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On the Kaup–Broer–Kupershmidt systems

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Abstract. The aim of this paper is to survey and complete, mostly by numerical simulations, results on a remarkable Boussinesq system describing weakly nonlinear, long surface water waves. It is the only member of the so-called $(abcd)$ family of Boussinesq systems known to be completely integrable.

In memoriam Thomas Kappeler (1953–2022)

1. Introduction

In this paper, we are interested in a particular case of the so-called $abcd$ Boussinesq systems for surface water waves; see [5–7]:

$$\begin{cases} \eta_t + \nabla\dot{\mathbf{v}} + \epsilon\nabla\cdot(\eta\mathbf{v}) + \mu[a\nabla\cdot\Delta\mathbf{v} - b\Delta\eta_t] = 0, \\ \mathbf{v}_t + \nabla\eta + \epsilon\dfrac{1}{2}\nabla|\mathbf{v}|^2 + \mu[c\nabla\Delta\eta - d\Delta\mathbf{v}_t] = 0. \end{cases} \tag{1.1}$$

We note that Boussinesq [8] was the first to derive a particular Boussinesq system, not in the class of those studied here though. We refer to [19–21] for details and for an excellent history of hydrodynamics in the nineteenth century. In the case above, $\eta = \eta(x,t)$, $x \in \mathbb{R}^d$, $d = 1,2$, $t \in \mathbb{R}$ is the elevation of the wave, $\mathbf{v} = \mathbf{v}(x,t)$ is a measure of the horizontal velocity, μ and ϵ are the small parameters (shallowness and nonlinearity parameters respectively) defined as

$$\mu = \frac{h^2}{\lambda^2}, \quad \epsilon = \frac{\alpha}{h},$$

where α is a typical amplitude of the wave, h a typical depth and λ a typical horizontal wavelength.

In the Boussinesq regime, ϵ and μ are supposed to be of same order, $\epsilon \sim \mu \ll 1$, and we will take for simplicity $\epsilon = \mu$, writing (1.1) as

$$\begin{cases} \eta_t + \nabla\cdot\mathbf{v} + \epsilon[\nabla\cdot(\eta\mathbf{v}) + a\nabla\cdot\Delta\mathbf{v} - b\Delta\eta_t] = 0, \\ \mathbf{v}_t + \nabla\eta + \epsilon\Big[\dfrac{1}{2}\nabla|\mathbf{v}|^2 + c\nabla\Delta\eta - d\Delta\mathbf{v}_t\Big] = 0. \end{cases} \tag{1.2}$$

Particular cases are formally derived in [9, 22, 47].

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The coefficients (a, b, c, d) are restricted by the condition

$$a + b + c + d = \frac{1}{3} - \tau,$$

where $\tau \geq 0$ is the surface tension coefficient. The reader is referred to [36] for a comprehensive review.

When restricted to one-dimensional, unidirectional motions, the system in (1.2) leads to the Korteweg–de Vries (KdV) equation; see [39]:

$$u_t + u_x + \epsilon\Big(\frac{1}{3} - \tau\Big)u_{xxx} + \epsilon u u_x = 0.$$

The class of systems (1.1), (1.2) model water waves on a flat bottom propagating in both directions in the aforementioned regime (see [5–7]).

It turns out that two particular one-dimensional cases of the $abcd$ systems have remarkable properties. The first one, we will refer to as the *Amick–Schonbek system*, can be viewed as a dispersive perturbation of the Saint-Venant (shallow water) system.[1] We refer to [35] for a review of known results together with new numerical simulations.

The second one, referred to as the *Kaup–Broer–Kupershmidt system* (KBK) was introduced in [9, 10] as a surface long wave model (see [9, equations (3.2), (3.3)]), and more recently in [12] as an internal wave model. On the other hand, Kaup and Kupershmidt introduced it as an integrable system [28,38]. It corresponds in the $abcd$ family to $a = \pm 1$, $b = c = d = 0$ and is written as follows:

$$\begin{cases} \eta_t + v_x + (\eta v)_x + \alpha v_{xxx} = 0, \\ \qquad\quad v_t + \eta_x + v v_x = 0, \end{cases} \tag{1.3}$$

where $\alpha = 1$ corresponds to the "bad" KBK system and $\alpha = -1$ to the "good" KBK system. The two systems turn out to be integrable in a sense that will be detailed below.

Remark 1.1. When viewed as water wave model, the variable η in (1.3) represents the elevation of the wave, so that physically the total depth $\zeta = 1 + \eta$ should be positive. The non-cavitation condition $\zeta > 0$ will be mostly ignored in what follows. Note that in terms of (v, ζ), (1.3) becomes

$$\begin{cases} \zeta_t + (\zeta v)_x + \alpha v_{xxx} = 0, \\ \qquad v_t + \zeta_x + v v_x = 0. \end{cases} \tag{1.4}$$

Setting $U = v_x$ in (1.4) and solving for ζ leads to the *Boussinesq-like* equation

$$U_{tt} + U_t U_{xx} + 2U_x U_{xt} + \frac{3}{2}U_x^2 U_{xx} - \alpha U_{xxxx} = 0.$$

[1] Actually, this system is a particular case of a system derived by Peregrine in [47], but Schonbek and Amick were the first to point out its remarkable mathematical properties.

Remark 1.2. The KBK system is sometimes written in terms of the velocity potential ϕ such that $\phi_x(x,t) = v(x,t)$:

$$\begin{cases} \eta_t + \phi_{xx} + (\eta\phi_x)_x + \alpha\phi_{xxxx} = 0, \\ \phi_t + \eta + \frac{1}{2}(\phi_x)^2 = 0. \end{cases}$$

Remark 1.3. Note that the KBK system should not be confused with the so-called Kaup–Kupershmidt equation

$$v_t = v_{xxxxx} + 10vv_{xxx} + 25v_xv_{xx} + 20v^2v_x$$

that was introduced in [29, 37], and which is the first equation in a hierarchy (different from the KdV hierarchy) of integrable equations with Lax operator $\partial_x^3 + 2u\partial_x + u_x$, see for instance [24]. We are not aware of a physical application of the Kaup–Kupershmidt equation.

The paper is organized as follows. In Section 2 we will describe general facts on the KBK system. The next section reviews the results obtained by partial differential equations (PDE) techniques while Section 4 focusses on the integrability side. In Section 5 we introduce a numerical approach to the KBK system and test it for the known soliton. In Section 6 the known stability of the solitons is illustrated. In Section 7 the long time behavior of KBK solutions for localized initial data is studied showing that the soliton resolution conjecture can be applied. In Section 8 we explore the formation of dispersive shock waves in the vicinity of shocks of the corresponding dispersionless Saint-Venant system. We add some concluding remarks in Section 9.

2. Generalities on the Kaup–Broer–Kupershmidt system

We recall here that the Kaup–Broer–Kupershmidt system is the one-dimensional version of the two-dimensional ($abcd$ Boussinesq) system

$$\begin{cases} \eta_t + \nabla\cdot\mathbf{v} + \epsilon\nabla\cdot(\eta\mathbf{v}) + \alpha\epsilon\Delta\nabla\cdot\mathbf{v} = 0, \ \alpha = \pm 1, \\ \mathbf{v}_t + \nabla\eta + \frac{\epsilon}{2}\nabla|\mathbf{v}|^2 = 0, \end{cases} \tag{2.1}$$

which is linearly well-posed when $\alpha = -1$. The local well-posedness of the Cauchy problem is established in [30] (see also [51]). The more difficult question of *long time existence* (that is on time scales of order $O(1/\epsilon)$) is established in [51] under the non-cavitation condition $1 + \epsilon\eta > 0$.

On the other hand, (2.1) is ill-posed when $\alpha = +1$. Note that the ill-posed version can be turned into a well-posed system by using the BBM trick, leading to

$$\begin{cases} \eta_t + \nabla\cdot\mathbf{v} + \epsilon\nabla\cdot(\eta\mathbf{v}) - \epsilon\Delta\eta_t = 0, \\ \mathbf{v}_t + \nabla\eta + \epsilon\frac{1}{2}\nabla|\mathbf{v}|^2 = 0, \end{cases}$$

for which the long-time existence is established in [51], see also [11].

From now on, we will restrict to the one-dimensional case with most of the time $\epsilon = 1$. The linearized "bad" KBK system ($\alpha = 1$ in (1.3)) is Hadamard ill-posed since the dispersion relation is $\omega^2 = k^2 - k^4$, showing the short wave ill-posedness. It was established in [1] that nonlinearity does not erase the problem, making the Cauchy problem ill-posed in all Sobolev spaces in the sense that arbitrary small smooth solutions can blow-up in arbitrary short time in Sobolev spaces norms.

