

Highest Weight Theory for Minimal Finite W -Superalgebras and Related Whittaker Categories

by

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Abstract

Let $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ be a basic classical Lie superalgebra over $\mathbb{C}$, and $e = e_\theta \in \mathfrak{g}_0$ with $-\theta$ being a minimal root of $\mathfrak{g}$. Set $U(\mathfrak{g}, e)$ to be the minimal finite W -superalgebras associated with the pair $(\mathfrak{g}, e)$. In this paper we study the highest weight theory for $U(\mathfrak{g}, e)$, introduce the Verma modules and give a complete isomorphism classification of finite-dimensional irreducible modules, via the parameter set consisting of pairs of weights and levels. Those Verma modules can be further described via parabolic induction from Whittaker modules for $\mathfrak{osp}(1|2)$ or $\mathfrak{sl}(2)$ respectively, depending on the detecting parity of $r := \dim \mathfrak{g}(-1)_1$. We then introduce and investigate the BGG category $\mathcal{O}$ for $U(\mathfrak{g}, e)$, establishing highest weight theory, as a counterpart for the works for finite W -algebras by Brundan–Goodwin–Kleshchev [Int. Math. Res. Not. IMRN 15 (2008), article no. rnn051] and Losev [in: Geometric methods in representation theory II (2012), 353–370]. In comparison with the non-super case, the significant difference here lies in the situation when r is odd, which is a completely new phenomenon. The difficulty and complicated computation arise from there.

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§0. Introduction

This work is a sequel to [49]. In [49] we obtained the PBW theorem of minimal refined W -superalgebras over $\mathbb{C}$, along with their generators and the commutator formulas. In this paper we continue to study the representations of minimal finite and refined W -superalgebras. Our purpose is to develop the highest weight theory for minimal finite and refined W -superalgebras, and to determine all simple objects in the corresponding BGG category $\mathcal{O}$.

§0.1.

A finite W -algebra (resp. superalgebra) $U(\mathfrak{g}, e)$ is a certain associative algebra (resp. superalgebra) associated with a complex semi-simple Lie algebra (resp. basic classical Lie superalgebra) $\mathfrak{g}$ and a nilpotent element $e \in \mathfrak{g}$ (resp. $e \in \mathfrak{g}_0$ for $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$). In the last two decades, finite W -algebras and finite W -superalgebras have been developed rapidly since Premet studied finite W -algebras in full generality in [37] (see [38, 39, 40], etc.).

In particular, Brundan–Goodwin–Kleshchev in [12] considered the highest weight theory for finite W -algebras and showed that the irreducible highest weight modules can be parameterized by some unknown set $\mathcal{L}$. In virtue of [12], Losev [30] further studied the so-called BGG category $\mathcal{O}$ for finite W -algebras first introduced by Brundan–Goodwin–Kleshchev, and established an equivalence of these categories and the generalized Whittaker categories studied by Miličić–Soergel [32] when the nilpotent element e is principal.

§0.2.

Let G be a connected semi-simple algebraic group over $\mathbb{C}$ with $\mathfrak{g} = \text{Lie}(G)$. When the element $e = e_\theta$, with $-\theta$ being a minimal root of $\mathfrak{g}$, the corresponding finite W -algebra $U(\mathfrak{g}, e)$ is called minimal. In the case of basic classical Lie superalgebras, one can also take (even) nilpotent elements associated with minimal roots, and then obtain minimal finite W -superalgebras. The study of minimal (affine) W -superalgebras can be traced back from [28] (see also [29, §5] for more details).

In the present paper, we will study the highest weight theory for minimal finite W -superalgebras and for refined minimal W -superalgebras. This work can be regarded as a counterpart of Premet’s work on the minimal finite W -algebras

in [38], and also a counterpart of Brundan–Goodwin–Kleshchev’s work [12] on highest weight theory for finite W -algebras. We will follow Premet’s strategy in [38, §7], and also the methods applied by Brundan–Goodwin–Kleshchev in [12, §4], with a lot of modifications.¹ Let us briefly introduce what we will do.

§0.3.

For a given complex basic classical Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ and a nilpotent element $e \in \mathfrak{g}_{\bar{0}}$, let $(\cdot, \cdot)$ be a non-degenerate even supersymmetric invariant bilinear form on $\mathfrak{g}$. Define $\chi \in \mathfrak{g}^*$ by letting $\chi(x) = (e, x)$ for all $x \in \mathfrak{g}$. Fix an $\mathfrak{sl}(2)$ -triple (e, h, f) in $\mathfrak{g}_{\bar{0}}$, and denote by $\mathfrak{g}^e := \ker(\text{ad } e)$ in $\mathfrak{g}$. The linear operator $\text{ad } h$ defines a $\mathbb{Z}$ -grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ with $e \in \mathfrak{g}(2)_{\bar{0}}$ and $f \in \mathfrak{g}(-2)_{\bar{0}}$, and we have $\mathfrak{g}^e = \bigoplus_{i \geq 0} \mathfrak{g}^e(i)$ by the $\mathfrak{sl}(2)$ -representation theory. Up to a scalar, we further assume that $(e, f) = 1$.

Choose $\mathbb{Z}_2$ -homogeneous bases $\{u_1, \dots, u_s\}$ of $\mathfrak{g}(-1)_{\bar{0}}$ and $\{v_1, \dots, v_r\}$ of $\mathfrak{g}(-1)_{\bar{1}}$ such that $\chi([u_i, u_j]) = i^* \delta_{i+j, s+1}$ for $1 \leq i, j \leq s$, where

$$i^* := \begin{cases} -1 & \text{if } 1 \leq i \leq \frac{s}{2}, \\ 1 & \text{if } \frac{s}{2} + 1 \leq i \leq s, \end{cases}$$

and $\chi([v_i, v_j]) = \delta_{i+j, r+1}$ for $1 \leq i, j \leq r$. Write $\mathfrak{g}(-1)'_{\bar{0}}$ for the $\mathbb{C}$ -span of $u_{\frac{s}{2}+1}, \dots, u_s$ and $\mathfrak{g}(-1)'_{\bar{1}}$ the $\mathbb{C}$ -span of $v_{\frac{r}{2}+1}, \dots, v_r$ (resp. $v_{\frac{r+3}{2}}, \dots, v_r$) when r is even (resp. odd). Denote $\mathfrak{g}(-1)' := \mathfrak{g}(-1)'_{\bar{0}} \oplus \mathfrak{g}(-1)'_{\bar{1}}$, and set the χ -admissible subalgebra and the extended χ -admissible subalgebra of $\mathfrak{g}$ to be

$$\mathfrak{m} := \bigoplus_{i \leq -2} \mathfrak{g}(i) \oplus \mathfrak{g}(-1)', \quad \mathfrak{m}' := \begin{cases} \mathfrak{m} & \text{if } r \text{ is even,} \\ \mathfrak{m} \oplus \mathbb{C}v_{\frac{r+1}{2}} & \text{if } r \text{ is odd,} \end{cases}$$

respectively. Then the generalized Gelfand–Graev $\mathfrak{g}$ -module associated with χ is defined by $Q_\chi := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi$, where $\mathbb{C}_\chi = \mathbb{C}1_\chi$ is a one-dimensional $\mathfrak{m}$ -module such that $x \cdot 1_\chi = \chi(x)1_\chi$ for all $x \in \mathfrak{m}$.

A finite W -superalgebra $U(\mathfrak{g}, e) := (\text{End}_{\mathfrak{g}} Q_\chi)^{\text{op}}$ is by definition isomorphic to $Q_\chi^{\text{ad } \mathfrak{m}}$, the invariants of Q_χ under the adjoint action of $\mathfrak{m}$. Recall that in [48, Thm. 4.5] we introduced the PBW theorem for finite W -superalgebras $U(\mathfrak{g}, e)$. As an important ingredient in our arguments, we will first introduce a refined version of the PBW theorem of $U(\mathfrak{g}, e)$, which will be present in Theorem 1.4 and Proposition 1.10.

¹Depending on the parity of the dimension for a particular subspace of $\mathfrak{g}$, one can observe critical distinctions for the construction of Verma modules for minimal finite and refined W -superalgebras, and also for the formulation of the corresponding BGG category $\mathcal{O}$, which never appears in the non-super case.

§0.4.

From now on, we will focus on the minimal case.

0.4.1. Recall that Cartan subalgebras of $\mathfrak{g}$ are conjugate, and are by definition just those of $\mathfrak{g}_{\bar{0}}$. Let $\mathfrak{h}$ be a standard Cartan subalgebra of $\mathfrak{g}$, and let Φ be the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. Denote by $\Phi_{\bar{0}}$ and $\Phi_{\bar{1}}$ the sets of all even roots and odd roots, respectively. From the detailed description of Φ in [25, §2.5.4] and [34, Thm. 3.10], we can choose a minimal root $-\theta$ of $\mathfrak{g}$, and then a positive root system Φ^+ of Φ with simple root system $\Delta = \{\alpha_1, \dots, \alpha_k\}$ satisfies that $\alpha_k = \theta$ when r is even, and $\alpha_k = \frac{\theta}{2}$ when r is odd (see the arguments for Conventions 2.1). Furthermore, we may choose root vectors $e = e_\theta$, $f = e_{-\theta}$ corresponding to roots θ and $-\theta$ such that $(e_\theta, [e_\theta, e_{-\theta}], e_{-\theta})$ is an $\mathfrak{sl}(2)$ -triple and put $h = h_\theta = [e_\theta, e_{-\theta}]$.

0.4.2. As explained in [48], the structure of finite W -superalgebras critically depends on the detecting parity of $r := \dim \mathfrak{g}(-1)_{\bar{1}}$. In the situation when r is even, the results concerned are very similar to those of the non-super case, despite the discussion here being more difficult. However, when r is odd, the finite W -superalgebra is significantly different from the finite W -algebra case. In this situation, the emergency of odd root $\frac{\theta}{2}$ makes the situation much more complicated, and an extra restriction (2.8) must be imposed to make the procedure go smoothly. So in the following we will mainly consider the case when r is odd.

Let $\Phi_e = \{\alpha \in \Phi \mid \alpha(h) = 0 \text{ or } 1\}$, and write $\Phi_e^\pm := \Phi_e \cap \Phi^\pm$, where $\Phi^- = -\Phi^+$. For $i = 0, 1$ set $\Phi_{e,i}^\pm := \{\alpha \in \Phi_e^\pm \mid \alpha(h) = i\}$, $(\Phi_{e,0}^+)_{\bar{0}} := \Phi_{e,0}^+ \cap \Phi_{\bar{0}}$. Set $\mathfrak{h}^e := \mathfrak{h} \cap \mathfrak{g}^e$ to be a Cartan subalgebra in $\mathfrak{g}^e(0)$ with $\{h_1, \dots, h_{k-1}\}$ being a basis such that $(h_i, h_j) = \delta_{i,j}$ for $1 \leq i, j \leq k-1$. Let δ , ρ , and $\rho_{e,0}$ be defined as in (2.7). Given a linear function λ on $\mathfrak{h}^e$ and $c \in \mathbb{C}$ satisfying equation (2.8), we call (λ, c) a matchable pair. Associated with such a pair, let $I_{\lambda,c}$ be a left ideal of $U(\mathfrak{g}, e)$ defined as in Section 2.2.2. We call the $U(\mathfrak{g}, e)$ -module $Z_{U(\mathfrak{g}, e)}(\lambda, c) := U(\mathfrak{g}, e)/I_{\lambda,c}$ the *Verma module of level c corresponding to λ* . Moreover, $Z_{U(\mathfrak{g}, e)}(\lambda, c)$ is proved to contain a unique maximal submodule which we denote $Z_{U(\mathfrak{g}, e)}^{\max}(\lambda, c)$ (see Section 2.2.3). Thus, to every matchable pair $(\lambda, c) \in (\mathfrak{h}^e)^* \times \mathbb{C}$, there corresponds an irreducible highest weight $U(\mathfrak{g}, e)$ -module $L_{U(\mathfrak{g}, e)}(\lambda, c) := Z_{U(\mathfrak{g}, e)}(\lambda, c)/Z_{U(\mathfrak{g}, e)}^{\max}(\lambda, c)$.

Recall that an irreducible module is of type M if its endomorphism ring is one-dimensional and it is of type Q if its endomorphism ring is two-dimensional. As the first main result of the paper, we have the following theorem.

Theorem 0.1. *Assume that r is odd. Given a matchable pair $(\lambda, c) \in (\mathfrak{h}^e)^* \times \mathbb{C}$, the following statements hold:*

- (1) $Z_{U(\mathfrak{g}, e)}^{\max}(\lambda, c)$ is the unique maximal submodule of the Verma module $Z_{U(\mathfrak{g}, e)}(\lambda, c)$, and $L_{U(\mathfrak{g}, e)}(\lambda, c)$ is a simple $U(\mathfrak{g}, e)$ -module of type Q .

- (2) The simple $U(\mathfrak{g}, e)$ -modules $L_{U(\mathfrak{g}, e)}(\lambda, c)$ and $L_{U(\mathfrak{g}, e)}(\lambda', c')$ are isomorphic if and only if $(\lambda, c) = (\lambda', c')$.
- (3) Any finite-dimensional simple $U(\mathfrak{g}, e)$ -module (up to parity switch) is isomorphic to one of the modules $L_{U(\mathfrak{g}, e)}(\lambda, c)$ for some $\lambda \in (\mathfrak{h}^e)^*$ satisfying $\lambda(h_\alpha) \in \mathbb{Z}_+$ for all $\alpha \in (\Phi_{e,0}^+)_{\bar{0}}$. We further have that c is a rational number in the case when $\mathfrak{g} = \mathfrak{spo}(2|m)$ with m being odd such that $\mathfrak{g}^e(0) = \mathfrak{so}(m)$, or when $\mathfrak{g} = \mathfrak{spo}(2m|1)$ with $m \geq 2$ such that $\mathfrak{g}^e(0) = \mathfrak{spo}(2m-2|1)$, or when $\mathfrak{g} = G(3)$ with $\mathfrak{g}^e(0) = G(2)$.

The proof of Theorem 0.1 will be given in Section 2.2.4.

0.4.3. For $\chi = (e, \cdot)$ we let $\mathcal{C}_\chi$ denote the category of Whittaker modules, i.e., all $\mathfrak{g}$ -modules on which $x - \chi(x)$ acts locally nilpotently for all $x \in \mathfrak{m}$. Given a $\mathfrak{g}$ -module M we set

$$\text{Wh}(M) = \{m \in M \mid x \cdot m = \chi(x)m, \forall x \in \mathfrak{m}\}.$$

We will link the Verma modules $Z_{U(\mathfrak{g}, e)}(\lambda, c)$ with $\mathfrak{g}$ -modules obtained by parabolic induction from Whittaker modules for $\mathfrak{osp}(1|2)$. Set $\mathfrak{s}_\theta = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f \oplus \mathbb{C}[v_{\frac{r+1}{2}}, e] \oplus \mathbb{C}v_{\frac{r+1}{2}}$, and put

$$\mathfrak{p}_\theta := \mathfrak{s}_\theta + \mathfrak{h} + \sum_{\alpha \in \Phi^+} \mathbb{C}e_\alpha, \quad \mathfrak{n}_\theta := \sum_{\alpha \in \Phi^+ \setminus \{\frac{\theta}{2}, \theta\}} \mathbb{C}e_\alpha, \quad \tilde{\mathfrak{s}}_\theta := \mathfrak{h}^e \oplus \mathfrak{s}_\theta,$$

where e_α denotes a root vector with $\alpha \in \Phi$. Let $C_\theta := 2ef + \frac{1}{2}h^2 - \frac{3}{2}h - 2v_{\frac{r+1}{2}}[v_{\frac{r+1}{2}}, e]$ be a central element of $U(\tilde{\mathfrak{s}}_\theta)$. Given $\lambda \in (\mathfrak{h}^e)^*$, write $I_\theta(\lambda)$ for the left ideal of $U(\mathfrak{p}_\theta)$ generated by $f - 1$, $[v_{\frac{r+1}{2}}, e] - \frac{3}{4}v_{\frac{r+1}{2}} + \frac{1}{2}v_{\frac{r+1}{2}}h$, $C_\theta + \frac{1}{8}$, all $t - \lambda(t)$ with $t \in \mathfrak{h}^e$, and all e_γ with $\gamma \in \Phi^+ \setminus \{\frac{\theta}{2}, \theta\}$. Set $Y(\lambda) := U(\mathfrak{p}_\theta)/I_\theta(\lambda)$ to be a $\mathfrak{p}_\theta$ -module with the trivial action of $\mathfrak{n}_\theta$, which is isomorphic to a Whittaker module for the Levi subalgebra $\tilde{\mathfrak{s}}_\theta$. Now define

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\theta)} Y(\lambda).$$

Since the restriction of $(\cdot, \cdot)$ to $\mathfrak{h}^e$ is non-degenerate, for any $\eta \in (\mathfrak{h}^e)^*$ there is a unique t_η in $\mathfrak{h}^e$ with $\eta = (t_\eta, \cdot)$. Hence $(\cdot, \cdot)$ induces a non-degenerate bilinear form on $(\mathfrak{h}^e)^*$ via $(\mu, \nu) := (t_\mu, t_\nu)$ for all $\mu, \nu \in (\mathfrak{h}^e)^*$. For a linear function $\varphi \in \mathfrak{h}^*$ we denote by $\bar{\varphi}$ the restriction of φ to $\mathfrak{h}^e$. Keep the notation ϵ as in (3.4). Under the twisted action of $U(\mathfrak{g}, e)$ on the Verma modules as defined in Section 3.2.3, we have the following second main result of the paper.

Theorem 0.2. *Assume that r is odd. Every $\mathfrak{g}$ -module $M(\lambda)$ is an object of the category $\mathcal{C}_\chi$. Furthermore, $\text{Wh}(M(\lambda)) \cong Z_{U(\mathfrak{g}, e)}(\lambda + \bar{\delta}, -\frac{1}{8} + (\lambda + 2\bar{\rho}, \lambda) + \epsilon)$ as $U(\mathfrak{g}, e)$ -modules.*

The proof of Theorem 0.2 will be given in Section 3.2.3. In virtue of Theorem 0.2 and Skryabin's equivalence in [48, Thm. 2.17] (see also Section 3.1 for more details), we can translate the problem of the computing of the composition multiplicities of the Verma modules $Z_{U(\mathfrak{g},e)}(\lambda, c)$ to that of the parabolic induced modules from Whittaker modules (i.e., the standard Whittaker modules) for $\mathfrak{osp}(1|2)$. These standard Whittaker modules have been studied in much detail in [14, 15], and it is known that their composition multiplicities can be determined by the composition factors of Verma modules for $U(\mathfrak{g})$ in the ordinary BGG category $\mathcal{O}$.

0.4.4. For $\alpha \in (\mathfrak{h}^e)^*$, let $\mathfrak{g}_\alpha = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_\alpha(i)$ denote the α -weight space of $\mathfrak{g}$ with respect to $\mathfrak{h}^e$. Then we have $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi'_e} \mathfrak{g}_\alpha$, where $\Phi'_e \subset (\mathfrak{h}^e)^*$ is the set of non-zero weights of $\mathfrak{h}^e$ on $\mathfrak{g}$. Let $(\Phi'_e)^+ := \Phi^+ \setminus \{\frac{\theta}{2}, \theta\}$ be a system of positive roots in the restricted root system Φ'_e . Setting $(\Phi'_e)^- := -(\Phi'_e)^+$, we define $\mathfrak{g}_\pm := \bigoplus_{\alpha \in (\Phi'_e)^\pm} \mathfrak{g}_\alpha$, so that $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$, $\mathfrak{q} = \mathfrak{g}_0 \oplus \mathfrak{g}_+$. The choice $(\Phi'_e)^+$ of positive roots induces a dominance ordering $\leq$ on $(\mathfrak{h}^e)^*$: $\mu \leq \lambda$ if $\lambda - \mu \in \mathbb{Z}_{\geq 0}(\Phi'_e)^+$.

Under the above settings, we can define the highest weight $U(\mathfrak{g}, e)$ -module $M_e(\lambda)$ with highest weight λ as in (4.16) and its irreducible quotient $L_e(\lambda)$ as in (4.17). Comparing Theorem 0.1 with Theorem 4.11, we can find that $M_e(\lambda)$ and $L_e(\lambda)$ share the same meaning as the Verma module $Z_{U(\mathfrak{g},e)}(\lambda, c)$ and its irreducible quotient $L_{U(\mathfrak{g},e)}(\lambda, c)$; see Remark 4.17 for more details.

Now we introduce an analogue of the BGG category $\mathcal{O}$. Let $\mathcal{O}(e) = \mathcal{O}(e; \mathfrak{h}, \mathfrak{q})$ denote the category of all finitely generated $U(\mathfrak{g}, e)$ -modules V , that are semi-simple over $\mathfrak{h}^e$ with finite-dimensional $\mathfrak{h}^e$ -weight spaces, such that the set $\{\lambda \in (\mathfrak{h}^e)^* \mid V_\lambda \neq \{0\}\}$ is contained in a finite union of sets of the form $\{\nu \in (\mathfrak{h}^e)^* \mid \nu \leq \mu\}$ for $\mu \in (\mathfrak{h}^e)^*$. Then we obtain the following theorem.

Theorem 0.3. *Assume that r is odd. For the category $\mathcal{O}(e)$, the following statements hold:*

- (1) *There is a complete set of isomorphism classes of simple objects which is $\{L_e(\lambda) \mid \lambda \in (\mathfrak{h}^e)^*\}$ as in (4.17).*
- (2) *The category $\mathcal{O}(e)$ is Artinian. In particular, every object has finite length of composition series.*
- (3) *The category $\mathcal{O}(e)$ has a block decomposition as $\mathcal{O}(e) = \bigoplus_{\psi^\lambda} \mathcal{O}_{\psi^\lambda}(e)$, where the direct sum is over all central characters $\psi^\lambda: Z(U(\mathfrak{g}, e)) \rightarrow \mathbb{C}$, and $\mathcal{O}_{\psi^\lambda}(e)$ denotes the Serre subcategory of $\mathcal{O}(e)$ generated by the irreducible modules $\{L_e(\mu) \mid \mu \in (\mathfrak{h}^e)^* \text{ such that } \psi^\mu = \psi^\lambda\}$.*

The proof of Theorem 0.3 will be given in Section 4.6, for which we roughly follow the strategy in [12], but the situation is quite different from the case of

finite W -algebras. The role of “Cartan subalgebra” is taken over by a finite W -superalgebra arising from the sum of a Lie subsuperalgebra which is isomorphic to $\mathfrak{osp}(1|2)$ and an abelian subalgebra which commutes with this Lie subsuperalgebra. The precise structural information of $U(\mathfrak{g}, e)$ previously presented enables us to successfully establish such a desired highest weight theory.

Similar theory can also be established for minimal refined W -superalgebras W'_χ (see Appendix A).

§0.5.

The paper is organized as follows. In Section 1 some basics on finite and refined W -superalgebras are recalled, and the PBW theorem of finite W -superalgebra $U(\mathfrak{g}, e)$ associated with an arbitrary nilpotent element is refined. In Section 2 we first study the topics of the Verma module $Z_{U(\mathfrak{g}, e)}(\lambda, c)$ and its simple quotient $L_{U(\mathfrak{g}, e)}(\lambda, c)$ for minimal finite W -superalgebras $U(\mathfrak{g}, e)$, modulo Lemma 2.7 whose lengthy proof is postponed until Appendix B. Then the Verma module $Z_{W'_\chi}(\lambda, c)$ and its simple quotient $L_{W'_\chi}(\lambda, c)$ for minimal refined W -superalgebras W'_χ are introduced. We finally demonstrate a complete set of isomorphism classes of irreducible *highest weight* modules. Section 3 is devoted to the correspondence between the Verma modules for finite W -superalgebras and their associated Whittaker categories, where the most important tool we use there is Skryabin’s equivalence in [48, Thm. 2.17]. In Section 4 we introduce the abstract universal highest weight modules for minimal finite W -superalgebras of type odd, consider their corresponding BGG category $\mathcal{O}$, and finally give a proof of Theorem 0.3. Appendix A is a counterpart of Section 4 for minimal refined W -superalgebras of both types. Appendix B is dedicated to the proof of Lemma 2.7.

§0.6.

Throughout the paper we work with the complex field $\mathbb{C}$ as the ground field. Let $\mathbb{Z}_+$ and $\mathbb{Q}_+$ be the sets of all the non-negative integers in $\mathbb{Z}$ and all the non-negative rational numbers in $\mathbb{Q}$ respectively, and denote

$$\begin{aligned}\mathbb{Z}_+^k &:= \{(i_1, \dots, i_k) \mid i_j \in \mathbb{Z}_+\}, \\ \Lambda_k &:= \{(i_1, \dots, i_k) \mid i_j \in \{0, 1\}\}, \\ \mathbf{a} &:= (a_1, \dots, a_k), \quad |\mathbf{a}| := \sum_{i=1}^k a_i.\end{aligned}$$

For any real number $a \in \mathbb{R}$, let $\lceil a \rceil$ denote the largest integer lower bound of a , and $\lfloor a \rfloor$ the least integer upper bound of a .

A superspace is a $\mathbb{Z}_2$ -graded vector space $V = V_0 \oplus V_1$, in which we call elements in V_0 and V_1 even and odd, respectively. Write $|v| \in \mathbb{Z}_2$ for the parity (or degree) of $v \in V$, which is implicitly assumed to be $\mathbb{Z}_2$ -homogeneous.

All Lie superalgebras $\mathfrak{g}$ will be assumed to be finite-dimensional. We consider vector spaces, subalgebras, ideals, modules, and submodules, etc. in the super sense unless otherwise specified, throughout the paper. A supermodule homomorphism is assumed to be a $\mathbb{Z}_2$ -graded parity-preserving linear map that is a homomorphism in the usual sense.

§1. A refined PBW theorem for finite W -superalgebras

In this section we give a refined version of the PBW theorem for finite W -superalgebras which will be very important to the subsequent arguments. For this, we will have a glance at basic classical Lie superalgebras, and recall some basic structure of finite W -superalgebras.

§1.1. Basic classical Lie superalgebras

We refer the readers to [20, 25, 26, 33] for basic classical Lie superalgebras, and [37, 45, 46, 48] for finite W -(super)algebras.

A complete list of basic classical simple Lie superalgebras consists of simple finite-dimensional Lie algebras and the Lie superalgebras $\mathfrak{sl}(m|n)$ ($= A(m-1|n-1)$) with $m, n \geq 1$, $m \neq n$; $\mathfrak{psl}(m|m)$ ($= A(m-1|m-1)$) with $m \geq 2$; $\mathfrak{osp}(m|2n) = \mathfrak{spo}(2n|m)$ (type B, C, D); $D(2, 1; \alpha)$ with $\alpha \in \mathbb{C}$, $\alpha \neq 0, -1$; $F(4)$; $G(3)$. These Lie superalgebras are divided into two types as shown in Table 1.

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a basic classical Lie superalgebra over $\mathbb{C}$ (we will simply call it a basic Lie superalgebra for short). Let $\mathfrak{h}$ be a standard Cartan subalgebra of $\mathfrak{g}$ and Φ be a root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. We take a simple root system $\Delta = \{\alpha_1, \dots, \alpha_k\}$. By [22, §3.3], we can choose a Chevalley basis $B = \{e_\gamma \mid \gamma \in \Phi\} \cup \{h_\alpha \mid \alpha \in \Delta\}$ of $\mathfrak{g}$. (In the case of $\mathfrak{g} = D(2, 1; \alpha)$ with $\alpha \notin \mathbb{Z}$ such that $\alpha \in \overline{\mathbb{Q}}$, one needs to adjust the definition of Chevalley basis by changing $\mathbb{Z}$ to $\mathbb{Z}[\alpha]$ (where $\mathbb{Z}[\alpha]$ denotes the $\mathbb{Z}$ -algebra generated by α) in the range of construction constants; see [23, §3.1]. If $\alpha \notin \overline{\mathbb{Q}}$, we just assume that B is a basis of $\mathfrak{g}$.) Denote by Φ^+ the positive system of Φ relative to Δ , and set $\Phi^- := -\Phi^+$. Denote by Φ_0^+ and Φ_1^+ the

Type I	$A(m n)$ ($m \neq n$)	$A(n n)$	$C(n)$		
Type II	$B(m n)$	$D(m n)$	$D(2, 1; \alpha)$	$F(4)$	$G(3)$

Table 1. The classification of basic classical Lie superalgebras

set of all even roots and odd roots, respectively. We always write $|\alpha| = \bar{0}$ for any $\alpha \in \Phi_{\bar{0}}$ and $|\alpha| = \bar{1}$ for any $\alpha \in \Phi_{\bar{1}}$. Set $\Phi_{\bar{0}}^{\pm} := \Phi^{\pm} \cap \Phi_{\bar{0}}$ and $\Phi_{\bar{1}}^{\pm} := \Phi^{\pm} \cap \Phi_{\bar{1}}$, respectively.

§1.2. Finite W -superalgebras

Given a non-zero nilpotent element $e \in \mathfrak{g}_{\bar{0}}$, by the Jacobson–Morozov theorem there is an $\mathfrak{sl}_2$ -triple (e, f, h) with $f, h \in \mathfrak{g}_{\bar{0}}$. Let $(\cdot, \cdot)$ be a non-degenerate even supersymmetric invariant bilinear form on $\mathfrak{g}$. Define $\chi \in \mathfrak{g}^*$ by letting $\chi(x) = (e, x)$ for all $x \in \mathfrak{g}$. Up to a scalar, we can further assume that $(e, f) = 1$.

The linear operator $\text{ad } h$ on $\mathfrak{g}$ defines a $\mathbb{Z}$ -grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ with $e \in \mathfrak{g}(2)_{\bar{0}}$ and $f \in \mathfrak{g}(-2)_{\bar{0}}$. Set $\mathfrak{p} := \bigoplus_{i \geq 0} \mathfrak{g}(i)$ to be a parabolic subalgebra of $\mathfrak{g}$. Denote by $\mathfrak{g}^e$ (resp. $\mathfrak{g}^f$) the centralizer of e (resp. f) in $\mathfrak{g}$. Then we have $\mathfrak{g}^e = \bigoplus_{i \geq 0} \mathfrak{g}^e(i)$ (resp. $\mathfrak{g}^f = \bigoplus_{i \leq 0} \mathfrak{g}^f(i)$) by the $\mathfrak{sl}(2)$ -representation theory. Define a symplectic (resp. symmetric) bilinear form $\langle \cdot, \cdot \rangle$ on the $\mathbb{Z}_2$ -graded subspace $\mathfrak{g}(-1)_{\bar{0}}$ (resp. $\mathfrak{g}(-1)_{\bar{1}}$) given by $\langle x, y \rangle = (e, [x, y]) = \chi([x, y])$ for all $x, y \in \mathfrak{g}(-1)_{\bar{0}}$ (resp. $x, y \in \mathfrak{g}(-1)_{\bar{1}}$). Set $s := \dim \mathfrak{g}(-1)_{\bar{0}}$ (note that s is an even number), and $r := \dim \mathfrak{g}(-1)_{\bar{1}}$. Choose $\mathbb{Z}_2$ -homogeneous bases $\{u_1, \dots, u_s\}$ of $\mathfrak{g}(-1)_{\bar{0}}$ and $\{v_1, \dots, v_r\}$ of $\mathfrak{g}(-1)_{\bar{1}}$ contained in $\mathfrak{g}$ such that $\langle u_i, u_j \rangle = i^* \delta_{i+j, s+1}$ for $1 \leq i, j \leq s$, where

$$i^* := \begin{cases} -1 & \text{if } 1 \leq i \leq \frac{s}{2}, \\ 1 & \text{if } \frac{s}{2} + 1 \leq i \leq s, \end{cases}$$

and $\langle v_i, v_j \rangle = \delta_{i+j, r+1}$ for $1 \leq i, j \leq r$. We further assume that the u_i with $1 \leq i \leq \frac{s}{2}$ and the v_i with $1 \leq i \leq \lceil \frac{r}{2} \rceil$ are root vectors corresponding to negative roots $\gamma_{\bar{0}i} \in \Phi_{\bar{0}}^-$ and $\gamma_{\bar{1}i} \in \Phi_{\bar{1}}^-$, respectively. When r is odd, we assume that the element $v_{\frac{r+1}{2}}$ is also a negative root vector in $\Phi_{\bar{1}}^-$.

