There exist sequences (ε_n) with $\lim_{n\to\infty} \varepsilon_n = 0$ and $(A_n) \in \mathfrak{A}((\Omega_n))$ such that

$$\frac{|A_n|}{|\Omega_n|} \le \varepsilon_n, \qquad \frac{\int_{A_n} v_{\Omega_n}}{\int_{\Omega_n} v_{\Omega_n}} \ge 1 - \varepsilon_n. \tag{1.12}$$

We conclude that if either (u_{Ω_n}) localises in L^2 or (v_{Ω_n}) localises in L^1, then, by (1.11) and (1.4), or (1.12) and (1.9),

$$\lim_{n\to\infty} \lambda_1(\Omega_n)|\Omega_n|^{2/m} = +\infty. \tag{1.13}$$

We arrive at the same conclusion in the case of κ-localisation by replacing $1 - \varepsilon$ by $\kappa(1 - \varepsilon)$ in the lines above. On the other hand, by considering a sequence of rectangles (R_n), $R_n = (0, 1) \times (0, n) \subseteq \mathbb{R}^2$, we see that (1.13) is clearly not sufficient for localisation of (u_{R_n}) or of (v_{Ω_n}).

It was shown in [27, Theorem 4] that if (v_{Ω_n}) either localises or κ-localises in L^1, then the corresponding sequence of eigenfunctions (u_{Ω_n}) localises in L^2. It was pointed out below [27, Theorem 4] that the torsion function does not localise for sequences of convex sets, while it was shown in [30] that there is a wide class of open, bounded, convex, elongating sequences of sets in $\mathbb{R}^m$ for which the sequence of first Dirichlet eigenfunctions localises. See [30, Examples 8, 9, 10]. In [30, Example 10], it was shown that the sequence $(u_{\Omega_{n,\alpha}})$ localises in L^2, where

$$\Omega_{n,\alpha} = \big\{(x_1, x') \in \mathbb{R}^m : -2^{-1}n < x_1 < 2^{-1}n, (2n^{-1}|x_1|)^\alpha + |x'|^\alpha < 1\big\}, \quad n \in \mathbb{N}, \tag{1.14}$$

where $\alpha \in [1, \infty)$ is fixed. The following localisation lemma (see [30, Lemma 3]) plays a crucial role in the proof of Theorem 5 below.

Lemma 1. *For $n \in \mathbb{N}$, let $f_n \in L^2(\Omega_n)$ with $\| f_n\|_2 > 0$, and $|\Omega_n| < \infty$. Then, (f_n) localises in L^2 if and only if*

$$\lim_{n\to\infty} \frac{1}{|\Omega_n|} \frac{\| f_n\|_1^2}{\| f_n\|_2^2} = 0.$$

Lemma 1 shows that a vanishing L^1-L^2 participation ratio is equivalent to localisation.

Definition 1 above was motivated by (1.1) and (1.5). We note that the very general definition of localisation above, or alternatively vanishing L^1-L^2 participation ratio in case $p = 2$, does not provide any information on where these sequences localise. However, in some concrete examples, such as in Example 4 below, it is possible to obtain this information.

Other ratios have been defined in [14, equations (7.1)–(7.3)]. We define the L^p-L^q with $p<q$ participation ratio of a function $u \in L^p(\Omega)\cap L^q(\Omega)$ as the number $|\Omega|^{\frac{1}{q}-\frac{1}{p}} \frac{\|u\|_p}{\|u\|_q}$. It was shown in [6] that for $\Omega \subseteq \mathbb{R}^m$ convex, there exist constants $k_m < \infty$ depending on m only such that

$$\|u_\Omega\|_\infty \le k_m \Big(\frac{\rho(\Omega)}{\mathrm{diam}(\Omega)}\Big)^{1/6} \rho(\Omega)^{-m/2} \|u_\Omega\|_2, \tag{1.15}$$

where $\rho(\Omega)$ denotes the inradius of Ω and $\mathrm{diam}(\Omega)$ its diameter. It follows by (1.15) that the L^2-L^∞ ratio is bounded from below by

$$\frac{1}{|\Omega|^{1/2}} \frac{\|u_\Omega\|_2}{\|u_\Omega\|_\infty} \ge k_m^{-1} \Big(\frac{\mathrm{diam}(\Omega)}{\rho(\Omega)}\Big)^{1/6} \frac{\rho(\Omega)^{m/2}}{|\Omega|^{1/2}}. \tag{1.16}$$

In order to get an upper bound for $|\Omega|$ in terms of $\mathrm{diam}(\Omega)$ and $\rho(\Omega)$, we use John's ellipsoid theorem [20]. The latter asserts that if $\Omega \subset \mathbb{R}^d$ is convex, then there exists an ellipsoid $E(a)$ with semi-axes $(a_1, a_2, \dots, a_m)$ such that $E(a) \subset \Omega \subset E(ma)$, where $E(ma)$ is a homothety of $E(a)$ with respect to its centre by a factor m. The ellipsoid $E(a)$ is of maximal measure. We may assume, by relabelling the axes, that $a_1 \ge a_2 \ge \dots \ge a_m$. Hence, $a_m \le \rho(\Omega) \le m a_m$, and $2a_1 \le \mathrm{diam}(\Omega) \le 2m a_1$. It follows that

$$\begin{aligned} |\Omega| &\le \omega_m m^m a_1 a_2 \cdots a_m \le \omega_m m^m a_1^{m-1} a_m \\ &\le \frac{\omega_m m^m}{2^{m-1}} \mathrm{diam}(\Omega)^{m-1} \rho(\Omega). \end{aligned} \tag{1.17}$$

By (1.16) and (1.17), there exists $\tilde{k}_m$ such that

$$\frac{1}{|\Omega|^{1/2}} \frac{\|u_\Omega\|_2}{\|u_\Omega\|_\infty} \geq \tilde{k}_m \left(\frac{\rho(\Omega)}{\mathrm{diam}(\Omega)} \right)^{(3m-4)/6}. \tag{1.18}$$

If (u_{Ω_n}) localises in L^2, then, for $\varepsilon \in (0,1)$ and all n sufficiently large, we have by (1.11) that

$$1 - \varepsilon \leq \int_{A_n} u_{\Omega_n}^2 \leq |A_n| \|u_{\Omega_n}\|_\infty^2 \leq \varepsilon |\Omega_n| \|u_{\Omega_n}\|_\infty^2. \tag{1.19}$$

If moreover Ω_n are convex, then by (1.18) and (1.19), we have for all n sufficiently large,

$$\frac{\rho(\Omega_n)}{\mathrm{diam}(\Omega_n)} \leq L_m \left(\frac{\varepsilon}{1-\varepsilon} \right)^{6/(3m-4)}$$

for some finite m-dependent constant L_m. This quantifies the elongation referred to in (1.4) and (1.9).

The rich interplay between localisation and the inverse of the torsion function has been studied in [2,9] and references therein.

The main results of this paper are the following. In Section 2, we construct a sequence of simply connected, planar, polygonal domains for which the corresponding sequence of first Dirichlet eigenfunctions κ-localises in L^2 (see as well [23] for a recent analysis of the eigenfunction localisation on dumbbell domains). In Section 3, we prove a simple sufficient geometric condition for localisation of a sequence of first Dirichlet eigenfunctions provided the corresponding Dirichlet Laplacians satisfy a uniform strong Hardy inequality. In Section 4, we prove localisation for a wide class of elongating horn-shaped domains. In the case of symmetric two-sided horn-shaped domains, we give a sufficient condition for localisation of the second Dirichlet eigenfunction. The results in that section vastly improve those presented in Theorem 6 and the various examples in [30]. In particular, no convexity hypotheses are made in Theorem 5 below. In Section 5, we construct a sequence of simply connected, planar, polygonal domains for which the first non-trivial Neumann eigenfunction κ-localises in L^2.

2. Example of κ-localisation for Dirichlet eigenfunctions

In this section, we construct a sequence of simply connected, planar, polygonal domains for which the corresponding sequence of first Dirichlet eigenfunctions κ-localises in L^2.

Let $\varepsilon \in (0,1), \delta > 0$ and let $\theta \in (0,\delta)$. Define the following planar open sets. The rectangle

$$R_\varepsilon = (-\varepsilon, \varepsilon) \times (-\varepsilon^{-1}, \varepsilon^{-1})$$

so that

$$\lambda_1(R_\varepsilon) = \frac{\pi^2}{4}(\varepsilon^2 + \varepsilon^{-2}).$$

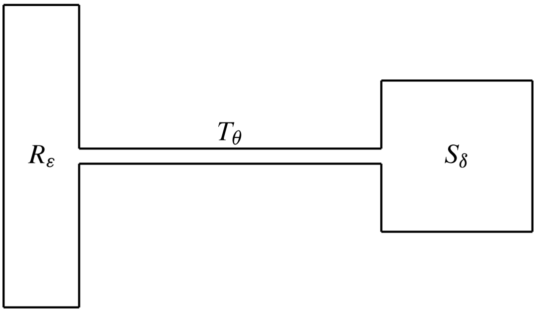

Figure 1. $\Omega_{\varepsilon,\theta,\delta} = R_\varepsilon \cup T_\theta \cup S_\delta$.