Remark 2.1. Following [38], the change of variable $v = v$, $\eta = h - v_x$ transforms the Kaup–Broer–Kupershmidt system with $\alpha = 1$ into the (ill-posed) system

$$\begin{cases} h_t + v_x + h_{xx} + (vh)_x = 0, \\ \quad v_t + h_x - v_{xx} + v v_x = 0, \end{cases} \tag{2.2}$$

with dispersion $\omega(k) = \pm i k\sqrt{1-k^2}$.

On the other hand, in the well-posed case $\alpha = -1$, the change of variable $v = v$, $\eta = h + i v_x$ leads to the (linearly well-posed) "Schrödinger-like" system

$$\begin{cases} h_t + v_x - i h_{xx} + (vh)_x = 0, \\ \quad v_t + h_x + i v_{xx} + v v_x = 0, \end{cases} \tag{2.3}$$

with dispersion $\omega(k) = \pm i k(1+k^2)^{1/2}$

As recalled in [46], Broer [9] and Kaup [28] derived the system in dimensional form

$$\begin{cases} \eta_t + h_0\phi_{xx} + (\eta\phi_x)_x + \left(\dfrac{h_0^3}{3} - \dfrac{h_0\tau}{\rho g}\right)\phi_{xxx} = 0, \\ \qquad\qquad\qquad\qquad \phi_t + \dfrac{1}{2}(\phi_x)^2 + g\eta = 0, \end{cases} \tag{2.4}$$

where η is the elevation of the wave, ϕ the velocity potential evaluated at the free surface, the constant h_0 is the quiescent water depth, $g > 0$ is the gravitational acceleration, $\tau \geq 0$ is the surface tension coefficient and ρ the density of the fluid.

The "bad" version corresponds to pure gravity waves ($\tau = 0$) or to gravity-capillarity waves with small surface tension ($\tau/(g\rho) < h_0^2/3$, corresponding to $b = d = c = 0, a > 0$ in (1.2)). The "good" version occurs with strong surface tension, that is, when $\tau/(g\rho) > h_0^2/3$, (a not too physical case ...), corresponding to $b = d = c = 0$, $a < 0$ in (1.2). The difference in nature of the equation depending on the capillarity is reminiscent of that of the Kadomtsev–Petviashvili equation (KP-II for gravity waves and KP-I for gravity-capillarity waves with strong surface tension) although both KP-II and KP-I are well-posed!

Remark 2.2. Despite the fact that the Kaup–Broer–Kupershmidt system with the plus sign is linearly ill-posed, it possesses solitary wave solutions $(\eta(x-kt), v(x-kt))$ when $k > 1$.[2] This has been found by Kaup [28] using the Inverse Scattering machinery, and

[2] In the classical version, $a = 1/3$.

who also proved the existence of N-solitons. Matveev and Yavor [41], exhibited a vast class of almost periodic solutions containing N-soliton solutions as a degenerate case; see also [46]. The existence of undular bores is proven in [23].

A direct approach to the existence of solitary waves is given in [13] with $a = 1/3$. Actually a solitary wave satisfies the equations

$$(v')^2 = R_1(u) \equiv \frac{3}{4}v^2(v-(2k-2))(v-(2k+2)), \quad \eta = v\Big(k - \frac{v}{2}\Big),$$

where the derivative is taken with respect to $\xi = x - kt$.

By studying the function R_1, one shows that there exists a unique solution $v(\xi)$, which is even and monotone for $\xi > 0$ for any $k > 1$. More precisely, the solution $v(\xi)$ reads

$$v(\xi) = \frac{2(k^2-1)}{\cosh(\sqrt{3(k^2-1)}\xi) + k}.$$

We also have

$$\eta(\xi) = \frac{2(k^2-1)(k\cosh(\sqrt{3(k^2-1)}\xi) + 1)}{(\cosh(\sqrt{3(k^2-1)}\xi) + k)^2}.$$

When $1 < k \leq 2$, the corresponding $\eta(\xi)$ is also monotonically decreasing for $\xi > 0$, while this is no longer true when $k > 2$. One notices that for $1 < k \leq 2$, we have

$$\|v\|_{L^\infty} = \|\eta\|_{L^\infty} = 2(k-1).$$

3. The Kaup–Broer–Kupershmidt system by PDE techniques

We recall that the Cauchy problem for (2.1) with $\alpha = -1$ is well-posed in both spatial dimensions one and two under the non-cavitation condition $1 + \epsilon\eta > 0$ on time scales of order $O(1/\epsilon)$; see [51].

We now focus on the one-dimensional "good" KBK system with $\epsilon = 1$, which has the Hamiltonian structure

$$\eta_t + \partial_x \frac{\delta \mathcal{H}}{\delta v} = 0, \quad v_t + \partial_x \frac{\delta \mathcal{H}}{\delta \eta} = 0,$$

where the Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H} = \frac{1}{2}\int [\eta^2 + (1+\eta)v^2 + (v_x^2)]\, dx.$$

Remark 3.1. The Hamiltonian of the "bad" KBK system is

$$\mathcal{H} = \frac{1}{2}\int [\eta^2 + (1+\eta)v^2 - (v_x^2)]\, dx.$$

Remark 3.2. Both the "bad" and "good" KBK system have the conserve quantity:

$$H_0 = \int_{\mathbb{R}} \eta v \, dx.$$

Angulo [2] proved that the Cauchy problem for the "good" KBK system is locally well-posed in $H^{s-1}(\mathbb{R}) \times H^s(\mathbb{R})$, $s > 3/2$.[3] Surprisingly, he also found an additional conservation law namely

$$\begin{aligned} I_3(\eta, v) = \frac{1}{8} \int \big[& 4(v_{xx}^2) + 8(v_x^2) + 4v^2 + 4(\eta_x^2) + 4\eta^2 + 6v^2(v_x)^2 \\ & - 16\eta v v_{xx} - 4\eta(v_x)^2 + 10\eta v^2 + 2\eta^3 + v^4 + 6\eta^2 v^2 + \eta v^4 \big] \, dx. \end{aligned}$$

The existence of such a conservation law suggests that the "good" KBK system is integrable in the sense of Inverse Scattering. We refer to the next section for further information on this issue. Actually, I_3 can be viewed as the Hamiltonian of the (linearly well-posed) higher order system

$$\begin{cases} v_t - \eta_{xxx} + \eta_x + \left\{ \frac{5}{4} v^2 + \frac{3}{4}\eta^2 + \frac{3}{2}\eta v^2 + \frac{1}{8} v^4 - 2v v_{xx} - \frac{1}{2} v_x^2 \right\}_x = 0, \\ \eta_t + v_{xxxxx} - 2v_{xxx} + v_x + \left\{ \frac{3}{2} v v_x^2 - \frac{3}{2}(v^2 v_x)_x \right. \\ \left. - 2\eta v_{xx} - 2(\eta v)_{xx} + \frac{1}{2}(\eta v_x)_x + \frac{5}{2} v\eta + \frac{1}{2} v^3 + \frac{3}{2}(v\eta^2) + \frac{1}{2} v^3 \eta \right\}_x = 0, \end{cases} \tag{3.1}$$

which can be written

$$\eta_t + \partial_x \frac{\delta I_3}{\delta v} = 0, \quad v_t + \partial_x \frac{\delta I_3}{\delta \eta} = 0.$$

We are not aware of results on the Cauchy problem for (3.1).

In order to get a further insight into the "good" KBK system, it is useful to diagonalize the linear part to obtain the equivalent system; see [51]:

$$\begin{cases} \eta_t + J_\epsilon \zeta_x + \frac{\epsilon}{2} N_1(\eta, w) = 0, \\ w_t - J_\epsilon w_x + \frac{\epsilon}{2} N_2(\eta, w) = 0, \end{cases} \tag{3.2}$$

where $J_\epsilon = (I - \epsilon \partial_x^2)^{1/2}$, and

$$\begin{aligned} N_1(\eta, w) &= \partial_x[(\zeta + w) J_\epsilon^{-1}(\eta - w)] + J_\epsilon[J_\epsilon^{-1}(\eta - w) J_\epsilon^{-1}(\eta_x - w_x)], \\ N_2(\eta, w) &= \partial_x[(\eta + w) J_\epsilon^{-1}(\zeta - w)] - J_\epsilon[J_\epsilon^{-1}(\eta - w) J_\epsilon^{-1}(\eta_x - w_x)]. \end{aligned}$$

Since

$$(1 + \epsilon\xi^2)^{1/2} - \epsilon^{1/2}|\xi| = \frac{1}{(1 + \epsilon\xi^2)^{1/2} + \epsilon^{1/2}|\xi|},$$

[3] A local well-posedness result in a weighted space with less regularity was obtained in [30].

the system (3.2) becomes

$$\begin{cases} \eta_t + \epsilon^{1/2}\mathcal{H}\eta_{xx} + R_\epsilon \eta + \dfrac{\epsilon}{2} N_1(\eta, w) = 0, \\ w_t - \epsilon^{1/2}\mathcal{H} w_{xx} - R_\epsilon w + \dfrac{\epsilon}{2} N_2(\eta, w) = 0, \end{cases} \tag{3.3}$$

where R_ϵ is the (order zero) skew-adjoint operator with symbol

$$\frac{i\xi}{(1+\epsilon\xi^2)^{1/2} + \epsilon^{1/2}|\xi|}.$$

Thus, the "good" KBK system is equivalent to a system having a decoupled "Benjamin–Ono-type" linear part. This property allows to apply the techniques used to obtain the local well-posedness of the Benjamin–Ono equation in low regularity spaces. Note that this approach would yield existence on the "short" time range $O(1/\sqrt{\epsilon})$; see the discussion in [51].