Let $z_{\alpha} := u_{\alpha}$ for $1 \leq \alpha \leq s$, and $z_{\alpha+s} := v_{\alpha}$ for $1 \leq \alpha \leq r$. Set $S(-1) := \{1, 2, \dots, s+r\}$, and then $\{z_{\alpha} \mid \alpha \in S(-1)\}$ is a basis of $\mathfrak{g}(-1)$. Set $z_{\alpha}^* := \alpha^{\natural} z_{s+1-\alpha}$ for $1 \leq \alpha \leq s$, where

$$\alpha^{\natural} := \begin{cases} 1 & \text{if } 1 \leq \alpha \leq \frac{s}{2}, \\ -1 & \text{if } \frac{s}{2} + 1 \leq \alpha \leq s, \end{cases}$$

and $z_{\alpha+s}^* := z_{r+1-\alpha+s}$ for $1 \leq \alpha \leq r$; i.e.,

$$u_i^* = \begin{cases} u_{s+1-i} & \text{if } 1 \leq i \leq \frac{s}{2}, \\ -u_{s+1-i} & \text{if } \frac{s}{2} + 1 \leq i \leq s, \end{cases}$$

and $v_i^* = v_{r+1-i}$ for $1 \leq i \leq r$. Then $\{z_{\alpha}^* \mid \alpha \in S(-1)\}$ is a dual basis of $\{z_{\alpha} \mid \alpha \in S(-1)\}$ such that $\langle z_{\alpha}^*, z_{\beta} \rangle = \delta_{\alpha, \beta}$ for $\alpha, \beta \in S(-1)$.

From now on, for any $\alpha \in S(-1)$ we will denote the parity of z_{α} by $|\alpha|$ for simplicity. It is straightforward that z_{α} and z_{α}^* have the same parity, and each u_i^*

(resp. v_i^*) with $1 \leq i \leq \frac{s}{2}$ (resp. $1 \leq i \leq \lceil \frac{r}{2} \rceil$) is a root vector in $\mathfrak{g}(-1)_{\bar{0}}$ (resp. $\mathfrak{g}(-1)_{\bar{1}}$) corresponding to $\gamma_{\bar{0}i}^* := -\theta - \gamma_{\bar{0}i} \in \Phi_0^+$ (resp. $\gamma_{\bar{1}i}^* := -\theta - \gamma_{\bar{1}i} \in \Phi_1^+$). Moreover, $\{u_1, \dots, u_{\frac{s}{2}}, u_1^*, \dots, u_{\frac{s}{2}}^*\}$ constitutes a $\mathbb{C}$ -basis of $\mathfrak{g}(-1)_{\bar{0}}$. On the other hand, $\{v_1, \dots, v_{\frac{r}{2}}, v_1^*, \dots, v_{\frac{r}{2}}^*\}$ (resp. $\{v_1, \dots, v_{\frac{r-1}{2}}, v_{\frac{r+1}{2}}, v_1^*, \dots, v_{\frac{r-1}{2}}^*\}$) is a $\mathbb{C}$ -basis of $\mathfrak{g}(-1)_{\bar{1}}$ for r being even (resp. odd). Write $\mathfrak{g}(-1)'_{\bar{0}}$ for the $\mathbb{C}$ -span of $u_{\frac{s}{2}+1}, \dots, u_s$ and $\mathfrak{g}(-1)'_{\bar{1}}$ the $\mathbb{C}$ -span of $v_{\frac{r}{2}+1}, \dots, v_r$ (resp. $v_{\frac{r+3}{2}}, \dots, v_r$) when r is even (resp. odd). Denote $\mathfrak{g}(-1)' := \mathfrak{g}(-1)'_{\bar{0}} \oplus \mathfrak{g}(-1)'_{\bar{1}}$.

Now we can introduce the so-called χ -admissible subalgebra of $\mathfrak{g}$ as

$$(1.1) \quad \mathfrak{m} := \bigoplus_{i \leq -2} \mathfrak{g}(i) \oplus \mathfrak{g}(-1)'.$$

Then χ vanishes on the derived subalgebra of $\mathfrak{m}$. We also have an *extended* χ -admissible subalgebra of $\mathfrak{g}$ as below:

$$(1.2) \quad \mathfrak{m}' := \begin{cases} \mathfrak{m} & \text{if } r \text{ is even,} \\ \mathfrak{m} \oplus \mathbb{C}v_{\frac{r+1}{2}} & \text{if } r \text{ is odd.} \end{cases}$$

Define the generalized Gelfand–Graev $\mathfrak{g}$ -module associated with χ as

$$(1.3) \quad Q_\chi := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi,$$

where $\mathbb{C}_\chi = \mathbb{C}1_\chi$ is a one-dimensional $\mathfrak{m}$ -module such that $x \cdot 1_\chi = \chi(x)1_\chi$ for all $x \in \mathfrak{m}$. The super structure of Q_χ is dependent on the parity of $\mathbb{C}_\chi$, which is indicated to be even hereafter. The finite W -superalgebra associated with the pair $(\mathfrak{g}, e)$ is defined as

$$U(\mathfrak{g}, e) = (\text{End}_{\mathfrak{g}} Q_\chi)^{\text{op}},$$

where $(\text{End}_{\mathfrak{g}} Q_\chi)^{\text{op}}$ denotes the opposite algebra of the endomorphism algebra of the $\mathfrak{g}$ -module Q_χ .

Let I_χ denote the $\mathbb{Z}_2$ -graded left ideal in $U(\mathfrak{g})$ generated by all $x - \chi(x)$ with $x \in \mathfrak{m}$. The fixed point space $(U(\mathfrak{g})/I_\chi)^{\text{ad } \mathfrak{m}}$ carries a natural algebra structure given by $(x + I_\chi) \cdot (y + I_\chi) := (xy + I_\chi)$ for $x, y \in U(\mathfrak{g})$ such that $[a, x], [a, y] \in I_\chi$, for all $a \in \mathfrak{m}$. Then $U(\mathfrak{g})/I_\chi \cong Q_\chi$ as $\mathfrak{g}$ -modules via the $\mathfrak{g}$ -module map sending $1 + I_\chi$ to 1_χ , and $U(\mathfrak{g}, e) \cong Q_\chi^{\text{ad } \mathfrak{m}}$ as $\mathbb{C}$ -algebras. Explicitly speaking, any element of $U(\mathfrak{g}, e)$ is uniquely determined by its effect on the generator $1_\chi \in Q_\chi$, and the canonical isomorphism between $U(\mathfrak{g}, e)$ and $Q_\chi^{\text{ad } \mathfrak{m}}$ is given by $u \mapsto u(1_\chi)$ for any $u \in U(\mathfrak{g}, e)$. In what follows we will often identify Q_χ with $U(\mathfrak{g})/I_\chi$ and $U(\mathfrak{g}, e)$ with $Q_\chi^{\text{ad } \mathfrak{m}}$.

Let $w_1, \dots, w_c$ be a basis of $\mathfrak{g}$ over $\mathbb{C}$. For any given $w_{i_1} \in \mathfrak{g}(j_1), \dots, w_{i_k} \in \mathfrak{g}(j_k)$, set $\text{wt}(w_{i_1} \cdots w_{i_k}) := j_1 + \cdots + j_k$ to be the weight of $w_{i_1} \cdots w_{i_k}$, and let $U(\mathfrak{g}) = \bigcup_{i \in \mathbb{Z}} F_i U(\mathfrak{g})$ be a filtration of $U(\mathfrak{g})$, where $F_i U(\mathfrak{g})$ is the $\mathbb{C}$ -span of all $w_{i_1} \cdots w_{i_k}$ with $(j_1 + 2) + \cdots + (j_k + 2) \leq i$. This filtration is called *Kazhdan*

filtration. The Kazhdan filtration on Q_χ is defined by $F_i Q_\chi := \text{Pr}(F_i U(\mathfrak{g}))$ with $\text{Pr}: U(\mathfrak{g}) \rightarrow U(\mathfrak{g})/I_\chi$ being the canonical homomorphism, which makes Q_χ into a filtered $U(\mathfrak{g})$ -module. Then there is an induced Kazhdan filtration $F_i U(\mathfrak{g}, e)$ on the subspace $U(\mathfrak{g}, e) = Q_\chi^{\text{ad } \mathfrak{m}}$ of Q_χ such that $F_j U(\mathfrak{g}, e) = 0$ unless $j \geq 0$. For any element $\Theta \in U(\mathfrak{g}, e)$, we will denote by $\deg_e(\Theta)$ the degree of Θ under the Kazhdan grading.

Denote by gr the corresponding graded algebras under the Kazhdan filtration as above. As $U(\mathfrak{g}, e) \subset U(\mathfrak{g})/I_\chi$ by definition, it is not hard to see that $\text{gr}(U(\mathfrak{g}))$ is supercommutative, and then $\text{gr}(U(\mathfrak{g}, e))$ is also supercommutative.

§1.3. Refined W -superalgebras

Recall that in [48, Def. 4.8] we introduced the so-called refined W -superalgebras W'_χ via

$$W'_\chi := (U(\mathfrak{g})/I_\chi)^{\text{ad } \mathfrak{m}'} \cong Q_\chi^{\text{ad } \mathfrak{m}'} \equiv \{\text{Pr}(y) \in U(\mathfrak{g})/I_\chi \mid [a, y] \in I_\chi, \forall a \in \mathfrak{m}'\},$$

and $\text{Pr}(y_1) \cdot \text{Pr}(y_2) := \text{Pr}(y_1 y_2)$ for all $\text{Pr}(y_1), \text{Pr}(y_2) \in W'_\chi$. By definition, W'_χ coincides with $U(\mathfrak{g}, e)$ if r is even, while W'_χ is a proper subalgebra of $U(\mathfrak{g}, e)$ if r is odd. Moreover, the PBW theorem of W'_χ was introduced in [49, Thm. 3.7], and the Kazhdan filtration on Q_χ induces a Kazhdan filtration $F_i W'_\chi$ on the subalgebra W'_χ of $U(\mathfrak{g}, e)$. The adoption of this notion enables us to conveniently prove the existence of Kac–Weisfeiler modules when $e = e_\theta$ with $-\theta$ being a minimal root in [49]. Apart from this achievement, the consideration of refined W -superalgebras can give rise to some other advantage in the arguments. In the sequel, we can see more about this point, owing to the isomorphism from W'_χ onto a quantum finite W -superalgebra introduced in [44], the latter of which will be used in our arguments immediately.

Set $\mathfrak{n} := \bigoplus_{i \leq -2} \mathfrak{g}(i)$ and $\mathfrak{n}' := \bigoplus_{i \leq -1} \mathfrak{g}(i)$ to be nilpotent subalgebras of $\mathfrak{g}$. Let I^{fin} be the left ideal of $U(\mathfrak{g})$ generated by the elements $\{x - \chi(x) \mid x \in \mathfrak{n}\}$, and let $Q_\chi^{\text{fin}} := U(\mathfrak{g})/I^{\text{fin}}$ be an induced $\mathfrak{g}$ -module. In what follows we denote by $\text{Pr}': U(\mathfrak{g}) \rightarrow U(\mathfrak{g})/I^{\text{fin}}$ the canonical projection. Suh [44] introduced the quantum finite W -superalgebra associated with the pair $(\mathfrak{g}, e)$ as

$$W^{\text{fin}}(\mathfrak{g}, e) := (Q_\chi^{\text{fin}})^{\text{ad } \mathfrak{n}'},$$

where $(Q_\chi^{\text{fin}})^{\text{ad } \mathfrak{n}'}$ denotes the invariant subspace of Q_χ^{fin} under the adjoint action of $\mathfrak{n}'$, and the associated product of $(Q_\chi^{\text{fin}})^{\text{ad } \mathfrak{n}'}$ is defined by

$$(x + I^{\text{fin}}) \cdot (y + I^{\text{fin}}) := xy + I^{\text{fin}}$$

for $x + I^{\text{fin}}, y + I^{\text{fin}} \in (Q_\chi^{\text{fin}})^{\text{ad } \mathfrak{n}'}$. The Kazhdan grading on the $U(\mathfrak{g})$ -module Q_χ^{fin} and the algebra $W^{\text{fin}}(\mathfrak{g}, e)$ can also be defined similarly to before.

Recall that in [49, Thm. 4.11] we showed that $W'_\chi \cong W^{\text{fin}}(\mathfrak{g}, e)$ as Kazhdan filtered algebras excluding the case of $\mathfrak{g} = D(2, 1; \alpha)$ with $\alpha \notin \overline{\mathbb{Q}}$. In fact, the above isomorphism is also valid for this special case. When the detecting parity of r is odd, by definition there must exist a root $\alpha \in \Phi_1^+$ of $\mathfrak{g}$ such that $2\alpha \in \Phi_0^+$. From the detailed description of the system of root Φ of $\mathfrak{g}$ in [25, §2.5.4], this happens only when $\mathfrak{g}$ is of type $B(m|n)$ or $G(3)$. Therefore, this special case corresponds to the situation when r is even. Let $\text{gr}(W'_\chi)$ and $\text{gr}(W^{\text{fin}}(\mathfrak{g}, e))$ denote the graded algebras of W'_χ and $W^{\text{fin}}(\mathfrak{g}, e)$ under the Kazhdan grading, respectively. On one hand, it follows from [42, Cor. 3.9(1)] that $\text{gr}(W'_\chi) \cong S(\mathfrak{g}^e)$ as $\mathbb{C}$ -algebras. On the other hand, we also have $\text{gr}(W^{\text{fin}}(\mathfrak{g}, e)) \cong S(\mathfrak{g}^e)$ by [49, Prop. 4.9]. Therefore, we obtain $\text{gr}(W'_\chi) \cong \text{gr}(W^{\text{fin}}(\mathfrak{g}, e))$, and then $W'_\chi \cong W^{\text{fin}}(\mathfrak{g}, e)$ as $\mathbb{C}$ -algebras.

From now on, we will consider the quantum finite W -superalgebra $W^{\text{fin}}(\mathfrak{g}, e)$ as the refined W -superalgebra W'_χ , which will cause no confusion.

§1.4. A variation of the definition of $U(\mathfrak{g}, e)$

In the following arguments, we need to vary the definition of finite W -superalgebras $U(\mathfrak{g}, e)$ for the convenience of arguments.

1.4.1. As we addressed in the previous subsection, the refined W -superalgebra W'_χ coincides with the finite W -superalgebra $U(\mathfrak{g}, e)$ if r is even. Therefore, by the discussion in Section 1.3 we can take the quantum finite W -superalgebra $W^{\text{fin}}(\mathfrak{g}, e)$ as $U(\mathfrak{g}, e)$ when r is even. However, the situation for r being odd is much more difficult. From now till the end of this section, we will always assume that r is odd, and the final statement will be given in Proposition 1.10.

1.4.2. Let $\mathfrak{l}$ be the $\mathbb{C}$ -span of $u_1, \dots, u_s$ and $v_1, \dots, v_{\frac{r-1}{2}}, v_{\frac{r+3}{2}}, \dots, v_r$. It is immediate that $\mathfrak{g}(-1) = \mathfrak{l} \oplus \mathbb{C}v_{\frac{r+1}{2}}$ as vector space. Set

$$(1.4) \quad \mathfrak{n}^0 := \bigoplus_{i \leq -2} \mathfrak{g}(i) \oplus \mathfrak{l},$$

which is also a nilpotent subalgebra of $\mathfrak{g}$.

Definition 1.1. Define the algebra $U^{\text{fin}}(\mathfrak{g}, e)$ associated with the pair $(\mathfrak{g}, e)$ by

$$(1.5) \quad U^{\text{fin}}(\mathfrak{g}, e) := (Q_\chi^{\text{fin}})^{\text{ad } \mathfrak{n}^0} \equiv \{ \text{Pr}'(y) \in U(\mathfrak{g})/I^{\text{fin}} \mid [a, y] \in I^{\text{fin}}, \forall a \in \mathfrak{n}^0 \},$$

where $\text{Pr}': U(\mathfrak{g}) \rightarrow U(\mathfrak{g})/I^{\text{fin}}$ is the canonical projection as defined in Section 1.3, and $(Q_\chi^{\text{fin}})^{\text{ad } \mathfrak{n}^0}$ denotes the invariant subspace of Q_χ^{fin} under the adjoint action of $\mathfrak{n}^0$. The associated product of $(Q_\chi^{\text{fin}})^{\text{ad } \mathfrak{n}^0}$ is defined by

$$(x + I^{\text{fin}}) \cdot (y + I^{\text{fin}}) := xy + I^{\text{fin}}$$

for $x + I^{\text{fin}}, y + I^{\text{fin}} \in (Q_\chi^{\text{fin}})^{\text{ad } \mathfrak{n}^0}$.

Obviously, $W^{\text{fin}}(\mathfrak{g}, e)$ is a subalgebra of $U^{\text{fin}}(\mathfrak{g}, e)$. There is also an induced Kazhdan filtration $F_i U^{\text{fin}}(\mathfrak{g}, e)$ on the subspace $(Q_\chi^{\text{fin}})^{\text{ad } \mathfrak{n}^0}$ of Q_χ^{fin} (see Section 1.3). To obtain the PBW theorem of $U^{\text{fin}}(\mathfrak{g}, e)$, we need some preparation. First note the following lemma.

Lemma 1.2. *Let $x \in \bigoplus_{i \geq -1} \mathfrak{g}(i)$. Then $\chi([\mathfrak{n}^0, x]) = 0$ if and only if $x \in \mathfrak{g}^e \oplus \mathbb{C}v_{\frac{r+1}{2}}$.*

Proof. Note that if $x \in \mathfrak{g}(i)$ and $Y \in \mathfrak{g}(j)$, then $\chi([Y, x]) \neq 0$ implies $i + j = -2$. Therefore, if $x \in \mathfrak{p}$, the condition $\chi([\mathfrak{n}^0, x]) = 0$ implies $\chi([\mathfrak{g}, x]) = 0$, and thus $x \in \mathfrak{g}^e$. If $x \in \mathfrak{g}(-1)$, it follows from $\chi([\mathfrak{n}^0, x]) = 0$ that $x \in \mathbb{C}v_{\frac{r+1}{2}}$. $\square$

By the PBW theorem, the graded algebra $\text{gr}(U(\mathfrak{g})/I^{\text{fin}})$ under the Kazhdan grading is isomorphic to $S(\mathfrak{p} \oplus \mathfrak{g}(-1))$ as vector space. The $\mathbb{Z}$ -grading of $\mathfrak{g}$ as defined at the beginning of Section 1.2 induces a grading on $S(\mathfrak{p} \oplus \mathfrak{g}(-1))$. For any $X \in S(\mathfrak{p} \oplus \mathfrak{g}(-1))$ we denote by $\bar{X}$ the element of highest degree under the $\mathbb{Z}$ -grading. Following Poletaeva–Serganova’s arguments in [36, §2.2], we can prove the following statement.

Proposition 1.3. *If $X \in \text{gr}(U^{\text{fin}}(\mathfrak{g}, e))$, then $\bar{X} \in S(\mathfrak{g}^e \oplus \mathbb{C}v_{\frac{r+1}{2}})$.*

Proof. Let $X \in \text{gr}(U^{\text{fin}}(\mathfrak{g}, e))$. Passing to the graded version of (1.5), for any $Y \in \mathfrak{n}^0$ we have

$$(1.6) \quad \text{Pr}'([Y, X]) = 0.$$

Define $\gamma: \mathfrak{n}^0 \otimes S(\mathfrak{p} \oplus \mathfrak{g}(-1)) \rightarrow S(\mathfrak{p} \oplus \mathfrak{g}(-1))$ by putting $\gamma(Y, Z) = \text{Pr}'([Y, Z])$ for all $Y \in \mathfrak{n}^0$, $Z \in S(\mathfrak{p} \oplus \mathfrak{g}(-1))$. It is easy to see that if $Y \in \mathfrak{g}(-i)$ with $i > 0$, and $Z \in S(\mathfrak{p} \oplus \mathfrak{g}(-1))(j)$, then $\gamma(Y, Z) \in S(\mathfrak{p} \oplus \mathfrak{g}(-1))(j-i) \oplus S(\mathfrak{p} \oplus \mathfrak{g}(-1))(j-i+2)$. Hence we can write $\gamma = \gamma_0 + \gamma_2$, where $\gamma_0(Y, Z)$ is the projection on $S(\mathfrak{p} \oplus \mathfrak{g}(-1))(j-i)$ and $\gamma_2(Y, Z)$ is the projection on $S(\mathfrak{p} \oplus \mathfrak{g}(-1))(j-i+2)$. The condition (1.6) implies that for any $X \in \text{gr}(U^{\text{fin}}(\mathfrak{g}, e))$,

$$(1.7) \quad \gamma_2(\mathfrak{n}^0, \bar{X}) = 0.$$

On the other hand, $\mathfrak{n}^0 \times S(\mathfrak{p} \oplus \mathfrak{g}(-1)) \rightarrow S(\mathfrak{p} \oplus \mathfrak{g}(-1))$ is a derivation with respect to the second argument defined by the condition $\gamma_2(Y, Z) = \chi([Y, Z])$ for any $Y \in \mathfrak{n}^0$, $Z \in \mathfrak{p} \oplus \mathfrak{g}(-1)$. Now by induction on the polynomial degree of $\bar{X}$ in $S(\mathfrak{p} \oplus \mathfrak{g}(-1))$, using Lemma 1.2, one can show that (1.7) implies $\bar{X} \in \mathfrak{g}^e \oplus \mathbb{C}v_{\frac{r+1}{2}}$. $\square$

1.4.3. Choose homogeneous elements $x_1, \dots, x_l, x_{l+1}, \dots, x_m \in \mathfrak{p}_{\bar{0}}, y_1, \dots, y_q, y_{q+1}, \dots, y_n \in \mathfrak{p}_{\bar{1}}$ as a basis of $\mathfrak{p}$ such that

- (1) $x_i \in \mathfrak{g}(k_i)_{\bar{0}}, y_j \in \mathfrak{g}(k'_j)_{\bar{1}}$, where $k_i, k'_j \in \mathbb{Z}_+$ with $1 \leq i \leq m$ and $1 \leq j \leq n$;

- (2) $x_1, \dots, x_l$ is a basis of $\mathfrak{g}_0^e$ and $y_1, \dots, y_q$ is a basis of $\mathfrak{g}_{\bar{1}}^e$;
 (3) $x_{l+1}, \dots, x_m \in [f, \mathfrak{g}_0]$ and $y_{q+1}, \dots, y_n \in [f, \mathfrak{g}_{\bar{1}}]$.

Also recall the bases $\{u_1, \dots, u_s\}$ of $\mathfrak{g}(-1)_0$ and $\{v_1, \dots, v_r\}$ of $\mathfrak{g}(-1)_{\bar{1}}$ as defined in Section 1.2.

Keep the notation in Section 0.6. Given $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \in \mathbb{Z}_+^m \times \Lambda_n \times \mathbb{Z}_+^s \times \Lambda_r$, let $x^{\mathbf{a}}y^{\mathbf{b}}u^{\mathbf{c}}v^{\mathbf{d}}$ denote the monomial $x_1^{a_1} \cdots x_m^{a_m} y_1^{b_1} \cdots y_n^{b_n} u_1^{c_1} \cdots u_s^{c_s} v_1^{d_1} \cdots v_r^{d_r}$ in $U(\mathfrak{g})$. It is obvious that the $\mathfrak{g}$ -module Q_χ^{fin} has a free basis $\{x^{\mathbf{a}}y^{\mathbf{b}}u^{\mathbf{c}}v^{\mathbf{d}} \otimes 1_\chi \mid (\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \in \mathbb{Z}_+^m \times \Lambda_n \times \mathbb{Z}_+^s \times \Lambda_r\}$. Write

$$|(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})|_e := \sum_{i=1}^m a_i(k_i + 2) + \sum_{i=1}^n b_i(k'_i + 2) + \sum_{i=1}^s c_i + \sum_{i=1}^r d_i,$$

which is exactly the Kazhdan degree of $x^{\mathbf{a}}y^{\mathbf{b}}u^{\mathbf{c}}v^{\mathbf{d}}$.

Set

$$(1.8) \quad Y_i := \begin{cases} x_i & \text{if } 1 \leq i \leq l, \\ y_{i-l} & \text{if } l+1 \leq i \leq l+q, \\ v_{\frac{l+1}{2}} & \text{if } i = l+q+1. \end{cases}$$

To simplify notation, we always assume that Y_i belongs to $\mathfrak{g}(m_i)$ for $1 \leq i \leq l+q$, and $Y_{l+q+1} \in \mathfrak{g}(-1)_{\bar{1}}$ by our earlier settings.

1.4.4. We are in a position to introduce the PBW theorem of $U^{\text{fin}}(\mathfrak{g}, e)$.

Theorem 1.4. *The following statements concerning the PBW structure of $U^{\text{fin}}(\mathfrak{g}, e)$ hold:*

- (1) *There exist homogeneous elements $\Theta_1, \dots, \Theta_l \in U^{\text{fin}}(\mathfrak{g}, e)_0$ and $\Theta_{l+1}, \dots, \Theta_{l+q+1} \in U^{\text{fin}}(\mathfrak{g}, e)_{\bar{1}}$ such that*

$$(1.9) \quad \begin{aligned} \Theta_k = & \left(Y_k + \sum_{\substack{|\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}|_e = m_k + 2, \\ |\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}| + |\mathbf{d}| \geq 2}} \lambda_{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}}^k x^{\mathbf{a}}y^{\mathbf{b}}u^{\mathbf{c}}v^{\mathbf{d}} \right. \\ & \left. + \sum_{|\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}|_e < m_k + 2} \lambda_{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}}^k x^{\mathbf{a}}y^{\mathbf{b}}u^{\mathbf{c}}v^{\mathbf{d}} \right) \otimes 1_\chi \end{aligned}$$

for $1 \leq k \leq l+q$ with $\lambda_{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}}^k \in \mathbb{Q}$, where $\lambda_{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}}^k = 0$ if $\mathbf{c} = \mathbf{d} = \mathbf{0}$ and $a_{l+1} = \dots = a_m = b_{q+1} = \dots = b_n = 0$, and $\Theta_{l+q+1} = v_{\frac{l+1}{2}} \otimes 1_\chi$.

- (2) *The monomials $\Theta_1^{a_1} \cdots \Theta_l^{a_l} \Theta_{l+1}^{b_1} \cdots \Theta_{l+q+1}^{b_{q+1}}$ with $a_i \in \mathbb{Z}_+$, $b_j \in \Lambda_1$ for $1 \leq i \leq l$ and $1 \leq j \leq q+1$ form a basis of $U^{\text{fin}}(\mathfrak{g}, e)$ over $\mathbb{C}$.*

- (3) For i, j satisfying $1 \leq i < j \leq l + q + 1$ and $l + 1 \leq i = j \leq l + q + 1$, there exist polynomial superalgebras $F_{i,j} \in \mathbb{Q}[X_1, \dots, X_l; X_{l+1}, \dots, X_{l+q+1}]$ with $X_1, \dots, X_l$ being even and $X_{l+1}, \dots, X_{l+q+1}$ being odd, such that

$$(1.10) \quad [\Theta_i, \Theta_j] = F_{i,j}(\Theta_1, \dots, \Theta_{l+q+1}),$$

where

$$(1.11) \quad F_{i,l+q+1}(\Theta_1, \dots, \Theta_{l+q+1}) = 0, \quad F_{l+q+1,l+q+1}(\Theta_1, \dots, \Theta_{l+q+1}) = 1$$

for $1 \leq i \leq l + q$. Moreover, if the elements $Y_i, Y_j \in \mathfrak{g}^e$ with $1 \leq i, j \leq l + q$ satisfy $[Y_i, Y_j] = \sum_{k=1}^{l+q} \alpha_{ij}^k Y_k$ in $\mathfrak{g}^e$, then

$$(1.12) \quad \begin{aligned} & F_{i,j}(\Theta_1, \dots, \Theta_{l+q+1}) \\ & \equiv \sum_{k=1}^{l+q} \alpha_{ij}^k \Theta_k + q_{ij}(\Theta_1, \dots, \Theta_{l+q+1}) \pmod{F_{m_i+m_j+1} U^{\text{fin}}(\mathfrak{g}, e)}, \end{aligned}$$

where q_{ij} is a polynomial superalgebra in $l+q+1$ variables in $\mathbb{Q}$ whose constant term and linear part are zero.

- (4) The algebra $U^{\text{fin}}(\mathfrak{g}, e)$ is generated by the $\mathbb{Z}_2$ -homogeneous elements $\Theta_1, \dots, \Theta_l \in U^{\text{fin}}(\mathfrak{g}, e)_{\bar{0}}$ and $\Theta_{l+1}, \dots, \Theta_{l+q+1} \in U^{\text{fin}}(\mathfrak{g}, e)_{\bar{1}}$ subject to the relations in (1.10) with $1 \leq i < j \leq l + q + 1$ and $l + 1 \leq i = j \leq l + q + 1$.

Proof. Since $\mathfrak{n}^0 \subset \mathfrak{n}'$ by definition, then $W^{\text{fin}}(\mathfrak{g}, e) = (Q_{\chi}^{\text{fin}})^{\text{ad } \mathfrak{n}'}$ is a subalgebra of $U^{\text{fin}}(\mathfrak{g}, e) = (Q_{\chi}^{\text{fin}})^{\text{ad } \mathfrak{n}^0}$. In virtue of [49, Thms 3.7, 4.11], we can choose the elements Θ_k for $1 \leq k \leq l + q$ as in (1.9), to be the generators of $W^{\text{fin}}(\mathfrak{g}, e)$ (all the coefficients $\lambda_{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}}^k \in \mathbb{Q}$ can be ensured by the knowledge of field theory as in the proof of [37, Thm. 4.6]). Moreover, it follows from the definition of $\mathfrak{n}^0$ in (1.4) that

$$[\mathfrak{n}^0, \Theta_{l+q+1}] = [\mathfrak{n}^0, v_{\frac{l+1}{2}} \otimes 1_{\chi}] = 0.$$

So all the elements in (1.9), together with $\Theta_{l+q+1} = v_{\frac{l+1}{2}} \otimes 1_{\chi}$, belong to $U^{\text{fin}}(\mathfrak{g}, e)$. Thanks to Proposition 1.3, we see that all these elements constitute a set of generators of $U^{\text{fin}}(\mathfrak{g}, e)$.

On one hand, since $\Theta_k \in (Q_{\chi}^{\text{fin}})^{\text{ad } \mathfrak{n}'}$ for $1 \leq k \leq l + q$, and $v_{\frac{l+1}{2}} \in \mathfrak{n}'$ by definition, we have

$$(1.13) \quad [\Theta_k, \Theta_{l+q+1}] = [\Theta_k, v_{\frac{l+1}{2}} \otimes 1_{\chi}] = 0,$$

which implies $F_{i,l+q+1}(\Theta_1, \dots, \Theta_{l+q+1}) = 0$ in (1.11). On the other hand, by assumption we have

$$[\Theta_{l+q+1}, \Theta_{l+q+1}] = [v_{\frac{l+1}{2}} \otimes 1_{\chi}, v_{\frac{l+1}{2}} \otimes 1_{\chi}] = [v_{\frac{l+1}{2}}, v_{\frac{l+1}{2}}] \otimes 1_{\chi} = 1 \otimes 1_{\chi}.$$

Thus $F_{l+q+1, l+q+1}(\Theta_1, \dots, \Theta_{l+q+1}) = 1$, which is just the second equation in (1.11). The remainder of the theorem can be obtained by the same discussion as in [48, Thms 4.5, 4.7]. $\square$

As an immediate consequence of Theorem 1.4, we have the following corollary.

Corollary 1.5. *Under the Kazhdan grading, $\text{gr}(U^{\text{fin}}(\mathfrak{g}, e)) \cong S(\mathfrak{g}^e) \otimes \mathbb{C}[\Lambda]$ as $\mathbb{C}$ -algebras, where Λ is the exterior algebra generated by one element Λ .*

1.4.5. Now we turn to finite W -superalgebra $U(\mathfrak{g}, e) \cong Q_\chi^{\text{ad m}}$. In [48, Thm. 0.1], we showed the following proposition.