The thin rectangle

$$T_\theta = (0,2) \times (-\theta, \theta).$$

The square

$$S_\delta = (2-\delta, 2+\delta) \times (-\delta, \delta)$$

so that

$$\lambda_1(S_\delta) = \frac{\pi^2}{2\delta^2}.$$

The values of δ, θ and ε will be chosen such that $\lambda_1(S_\delta) \approx \lambda_1(R_\varepsilon)$ and $\theta \ll \varepsilon$.

Let

$$\Omega_{\varepsilon,\theta,\delta} = R_\varepsilon \cup T_\theta \cup S_\delta.$$

See Figure 1.

Since $\Omega_{\varepsilon,\theta,\delta}$ is connected, $\lambda_1(\Omega_{\varepsilon,\theta,\delta})$ is simple. Let $u_{\Omega_{\varepsilon,\theta,\delta}}$ be the corresponding positive, L^2-normalised eigenfunction.

Theorem 2. *Let $\kappa \in (0,1)$ be fixed. There exists a sequence of sets of the form $\Omega_{\varepsilon,\theta,\delta}$ for which the first Dirichlet eigenfunction κ-localises.*

Proof. Step 1. Fix $\varepsilon > 0$, and choose

$$\delta_\varepsilon = \frac{\sqrt{2}\varepsilon}{\sqrt{1+\varepsilon^4}}. \tag{2.1}$$

Then,

$$\lambda_1(S_{\delta_\varepsilon}) = \lambda_1(R_\varepsilon) = \frac{\pi^2}{2\delta_\varepsilon^2} = \frac{\pi^2}{4}(\varepsilon^2 + \varepsilon^{-2}).$$

Step 2. For $n \in \mathbb{N}$, $n \ge \frac{4}{\delta_\varepsilon}$, $\delta \in [\delta_\varepsilon - \frac{1}{n}, \delta_\varepsilon + \frac{1}{n}]$, and $\theta \in (0, \frac{\delta_\varepsilon}{4})$, we define

$$F(\theta,\delta) = \int_{T_\theta \cup S_\delta} u^2_{\Omega_{\varepsilon,\theta,\delta}}.$$

Since $\Omega_{\varepsilon,\theta,\delta}$ is simply connected, the perturbation of the parameters θ, δ is γ-continuous (see, for instance, [8, Chapter 4]). Hence, F is continuous on

$$(0, \delta_\varepsilon/4) \times [\delta_\varepsilon - n^{-1}, \delta_\varepsilon + n^{-1}].$$

Moreover, we observe that

$$\lim_{\theta \downarrow 0} F(\theta, \delta_\varepsilon - n^{-1}) = 0$$

and

$$\lim_{\theta \downarrow 0} F(\theta, \delta_\varepsilon + n^{-1}) = 1.$$

Setting

$$\eta = \eta_{n,\varepsilon} := \frac{1}{2} \min\left\{\frac{1}{n}, \frac{\delta_\varepsilon}{4}\right\} = \frac{1}{2n},$$

we define the curve $C_\eta : [0, \pi] \to \mathbb{R}^2$ by

$$C_\eta(t) = \left(\eta \sin t, \delta_\varepsilon - \frac{1}{n} + \frac{2t}{\pi n}\right), \quad 0 \le t \le \pi.$$

The function F is continuous along C_η and takes the value 0 at $t = 0$ and 1 at $t = \pi$. By continuity, there exists $t^* \in (0, \pi)$ such that

$$F(C_\eta(t^*)) = \kappa.$$

Let $C_\eta(t^*) = (\theta_{n,\varepsilon}, \delta_{n,\varepsilon})$.

Step 3. In this step, we keep ε constant, and let $n \to +\infty$. We have that

$$\Omega_{\varepsilon,\theta_{n,\varepsilon},\delta_{n,\varepsilon}} \xrightarrow{\gamma} R_\varepsilon \cup S_{\delta_\varepsilon}$$

γ-converges. We get

$$\lim_{n\to\infty} \lambda_1(\Omega_{\varepsilon,\theta_{n,\varepsilon},\delta_{n,\varepsilon}}) = \lambda_1(R_\varepsilon \cup S_{\delta_\varepsilon}) = \frac{\pi^2}{4}(\varepsilon^2 + \varepsilon^{-2}).$$

Moreover, $u_{\Omega_{\varepsilon,\theta_{n,\varepsilon},\delta_{n,\varepsilon}}}$ converges in $H^1(\mathbb{R}^2)$ to an eigenfunction $u \in H^1_0(R_\varepsilon \cup S_{\delta_\varepsilon})$ corresponding to the first eigenvalue of $R_\varepsilon \cup S_{\delta_\varepsilon}$. By our choice of t^*, we get

$$\int_{S_{\delta_\varepsilon}} u^2 = \kappa, \quad \int_{R_\varepsilon} u^2 = 1 - \kappa. \tag{2.2}$$

We now keep track of the L^∞-norm of $u_{\Omega_{\varepsilon,\theta_{n,\varepsilon},\delta_{n,\varepsilon}}}$ on R_ε, and claim that

$$\lim_{n\to+\infty} \|u_{\Omega_{\varepsilon,\theta_{n,\varepsilon},\delta_{n,\varepsilon}}}\|_{L^\infty(R_\varepsilon)} = \|u\|_{L^\infty(R_\varepsilon)}. \tag{2.3}$$

By the a.e. pointwise convergence, we have that

$$\|u\|_{L^\infty(R_\varepsilon)} \le \liminf_{n\to+\infty} \|u_{\Omega_{\varepsilon,\theta_{n,\varepsilon},\delta_{n,\varepsilon}}}\|_{L^\infty(R_\varepsilon)}.$$

In order to prove the converse inequality, we follow a classical strategy (see, for instance, [17, Theorem 2.2] or [27] and references therein). From the eigenvalue monotonicity with respect to inclusions, we obtain by (1.3)

$$-\Delta u_{\Omega_{\varepsilon,\theta_{n,\varepsilon},\delta_{n,\varepsilon}}} \leq \lambda_1^{3/2}(S_{\delta_{n,\varepsilon}/4}) := M_\varepsilon \quad \text{in } \mathcal{D}'(\mathbb{R}^2).$$

Then, for every point $x_n \in \mathbb{R}^2$, we get

$$-\Delta\left(u_{\Omega_{\varepsilon,\theta_{n,\varepsilon},\delta_{n,\varepsilon}}} + M_\varepsilon \frac{|\cdot - x_n|^2}{4}\right) \leq 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2).$$

So, by subharmonicity,

$$u_{\Omega_{\varepsilon,\theta_{n,\varepsilon},\delta_{n,\varepsilon}}}(x_n) \leq \frac{\int_{B(x_n;\rho)} dx \left(u_{\Omega_{\varepsilon,\theta_{n,\varepsilon},\delta_{n,\varepsilon}}}(x) + M_\varepsilon \frac{|x-x_n|^2}{4}\right)}{|B(x_n;\rho)|},$$

where $B(p;r) = \{x \in \mathbb{R}^m : |p-x| < r\}$ for $p \in \mathbb{R}^m, r > 0$. Let $x_n \in R_\varepsilon$ be such that

$$\|u_{\Omega_{\varepsilon,\theta_{n,\varepsilon},\delta_{n,\varepsilon}}}\|_{L^\infty(R_\varepsilon)} - \frac{1}{n} \leq u_{\Omega_{\varepsilon,\theta_{n,\varepsilon},\delta_{n,\varepsilon}}}(x_n).$$

Taking the limit $n \to +\infty$, and assuming without loss of generality that $x_n \to x_0$, we get

$$\begin{aligned}\limsup_{n\to+\infty} \|u_{\Omega_{\varepsilon,\theta_{n,\varepsilon},\delta_{n,\varepsilon}}}\|_{L^\infty(R_\varepsilon)} &\leq \frac{\int_{B(x_0;\rho)} dx \left(u(x) + M_\varepsilon \frac{|x-x_0|^2}{4}\right)}{|B(x_0;\rho)|} \\ &\leq \|u\|_{L^\infty(R_\varepsilon)} + M_\varepsilon \frac{\rho^2}{8}.\end{aligned}$$

Taking the limit $\rho \downarrow 0$, we obtain (2.3).

Since u is a first eigenfunction on R_ε, we have that $\frac{\|u\|_\infty}{\|u\|_2} = \frac{2}{|R_\varepsilon|^{\frac{1}{2}}}$. Consequently, from (2.2), we get

$$\|u\|_{L^\infty(R_\varepsilon)} = 2\sqrt{1-\kappa}.$$

Step 4. Now, let $\varepsilon \downarrow 0$. For every such ε, we pick up from Step 3 some $n = n_\varepsilon$ such that

$$\|u_{\Omega_{\varepsilon,\theta_{n_\varepsilon},\delta_{n_\varepsilon}}}\|_{L^\infty(R_\varepsilon)} \leq 2\sqrt{1-\kappa} + \varepsilon. \tag{2.4}$$

This sequence κ-localises on $T_{\theta_{n,\varepsilon}} \cup S_{\delta_{n,\varepsilon}}$. ∎

The data in Figure 2 have been obtained with the MATLAB PDE toolbox and illustrate the mass distribution of the first eigenfunction.