Remark 3.3. In a recent paper, Melinand [42] derived various dispersive estimates (for example, Strichartz, local Kato smoothing, and Morawetz) for a large class of $(abcd)$ systems including the Kaup–Broer–Kupershmidt system, both in one and two spatial dimensions. These estimates should play an important role for the local resolution of the Cauchy problem in "large" functional spaces and in the proof of the possible scattering of small solutions.

We conclude this section by alluding to results in [16], concerning the evolution of discontinuous initial data under the flow of the "good" KBK system. Actually, a Riemann-type problem is considered with initial data

$$\begin{aligned} \eta(x,t=0) &= \eta_L \text{ and } v(x,t=0) = v_L \quad \text{for } x<0, \\ \eta(x,t=0) &= \eta_R \text{ and } v(x,t=0) = v_R \quad \text{for } x>0, \end{aligned}$$

and one finds in [16] a classification of the wave patterns evolving from these initial discontinuities.

3.1. Traveling wave solutions

Following Angulo [2] who considered the case $\epsilon = 1$, we now consider the traveling wave solutions of (1.3) that is solutions of the form $\mathbf{U}_c = (\eta, v)$ with

$$\eta(x,t) = n(x-ct), \quad v(x,t) = \phi(x-ct)$$

with $c \in \mathbb{R}$ constant so that (n,ϕ) satisfies the system (we have assumed that (n,ϕ) vanish at infinity which is natural in the water waves context):

$$\begin{cases} -cn + \phi + \epsilon n\phi - \epsilon\phi'' = 0, \\ -c\phi + n + \epsilon\dfrac{\phi^2}{2} = 0, \end{cases}$$

where $'$ denotes the derivative with respective to $\xi = x - ct$.

Eliminating n from the second equation yields

$$-\epsilon\phi'' + (1-c^2)\phi + \frac{3}{2}C\epsilon\phi^2 - \frac{\epsilon^2}{2}\phi^3 = 0. \tag{3.4}$$

From [4, Theorem 5] non-trivial solitary waves exist if and only if $|c| < 1$. Actually, (3.4) can be integrated by standard methods, leading to, for $|c| < 1$, a unique even solution (up to translations), which reads when $\epsilon = 1$ as; see [2]:

$$v_{c,1}(\xi) = \frac{2(1-c^2)}{\cosh(\sqrt{1-c^2}\xi) - c}, \quad \eta_{c,1}(\xi) = c v_{c,1}(\xi) - \frac{1}{2}v_{c,1}(\xi) \tag{3.5}$$

from which we deduce the expression for $v_{c,\epsilon}$:

$$v_{c,\epsilon}(\xi) = \frac{2(1-c^2)}{\epsilon[\cosh(\sqrt{1-c^2}\epsilon^{-1/2}\xi) - c]}.$$

We now turn to stability issues for the above solitary wave, following Angulo [2] who solved the problem with $\epsilon = 1$ and actually proved using the method in [25] that the solitary wave is orbitally stable in $L^2(\mathbb{R}) \times H^1(\mathbb{R})$ for $|c| < 1$. This uses the fact that $\mathbf{U}_c$ can be viewed as a critical point of the functional $F = \Phi_1 - c\Phi_0$, that is, it satisfies

$$(\Phi_1 - c\Phi_0)'(\mathbf{U}_c) = 0,$$

where

$$\Phi_1(v,\eta) = \frac{1}{2}\int_{\mathbb{R}} [v^2 + \eta^2 + v^2\eta + (v_x)^2]\, dx,$$
$$\Phi_0(v,\eta) = \int_{\mathbb{R}} v\eta\, dx.$$

Angulo proved, moreover, using the higher order conservation law, that the Cauchy problem is globally well-posed in $H^{s-1}(\mathbb{R}) \times H^s(\mathbb{R}), s \geq 2$ provided the initial data (η_0, v_0) are $L^2(\mathbb{R}) \times H^1(\mathbb{R})$-close to some translation of the solitary wave. This is one of the few known results on the global well-posedness of the Cauchy problem for $abcd$ Boussinesq systems.

Remark 3.4. We are not aware of explicit formulas for N-solitons of the "good" KBK system.

3.2. Remarks on the stationary solutions

We comment here on the stationary solutions of the "good" KBK system (here we keep the dependence in ϵ). Assuming that such a solution vanishes at infinity. In one spatial dimension (η, v) should satisfy

$$-v'' + v - \frac{\epsilon^2}{2}v^3 = 0, \quad \eta = -\frac{\epsilon}{2}v^2,$$

and v can be expressed in terms of the profile of the soliton of the focusing cubic nonlinear Schrödinger equation, leading to the explicit form

$$v(x) = \frac{2}{\epsilon \cosh(\epsilon^{-1/2} x)}.$$

In the two-dimensional case, one has

$$\nabla \cdot \Big[-\Delta \mathbf{v} + \mathbf{v} - \frac{\epsilon^2}{2}|\mathbf{v}|^2\mathbf{v}\Big] = 0, \quad \eta = -\frac{\epsilon}{2}|\mathbf{v}|^2. \tag{3.6}$$

Any solution $\mathbf{v}$ of

$$-\Delta \mathbf{v} + \mathbf{v} - \frac{\epsilon^2}{2}|\mathbf{v}|^2\mathbf{v} = 0, \tag{3.7}$$

solves (3.6).

Equation (3.7) is a particular case of the equation of a bound state solution of a vector nonlinear Schrödinger equation. It is proven in [15] that (3.7) has a solution whose components are constant multiples of the ground state of the corresponding *scalar* nonlinear Schrödinger equation. In both cases it would be interesting to investigate the stability of those solutions with respect to the "good" KBK system.

4. The Kaup–Broer–Kupershmidt system by IST techniques

The connection of (1.3) with the theory of integrable systems was first noticed in [28, 38] for both the "bad" and "good" cases. Actually, Kaup [28], proved that the KBK system is the compatibility condition for a pair of linear equations.

Writing the KBK system as

$$\zeta_t + (\zeta v)_x \pm v_{xxx} = 0, \quad v_t + v v_x + \zeta_x = 0, \quad \zeta = 1 + \eta,$$

the two aforementioned linear equations are

$$\begin{cases} 4\psi_{xx} = \pm\Big[\Big(\lambda - \frac{1}{2}v\Big)^2 - \zeta\Big]\psi, \\ \psi_t = \frac{1}{4}v_x\psi - \Big(\lambda + \frac{1}{2}v\Big)\psi_x. \end{cases}$$

Kaup also wrote the formulation of the direct and inverse scattering problems and the formula for the soliton solution. Kupershmidt [38] showed how the KBK system can be derived in bi-Hamiltonian form. He considered in fact a more general class of systems that he described as the "richest integrable known system known to date":

$$\begin{cases} u_t + h_x + \beta u_{xx} + u u_x = 0, \\ h_t + \alpha u_{xxx} - \beta h_{xx} = 0, \end{cases}$$

where α and β are arbitrary real constants. Note that the system is linearly well-posed if and only if $\alpha < -\beta^2$. The KBK systems correspond to $\beta = 0$, $\alpha > 0$ for the "bad" one, $\alpha < 0$ for the "good" one. Kupershmidt derived a corresponding hierarchy by making use of the theory of non-standard integrable systems.

As already alluded to, the integrable system structure of the KBK systems yields existence of various special solutions, both in the "good" and the "bad" case. For instance, Sachs [50] wrote an infinite family of rational solutions of the "bad" KBK system. In [14], Clarkson obtains a larger class of rational solutions in terms of the generalized Hermite and generalized Okamoto polynomials for both the "good" and "bad" version of the KBK system. Furthermore, Ito in [27] (see also [50]) exhibited an infinite family of commuting Hamiltonian flows associated to an infinite set of commuting integrals F_n such that

$$\{F_n, \mathcal{H}\} = 0,$$

where $\{\,,\}$ is the Poisson bracket and $\mathcal{H}$ the Hamiltonian of the KBK system. This leads to a hierarchy of KBK Boussinesq systems.