Proposition 1.6 ([48]). *When r is odd, $\text{gr}(U(\mathfrak{g}, e)) \cong S(\mathfrak{g}^e) \otimes \mathbb{C}[\Lambda]$ as vector space under the Kazhdan grading, where Λ is the exterior algebra generated by Λ .*

We will improve Proposition 1.6, by adopting a new approach which is different from the one used by Shu–Xiao under the settings of Poisson geometric realization of finite W -superalgebras in [42, Cor. 3.9(2)]. We also refer to [36, §2.2] for more details.

Recall that we introduced the PBW theorem of refined W -superalgebra W'_χ in [49, Thm. 3.7]. Since W'_χ is a subalgebra of $U(\mathfrak{g}, e)$, by [48, Lem. 4.3] we can choose the elements given in [49, Thm. 3.7(1)] and also $\Theta_{l+q+1} = v_{\frac{r+1}{2}} \otimes 1_\chi$ as the generators of $U(\mathfrak{g}, e)$. In particular, such a choice ensures the equation (1.11) is still valid in this case. Now, with the same discussion as in the proofs of Theorem 1.4 and Corollary 1.5, one can conclude the following theorem.

Theorem 1.7 ([42]). *The isomorphism of vector spaces in Proposition 1.6 is in fact an isomorphism of $\mathbb{C}$ -algebras.*

Remark 1.8. In the proof of Theorem 1.7 as above, we can observe that the construction of the generators of W'_χ introduced in [49, Thm. 3.7] plays a key role. In fact, in the procedure of formulating the PBW theorem for W'_χ there, since the “admissible” procedure from the modular finite W -superalgebras is employed, we have always assumed that the associated $\mathfrak{g}$ is a basic Lie superalgebra excluding the case of $D(2, 1; \alpha)$ with $\alpha \notin \overline{\mathbb{Q}}$. However, this does not affect the proof here. As mentioned at the end of Section 1.3, the case of r being odd appears only when $\mathfrak{g}$ is of type $B(m|n)$ or $G(3)$. Then $\mathfrak{g}$ cannot be of type $D(2, 1; \alpha)$.

Combining Theorem 1.7 with Corollary 1.5, we now obtain the following proposition.

Proposition 1.9. *When r is odd, $\text{gr}(U^{\text{fin}}(\mathfrak{g}, e)) \cong \text{gr}(U(\mathfrak{g}, e))$ as $\mathbb{C}$ -algebras under the Kazhdan grading.*

Translating Proposition 1.9 into the corresponding Kazhdan-filtrated algebras, we have the following proposition.

Proposition 1.10. *When r is odd, there is an isomorphism $U^{\text{fin}}(\mathfrak{g}, e) \cong U(\mathfrak{g}, e)$ as $\mathbb{C}$ -algebras.*

Owing to Proposition 1.10, from now on we will regard $U^{\text{fin}}(\mathfrak{g}, e)$ in Definition 1.1 as a finite W -superalgebra $U(\mathfrak{g}, e)$, i.e., $U(\mathfrak{g}, e) \triangleq U^{\text{fin}}(\mathfrak{g}, e)$.

Remark 1.11. By all the discussion above, Theorem 1.4 can be considered as the PBW theorem of finite W -superalgebra $U(\mathfrak{g}, e)$. Compared with [48, Thm. 4.5], the most important difference lies in (1.11), where $F_{i, l+q+1}(\Theta_1, \dots, \Theta_{l+q+1})$ for $1 \leq i \leq l+q$ cannot be easily determined in [48, equality (4.2)], while $F_{i, l+q+1}(\Theta_1, \dots, \Theta_{l+q+1}) = 0$ in our case, and such a choice makes the construction of $U(\mathfrak{g}, e)$ much easier to determine.

For further discussion, we need the following ring-theoretic property of finite and refined W -superalgebras, which is parallel to the non-super case in [43, Lem. 1.1(2)].

Proposition 1.12. *Both $U(\mathfrak{g}, e)$ and W'_χ are Noetherian rings.*

Proof. Under the Kazhdan grading, we showed in Theorem 1.7 that $\text{gr}(U(\mathfrak{g}, e)) \cong S(\mathfrak{g}^e) \otimes \mathbb{C}[\Lambda]$ as $\mathbb{C}$ -algebra, and $\text{gr}(W'_\chi) \cong S(\mathfrak{g}^e)$ by [49, Cor. 3.8]. So the gradation of $U(\mathfrak{g}, e)$ and W'_χ are isomorphic to polynomial superalgebras. Then it follows from [31, Thm. 1.6.9] that both the filtration algebras $U(\mathfrak{g}, e)$ and W'_χ are Noetherian, as the Noetherian property holds for their associated graded algebras. $\square$

§2. Verma modules and isomorphism classes of their irreducible quotients for minimal finite and refined W -superalgebras

In this section we will study Verma modules for minimal finite W -superalgebras $U(\mathfrak{g}, e)$.

§2.1.

We first recall some basics on minimal finite and refined W -(super)algebras in this subsection. We refer the readers to [1, 2, 3, 4, 27, 38, 44, 49].

2.1.1. A root $-\theta$ is called minimal if it is even and there exists an additive function $\varphi: \Phi \rightarrow \mathbb{R}$ such that $\varphi|_\Phi \neq 0$ and $\varphi(\theta) > \varphi(\eta)$ for all $\eta \in \Phi \setminus \{\theta\}$. In the ordering defined by φ , a minimal root $-\theta$ is the lowest root of $\mathfrak{g}_0$. Conversely, it is easy to see, using the description of Φ given in [25], that a long root of $\mathfrak{g}_0$ (with respect

to the bilinear form $(\cdot, \cdot)$ is minimal except when $\mathfrak{g} = \mathfrak{osp}(3|n)$ and the simple component of $\mathfrak{g}_{\bar{0}}$ is $\mathfrak{so}(3)$.

For a fixed minimal root $-\theta$, we can choose a simple root system Δ of Φ such that it contains $\frac{\theta}{2}$ whenever this is a root. Otherwise, we can choose a simple root system such that it contains θ (see, e.g., [34, Thm. 3.10]). Thus we can make the following convention for our later arguments.

Conventions 2.1. For the minimal root $-\theta$ of $\mathfrak{g}$, fix a simple root system $\Delta = \{\alpha_1, \dots, \alpha_k\}$ satisfying $\alpha_k = \theta$ when r is even, and $\alpha_k = \frac{\theta}{2}$ when r is odd.

Fix a minimal root $-\theta$ of $\mathfrak{g}$. We may choose root vectors $e := e_\theta$ and $f := e_{-\theta}$ such that

$$[e, f] = h := h_\theta \in \mathfrak{h}, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

As $(e, f) = 1$ by our earlier assumption, we have $(\theta, \theta) = 2$. It is well known that the eigenspace decomposition of $\text{ad } h$ gives rise to a short $\mathbb{Z}$ -grading

$$(2.1) \quad \mathfrak{g} = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2).$$

Moreover, $\mathfrak{g}(2) = \mathbb{C}e$ and $\mathfrak{g}(-2) = \mathbb{C}f$, with $\mathfrak{g}(1) \oplus \mathfrak{g}(2)$ and $\mathfrak{g}(-1) \oplus \mathfrak{g}(-2)$ being Heisenberg Lie superalgebras. We thus have a bijective correspondence between short $\mathbb{Z}$ -gradings (up to an automorphism of $\mathfrak{g}$) and minimal roots (up to the action of the Weyl group). Furthermore, one has

$$\mathfrak{g}^e = \mathfrak{g}(0)^\sharp \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2),$$

where $\mathfrak{g}(0)^\sharp = \mathfrak{g}^e(0) = \{x \in \mathfrak{g}(0) \mid [x, e] = 0\}$. Note that $\mathfrak{g}(0)^\sharp$ is the centralizer of the triple (e, f, h) by $\mathfrak{sl}(2)$ -theory. Moreover, $\mathfrak{g}(0)^\sharp$ is the orthogonal complement to $\mathbb{C}h$ in $\mathfrak{g}(0)$, and coincides with the image of the Lie superalgebra endomorphism

$$(2.2) \quad \sharp: \mathfrak{g}(0) \rightarrow \mathfrak{g}(0), \quad x \mapsto x - \frac{1}{2}(h, x)h.$$

Obviously $\mathfrak{g}(0)^\sharp$ is an ideal of codimension 1 in the Levi subalgebra $\mathfrak{g}(0)$.

Denote by Φ_e the set of all $\alpha \in \Phi$ with $\alpha(h) \in \{0, 1\}$, and write $\Phi_e^\pm := \Phi_e \cap \Phi^\pm$, $\Phi_{e,i}^\pm := \{\alpha \in \Phi_e^\pm \mid \alpha(h) = i\}$ for $i = 0, 1$, $(\Phi_{e,0}^+)_{\bar{0}} := \Phi_{e,0}^+ \cap \Phi_{\bar{0}}$. Write $\mathfrak{h}^e := \mathfrak{h} \cap \mathfrak{g}^e$, a Cartan subalgebra in $\mathfrak{g}(0)^\sharp$. Then $\mathfrak{g}^e$ is spanned by $\mathfrak{h}^e \cup \{e_\alpha \mid \alpha \in \Phi_e\} \cup \{e\}$. Note that the restrictions of $(\cdot, \cdot)$ to $\mathfrak{g}(0)^\sharp$ and $\mathfrak{h}^e$ are both non-degenerate. Take a basis $\{h_1, \dots, h_{k-1}\}$ of $\mathfrak{h}^e$ such that $(h_i, h_j) = \delta_{i,j}$ for $1 \leq i, j \leq k-1$, and denote by $\mathfrak{n}^\pm(i)$ the span of all e_α with $\alpha \in \Phi_{e,i}^\pm$. Then $\mathfrak{n}^+(0)$ and $\mathfrak{n}^-(0)$ are maximal subalgebras of $\mathfrak{g}(0)^\sharp$. Take bases $\{x_1, \dots, x_w\}$ and $\{x_1^*, \dots, x_w^*\}$ of $\mathfrak{n}_0^-(0)$ and $\mathfrak{n}_0^+(0)$ respectively, such that $(x_i, x_j) = (x_i^*, x_j^*) = 0$ and $(x_i^*, x_j) = \delta_{i,j}$ for $1 \leq i, j \leq w$. Furthermore, we can assume that each x_i (resp. x_i^*) with $1 \leq i \leq w$ is

a root vector for $\mathfrak{h}$ corresponding to $-\beta_{\bar{0}i} \in \Phi_{\bar{0}}^-$ (resp. $\beta_{\bar{0}i} \in \Phi_{\bar{0}}^+$). Set $\{y_1, \dots, y_\ell\}$ and $\{y_1^*, \dots, y_\ell^*\}$ to be bases of $\mathfrak{n}_{\bar{1}}^-(0)$ and $\mathfrak{n}_{\bar{1}}^+(0)$ such that $(y_i, y_j) = (y_i^*, y_j^*) = 0$ and $(y_i^*, y_j) = \delta_{i,j}$ for $1 \leq i, j \leq \ell$. We also assume that each y_i (resp. y_i^*) with $1 \leq i \leq \ell$ is a root vector for $\mathfrak{h}$ corresponding to $-\beta_{\bar{1}i} \in \Phi_{\bar{1}}^-$ (resp. $\beta_{\bar{1}i} \in \Phi_{\bar{1}}^+$). Recall that in Section 1.2 we assumed that u_i, u_j^* (resp. v_i, v_j^*) with $1 \leq i, j \leq \frac{s}{2}$ (resp. $1 \leq i, j \leq \lceil \frac{r}{2} \rceil$) are elements in $\mathfrak{g}(-1)_{\bar{0}}$ (resp. $\mathfrak{g}(-1)_{\bar{1}}$). For $1 \leq i \leq \frac{s}{2}$, set $f_i = [e, u_i]$ and $f_i^* = [e, u_i^*]$. Then f_i (resp. f_i^*) is a root vector for $\mathfrak{h}$ corresponding to the root $\theta + \gamma_{\bar{0}i} \in \Phi_{e,1}^-$ (resp. $\theta + \gamma_{\bar{0}i}^* \in \Phi_{e,1}^+$). For $1 \leq i \leq \lceil \frac{r}{2} \rceil$, write $g_i = [e, v_i]$ and $g_i^* = [e, v_i^*]$. Then g_i (resp. g_i^*) is a root vector for $\mathfrak{h}$ corresponding to the root $\theta + \gamma_{\bar{1}i} \in \Phi_{e,1}^-$ (resp. $\theta + \gamma_{\bar{1}i}^* \in \Phi_{e,1}^+$). When $r = \dim \mathfrak{g}(-1)_{\bar{1}}$ is odd, by our discussion in Section 1.2 the elements $v_{\frac{r+1}{2}}$ and $[v_{\frac{r+1}{2}}, e]$ are root vectors corresponding to the negative root $-\frac{\theta}{2} \in \Phi_{\bar{1}}^-$ and positive root $\frac{\theta}{2} \in \Phi_{e,1}^+$, respectively.

2.1.2. Since $\mathfrak{g}^e(0) = \mathfrak{h}^e \oplus \mathfrak{n}_{\bar{0}}^-(0) \oplus \mathfrak{n}_{\bar{0}}^+(0) \oplus \mathfrak{n}_{\bar{1}}^-(0) \oplus \mathfrak{n}_{\bar{1}}^+(0)$ as vector space, then

$$(2.3) \quad \{h_1, \dots, h_{k-1}, x_1, \dots, x_w, x_1^*, \dots, x_w^*, y_1, \dots, y_\ell, y_1^*, \dots, y_\ell^*\}$$

is a base of $\mathfrak{g}^e(0)$ by our earlier assumption. Now let

$$(2.4) \quad C_0 := \sum_{i=1}^{k-1} h_i^2 + \sum_{i=1}^w x_i x_i^* + \sum_{i=1}^w x_i^* x_i + \sum_{i=1}^\ell y_i y_i^* - \sum_{i=1}^\ell y_i^* y_i$$

be the corresponding Casimir element of $U(\mathfrak{g}^e(0))$.

Recall that in [49, Prop. 1.2], we introduced a set of generators for minimal refined W -superalgebras as below.

Theorem 2.2 ([49]). *Let $-\theta$ be a minimal root, and $e = e_\theta \in \mathfrak{g}$ a root vector for θ . Suppose $v \in \mathfrak{g}^e(0)$, $w \in \mathfrak{g}^e(1)$, C is a central element of W'_χ , and set $s = \dim \mathfrak{g}(-1)_{\bar{0}}$, $r = \dim \mathfrak{g}(-1)_{\bar{1}}$. Then the following are $\mathbb{Z}_2$ -homogeneous generators of minimal refined W -superalgebra W'_χ :*

$$\begin{aligned} \Theta_v &= \left(v - \frac{1}{2} \sum_{\alpha \in S(-1)} z_\alpha [z_\alpha^*, v] \right) \otimes 1_\chi, \\ \Theta_w &= \left(w - \sum_{\alpha \in S(-1)} z_\alpha [z_\alpha^*, w] + \frac{1}{3} \left(\sum_{\alpha, \beta \in S(-1)} z_\alpha z_\beta [z_\beta^*, [z_\alpha^*, w]] - 2[w, f] \right) \right) \otimes 1_\chi, \\ C &= \left(2e + \frac{h^2}{2} - \left(1 + \frac{s-r}{2} \right) h + C_0 + 2 \sum_{\alpha \in S(-1)} (-1)^{|\alpha|} [e, z_\alpha^*] z_\alpha \right) \otimes 1_\chi. \end{aligned}$$

For any $\mathbb{Z}_2$ -homogeneous element $w \in \mathfrak{g}$, denote by $|w|$ the parity of w . Set

$$(2.5) \quad \Theta_{\text{Cas}} := \sum_{i=1}^{k-1} \Theta_{h_i}^2 + \sum_{i=1}^w \Theta_{x_i} \Theta_{x_i^*} + \sum_{i=1}^w \Theta_{x_i^*} \Theta_{x_i} + \sum_{i=1}^\ell \Theta_{y_i} \Theta_{y_i^*} - \sum_{i=1}^\ell \Theta_{y_i^*} \Theta_{y_i},$$

an element that commutes with all operators Θ_v for $v \in \mathfrak{g}^e(0)$ (see [49, Prop. 5.7] for more details). Then the commutators between the generators in Theorem 2.2 are presented in [49, Thm. 1.3], i.e., we have the following theorem.

Theorem 2.3 ([49]). *The minimal refined W -superalgebra W'_χ is generated by the Casimir element C and the subspaces $\Theta_{\mathfrak{g}^e(i)}$ for $i = 0, 1$, as described in Theorem 2.2, subject to the following relations:*

- (1) $[\Theta_{v_1}, \Theta_{v_2}] = \Theta_{[v_1, v_2]}$ for all $v_1, v_2 \in \mathfrak{g}^e(0)$;
- (2) $[\Theta_v, \Theta_w] = \Theta_{[v, w]}$ for all $v \in \mathfrak{g}^e(0)$ and $w \in \mathfrak{g}^e(1)$;
- (3) $[\Theta_{w_1}, \Theta_{w_2}] = \frac{1}{2}([w_1, w_2], f)(C - \Theta_{\text{Cas}} - c_0) - \frac{1}{2} \sum_{\alpha \in S(-1)} (\Theta_{[w_1, z_\alpha]}^\# \Theta_{[z_\alpha^*, w_2]}^\# - (-1)^{|w_1| |w_2|} \Theta_{[w_2, z_\alpha]}^\# \Theta_{[z_\alpha^*, w_1]}^\#)$ for all $w_1, w_2 \in \mathfrak{g}^e(1)$;
- (4) $[C, W'_\chi] = 0$.

In (3), the notation $\#$ is defined as in (2.2), and the constant c_0 is decided by the following equation:

$$(2.6) \quad c_0([w_1, w_2], f) = \frac{1}{12} \sum_{\alpha, \beta \in S(-1)} [[w_1, z_\alpha], z_\beta], [z_\beta^*, [z_\alpha^*, w_2]] \otimes 1_\chi - \frac{3(s-r)+4}{12} ([w_1, w_2], f),$$

where $s = \dim \mathfrak{g}(-1)_{\bar{0}}$ and $r = \dim \mathfrak{g}(-1)_{\bar{1}}$.

2.1.3. When r is even, the refined W -superalgebra W'_χ coincides with the finite W -superalgebra $U(\mathfrak{g}, e)$ by definition. Therefore, Theorems 2.2 and 2.3 can also be considered as the PBW theorem of minimal finite W -superalgebra $U(\mathfrak{g}, e)$ in this case.

Now we assume that r is odd. In fact, taking Theorems 2.2, 2.3, 1.4 and Proposition 1.10 into consideration, we have the following theorem.

Theorem 2.4. *Let $-\theta$ be a minimal root, and $e = e_\theta \in \mathfrak{g}$ a root vector for θ . Suppose $v \in \mathfrak{g}^e(0)$, $w \in \mathfrak{g}^e(1)$. Let Θ_v, Θ_w, C be as defined in Theorem 2.2, and Θ_{l+q+1} as in Theorem 1.4(1). When r is odd, the minimal finite W -superalgebra $U(\mathfrak{g}, e)$ is generated by $\Theta_v, \Theta_w, \Theta_{l+q+1}$, and the Casimir element C , subject to the relations in Theorem 2.3(1)–(4), $[\Theta_{l+q+1}, \Theta_v] = [\Theta_{l+q+1}, \Theta_w] = [\Theta_{l+q+1}, C] = 0$, and also $[\Theta_{l+q+1}, \Theta_{l+q+1}] = 1 \otimes 1_\chi$.*

In order to highlight the key role that the odd element $v_{\frac{r+1}{2}}$ plays, from now on we will write $\Theta_{v_{\frac{r+1}{2}}}$ instead of Θ_{l+q+1} in Theorems 1.4 and 2.4.

Remark 2.5. For the convenience of following discussion, we will make some conventions. To be explicit, from the description of the system of roots Φ and of

simple roots Δ relative to $\mathfrak{h}$ of $\mathfrak{g}$ in [25, §2.5.4], and also the description of the corresponding $\mathfrak{g}^e(0)$ -module $\mathfrak{g}(1)$ ($\cong \mathfrak{g}^*(-1)$) given in [29, Tables 1–3] (note that $\mathfrak{g}^f(0) = \mathfrak{g}^e(0)$ is denoted by $\mathfrak{g}^\natural$, and $\mathfrak{g}(1)$ is written as $\mathfrak{g}_{\frac{1}{2}}$ in their settings), taking Conventions 2.1 into consideration, one can observe the following:

- (1) Let $\mathfrak{g}$ be a Lie algebra, or a Lie superalgebra of type $A(m|n)$ ($m \neq n$), $A(n|n)$, $C(n)$, $D(m|n)$, $D(2, 1; \alpha)$, $F(4)$. Then θ is a simple root of $\mathfrak{g}$ relative to Φ . Or let $\mathfrak{g}$ be a certain subclass of type $B(m|n)$ or $G(3)$ such that θ is a simple root of $\mathfrak{g}$ relative to Φ (see Table 2 for more details). Since $-\frac{\theta}{2}$ is not a root in Φ , we have $\mathfrak{m}' = \mathfrak{m}$ with r being an even number. Then the corresponding minimal refined W -superalgebra W'_χ coincides with the minimal finite W -superalgebra $U(\mathfrak{g}, e)$, which will be called the minimal finite (refined) W -superalgebra of type even.

$\mathfrak{g}$	$\mathfrak{g}^e(0)$	$\mathfrak{g}(1)$
Simple Lie algebras	See [29, Table 1]	See [29, Table 1]
$\mathfrak{sl}(2 m)$ ($m \neq 2$)	$\mathfrak{gl}(m)$	$\mathbb{C}^m \oplus \mathbb{C}^{m*}$
$\mathfrak{sl}(m n)$ ($m \neq n, m > 2$)	$\mathfrak{gl}(m-2 n)$	$\mathbb{C}^{m-2 n} \oplus \mathbb{C}^{m-2 n*}$
$\mathfrak{psl}(2 2)$	$\mathfrak{sl}(2)$	$\mathbb{C}^2 \oplus \mathbb{C}^2$
$\mathfrak{psl}(m m)$ ($m > 2$)	$\mathfrak{sl}(m-2 m)$	$\mathbb{C}^{m-2 m} \oplus \mathbb{C}^{m-2 m*}$
$\mathfrak{spo}(2 m)$ (m even)	$\mathfrak{so}(m)$	$\mathbb{C}^m$
$\mathfrak{osp}(4 2m)$	$\mathfrak{sl}(2) \oplus \mathfrak{sp}(2m)$	$\mathbb{C}^2 \otimes \mathbb{C}^{2m}$
$\mathfrak{spo}(2n m)$ ($n \geq 2, m$ even)	$\mathfrak{spo}(2n-2 m)$	$\mathbb{C}^{2n-2 m}$
$\mathfrak{osp}(m 2n)$ ($m \geq 5$)	$\mathfrak{osp}(m-4 2n) \oplus \mathfrak{sl}(2)$	$\mathbb{C}^{m-4} \otimes \mathbb{C}^2$
$D(2, 1; \alpha)$	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$	$\mathbb{C}^2 \otimes \mathbb{C}^2$
$F(4)$	$\mathfrak{so}(7)$	$\mathfrak{spin}(7)$
$F(4)$	$D(2, 1; 2)$	$\overset{1}{\circ} \leftarrow \otimes \rightarrow \overset{1}{\circ}$ (6 4)-dim
$G(3)$	$\mathfrak{osp}(3 2)$	$\overset{-3}{\otimes} \Rightarrow \overset{1}{\circ}$ (4 4)-dim

Table 2. The classification of $\mathfrak{g}$ involving minimal finite (refined) W -superalgebras of type even

- (2) Let $\mathfrak{g}$ be a certain subclass of type $B(m|n)$ or $G(3)$ such that $\frac{\theta}{2}$ is a simple root in $\Phi_{\bar{1}}$ (see Table 3 for more details), and then $\mathfrak{m}$ is a proper subalgebra of $\mathfrak{m}'$ with r being odd. So W'_χ is also a proper subalgebra of $U(\mathfrak{g}, e)$. In this case we will refer to them as the minimal refined W -superalgebra of type odd and minimal finite W -superalgebra of type odd, respectively.

$\mathfrak{g}$	$\mathfrak{g}^e(0)$	$\mathfrak{g}(1)$
$\mathfrak{spo}(2 m)$ (m odd)	$\mathfrak{so}(m)$	$\mathbb{C}^m$
$\mathfrak{spo}(2n m)$ ($n \geq 2, m$ odd)	$\mathfrak{spo}(2n-2 m)$	$\mathbb{C}^{2n-2 m}$
$G(3)$	$G(2)$	7-dim

Table 3. The classification of $\mathfrak{g}$ involving minimal refined W -superalgebras of type odd & minimal finite W -superalgebras of type odd

In the paper, when we refer to minimal finite (refined) W -superalgebras, we will always keep the conventions in Remark 2.5.

§2.2.

By the classification of minimal W -superalgebras in Remark 2.5, the most complicated, and also significantly different from the non-super situation, is the case in Table 3. So we dedicate this subsection to introducing the Verma modules for minimal finite W -superalgebras $U(\mathfrak{g}, e)$ of type odd, and other cases will be considered in Section 2.3.

2.2.1. We begin with the following observation.

Lemma 2.6. *The following statements hold:*

- (1) *When $\mathfrak{g}$ is a simple Lie algebra not of type A , $\mathfrak{g}^e(0)$ is a semi-simple Lie algebra.*
- (2) *When $\mathfrak{g}$ is not a Lie algebra, but $\mathfrak{g}^e(0)$ is and $\mathfrak{g}(1)$ is purely odd, then $\mathfrak{g}^e(0)$ is a semi-simple Lie algebra if*
 - (i) *$\mathfrak{g}$ is of type $\mathfrak{psl}(2|2)$, $\mathfrak{spo}(2|m)$ with m being even such that $\mathfrak{g}^e(0) = \mathfrak{so}(m)$, $\mathfrak{osp}(4|2m)$ with $\mathfrak{g}^e(0) = \mathfrak{sl}(2) \oplus \mathfrak{sp}(2m)$, $D(2, 1; \alpha)$, or $F(4)$ with $\mathfrak{g}^e(0) = \mathfrak{so}(7)$; in these cases r is always even;*
 - (ii) *$\mathfrak{g}$ is of type $\mathfrak{spo}(2|m)$ with m being odd such that $\mathfrak{g}^e(0) = \mathfrak{so}(m)$, or $G(3)$ with $\mathfrak{g}^e(0) = G(2)$; in these cases r is always odd.*
- (3) *When $\mathfrak{g}$ and $\mathfrak{g}^e(0)$ are not Lie algebras, all $\mathfrak{g}^e(0)$ -modules are completely reducible if and only if*
 - (i) *$\mathfrak{g}$ is of type $\mathfrak{osp}(5|2m)$ with $\mathfrak{g}^e(0) = \mathfrak{osp}(1|2m) \oplus \mathfrak{sl}(2)$; in this case r is even;*
 - (ii) *$\mathfrak{g}$ is of type $\mathfrak{spo}(2m|1)$ with $m \geq 2$ such that $\mathfrak{g}^e(0) = \mathfrak{spo}(2m-2|1)$; in this case r is odd.*

- (4) *In other cases, not all finite-dimensional representations of $\mathfrak{g}^e(0)$ are completely reducible.*

Proof. Recall that r is the dimension of $\mathfrak{g}(-1)_{\bar{1}}$, and by the non-degeneracy of the bilinear form $(\cdot, \cdot)$ we know that r has the same parity as $\mathfrak{g}(1)_{\bar{1}}$. Then statements (1) and (2) are immediate consequences of [29, Tables 1–2].

Recall that all finite-dimensional representations of a Lie (super)algebra $\mathfrak{L}$ are completely reducible if and only if $\mathfrak{L}$ is isomorphic to the direct product of a semi-simple Lie algebra with finitely many Lie superalgebras of the type $\mathfrak{osp}(1|2m)$ with $m \geq 1$ (see, e.g., [41, Chap. III, §3.1, Thm. 1]). Applying [29, Table 3] again, we see that when $\mathfrak{g} = \mathfrak{spo}(2m|1)$ with $m \geq 2$ such that $\mathfrak{g}^e(0) = \mathfrak{spo}(2m-2|1)$, the $\mathfrak{g}^e(0)$ -module $\mathfrak{g}(1)$ is isomorphic to $\mathbb{C}^{2m-2|1}$, thus $\dim \mathfrak{g}(1)_{\bar{1}} = 1$. We also observe that when $\mathfrak{g} = \mathfrak{osp}(5|2m)$ with $\mathfrak{g}^e(0) = \mathfrak{osp}(1|2m) \oplus \mathfrak{sl}(2)$, the $\mathfrak{g}^e(0)$ -module $\mathfrak{g}(1)$ is isomorphic to $\mathbb{C}^{1|2m} \otimes \mathbb{C}^2$, thus $\dim \mathfrak{g}(1)_{\bar{1}}$ is even. For other cases in [29, Table 3], $\mathfrak{g}^e$ is not isomorphic to the direct product of a semi-simple Lie algebra with finitely many $\mathfrak{osp}(1|2m)$. Then statement (3) is proved. Statement (4) is just an immediate consequence of statements (1)–(3) and [41, Chap. III, §3.1, Thm. 1]. $\square$

2.2.2. Keep the notation in Section 0.6. For any $\alpha, \beta \in \mathfrak{g}^*$ and $x \in \mathfrak{g}$, we will write $(\alpha + \beta)(x) := \alpha(x) + \beta(x)$ and $(\alpha \cdot \beta)(x) := \alpha(x) \cdot \beta(x)$ for simplicity. Put

$$\begin{aligned}
 \delta &= \frac{1}{2} \left(\sum_{i=1}^{\frac{s}{2}} \gamma_{0i}^* - \sum_{i=1}^{\frac{r-1}{2}} \gamma_{1i}^* \right) \\
 &= \frac{1}{2} \left(\sum_{i=1}^{\frac{s}{2}} (-\theta - \gamma_{0i}) + \sum_{i=1}^{\frac{r-1}{2}} (\theta + \gamma_{1i}) \right) \\
 &= \frac{1}{2} \left(-\sum_{i=1}^{\frac{s}{2}} \gamma_{0i} + \sum_{i=1}^{\frac{r-1}{2}} \gamma_{1i} \right) - \frac{s-r+1}{4} \theta, \\
 (2.7) \quad \rho &= \frac{1}{2} \sum_{\alpha \in \Phi^+} (-1)^{|\alpha|} \alpha, \\
 \rho_{e,0} &= \rho - 2\delta - \left(\frac{s-r}{4} + \frac{1}{2} \right) \theta = \frac{1}{2} \sum_{\alpha \in \Phi_{e,0}^+} (-1)^{|\alpha|} \alpha \\
 &= \frac{1}{2} \left(\sum_{j=1}^w \beta_{0j} - \sum_{j=1}^\ell \beta_{1j} \right),
 \end{aligned}$$

where $\gamma_{0i}^* \in \Phi_0^+$, $\gamma_{1j}^* \in \Phi_1^+$, $\gamma_{0i} \in \Phi_0^-$, $\gamma_{1j} \in \Phi_1^-$ for $1 \leq i \leq \frac{s}{2}$ and $1 \leq j \leq \frac{r-1}{2}$ are defined in Section 1.2, $\beta_{0i} \in \Phi_0^+$, $\beta_{1j} \in \Phi_1^+$ for $1 \leq i \leq w$ and $1 \leq j \leq \ell$ are defined in Section 2.1.1, and $|\alpha|$ denotes the parity of α . For a linear function λ on $\mathfrak{h}^e$ and

$c \in \mathbb{C}$, we will call (λ, c) a matchable pair if they satisfy the following equation:

$$(2.8) \quad \begin{aligned} c &= c_0 + \sum_{i=1}^{k-1} \lambda(h_i)^2 + \sum_{i=1}^{k-1} \left(\lambda \cdot \left(\sum_{j=1}^w \beta_{\bar{0}j} - \sum_{j=1}^{\ell} \beta_{\bar{1}j} - \sum_{j=1}^{\frac{s}{2}} \gamma_{\bar{0}j} + \sum_{j=1}^{\frac{r-1}{2}} \gamma_{\bar{1}j} \right) (h_i) \right) \\ &= c_0 + \sum_{i=1}^{k-1} \lambda(h_i)^2 + 2 \sum_{i=1}^{k-1} (\lambda \cdot (\rho_{e,0} + \delta))(h_i), \end{aligned}$$

where c_0 has the same meaning as in (2.6). Given a matchable pair $(\lambda, c) \in (\mathfrak{h}^e)^* \times \mathbb{C}$, denote by $I_{\lambda,c}$ the linear span in $U(\mathfrak{g}, e)$ of all PBW monomials of the form

$$\begin{aligned} & \prod_{i=1}^w \Theta_{x_i}^{a_i} \cdot \prod_{i=1}^{\ell} \Theta_{y_i}^{c_i} \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i}^{m_i} \cdot \prod_{i=1}^{\frac{r-1}{2}} \Theta_{g_i}^{p_i} \cdot \Theta_{v_{\frac{r+1}{2}}}^{\iota} \cdot \prod_{i=1}^{k-1} (\Theta_{h_i} - \lambda(h_i))^{t_i} \\ & \cdot (C - c)^{t_k} \cdot \Theta_{[v_{\frac{r+1}{2}}, e]}^{\varepsilon} \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i^*}^{n_i} \cdot \prod_{i=1}^{\frac{r-1}{2}} \Theta_{g_i^*}^{q_i} \cdot \prod_{i=1}^w \Theta_{x_i^*}^{b_i} \cdot \prod_{i=1}^{\ell} \Theta_{y_i^*}^{d_i}, \end{aligned}$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_+^w$, $\mathbf{c}, \mathbf{d} \in \Lambda_{\ell}$, $\mathbf{m}, \mathbf{n} \in \mathbb{Z}_+^{\frac{s}{2}}$, $\mathbf{p}, \mathbf{q} \in \Lambda_{\frac{r-1}{2}}$, $\iota, \varepsilon \in \Lambda_1$, $\mathbf{t} \in \mathbb{Z}_+^k$ with

$$\sum_{i=1}^k t_i + \varepsilon + \sum_{i=1}^{\frac{s}{2}} n_i + \sum_{i=1}^{\frac{r-1}{2}} q_i + \sum_{i=1}^w b_i + \sum_{i=1}^{\ell} d_i > 0.$$

In what follows, when we refer to the notation $I_{\lambda,c}$, we will always assume that (λ, c) is a matchable pair.