We make the following observation. Given $\varepsilon > 0$, and assume that δ is chosen slightly higher than the critical value in equation (2.1). In this case, the first eigenfunction will be (almost) supported by the square, while the second by the rectangle, provided the connecting tube is thin enough. In such a way we can construct a sequence of domains for which the first eigenfunctions localise (on the squares) while the second eigenfunctions

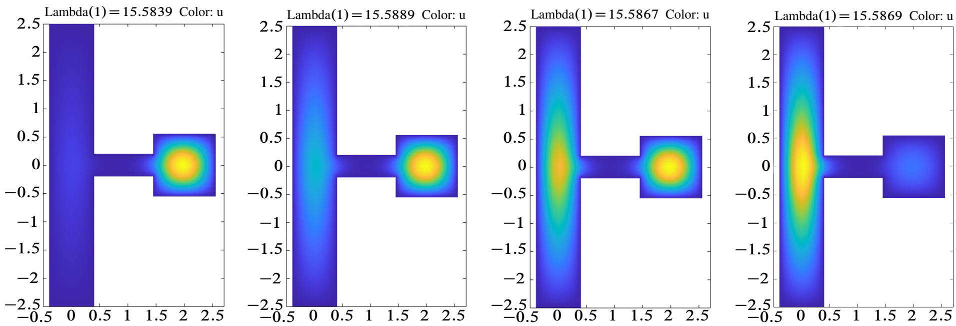

Figure 2. The mass distribution of u_1 when perturbing the size of the square on the right: $\varepsilon = 0.4$, $\theta = 0.2$, $\delta = \frac{\sqrt{2\varepsilon}}{\sqrt{1+\varepsilon^4}} - c$, for $c = 0.00281$, $c = 0.00286$, $c = 0.00287$, $c = 0.00292$, respectively.

do not localise. Assume now that δ is chosen slightly smaller than the critical value in equation (2.1). In this case, the second eigenfunction will be (almost) supported by the square, while the first one by the rectangle, provided the connecting tube is thin enough. In such a way, we construct a sequence of domains for which the second eigenfunctions localise (on the squares) while the first eigenfunctions do not localise. We conclude that there is no direct relationship between localisations of the first and second eigenfunctions, respectively. It is also possible to construct a sequence of domains for which both the first and the second Dirichlet eigenfunctions localise in L^2. Let Ω_n be a rhombus with four sides of length n and one diagonal of length 1, with $n \to +\infty$. The corresponding first eigenfunctions localise in L^2 (see [15, 16, 18] or Theorem 5 in this paper). The nodal line of the second eigenfunction is the shortest diagonal, so the second eigenfunction is a first eigenfunction on an elongating triangle and localises in L^2 as well.

3. Localisation of the first Dirichlet eigenfunction and Hardy's inequality

The results in this section are obtained under the hypothesis that the Dirichlet Laplacian satisfies the strong Hardy inequality. The mechanism for localisation is that the distance function is small on a very large set. The Hardy inequality implies that the boundary of this set is not thin, in terms of potential theory (see [1]). This in turn implies that the eigenfunction is small on this large set and has most of its L^2 mass on the complement.

Definition 2. The Dirichlet Laplacian $-\Delta$ acting in $L^2(\Omega)$ satisfies the strong Hardy inequality, with constant $c_\Omega \in (0, \infty)$, if

$$\|\nabla w\|_2^2 \geq \frac{1}{c_\Omega} \int_\Omega \frac{w^2}{d_\Omega^2} \quad \forall\, w \in C_c^\infty(\Omega), \tag{3.1}$$

where d_Ω is the distance to the boundary function,

$$d_\Omega(x) = \inf\left\{|x-y| : y \in \mathbb{R}^m \setminus \Omega\right\}, \quad x \in \Omega.$$

Both the validity and applications of inequalities like (3.1) to spectral theory and partial differential equations have been investigated in depth. See, for example, [1, 10–13]. In particular, it was shown in [1, page 208], that for any proper simply connected open subset Ω in $\mathbb{R}^2$, inequality (3.1) holds with $c_\Omega = 16$. The following was proved in [32].

Let (Ω_n) be a sequence of open sets in $\mathbb{R}^m$ with $0 < |\Omega_n| < \infty, n \in \mathbb{N}$, which satisfy (3.1) with strong Hardy constants c_{Ω_n}. Suppose

$$\mathrm{c} = \sup\{c_{\Omega_n} : n \in \mathbb{N}\} < \infty. \tag{3.2}$$

(i) If (η_n) is a sequence of strictly positive real numbers such that

$$\lim_{n\to\infty} \frac{|\{d_{\Omega_n} \ge \eta_n\}|}{|\Omega_n|} = 0 \tag{3.3}$$

and

$$\lim_{n\to\infty} \frac{\eta_n^2 |\Omega_n|}{\int_{\{d_{\Omega_n} \ge \eta_n\}} d_{\Omega_n}^2} = 0, \tag{3.4}$$

then (v_{Ω_n}) localises along the sequence (A_n), where $A_n = \{x \in \Omega_n : d_{\Omega_n} \ge \eta_n\}$.

(ii) If any sequence (A_n) of measurable sets, $A_n \subset \Omega_n$, $n \in \mathbb{N}$, with

$$\lim_{n\to\infty} \frac{|A_n|}{|\Omega_n|} = 0,$$

satisfies

$$\lim_{n\to\infty} \frac{\int_{A_n} d_{\Omega_n}^2}{\int_{\Omega_n} d_{\Omega_n}^2} = 0,$$

then (v_{Ω_n}) does not localise.

In [27, Theorem 4], it was shown that if (v_{Ω_n}) localises in L^1, then (u_{Ω_n}) localises in L^2. This, together with the assertion under (i) above, implies localisation of (u_{Ω_n}) provided (3.3) and (3.4) hold. The following result asserts localisation of (u_{Ω_n}) under weaker assumptions.

Theorem 3. *Let (Ω_n) be a sequence of open sets in $\mathbb{R}^m$ with $0 < |\Omega_n| < \infty, n \in \mathbb{N}$, which satisfies (3.2). If there exists a sequence (A_n) of measurable sets, $A_n \subset \Omega_n, n \in \mathbb{N}$, with*

$$\lim_{n\to\infty} \frac{|A_n|}{|\Omega_n|} = 1, \tag{3.5}$$

and which satisfies

$$\lim_{n\to\infty} \frac{\sup_{A_n} d_{\Omega_n}}{\max_{\Omega_n} d_{\Omega_n}} = 0, \tag{3.6}$$

then, for every $k \in \mathbb{N}$, (u_{k,Ω_n}) localises in L^2, where u_{k,Ω_n} is an L^2-normalised eigenfunction corresponding to the kth eigenvalue on Ω_n.

Proof. For an open set with finite measure Ω and with a Hardy constant c_Ω, let $u_{k,\Omega}$ be a kth Dirichlet eigenfunction normalised in $L^2(\Omega)$. By Cauchy–Schwarz and (2.4), we have, for any measurable set $A \subset \Omega$,

$$\begin{aligned}\int_A u_{k,\Omega}^2 &\le \int_A \frac{u_{k,\Omega}^2}{d_\Omega^2}\big(\sup_A d_\Omega\big)^2 \\ &\le \big(\sup_A d_\Omega\big)^2 \int_\Omega \frac{u_{k,\Omega}^2}{d_\Omega^2} \\ &\le c_\Omega \big(\sup_A d_\Omega\big)^2 \int_\Omega |\nabla u_{k,\Omega}|^2 \\ &= c_\Omega \lambda_k(\Omega)\big(\sup_A d_\Omega\big)^2. \end{aligned} \tag{3.7}$$

Since Ω contains a ball of radius $\frac{1}{2}\sup_\Omega d_\Omega$, we have by monotonicity of Dirichlet eigenvalues that

$$\lambda_k(\Omega) \le 4\lambda_k(B_1)\big(\sup_\Omega d_\Omega\big)^{-2}. \tag{3.8}$$

By (3.7) and (3.8), we have

$$\int_A u_{k,\Omega}^2 \le 4c_\Omega \lambda_k(B_1)\left(\frac{\sup_A d_\Omega}{\sup_\Omega d_\Omega}\right)^2.$$

This implies the assertion in Theorem 3, since $\lim_{n\to\infty}\int_{A_n} u_{k,\Omega_n}^2 = 0$, and so,

$$\lim_{n\to\infty}\int_{\Omega_n\setminus A_n} u_{k,\Omega_n}^2 = 1.$$

By (3.5), $\lim_{n\to\infty} |\Omega_n \setminus A_n|/|\Omega_n| = 0$. Hence, (u_{k,Ω_n}) localises in L^2. ∎

Below, we show that the hypotheses (3.3)–(3.4) imply those of Theorem 3. Let $A_n = \{x \in \Omega_n : d_{\Omega_n} < \eta_n\}$. Hence, (3.3) implies (3.5). Furthermore,

$$\begin{aligned}\frac{\eta_n^2|\Omega_n|}{\int_{\{d_{\Omega_n}\ge\eta_n\}} d_{\Omega_n}^2} &\ge \frac{\eta_n^2|\Omega_n|}{\int_{\{d_{\Omega_n}\ge\eta_n\}} \sup_{\Omega_n} d_{\Omega_n}^2} \\ &\ge \frac{\eta_n^2}{\sup_{\Omega_n} d_{\Omega_n}^2} \\ &\ge \left(\frac{\sup_{A_n} d_{\Omega_n}}{\sup_{\Omega_n} d_{\Omega_n}}\right)^2. \end{aligned}$$

Hence, (3.4) implies (3.6).