The second one in the "bad" case is

$$\begin{cases} v_t + \dfrac{1}{4}(v^3 + 6v\zeta + 4v_{xx})_x = 0, \\ \zeta_t + \dfrac{1}{4}(3v^2\zeta + 3\zeta^2 + 3v_x^2 + 6vv_{xx} + 4\zeta_{xx})_x = 0, \end{cases} \tag{4.1}$$

where again $\zeta = 1 + \eta$. Note that, contrary to the "bad" KBK system, the linear part in (4.1) is decoupled and well-posed. The Hamiltonian corresponding to (4.1) is

$$H_2(v,\zeta) = \frac{1}{4}\int_{\mathbb{R}} \left\{ v^3\zeta + 3v\zeta^2 + 2v_{xx}\zeta + 2v\zeta_{xx} + \frac{3}{2}v^2 v_{xx} \right\} dx.$$

Remark 4.1. The system (4.1) is different from another integrable system introduced by Kupershmidt [37] as the *s-mKdV system* and which, in the notation of [53] where one also finds a bi-Hamiltonian formulation of the equation, is written

$$\begin{cases} v_t - v_{xxx} + 6v^2 v_x - 3(\eta_{xx}\eta)_x - 6(\eta_x\eta v)_x = 0, \\ \eta_t - 4\eta_{xxx} + 6(v^2 - v_x) + 3(2v_x v - v_{xx})\eta = 0. \end{cases} \tag{4.2}$$

Actually, (4.2) is obtained after a Miura-type transform

$$u = -v_x - v^2 + \eta_x\eta, \quad \xi = \eta_x + \eta v$$

from the so-called super-KdV equation (s-KdV); see [37,53]:

$$\begin{cases} u_t - u_{xxx} - 6uu_x + 12\xi\xi_{xx} = 0, \\ \xi_t - 4\xi_{xxx} - 6u\xi_x - 3\xi u_x = 0. \end{cases} \tag{4.3}$$

We refer to [3,43,48] for ill-posedness and well-posedness results on the Cauchy problem for a variant of (4.3). On the other hand, we are not aware of mathematical results on the Cauchy problem for (4.2).

The next system in the "bad" KBK hierarchy is in the (v, ζ) variables

$$\begin{cases} v_t + \dfrac{1}{8}(v^4 + 12v^2\zeta + 6\zeta^2 + 6v_x^2 + 16vv_{xx} + 8\zeta_{xx})_x = 0, \\ \zeta_t + \dfrac{1}{8}(4v^3\zeta + 12v\zeta^2 + 16v\zeta_{xx} \\ \qquad + 20v_x\zeta_x + 20v_{xx}\zeta + 8v_{xxxx} + 12vv_x^2 + 12v^2v_{xx})_x = 0, \end{cases}$$

which is linearly ill-posed. More generally, the higher order flows are defined by the equations

$$\begin{pmatrix} v \\ \zeta \end{pmatrix}_t + J\nabla H_m = 0,$$

where

$$J = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix},$$

and where the Hamiltonian H_m are defined inductively by

$$H_2 = \frac{1}{2}\int_{\mathbb{R}} (v^2\zeta + \zeta^2 - v_x^2)\, dx, \quad J\nabla H_m = K\nabla H_{m-1},$$

where K is the skew-symmetric operator

$$K = \begin{pmatrix} \partial_x & \frac{1}{2}\partial_x v \\ \frac{1}{2}v\partial_x & \partial_x^3 + \zeta\partial_x + \frac{1}{2}\zeta_x \end{pmatrix}.$$

The same strategy can be applied to the "good" KBK system, starting now from the Hamiltonian

$$H_2 = \frac{1}{2}\int_{\mathbb{R}} (v^2\zeta + \zeta^2 + v_x^2)\, dx,$$

and the new K defined as

$$K = \begin{pmatrix} \partial_x & \frac{1}{2}\partial_x v \\ \frac{1}{2}v\partial_x & -\partial_x^3 + \zeta\partial_x + \frac{1}{2}\zeta_x \end{pmatrix}.$$

The second equation in the hierarchy is thus

$$\begin{cases} v_t + \dfrac{1}{4}(v^3 + 6v\zeta - 4v_{xx})_x = 0, \\ \zeta_t + \dfrac{1}{4}(3v^2\zeta + 3\zeta^2 - 3v_x^2 - 6vv_{xx} - 4\zeta_{xx})_x = 0, \end{cases} \tag{4.4}$$

which again is linearly well-posed and correspond to the Hamiltonian

$$H_2(v,\zeta) = \frac{1}{4}\int_{\mathbb{R}} \left\{ v^3\zeta + 3v\zeta^2 - 2v_{xx}\zeta - 2v\zeta_{xx} - \frac{3}{2}v^2v_{xx} \right\} dx.$$

Remark 4.2. In terms of the variables (v, η), (4.4) becomes

$$\begin{cases} v_t + \frac{1}{4}(v^3 + 6v + 6v\eta - 4v_{xx})_x = 0, \\ \zeta_t + \frac{1}{4}(3v^2 + 3v^2\eta + 6\eta + 3\eta^2 - 3v_x^2 - 6vv_{xx} - 4\eta_{xx})_x = 0. \end{cases} \tag{4.5}$$

Remark 4.3. Equations (4.1), (4.4) and (4.5) are systems of KdV type involving second and third order nonlinear terms making the study of the local Cauchy problem delicate. We refer to [18, 40] for a study of the Cauchy problem for scalar dispersive equations involving a nonlinear dispersive third order term.

The next system in the "good" KBK hierarchy is

$$\begin{cases} v_t + \frac{1}{8}(v^4 + 12v^2\zeta + 6\zeta^2 - 6v_x^2 - 16vv_{xx} - 8\zeta_{xx})_x = 0, \\ \zeta_t + \frac{1}{8}(4v^3\zeta + 12v\zeta^2 - 16v\zeta_{xx} \\ \qquad - 20v_x\zeta_x - 20v_{xx}\zeta + 8v_{xxxx} - 12vv_x^2 - 12v^2v_{xx})_x = 0, \end{cases}$$

which is linearly well-posed and the (v, ζ) version of (3.1). It corresponds to the Hamiltonian

$$H(v, \zeta) = \frac{1}{8}\int_{\mathbb{R}} (v^4\zeta + 6v^2\zeta^2 - 8v^2\zeta_{xx} + 4v_{xx}^2 + 10\zeta v_x^2 + 4\zeta_x^2 - 2v^3v_{xx})\, dx.$$

We now briefly describe some recent progress on the Kaup–Broer–Kupershmidt system. The paper [45] focusses on the periodic "good" KBK system and provides an explicit form of the periodic traveling waves in terms of the Weierstrass elliptic function $\wp$:

$$v(\xi) = \frac{a_{11}\wp(\xi + \phi) + a_{12}}{a_{21}\wp(\xi + \phi) + a_{22}}, \quad \eta(\xi) = cv(\xi) - \frac{1}{2}v(\xi)^2, \qquad \xi = (x - ct).$$

The authors also exhibit a matrix Lax pair that is used it to prove the existence of an infinite number of complex conservation laws.

Theorem 4.1 ([45]). *The following equation holds:*

$$(\rho_n)_t + (j_n)_x = 0,$$

where the conserved densities ρ_n are determined by the recursion

$$\rho_1 = \frac{1}{2}\eta + \frac{i}{2}v_x, \quad \rho_2 = iv\rho_1 - 2\rho_{1x} + \frac{i}{2}v,$$

$$\rho_{n+1} = iv\rho_n - \rho_{n-1} - 2\rho_{nx} - 2\sum_{k=1}^{n-1}\rho_k\rho_{n-k}, \quad n > 1,$$

and the conserved currents are determined by the conserved densities

$$j_1 = \frac{1}{4}v + \frac{1}{2}v\rho_1 - \frac{i}{2}\rho - 2,$$
$$j_n = \frac{1}{2}v\rho_n - \frac{i}{2}\rho_{n+1} + \frac{i}{2}\rho_{n-1}, \quad n > 1.$$

Each complex conservation law gives rise to two real conservation laws that are polynomials in η, v and their higher order derivatives with respect to x.

The Lax pair is also used to construct the forward scattering transform for periodic solutions and to find exact formulas for finite gap solutions.

Two-dimensional integrable generalizations of the Kaup–Broer–Kupershmidt system are given in [44, 46, 49]. None of them seem physically relevant, however. On the other hand, for capillary waves such that $\tau/(g\rho) > h_0^2/3$, (2.4) is analogous to the "good" BSK system.