By the same strategy as in [38, Lem. 7.1], we have the following lemma.

Lemma 2.7. *The subspace $I_{\lambda,c}$ is a left ideal of a minimal finite W -superalgebra $U(\mathfrak{g}, e)$ of type odd.*

Since the proof of Lemma 2.7 is rather lengthy, we will postpone it till Appendix B.

2.2.3. Put $Z_{U(\mathfrak{g}, e)}(\lambda, c) := U(\mathfrak{g}, e)/I_{\lambda,c}$, and let v_0 denote the image of 1 in $Z_{U(\mathfrak{g}, e)}(\lambda, c)$. Clearly, $Z_{U(\mathfrak{g}, e)}(\lambda, c)$ is a cyclic $U(\mathfrak{g}, e)$ -module generated by v_0 . We will call $Z_{U(\mathfrak{g}, e)}(\lambda, c)$ the Verma module of level c corresponding to λ . By Lemma 2.7, the vectors

$$\begin{aligned} & \left\{ \prod_{i=1}^w \Theta_{x_i}^{a_i} \cdot \prod_{i=1}^{\ell} \Theta_{y_i}^{c_i} \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i}^{m_i} \cdot \prod_{i=1}^{\frac{r-1}{2}} \Theta_{g_i}^{p_i} \cdot \Theta_{v_{\frac{r+1}{2}}}^{\iota} (v_0) \mid (\mathbf{a}, \mathbf{c}, \mathbf{m}, \mathbf{p}, \iota) \in \mathbb{Z}_+^w \times \Lambda_{\ell} \right. \\ & \quad \left. \times \mathbb{Z}_+^{\frac{s}{2}} \times \Lambda_{\frac{r-1}{2}} \times \Lambda_1 \right\} \end{aligned}$$

form a $\mathbb{C}$ -basis of the Verma module $Z_{U(\mathfrak{g},e)}(\lambda, c)$ over $\mathbb{C}$. Denote by $Z_{U(\mathfrak{g},e)}^+(\lambda, c)$ the $\mathbb{C}$ -span of all $\prod_{i=1}^w \Theta_{x_i}^{a_i} \cdot \prod_{i=1}^\ell \Theta_{y_i}^{c_i} \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i}^{m_i} \cdot \prod_{i=1}^{\frac{r-1}{2}} \Theta_{g_i}^{p_i} \cdot \Theta_{v_{\frac{r+1}{2}}}^\ell(v_0)$ with

$$\sum_{i=1}^w a_i + \sum_{i=1}^\ell c_i + \sum_{i=1}^{\frac{s}{2}} m_i + \sum_{i=1}^{\frac{r-1}{2}} p_i > 0.$$

Set $Z_{U(\mathfrak{g},e)}^{\max}(\lambda, c)$ to be the sum of all $U(\mathfrak{g}, e)$ -submodules of $Z_{U(\mathfrak{g},e)}(\lambda, c)$ contained in $Z_{U(\mathfrak{g},e)}^+(\lambda, c)$, and let

$$L_{U(\mathfrak{g},e)}(\lambda, c) := Z_{U(\mathfrak{g},e)}(\lambda, c) / Z_{U(\mathfrak{g},e)}^{\max}(\lambda, c).$$

2.2.4. Proof of Theorem 0.1 Now we are in a position to prove Theorem 0.1. Generally speaking, we can repeat the proof of [38, Prop. 7.1], with a lot of modifications. We will complete the proof by steps.

(1) Given a root $\alpha = \sum_{i=1}^k n_i \alpha_i \in \Phi$, set

$$\text{ht}_\theta(\alpha) := \sum_{\alpha_i \neq \frac{\theta}{2}} n_i.$$

Since $\frac{\theta}{2}$ is an odd simple root by Conventions 2.1, then $\text{ht}_\theta(\alpha) = 0$ if and only if $\alpha = \pm \frac{\theta}{2}, \pm \theta$. By [25, Prop. 5.1.2] we know that all derivations of $\mathfrak{g}$ are inner. Therefore, we can find a unique $h_0 \in \mathfrak{h}$ such that $[h_0, e_\alpha] = \text{ht}_\theta(\alpha) e_\alpha$ for any $\alpha \in \Phi$. By definition we have $[h_0, e_{\pm\theta}] = 0$ (recall that $e_\theta = e$ and $e_{-\theta} = f$), thus $h_0 \in \mathfrak{h}^e$. It is obvious that $\Theta_{h_0}(v_0) = \lambda(h_0)v_0$ by definition, and we have the decomposition $Z_{U(\mathfrak{g},e)}(\lambda, c) = \mathbb{C}v_0 \oplus \mathbb{C}\Theta_{v_{\frac{r+1}{2}}}(v_0) \oplus Z_{U(\mathfrak{g},e)}^+(\lambda, c)$ as $\mathbb{C}$ -vector space. As all x_i, y_i, f_i , and g_i are root vectors for $\mathfrak{h}$, corresponding to negative roots different from $-\frac{\theta}{2}$ and $-\theta$, it follows from Theorem 2.4 that the subspace $Z_{U(\mathfrak{g},e)}^+(\lambda, c)$ decomposes into eigenspaces for Θ_{h_0} , and the eigenvalues of Θ_{h_0} on $Z_{U(\mathfrak{g},e)}^+(\lambda, c)$ are of the form $\lambda(h_0) - k$ with k being a positive integer.

Let V be a non-zero $\mathbb{Z}_2$ -graded submodule of the $U(\mathfrak{g}, e)$ -module $Z_{U(\mathfrak{g},e)}(\lambda, c)$. If $V \not\subseteq Z_{U(\mathfrak{g},e)}^+(\lambda, c)$, it follows from the discussion above that $v_0, \Theta_{v_{\frac{r+1}{2}}}(v_0) \in V$, which entails that $V = Z_{U(\mathfrak{g},e)}(\lambda, c)$. Therefore, any proper submodule of $Z_{U(\mathfrak{g},e)}(\lambda, c)$ is contained in $Z_{U(\mathfrak{g},e)}^+(\lambda, c)$, and $Z_{U(\mathfrak{g},e)}^{\max}(\lambda, c)$ is a unique maximal submodule of $Z_{U(\mathfrak{g},e)}(\lambda, c)$. Obviously, $L_{U(\mathfrak{g},e)}(\lambda, c)$ is a simple module of type $\mathcal{Q}$, for which the odd endomorphism is induced by the element $\Theta_{v_{\frac{r+1}{2}}} \in U(\mathfrak{g}, e)$. Now we complete the proof of statement (1).

(2) From the discussion in (1) we know that each $U(\mathfrak{g}, e)$ -module $L_{U(\mathfrak{g},e)}(\lambda, c)$ decomposed into eigenspaces for Θ_{h_0} , and the eigenvalues of Θ_{h_0} are in the set $\lambda(h_0) - \mathbb{Z}_+$. Moreover, the eigenspace $L_{U(\mathfrak{g},e)}(\lambda, c)_{\lambda(h_0)}$ of the $U(\mathfrak{g}, e)$ -module

$L_{U(\mathfrak{g},e)}(\lambda, c)$ is spanned by the elements v_0 and $\Theta_{v_{\frac{r+1}{2}}}(v_0)$. If $L_{U(\mathfrak{g},e)}(\lambda, c) \cong L_{U(\mathfrak{g},e)}(\lambda', c')$ as $U(\mathfrak{g}, e)$ -modules, it follows from the discussion above that $\lambda(h_0) \in \lambda'(h_0) - \mathbb{Z}_+$ and $\lambda'(h_0) \in \lambda(h_0) - \mathbb{Z}_+$. This implies that $\lambda(h_0) = \lambda'(h_0)$ and $L_{U(\mathfrak{g},e)}(\lambda, c)_{\lambda(h_0)} \cong L_{U(\mathfrak{g},e)}(\lambda', c')_{\lambda'(h_0)}$ as modules over the commutative subalgebra $\Theta_{\mathfrak{h}^e} \oplus \mathbb{C}C$ of $U(\mathfrak{g}, e)$. So we have $\lambda = \lambda'$ and $c = c'$, and the proof of statement (2) is completed.

(3) Let M be a finite-dimensional simple $U(\mathfrak{g}, e)$ -module. As the even element C is in the center of $U(\mathfrak{g}, e)$, Schur's lemma entails that C acts on M as $c \text{id}$ for some $c \in \mathbb{C}$. As $\Theta_{\mathfrak{h}^e}$ is an abelian by Theorem 2.4 (more precisely, Theorem 2.3(1)), by the knowledge of linear algebra we know that M contains at least one weight subspace for $\Theta_{\mathfrak{h}^e}$. Applying Theorem 2.4 again we know that the vector space $\bigoplus_{\mu \in (\mathfrak{h}^e)^*} M_\mu$ of all weight subspaces of M is a $U(\mathfrak{g}, e)$ -submodule of M . From the irreducibility of the $U(\mathfrak{g}, e)$ -module M we know that M decomposes into weight spaces relative to $\Theta_{\mathfrak{h}^e}$.

Since $\mathfrak{h} = \mathbb{C}h \oplus \mathfrak{h}^e$ as vector space, and $[e_\theta, e_{-\theta}] = [e, f] = h$ by definition, one can easily conclude that any linear function vanishing on $\mathfrak{h}^e$ is a scalar multiple of θ . As r is odd, $\frac{\theta}{2}$ is a simple root by Conventions 2.1. Then any sum of roots from $\Phi_e^+ \setminus \{\frac{\theta}{2}, \theta\}$ restricts to a non-zero function on $\mathfrak{h}^e$. Therefore, we can define a partial ordering on $(\mathfrak{h}^e)^*$ by

$$(2.9) \quad \psi \leq \phi \Leftrightarrow \phi = \psi + \left(\sum_{\gamma \in \Phi_e^+ \setminus \{\frac{\theta}{2}, \theta\}} r_\gamma \gamma \right)_{|\mathfrak{h}^e}, \quad r_\gamma \in \mathbb{Z}_+ \quad (\forall \phi, \psi \in (\mathfrak{h}^e)^*).$$

Recall that the set of $\Theta_{\mathfrak{h}^e}$ -weights of M is finite. Then it contains at least one maximal element with the ordering we just defined above, and we call it λ . For a non-zero vector m in M_λ , we have $\Theta_{x_i^*} \cdot m = \Theta_{y_i^*} \cdot m = \Theta_{f_i^*} \cdot m = \Theta_{g_i^*} \cdot m = 0$ for all admissible i (since M is finite-dimensional, we can further assume that $\Theta_{[v_{\frac{r+1}{2}}, e]} \cdot m = 0$). So there must exist a $U(\mathfrak{g}, e)$ -module homomorphism ξ from either $Z_{U(\mathfrak{g},e)}(\lambda, c)$ or $\prod Z_{U(\mathfrak{g},e)}(\lambda, c)$ (here, $\prod$ denotes the parity switching functor) to M such that $\xi(v_0) = m$. Moreover, the simplicity of M entails that ξ is surjective, and statement (1) yields $\ker \xi = Z_{U(\mathfrak{g},e)}^{\max}(\lambda, c)$. Taking Theorem 2.4 (more precisely, Theorem 2.3(1)) into consideration, when we restrict M to the $\mathfrak{sl}(2)$ -triple $(\Theta_{e_\alpha}, \Theta_{h_\alpha}, \Theta_{e_{-\alpha}}) \subset U(\mathfrak{g}, e)$ with $\alpha \in (\Phi_{e,0}^+)_{\bar{0}}$, one can easily conclude that $\lambda(h_\alpha) \in \mathbb{Z}_+$ for any $\alpha \in (\Phi_{e,0}^+)_{\bar{0}}$.

Finally, let $\mathfrak{g} = \mathfrak{spo}(2|m)$ with m being odd such that $\mathfrak{g}^e(0) = \mathfrak{so}(m)$, or $\mathfrak{g} = \mathfrak{spo}(2m|1)$ with $m \geq 2$ such that $\mathfrak{g}^e(0) = \mathfrak{spo}(2m-2|1)$, or $\mathfrak{g} = G(3)$ with $\mathfrak{g}^e(0) = G(2)$ in Table 3. Then all finite-dimensional representations of $\mathfrak{g}^e(0)$ are completely reducible by Lemma 2.6. It follows from Theorem 2.4 (more precisely, Theorem 2.3(1)) that the linear map $\Theta: \mathfrak{g}^e(0) \rightarrow \Theta_{\mathfrak{g}^e(0)}, x \mapsto \Theta_x$ is a Lie superalgebra

isomorphism. Let M be a finite-dimensional simple $U(\mathfrak{g}, e)$ -module. Then M is completely reducible as a $\Theta_{\mathfrak{g}^e(0)}$ -module. Let $\mathfrak{g}_{\mathbb{Q}}$ be the $\mathbb{Q}$ -form in $\mathfrak{g}$ spanned by the Chevalley basis from Section 1.1, and write $\mathfrak{g}_{\mathbb{Q}}^e(i) := \mathfrak{g}_{\mathbb{Q}} \cap \mathfrak{g}^e(i)$ with $i = 0, 1$. Choose $u, v \in \mathfrak{g}_{\mathbb{Q}}^e(1)$ with $([u, v], f) = 2$, and also assume that $z_{\alpha}, z_{\alpha}^* \in \mathfrak{g}_{\mathbb{Q}}(-1)$ for all $\alpha \in S(-1)$. Then $[u, z_{\alpha}]^{\sharp}, [z_{\alpha}^*, u]^{\sharp}, [v, z_{\alpha}]^{\sharp}, [z_{\alpha}^*, v]^{\sharp} \in \mathfrak{g}_{\mathbb{Q}}^e$. The highest weight theory implies that there is a $\mathbb{Q}$ -form in M stable under the action of $\Theta_{\mathfrak{g}_{\mathbb{Q}}^e(0)}$. So we have $\text{tr}_M(\Theta_{[u, z_{\alpha}]^{\sharp}} \Theta_{[z_{\alpha}^*, v]^{\sharp}}), \text{tr}_M(\Theta_{[v, z_{\alpha}]^{\sharp}} \Theta_{[z_{\alpha}^*, u]^{\sharp}}) \in \mathbb{Q}$ for all $\alpha \in S(-1)$. As $\text{tr}_M[\Theta_u, \Theta_v] = 0$, it follows from Theorem 2.4 (more precisely, Theorem 2.3(3)) that $(c - c_0) \dim M \in \mathbb{Q}$. Since $c_0 \in \mathbb{Q}$ by (2.6), we have $c \in \mathbb{Q}$. $\square$

Remark 2.8. By the same discussion as in Lemma 2.7, one can conclude that the linear span in $U(\mathfrak{g}, e)$ of all PBW monomials as in (2.10) with

$$\sum_{i=1}^k t_i + \sum_{i=1}^{\frac{s}{2}} n_i + \sum_{i=1}^{\frac{r-1}{2}} q_i + \sum_{i=1}^w b_i + \sum_{i=1}^{\ell} d_i > 0$$

is also a left ideal of minimal finite W -superalgebra $U(\mathfrak{g}, e)$ of type odd (note that there is no restriction on the pair (λ, c) as in (2.8)), and write it as $I'_{\lambda, c}$. Then we can introduce the $U(\mathfrak{g}, e)$ -module $Z'_{U(\mathfrak{g}, e)}(\lambda, c) := U(\mathfrak{g}, e)/I'_{\lambda, c}$ as in Section 2.2.3 correspondingly. The reason we did not consider $Z'_{U(\mathfrak{g}, e)}(\lambda, c)$ lies in the fact that for the vector $[v_{\frac{r+1}{2}}, e]$ associated with the simple odd root $\frac{\theta}{2}$, we may have $\Theta_{[v_{\frac{r+1}{2}}, e]} \cdot m \neq 0$ for every $m \in Z'_{U(\mathfrak{g}, e)}(\lambda, c)$, and then $Z'_{U(\mathfrak{g}, e)}(\lambda, c)$ is not necessarily a highest weight module for $U(\mathfrak{g}, e)$.

§2.3.

In this subsection we will consider Verma modules for other cases as in Remark 2.5, which is parallel to those in Section 2.2. Recall that for the type even case, $U(\mathfrak{g}, e)$ coincides with W'_{χ} . So we just need to consider W'_{χ} for both types. Since the proofs are similar, we will just sketch them.

2.3.1. Keep the notation in Section 0.6. For a linear function λ on $\mathfrak{h}^e$ and $c \in \mathbb{C}$, we denote by $J_{\lambda, c}$ the linear span in W'_{χ} of all PBW monomials of the form

$$(2.10) \quad \prod_{i=1}^w \Theta_{x_i}^{a_i} \cdot \prod_{i=1}^{\ell} \Theta_{y_i}^{c_i} \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i}^{m_i} \cdot \prod_{i=1}^{\lceil \frac{r}{2} \rceil} \Theta_{g_i}^{p_i} \cdot \prod_{i=1}^{k-1} (\Theta_{h_i} - \lambda(h_i))^{t_i} \\ \cdot (C - c)^{t_k} \cdot \Theta_{[v_{\frac{r+1}{2}}, e]}^{\varepsilon} \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i^*}^{n_i} \cdot \prod_{i=1}^{\lceil \frac{r}{2} \rceil} \Theta_{g_i^*}^{q_i} \cdot \prod_{i=1}^w \Theta_{x_i^*}^{b_i} \cdot \prod_{i=1}^{\ell} \Theta_{y_i^*}^{d_i},$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_+^w$, $\mathbf{c}, \mathbf{d} \in \Lambda_\ell$, $\mathbf{m}, \mathbf{n} \in \mathbb{Z}_+^{\frac{s}{2}}$, $\mathbf{p}, \mathbf{q} \in \Lambda_{\lceil \frac{r}{2} \rceil}$, $\mathbf{t} \in \mathbb{Z}_+^k$, $\varepsilon \in \Lambda_1$, with

$$\sum_{i=1}^k t_i + \varepsilon + \sum_{i=1}^{\frac{s}{2}} n_i + \sum_{i=1}^{\lceil \frac{r}{2} \rceil} q_i + \sum_{i=1}^w b_i + \sum_{i=1}^{\ell} d_i > 0.$$

(The term $\Theta_{[v_{\frac{r+1}{2}}, e]}$ occurs if and only if r is odd. In this subsection, when we consider the type-even case, we always assume that $\varepsilon = 0$.)

Lemma 2.9. *Under the above settings, we have the following results:*

- (1) *The subspace $J_{\lambda, c}$ is a left ideal of the minimal refined W -superalgebra W'_χ of type even.*
- (2) *The subspace $J_{\lambda, c}$ with the pair (λ, c) satisfying (2.8) is a left ideal of the minimal refined W -superalgebra W'_χ of type odd.*

Proof. Since the proof is much the same as that for Lemma 2.7, we will just sketch the differences. In fact, one can observe that all the considerations in Appendix B are still valid for statement (1) except Appendix B.3, where the emergence of $\Theta_{[v_{\frac{r+1}{2}}, e]}^2$ in (B.28) makes it necessary to impose the restriction (2.8). Since the element $\Theta_{[v_{\frac{r+1}{2}}, e]}$ will never appear when W'_χ is of type even, then $\lambda \in (\mathfrak{h}^e)^*$ and $c \in \mathbb{C}$ in (1) can be chosen arbitrarily, which is much the same as the minimal finite W -algebra case. On the other hand, repeating verbatim as in Appendix B we can obtain statement (2). $\square$

Remark 2.10. Lemma 2.9 will play a key role for the exposition on W'_χ below. To ease notation, from now on when we consider the pair (λ, c) for $J_{\lambda, c}$, we always assume that $\lambda \in (\mathfrak{h}^e)^*$ and $c \in \mathbb{C}$ are arbitrarily chosen for the minimal refined W -superalgebras of type even, and λ, c should satisfy (2.8) (i.e., (λ, c) is a matchable pair as defined in Section 2.2.2) for the minimal refined W -superalgebras of type odd, unless otherwise specified.

2.3.2. Retain the conventions in Remark 2.10. Write $Z_{W'_\chi}(\lambda, c) := W'_\chi / J_{\lambda, c}$, and denote by v_0 the image of 1 in $Z_{W'_\chi}(\lambda, c)$. By definition we know that $Z_{W'_\chi}(\lambda, c)$ is a cycle W'_χ -module generated by v_0 . We will call $Z_{W'_\chi}(\lambda, c)$ the Verma module of level c corresponding to λ . Moreover, Lemma 2.9 entails that the vectors

$$\left\{ \prod_{i=1}^w \Theta_{x_i}^{a_i} \cdot \prod_{i=1}^{\ell} \Theta_{y_i}^{c_i} \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i}^{m_i} \cdot \prod_{i=1}^{\lceil \frac{r}{2} \rceil} \Theta_{g_i}^{p_i}(v_0) \mid (\mathbf{a}, \mathbf{c}, \mathbf{m}, \mathbf{p}) \in \mathbb{Z}_+^w \times \Lambda_\ell \times \mathbb{Z}_+^{\frac{s}{2}} \times \Lambda_{\lceil \frac{r}{2} \rceil} \right\}$$

form a basis of the Verma module $Z_{W'_\chi}(\lambda, c)$ over $\mathbb{C}$. Denote by $Z_{W'_\chi}^+(\lambda, c)$ the $\mathbb{C}$ -span of all

$$\prod_{i=1}^w \Theta_{x_i}^{a_i} \cdot \prod_{i=1}^{\ell} \Theta_{y_i}^{c_i} \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i}^{m_i} \cdot \prod_{i=1}^{\lceil \frac{r}{2} \rceil} \Theta_{g_i}^{p_i}(v_0)$$

with

$$\sum_{i=1}^w a_i + \sum_{i=1}^{\ell} c_i + \sum_{i=1}^{\frac{s}{2}} m_i + \sum_{i=1}^{\lceil \frac{r}{2} \rceil} p_i > 0.$$

Set $Z_{W'_x}^{\max}(\lambda, c)$ to be the sum of all W'_x -submodules of $Z_{W'_x}(\lambda, c)$ contained in $Z_{W'_x}^+(\lambda, c)$, and let

$$L_{W'_x}(\lambda, c) := Z_{W'_x}(\lambda, c) / Z_{W'_x}^{\max}(\lambda, c).$$

Under the settings above, we can introduce the main result of this subsection.

Theorem 2.11. *Keep the conventions above. The following statements hold:*

- (1) $Z_{W'_x}^{\max}(\lambda, c)$ is the unique maximal submodule of the Verma module $Z_{W'_x}(\lambda, c)$, and $L_{W'_x}(\lambda, c)$ is a simple W'_x -module of type M .
- (2) The simple W'_x -modules $L_{W'_x}(\lambda, c)$ and $L_{W'_x}(\lambda', c')$ are isomorphic if and only if $(\lambda, c) = (\lambda', c')$.
- (3) Any finite-dimensional simple W'_x -module is isomorphic to one of the modules $L_{W'_x}(\lambda, c)$ for some $\lambda \in (\mathfrak{h}^e)^*$ satisfying $\lambda(h_\alpha) \in \mathbb{Z}_+$ for all $\alpha \in (\Phi_{e,0}^+)_{\bar{0}}$. We further have that c is a rational number in the case when $\mathfrak{g}$ is a simple Lie algebra except type $A(m)$, or when $\mathfrak{g} = \mathfrak{psl}(2|2)$, $\mathfrak{g} = \mathfrak{spo}(2m|1)$ with $m \geq 2$ such that $\mathfrak{g}^e(0) = \mathfrak{spo}(2m-2|1)$, or when $\mathfrak{g} = \mathfrak{spo}(2|m)$ with $\mathfrak{g}^e(0) = \mathfrak{so}(m)$, or when $\mathfrak{osp}(4|2m)$ with $\mathfrak{g}^e(0) = \mathfrak{sl}(2) \oplus \mathfrak{sp}(2m)$, or when $\mathfrak{g} = \mathfrak{osp}(5|2m)$ with $\mathfrak{g}^e(0) = \mathfrak{osp}(1|2m) \oplus \mathfrak{sl}(2)$, or when $\mathfrak{g} = D(2, 1; \alpha)$ with $\alpha \in \overline{\mathbb{Q}}$, or when $\mathfrak{g} = F(4)$ with $\mathfrak{g}^e(0) = \mathfrak{so}(7)$, or when $\mathfrak{g} = G(3)$ with $\mathfrak{g}^e(0) = G(2)$.

Proof. Take Lemmas 2.6 and 2.9 into consideration, and repeat verbatim the proof of Theorem 0.1. Then the theorem can be proved. $\square$

§3. The associated Whittaker categories of $\mathfrak{g}$

In this section we will relate Verma modules $Z_{U(\mathfrak{g}, e)}(\lambda, c)$ for minimal finite W -superalgebras $U(\mathfrak{g}, e)$ of both types to $\mathfrak{g}$ -modules obtained by parabolic induction from Whittaker modules for $\mathfrak{osp}(1|2)$ or $\mathfrak{sl}(2)$ separately, depending on the parity of r as discussed in Remark 2.5. Combining this with the related results on Whittaker categories in [14, 15], we obtain a complete solution to the problem of determining the composition multiplicities of Verma modules $Z_{U(\mathfrak{g}, e)}(\lambda, c)$ in terms of composition factors of Verma modules for $U(\mathfrak{g})$ in the ordinary BGG category $\mathcal{O}$.

Although the tools we applied for both types are similar, the discussion for minimal finite W -superalgebras of type odd is much more difficult. Therefore, we will give a detailed exposition for the case of type odd, and then sketch the case of type even.

§3.1.

Recall that a $\mathfrak{g}$ -module M is called a Whittaker module if $a - \chi(a)$ acts on M locally nilpotently for each $a \in \mathfrak{m}$. A Whittaker vector in a Whittaker $\mathfrak{g}$ -module M is a vector $v \in M$ which satisfies $(a - \chi(a))v = 0$ for all $a \in \mathfrak{m}$. Let $\mathcal{C}_\chi$ denote the category of finitely generated Whittaker $\mathfrak{g}$ -modules. Write

$$\text{Wh}(M) = \{v \in M \mid (a - \chi(a))v = 0, \forall a \in \mathfrak{m}\}$$

for the subspace of all Whittaker vectors in M . For $M \in \mathcal{C}_\chi$, it is obvious that $\text{Wh}(M) = 0$ if and only if $M = 0$.

For any $y \in U(\mathfrak{g})$, denote by $\text{Pr}(y) \in U(\mathfrak{g})/I_\chi$ the coset associated to y . Given a Whittaker $\mathfrak{g}$ -module M with an action map ρ , since $U(\mathfrak{g}, e) \cong Q_\chi^{\text{ad } \mathfrak{m}}$ as $\mathbb{C}$ -algebras, $\text{Wh}(M)$ is naturally a $U(\mathfrak{g}, e)$ -module by letting $\text{Pr}(y) \cdot v = \rho(y)v$ for $v \in \text{Wh}(M)$ and $\text{Pr}(y) \in U(\mathfrak{g})/I_\chi$. For a $U(\mathfrak{g}, e)$ -module M , $Q_\chi \otimes_{U(\mathfrak{g}, e)} M$ is a Whittaker $\mathfrak{g}$ -module by letting $y \cdot (q \otimes v) = (y \cdot q) \otimes v$ for $y \in U(\mathfrak{g})$ and $q \in Q_\chi$, $v \in M$.

Let $U(\mathfrak{g}, e)\text{-mod}$ be the category of finitely generated $U(\mathfrak{g}, e)$ -modules (here $U(\mathfrak{g}, e)$ denotes a finite W -superalgebra in the general case, not just for minimal ones). In [48, Thm. 2.17] we introduced Skryabin's equivalence between the finitely generated Whittaker $\mathfrak{g}$ -modules and finitely generated $U(\mathfrak{g}, e)$ -modules, i.e., we have the following theorem.

Theorem 3.1. *The functor $Q_\chi \otimes_{U(\mathfrak{g}, e)} -: U(\mathfrak{g}, e)\text{-mod} \rightarrow \mathcal{C}_\chi$ is an equivalence of categories, with $\text{Wh}: \mathcal{C}_\chi \rightarrow U(\mathfrak{g}, e)\text{-mod}$ as its quasi-inverse.*

§3.2.

In this part we consider minimal finite W -superalgebras $U(\mathfrak{g}, e)$ of type odd, with $\mathfrak{g}$ given in Table 3.

3.2.1. To describe the composition factors of the Verma modules $Z_{U(\mathfrak{g}, e)}(\lambda, c)$ with their multiplicities, we are going to establish a link between these $U(\mathfrak{g}, e)$ -modules and the $\mathfrak{g}$ -modules obtained by parabolic induction from Whittaker modules for $\mathfrak{osp}(1|2)$. The Skryabin equivalence in Theorem 3.1 will be relied on; the Kazhdan filtration of $U(\mathfrak{g}, e)$ will play an important role too.

There have been many results on the topic of Whittaker modules for Lie superalgebras. Whittaker categories for Lie superalgebras were defined and a category decomposition was presented by Bagci-Christodouloupoulou-Wiesner [5]. In further work, Chen [14] classified simple Whittaker modules for classical Lie superalgebras in terms of their parabolic decompositions, and established a type of Milićić-Soergel equivalence of a category of Whittaker modules and a category of

Harish-Chandra bimodules. Furthermore, for classical Lie superalgebras of type I, the problem of composition factors of standard Whittaker modules (i.e., the parabolic induced modules from Whittaker modules) was reduced to that of Verma modules in their BGG category $\mathcal{O}$ there. For any quasi-reductive Lie superalgebra (including all the basic classical ones), Chen–Cheng [15, Thm. 1] recently gave a complete solution to the problem of determining the composition factors of the standard Whittaker modules in terms of composition factors of Verma modules for $U(\mathfrak{g})$ in the ordinary BGG category $\mathcal{O}$. In most cases (including all basic Lie superalgebras of type A, B, C, D), the latter can be computed by related works (e.g., [6, 7, 8, 9, 16, 17, 18, 19]).