To prove that the hypotheses in Theorem 3 are strictly weaker than (3.3), (3.4), we have the following.

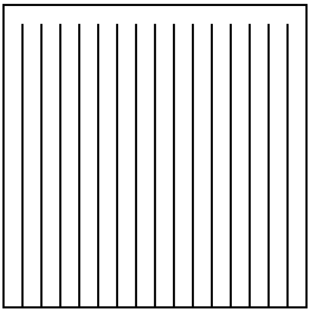

Figure 3. $\Omega_{n,\alpha,d}$ with $n-1$ parallel vertical line segments at distance n^{-1} of length $1-dn^{-\alpha}$ in the open unit square Q.

Example 4. Let Q be the open unit square in $\mathbb{R}^2$ with vertices $(0,0), (1,0), (1,1)$ and $(0,1)$. Let $0<\alpha<1, 0<d<\infty$, and let $n\in\mathbb{N}$ be such that $dn^{-\alpha}<1$. For $a,b\in\mathbb{R}^2$, we denote by $L_{a,b}$ the closed line segment with endpoints a and b, respectively. For $i=1,\dots,n-1$, let $a_i=(\frac{i}{n},0), b_i=(\frac{i}{n},1-dn^{-\alpha})$. The set

$$\Omega_{n,\alpha,d}\Big\backslash\bigcup_{i=1}^{n-1}L_{a_i,b_i}$$

is open, simply connected with $|\Omega_{n,\alpha,d}|=1$. See Figure 3. Hardy's inequality holds with $c=c_{\Omega_{n,\alpha,d}}=16$. It was shown in [32] that $(v_{\Omega_{n,\alpha,d}})$ localises in L^1 for $0<\alpha<\frac{2}{3}$, and does not localise for $\frac{2}{3}<\alpha<1$. The proof that $(v_{\Omega_{n,\frac{2}{3},d}})$ κ-localises with $\kappa=\frac{d^3}{1+d^3}$ is quite involved (see [32]). To prove that $(u_{\Omega_{n,\alpha,d}})$ localises in L^2 for all $0<\alpha<1$, we choose $A_n=\{x\in\Omega_{n,\alpha,d}:d_{\Omega_{n,\alpha,d}}<\frac{1}{2n}\}$. Then, $\sup_{A_n}d_{\Omega_{n,\alpha,d}}\le\frac{1}{2n}$, and

$$\max_{\Omega_{n,\alpha,d}}d_{\Omega_{n,\alpha,d}}\ge\frac{1}{2}dn^{-\alpha}.$$

Hence, (3.6) is satisfied. Also, $|A_n|\ge 1-dn^{-\alpha}$, which implies (3.5). This implies localisation by Theorem 3. We see that localisation takes place in a neighbourhood of the rectangle $\Omega_{n,\alpha,d}\cap\{x_2>1-dn^{-\alpha}\}$.

4. Localisation of the first Dirichlet eigenfunction for elongated horn-shaped regions

Below, we obtain localisation results for sequences of sets in $\mathbb{R}^m$ which satisfy a monotonicity property in the x_1-direction along which elongation takes place. This monotonicity property is known in the literature as horn-shaped. The Dirichlet spectrum and eigenfunctions of horn-shaped open sets have been studied extensively in the non-compact setting in, for example, [5,24,29] and references there in. In [30], it was used to prove localisation for various examples such as (1.14) mentioned above. We recall the setup and notation.

Definition 3. Points in $\mathbb{R}^m$ are denoted by a Cartesian pair (x_1, x') with $x_1 \in \mathbb{R}, x' \in \mathbb{R}^{m-1}$. If Ω is an open set in $\mathbb{R}^m$, then we define its cross-section at x_1 by $\Omega(x_1) = \{x' \in \mathbb{R}^{m-1} : (x_1, x') \in \Omega\}$. A set $\Omega \subset \mathbb{R}^m$ is horn-shaped if it is non-empty, open, and connected, $x_1 > x_2 > 0$ implies $\Omega(x_1) \subset \Omega(x_2)$, and $x_1 < x_2 < 0$ implies $\Omega(x_1) \subset \Omega(x_2)$.

Let Ω' be an open set in $\mathbb{R}^{m-1}$. Its first $(m-1)$-dimensional Dirichlet eigenvalue is denoted by $\mu(\Omega')$, and its $(m-1)$-dimensional Lebesgue measure is denoted by $|\Omega'|_{m-1}$. For $a > 0$, we let $a\Omega'$ be the homothety of Ω' by a factor a with respect to that origin.

Let $-\infty < c_- \le 0 < c_+ < \infty$. We consider the following class of monotone functions.

$$\begin{aligned}\mathfrak{F} = \big\{ f : [c_-, c_+] \to [0,1],\ &\text{non-increasing, and right-continuous on } [0, c_+],\\ &\text{non-decreasing, and left-continuous on } [c_-, 0],\ f(0) = 1,\ f(x_1) < 1 \text{ for } x_1 \ne 0 \big\}.\end{aligned}$$

Given $f \in \mathfrak{F}$, let

$$f_n : [nc_-, nc_+] \to [0,1], \quad f_n(x_1) = f(x_1/n),$$

let $\Omega' \subset \mathbb{R}^{m-1}$ be a non-empty, open, bounded, and convex set containing the origin, and let

$$\Omega_{f_n,\Omega'} = \big\{(x_1, x') \in \mathbb{R}^m : c_- n < x_1 < c_+ n, x' \in f(x_1/n)\Omega'\big\}.$$

Theorem 5. (i) *If f and Ω' satisfy the hypotheses above, then $(u_{\Omega_{f_n,\Omega'}})$ localises in L^2.*

(ii) *If $m = 2$, if $f \in \mathfrak{F}$ is concave such that $-c_- = c_+$, $f(x_1) = f(-x_1), 0 \le x_1 < c_+$, and if Ω' is an interval of length 1 containing the origin, then $(u_{2,\Omega_{f_n,\Omega'}})$ localises in L^2.*

The proof requires some lemmas which are given below.

The following is a generalisation of a two-dimensional bound. See [26, Theorem 2].

Lemma 6. *Let Ω' be a non-empty open, bounded, and convex set in $\mathbb{R}^{m-1}$ which contains the origin, let $f \in \mathfrak{F}$, and let*

$$N^* = \min\big\{n \in \mathbb{N} : n \ge 1, f(c_+ n^{-1/2}) \ge 2^{-1}\big\}. \tag{4.1}$$

If $n \ge N^*$, then

$$\lambda_1(\Omega_{f_n,\Omega'}) \le \mu(\Omega') + \frac{\pi^2}{c_+^2 n} + 6\mu(\Omega')(1 - f(c_+ n^{-1/2})). \tag{4.2}$$

The proof is similar in spirit to the one in [31, page 2095] and runs as follows.

Proof. Consider the cylinder $C_{f_n,\delta}$ with base $f_n(\delta)\Omega'$ and height δ with $\delta < c_+ n$. By separation of variables,

$$\lambda_1(C_{f_n,\delta}) = \frac{\pi^2}{\delta^2} + (f_n(\delta))^{-2}\mu(\Omega').$$

By monotonicity of Dirichlet eigenvalues under inclusion,

$$\begin{aligned}\lambda_1(\Omega_{f_n,\Omega'}) &\le \lambda_1(C_{f_n,\delta})\\ &= \frac{\pi^2}{\delta^2} + (f(\delta/n))^{-2}\mu(\Omega')\\ &= \mu(\Omega') + \frac{\pi^2}{\delta^2} + (1-f(\delta/n))\frac{1+f(\delta/n)}{(f(\delta/n))^2}\mu(\Omega').\end{aligned}$$

Choose $\delta = n^{1/2}c_+$ so that

$$\lambda_1(\Omega_{f_n,\Omega'}) \le \mu(\Omega') + \frac{\pi^2}{c_+^2 n} + \frac{1+f(c_+n^{-1/2})}{(f(c_+n^{-1/2}))^2}(1-f(c_+n^{-1/2}))\mu(\Omega').$$

Since f is right-continuous at 0, $N^* < \infty$. Furthermore, since f is non-increasing on $[0, c_+]$ and $(1+f)f^{-2}$ is non-decreasing for $f > 0$, we have by (4.1) that

$$\frac{1+f(c_+n^{-1/2})}{(f(c_+n^{-1/2}))^2} \le 6,\ n \ge N^*. \qquad \blacksquare$$

The Dirichlet heat kernel for an open set Ω is denoted by $p_\Omega(x, y; t)$, $x \in \Omega$, $y \in \Omega$, $t > 0$. If $|\Omega| < \infty$, then the spectrum of the Dirichlet Laplacian is discrete, and the corresponding Dirichlet heat kernel has an L^2-eigenfunction expansion given by

$$p_\Omega(x, y; t) = \sum_{j=1}^{\infty} e^{-t\lambda_j(\Omega)} u_{j,\Omega}(x) u_{j,\Omega}(y).$$

Recall that

$$w_\Omega(x; t) = \int_\Omega dy\ p_\Omega(x, y; t)$$

is the solution of the heat equation

$$\Delta w = \frac{\partial w}{\partial t}, \quad x \in \Omega,\ t > 0,$$

with Dirichlet boundary condition

$$w(\cdot; t) \in H_0^1(\Omega; t)$$

and initial condition

$$w(x; 0) = 1, \quad x \in \Omega.$$

The heat content for an open set $\Omega \subset \mathbb{R}^m$ with finite Lebesgue measure at t is given by

$$Q_\Omega(t) = \int_\Omega \int_\Omega dx\, dy\ p_\Omega(x, y; t).$$

We denote by Ω' an open set in $\mathbb{R}^{m-1}$. Its heat content (in dimension $m-1$) is also denoted by $Q_{\Omega'}(t)$.