Remark 4.4. The "bad" Kaup–Broer–Kupershmidt system is studied from the Inverse Scattering point of view in [54] under the form

$$\begin{cases} v_t = \frac{1}{2}(v^2 + 2w - v_x)_x, \\ w_t = \left(vw + \frac{1}{2}w_x\right)_x, \end{cases} \tag{4.6}$$

which is an alternative form of (2.2). Using this version of the system, (4.6) is the compatibility condition of the following Lax pair

$$\psi_x = U\psi, \quad \psi_t = V\psi,$$

with

$$\begin{cases} U = -ik\sigma_3 + Q, \;\; Q = \frac{v}{2}\sigma_3 + \sigma_+ - w\sigma_-, \\ V = k^2\sigma_3 + \widehat{Q}, \quad \widehat{Q} = -\frac{1}{4}(v_x - v^2)\sigma_3 + \left(ik + \frac{v}{2}\right)\sigma_+ - \frac{1}{2}(w_x + wv)\sigma_-, \end{cases}$$

where k is a spectral parameter, σ_j, $j = 1, 2, 3$, are the classical Pauli matrices and

$$\sigma_+ = \frac{1}{2}(\sigma_1 + i\sigma_2), \quad \sigma_- = \frac{1}{2}(\sigma_1 - i\sigma_2).$$

The Inverse Scattering mechanism described in [54] allows the construction of special solutions such as kinks; see [54] for details.

Remark 4.5. There is so far no rigorous result on the complete resolution of the Cauchy problem for the well-posed Kaup–Broer–Kupershmidt system ($\alpha = -1$) under an appropriate functional setting for the initial data and the solution by using inverse scattering techniques, and giving possibly insight into the qualitative behavior of the solutions.This would include a rigorous theory for the direct and inverse scattering problem. We refer to [45] for progress in this direction.

5. Numerical approach to the Kaup–Broer–Kupershmidt system

In this section, we will detail the numerical approach used to solve the "good" KBK system in the numerical experiments, similarly to [33, 34].

We recall the conserved energy for this equation is

$$E = \frac{1}{2}\int_{\mathbb{R}} \left(\eta^2 + (1+\eta)v^2 + v_x^2\right) dx. \tag{5.1}$$

Solitons of the KBK system (with location x_0 of the maximum for $t = 0$) can be written for the velocity $C \in \mathbb{R}$, $|C| < 1$ with (3.5) in the form

$$v = \frac{2(1-C^2)}{\cosh(\sqrt{1-C^2}(x - Ct - x_0)) - C}, \quad \eta = Cv - \frac{1}{2}v^2. \tag{5.2}$$

To numerically solve the system (1.3) we essentially use the diagonalization approach for the linear part of (3.2), (3.3). We consider the KBK system in Fourier space,

$$\widehat{\eta}_t = -ik(1+k^2)\widehat{v} - ik\widehat{\eta v}, \quad \widehat{v}_t = -ik\widehat{\eta} - \frac{ik}{2}\widehat{v}^2 \tag{5.3}$$

Here we use the standard definition for a Fourier transform for integrable functions $u(x)$ denoted by $\widehat{u}(k)$ with dual variable k and its inverse (in the sense of tempered distributions),

$$\widehat{u}(k) = \int_{\mathbb{R}} u(x)e^{-ikx}\,dx, \qquad k \in \mathbb{R},$$
$$u(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{u}(k)e^{ikx}\,dk, \quad x \in \mathbb{R}.$$

Introducing

$$\widehat{u}_{\pm} = \widehat{v} \pm \frac{\widehat{\eta}}{\sqrt{1+k^2}},$$

we can write the system (5.3) in the form

$$(u_{\pm})_t = \mp u_{\pm} - ik\Big(\frac{1}{2}\widehat{v}^2 \pm \frac{\widehat{\eta v}}{\sqrt{1+k^2}}\Big). \tag{5.4}$$

Obviously the dispersion relation is as in (2.3), i.e., for large $|k|$ as in the Schrödinger equation.

For the numerical treatment, the Fourier transform in (5.4) will be approximated in standard way by the Discrete Fourier Transform (DFT) which can be conveniently computed by the *Fast Fourier Transform* (FFT). This is a *spectral method* which means that the numerical error in approximating smooth periodic functions with N modes in the DFT leads to a numerical error exponentially decreasing with N, see the discussion in [52] and references therein. Thus we will in the following always work on a torus of period $2\pi L$ with $L > 0$, i.e., we will consider values of $x \in L[-\pi, \pi]$, where we will apply N DFT modes. In an abuse of notation, we will denote the discrete Fourier transform with the same symbol as the standard Fourier transform. Since the numerical error in approximating a function with a DFT is of the order of the highest DFT coefficients, we always use values of L and N such that the DFT coefficients decrease to machine precision which is here of the order of 10^{-16}.

The resulting $2N$-dimensional system of equations (5.4) are of the form

$$u_t = \mathcal{L}u + \mathcal{N}(u), \tag{5.5}$$

where $\mathcal{L}$ is a linear diagonal operator, here proportional to $\pm ik\sqrt{1+k^2}$, whereas $\mathcal{N}(u)$ is a nonlinear term in the $u_\pm$. Due to $|\mathcal{L}|$ being rapidly increasing with $|k|$, the system is *stiff*, which means that explicit time integration schemes are not efficient for such systems due to stability conditions; see, for instance, the discussion in [26] and references therein.

However, there are efficient time integration schemes to address systems of the form given in (5.5) with a stiff diagonal linear term; see [26] for so-called exponential time differencing schemes (ETD). The idea of ETD schemes is to use equidistant time steps h and to integrate equation (5.5) between the time steps t_n and t_{n+1}, $n = 1, 2, \dots,$ with an exponential integrator with respect to t. We get

$$u(t_{n+1}) = e^{\mathcal{L}h}u(t_n) + \int_0^h e^{\mathcal{L}(h-\tau)}\mathcal{N}\big(u(t_n+\tau), t_n+\tau\big)\, d\tau.$$

The integral will be computed in an approximate way for which different schemes exist.

In [32], we have compared for dispersive PDEs various Runge–Kutta schemes of classical order 4 which all showed similar performance. Therefore, we apply in the following the one by Cox–Matthews [17]. Since the method is explicit (only information of the functions at the previous time step t_n is needed), it is not important that the nonlinear part is not diagonal in (5.4) as the linear part. As discussed in [31, 32], the exactly conserved energy can be used to control the numerical error in the time integration. Whereas the energy (5.1) is exactly conserved for the KBK system, the numerically computed energy will depend on time due to unavoidable numerical errors. The relative energy

$$\Delta := |E(t)/E(0) - 1|$$

typically overestimates the numerical error by one to two orders of magnitude. We will always aim at a Δ considerably smaller than 10^{-3}.

To test the code, we consider the soliton (5.2) for $C = 0.8$ as initial data. We use $N = 2^{11}$ DFT modes for $x \in 15[-\pi, \pi]$ with $N_t = 4000$ time steps for $t \le 1$. The DFT coefficients decrease to machine precision during the whole computation in this case, and the relative energy is conserved to better than 10^{-12}. In Figure 1 we show the difference between numerical and exact solution at the final time. It can be seen that it is of the order of 10^{-12}, which shows that the code can propagate the soliton with essentially machine precision.

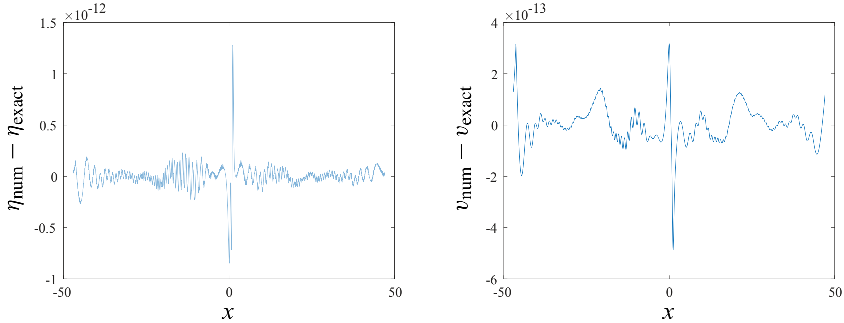

Figure 1. Difference between numerical and exact solution for soliton initial data (5.2) with $C = 0.8$ for $t = 1$, on the left η, on the right v.

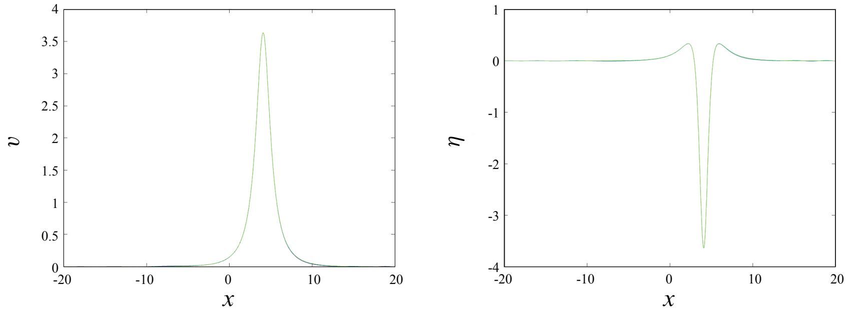

Figure 2. The solution to the KBK system for perturbed soliton initial data of the form (6.1) with $\lambda = 1.01$ and $\mu = 1$ at the final time $t = 5$ in blue and a fitted soliton in green, on the left v, on the right η.