3.2.2. Denote by $\mathfrak{s}_\theta$ the subalgebra of $\mathfrak{g}$ spanned by

$$(3.1) \quad (e, h, f, E, F) := (e_\theta, h_\theta, e_{-\theta}, [\sqrt{-2}v_{\frac{r+1}{2}}, e_\theta], \sqrt{-2}v_{\frac{r+1}{2}}).$$

Taking Theorem 2.4, (B.14), and (B.22) into account, the following lemma is readily checked.

Lemma 3.2. *The subalgebra $\mathfrak{s}_\theta$ of $\mathfrak{g}$ is isomorphic to Lie superalgebra $\mathfrak{osp}(1|2)$ with even subalgebra generated by $\{e, h, f\}$ and odd subalgebra generated by $\{E, F\}$. The commutators of these basis elements are given by*

$$\begin{aligned} [h, e] &= 2e, & [h, f] &= -2f, & [e, f] &= h, & [h, E] &= E, \\ [h, F] &= -F, & [e, E] &= 0, & [e, F] &= -E, & [f, E] &= -F, \\ [f, F] &= 0, & [E, E] &= 2e, & [E, F] &= h, & [F, F] &= -2f. \end{aligned}$$

Put

$$(3.2) \quad \mathfrak{p}_\theta := \mathfrak{s}_\theta + \mathfrak{h} + \sum_{\alpha \in \Phi^+} \mathbb{C}e_\alpha, \quad \mathfrak{n}_\theta := \sum_{\alpha \in \Phi^+ \setminus \{\frac{\theta}{2}, \theta\}} \mathbb{C}e_\alpha, \quad \tilde{\mathfrak{s}}_\theta := \mathfrak{h}^e \oplus \mathfrak{s}_\theta.$$

It is obvious that $\mathfrak{p}_\theta = \tilde{\mathfrak{s}}_\theta \oplus \mathfrak{n}_\theta$ is a parabolic subalgebra of $\mathfrak{g}$ with nilradical $\mathfrak{n}_\theta$ and $\tilde{\mathfrak{s}}_\theta$ is a Levi subalgebra of $\mathfrak{p}_\theta$. Set

$$(3.3) \quad C_\theta := ef + fe + \frac{1}{2}h^2 - \frac{1}{2}EF + \frac{1}{2}FE = 2ef + \frac{1}{2}h^2 - \frac{3}{2}h + FE$$

to be a Casimir element of $U(\mathfrak{s}_\theta)$. Given $\lambda \in (\mathfrak{h}^e)^*$, write $I_\theta(\lambda)$ for the left ideal of $U(\mathfrak{p}_\theta)$ generated by $f - 1$, $E - \frac{3}{4}F + \frac{1}{2}Fh$ (this requirement will be explained in (3.17)), $C_\theta + \frac{1}{8}$ (this requirement will be explained in (3.19)), all $t - \lambda(t)$ with $t \in \mathfrak{h}^e$, and all e_γ with $\gamma \in \Phi^+ \setminus \{\frac{\theta}{2}, \theta\}$.

Set $Y(\lambda) := U(\mathfrak{p}_\theta)/I_\theta(\lambda)$ to be a $\mathfrak{p}_\theta$ -module with the trivial action of $\mathfrak{n}_\theta$, and let 1_λ denote the image of 1 in $Y(\lambda)$. Since $f \cdot 1_\lambda = 1_\lambda$ by definition, then

$$\begin{aligned} e \cdot 1_\lambda &= \frac{1}{2} \left(C_\theta - \frac{1}{2} h^2 + \frac{3}{2} h - FE \right) \cdot 1_\lambda \\ &= \left(-\frac{1}{4} h^2 + \frac{3}{4} h + \frac{1}{2} \left(-\frac{3}{4} F^2 + \frac{1}{2} F^2 h \right) - \frac{1}{16} \right) \cdot 1_\lambda \\ &= \left(-\frac{1}{4} h^2 + \frac{1}{4} h - \frac{5}{16} \right) \cdot 1_\lambda. \end{aligned}$$

Combining this with the PBW theorem we see that the vectors $\{F^k h^l \cdot 1_\lambda \mid k \in \Lambda_1, l \in \mathbb{Z}_+\}$ form a $\mathbb{C}$ -basis of $Y(\lambda)$ (the independence of these vectors follows from the fact that $Y(\lambda)$ is infinite-dimensional). Moreover, one can easily conclude that $Y(\lambda)$ is isomorphic to a Whittaker module for $\mathfrak{s}_\theta \cong \mathfrak{osp}(1|2)$.

It follows from the discussion above that the vectors

$$\begin{aligned} m(\mathbf{i}, \mathbf{j}, \iota, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, t) &:= u_1^{i_1} \cdots u_{\frac{s}{2}}^{i_{\frac{s}{2}}} \cdot v_1^{j_1} \cdots v_{\frac{r-1}{2}}^{j_{\frac{r-1}{2}}} \cdot v_{\frac{r+1}{2}}^{\iota} \cdot x_1^{k_1} \cdots x_w^{k_w} \cdot y_1^{l_1} \cdots y_\ell^{l_\ell} \\ &\quad \cdot f_1^{m_1} \cdots f_{\frac{s}{2}}^{m_{\frac{s}{2}}} \cdot g_1^{n_1} \cdots g_{\frac{r-1}{2}}^{n_{\frac{r-1}{2}}} \cdot h^t(1_\lambda) \end{aligned}$$

with $\mathbf{i}, \mathbf{m} \in \mathbb{Z}_+^{\frac{s}{2}}$, $\mathbf{j}, \mathbf{n} \in \Lambda_{\frac{r-1}{2}}$, $\iota \in \Lambda_1$, $\mathbf{k} \in \mathbb{Z}_+^w$, $\mathbf{l} \in \Lambda_\ell$, and $t \in \mathbb{Z}_+$ form a $\mathbb{C}$ -basis of the induced $\mathfrak{g}$ -module

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\theta)} Y(\lambda).$$

3.2.3. Keep the notation in (2.7). Denote

$$(3.4) \quad \epsilon := c_0 + \frac{1}{8} + 2 \sum_{i=1}^{k-1} \rho_{e,0}(h_i) \delta(h_i) + 3 \sum_{i=1}^{k-1} \delta(h_i)^2.$$

Since the element C lies in the center of $U(\mathfrak{g}, e)$, and the Verma module $Z_{U(\mathfrak{g}, e)}(\lambda, c)$ is a cycle $U(\mathfrak{g}, e)$ -module, then C acts on $Z_{U(\mathfrak{g}, e)}(\lambda, c)$ as the scalar c . We introduce the twisted action of $U(\mathfrak{g}, e)$ on $Z_{U(\mathfrak{g}, e)}(\lambda, c)$ as follows: for any $v \in Z_{U(\mathfrak{g}, e)}(\lambda, c)$, set

$$(3.5) \quad C \cdot v = \text{tw}(C)(v) := (C - \epsilon)(v) = (c - \epsilon)(v),$$

while keeping the action of all the other generators of $U(\mathfrak{g}, e)$ (as defined in Theorem 2.4) on v as usual.

Since the restriction of $(\cdot, \cdot)$ to $\mathfrak{h}^e$ is non-degenerate, for any $\eta \in (\mathfrak{h}^e)^*$ there exists a unique t_η in $\mathfrak{h}^e$ with $\eta = (t_\eta, \cdot)$. Hence $(\cdot, \cdot)$ induces a non-degenerate bilinear form on $(\mathfrak{h}^e)^*$ via $(\mu, \nu) := (t_\mu, t_\nu)$ for all $\mu, \nu \in (\mathfrak{h}^e)^*$. For a linear function φ on $\mathfrak{h}$ we denote by $\bar{\varphi}$ the restriction of φ to $\mathfrak{h}^e$.

Under the above settings, we can introduce the proof of Theorem 0.2. It is remarkable that there exists a great distinction between the structure theory of finite W -algebras in [37] and its super case in [48]. Therefore, as a super version of [38, Thm. 7.1], one can observe significant differences not only in the exposition, but also in the proofs.

Proof of Theorem 0.2. We split the proof into steps.

(1) Set $M := M(\lambda)$, and let M_0 and M_1 denote the $\mathbb{C}$ -span of all $m(\mathbf{i}, \mathbf{j}, \ell, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, t)$ in M with $|\mathbf{i}| + |\mathbf{j}| + t = 0$ and $|\mathbf{i}| + |\mathbf{j}| + t > 0$, respectively. Then $M = M_0 \oplus M_1$ as vector space. Denote by $\text{pr}: M = M_0 \oplus M_1 \rightarrow M_0$ the first projection.

Let M^k denote the $\mathbb{C}$ -span in M of all $m(\mathbf{i}, \mathbf{j}, \ell, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, t)$ of Kazhdan degree $\leq k$. Then $\{M^k \mid k \in \mathbb{Z}_+\}$ is an increasing filtration in M and $M^0 = \mathbb{C}1_\lambda$. Taking $U(\mathfrak{g})$ with its Kazhdan filtration we thus regard M as a filtrated $U(\mathfrak{g})$ -module.

Set

$$z = \lambda f + \sum_{i=1}^{\frac{s}{2}} \mu_i u_i^* + \sum_{i=1}^{\frac{r-1}{2}} \nu_i v_i^* \in \mathfrak{m},$$

where $\lambda, \mu_i, \nu_i \in \mathbb{C}$. Since $u_i^*, v_j^* \in \mathfrak{n}_\theta$ for $1 \leq i \leq \frac{s}{2}$ and $1 \leq j \leq \frac{r-1}{2}$, and $f \cdot 1_\lambda = 1_\lambda$ by definition, we have $z \cdot 1_\lambda = \lambda \cdot 1_\lambda = \chi(z) \cdot 1_\lambda$. As z acts locally nilpotently on $U(\mathfrak{g})$, we can deduce that $z - \chi(z)$ acts locally nilpotently on M for all $z \in \mathfrak{m}$. Therefore, M is an object of $\mathcal{C}_\chi$. It follows from the discussion in Section 3.1 that $\text{Wh}(M) \neq 0$, the algebra $U(\mathfrak{g}, e)$ acts on $\text{Wh}(M)$, and $M \cong Q_\chi \otimes_{U(\mathfrak{g}, e)} \text{Wh}(M)$ as $\mathfrak{g}$ -modules.

(2) For $1 \leq l \leq \frac{r-1}{2}$, since $(v_l^2) \cdot 1_\lambda = \frac{1}{2}[v_l, v_l] \cdot 1_\lambda = 0$, now observe that

$$\begin{aligned} u_k^* \cdot m(\mathbf{i}, \mathbf{j}, \ell, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, t) &\in \sum_{j=0}^t i_k \cdot C_t^j 2^j \cdot m(\mathbf{i} - \mathbf{e}_k, \mathbf{j}, \ell, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, t-j) \\ &\quad + \text{span}\{m(\mathbf{i}', \mathbf{j}', \ell', \mathbf{k}', \mathbf{l}', \mathbf{m}', \mathbf{n}', t') \mid |\mathbf{i}'| + |\mathbf{j}'| \geq |\mathbf{i}| + |\mathbf{j}|\}, \\ v_l^* \cdot m(\mathbf{i}, \mathbf{j}, \ell, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, t) &\in \sum_{i=0}^t (-1)^{\sum_{r=1}^{l-1} j_r} j_l \cdot C_t^i 2^i \cdot m(\mathbf{i}, \mathbf{j} - \mathbf{e}_l, \ell, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, t-i) \\ &\quad + \text{span}\{m(\mathbf{i}', \mathbf{j}', \ell', \mathbf{k}', \mathbf{l}', \mathbf{m}', \mathbf{n}', t') \mid |\mathbf{i}'| + |\mathbf{j}'| \geq |\mathbf{i}| + |\mathbf{j}|\} \end{aligned}$$

for all $1 \leq k \leq \frac{s}{2}$, $1 \leq l \leq \frac{r-1}{2}$, and for $t > 0$ we have

$$\begin{aligned} (f-1) \cdot m(\mathbf{i}, \mathbf{j}, \ell, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, t) &\in 2^t \cdot m(\mathbf{i}, \mathbf{j}, \ell, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, 0) \\ &\quad + \text{span}\{m(\mathbf{i}', \mathbf{j}', \ell', \mathbf{k}', \mathbf{l}', \mathbf{m}', \mathbf{n}', t') \mid t' > 0\}. \end{aligned}$$

From all these it is immediate that the map $\text{pr}: \text{Wh}(M) \rightarrow M_0$ is injective.

(3)(i) Note that $1_\lambda \in \text{Wh}(M)$, and for all $t \in \mathfrak{h}^e$ we have

$$\begin{aligned}
 \Theta_t(1_\lambda) &= \left(t - \frac{1}{2} \sum_{\alpha \in S(-1)} z_\alpha [z_\alpha^*, t] \right) (1_\lambda) \\
 &= \left(t - \frac{1}{2} \sum_{i=\frac{s}{2}+1}^s u_i [u_i^*, t] - \frac{1}{2} \sum_{i=\frac{r+1}{2}}^r v_i [v_i^*, t] \right) (1_\lambda) \\
 &= \left(t + \frac{1}{2} \sum_{i=1}^{\frac{s}{2}} [u_i^*, [u_i, t]] - \frac{1}{2} \sum_{i=1}^{\frac{r-1}{2}} [v_i^*, [v_i, t]] - \frac{1}{2} v_{\frac{r+1}{2}} [v_{\frac{r+1}{2}}, t] \right) (1_\lambda) \\
 &= \left(\lambda(t) - \frac{1}{2} \sum_{i=1}^{\frac{s}{2}} \gamma_{\bar{0}i}(t) f + \frac{1}{2} \sum_{i=1}^{\frac{r-1}{2}} \gamma_{\bar{1}i}(t) f - \frac{1}{8} \theta(t) f \right) (1_\lambda) \\
 &= \left(\lambda + \delta + \frac{2s-2r+1}{8} \theta \right) (t) \cdot (1_\lambda) \\
 (3.6) \quad &= (\lambda + \delta)(t) \cdot 1_\lambda.
 \end{aligned}$$

Suppose $v \in \mathfrak{g}^e(0)$ is a root vector for $\mathfrak{h}$ corresponding to root $\gamma \in \Phi_{e,0}^+$. As $\frac{\theta}{2}$ is a simple root by Conventions 2.1, and $0 = [h_\theta, v] = \gamma(h_\theta)v$, then $v, [u_i^*, [u_i, v]] \in \mathfrak{n}_\theta$ for all $1 \leq i \leq \frac{s}{2}$, and $[v_i^*, [v_i, v]] \in \mathfrak{n}_\theta$ for all $1 \leq i \leq \frac{r-1}{2}$. As we also have $[v_{\frac{r+1}{2}}, v] \in \mathfrak{n}_\theta$, then

$$\begin{aligned}
 \Theta_v(1_\lambda) &= \left(v - \frac{1}{2} \sum_{\alpha \in S(-1)} z_\alpha [z_\alpha^*, v] \right) (1_\lambda) \\
 &= \left(v - \frac{1}{2} \sum_{i=\frac{s}{2}+1}^s u_i [u_i^*, v] - \frac{1}{2} \sum_{i=\frac{r+1}{2}}^r v_i [v_i^*, v] \right) (1_\lambda) \\
 &= \left(v + \frac{1}{2} \sum_{i=1}^{\frac{s}{2}} [u_i^*, [u_i, v]] - \frac{1}{2} \sum_{i=1}^{\frac{r-1}{2}} [v_i^*, [v_i, v]] - \frac{1}{2} v_{\frac{r+1}{2}} [v_{\frac{r+1}{2}}, v] \right) (1_\lambda) \\
 &= 0.
 \end{aligned}$$

(3)(ii) Now let $w \in \mathfrak{g}^e(1)$ be a root vector for $\mathfrak{h}$ corresponding to root $\gamma' \in \Phi_{e,1}^+$. Then

$$\begin{aligned}
 \Theta_w(1_\lambda) &= \left(w - \sum_{\alpha \in S(-1)} z_\alpha [z_\alpha^*, w] + \frac{1}{3} \left(\sum_{\alpha, \beta \in S(-1)} z_\alpha z_\beta [z_\beta^*, [z_\alpha^*, w]] - 2[w, f] \right) \right) (1_\lambda) \\
 &= \left(w - \sum_{i=\frac{s}{2}+1}^s u_i [u_i^*, w] - \sum_{i=\frac{r+1}{2}}^r v_i [v_i^*, w] \right.
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{3} \left(\sum_{i,j=\frac{s}{2}+1}^s u_i u_j [u_j^*, [u_i^*, w]] + \sum_{i=\frac{s}{2}+1}^s \sum_{j=1}^{\frac{s}{2}} u_i u_j [u_j^*, [u_i^*, w]] \right. \\
& + \sum_{i=1}^{\frac{s}{2}} \sum_{j=\frac{s}{2}+1}^s u_i u_j [u_j^*, [u_i^*, w]] + 2 \sum_{i=1}^{\frac{s}{2}} \sum_{j=\frac{r+1}{2}}^r u_i v_j [v_j^*, [u_i^*, w]] \\
& + 2 \sum_{i=\frac{s}{2}+1}^s \sum_{j=1}^{\frac{r-1}{2}} u_i v_j [v_j^*, [u_i^*, w]] + 2 \sum_{i=\frac{s}{2}+1}^s \sum_{j=\frac{r+1}{2}}^r u_i v_j [v_j^*, [u_i^*, w]] \\
& + \sum_{i=1}^{\frac{r-1}{2}} \sum_{j=\frac{r+1}{2}}^r v_i v_j [v_j^*, [v_i^*, w]] + \sum_{i=\frac{r+1}{2}}^r \sum_{j=1}^{\frac{r-1}{2}} v_i v_j [v_j^*, [v_i^*, w]] \\
& \left. + \sum_{i=\frac{r+1}{2}}^r \sum_{j=\frac{r+1}{2}}^r v_i v_j [v_j^*, [v_i^*, w]] - 2[w, f] \right) (1_\lambda) \\
& = \left(w + \sum_{i=1}^{\frac{s}{2}} [u_i^*, [u_i, w]] - \left(\sum_{i=1}^{\frac{r-1}{2}} [v_i^*, [v_i, w]] + v_{\frac{r+1}{2}} [v_{\frac{r+1}{2}}^*, w] \right) \right. \\
& + \frac{1}{3} \left(\sum_{i,j=1}^{\frac{s}{2}} u_i^* u_j^* [u_j, [u_i, w]] - \sum_{i,j=1}^{\frac{s}{2}} u_i^* u_j [u_j^*, [u_i, w]] \right. \\
& - \sum_{i,j=1}^{\frac{s}{2}} u_i u_j^* [u_j, [u_i^*, w]] + 2 \sum_{i=1}^{\frac{s}{2}} \sum_{j=1}^{\frac{r-1}{2}} u_i v_j^* [v_j, [u_i^*, w]] \\
& + 2 \sum_{i=1}^{\frac{s}{2}} u_i v_{\frac{r+1}{2}} [v_{\frac{r+1}{2}}^*, [u_i^*, w]] - 2 \sum_{i=1}^{\frac{s}{2}} \sum_{j=1}^{\frac{r-1}{2}} u_i^* v_j [v_j^*, [u_i, w]] \\
& - 2 \sum_{i=1}^{\frac{s}{2}} \sum_{j=1}^{\frac{r-1}{2}} u_i^* v_j^* [v_j, [u_i, w]] - 2 \sum_{i=1}^{\frac{s}{2}} u_i^* v_{\frac{r+1}{2}} [v_{\frac{r+1}{2}}^*, [u_i, w]] \\
& + \sum_{i,j=1}^{\frac{r-1}{2}} v_i v_j^* [v_j, [v_i^*, w]] + \sum_{i=1}^{\frac{r-1}{2}} v_i v_{\frac{r+1}{2}} [v_{\frac{r+1}{2}}^*, [v_i^*, w]] \\
& + \sum_{i,j=1}^{\frac{r-1}{2}} v_i^* v_j [v_j^*, [v_i, w]] + \sum_{i=1}^{\frac{r-1}{2}} v_{\frac{r+1}{2}} v_i [v_i^*, [v_{\frac{r+1}{2}}^*, w]] \\
& + \sum_{i,j=1}^{\frac{r-1}{2}} v_i^* v_j^* [v_j, [v_i, w]] + \sum_{i=1}^{\frac{r-1}{2}} v_{\frac{r+1}{2}} v_i^* [v_i, [v_{\frac{r+1}{2}}^*, w]] \\
& \left. + \sum_{i=1}^{\frac{r-1}{2}} v_i^* v_{\frac{r+1}{2}} [v_{\frac{r+1}{2}}^*, [v_i, w]] + v_{\frac{r+1}{2}}^2 [v_{\frac{r+1}{2}}^*, [v_{\frac{r+1}{2}}^*, w]] - 2[w, f] \right) (1_\lambda)
\end{aligned}$$

$$\begin{aligned}
&= \left(w + \sum_{i=1}^{\frac{s}{2}} [u_i^*, [u_i, w]] - \left(\sum_{i=1}^{\frac{r-1}{2}} [v_i^*, [v_i, w]] + v_{\frac{r+1}{2}} [v_{\frac{r+1}{2}}, w] \right) \right. \\
&\quad + \frac{1}{3} \left(\sum_{i,j=1}^{\frac{s}{2}} [u_i^*, [u_j^*, [u_j, [u_i, w]]]] - \sum_{i=1}^{\frac{s}{2}} [u_i^*, [u_i, w]] f \right. \\
&\quad - \sum_{i,j=1}^{\frac{s}{2}} u_j [u_i^*, [u_j^*, [u_i, w]]] - \sum_{i,j=1}^{\frac{s}{2}} u_i [u_j^*, [u_j, [u_i^*, w]]] \\
&\quad + 2 \sum_{i=1}^{\frac{s}{2}} \sum_{j=1}^{\frac{r-1}{2}} u_i [v_j^*, [v_j, [u_i^*, w]]] + 2 \sum_{i=1}^{\frac{s}{2}} u_i v_{\frac{r+1}{2}} [v_{\frac{r+1}{2}}, [u_i^*, w]] \\
&\quad - 2 \sum_{i=1}^{\frac{s}{2}} \sum_{j=1}^{\frac{r-1}{2}} v_j [u_i^*, [v_j^*, [u_i, w]]] - 2 \sum_{i=1}^{\frac{s}{2}} \sum_{j=1}^{\frac{r-1}{2}} [u_i^*, [v_j^*, [v_j, [u_i, w]]]] \\
&\quad - 2 \sum_{i=1}^{\frac{s}{2}} v_{\frac{r+1}{2}} [u_i^*, [u_i, [v_{\frac{r+1}{2}}, w]]] + \sum_{i,j=1}^{\frac{r-1}{2}} v_i [v_j^*, [v_j, [v_i^*, w]]] \\
&\quad + 2 \sum_{i=1}^{\frac{r-1}{2}} v_i v_{\frac{r+1}{2}} [v_{\frac{r+1}{2}}, [v_i^*, w]] + \sum_{i=1}^{\frac{r-1}{2}} [v_i^*, [v_i, w]] f \\
&\quad - \sum_{i,j=1}^{\frac{r-1}{2}} v_j [v_i^*, [v_j^*, [v_i, w]]] + \sum_{i,j=1}^{\frac{r-1}{2}} [v_i^*, [v_j^*, [v_j, [v_i, w]]]] \\
&\quad + 2 \sum_{i=1}^{\frac{r-1}{2}} v_{\frac{r+1}{2}} [v_i^*, [v_i, [v_{\frac{r+1}{2}}, w]]] + \frac{1}{2} [v_{\frac{r+1}{2}}, [v_{\frac{r+1}{2}}, w]] f \\
&\quad \left. \left. - 2[w, f] \right) \right) (1_\lambda).
\end{aligned}
\tag{3.7}$$

Now we will discuss the terms in (3.7). To ease notation, we will call i admissible for u_i if $1 \leq i \leq \frac{s}{2}$, and also for v_i if $1 \leq i \leq \frac{r-1}{2}$. For any admissible i, j , by degree considerations we see that

$$[u_i^*, [u_j^*, [u_j, [u_i, w]]]] \in \mathfrak{g}(-3) = \{0\}.$$

The same discussion entails that $[u_i^*, [v_j^*, [v_j, [u_i, w]]]] = [v_i^*, [v_j^*, [v_j, [v_i, w]]]] = 0$ for all admissible i, j .

On the other hand, one can easily conclude that $[u_i^*, [u_j^*, [u_i, w]]] \in \mathfrak{g}(-2)$ for all admissible i, j , thus $[u_i^*, [u_j^*, [u_i, w]]] = kf$ for some $k \in \mathbb{C}$. As $[u_i^*, [u_j^*, [u_i, w]]]$ is a root vector for $\mathfrak{h}$ corresponding to root $\gamma' + \gamma_{0j}^* - \theta$, it follows from $f = e_{-\theta}$ that $k \neq 0$ if and only if $\gamma' + \gamma_{0j}^* - \theta = -\theta$, which is impossible. Then $[u_i^*, [u_j^*, [u_i, w]]] = 0$.

By the same discussion we can conclude that

$$\begin{aligned} [u_j^*, [u_j, [u_i^*, w]]] &= [v_j^*, [v_j, [u_i^*, w]]] = [u_i^*, [v_j^*, [u_i, w]]] \\ &= [v_j^*, [v_j, [v_i^*, w]]] = [v_i^*, [v_j^*, [v_i, w]]] = 0 \end{aligned}$$

for all admissible i, j . The same consideration also applies for $[u_i^*, [u_i, [v_{\frac{r+1}{2}}^*, w]]]$ and $[v_i^*, [v_i, [v_{\frac{r+1}{2}}^*, w]]]$, which are non-zero if and only if $\gamma' = \frac{1}{2}\theta$. Moreover, since $\frac{\theta}{2}$ is a simple root by Conventions 2.1, for all admissible i, j we have $[v_{\frac{r+1}{2}}^*, [u_i^*, w]]$, $[v_{\frac{r+1}{2}}^*, [v_i^*, w]] \in \mathfrak{n}_\theta$ by weight considerations.

Note that w is a linear combination of $f_1^*, \dots, f_{\frac{s}{2}}^*, g_1^*, \dots, g_{\frac{r-1}{2}}^*, [v_{\frac{r+1}{2}}^*, e]$, where $[v_{\frac{r+1}{2}}^*, e]$ is a root vector for $\mathfrak{h}$ corresponding to simple root $\frac{\theta}{2}$. For $w' \in \{f_1^*, \dots, f_{\frac{s}{2}}^*, g_1^*, \dots, g_{\frac{r-1}{2}}^*\} \subset \mathfrak{n}_\theta$, by weight considerations we have $[u_i^*, [u_i, w']]$, $[v_i^*, [v_i, w']]$, $[v_{\frac{r+1}{2}}^*, w']$, $[v_{\frac{r+1}{2}}^*, [v_{\frac{r+1}{2}}^*, w']]$, $[w', f] \in \mathfrak{n}_\theta$ for all admissible i . Taking all the above into consideration, we know that (3.7) equals zero in this situation, and then $\Theta_{w'}(1_\lambda) = 0$.

It remains to consider the case with $w = [v_{\frac{r+1}{2}}^*, e]$. For any $1 \leq \alpha \leq \frac{s}{2}$ or $s+1 \leq \alpha \leq s + \frac{r-1}{2}$ (i.e., for all admissible i of u_i and v_i), since

$$[[z_\alpha^*, [z_\alpha, e]], f] = [z_\alpha^*, [z_\alpha, [e, f]]] = [z_\alpha^*, [z_\alpha, h]] = f,$$

we have $[z_\alpha^*, [z_\alpha, e]] + \frac{1}{2}h \in \mathfrak{h}^e$. Set

$$\gamma_{|\alpha|, \alpha}^* := \begin{cases} \gamma_{0\alpha}^* & \text{if } 1 \leq \alpha \leq \frac{s}{2}, \\ \gamma_{1\alpha-s}^* & \text{if } s+1 \leq \alpha \leq s + \frac{r-1}{2}, \end{cases}$$

where γ_{0i}^* for $1 \leq i \leq \frac{s}{2}$ and γ_{1j}^* for $1 \leq j \leq \frac{r-1}{2}$ are defined as in Section 1.2. Let x be arbitrary element in $\mathfrak{h}^e$. Then $(x, h) = 0$, and

$$(3.8) \quad \left(x, [z_\alpha^*, [z_\alpha, e]] + \frac{1}{2}h\right) = ([x, z_\alpha^*], [z_\alpha, e]) = \gamma_{|\alpha|, \alpha}^*(x)(z_\alpha^*, [z_\alpha, e]) = \gamma_{|\alpha|, \alpha}^*(x),$$

that is,

$$(3.9) \quad [z_\alpha^*, [z_\alpha, e]] = -\frac{1}{2}h + t_{\bar{\gamma}_{|\alpha|, \alpha}^*}.$$

Therefore, we have

$$\begin{aligned} (3.10) \quad [z_\alpha^*, [z_\alpha, [v_{\frac{r+1}{2}}^*, e]]] &= -[[z_\alpha^*, [z_\alpha, e]], v_{\frac{r+1}{2}}^*] = \left[\frac{1}{2}h - t_{\bar{\gamma}_{|\alpha|, \alpha}^*}, v_{\frac{r+1}{2}}^*\right] \\ &= \left(-\frac{1}{2} + \frac{1}{2}\theta(t_{\bar{\gamma}_{|\alpha|, \alpha}^*})\right)v_{\frac{r+1}{2}}^* \end{aligned}$$

For any $t \in \mathfrak{h}^{e\theta}$, we have $\theta(t) = 0$ by definition. Then it follows from $\theta(t_{\tilde{\gamma}_{|\alpha|,\alpha}}^*) = 0$ and (3.10) that

$$(3.11) \quad [z_\alpha^*, [z_\alpha, [v_{\frac{r+1}{2}}, e]]] = -\frac{1}{2}v_{\frac{r+1}{2}}.$$

Since

$$(3.12) \quad [v_{\frac{r+1}{2}}, [v_{\frac{r+1}{2}}, e]] = [[v_{\frac{r+1}{2}}, e], v_{\frac{r+1}{2}}] = -\frac{1}{2}h$$

by (B.22), then

$$(3.13) \quad [v_{\frac{r+1}{2}}, [v_{\frac{r+1}{2}}, [v_{\frac{r+1}{2}}, e]]] = [v_{\frac{r+1}{2}}, -\frac{1}{2}h] = -\frac{1}{2}v_{\frac{r+1}{2}},$$

and for any $1 \leq \alpha \leq \frac{s}{2}$ or $s+1 \leq \alpha \leq s + \frac{r-1}{2}$, we have

$$(3.14) \quad [z_\alpha^*, [z_\alpha, [v_{\frac{r+1}{2}}, [v_{\frac{r+1}{2}}, e]]]] = -[z_\alpha^*, [z_\alpha, \frac{1}{2}h]] = -\frac{1}{2}[z_\alpha^*, z_\alpha] = -\frac{1}{2}f.$$

Also note that

$$(3.15) \quad [[v_{\frac{r+1}{2}}, e], f] = [v_{\frac{r+1}{2}}, [e, f]] = [v_{\frac{r+1}{2}}, h] = v_{\frac{r+1}{2}}.$$

Combining (3.7), (3.11)–(3.15) with our earlier discussion, one can conclude that

$$(3.16) \quad \begin{aligned} \Theta_{[v_{\frac{r+1}{2}}, e]}(1_\lambda) &= \left([v_{\frac{r+1}{2}}, e] - \frac{s}{4}v_{\frac{r+1}{2}} - \left(-\frac{r-1}{4}v_{\frac{r+1}{2}} - \frac{1}{2}v_{\frac{r+1}{2}}h \right) \right. \\ &\quad \left. + \frac{1}{3} \left(\frac{s}{4}v_{\frac{r+1}{2}} + \frac{s}{2}v_{\frac{r+1}{2}} - \frac{r-1}{4}v_{\frac{r+1}{2}} - \frac{r-1}{2}v_{\frac{r+1}{2}} - \frac{1}{4}v_{\frac{r+1}{2}} - 2v_{\frac{r+1}{2}} \right) \right)(1_\lambda) \\ &= \left([v_{\frac{r+1}{2}}, e] - \frac{3}{4}v_{\frac{r+1}{2}} + \frac{1}{2}v_{\frac{r+1}{2}}h \right)(1_\lambda). \end{aligned}$$