Lemma 7. *If* $|\Omega| < \infty$*, then*

$$Q_\Omega(t) \le e^{-t\lambda_1(\Omega)}|\Omega| \tag{4.3}$$

and

$$\frac{1}{|\Omega|}\left(\int_\Omega u_\Omega\right)^2 \le \frac{e^{t\lambda_1(\Omega)}}{|\Omega|}Q_\Omega(t). \tag{4.4}$$

If $|\Omega'|_{m-1} < \infty$*, then*

$$Q_{\Omega'}(t) \le e^{-t\mu(\Omega')}|\Omega'|_{m-1}. \tag{4.5}$$

Proof. It follows from Parseval's identity that

$$Q_\Omega(t) = \sum_{j\in\mathbb{N}} e^{-t\lambda_j(\Omega)}\left(\int_\Omega u_{j,\Omega}\right)^2 \le e^{-t\lambda_1(\Omega)}\sum_{j\in\mathbb{N}}\left(\int_\Omega u_{j,\Omega}\right)^2 = e^{-t\lambda_1(\Omega)}|\Omega|. \tag{4.6}$$

This proves (4.3). The first equality in (4.6) implies (4.4). Inequality (4.5) is the $(m-1)$-dimensional version of (4.3). ∎

Let $(B(s), s \ge 0; \mathbb{P}_x, x \in \mathbb{R}^m)$ be Brownian motion on $\mathbb{R}^m$ with generator Δ. For $x \in \Omega$, we denote the first exit time of Brownian motion by

$$T_\Omega = \inf\{s \ge 0 : B(s) \notin \Omega\}.$$

It is a standard fact that

$$w_\Omega(x;t) = \mathbb{P}_x[T_\Omega > t]. \tag{4.7}$$

So, this gives

$$\frac{1}{|\Omega|}\left(\int_\Omega u_\Omega\right)^2 \le \frac{e^{t\lambda_1(\Omega)}}{|\Omega|}\int_\Omega dx\, \mathbb{P}_x[T_\Omega > t].$$

The lemma below extends [29, Theorem 5.3] to two-sided, horn-shaped regions.

Lemma 8. *Let* Ω *be horn-shaped in* $\mathbb{R}^m$*, and let both* $|\Omega| < \infty$*, and* $|\Omega'|_{m-1} < \infty$*. If* $t > 0$*, then*

$$Q_\Omega(t) \le \int_{[c_-,c_+]} dx_1\, Q_{\Omega(x_1)}(t) + 4\left(\frac{t}{\pi}\right)^{1/2} Q_{\Omega'}(t). \tag{4.8}$$

Proof. It is convenient to define for horn-shaped sets,

$$\Omega^- = \Omega \cup \{(x_1, x') \in \mathbb{R}^m : x_1 \le 0, x' \in \Omega'\} \tag{4.9}$$

and

$$\Omega^+ = \Omega \cup \{(x_1, x') \in \mathbb{R}^m : x_1 \ge 0, x' \in \Omega'\}.$$

For $x \in \Omega, x_1 > 0$, we have by (4.9)

$$\mathbb{P}_x[T_\Omega > t] \le \mathbb{P}_x[T_{\Omega^-} > t].$$

Let $(B_1(s),\ s \geq 0)$ be 1-dimensional Brownian motion in the x_1-direction, and let $(B'(s),\ s \geq 0)$ be an independent $(m-1)$-dimensional Brownian motion in the x'-plane. Then, $B = (B_1, B')$. By solving the heat equation on $(-\infty, \xi) \times (0, \infty)$ with $\xi > 0$, we have by (4.7) and the preceding lines,

$$\mathbb{P}_0[T_{(-\infty,\xi)} > t] = \int_{(0,\xi)} d\eta (\pi t)^{-1/2} e^{-\eta^2/(4t)}.$$

Hence, the density of the random variable $\max_{0 \leq s \leq t} B_1(s)$ with $B_1(0) = 0$ is given by

$$\rho(\xi; t) = (\pi t)^{-1/2} e^{-\xi^2/(4t)} 1_{(0,\infty)}(\xi),$$

with a similar expression for $\min_{0 \leq s \leq t} B_1(s)$. For $x \in \Omega$, $x_1 > 0$,

$$\begin{aligned}
&\mathbb{P}_x[T_{\Omega^-} > t] \\
&\quad \leq \int_{\mathbb{R}^+} d\xi\, \rho(\xi; t) \mathbb{P}_{x'}[T_{\Omega^-(x_1-\xi)} > t] \\
&\quad = \int_{(0,x_1)} d\xi\, \rho(\xi; t) \mathbb{P}_{x'}[T_{\Omega^-(x_1-\xi)} > t] + \int_{(x_1,\infty)} d\xi\, \rho(\xi; t) \mathbb{P}_{x'}[T_{\Omega^-(x_1-\xi)} > t] \\
&\quad = \int_{(0,x_1)} d\xi\, \rho(\xi; t) \mathbb{P}_{x'}[T_{\Omega(x_1-\xi)} > t] + \int_{(x_1,\infty)} d\xi\, \rho(\xi; t) \mathbb{P}_{x'}[T_{\Omega'} > t].
\end{aligned}$$

We obtain that

$$\begin{aligned}
&\int_{\Omega \cap \{0 \leq x_1 \leq c_+\}} dx\, w_\Omega(x; t) \\
&\quad \leq \int_{[0,c_+]} dx_1 \int_{\Omega(x_1)} dx' \int_{(0,x_1)} d\xi\, \rho(\xi; t) \mathbb{P}_{x'}[T_{\Omega(x_1-\xi)} > t] \\
&\qquad + \int_{[0,c_+]} dx_1 \int_{(x_1,\infty)} d\xi\, \rho(\xi; t) \int_{\Omega'} dx'\, \mathbb{P}_{x'}[T_{\Omega'} > t].
\end{aligned} \tag{4.10}$$

By Tonelli's theorem, we obtain that the first term on the right-hand side of (4.10) equals

$$\begin{aligned}
&\int_{[0,c_+]} dx_1 \int_{(0,x_1)} d\xi\, \rho(\xi; t) \int_{\Omega(x_1)} dx'\, \mathbb{P}_{x'}[T_{\Omega(x_1-\xi)} > t] \\
&\quad \leq \int_{[0,c_+]} dx_1 \int_{(0,x_1)} d\xi\, \rho(\xi; t) \int_{\Omega(x_1-\xi)} dx'\, \mathbb{P}_{x'}[T_{\Omega(x_1-\xi)} > t] \\
&\quad = \int_{[0,c_+]} dx_1 \int_{(0,x_1)} d\xi\, \rho(\xi; t) Q_{\Omega(x_1-\xi)}(t) \\
&\quad = \int_{[0,c_+]} dx_1\, Q_{\Omega(x_1)}(t),
\end{aligned} \tag{4.11}$$

where we have used in the last equality that the integral of a convolution is the product of the integrals, and that the integral of a probability density equals 1. For the second term on the right-hand side of (4.10), we obtain by an integration by parts that

$$\begin{aligned}\int_{[0,c_+]} dx_1 \int_{(x_1,\infty)} d\xi\, \rho(\xi;t) \int_{\Omega'} dx'\, \mathbb{P}_{x'}[T_{\Omega'} > t] \\ \le \int_{[0,\infty)} dx_1 \int_{(x_1,\infty)} d\xi\, \rho(\xi;t) \int_{\Omega'} dx'\, \mathbb{P}_{x'}[T_{\Omega'} > t] \\ = \left(\frac{4t}{\pi}\right)^{1/2} Q_{\Omega'}(t).\end{aligned} \tag{4.12}$$

By (4.11) and (4.12), we have

$$\int_{\Omega\cap\{0\le x_1\le c_+\}} dx\, w_\Omega(x;t) \le \int_{[0,c_+]} dx_1 Q_{\Omega(x_1)}(t) + \left(\frac{4t}{\pi}\right)^{1/2} Q_{\Omega'}(t). \tag{4.13}$$

Similarly,

$$\int_{\Omega\cap\{c_-\le x_1\le 0\}} dx\, w_\Omega(x;t) \le \int_{[c_-,0]} dx_1 Q_{\Omega(x_1)}(t) + \left(\frac{4t}{\pi}\right)^{1/2} Q_{\Omega'}(t). \tag{4.14}$$