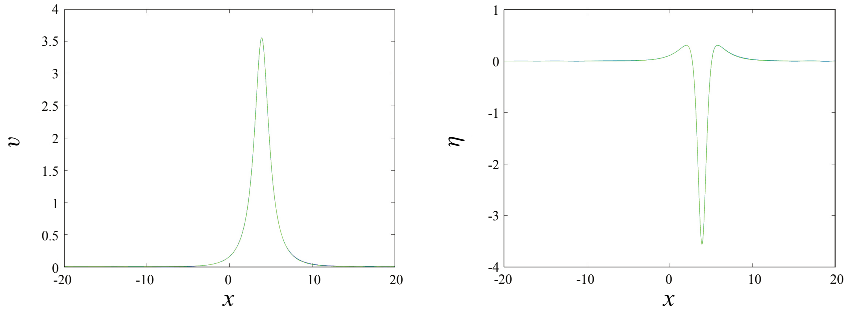

Figure 3. The solution to the KBK system for perturbed soliton initial data of the form (6.1) with $\lambda = 0.99$ and $\mu = 1$ at the final time $t = 5$ in blue and a fitted soliton in green, on the left v, on the right η.

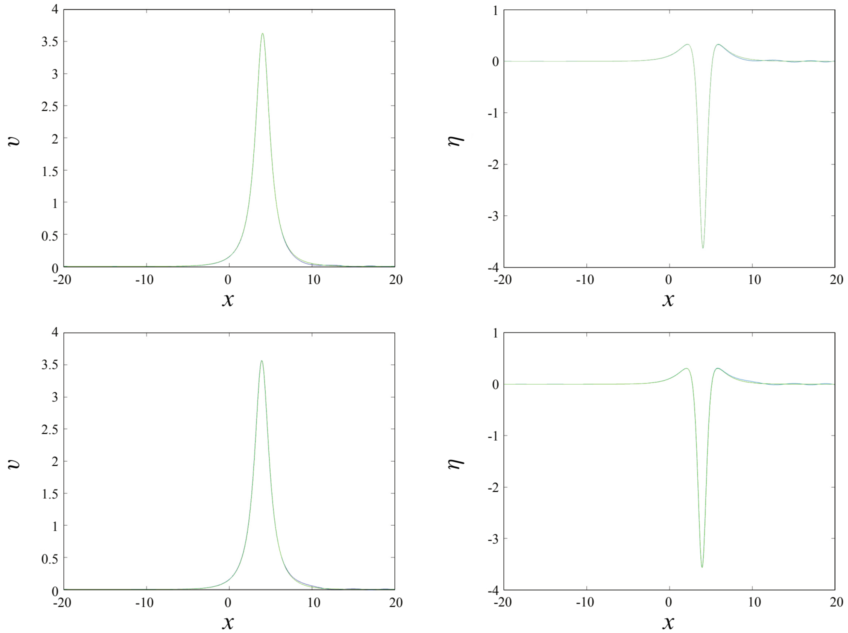

Figure 4. The solution to the KBK system for perturbed soliton initial data of the form (6.1) with $\lambda = 1$ and $\mu = 1.01$ in the upper row and $\mu = 0.99$ in the lower row fitted soliton in green, on the left v, on the right η.

6. Perturbed solitons

Angulo [2] showed that the solitons of the KBK system are stable. In this section, we illustrate this by considering perturbations of the solitons.

We study perturbations of the form

$$v(x,0) = \lambda v_C(x), \quad \eta(x,0) = \mu \eta_C(x), \tag{6.1}$$

where v_C, η_C are the solitons (5.2) for a given real velocity C with $|C| < 1$, and where λ, μ are real numbers in the vicinity of 1. We use $N = 2^{12}$ DFT modes for $x \in 30[-\pi, \pi]$ and $N_t = 4000$ time steps for $t \in [0,5]$. Note that the maximum of v in (5.2) is given by $1 + C$, and thus strictly smaller than 2. This means that the value of λ in (6.1) has to be chosen such that this condition is satisfied. To study numerically perturbations and to see an effect in finite time, one has to consider values of λ and or μ that have a finite difference to 1. This implies that the resulting solution will be close to the original soliton, but not of identical velocity. We fit the soliton in the following way: in the numerical solution for v at the final time $t = 5$, we identify the location x_0 and the value v_0 of the maximum. Since the maximum value for v in (5.2) is given by $2(C + 1)$, we can get the value of C from this maximum.

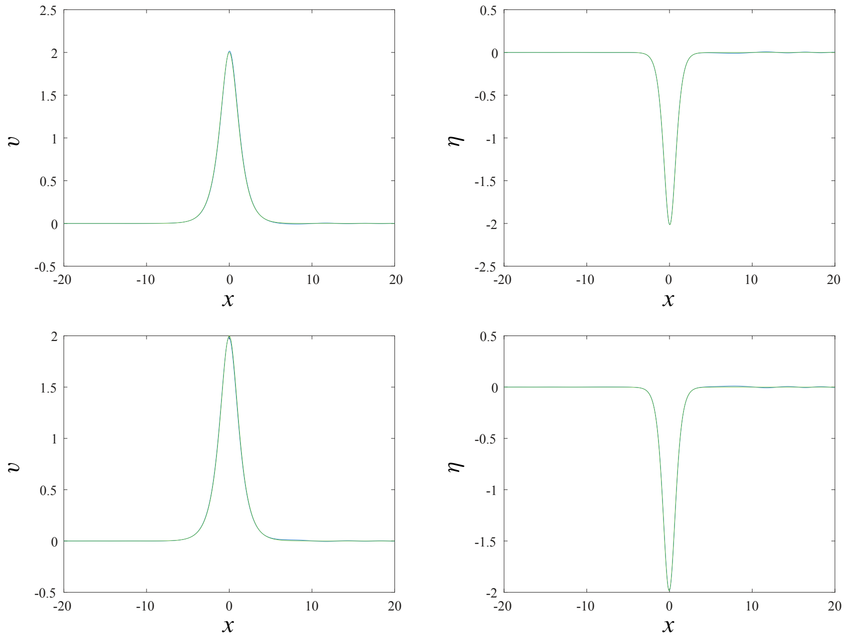

Figure 5. The solution to the KBK system for the perturbed stationary solution of the form (6.1) with $\lambda = 1$ and $\mu = 1.01$ in the upper row and $\mu = 0.99$ in the lower row at the final time $t = 5$ in blue and a fitted stationary solution in green, on the left v, on the right η.

We show the solution to the KBK system for $C = 0.8$, $\lambda = 1.01$ and $\mu = 1$ at the final time together with the fitted soliton in green (the fitted velocity is 0.819) in Figure 2. It can be seen that the solution is very close to the fitted soliton, but that there is also some small radiation. The soliton is thus as expected stable.

The situation is very similar for initial data of the form (6.1) with $\lambda = 0.99$ and $\mu = 1$ as can be seen in Figure 3. The fitted soliton has velocity 0.7811. Again the final state is a soliton plus radiation.

The same stability aspects are observed if perturbations of the form (6.1) with $\lambda = 1$ and $\mu \sim 1$ are studied as shown in Figure 4. The fitted values of the velocity are 0.8155 for $\mu = 1.01$ and 0.7846 for $\mu = 0.99$. In all cases the final state of the solution is a soliton plus radiation.

For $C = 0$, the solutions (5.2) become stationary. If we apply the same perturbations (6.1) to this solution, we find that the stationary solution is also stable. For $\mu = 1$ in the initial data (6.1), we get the solutions shown in Figure 5.

The situation is similar for $\lambda = 1$ as shown in Figure 6.