Since

$$(3.17) \quad \left(E - \frac{3}{4}F + \frac{1}{2}Fh \right) \cdot 1_\lambda = 0$$

by our assumption (see Section 3.2.2), i.e.,

$$\left([v_{\frac{r+1}{2}}, e] - \frac{3}{4}v_{\frac{r+1}{2}} + \frac{1}{2}v_{\frac{r+1}{2}}h \right)(1_\lambda) = 0,$$

then (3.16) entails that $\Theta_{[v_{\frac{r+1}{2}}, e]}(1_\lambda) = 0$. Thus we have $\Theta_w(1_\lambda) = 0$ for all positive root vectors $w \in \mathfrak{g}^e(1)$. Moreover, it follows from

$$\begin{aligned} &\left[E - \frac{3}{4}F + \frac{1}{2}Fh, E - \frac{3}{4}F + \frac{1}{2}Fh \right] \otimes 1_\chi \\ &= \left([E, E] - \frac{3}{4}[F, E] + \frac{1}{2}[Fh, E] - \frac{3}{4}[E, F] + \frac{9}{16}[F, F] - \frac{3}{8}[Fh, F] \right. \\ &\quad \left. + \frac{1}{2}[E, Fh] - \frac{3}{8}[F, Fh] + \frac{1}{4}[Fh, Fh] \right) \otimes 1_\chi \end{aligned}$$

$$\begin{aligned}
&= \left([E, E] - \frac{3}{2}[F, E] + [F, E]h + F[h, E] + \frac{9}{16}[F, F] - \frac{3}{4}[F, F]h \right. \\
&\quad \left. - \frac{3}{4}F[h, F] + \frac{1}{4}[F, F]h^2 + \frac{1}{4}F[h, F]h - \frac{1}{4}F[F, h]h \right) \otimes 1_\chi \\
&= \left(2e - \frac{3}{2}h + \frac{1}{2}h^2 + FE + \frac{1}{8} \right) \otimes 1_\chi \\
(3.18) \quad &= \left(C_\theta + \frac{1}{8} \right) \otimes 1_\chi
\end{aligned}$$

that

$$(3.19) \quad C_\theta \cdot 1_\lambda = \left(\left[E - \frac{3}{4}F + \frac{1}{2}Fh, E - \frac{3}{4}F + \frac{1}{2}Fh \right] - \frac{1}{8} \right) \cdot 1_\lambda = -\frac{1}{8} \cdot 1_\lambda.$$

(3)(iii) Let C_0 be the Casimir element of $U(\mathfrak{g}^e(0))$ as defined in (2.4). In virtue of Theorem 2.4, (3.19), and (B.22), we have

$$\begin{aligned}
C(1_\lambda) &= \left(2e + \frac{h^2}{2} - \left(1 + \frac{s-r}{2} \right) h + C_0 + 2 \sum_{\alpha \in S(-1)} (-1)^{|\alpha|} [e, z_\alpha^*] z_\alpha \right) (1_\lambda) \\
&= \left(2e + \frac{h^2}{2} - \left(1 + \frac{s-r}{2} \right) h + C_0 + 2 \sum_{i=1}^{\frac{s}{2}} [[e, u_i^*], u_i] - 2 \sum_{i=1}^{\frac{r-1}{2}} [[e, v_i^*], v_i] \right. \\
&\quad \left. - 2[e, v_{\frac{r+1}{2}}] v_{\frac{r+1}{2}} \right) (1_\lambda) \\
&= \left(C_\theta + C_0 - \frac{s-r+1}{2} h + 2 \sum_{i=1}^{\frac{s}{2}} [[e, u_i^*], u_i] - 2 \sum_{i=1}^{\frac{r-1}{2}} [[e, v_i^*], v_i] \right) (1_\lambda) \\
(3.20) \quad &= \left(-\frac{1}{8} + C_0 - \frac{s-r+1}{2} h + 2 \sum_{i=1}^{\frac{s}{2}} [[e, u_i^*], u_i] - 2 \sum_{i=1}^{\frac{r-1}{2}} [[e, v_i^*], v_i] \right) (1_\lambda).
\end{aligned}$$

For any $1 \leq \alpha \leq \frac{s}{2}$ or $s+1 \leq \alpha \leq s + \frac{r-1}{2}$, since $[[[e, z_\alpha^*], z_\alpha], f] = [[e, f], z_\alpha^*], z_\alpha] = [[h, z_\alpha^*], z_\alpha] = -f$, one can conclude that $[[e, z_\alpha^*], z_\alpha] - \frac{1}{2}h \in \mathfrak{h}^e$. Let x be an arbitrary element in $\mathfrak{h}^e$. Then $(x, h) = 0$, $\theta(x) = 0$, and

$$\left(x, [[e, z_\alpha^*], z_\alpha] - \frac{1}{2}h \right) = ([x, [e, z_\alpha^*]], z_\alpha) = \gamma_{|\alpha|, \alpha}^*(x)([e, z_\alpha^*], z_\alpha) = \gamma_{|\alpha|, \alpha}^*(x),$$

that is,

$$(3.21) \quad [[e, z_\alpha^*], z_\alpha] - \frac{1}{2}h = t_{\bar{\gamma}_{|\alpha|, \alpha}^*}.$$

Then

$$\begin{aligned}
 & \left(2 \sum_{i=1}^{\frac{s}{2}} [[e, u_i^*], u_i] - 2 \sum_{i=1}^{\frac{r-1}{2}} [[e, v_i^*], v_i] - \frac{s-r+1}{2} h \right) (1_\lambda) \\
 &= \left(2 \sum_{i=1}^{\frac{s}{2}} t_{\bar{\gamma}_{0i}^*} - 2 \sum_{i=1}^{\frac{r-1}{2}} t_{\bar{\gamma}_{1i}^*} \right) (1_\lambda) \\
 &= 2 \left(\sum_{i=1}^{\frac{s}{2}} \lambda(t_{\bar{\gamma}_{0i}^*}) - \sum_{i=1}^{\frac{r-1}{2}} \lambda(t_{\bar{\gamma}_{1i}^*}) \right) \cdot 1_\lambda \\
 &= 2 \left(\lambda, \sum_{i=1}^{\frac{s}{2}} \bar{\gamma}_{0i}^* - \sum_{i=1}^{\frac{r-1}{2}} \bar{\gamma}_{1i}^* \right) \cdot 1_\lambda \\
 (3.22) \quad &= 4(\lambda, \bar{\delta}) \cdot 1_\lambda.
 \end{aligned}$$

As C_0 is a Casimir element in $U(\mathfrak{g}^e(0))$, and all positive vectors in $\mathfrak{g}^e(0)$ annihilate 1_λ , by the same discussion as in [33, Lem. 8.5.3] we see that

$$(3.23) \quad C_0(1_\lambda) = \left(\sum_{i=1}^{k-1} \lambda(h_i)^2 + \sum_{\alpha \in \Phi_{e,0}^+} (-1)^{|\alpha|} \lambda(h_\alpha) \right) (1_\lambda) = (\lambda, \lambda + 2\bar{\rho}_{e,0}) \cdot 1_\lambda,$$

where $|\alpha|$ denotes the parity of α .

We can conclude from (3.20), (3.22), and (3.23) that

$$(3.24) \quad C(1_\lambda) = \left(-\frac{1}{8} + (\lambda, \lambda + 2\bar{\rho}_{e,0}) + 4(\lambda, \bar{\delta}) \right) \cdot 1_\lambda = \left(-\frac{1}{8} + (\lambda, \lambda + 2\bar{\rho}) \right) \cdot 1_\lambda.$$

(3)(iv) In virtue of (3.6) and (3.24), set

$$(3.25) \quad \lambda' := \lambda + \bar{\delta}, \quad c' := -\frac{1}{8} + (\lambda, \lambda + 2\bar{\rho}) + \epsilon.$$

Under the twisted action of $U(\mathfrak{g}, e)$ on $Z_{U(\mathfrak{g}, e)}(\lambda', c')$, we have

$$C \cdot v = \text{tw}(C)(v) = (C - \epsilon)(v) = (c' - \epsilon)(v) = \left(-\frac{1}{8} + (\lambda, \lambda + 2\bar{\rho}) \right) (v)$$

for any $v \in Z_{U(\mathfrak{g}, e)}(\lambda', c')$. To show that $(\lambda', c') \in (\mathfrak{h}^e)^* \times \mathbb{C}$ is a matchable pair as in (2.8), we need another description of c' in (3.25). Since $h_1, \dots, h_{k-1}$ is an orthogonal basis of $\mathfrak{h}^e$ with respect to $(\cdot, \cdot)$, we can write $t_{\bar{\gamma}_{ai}^*} = \sum_{j=1}^{k-1} l_j h_j$ for $a \in \{\bar{0}, \bar{1}\}$, and then we have

$$\gamma_{ai}^*(h_m) = (t_{\bar{\gamma}_{ai}^*}, h_m) = \sum_{j=1}^{k-1} l_j (h_j, h_m) = l_m,$$

thus

$$(3.26) \quad t_{\bar{\gamma}_{ai}}^* = \sum_{i=1}^{k-1} \gamma_{ai}^*(h_j) h_j = - \sum_{i=1}^{k-1} \gamma_{ai}(h_j) h_j.$$

So we have

$$(3.27) \quad \begin{aligned} (\lambda, \bar{\delta}) &= \frac{1}{2} \lambda \left(\sum_{j=1}^{\frac{s}{2}} t_{\bar{\gamma}_{0j}}^* - \sum_{j=1}^{\frac{r-1}{2}} t_{\bar{\gamma}_{1j}}^* \right) = \frac{1}{2} \sum_{i=1}^{k-1} \left(- \sum_{j=1}^{\frac{s}{2}} \gamma_{0j}(h_i) + \sum_{j=1}^{\frac{r-1}{2}} \gamma_{1j}(h_i) \right) \lambda(h_i) \\ &= \sum_{i=1}^{k-1} \delta(h_i) \lambda(h_i). \end{aligned}$$

Thanks to the definition of C_0 in (2.4), and also (B.17), (B.19), we obtain

$$(3.28) \quad \begin{aligned} C_0(1_\lambda) &= \left(\sum_{i=1}^{k-1} h_i^2 + \sum_{i=1}^w x_i x_i^* + \sum_{i=1}^w x_i^* x_i + \sum_{i=1}^\ell y_i y_i^* - \sum_{i=1}^\ell y_i^* y_i \right) (1_\lambda) \\ &= \left(\sum_{i=1}^{k-1} h_i^2 + \sum_{i=1}^w [x_i^*, x_i] - \sum_{i=1}^\ell [y_i^*, y_i] \right) (1_\lambda) \\ &= \left(\sum_{i=1}^{k-1} h_i^2 + \sum_{i=1}^{k-1} \sum_{j=1}^w \beta_{0j}(h_i) h_i - \sum_{i=1}^{k-1} \sum_{j=1}^\ell \beta_{1j}(h_i) h_i \right) (1_\lambda) \\ &= \left(\sum_{i=1}^{k-1} \lambda(h_i)^2 + 2 \sum_{i=1}^{k-1} \rho_{e,0}(h_i) \lambda(h_i) \right) (1_\lambda). \end{aligned}$$

From (3.20), (3.22), (3.27), and (3.28), we get

$$(3.29) \quad C(1_\lambda) = \left(-\frac{1}{8} + \sum_{i=1}^{k-1} \lambda(h_i)^2 + 2 \sum_{i=1}^{k-1} \rho_{e,0}(h_i) \lambda(h_i) + 4 \sum_{i=1}^{k-1} \delta(h_i) \lambda(h_i) \right) (1_\lambda).$$

Taking (3.24) and (3.29) into consideration, we have

$$\begin{aligned} c' &= -\frac{1}{8} + (\lambda, \lambda + 2\bar{\rho}) + \epsilon \\ &= -\frac{1}{8} + \sum_{i=1}^{k-1} \lambda(h_i)^2 + 2 \sum_{i=1}^{k-1} \rho_{e,0}(h_i) \lambda(h_i) + 4 \sum_{i=1}^{k-1} \delta(h_i) \lambda(h_i) + c_0 + \frac{1}{8} \\ &\quad + 2 \sum_{i=1}^{k-1} \rho_{e,0}(h_i) \delta(h_i) + 3 \sum_{i=1}^{k-1} \delta(h_i)^2 \end{aligned}$$

$$\begin{aligned}
&= c_0 + \sum_{i=1}^{k-1} ((\lambda + \delta)(h_i))^2 + 2 \sum_{i=1}^{k-1} ((\lambda + \delta)(\rho_{e,0} + \delta))(h_i) \\
(3.30) \quad &= c_0 + \sum_{i=1}^{k-1} \lambda'(h_i)^2 + 2 \sum_{i=1}^{k-1} (\lambda' \cdot (\rho_{e,0} + \delta))(h_i),
\end{aligned}$$

which verifies equation (2.8).

Denote by V_0 the $U(\mathfrak{g}, e)$ -submodule of M generated by 1_λ , and let $I_{\lambda', c'}$ be the left ideal of $U(\mathfrak{g}, e)$ as defined in Section 2.2.2. From all the discussion above we know that the left ideal $I_{\lambda', c'}$ of $U(\mathfrak{g}, e)$ annihilates 1_λ . Then V_0 is a homomorphic image of the Verma module $Z_{U(\mathfrak{g}, e)}(\lambda', c')$.

(3)(v) Now we claim that the restriction $\text{pr}: \text{Wh}(M) \twoheadrightarrow M_0$ to V_0 is surjective. Recall that M_0 is spanned by all $m(\mathbf{0}, \mathbf{0}, \iota, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, 0)$ in M with $\iota \in \Lambda_1$, $\mathbf{k} \in \mathbb{Z}_+^w$, $\mathbf{l} \in \Lambda_\ell$, $\mathbf{m} \in \mathbb{Z}_+^{\frac{s}{2}}$, and $\mathbf{n} \in \Lambda_{\frac{r-1}{2}}$. It is obvious that $m(\mathbf{0}, \mathbf{0}, 0, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, 0) = 1_\lambda \in \text{pr}(V_0)$. Assume that all the vectors $m(\mathbf{0}, \mathbf{0}, \iota, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, 0)$ of Kazhdan degree $\iota + 2(|\mathbf{k}| + |\mathbf{l}|) + 3(|\mathbf{m}| + |\mathbf{n}|) < p$ are in $\text{pr}(V_0)$. Set $m(\mathbf{0}, \mathbf{0}, \iota, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, 0) \in M_0$ to be such that $\iota + 2(|\mathbf{a}| + |\mathbf{b}|) + 3(|\mathbf{c}| + |\mathbf{d}|) = p$ and $\iota + |\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}| + |\mathbf{d}| = q$, and denote by $M_{p,q}$ the span of all $m(\mathbf{i}, \mathbf{j}, \iota, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, t)$ of Kazhdan degree p with $|\mathbf{i}| + |\mathbf{j}| + \iota + |\mathbf{k}| + |\mathbf{l}| + |\mathbf{m}| + |\mathbf{n}| + t > q$. Assume that all vectors $m(\mathbf{0}, \mathbf{0}, \iota, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, 0)$ of Kazhdan degree p with $\iota + |\mathbf{k}| + |\mathbf{l}| + |\mathbf{m}| + |\mathbf{n}| > q$ are in $\text{pr}(V_0)$. Since M is a filtrated $U(\mathfrak{g})$ -module, and also

$$v_{\frac{r+1}{2}}^2(1_\lambda) = \frac{1}{2} \cdot 1_\lambda, \quad y_i^2(1_\lambda) = \frac{1}{2} [y_i, y_i](1_\lambda), \quad g_j^2(1_\lambda) = \frac{1}{2} [g_j, g_j](1_\lambda)$$

for $1 \leq i \leq \ell$ and $1 \leq j \leq \frac{r-1}{2}$, then it follows from Theorem 2.4 that

$$\begin{aligned}
&\Theta_{v_{\frac{r+1}{2}}}^\iota \cdot \prod_{i=1}^w \Theta_{x_i}^{a_i} \cdot \prod_{i=1}^\ell \Theta_{y_i}^{b_i} \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i}^{c_i} \cdot \prod_{i=1}^{\frac{r-1}{2}} \Theta_{g_i}^{d_i}(1_\lambda) \\
(3.31) \quad &\in m(\mathbf{0}, \mathbf{0}, \iota, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, 0) + M_{p,q} + M^{p-1}.
\end{aligned}$$

By our assumptions on p and q we can obtain $m(\mathbf{0}, \mathbf{0}, \iota, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, 0) \in \text{pr}(V_0 + M_{p,q} + M^{p-1}) = \text{pr}(V_0)$. Then our claim follows by double induction on p and q . Recall that in (2) we established that $\text{pr}: \text{Wh}(M) \rightarrow M_0$ is injective; this yields $\text{Wh}(M) = V_0$.

(4) Applying (3.31) it is easy to observe that

$$\Theta_{v_{\frac{r+1}{2}}}^\iota \cdot \prod_{i=1}^w \Theta_{x_i}^{a_i} \cdot \prod_{i=1}^\ell \Theta_{y_i}^{b_i} \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i}^{c_i} \cdot \prod_{i=1}^{\frac{r-1}{2}} \Theta_{g_i}^{d_i}(1_\lambda)$$

with $\iota \in \Lambda_1$, $\mathbf{a} \in \mathbb{Z}_+^w$, $\mathbf{b} \in \Lambda_\ell$, $\mathbf{c} \in \mathbb{Z}_+^{\frac{s}{2}}$, and $\mathbf{d} \in \Lambda_{\frac{\epsilon-1}{2}}$ are linearly independent over $\mathbb{C}$. Hence it follows from Lemma 2.7, (1.13), and the discussion at the beginning of Section 2.2.3 that $V_0 \cong Z_{U(\mathfrak{g},e)}(\lambda + \bar{\delta}, -\frac{1}{8} + (\lambda + 2\bar{\rho}, \lambda) + \epsilon)$ as $U(\mathfrak{g}, e)$ -modules. $\square$

Remark 3.3. We guess that the shift $\epsilon \neq 0$ for all basic Lie superalgebras. For example, let $\mathfrak{g} = \mathfrak{osp}(1|2)$ be as in Lemma 3.2. Then $\mathfrak{g}(-1)_{\bar{1}} = \mathbb{C}F$, $s = \dim \mathfrak{g}(-1)_{\bar{0}} = 0$, $r = \dim \mathfrak{g}(-1)_{\bar{1}} = 1$, $F^* = -\frac{1}{2}F$, and $([E, E], f) = 2(e, f) = 2$. It follows from (2.6) that

$$\begin{aligned} c_0 &= \frac{1}{([E, E], f)} \left(\frac{1}{12} \left([[E, F], F], \left[-\frac{1}{2}F, \left[-\frac{1}{2}F, E \right] \right] \right) \right) \otimes 1_\chi - \frac{1}{12}([E, E], f) \\ &= \frac{1}{24} \left(\frac{1}{4} [[h, F], [F, h]] \otimes 1_\chi - 2(e, f) \right) \\ &= \frac{1}{24} \left(-\frac{1}{4} [F, F] \otimes 1_\chi - 2 \right) = \frac{1}{24} \left(\frac{1}{2} - 2 \right) = -\frac{1}{16}. \end{aligned}$$

As $\mathfrak{h}^e = 0$, we get $\epsilon = c_0 + \frac{1}{8} = -\frac{1}{16} + \frac{1}{8} = \frac{1}{16} \neq 0$. However, the complication for the calculation of c_0 in (2.6) makes it difficult to verify $\epsilon \neq 0$ in general.

§3.3.

This subsection is devoted to minimal finite W -superalgebras $U(\mathfrak{g}, e)$ of type even. All $\mathfrak{g}$ corresponding to this type are listed in Table 2. In this type, both $U(\mathfrak{g}, e)$ and W'_χ are identical.

To determine the composition factors of the Verma modules $Z_{U(\mathfrak{g},e)}(\lambda, c)$ with their multiplicities, we will express these $U(\mathfrak{g}, e)$ -modules in terms of the $\mathfrak{g}$ -modules obtained by parabolic induction from Whittaker modules for $\mathfrak{sl}(2)$. As mentioned in Section 3.2.1, the latter modules have been studied in much detail in [14, 15], and Chen–Cheng [15, Thm. 1] gave a complete solution to the problem of determining the composition factors of the standard Whittaker modules in terms of composition factors of Verma modules in the category $\mathcal{O}$. As a special case, if $\mathfrak{g}$ is a basic Lie superalgebra of type I (in other words, if $\mathfrak{g}$ is a simple Lie algebra, or $\mathfrak{sl}(m|n)$ with $m \neq n$, $m \geq 2$, or $\mathfrak{psl}(m|m)$ with $m \geq 2$, or $\mathfrak{spo}(2m|2)$; see Tables 1 and 2), then the corresponding minimal finite W -superalgebra $U(\mathfrak{g}, e)$ is of type even. It follows from [14, Thm. C] that the composition of standard Whittaker modules can be computed by some already known results (e.g., [6, 7, 8, 17, 18, 19]) on the irreducible characters of the BGG category $\mathcal{O}$.

Denote by $\mathfrak{s}_\theta$ the subalgebra of $\mathfrak{g}$ spanned by $(e, h, f) = (e_\theta, h_\theta, e_{-\theta})$, and put

$$\mathfrak{p}_\theta := \mathfrak{s}_\theta + \mathfrak{h} + \sum_{\alpha \in \Phi^+} \mathbb{C}e_\alpha, \quad \mathfrak{n}_\theta := \sum_{\alpha \in \Phi^+ \setminus \{\theta\}} \mathbb{C}e_\alpha, \quad \tilde{\mathfrak{s}}_\theta := \mathfrak{h}^e \oplus \mathfrak{s}_\theta.$$

It is obvious that $\mathfrak{p}_\theta = \tilde{\mathfrak{s}}_\theta \oplus \mathfrak{n}_\theta$ is a parabolic subalgebra of $\mathfrak{g}$ with nilradical $\mathfrak{n}_\theta$ and $\tilde{\mathfrak{s}}_\theta$ is a Levi subalgebra of $\mathfrak{p}_\theta$. Set $C_\theta := ef + fe + \frac{1}{2}h^2 = 2ef + \frac{1}{2}h^2 - h$ to be a Casimir element of $U(\mathfrak{s}_\theta)$. For $\lambda \in (\mathfrak{h}^e)^*$ and $c \in \mathbb{C}$, write $I_\theta(\lambda, c)$ for the left ideal of $U(\mathfrak{p}_\theta)$ generated by $f - 1$, $C_\theta - c$, all $t - \lambda(t)$ with $t \in \mathfrak{h}^e$, and all e_γ with $\gamma \in \Phi^+ \setminus \{\theta\}$.

Set $Y(\lambda, c) := U(\mathfrak{p}_\theta)/I_\theta(\lambda, c)$ to be a $\mathfrak{p}_\theta$ -module with the trivial action of $\mathfrak{n}_\theta$, and let $1_{\lambda, c}$ denote the image of 1 in $Y(\lambda, c)$. Since $f \cdot 1_{\lambda, c} = 1_{\lambda, c}$ by definition, then

$$e \cdot 1_{\lambda, c} = \frac{1}{2} \left(C_\theta - \frac{1}{2}h^2 + h \right) \cdot 1_{\lambda, c} = \left(-\frac{1}{4}h^2 + \frac{1}{2}h + \frac{1}{2}c \right) \cdot 1_{\lambda, c}.$$

Combining this with the PBW theorem, we see that the vectors $\{h^k \cdot 1_{\lambda, c} \mid k \in \mathbb{Z}_+\}$ form a $\mathbb{C}$ -basis of $Y(\lambda, c)$. Moreover, one can easily conclude that $Y(\lambda, c)$ is isomorphic to a Whittaker module for $\mathfrak{s}_\theta \cong \mathfrak{sl}(2)$.

It follows from the discussion above that the vectors

$$\begin{aligned} m(\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, t) &:= u_1^{i_1} \cdots u_{\frac{s}{2}}^{i_{\frac{s}{2}}} \cdot v_1^{j_1} \cdots v_{\frac{r}{2}}^{j_{\frac{r}{2}}} \cdot x_1^{k_1} \cdots x_w^{k_w} \cdot y_1^{l_1} \cdots y_\ell^{l_\ell} \\ &\quad \cdot f_1^{m_1} \cdots f_{\frac{s}{2}}^{m_{\frac{s}{2}}} \cdot g_1^{n_1} \cdots g_{\frac{r}{2}}^{n_{\frac{r}{2}}} \cdot h^t(1_{\lambda, c}) \end{aligned}$$

with $\mathbf{i}, \mathbf{m} \in \mathbb{Z}_+^{\frac{s}{2}}$, $\mathbf{j}, \mathbf{n} \in \Lambda_{\frac{r}{2}}$, $\mathbf{k} \in \mathbb{Z}_+^w$, $\mathbf{l} \in \Lambda_\ell$, and $t \in \mathbb{Z}_+$ form a $\mathbb{C}$ -basis of the induced $\mathfrak{g}$ -module

$$M(\lambda, c) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\theta)} Y(\lambda, c).$$

Put

$$\begin{aligned} \delta &= \frac{1}{2} \left(\sum_{i=1}^{\frac{s}{2}} \gamma_{\bar{0}i}^* - \sum_{i=1}^{\frac{r}{2}} \gamma_{\bar{1}i}^* \right) = \frac{1}{2} \left(\sum_{i=1}^{\frac{s}{2}} (-\theta - \gamma_{\bar{0}i}) + \sum_{i=1}^{\frac{r}{2}} (\theta + \gamma_{\bar{1}i}) \right) \\ &= \frac{1}{2} \left(-\sum_{i=1}^{\frac{s}{2}} \gamma_{\bar{0}i} + \sum_{i=1}^{\frac{r}{2}} \gamma_{\bar{1}i} \right) - \frac{s-r}{4} \theta, \\ (3.32) \quad \rho &= \frac{1}{2} \sum_{\alpha \in \Phi^+} (-1)^{|\alpha|} \alpha, \\ \rho_{e,0} &= \rho - 2\delta - \left(\frac{s-r}{4} + \frac{1}{2} \right) \theta = \frac{1}{2} \sum_{\alpha \in \Phi_{e,0}^+} (-1)^{|\alpha|} \alpha \\ &= \frac{1}{2} \left(\sum_{j=1}^w \beta_{\bar{0}j} - \sum_{j=1}^\ell \beta_{\bar{1}j} \right), \end{aligned}$$

where $\gamma_{\bar{0}i}^* \in \Phi_0^+$, $\gamma_{\bar{1}j}^* \in \Phi_1^+$, $\gamma_{\bar{0}i} \in \Phi_0^-$, $\gamma_{\bar{1}j} \in \Phi_1^-$ for $1 \leq i \leq \frac{s}{2}$ and $1 \leq j \leq \frac{r}{2}$ are defined in Section 1.2, $\beta_{\bar{0}i} \in \Phi_0^+$, $\beta_{\bar{1}j} \in \Phi_1^+$ for $1 \leq i \leq w$ and $1 \leq j \leq \ell$ are defined in Section 2.1.1, and $|\alpha|$ denotes the parity of α . For any $\eta \in (\mathfrak{h}^e)^*$, there exist a

unique t_η in $\mathfrak{h}^e$ with $\eta = (t_\eta, \cdot)$, and also a non-degenerate bilinear form on $(\mathfrak{h}^e)^*$ via $(\mu, \nu) := (t_\mu, t_\nu)$ for all $\mu, \nu \in (\mathfrak{h}^e)^*$. Given a linear function φ on $\mathfrak{h}$ we denote by $\bar{\varphi}$ the restriction of φ to $\mathfrak{h}^e$.

Now we have the following theorem, which is a parallel result to Theorem 0.2.

Theorem 3.4. *Assume that r is even. Every $\mathfrak{g}$ -module $M(\lambda, c)$ is an object of the category $\mathcal{C}_\chi$. Furthermore, $\text{Wh}(M(\lambda, c)) \cong Z_{U(\mathfrak{g}, e)}(\lambda + \bar{\delta}, c + (\lambda + 2\bar{\rho}, \lambda))$ as $U(\mathfrak{g}, e)$ -modules.*

Proof. The proof of the theorem is the same as that of Theorem 0.2, while the lack of the element $v_{\frac{r+1}{2}} \in \mathfrak{g}(-1)_{\bar{1}}$ here makes the discussion much easier. In particular, from Theorem 2.11 there is no restriction on $\lambda \in (\mathfrak{h}^e)^*$ and $c \in \mathbb{C}$. Then the action of $U(\mathfrak{g}, e)$ on Verma modules need not be twisted, so the shift $-\epsilon$ on C as in (3.5) is redundant. The proof will be omitted. $\square$

Remark 3.5. Here we omit the arguments on the minimal refined W -superalgebra W'_χ of type odd because there is a lack of Skryabin's equivalence for this case.

§4. On the category $\mathcal{O}$ for minimal finite W -superalgebras of type odd

All the discussion in previous sections is concentrated on the minimal finite (refined) W -superalgebras, for which the generators and their relationship are given explicitly, and their Verma modules are introduced, which is much like the highest weight theory for $U(\mathfrak{g})$. However, the theory cannot be applied directly in the general settings. We will manage to develop the BGG category $\mathcal{O}$ for minimal finite W -superalgebras $U(\mathfrak{g}, e)$, by exploiting the arguments in [12] on highest weight theory of finite W -algebras to the super case. During the expositions, we mainly follow the strategy in [12, §4], with a lot of modifications. It should be expected this is an effective attempt for the general situation.

In this section we will only consider minimal finite W -superalgebras of type odd. Then $\frac{\theta}{2}$ is an odd root of $\mathfrak{g}$.

§4.1.

Keep the notation used in previous sections. Recall that (e, h, f) is an $\mathfrak{sl}(2)$ -triple in $\mathfrak{g}_{\bar{0}}$, and $\mathfrak{g}^e$ is the centralizer of e in $\mathfrak{g}$. Write $\mathfrak{g}^h$ for the centralizer of h in $\mathfrak{g}$, which is equal to $\mathfrak{g}(0)$ by definition. Then $\mathfrak{g}^h \cap \mathfrak{g}^e = \mathfrak{g}^e(0) = \mathfrak{g}(0)^\sharp$ is a Levi factor of $\mathfrak{g}^e$, and $\mathfrak{h}^e = \mathfrak{h} \cap \mathfrak{g}^e$ is a Cartan subalgebra of this Levi factor. As in Section 2.1.1, $\{h_1, \dots, h_{k-1}\}$ is a basis of $\mathfrak{h}^e$, and $\mathfrak{g}^e(0)$ has a basis as in (2.3).

For $\alpha \in (\mathfrak{h}^e)^*$, let $\mathfrak{g}_\alpha = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_\alpha(i)$ denote the α -weight space of $\mathfrak{g}$ with respect to $\mathfrak{h}^e$. So

$$(4.1) \quad \mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi'_e} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_0$ is the centralizer of $\mathfrak{h}^e$ in $\mathfrak{g}$, and $\Phi'_e \subset (\mathfrak{h}^e)^*$ is the set of non-zero weights of $\mathfrak{h}^e$ on $\mathfrak{g}$. Since the eigenspace decomposition of $\text{ad } h$ gives rise to a short $\mathbb{Z}$ -grading of $\mathfrak{g}$ as in (2.1), and only $\theta(h) = 2$ by definition, it is immediate that for all $\alpha \in \Phi \setminus \{\pm\theta\}$ we have $\alpha(h) \in \{-1, 0, 1\}$. Keep in mind that $\mathfrak{h} = \mathfrak{h}^e \oplus \mathbb{C}h$; then $\mathfrak{g}_0$ is equal to $\tilde{\mathfrak{s}}_\theta$ as defined in (3.2). Since $\frac{\theta}{2} = \alpha_k$ is a simple root in $\Delta = \{\alpha_1, \dots, \alpha_k\}$ of Φ by Conventions 2.1, then $\Phi_k := \Phi \cap \mathbb{Z}\alpha_k = \{\pm\alpha_k, \pm 2\alpha_k\}$ is a closed subsystem of Φ with base $\Delta_k = \{\alpha_k\}$, which entails the following lemma.