Adding the contributions from (4.13) and (4.14) gives (4.8). Note that the hypotheses on $|\Omega|$ and $|\Omega'|_{m-1}$ guarantee that the right-hand side of (4.8) is finite for all $t > 0$. ■

Proof of Theorem 5. (i) Since $f \in \mathfrak{F}$, and Ω' is convex containing the origin, $\Omega_{f,\Omega'}$ is horn-shaped. By Lemma 7 applied to the $(m-1)$-dimensional set $f_n(x_1)\Omega'$, we have

$$\begin{aligned}Q_{\Omega_{f_n,\Omega'}(x_1)}(t) &= Q_{f(x_1/n)\Omega'}(t) \\ &\le (f(x_1/n))^{m-1}|\Omega'|_{m-1}e^{-t\mu(\Omega')(f(x_1/n))^{-2}} \\ &\le |\Omega'|_{m-1}e^{-t\mu(\Omega')(f(x_1/n))^{-2}}.\end{aligned} \tag{4.15}$$

Furthermore,

$$|\Omega_{f_n,\Omega'}| = n|\Omega_{f,\Omega'}|. \tag{4.16}$$

By (4.4), (4.8), and (4.16), we have

$$\begin{aligned}&\frac{1}{|\Omega_{f_n,\Omega'}|}\left(\int_{\Omega_{f_n,\Omega'}} u_{\Omega_{f_n,\Omega'}}\right)^2 \\ &\le \frac{e^{t\lambda_1(\Omega_{f_n,\Omega'})}}{n|\Omega_{f,\Omega'}|}\left(\int_{[nc_-,nc_+]} dx_1\, Q_{\Omega_{f_n,\Omega'}(x_1)}(t) + 4\Big(\frac{t}{\pi}\Big)^{1/2} Q_{\Omega'}(t)\right) \\ &\le \frac{e^{t\lambda_1(\Omega_{f_n,\Omega'})}|\Omega'|_{m-1}}{n|\Omega_{f,\Omega'}|}\left(\int_{[nc_-,nc_+]} dx_1 e^{-t\mu(\Omega')(f(x_1/n))^{-2}} + 4e^{-t\mu(\Omega')}\Big(\frac{t}{\pi}\Big)^{1/2}\right) \\ &= \frac{e^{t\lambda_1(\Omega_{f_n,\Omega'})}|\Omega'|_{m-1}}{|\Omega_{f,\Omega'}|}\left(\int_{[c_-,c_+]} dx_1 e^{-t\mu(\Omega')(f(x_1))^{-2}} + \frac{4}{n}e^{-t\mu(\Omega')}\Big(\frac{t}{\pi}\Big)^{1/2}\right),\end{aligned} \tag{4.17}$$

where we have used (4.15) and (4.5) in the third line above. By (4.2) and (4.17), we have

$$\frac{1}{|\Omega_{f_n,\Omega'}|}\left(\int_{\Omega_{f_n,\Omega'}} u_{\Omega_{f_n,\Omega'}}\right)^2 \le \frac{e^{t\left(\frac{\pi^2}{nc_+^2}+6\mu(\Omega')(1-f(n^{-1/2}c_+))\right)}|\Omega'|_{m-1}}{|\Omega_{f,\Omega'}|} \times\left(\int_{[c_-,c_+]} dx_1 e^{t\mu(\Omega')(1-(f(x_1))^{-2})}+\frac{4}{n}\left(\frac{t}{\pi}\right)^{1/2}\right). \tag{4.18}$$

To complete the proof, we choose

$$t=t_n=\left(\frac{\pi^2}{nc_+^2}+6\mu(\Omega')(1-f(n^{-1/2}c_+))\right)^{-1}. \tag{4.19}$$

Substituting this into (4.18) gives

$$\frac{1}{|\Omega_{f_n,\Omega'}|}\left(\int_{\Omega_{f_n,\Omega'}} u_{\Omega_{f_n,\Omega'}}\right)^2 \le \frac{e|\Omega'|_{m-1}}{|\Omega_{f,\Omega'}|}\left(\int_{[c_-,c_+]} dx_1\, e^{t_n\mu(\Omega')(1-(f(x_1))^{-2})}+\left(\frac{16t_n}{\pi n^2}\right)^{1/2}\right). \tag{4.20}$$

The integrand in the first term on the right-hand side of (4.20) side is bounded by 1, and is integrable on $[c_-,c_+]$. This term goes to 0 as $n\to\infty$ by Lebesgue's dominated convergence theorem since $t_n\to\infty$ and $1-(f(x_1))^{-2}<0$ for all $x_1\ne 0$. The second term is $O(n^{-1/2})$ by (4.19). Localisation in L^2 follows by Lemma 1. This proves (i).

(ii) Under the hypotheses of (ii), the sets $\Omega_{f_n,\Omega'}$ are convex and symmetric with respect to the vertical axis. Following the results of Jerison [18, Theorem B], and Grieser and Jerison [15, Theorem 1], the second eigenfunction has to be odd in the x variable, hence to have the nodal line on the vertical axis. Indeed, assume there is a second eigenfunction $u_{2,n}$ which is not odd in the x variable. Then, $v(x,y)=u_{2,n}(x,y)+u_{2,n}(-x,y)$ is a non trivial second eigenfunction which is even in the x variable, thus having a nodal line symmetric about the vertical axis. Following [15, Theorem 1], the nodal line is contained in a vertical strip of width of order $\frac{1}{n}$. There are two possibilities: (i) the nodal line intersects the upper and lower boundary and, from symmetry, we get more than two nodal domains, thus ending up with a contradiction, (ii) the nodal line intersects only one of the boundaries enclosing a nodal domain with first eigenvalue of order n^2 contradicting that the eigenvalues do converge to π^2. ■

5. Example of κ-localisation for Neumann eigenfunctions

In this section, we construct a sequence of simply connected, planar, polygonal domains for which the corresponding sequence of first Neumann eigenfunctions κ-localises in L^2.

Localisation of the first Neumann eigenfunction has been implicitly noted in [4, Theorem 4.1] based on the following (Courant–Hilbert) example, with the geometry similar

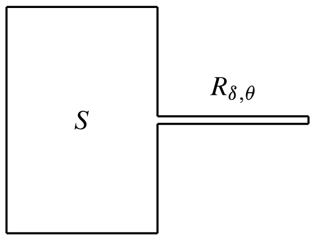

Figure 4. $S \cup R_{\delta,\theta}$.

to Figure 1. Let $\eta > 0$ and define for $\varepsilon > 0$ small

$$R = (-1, 0) \times (-1, 1),$$
$$T_{\varepsilon,\eta} = [0, \varepsilon] \times (-\varepsilon^\eta, \varepsilon^\eta),$$
$$S_\varepsilon = (\varepsilon, 2\varepsilon) \times (-\varepsilon, \varepsilon)$$

and

$$\Omega_\varepsilon := R \cup T_{\varepsilon,\eta} \cup S_\varepsilon.$$

Consider the Neumann eigenvalue problem in Ω_ε, and denote by $\mu_1(\Omega_\varepsilon)$ the first non-zero Neumann eigenvalue of the Laplace operator. Let u_ε be a first L^2-normalised corresponding eigenfunction. The following result was proved in [4, Theorem 4.1] (see also [3]): let $\eta > 3$ and $\varepsilon \to 0$, then $\mu_1(\Omega_\varepsilon) \to 0$ and $\int_{S_\varepsilon} u_\varepsilon^2 dx \to 1$. In other words, the sequence of the first Neumann eigenfunctions localises.

We introduce the following geometry. For every small $\theta > 0$ and δ in a neighbourhood of 0, we define the following sets. The open rectangle

$$S = (-1, 0) \times (-1, 1) \subset \mathbb{R}^2,$$

with $\mu_1(S) = \frac{\pi^2}{4}$ simple, and the rectangle

$$R_{\delta,\theta} = [0, 1 + \delta) \times (-\theta, \theta).$$

Note that the first eigenvalue of the segment of length 1 and with Dirichlet boundary conditions at one vertex and Neumann boundary conditions at the opposite vertex is equal to $\frac{\pi^2}{4}$, and is also simple.

Let

$$\Omega_{\delta,\theta} = S \cup R_{\delta,\theta},$$

and let $u^1_{\delta,\theta}$ be a first eigenfunction. See Figure 4.

Theorem 9. *Let $\kappa \in (0, 1)$ be fixed. There exists a sequence of sets of the form $\Omega_{\delta,\theta}$ for which the first Neumann eigenfunction κ-localises.*

Following Jimbo [19] and Arrieta [3], when $\delta \neq 0$ is fixed and $\theta \to 0$, the eigenvalues of the Neumann Laplacian on $\Omega_{\delta,\theta}$ converge to the union of eigenvalues of the segment of length $1+\delta$ and mixed Dirichlet–Neumann boundary conditions and the Neumann spectrum of S.

The idea is to identify suitable pairs $(\delta_n, \theta_n) \to (0,0)$ either with double first non-zero eigenvalue or with a simple first non-zero eigenvalue having an eigenfunction with balanced mass between S and R_{δ_n,θ_n}. Both situations will lead to κ-localisation.