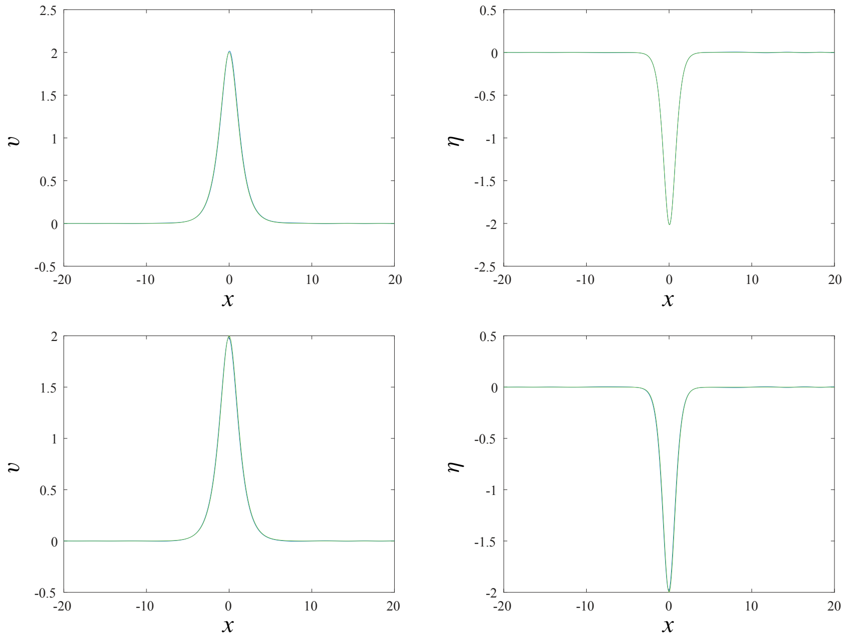

Figure 6. The solution to the KBK system for the perturbed stationary solution of the form (6.1) with $\lambda = 1$ and $\mu = 1.01$ in the upper row and $\mu = 0.99$ in the lower row fitted stationary solution in green, on the left v, on the right η.

7. Localized initial data

In this section, we will study KBK solutions for localized initial data. The results confirm the applicability of the soliton resolution conjecture to the KBK system, that the long time behavior of solutions is given by solitons plus radiation.

In this section, we always use $N = 2^{12}$ DFT modes for $x \in 30[-\pi, \pi]$ and $N_t = 4000$ time steps. The DFT coefficients decrease to machine precision in all examples, and the relative conservation of the energy is of the order of 10^{-10} and better during the computations. First we consider initial data of the form

$$\eta(x,0) = 0, \quad v(x,0) = 3\exp(-x^2). \tag{7.1}$$

The resulting solution for v can be seen in Figure 7. There are strong oscillations propagating to the right and smaller ones propagating to the left. And there is a solitary wave traveling towards $-\infty$ to be discussed in more detail below.

The corresponding solution η can be seen in Figure 8. There is similar radiation as in the solution v, and again a solitary structure traveling to the left.

To check whether the solitary structure near the origin is in fact evolving into a soliton, we fit it to the soliton (5.2) as described in the previous section. We show the result of this

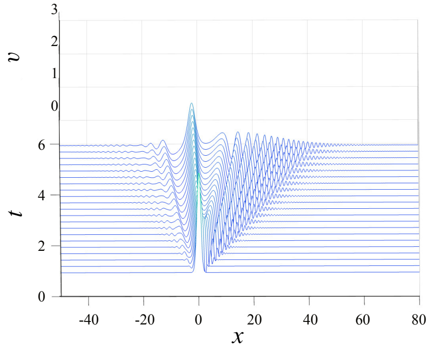

Figure 7. Solution v to the KBK system for the initial data (7.1).

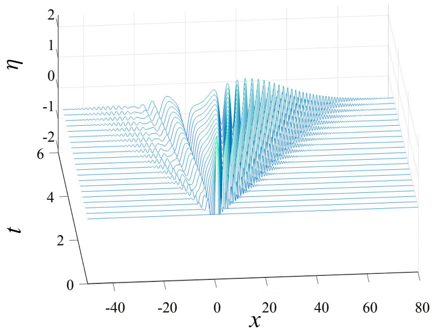

Figure 8. Solution η to the KBK system for the initial data (7.1).

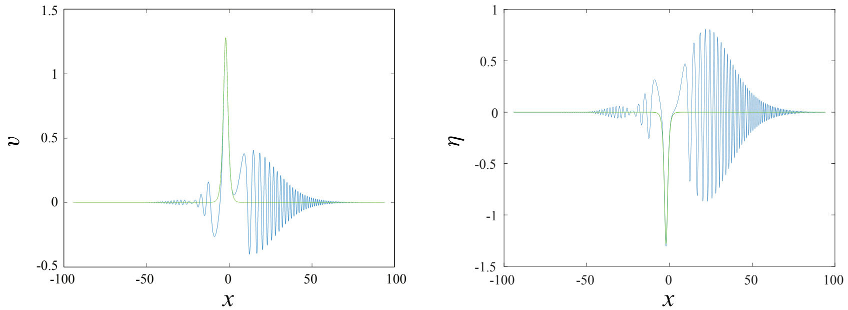

Figure 9. The solutions shown in Figure 7 and Figure 8 at the final time $t = 5$ in blue and a fitted soliton in green.

fitting for v on left of Figure 9. The numerical solution is given in blue, the fitted soliton in green. The corresponding plot for η is shown on the right of the same figure. It can be seen that the agreement is already very good though there is still a considerable amount of radiation in the vicinity of the soliton.

Note that there is no criterion known which initial data lead to which number of KBK solitons for large times. If we consider for instance the initial data

$$\eta(x,0) = A\exp(-x^2), \quad v(x,0) = 0, \qquad A \in \mathbb{R}, \tag{7.2}$$

we get with $A = 3$ for v the solution shown in Figure 10. In this case there is no indication of solitons.

The corresponding solution η is shown in Figure 11. It appears that the initial data are simply dispersed in this case. Note that the solution looks qualitatively similar for larger values for A, for instance $A = 20$, there is no indication of solitons in this class of initial data.

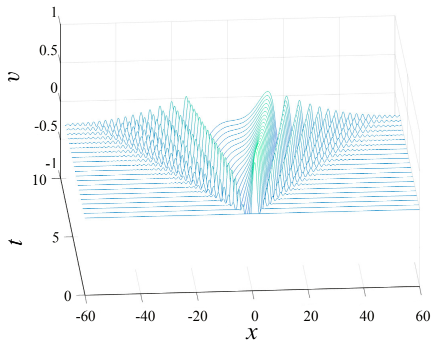

Figure 10. Solution v to the KBK system for the initial data (7.2) with $A = 3$.

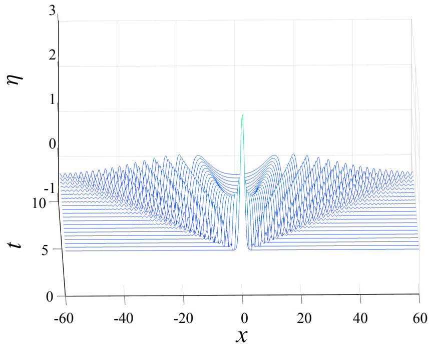

Figure 11. Solution η to the KBK system for the initial data (7.2) with $A = 3$.

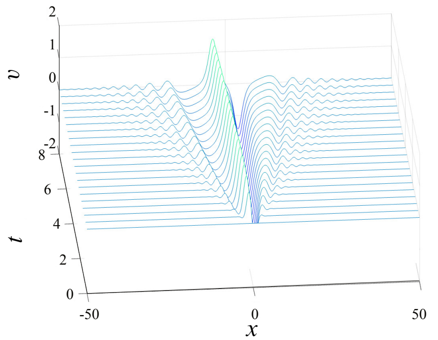

Figure 12. Solution v to the KBK system for the initial data (7.2) with $A = -3$.

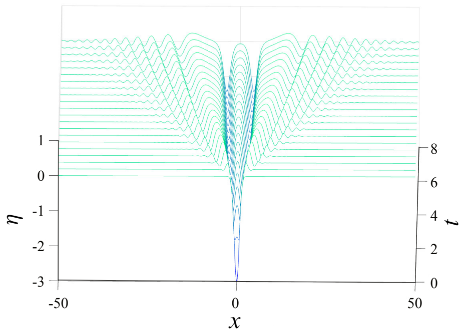

Figure 13. Solution η to the KBK system for the initial data (7.2) with $A = -3$.

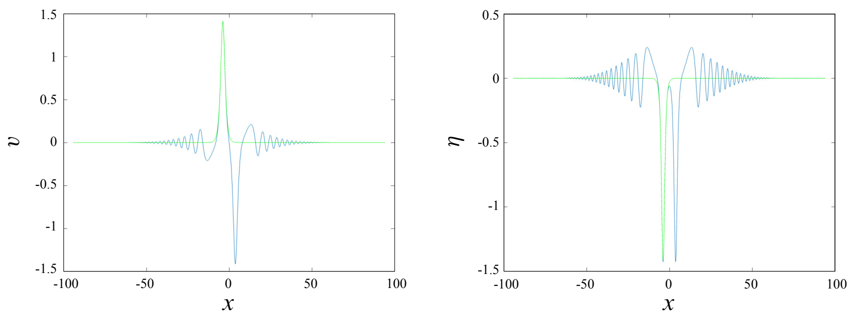

Figure 14. The solutions shown in Figure 12 and Figure 13 at the final time $t = 8$ in blue and a fitted soliton in green.