Lemma 4.1. $\Phi'_e = (\Phi \setminus \{\pm\frac{\theta}{2}, \pm\theta\})|_{(\mathfrak{h}^e)^*} = (\Phi \setminus \Phi_k)|_{(\mathfrak{h}^e)^*}.$

Therefore, Φ'_e is a restricted root system in the same sense of the non-super case [10, §2], namely, the set of non-zero restrictions of roots $\alpha \in \Phi$ to $\mathfrak{h}^e$. It is not a root system in the usual sense; for example, since $\theta(\mathfrak{h}^e) = 0$, for $\alpha \in \Phi'_e$ there may be $\theta \pm \alpha$ that belong to Φ'_e . Then for $\alpha \in \Phi'_e$, we can write $\mathfrak{g}_\alpha = \bigoplus_{i=1}^{I(\alpha)} \mathbb{C}e_{\alpha,i}$, where $e_{\alpha,i}$ with $i \in I(\alpha)$ are the linear independent restricted root vectors which span $\mathfrak{g}_\alpha$. Denote by $(\Phi'_e)_{\bar{0}}$ and $(\Phi'_e)_{\bar{1}}$ the set of all restricted even roots and odd roots, respectively. Similarly, each of the spaces $\mathfrak{g}(-1)$, $\mathfrak{g}(0)$, and $\mathfrak{g}(1)$ decomposes into $\mathfrak{h}^e$ -weight spaces. There is an induced restricted root decomposition

$$(4.2) \quad \mathfrak{g}^e = \mathfrak{g}_0^e \oplus \bigoplus_{\alpha \in \Phi'_e} \mathfrak{g}_\alpha^e$$

of the centralizer $\mathfrak{g}^e$, where $\mathfrak{g}_0^e$ is the centralizer of e in $\mathfrak{g}_0$, and $\mathfrak{g}_\alpha^e$ is the centralizer of e in $\mathfrak{g}_\alpha$. Writing $\tilde{\mathfrak{s}}_\theta^e$ for the centralizer of e in $\tilde{\mathfrak{s}}_\theta$, the next lemma follows from (3.2).

Lemma 4.2. $\mathfrak{g}_0^e = \tilde{\mathfrak{s}}_\theta^e = \mathfrak{h}^e \oplus \mathbb{C}e \oplus \mathbb{C}E.$

Moreover, by $\mathfrak{sl}(2)$ -representation theory we have $\mathfrak{g}^e \in \bigoplus_{i \geq 0} \mathfrak{g}(i)$, thus the second summands in (4.2) can also be considered as chosen in $\Phi_e \setminus \{\frac{\theta}{2}\}$ as defined in Section 2.1.1. By the same discussion as in [10, Lem. 13], $\Phi_e \setminus \{\frac{\theta}{2}\}$ is also the set of non-zero weights of $\mathfrak{h}^e$ on $\mathfrak{g}^e$, so all the subspaces $\mathfrak{g}_\alpha^e = \bigoplus_{i \geq 0} \mathfrak{g}_\alpha^e(i)$ in this decomposition are non-zero.

Recall that the restricted root system Φ'_e is the set of non-zero weights of $\mathfrak{h}^e$ on $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi'_e} \mathfrak{g}_\alpha$, and the zero weight space $\mathfrak{g}_0$ is the centralizer of the toral subalgebra $\mathfrak{h}^e$ in $\mathfrak{g}$, so it is a Levi factor of a parabolic subalgebra of $\mathfrak{g}$. By Bala–Carter theory [13, Props 5.9.3–5.9.4], e is a distinguished nilpotent element of $(\mathfrak{g}_0)_{\bar{0}}$, i.e., the only semi-simple elements of $(\mathfrak{g}_0)_{\bar{0}}$ that centralize e belong to

the center of $(\mathfrak{g}_0)_{\bar{0}}$, and h, f also lie in $(\mathfrak{g}_0)_{\bar{0}}$. Moreover, by [13, Prop. 5.7.6] the grading of $(\mathfrak{g}_0)_{\bar{0}}$ under the action of $\text{ad } h$ is even, i.e., $(\mathfrak{g}_0(-1))_{\bar{0}} = (\mathfrak{g}_0(1))_{\bar{0}} = 0$. In our case, we have $\mathfrak{g}_0 = \tilde{\mathfrak{s}}_\theta$ is a Levi factor of the parabolic subalgebra $\mathfrak{p}_\theta$ (defined in (3.2)) of $\mathfrak{g}$, and $(\mathfrak{g}_0)_{\bar{0}} = \mathfrak{h}^e \oplus \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$.

§4.2.

For the Lie superalgebra $\mathfrak{g}_0 = \tilde{\mathfrak{s}}_\theta = \mathfrak{h}^e \oplus \mathfrak{s}_\theta$ as in Section 3.2.2, one can observe that it is also a direct sum decomposition of ideals of $\mathfrak{g}_0$ with $\mathfrak{h}^e$ being abelian, and $\mathfrak{s}_\theta \cong \mathfrak{osp}(1|2)$ by Lemma 3.2. Let $\mathfrak{m}_0 := \mathbb{C}f$ be the χ -admissible subalgebra of $\mathfrak{g}_0$, and define the corresponding extended χ -admissible subalgebra by $\mathfrak{m}'_0 := \mathbb{C}F \oplus \mathbb{C}f$. Define the generalized Gelfand–Graev $\mathfrak{g}_0$ -module associated with χ by $(Q_0)_\chi := U(\mathfrak{g}_0) \otimes_{U(\mathfrak{m}_0)} \mathbb{C}_\chi$, where $\mathbb{C}_\chi = \mathbb{C}1_\chi$ is a one-dimensional $\mathfrak{m}_0$ -module such that $x \cdot 1_\chi = \chi(x)1_\chi$ for all $x \in \mathfrak{m}_0$. Let $(I_0)_\chi$ denote the $\mathbb{Z}_2$ -graded left ideal in $U(\mathfrak{g}_0)$ generated by all $x - \chi(x)$ with $x \in \mathfrak{m}_0$, and write $\text{Pr}_0: U(\mathfrak{g}_0) \rightarrow U(\mathfrak{g}_0)/(I_0)_\chi$ for the canonical homomorphism. Now we can define the finite W -superalgebra $U(\mathfrak{g}_0, e)$ associated to $e \in \mathfrak{g}_0$ by

$$U(\mathfrak{g}_0, e) := (\text{End}_{\mathfrak{g}_0}(Q_0)_\chi)^{\text{op}} \cong (Q_0)_\chi^{\text{ad } \mathfrak{m}_0}.$$

Recall that $-\theta$ is a minimal root. As $e_\theta \in \mathfrak{g}_0$ is a root vector for θ , $U(\mathfrak{g}_0, e)$ is also a minimal finite W -superalgebra, which plays a role similar to a “Cartan subalgebra” in the classical BGG category. This will be important in the formulation of our BGG category $\mathcal{O}$ for $U(\mathfrak{g}, e)$.

4.2.1. Let us first look at the structure of $U(\mathfrak{g}_0, e)$. We have the following proposition.

Proposition 4.3. *The minimal finite W -superalgebra $U(\mathfrak{g}_0, e)$ is generated by*

- (1) $\Theta'_{h_i} = h_i \otimes 1_\chi$ for $1 \leq i \leq k-1$;
- (2) $\Theta'_E = (E + \frac{1}{2}Fh - \frac{3}{4}F) \otimes 1_\chi$;
- (3) $C'_\theta = (2e + \frac{1}{2}h^2 - \frac{3}{2}h + FE) \otimes 1_\chi$;
- (4) $\Theta'_F = F \otimes 1_\chi$;

subject to the following relations:

- (i) $[\Theta'_E, \Theta'_E] = C'_\theta + \frac{1}{8} \otimes 1_\chi$;
- (ii) $[\Theta'_F, \Theta'_F] = -2 \otimes 1_\chi$;

and the commutators between the other generators are all zero.

Proof. The proposition comes as a special case of Theorem 2.4. For the generators of $U(\mathfrak{g}_0, e)$, we can obtain (1), (2), and (4) by direct computation. The element C_θ

in (3.3) is the Casimir element of $\mathfrak{s}_\theta$, then $C'_\theta = \text{Pr}_0(C_\theta)$ is in the center of $U(\mathfrak{g}_0, e)$. Since $\mathfrak{g}_0^e = \mathfrak{h}^e \oplus \mathbb{C}e \oplus \mathbb{C}E$ by Lemma 4.2, then the first part of the proposition follows.

For the second part of the proposition, the commutators of these generators can be calculated directly. In particular, $[\Theta'_E, \Theta'_E]$ has been calculated in (3.18). $\square$

4.2.2. We can describe the structure of the center of $U(\mathfrak{g}_0, e)$ as follows.

Proposition 4.4. *The center $Z(U(\mathfrak{g}_0, e))$ of the minimal finite W -superalgebra $U(\mathfrak{g}_0, e)$ is generated by Θ'_{h_i} for $1 \leq i \leq k-1$ and C'_θ .*

Proof. One can easily conclude from Proposition 4.3 that C'_θ and Θ'_t with $t \in \mathfrak{h}^e$ lie in the center of $U(\mathfrak{g}_0, e)$. On the other hand, assume that

$$(4.3) \quad C' := \sum_{\mathbf{a}} \lambda_{\mathbf{a}} (\Theta'_F)^{a_1} (\Theta'_t)^{\mathbf{a}_2} (C'_\theta)^{a_3} (\Theta'_E)^{a_4} \in Z(U(\mathfrak{g}_0, e))$$

for $\lambda_{\mathbf{a}} \in \mathbb{C}$ with $\mathbf{a} = (a_1, \mathbf{a}_2, a_3, a_4) \in \Lambda_1 \times \mathbb{Z}_+^{k-1} \times \mathbb{Z}_+ \times \Lambda_1$ such that $(\Theta'_t)^{\mathbf{a}_2} := (\Theta'_{h_1})^{a_{21}} \cdots (\Theta'_{h_{k-1}})^{a_{2,k-1}}$ is the linear span of the PBW basis of $U(\mathfrak{g}_0, e)$. In virtue of Proposition 4.3, we get

$$\begin{aligned} 0 &= \left[\sum_{\mathbf{a}} \lambda_{\mathbf{a}} (\Theta'_F)^{a_1} (\Theta'_t)^{\mathbf{a}_2} (C'_\theta)^{a_3} (\Theta'_E)^{a_4}, \Theta'_F \right] \\ &= \left[\sum_{\mathbf{a}_2, a_3} \lambda_{(0, \mathbf{a}_2, a_3, 0)} (\Theta'_t)^{\mathbf{a}_2} (C'_\theta)^{a_3}, \Theta'_F \right] \\ &\quad + \left[\sum_{\mathbf{a}_2, a_3} \lambda_{(1, \mathbf{a}_2, a_3, 0)} \Theta'_F (\Theta'_t)^{\mathbf{a}_2} (C'_\theta)^{a_3}, \Theta'_F \right] \\ &\quad + \left[\sum_{\mathbf{a}_2, a_3} \lambda_{(0, \mathbf{a}_2, a_3, 1)} (\Theta'_t)^{\mathbf{a}_2} (C'_\theta)^{a_3} \Theta'_E, \Theta'_F \right] \\ &\quad + \left[\sum_{\mathbf{a}_2, a_3} \lambda_{(1, \mathbf{a}_2, a_3, 1)} \Theta'_F (\Theta'_t)^{\mathbf{a}_2} (C'_\theta)^{a_3} \Theta'_E, \Theta'_F \right] \\ &= \sum_{\mathbf{a}_2, a_3} \lambda_{(1, \mathbf{a}_2, a_3, 0)} [\Theta'_F, \Theta'_F] (\Theta'_t)^{\mathbf{a}_2} (C'_\theta)^{a_3} \\ &\quad - \sum_{\mathbf{a}_2, a_3} \lambda_{(1, \mathbf{a}_2, a_3, 1)} [\Theta'_F, \Theta'_F] (\Theta'_t)^{\mathbf{a}_2} (C'_\theta)^{a_3} \Theta'_E \\ (4.4) \quad &= -2 \sum_{\mathbf{a}_2, a_3} \lambda_{(1, \mathbf{a}_2, a_3, 0)} (\Theta'_t)^{\mathbf{a}_2} (C'_\theta)^{a_3} + 2 \sum_{\mathbf{a}_2, a_3} \lambda_{(1, \mathbf{a}_2, a_3, 1)} (\Theta'_t)^{\mathbf{a}_2} (C'_\theta)^{a_3} \Theta'_E. \end{aligned}$$

So all the coefficients $\lambda_{\mathbf{a}}$ with $a_1 = 1$ in (4.3) equal zero. Taking this into consideration, by the same discussion as in (4.4) we have

$$(4.5) \quad \begin{aligned} 0 &= \left[\sum_{\mathbf{a}} \lambda_{\mathbf{a}} (\Theta'_F)^{a_1} (\Theta'_t)^{a_2} (C'_\theta)^{a_3} (\Theta'_E)^{a_4}, \Theta'_E \right] \\ &= \sum_{\mathbf{a}_2, a_3} \lambda_{(0, \mathbf{a}_2, a_3, 1)} (\Theta'_t)^{a_2} (C'_\theta)^{a_3+1} + \frac{1}{8} \sum_{\mathbf{a}_2, a_3} \lambda_{(0, \mathbf{a}_2, a_3, 1)} (\Theta'_t)^{a_2} (C'_\theta)^{a_3}. \end{aligned}$$

If there exists some $\lambda_{(0, \mathbf{a}_2, a_3, 1)} \neq 0$ in (4.3), set $a'_3 \in \mathbb{Z}_+$ to be the largest number with this property. Then we have $\sum_{\mathbf{a}_2} \lambda_{(0, \mathbf{a}_2, a'_3, 1)} (\Theta'_t)^{a_2} (C'_\theta)^{a'_3+1} = 0$ by (4.5), which means that $\lambda_{(0, \mathbf{a}_2, a'_3, 1)} = 0$, a contradiction. Combining this with our earlier discussion, we see that the coefficients $\lambda_{\mathbf{a}}$ with $a_1 \neq 0$ or $a_4 \neq 0$ in (4.3) are all zeros. Then any element in $Z(U(\mathfrak{g}_0, e))$ can be written as a linear span of $(\Theta'_{h_1})^{a_{21}} \dots (\Theta'_{h_{k-1}})^{a_{2k-1}} (C'_\theta)^{a_3}$, completing the proof. $\square$

Let $Z(U(\mathfrak{g}_0))$ denote the center of $U(\mathfrak{g}_0)$. The canonical homomorphism $\text{Pr}_0: U(\mathfrak{g}_0) \twoheadrightarrow U(\mathfrak{g}_0)/(I_0)_\chi$ we introduced earlier induces an algebra homomorphism from $Z(U(\mathfrak{g}_0))$ to $Z(U(\mathfrak{g}_0, e))$. In fact, we further have the following proposition.

Proposition 4.5. *The map Pr_0 sends $Z(U(\mathfrak{g}_0))$ isomorphically onto the center of $U(\mathfrak{g}_0, e)$.*

Proof. Recall that $\mathfrak{g}_0 = \mathfrak{h}^e \oplus \mathfrak{s}_\theta$ is a direct sum decomposition of ideals, with $\mathfrak{h}^e$ being abelian. It is well known that the center of $U(\mathfrak{osp}(1|2))$ is generated by its Casimir element, and $\mathfrak{s}_\theta \cong \mathfrak{osp}(1|2)$ by our earlier remark. Then $Z(U(\mathfrak{s}_\theta))$ is also generated by the Casimir element C_θ defined in (3.3). Now we conclude that $Z(U(\mathfrak{g}_0))$ is generated by the algebraically independent elements $h_1, \dots, h_{k-1}$ and C_θ .

By the definition of the map Pr_0 , it is readily checked that

$$\begin{aligned} \text{Pr}_0(h_i) &= h_i \otimes 1_\chi = \Theta'_{h_i} \quad \text{for } 1 \leq i \leq k-1, \\ \text{Pr}_0(C_\theta) &= \left(2ef + \frac{1}{2}h^2 - \frac{3}{2}h + FE \right) \otimes 1_\chi = \left(2e + \frac{1}{2}h^2 - \frac{3}{2}h + FE \right) \otimes 1_\chi = C'_\theta. \end{aligned}$$

Since Θ'_{h_i} for $1 \leq i \leq k-1$ and C'_θ are also algebraically independent, then the proposition follows from Proposition 4.4. $\square$

4.2.3. We will describe the finite-dimensional irreducible modules for $U(\mathfrak{g}_0, e)$, which will be used in a later discussion. Let $V_\lambda := \mathbb{C}v_\lambda \oplus \mathbb{C}\Theta'_F(v_\lambda)$ with $\lambda \in (\mathfrak{h}^e)^*$ be a vector space spanned by $v_\lambda \in (V_\lambda)_0$ and $\Theta'_F(v_\lambda) \in (V_\lambda)_1$ satisfying $\Theta'_E(v_\lambda) = 0$, and $\Theta'_E \cdot \Theta'_F(v_\lambda) = 0$, $\Theta'_F \cdot \Theta'_F(v_\lambda) = -v_\lambda$, $C'_\theta(v_\lambda) = -\frac{1}{8}v_\lambda$, $C'_\theta \cdot \Theta'_F(v_\lambda) =$

$-\frac{1}{8}\Theta'_F(v_\lambda)$ (the above four equations are derived from Proposition 4.3(i)–(ii)), $\Theta'_t(v_\lambda) = \lambda(t)v_\lambda$, $\Theta'_t \cdot \Theta'_F(v_\lambda) = \lambda(t)\Theta'_F(v_\lambda)$ for all $t \in \mathfrak{h}^e$. In fact, we have the following theorem.

Theorem 4.6. *The set $\{V_\lambda \mid \lambda \in (\mathfrak{h}^e)^*\}$ forms a complete set of pairwise inequivalent finite-dimensional irreducible $U(\mathfrak{g}_0, e)$ -modules (up to parity switch), all of which are of type Q .*

Proof. Due to Proposition 4.3 and the fact that $(\Theta'_F)^2 = \frac{1}{2}[\Theta'_F, \Theta'_F] = -1 \otimes 1_\chi$, it is readily checked that V_λ is an irreducible $U(\mathfrak{g}_0, e)$ -module. Obviously, V_λ is a simple module of type Q , for which the odd endomorphism is induced by the element Θ'_F .

If the simple $U(\mathfrak{g}_0, e)$ -modules V_λ and $V_{\lambda'}$ are isomorphic, then by parity considerations $\mathbb{C}v_\lambda \cong \mathbb{C}v_{\lambda'}$ as modules over the commutative subalgebra $\Theta'_{\mathfrak{h}^e}$ of $U(\mathfrak{g}_0, e)$. So we have $\lambda = \lambda'$.

Let M be a finite-dimensional simple $U(\mathfrak{g}_0, e)$ -module. By the same discussion as the first two paragraphs in step (3) of the proof of Theorem 0.1, M decomposes into weight spaces relative to $\Theta_{\mathfrak{h}^e}$, and it contains at least one maximal weight element, which we call μ . For a non-zero vector m in M_μ , we have $\Theta'_t(m) = \mu(t)m$ for $t \in \mathfrak{h}^e$. Since M is finite-dimensional, we can further assume that $\Theta'_E(m) = 0$. Then there must exist a $U(\mathfrak{g}_0, e)$ -module homomorphism ξ from either V_μ or $\prod V_\mu$ (here $\prod$ denotes the parity switching functor) to M such that $\xi(v_\mu) = m$. Moreover, we have $\Theta'_E \cdot \Theta'_F(m) = 0$ by Proposition 4.3. Then the simplicity of M entails that ξ is surjective, thus also injective by the knowledge of linear algebras. $\square$

Remark 4.7. In [36, Lem. 3.4], Poletaeva–Serganova gave another description of the PBW theorem and the irreducible representations of $U(\mathfrak{osp}(1|2), e)$ with e being regular nilpotent (in this case we can also write $e = e_\theta$ with $-\theta$ being a minimal root). Since $\mathfrak{g}_0 \cong \mathfrak{h}^e \oplus \mathfrak{osp}(1|2)$ as a direct sum decomposition of ideals, one can compare their results with Proposition 4.3 and Theorem 4.6.

The finite W -superalgebra $U(\mathfrak{g}_0, e)$ is going to play the role of a Cartan subalgebra in the highest theory. However, just like the non-super case, it does not embed obviously as a subalgebra of the minimal finite W -superalgebra $U(\mathfrak{g}, e)$; instead we will realize it in another way, which is also different from the one applied in the non-super case. We will put it in the next subsection.

§4.3.

Let us turn back to the minimal finite W -superalgebra $U(\mathfrak{g}, e)$. Let $(\Phi'_e)^+ := \Phi^+ \setminus \{\frac{\theta}{2}, \theta\}$ be a system of positive roots in the restricted root system Φ'_e . Setting

$(\Phi'_e)^- := -(\Phi'_e)^+$, we define $\mathfrak{g}_\pm := \bigoplus_{\alpha \in (\Phi'_e)^\pm} \mathfrak{g}_\alpha$, so that

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+, \quad \mathfrak{q} = \mathfrak{g}_0 \oplus \mathfrak{g}_+.$$

The choice $(\Phi'_e)^+$ of positive roots induces a dominance ordering $\leq$ on $(\mathfrak{h}^e)^*$: $\mu \leq \lambda$ if $\lambda - \mu \in \mathbb{Z}_{\geq 0}(\Phi'_e)^+$, which is exactly the one we defined in (2.9). Furthermore, since $\theta(\mathfrak{h}^e) = 0$, and $\frac{\theta}{2}$ is a simple root in Φ by Conventions 2.1, we can always assume that $\mu = \lambda$ on $(\mathfrak{h}^e)^*$ when $\lambda = \mu + k\theta$ for some $k \in \mathbb{C}$. Denote $(\Phi'_e)_0^+ := (\Phi'_e)^+ \cap (\Phi'_e)_0$ and $(\Phi'_e)_1^+ := (\Phi'_e)^+ \cap (\Phi'_e)_1$, respectively.

In this paragraph, we write $\mathfrak{a}$ for $\mathfrak{g}$ or $\mathfrak{g}^e$. Recall from Section 4.1 that the adjoint actions of $\mathfrak{h}^e$ on $\mathfrak{a}$ and its universal enveloping algebra $U(\mathfrak{a})$ induce decompositions $\mathfrak{a} = \mathfrak{a}_0 \oplus \bigoplus_{\alpha \in \Phi'_e} \mathfrak{a}_\alpha$ and $U(\mathfrak{a}) = \bigoplus_{\alpha \in \mathbb{Z}\Phi'_e} U(\mathfrak{a})_\alpha$. In particular, $U(\mathfrak{a})_0$, the zero weight space of $U(\mathfrak{a})$ with respect to the adjoint action, is a subalgebra of $U(\mathfrak{a})$. Let $U(\mathfrak{a})_\#$ (resp. $U(\mathfrak{a})_b$) denote the left (resp. right) ideal of $U(\mathfrak{a})$ generated by the root spaces $\mathfrak{a}_\alpha$ for $\alpha \in (\Phi'_e)^+$ (resp. $\alpha \in (\Phi'_e)^-$). Let

$$U(\mathfrak{a})_{0,\#} := U(\mathfrak{a})_0 \cap U(\mathfrak{a})_\#, \quad U(\mathfrak{a})_{b,0} := U(\mathfrak{a})_b \cap U(\mathfrak{a})_0,$$

which are left and right ideals of $U(\mathfrak{a})_0$, respectively. By the PBW theorem for Lie superalgebras, we actually have that $U(\mathfrak{a})_{0,\#} = U(\mathfrak{a})_{b,0}$, hence $U(\mathfrak{a})_{0,\#}$ is a two-sided ideal of $U(\mathfrak{a})_0$. Moreover, $\mathfrak{a}_0$ is a subalgebra of $\mathfrak{a}$, and by the PBW theorem again we have that $U(\mathfrak{a})_0 = U(\mathfrak{a}_0) \oplus U(\mathfrak{a})_{0,\#}$. The projection along this decomposition defines a surjective algebra homomorphism

$$(4.6) \quad \pi: U(\mathfrak{a})_0 \twoheadrightarrow U(\mathfrak{a}_0)$$

with $\ker \pi = U(\mathfrak{a})_{0,\#}$. Hence $U(\mathfrak{a})_0/U(\mathfrak{a})_{0,\#} \cong U(\mathfrak{a}_0)$.

Recall that in Sections 1.2 and 2.1.1 we chose a basis consisting of $\mathfrak{h}^e$ -weight vectors

$$(4.7) \quad \begin{aligned} & x_1, \dots, x_w, y_1, \dots, y_\ell, f_1, \dots, f_{\frac{s}{2}}, g_1, \dots, g_{\frac{r-1}{2}}, h_1, \dots, h_{k-1}, \\ & e, [v_{\frac{r+1}{2}}, e], f_1^*, \dots, f_{\frac{s}{2}}^*, g_1^*, \dots, g_{\frac{r-1}{2}}^*, x_1^*, \dots, x_w^*, y_1^*, \dots, y_\ell^* \end{aligned}$$

of $\mathfrak{g}^e$ so that the weights of x_i, y_j, f_k, g_l are respectively $-\beta_{0i}, -\beta_{1j}, \theta + \gamma_{0k}, \theta + \gamma_{1l} \in (\Phi'_e)^-$, and the weights of $f_k^*, g_l^*, x_i^*, y_j^*$ are respectively $\theta + \gamma_{0k}^*, \theta + \gamma_{1l}^*, \beta_{0i}, \beta_{1j} \in (\Phi'_e)^+$, while $h_i, e, [v_{\frac{r+1}{2}}, e] \in \mathfrak{g}_0^e$. Moreover, we have the following PBW basis for $U(\mathfrak{g}, e)$:

$$(4.8) \quad \begin{aligned} & \prod_{i=1}^w \Theta_{x_i}^{a_i} \cdot \prod_{i=1}^\ell \Theta_{y_i}^{c_i} \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i}^{m_i} \cdot \prod_{i=1}^{\frac{r-1}{2}} \Theta_{g_i}^{p_i} \cdot \Theta_{v_{\frac{r+1}{2}}}^t \cdot \prod_{i=1}^{k-1} \Theta_{h_i}^{t_i} \\ & \cdot C^{t_k} \cdot \Theta_{[v_{\frac{r+1}{2}}, e]}^\varepsilon \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i^*}^{n_i} \cdot \prod_{i=1}^{\frac{r-1}{2}} \Theta_{g_i^*}^{q_i} \cdot \prod_{i=1}^w \Theta_{x_i^*}^{b_i} \cdot \prod_{i=1}^\ell \Theta_{y_i^*}^{d_i}, \end{aligned}$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_+^w$, $\mathbf{c}, \mathbf{d} \in \Lambda_\ell$, $\mathbf{m}, \mathbf{n} \in \mathbb{Z}_+^{\frac{s}{2}}$, $\mathbf{p}, \mathbf{q} \in \Lambda_{\frac{r-1}{2}}$, $\iota, \varepsilon \in \Lambda_1$, $\mathbf{t} \in \mathbb{Z}_+^k$. Let v be any element in (4.7) excluding e , or let $v = v_{\frac{r+1}{2}}$. Since $\theta(\mathfrak{h}^e) = 0$ by definition, from the explicit description of Θ_v in Theorem 2.2, we see that v and Θ_v have the same $\mathfrak{h}^e$ -weight. Also note that the $\mathfrak{h}^e$ -weight of C is zero. Then the subspace $U(\mathfrak{g}, e)_\alpha$ in the restricted root space decomposition has a basis given by all the PBW monomials as in (4.8) such that $\sum_i (-a_i + b_i)\beta_{\bar{0}i} + \sum_i (-c_i + d_i)\beta_{\bar{1}i} + \sum_i (m_i - n_i)\gamma_{\bar{0}i} + \sum_i (p_i - q_i)\gamma_{\bar{1}i} = \alpha$.

Set $U(\mathfrak{g}, e)_\#$ (resp. $U(\mathfrak{g}, e)_b$) to be the left (resp. right) ideal of $U(\mathfrak{g}, e)$ generated by

$$\begin{aligned} & \Theta f_1^*, \dots, \Theta f_{\frac{s}{2}}^*, \Theta g_1^*, \dots, \Theta g_{\frac{r-1}{2}}^*, \Theta x_1^*, \dots, \Theta x_w^*, \Theta y_1^*, \dots, \Theta y_\ell^* \\ & \text{(resp. } \Theta x_1, \dots, \Theta x_w, \Theta y_1, \dots, \Theta y_\ell, \Theta f_1, \dots, \Theta f_{\frac{s}{2}}, \Theta g_1, \dots, \Theta g_{\frac{r-1}{2}} \text{)}. \end{aligned}$$

Note that $U(\mathfrak{g}, e)_\#$ (resp. $U(\mathfrak{g}, e)_b$) is equivalently the left (resp. right) ideal of $U(\mathfrak{g}, e)$ generated by all $U(\mathfrak{g}, e)_\alpha$ for $\alpha \in (\Phi'_e)^+$ (resp. $\alpha \in (\Phi'_e)^-$), and it does not depend on the explicit choice of the basis. Set

$$U(\mathfrak{g}, e)_{0,\#} := U(\mathfrak{g}, e)_0 \cap U(\mathfrak{g}, e)_\#, \quad U(\mathfrak{g}, e)_{b,0} := U(\mathfrak{g}, e)_b \cap U(\mathfrak{g}, e)_0,$$

which are obviously left and right ideals of the zero weight space $U(\mathfrak{g}, e)_0$, respectively. The PBW basis of $U(\mathfrak{g}, e)_\#$ (resp. $U(\mathfrak{g}, e)_b$) is the monomials as in (4.8) with $(\mathbf{n}, \mathbf{q}, \mathbf{b}, \mathbf{d}) \neq \mathbf{0}$ (resp. $(\mathbf{a}, \mathbf{c}, \mathbf{m}, \mathbf{p}) \neq \mathbf{0}$), and the PBW basis of $U(\mathfrak{g}, e)_0$ is the monomials as in (4.8) with $\sum_i (-a_i + b_i)\beta_{\bar{0}i} + \sum_i (-c_i + d_i)\beta_{\bar{1}i} + \sum_i (m_i - n_i)\gamma_{\bar{0}i} + \sum_i (p_i - q_i)\gamma_{\bar{1}i} = 0$. We also have $U(\mathfrak{g}, e)_{0,\#} = U(\mathfrak{g}, e)_{b,0}$ by the PBW theorem, hence it is a two-sided ideal of $U(\mathfrak{g}, e)_0$. However, the cosets of the PBW monomials of the form $\Theta_{v_{\frac{r+1}{2}}}^\iota \cdot \prod_{i=1}^{k-1} \Theta_{h_i}^{t_i} \cdot C^{t_k} \cdot \Theta_{[v_{\frac{r+1}{2}}, e]}^\varepsilon$ need not span a subalgebra of $U(\mathfrak{g}, e)$, unlike the situation for the algebras $U(\mathfrak{a})$ discussed earlier.