In Lemma 10 below, we give some information of the behaviour of a sequence of eigenfunctions on $\Omega_{\delta,\theta}$ when $(\delta,\theta) \to (0,0)$. For further details concerning the spectrum on these kinds of geometries, we refer to [3].

Lemma 10. *Let $(\delta_n, \theta_n) \to (0,0)$ and (u_n, μ_n) be an eigenpair on $\Omega_n := \Omega_{\delta_n,\theta_n}$ such that $\int_{\Omega_n} u_n^2 = 1$ and $\limsup_{n\to+\infty} \mu_n < +\infty$. Then, there exist $\mu \geq 0$ and a subsequence (still denoted with the same index) such that the following hold.*

(i) *$u_n|_S \to u$, weakly in $H^1(S)$, strongly in $L^2(S)$ with $\int_S u\,dx = 0$, and*

$$\begin{cases} -\Delta u = \mu u & \text{in } S, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial S. \end{cases}$$

(ii) *Denoting $v_n(x,y) := \sqrt{\theta_n} u_n(-x^- + \frac{x^+}{1+\delta_n}, \theta_n y)$, $\widetilde{S} = (-1,1)\times(-1,1)$, we have $v_n \to v$ weakly in $H^1(\widetilde{S})$, strongly in $L^2(\widetilde{S})$, with $v(x,y) = v(x) \in H^1(-1,1)$ and*

$$\begin{cases} -v'' = \mu v & \text{in } (0,1), \\ v(0) = 0, v'(1) = 0. \end{cases}$$

Note that u or v in the above may be the 0-function.

Proof. For a subsequence, we can assume $\mu_n \to \mu$ and $u_n|_{\S} \to u$, weakly in $H^1(S)$. Let $\varphi \in H^1_{loc}(\mathbb{R}^2)$. Note that

$$\left|\int_{R_n} u_n\varphi\right| + \left|\int_{R_n} \nabla u_n \nabla\varphi\right| \\ \leq \|u_n\|_{L^2(\Omega_n)}\left(\int_{R_n}\varphi^2\right)^{\frac12} + \|\nabla u_n\|_{L^2(\Omega_n)}\left(\int_{R_n}|\nabla\varphi|^2\right)^{\frac12} \to 0.$$

This implies, in particular, $\int_S u_n \to 0$, and hence, $\int_S u = 0$. Taking $\varphi|_{\Omega_n}$ as a test function in $H^1(\Omega_n)$, we get

$$\int_{\Omega_n} \nabla u_n \nabla\varphi = \mu_n \int_{\Omega_n} u_n\varphi.$$

Splitting the sums over $\Omega_n = S \cup \mathcal{R}_n$, and using the weak convergence in $H^1(S)$, we get

$$\int_S \nabla u \nabla \varphi = \mu \int_S u\varphi.$$

Since $H^1_{loc}(\mathbb{R}^2)|_S$ coincides with $H^1(S)$, Lemma 10 part (i) is proved.

To prove Lemma 10 part (ii), we note that

$$\int_{\widetilde{S}} v_n^2 \leq 1 + |\delta_n|, \int_{\widetilde{S}} \left(\frac{\partial v_n}{\partial x}\right)^2 \leq (1 + 2|\delta_n|)\mu_n, \int_{\widetilde{S}} \left(\frac{\partial v_n}{\partial y}\right)^2 \leq (1 + |\delta_n|)\theta_n^2.$$

Then, for a subsequence, (v_n), $v_n \to v$ weakly in $H^1(\widetilde{S})$ with $\frac{\partial v}{\partial y} = 0$ in $\widetilde{S}$. So, the function v depends only on the variable x. Moreover, v is continuous and $v = 0$ on $(-1, 0]$. This is a consequence of the trace theorem on $(-1, 0) \times \{0\}$ applied to u_n giving that $\int_{-1}^{0} u_n(x, 0)^2 dx$ is bounded. This implies that $\sqrt{\theta_n} u_n(\cdot, 0)$ converges strongly to 0 on $(-1, 0)$. This also implies that the convergence is strong in $L^2(\widetilde{S})$.

Taking a test function $\varphi \in H^1(0, 1)$ with $\varphi(0) = 0$, that we extend by zero on $(-1, 0)$ and constant in y on $(-1, 1)$ in the equation satisfied by u_n, we get

$$\int_{R_n} \nabla u_n \nabla \varphi = \mu_n \int_{R_n} u_n \varphi,$$

and in terms of v_n

$$\int_{(0,1)\times(-1,1)} \partial_x v_n \partial_x \varphi = \mu_n (1 + \delta_n) \int_{(0,1)\times(-1,1)} v_n \varphi,$$

that we pass to the limit to get the equation. ∎

Proof. Fix $\kappa \in (0, 1)$. Let $\delta_1 > 0$. Following [3], we know that for $\theta \to 0$

$$\mu_1(\Omega_{\delta_1,\theta}) \to \left(\frac{\pi}{2 + 2\delta_1}\right)^2, \quad \mu_2(\Omega_{\delta_1,\theta}) \to \frac{\pi^2}{4},$$

with convergence of eigenfunctions given by the preceding lemma. Hence,

$$\int_{R_{\delta_1,\theta}} (u^1_{\delta_1,\theta})^2 \to 1.$$

At the same time,

$$\mu_2(\Omega_{-\delta_1,\theta}) \to \left(\frac{\pi}{2 - 2\delta_1}\right)^2, \quad \mu_1(\Omega_{-\delta_1,\theta}) \to \frac{\pi^2}{4}.$$

Hence, $\int_{R_{\delta_1,\theta}} (u^1_{-\delta_1,\theta})^2 \to 0$.

We choose θ small enough such that

$$\int_{R_{\delta_1,\theta}} (u^1_{\delta_1,\theta})^2 \geq \frac{1 + \kappa}{2} \quad \text{and} \quad \int_{R_{\delta_1,\theta}} (u^1_{-\delta_1,\theta})^2 \leq \frac{\kappa}{2}.$$

For this value of θ, denoted by θ_1, we vary δ continuously from $-\delta_1$ to δ_1. The spectrum of the Neumann Laplacian varies continuously along this trajectory, and the eigenfunctions corresponding to simple eigenvalues are continuous. In particular, if the first eigenvalue is

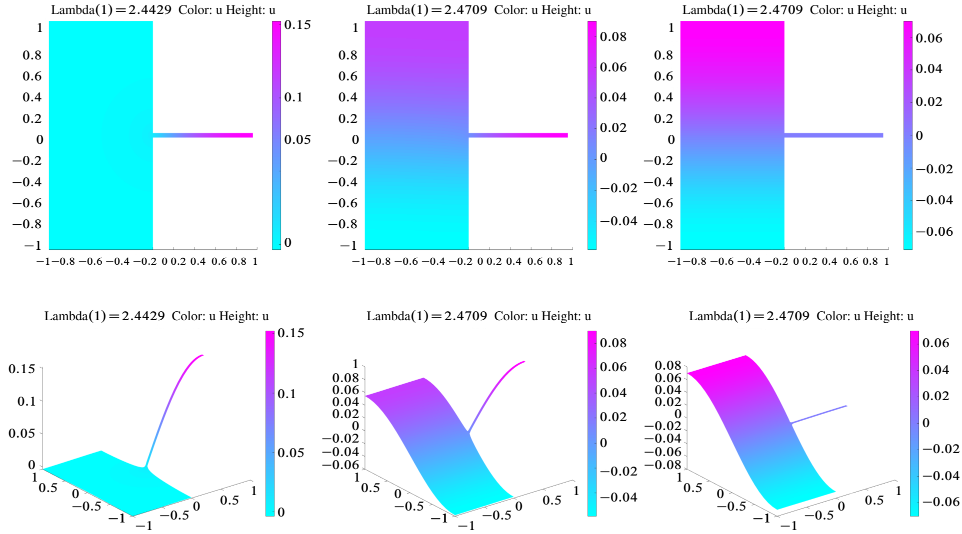

Figure 5. The graph of $u^1_{\delta,\theta}$, from localisation to non localisation, when perturbing the length of the thin rectangle: $\theta = 0.02$ and $\delta = -0.039$, $\delta = -0.04491$, $\delta = -0.05$, respectively.

always simple, then the mass of the corresponding eigenfunction varies continuously on S (and its complement).

There are two situations: either the first eigenvalue is simple along the entire trajectory, or not. In the latter case, we stop at the point when the eigenvalue becomes double.

We now repeat this procedure, taking $\delta_2 = \delta_1/2$, choosing $\theta_2 \le \theta_1/2$, and so on. In this way we find either a sequence of sets (Ω_n) either with simple first eigenvalues and with balanced mass $1-\kappa$ on S and κ on R_n, or a sequence of sets (Ω_n) with double first eigenvalues.

If the first situation occurs, the sequence of eigenfunctions κ-localises. Indeed, on S the sequence converges to a first eigenfunction of S which has the mass $1-\kappa$ and no localisation can occur on S. For $A_n \subseteq S$, we have

$$\int_{A_n} u_n^2 \le |1_{A_n}|_{L^2} |u_n^2|_{L^2} \to 0,$$

from the continuous injection $H^1(S) \subseteq L^4(S)$.