However, the situation is different for negative A in (7.2) as can be seen for $A = -3$ in Figure 12. There appears to be a solitary structure near the origin.

The corresponding solution η can be seen in Figure 11.

As in Figure 9, we fit the left maximum to the soliton (5.2). We show the solution for v at $t = 8$ with the fitted soliton in green on the left of Figure 14. The corresponding plot for η can be seen on the right of the same figure. There is clearly a second soliton moving to the right in this case for symmetry reasons.

Note that these initial data do not satisfy the non-cavitation condition, but that there is no indication of a blow-up in this case. The solution appears to be smooth for all times.

8. Dispersive shock waves

Dispersive shock waves (DSWs) are zones of rapid modulated oscillations in solutions to dispersive nonlinear PDEs in the vicinity of shocks of the corresponding dispersionless system, here the Saint-Venant system. A possible way to study such zones for the KBK system is to consider solutions on x scales of order $1/\varepsilon$ for $\varepsilon \ll 1$ on time scales of order $1/\varepsilon$. This can be conveniently done by rescaling x and t by a factor $1/\varepsilon$. This leads for (1.3) (we use the same notation as before):

$$\begin{cases} \eta_t + v_x + (\eta v)_x + \varepsilon^2 v_{xxx} = 0, \\ v_t + \eta_x + v v_x = 0. \end{cases} \tag{8.1}$$

In the formal limit $\varepsilon \to 0$, this leads to the Saint-Venant system

$$\begin{cases} \eta_t + v_x + (\eta v)_x = 0, \\ v_t + \eta_x + v v_x = 0, \end{cases} \tag{8.2}$$

the solutions of which will have shocks in finite time for hump-like initial data. The system (8.1) can be seen as a dispersive regularization of the system (8.2). For the same initial data leading to shocks for the Saint-Venant system, DSWs are expected in the vicinity of the former for the system (8.1).

We use $N = 2^{14}$ DFT modes for $x \in 3[-\pi, \pi]$ and 10^4 time steps for $t \leq 3$ for the initial data $\eta(x, 0) = \exp(-x^2)$, $v(x, 0) = 0$. The solution η for $\varepsilon = 0.1$ can be seen in Figure 15. The initial hump splits into two humps developing strong gradients at the outer edges where oscillations can be observed.

The corresponding solution v is shown in Figure 16. The solution is odd in x, but shows oscillations at the same x-values as η.

We show a close up of the oscillatory zone in Figure 17. The smaller ε, the more rapid the oscillations and the more they are localized to a zone sometimes called the Whitham zone.

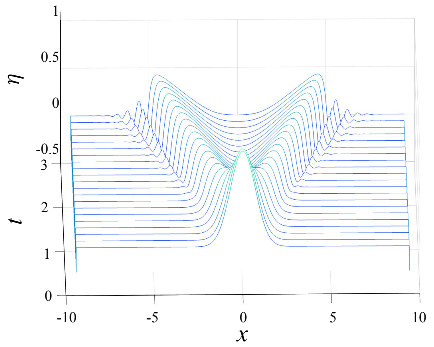

Figure 15. Solution η to the KBK system (8.1) for $\varepsilon = 0.1$ and for the initial data $\eta(x, 0) = \exp(-x^2)$, $v(x, 0) = 0$.

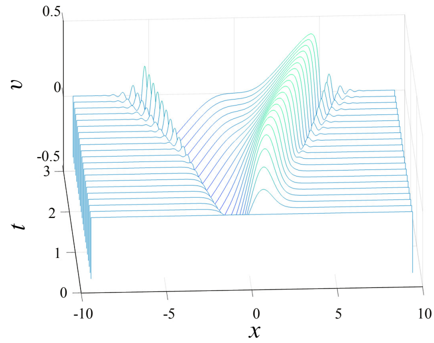

Figure 16. Solution v to the KBK system (8.1) for $\varepsilon = 0.1$ and for the initial data $\eta(x, 0) = \exp(-x^2)$, $v(x, 0) = 0$.

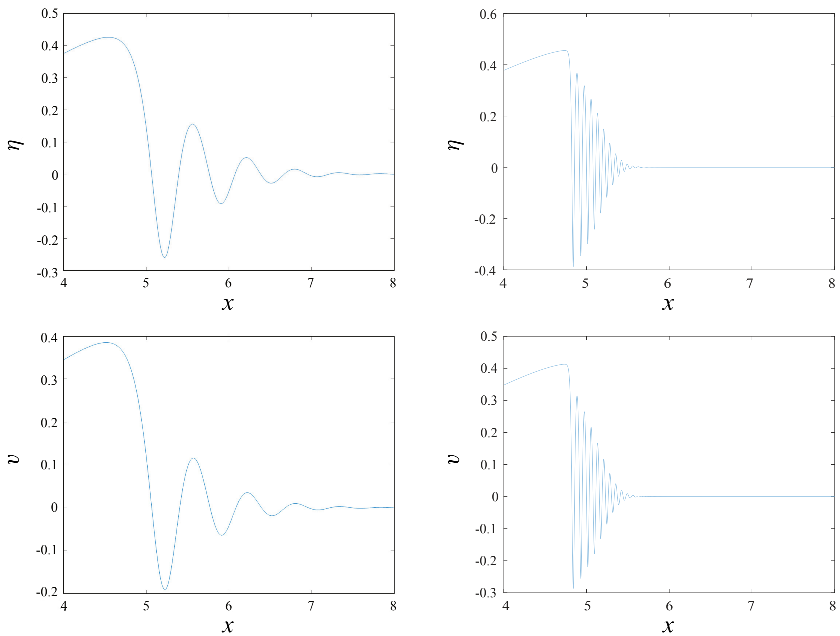

Figure 17. On the left, a close-up of the solutions in Figure 15 and Figure 16 at the final time $t = 3$, in the upper row η, in the lower row v; on the right, the oscillatory zone for the same initial data for $\varepsilon = 0.01$.

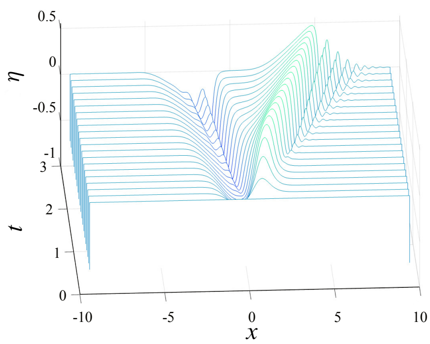

Figure 18. Solution η to the KBK system (8.1)) for $\varepsilon = 0.1$ and for the initial data $v(x,0) = \exp(-x^2)$, $\eta(x,0) = 0$.

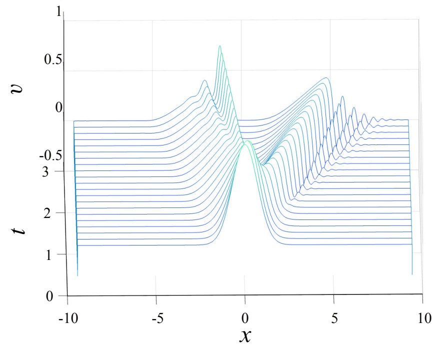

Figure 19. Solution v to the KBK system (8.1) for $\varepsilon = 0.1$ and for the initial data $v(x,0) = \exp(-x^2)$, $\eta(x,0) = 0$.

The situation is similar for other hump-like initial data as $\eta(x,0) = 0$ and $v(x,0) = \exp(-x^2)$. The solution η for $\varepsilon = 0.1$ can be seen in Figure 18. The solution is not symmetric, but there are oscillatory zones as before on both sides of the humps.

The corresponding solution v is shown in Figure 19. Note that due to the rescaling of the original KBK system (1.3), the form of the solitons of (8.1) is slightly different. It appears that the oscillations traveling to the left in Figure 19 are solitons.

9. Conclusion

In this paper we have presented a detailed numerical study of solutions to the "good" KBK system (1.3). The stability of the solitons as proven by Angulo [2] was illustrated for several examples, also for the stationary solution. It was shown that the long time behavior of solutions for localized initial data is given by solitons plus radiation. In the vicinity of shocks to the corresponding dispersionless Saint-Venant system, dispersive shock waves were observed. No indication of a blow-up was found even in cases where the non-cavitation condition is not satisfied.

It is an interesting question whether these features can also be observed in the two-dimensional variant of the KBK system which is most probably not integrable. An important point is whether there are two-dimensional localized solitary waves called lumps as in the KP-I equation, or whether there are no localized two-dimensional structures as for KP-II. The one-dimensional solitons discussed in the present paper are infinitely extended exact solutions to the two-dimensional KBK system, so-called line solitons. Their stability in the two-dimensional equation has to be studied as well as the question whether there can be blow-up in two-dimensional. This will be the subject of future work.

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