If the second situation occurs, let us denote u_n^1, u_n^2 two normalised L^2-orthogonal eigenfunctions corresponding to the first (double) eigenvalue. We follow the masses of the eigenfunctions: assume (for a subsequence) that

$$\int_S (u_n^1)^2 dx \to a, \quad \int_S (u_n^2)^2 dx \to b.$$

If both $a \neq 0$, $b \neq 0$, then we consider the weak $H^1(S)$-limits of $u_n^1|_S$ and $u_n^2|_S$, denoted u^1, u^2, respectively. Both of them are non-zero eigenfunctions corresponding to the first eigenvalue on S. This being simple, there exists $\lambda \in \mathbb{R}$ such that $u^1 + \lambda u^2 = 0$. This implies that the sequence given by $\tilde{u}_n = \frac{1}{\sqrt{1+\lambda^2}}(u_n^1 + \lambda u_n^2)$ is a sequence of normalised first eigenfunctions converging to 0 on S. In other words, we can assume that $a = 0$ and relabel $u_n^1 = \tilde{u}_n$.

A similar argument applied to R_{δ_n,θ_n} gives that $b = 1$. Indeed, if $b \neq 1$, then the sequences v_n^1, v_n^2 constructed in Lemma 10 part (ii) would converge to a non-zero first eigenfunction on the segment $(0, 1)$, so that the previous argument can be used again.

Since we know now that for suitable sequences of eigenfunctions we have $a = 0, b = 1$, we consider the sequence $\kappa u_n^1 + \sqrt{1-\kappa} u_n^2$ of normalised first eigenfunctions on Ω_n which κ-localises. ■

The data in Figure 5 have been obtained with the MATLAB PDE toolbox, and illustrate the mass distribution of the first eigenfunction.

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References

[1] A. Ancona, On strong barriers and an inequality of Hardy for domains in $\mathbf{R}^n$. *J. London Math. Soc. (2)* **34** (1986), no. 2, 274–290 Zbl 0629.31002 MR 0856511

[2] D. N. Arnold, G. David, M. Filoche, D. Jerison, and S. Mayboroda, Localization of eigenfunctions via an effective potential. *Comm. Partial Differential Equations* **44** (2019), no. 11, 1186–1216 Zbl 1432.35061 MR 3995095

[3] J. M. Arrieta, Neumann eigenvalue problems on exterior perturbations of the domain. *J. Differential Equations* **118** (1995), no. 1, 54–103 Zbl 0860.35086 MR 1329403

[4] J. M. Arrieta, J. K. Hale, and Q. Han, Eigenvalue problems for non-smoothly perturbed domains. *J. Differential Equations* **91** (1991), no. 1, 24–52 Zbl 0736.35073 MR 1106116

[5] R. Bañuelos and M. van den Berg, Dirichlet eigenfunctions for horn-shaped regions and Laplacians on cross sections. *J. London Math. Soc. (2)* **53** (1996), no. 3, 503–511 Zbl 0863.35070 MR 1396714

[6] T. Beck, Localisation of the first eigenfunction of a convex domain. *Comm. Partial Differential Equations* **46** (2021), no. 3, 395–412 Zbl 1466.35271 MR 4232499

[7] L. Brasco, On torsional rigidity and principal frequencies: an invitation to the Kohler–Jobin rearrangement technique. *ESAIM Control Optim. Calc. Var.* **20** (2014), no. 2, 315–338 Zbl 1290.35160 MR 3264206

[8] D. Bucur and G. Buttazzo, *Variational methods in shape optimization problems*. Progr. Nonlinear Differential Equations Appl. 65, Birkhäuser, Boston, MA, 2005 Zbl 1117.49001 MR 2150214

[9] G. David, M. Filoche, and S. Mayboroda, The landscape law for the integrated density of states. *Adv. Math.* **390** (2021), article no. 107946 Zbl 1479.35267 MR 4298594

[10] E. B. Davies, *Heat kernels and spectral theory*. Cambridge Tracts in Math. 92, Cambridge University Press, Cambridge, 1990 Zbl 0699.35006 MR 1103113

[11] E. B. Davies, The Hardy constant. *Quart. J. Math. Oxford Ser. (2)* **46** (1995), no. 184, 417–431 Zbl 0857.26005 MR 1366614

[12] E. B. Davies, A review of Hardy inequalities. In *The Maz'ya anniversary collection, Vol. 2 (Rostock, 1998)*, pp. 55–67, Oper. Theory Adv. Appl. 110, Birkhäuser, Basel, 1999 Zbl 0936.35121 MR 1747888

[13] E. B. Davies, Sharp boundary estimates for elliptic operators. *Math. Proc. Cambridge Philos. Soc.* **129** (2000), no. 1, 165–178 Zbl 0963.35146 MR 1757786

[14] D. S. Grebenkov and B.-T. Nguyen, Geometrical structure of Laplacian eigenfunctions. *SIAM Rev.* **55** (2013), no. 4, 601–667 Zbl 1290.35157 MR 3124880

[15] D. Grieser and D. Jerison, Asymptotics of the first nodal line of a convex domain. *Invent. Math.* **125** (1996), no. 2, 197–219 Zbl 0857.31002 MR 1395718

[16] D. Grieser and D. Jerison, The size of the first eigenfunction of a convex planar domain. *J. Amer. Math. Soc.* **11** (1998), no. 1, 41–72 Zbl 0896.35092 MR 1470858

[17] A. Henrot, I. Lucardesi, and G. Philippin, On two functionals involving the maximum of the torsion function. *ESAIM Control Optim. Calc. Var.* **24** (2018), no. 4, 1585–1604 Zbl 1442.35281 MR 3922448

[18] D. Jerison, The diameter of the first nodal line of a convex domain. *Ann. of Math. (2)* **141** (1995), no. 1, 1–33 Zbl 0831.35115 MR 1314030

[19] S. Jimbo, The singularly perturbed domain and the characterization for the eigenfunctions with Neumann boundary condition. *J. Differential Equations* **77** (1989), no. 2, 322–350 Zbl 0703.35138 MR 0983298

[20] F. John, Extremum problems with inequalities as subsidiary conditions. In *Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948*, pp. 187–204, Interscience, New York, 1948 Zbl 0034.10503 MR 0030135

[21] M.-T. Kohler-Jobin, Une méthode de comparaison isopérimétrique de fonctionnelles de domaines de la physique mathématique. I. Une démonstration de la conjecture isopérimétrique $P\lambda^2 \geq \pi j_0^4/2$ de Pólya et Szegő. *Z. Angew. Math. Phys.* **29** (1978), no. 5, 757–766 Zbl 0427.73056 MR 0511908

[22] M.-T. Kohler-Jobin, Une méthode de comparaison isopérimétrique de fonctionnelles de domaines de la physique mathématique. II. Cas inhomogène: une inégalité isopérimétrique entre la fréquence fondamentale d'une membrane et l'énergie d'équilibre d'un problème de Poisson. *Z. Angew. Math. Phys.* **29** (1978), no. 5, 767–776 Zbl 0427.73057 MR 0511909

[23] S. Maji and S. Saha, Eigenfunction localization and nodal geometry on dumbbell domains. 2023, arXiv:2309.11441v1

[24] M. van den Berg, Dirichlet–Neumann bracketing for horn-shaped regions. *J. Funct. Anal.* **104** (1992), no. 1, 110–120 Zbl 0763.35071 MR 1152461

[25] M. van den Berg, Estimates for the torsion function and Sobolev constants. *Potential Anal.* **36** (2012), no. 4, 607–616 Zbl 1246.60108 MR 2904636

[26] M. van den Berg, Spectral bounds for the torsion function. *Integral Equations Operator Theory* **88** (2017), no. 3, 387–400 Zbl 1378.58024 MR 3682197

[27] M. van den Berg, D. Bucur, and T. Kappeler, On efficiency and localisation for the torsion function. *Potential Anal.* **57** (2022), no. 4, 571–600 Zbl 1505.35110 MR 4512658

[28] M. van den Berg and T. Carroll, Hardy inequality and L^p estimates for the torsion function. *Bull. Lond. Math. Soc.* **41** (2009), no. 6, 980–986 Zbl 1180.35396 MR 2575328

[29] M. van den Berg and E. B. Davies, Heat flow out of regions in $\mathbf{R}^m$. *Math. Z.* **202** (1989), no. 4, 463–482 Zbl 0661.35040 MR 1022816

[30] M. van den Berg, F. Della Pietra, G. di Blasio, and N. Gavitone, Efficiency and localisation for the first Dirichlet eigenfunction. *J. Spectr. Theory* **11** (2021), no. 3, 981–1003 Zbl 1485.35166 MR 4322028

[31] M. van den Berg, V. Ferone, C. Nitsch, and C. Trombetti, On a Pólya functional for rhombi, isosceles triangles, and thinning convex sets. *Rev. Mat. Iberoam.* **36** (2020), no. 7, 2091–2105 Zbl 1460.52008 MR 4163993

[32] M. van den Berg and T. Kappeler, Localization for the torsion function and the strong Hardy inequality. *Mathematika* **67** (2021), no. 2, 514–531 Zbl 1546.35036 MR 4232983

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