The goal now is to prove the quotient algebra $U(\mathfrak{g}, e)_0/U(\mathfrak{g}, e)_{0,\#}$ is canonically isomorphic to $U(\mathfrak{g}_0, e)$. As in Section 3.2.3, for a linear function φ on $\mathfrak{h}$ we still denote by $\bar{\varphi}$ the restriction of φ to $\mathfrak{h}^e$. Recall that in (2.7) and (3.4) we denote

$$\begin{aligned} \bar{\delta} &= \frac{1}{2} \left(\sum_{i=1}^{\frac{s}{2}} \gamma_{\bar{0}i}^* - \sum_{i=1}^{\frac{r-1}{2}} \gamma_{\bar{1}i}^* \right) \Big|_{\mathfrak{h}^e} = \frac{1}{2} \left(- \sum_{i=1}^{\frac{s}{2}} \gamma_{\bar{0}i} + \sum_{i=1}^{\frac{r-1}{2}} \gamma_{\bar{1}i} \right) \Big|_{\mathfrak{h}^e}, \\ \bar{\rho}_{e,0} &= \frac{1}{2} \left(\sum_{i=1}^w \beta_{\bar{0}i} - \sum_{i=1}^\ell \beta_{\bar{1}i} \right) \Big|_{\mathfrak{h}^e}, \\ \epsilon &= c_0 + \frac{1}{8} + \sum_{i=1}^{k-1} (2\rho_{e,0}\delta + 3\delta^2)(h_i), \end{aligned}$$

where $\gamma_{\bar{0}i}^* \in \Phi_0^+$, $\gamma_{\bar{1}j}^* \in \Phi_1^+$, $\gamma_{\bar{0}i} \in \Phi_0^-$, $\gamma_{\bar{1}j} \in \Phi_1^-$ for $1 \leq i \leq \frac{s}{2}$ and $1 \leq j \leq \frac{r-1}{2}$ are defined in Section 1.2, $\beta_{\bar{0}i} \in \Phi_0^+$, $\beta_{\bar{1}j} \in \Phi_1^+$ for $1 \leq i \leq w$ and $1 \leq j \leq \ell$ are

defined in Section 2.1.1, and $|\alpha|$ denotes the parity of α . Now we introduce a shift S_ϵ on $U(\mathfrak{g}_0, e)$ by keeping the other generators as in Proposition 4.3 invariant and sending C'_θ to $C'_\theta + \epsilon$.

Keeping the notation above, we have the following proposition.

Proposition 4.8. *The projection $\pi: U(\mathfrak{g})_0 \twoheadrightarrow U(\mathfrak{g}_0)$ as in (4.6) induces a surjective homomorphism*

$$\pi: U(\mathfrak{g}, e)_0 \twoheadrightarrow U(\mathfrak{g}_0, e)$$

with $\ker \pi = U(\mathfrak{g}, e)_{0, \#}$. Moreover, there exists an algebra isomorphism

$$\pi_\epsilon := S_\epsilon \circ \pi: U(\mathfrak{g}, e)_0 / U(\mathfrak{g}, e)_{0, \#} \cong U(\mathfrak{g}_0, e).$$

Proof. (1) Under the linear projection π , we first discuss the images of $U(\mathfrak{g}, e)_0$ with respect to the PBW basis of minimal finite W -superalgebras $U(\mathfrak{g}, e)$.

We consider the generators of $U(\mathfrak{g}, e)$ as in Theorem 2.4. It is obvious that $\pi(\Theta_{v_{\frac{r+1}{2}}}) = \pi(v_{\frac{r+1}{2}}) = v_{\frac{r+1}{2}} \otimes 1_\chi$. For $1 \leq j \leq k-1$, note that

$$\pi(u_i[u_i^*, h_j]) = \pi(v_t[v_t^*, h_j]) = 0 \quad \text{for } 1 \leq i \leq \frac{s}{2}, 1 \leq t \leq \frac{r+1}{2},$$

and

$$\pi([u_i^*, h_j]u_i) = \pi([v_t^*, h_j]v_t) = 0 \quad \text{for } \frac{s}{2} + 1 \leq i \leq s, \frac{r+3}{2} \leq t \leq r,$$

and also $\pi(h_j) = h_j$, $\pi(e) = e$, $\pi(h) = h$, $\pi(f) = f$ by definition. Then we have

$$\begin{aligned} \pi(\Theta_{h_j}) &= \pi(h_j) - \frac{1}{2} \left(\sum_{i=\frac{s}{2}+1}^s \pi([u_i, [u_i^*, h_j]]) + \sum_{i=\frac{r+3}{2}}^r \pi([v_i, [v_i^*, h_j]]) \right) \\ &= \pi(h_j) + \frac{1}{2} \sum_{i=1}^{\frac{s}{2}} \pi([u_i^*, [u_i, h_j]]) - \frac{1}{2} \sum_{i=1}^{\frac{r-1}{2}} \pi([v_i^*, [v_i, h_j]]) \\ &= \left(h_j + \frac{1}{2} (-\gamma_{01} - \cdots - \gamma_{0\frac{s}{2}} + \gamma_{11} + \cdots + \gamma_{1\frac{r-1}{2}})(h_j) \right) \otimes 1_\chi \\ (4.9) \quad &= (h_j + \delta(h_j)) \otimes 1_\chi. \end{aligned}$$

By a similar calculation to (3.7), we obtain

$$\begin{aligned} \pi(\Theta_{[v_{\frac{r+1}{2}}, e]}) &= \pi([v_{\frac{r+1}{2}}, e]) + \frac{2}{3} \sum_{i=1}^{\frac{s}{2}} \pi([u_i^*, [u_i, [v_{\frac{r+1}{2}}, e]]]) \\ &\quad - \left(\frac{2}{3} \sum_{i=1}^{\frac{r-1}{2}} \pi([v_i^*, [v_i, [v_{\frac{r+1}{2}}, e]]]) + v_{\frac{r+1}{2}} [v_{\frac{r+1}{2}}, \pi([v_{\frac{r+1}{2}}, e])] \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{3} \left(-2 \sum_{i=1}^{\frac{s}{2}} v_{\frac{r+1}{2}} \pi([u_i^*, [u_i, [v_{\frac{r+1}{2}}, [v_{\frac{r+1}{2}}, e]]]]) \right. \\
& \quad + 2 \sum_{i=1}^{\frac{r-1}{2}} v_{\frac{r+1}{2}} \pi([v_i^*, [v_i, [v_{\frac{r+1}{2}}, [v_{\frac{r+1}{2}}, e]]]]) \\
& \quad \left. + \frac{1}{2} [v_{\frac{r+1}{2}}, [v_{\frac{r+1}{2}}, \pi([v_{\frac{r+1}{2}}, e])]] - 2[\pi([v_{\frac{r+1}{2}}, e]), f] \right).
\end{aligned}
\tag{4.10}$$

Taking (3.11)–(3.15) into consideration, by (4.10) we have

$$\pi(\Theta_{[v_{\frac{r+1}{2}}, e]}) = \left([v_{\frac{r+1}{2}}, e] - \frac{3}{4} v_{\frac{r+1}{2}} + \frac{1}{2} v_{\frac{r+1}{2}} h \right) \otimes 1_\chi,
\tag{4.11}$$

For the Casimir element C of $U(\mathfrak{g})$ corresponding to the invariant form $(\cdot, \cdot)$, by a similar discussion to (3.20), it follows from (3.12), (2.4), (B.17), (B.19), (3.21), and (3.26) that

$$\begin{aligned}
\pi(C) &= \left(2e + \frac{h^2}{2} - \left(1 + \frac{s-r}{2} \right) h + C_0 + 2 \sum_{\alpha \in S(-1)} (-1)^{|\alpha|} [e, z_\alpha^*] z_\alpha \right) \otimes 1_\chi \\
&= \left(2e + \frac{h^2}{2} - \left(2 + \frac{s-r}{2} \right) h - 2v_{\frac{r+1}{2}} [v_{\frac{r+1}{2}}, e] \right. \\
&\quad + \sum_{i=1}^{k-1} h_i^2 + \sum_{i=1}^w \pi([x_i^*, x_i]) - \sum_{i=1}^\ell \pi([y_i^*, y_i]) \\
&\quad \left. + 2 \sum_{i=1}^{\frac{s}{2}} \pi([e, u_i^*], u_i) - 2 \sum_{i=1}^{\frac{r-1}{2}} \pi([e, v_i^*], v_i) \right) \otimes 1_\chi \\
&= \left(2e + \frac{h^2}{2} - \frac{3}{2} h - 2v_{\frac{r+1}{2}} [v_{\frac{r+1}{2}}, e] \right. \\
&\quad + \sum_{i=1}^{k-1} h_i^2 + \sum_{i=1}^{k-1} \sum_{j=1}^w \beta_{\bar{0}j}(h_i) h_i - \sum_{i=1}^{k-1} \sum_{j=1}^\ell \beta_{\bar{1}j}(h_i) h_i \\
&\quad \left. + 2 \sum_{i=1}^{\frac{s}{2}} t_{\bar{\gamma}_{0i}^*} - 2 \sum_{i=1}^{\frac{r-1}{2}} t_{\bar{\gamma}_{1i}^*} \right) \otimes 1_\chi \\
&= \left(2e + \frac{h^2}{2} - \frac{3}{2} h + \sqrt{-2} v_{\frac{r+1}{2}} [\sqrt{-2} v_{\frac{r+1}{2}}, e] + \sum_{i=1}^{k-1} h_i^2 \right. \\
&\quad \left. + \sum_{i=1}^{k-1} \left(\sum_{j=1}^w \beta_{\bar{0}j} - \sum_{j=1}^\ell \beta_{\bar{1}j} - 2 \sum_{j=1}^{\frac{s}{2}} \gamma_{\bar{0}j} + 2 \sum_{j=1}^{\frac{r-1}{2}} \gamma_{\bar{1}j} \right) (h_i) h_i \right) \otimes 1_\chi.
\end{aligned}
\tag{4.12}$$

Recall that in Proposition 4.3 we introduced the PBW theorem of $U(\mathfrak{g}_0, e)$. Keeping the notation in (3.1), it follows from (4.9)–(4.12) that

$$\begin{aligned}
 \pi(\Theta_{h_i}) &= \Theta'_{h_i} + \delta(h_i) \quad \text{for } 1 \leq i \leq k-1, \\
 \pi(\sqrt{-2}\Theta_{[v_{\frac{r+1}{2}}, e]}) &= \left([\sqrt{-2}v_{\frac{r+1}{2}}, e] - \frac{3}{4}\sqrt{-2}v_{\frac{r+1}{2}} + \frac{1}{2}\sqrt{-2}v_{\frac{r+1}{2}}h \right) \otimes 1_\chi \\
 &= \Theta'_E, \\
 (4.13) \quad \pi(C) &= C'_\theta + \sum_{i=1}^{k-1} (\Theta'_{h_i})^2 \\
 &\quad + \sum_{i=1}^{k-1} \left(\sum_{j=1}^w \beta_{\bar{0}j} - \sum_{j=1}^\ell \beta_{\bar{1}j} - 2 \sum_{j=1}^{\frac{s}{2}} \gamma_{\bar{0}j} + 2 \sum_{j=1}^{\frac{r-1}{2}} \gamma_{\bar{1}j} \right) (h_i) \Theta'_{h_i}, \\
 \pi(\sqrt{-2}\Theta_{v_{\frac{r+1}{2}}}) &= \sqrt{-2}v_{\frac{r+1}{2}} \otimes 1_\chi = \Theta'_F.
 \end{aligned}$$

(2) Recall that any $\Theta \in U(\mathfrak{g}, e)_0$ can be written as a linear combination of the monomials as in (4.8) with

$$\sum_i (-a_i + b_i) \beta_{\bar{0}i} + \sum_i (-c_i + d_i) \beta_{\bar{1}i} + \sum_i (m_i - n_i) \gamma_{\bar{0}i} + \sum_i (p_i - q_i) \gamma_{\bar{1}i} = 0.$$

Now it follows from the definition of π in (4.6), (4.13), and Proposition 4.3 that the restriction π defines a surjective homomorphism $\pi: U(\mathfrak{g}, e)_0 \rightarrow U(\mathfrak{g}_0, e)$, and $U(\mathfrak{g}, e)_{0, \#} \subset \ker \pi$ by definition. Then the quotient $U(\mathfrak{g}, e)_0 / U(\mathfrak{g}, e)_{0, \#}$ has a basis given by the coset of the PBW monomials

$$\Theta_{v_{\frac{r+1}{2}}}^\iota \left(\prod_{i=1}^{k-1} \Theta_{h_i}^{t_i} \right) C^{t_k} \Theta_{[v_{\frac{r+1}{2}}, e]}^\varepsilon$$

with $\iota, \varepsilon \in \Lambda_1$, $\mathbf{t} \in \mathbb{Z}_+^k$. Moreover, (4.13) and Proposition 4.3 entail that the monomials

$$\pi(\Theta_{v_{\frac{r+1}{2}}})^\iota \left(\prod_{i=1}^{k-1} \pi(\Theta_{h_i})^{t_i} \right) \pi(C)^{t_k} \pi(\Theta_{[v_{\frac{r+1}{2}}, e]})^\varepsilon$$

with $\iota, \varepsilon \in \Lambda_1$, $\mathbf{t} \in \mathbb{Z}_+^k$ actually form a basis for $U(\mathfrak{g}_0, e)$, then the kernel of π is no bigger than $U(\mathfrak{g}, e)_{0, \#}$.

(3) Since S_ϵ is a shift on $U(\mathfrak{g}_0, e)$, by the discussion in Step (2) it remains to show that $\pi_\epsilon = S_\epsilon \circ \pi: U(\mathfrak{g}, e)_0 / U(\mathfrak{g}, e)_{0, \#} \rightarrow U(\mathfrak{g}_0, e)$ is an algebra homomorphism.

Keeping the notation in (2.7), it follows from (B.27), (4.13) and Theorems 2.4, 4.3 that

$$\begin{aligned}
& \pi_\epsilon([\Theta_{[v_{\frac{r+1}{2}}, e]}, \Theta_{[v_{\frac{r+1}{2}}, e]}]) \\
&= \pi_\epsilon \left(-\frac{1}{2}C + \frac{1}{2}c_0 + \frac{1}{2} \sum_{i=1}^{k-1} \Theta_{h_i}^2 + \sum_{i=1}^w \Theta_{x_i} \Theta_{x_i^*} + \frac{1}{2} \sum_{i=1}^{k-1} \sum_{j=1}^w \beta_{\bar{0}j}(h_i) \Theta_{h_i} \right. \\
&\quad - \frac{1}{2} \sum_{i=1}^{k-1} \sum_{j=1}^\ell \beta_{\bar{1}j}(h_i) \Theta_{h_i} + \sum_{i=1}^\ell \Theta_{y_i} \Theta_{y_i^*} \\
&\quad - 2 \sum_{i=1}^{\frac{s}{2}} \Theta_{[[v_{\frac{r+1}{2}}, e], u_i]^\#} \Theta_{[u_i^*, [v_{\frac{r+1}{2}}, e]]^\#} - \frac{1}{2} \sum_{i=1}^{k-1} \sum_{j=1}^{\frac{s}{2}} \gamma_{\bar{0}j}(h_i) \Theta_{h_i} \\
&\quad \left. - 2 \sum_{i=1}^{\frac{r-1}{2}} \Theta_{[[v_{\frac{r+1}{2}}, e], v_i]^\#} \Theta_{[v_i^*, [v_{\frac{r+1}{2}}, e]]^\#} + \frac{1}{2} \sum_{i=1}^{k-1} \sum_{j=1}^{\frac{r-1}{2}} \gamma_{\bar{1}j}(h_i) \Theta_{h_i} \right) \\
&= -\frac{1}{2}C'_\theta - \frac{1}{2} \sum_{i=1}^{k-1} (\Theta'_{h_i})^2 \\
&\quad - \frac{1}{2} \sum_{i=1}^{k-1} \left(\sum_{j=1}^w \beta_{\bar{0}j} - \sum_{j=1}^\ell \beta_{\bar{1}j} - 2 \sum_{j=1}^{\frac{s}{2}} \gamma_{\bar{0}j} + 2 \sum_{j=1}^{\frac{r-1}{2}} \gamma_{\bar{1}j} \right) (h_i) \Theta'_{h_i} - \frac{1}{2} \epsilon \\
&\quad + \frac{1}{2} c_0 + \frac{1}{2} \sum_{i=1}^{k-1} (\Theta'_{h_i} + \delta(h_i))^2 \\
&\quad + \frac{1}{2} \sum_{i=1}^{k-1} \left(\sum_{j=1}^w \beta_{\bar{0}j} - \sum_{j=1}^\ell \beta_{\bar{1}j} - \sum_{j=1}^{\frac{s}{2}} \gamma_{\bar{0}j} + \sum_{j=1}^{\frac{r-1}{2}} \gamma_{\bar{1}j} \right) (\Theta'_{h_i} + \delta(h_i)) \\
&= -\frac{1}{2}C'_\theta - \frac{1}{2} \epsilon + \frac{1}{2} c_0 + \sum_{i=1}^{k-1} \left(\rho_{e,0} \delta + \frac{3}{2} \delta^2 \right) (h_i) \\
&= -\frac{1}{2}C'_\theta + \frac{1}{2} \left(-\epsilon + c_0 + \sum_{i=1}^{k-1} (2\rho_{e,0} \delta + 3\delta^2) (h_i) \right) \\
(4.14) \quad &= -\frac{1}{2}C'_\theta - \frac{1}{16} = -\frac{1}{2}[\Theta'_E, \Theta'_E] = [\pi_\epsilon(\Theta_{[v_{\frac{r+1}{2}}, e]}), \pi_\epsilon(\Theta_{[v_{\frac{r+1}{2}}, e]})],
\end{aligned}$$

and

$$\pi_\epsilon([\Theta_{v_{\frac{r+1}{2}}}, \Theta_{v_{\frac{r+1}{2}}}]) = 1 \otimes 1_\chi = -\frac{1}{2}[\Theta'_F, \Theta'_F] = [\pi_\epsilon(\Theta_{v_{\frac{r+1}{2}}}), \pi_\epsilon(\Theta_{v_{\frac{r+1}{2}}})],$$

with the other commutators being trivial. $\square$

Remark 4.9. In the procedure of constructing the isomorphism in Proposition 4.8, we involve a shift S_ϵ which has an important distinction from the one in [12, Thm. 4.3] for the finite W -algebra case. Recall that in [13, Prop. 5.7.6] the grading of $\mathfrak{g}_0$ under the action of $\text{ad } h$ is even for any semi-simple Lie algebra $\mathfrak{g}$, which means that $\mathfrak{g}_0(i) = \{0\}$ for all odd $i \in \mathbb{Z}$. Then $\gamma \in (\mathfrak{h}^e)^*$ defined in [12, §4.1] extends uniquely to a character of the parabolic subalgebra $\mathfrak{p}_0 := \bigoplus_{i \geq 0} \mathfrak{g}_0(i)$ of $\mathfrak{g}_0$ by [12, Lem. 4.1]. In the meanwhile, the finite W -algebra $U(\mathfrak{g}_0, e)$ can be realized as a subalgebra of $U(\mathfrak{p}_0)$. Under the above settings, the proof for [12, Thm. 4.3] goes through. However, for the basic Lie superalgebra $\mathfrak{g}_0$ and a minimal root $-\theta$, under the action of $\text{ad } h_\theta$ we have $\mathfrak{g}_0(-1)_{\bar{1}} = \mathbb{C}v_{\frac{r+1}{2}} \neq \{0\}$ and $\mathfrak{g}_0(1)_{\bar{1}} = \mathbb{C}[v_{\frac{r+1}{2}}, e] \neq \{0\}$ under our settings. So the grading of $\mathfrak{g}_0$ under the action of $\text{ad } h$ is not even. Moreover, the proper subspace $\tilde{\mathfrak{p}}_0 := \bigoplus_{i \geq -1} \mathfrak{g}_0(i) \supset \mathfrak{p}_0$ of $\mathfrak{g}_0$ is not a good choice. Therefore, the technique in [12] is not available here. Fortunately, in the case when minimal roots are concerned, we have already obtained the precise generators and their relations for $U(\mathfrak{g}, e)$. This enables us to construct the desired isomorphism. However, there should be a requirement to develop a general technique for dealing with finite W -superalgebras $U(\mathfrak{g}, e)$ associated with an arbitrary nilpotent element $e \in \mathfrak{g}_0$.

§4.4.

For a $U(\mathfrak{g}, e)$ -module M and $\alpha \in (\mathfrak{h}^e)^*$, we define the α -weight space

$$(4.15) \quad M_\alpha := \{m \in M \mid (\Theta_t - \delta(t))(m) = \alpha(t)m \text{ for all } t \in \mathfrak{h}^e\}.$$

By Theorem 2.4 we have that $U(\mathfrak{g}, e)_\beta M_\alpha \subset M_{\beta+\alpha}$. So each M_α is invariant under the action of the subalgebra $U(\mathfrak{g}, e)_0$. We call M_α a maximal weight space of V if $U(\mathfrak{g}, e)_\# V_\alpha = \{0\}$.

Set M_α to be a maximal weight space in a $U(\mathfrak{g}, e)$ -module M . By Proposition 4.8, the action of $U(\mathfrak{g}, e)_0$ on M_α factors through the map π_ϵ to make M_α into a $U(\mathfrak{g}_0, e)$ -module such that $u \cdot m = \pi_\epsilon(u)m$ for $u \in U(\mathfrak{g}, e)_0$ and $m \in M_\alpha$. It follows from Proposition 4.3 that $\Theta'_{\mathfrak{h}^e}$ lies in the center of $U(\mathfrak{g}_0, e)$, then $\mathfrak{h}^e$ can be considered as a Lie subalgebra of $U(\mathfrak{g}_0, e)$. So by Proposition 4.3(1), (4.13), and (4.15) we get the action of $\mathfrak{h}^e$ on M_α via

$$t(m) = \Theta'_t(m) = \alpha(t)m$$

for all $t \in \mathfrak{h}^e$, which explains why the additional shift by $-\delta$ in the definition of the α -weight space of a $U(\mathfrak{g}, e)$ -module is necessary.

A $U(\mathfrak{g}, e)$ -module M is called a highest weight module if it is generated by a maximal weight space M_λ such that M_λ is finite-dimensional and irreducible

as a $U(\mathfrak{g}_0, e)$ -module. In that case, as we will see shortly, λ is the unique maximal weight of M in the dominance ordering. Recall that pairwise inequivalent finite-dimensional irreducible $U(\mathfrak{g}_0, e)$ -modules V_λ with $\lambda \in (\mathfrak{h}^e)^*$ are given in Theorem 4.6.

By definition we see that $U(\mathfrak{g}, e)_\#$ is invariant under left multiplication by $U(\mathfrak{g}, e)$ and right multiplication by $U(\mathfrak{g}, e)_0$. Then $U(\mathfrak{g}, e)/U(\mathfrak{g}, e)_\#$ is a $(U(\mathfrak{g}, e), U(\mathfrak{g}, e)_0)$ -bimodule. And then the right action of $U(\mathfrak{g}, e)_0$ factors through the map π_ϵ from Proposition 4.8 to make $U(\mathfrak{g}, e)/U(\mathfrak{g}, e)_\#$ into a $(U(\mathfrak{g}, e), U(\mathfrak{g}_0, e))$ -bimodule. We introduce the highest weight $U(\mathfrak{g}, e)$ -module with highest weight λ as

$$(4.16) \quad M_e(\lambda) := (U(\mathfrak{g}, e)/U(\mathfrak{g}, e)_\#) \otimes_{U(\mathfrak{g}_0, e)} V_\lambda.$$

We will show that $M_e(\lambda)$ is universal, meaning that if M is another highest weight module generated by a maximal weight space M_μ and $f: V_\lambda \xrightarrow{\sim} M_\mu$ is an even $U(\mathfrak{g}_0, e)$ -module isomorphism, then there is a unique $U(\mathfrak{g}, e)$ -module homomorphism $\tilde{f}: M_e(\lambda) \rightarrow M$ extending f . First observe the following lemma.

Lemma 4.10. *As a right $U(\mathfrak{g}_0, e)$ -module, $U(\mathfrak{g}, e)/U(\mathfrak{g}, e)_\#$ is free with basis*

$$\left\{ \prod_{i=1}^w \Theta_{x_i}^{a_i} \cdot \prod_{i=1}^\ell \Theta_{y_i}^{b_i} \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i}^{c_i} \cdot \prod_{i=1}^{\frac{r-1}{2}} \Theta_{g_i}^{d_i} \mid \mathbf{a} \in \mathbb{Z}_+^w, \mathbf{b} \in \Lambda_\ell, \mathbf{c} \in \mathbb{Z}_+^{\frac{s}{2}}, \mathbf{d} \in \Lambda_{\frac{r-1}{2}} \right\}.$$

Proof. This follows because the cosets of the PBW monomials of the form

$$\prod_{i=1}^w \Theta_{x_i}^{a_i} \cdot \prod_{i=1}^\ell \Theta_{y_i}^{c_i} \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i}^{m_i} \cdot \prod_{i=1}^{\frac{r-1}{2}} \Theta_{g_i}^{p_i} \cdot \Theta_{v_{\frac{r+1}{2}}}^\iota \cdot \prod_{i=1}^{k-1} \Theta_{h_i}^{t_i} \cdot C^{t_k} \cdot \Theta_{[v_{\frac{r+1}{2}}, e]}^\varepsilon$$

with $\mathbf{a} \in \mathbb{Z}_+^w$, $\mathbf{c} \in \Lambda_\ell$, $\mathbf{m} \in \mathbb{Z}_+^{\frac{s}{2}}$, $\mathbf{p} \in \Lambda_{\frac{r-1}{2}}$, $\iota, \varepsilon \in \Lambda_1$, $\mathbf{t} \in \mathbb{Z}_+^k$, form a basis for the quotient $U(\mathfrak{g}, e)/U(\mathfrak{g}, e)_\#$, and the cosets of the monomials of the form

$$\Theta_{v_{\frac{r+1}{2}}}^\iota \cdot \prod_{i=1}^{k-1} \Theta_{h_i}^{t_i} \cdot C^{t_k} \cdot \Theta_{[v_{\frac{r+1}{2}}, e]}^\varepsilon$$

with $\iota, \varepsilon \in \Lambda_1$, $\mathbf{t} \in \mathbb{Z}_+^k$, form a basis for $U(\mathfrak{g}, e)_0/U(\mathfrak{g}, e)_{0,\#} \cong U(\mathfrak{g}_0, e)$ by step (2) in the proof of Proposition 4.8. $\square$

Now we will introduce the main result of this subsection. We mainly follow the strategy for the non-super case as in [12, Thm. 4.5], with a lot of modifications.

Theorem 4.11. *For $\lambda \in (\mathfrak{h}^e)^*$, let v_λ and $\Theta'_F(v_\lambda) = \sqrt{-2}\Theta_{v_{\frac{r+1}{2}}}(v_\lambda)$ be a basis for $U(\mathfrak{g}_0, e)$ -modules V_λ with $\mathfrak{h}^e$ -weight λ as in Section 4.2.3.*

- (1) The vectors $\{\prod_{i=1}^w \Theta_{x_i}^{a_i} \cdot \prod_{i=1}^\ell \Theta_{y_i}^{b_i} \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i}^{c_i} \cdot \prod_{i=1}^{\frac{r-1}{2}} \Theta_{g_i}^{d_i} \cdot \Theta_{v_{\frac{r+1}{2}}}^\iota (v_\lambda) \mid \mathbf{a} \in \mathbb{Z}_+^w, \mathbf{b} \in \Lambda_\ell, \mathbf{c} \in \mathbb{Z}_+^{\frac{s}{2}}, \mathbf{d} \in \Lambda_{\frac{r-1}{2}}, \iota \in \Lambda_1\}$ form a basis of $M_e(\lambda)$.
- (2) The weight λ is the unique maximal weight of $M_e(\lambda)$ in the dominance ordering, $M_e(\lambda)$ is generated by the maximal weight space $M_e(\lambda)_\lambda$, and $M_e(\lambda)_\lambda \cong V_\lambda$ as $U(\mathfrak{g}_0, e)$ -modules.
- (3) The module $M_e(\lambda)$ is a universal highest weight module with highest weight λ .
- (4) There is a unique maximal proper submodule $M_e^{\max}(\lambda)$ in $M_e(\lambda)$,

$$(4.17) \quad L_e(\lambda) := M_e(\lambda) / M_e^{\max}(\lambda)$$

is an irreducible module type Q , and $\{L_e(\lambda) \mid \lambda \in (\mathfrak{h}^e)^*\}$ is a complete set of pairwise inequivalent irreducible highest weight modules over $U(\mathfrak{g}, e)$ (up to parity switch). Moreover, any finite-dimensional simple $U(\mathfrak{g}, e)$ -module (up to parity switch) is isomorphic to one of the modules $L_e(\lambda)$ for $\lambda \in \Lambda_0^+ := \{\lambda \in (\mathfrak{h}^e)^* \mid \lambda(h_\alpha) \in \mathbb{Z}_+ \text{ for } \alpha \in (\Phi_{e,0}^+)_{\bar{0}}\}$.

Proof. (1) This is clear from Lemma 4.10.

(2) Since $\theta(\mathfrak{h}^e) = 0$ by definition, then

$$\prod_{i=1}^w \Theta_{x_i}^{a_i} \cdot \prod_{i=1}^\ell \Theta_{y_i}^{b_i} \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i}^{c_i} \cdot \prod_{i=1}^{\frac{r-1}{2}} \Theta_{g_i}^{d_i} \cdot \Theta_{v_{\frac{r+1}{2}}}^\iota (v_\lambda)$$

is of $\mathfrak{h}^e$ -weight

$$\lambda - \sum_i a_i \beta_{\bar{0}i} - \sum_i b_i \beta_{\bar{1}i} + \sum_i c_i \gamma_{\bar{0}i} + \sum_i d_i \gamma_{\bar{1}i}$$

(keep in mind that $\gamma_{\bar{0}i}, \gamma_{\bar{1}j} \in (\Phi'_e)^-$). Hence the λ -weight space of $M_e(\lambda)$ is $1 \otimes V_\lambda$ and all other weights of $M_e(\lambda)$ are strictly smaller in the dominance ordering.

(3) It follows from (1) and (2) that $M_e(\lambda)$ is a highest weight module with highest weight λ . Let M be another highest weight module generated by a maximal weight space M_μ and $f: V_\lambda \rightarrow M_\mu$ be an even $U(\mathfrak{g}_0, e)$ -module isomorphism. By Theorem 4.6 we obtain $\mu = \lambda$. By the adjointness of tensor and hom, f extends uniquely to a $U(\mathfrak{g}, e)$ -module homomorphism $\tilde{f}: M_e(\lambda) \rightarrow M$ such that $\tilde{f}(1 \otimes v_\lambda) = f(v_\lambda)$. As $\tilde{f}(1 \otimes V_\lambda) = f(V_\lambda)$ generates M , then $\tilde{f}$ is surjective.

(4) Let M be a submodule of $M_e(\lambda)$. Then M is the direct sum of its $\mathfrak{h}^e$ -weight spaces. If $M_\lambda \neq 0$ then M_λ generates all of $1 \otimes V_\lambda$ as a $U(\mathfrak{g}_0, e)$ -module, hence it generates all of $M_e(\lambda)$ as a $U(\mathfrak{g}, e)$ -module. This shows that if M is a proper submodule then it is contained in $M_e(\lambda)_- := \bigoplus_{\mu < \lambda} M_e(\lambda)_\mu$. Hence the sum of all

proper submodules of $M_e(\lambda)$ is still a proper submodule, so $M_e(\lambda)$ has a unique maximal submodule $M_e^{\max}(\lambda)$ as claimed. By (3) any irreducible highest weight module N of type λ is a quotient of $M_e(\lambda)$, hence $N \cong L_e(\lambda)$ or $\prod L_e(\lambda)$ (here $\prod$ denotes the parity switching functor). Moreover, λ is the unique maximal weight of $L_e(\lambda)$ by (2) and $L_e(\lambda)_\lambda$ is isomorphic to N_λ or $\prod N_\lambda$ as $U(\mathfrak{g}_0, e)$ -modules. Thus λ is uniquely determined by N . Since N_λ is an irreducible $U(\mathfrak{g}_0, e)$ -module of type Q by Theorem 4.6, then $L_e(\lambda)$ is also of type Q .

By Theorem 2.4 (more precisely, Theorem 2.3(1)), when we restrict $L_e(\lambda)$ to the $\mathfrak{sl}(2)$ -triple $(\Theta_{e_\alpha}, \Theta_{h_\alpha}, \Theta_{e_{-\alpha}}) \subset U(\mathfrak{g}, e)$ with $\alpha \in (\Phi_{e,0}^+)_{\bar{0}}$, we get that $\lambda(h_\alpha) \in \mathbb{Z}_+$ for any $\alpha \in (\Phi_{e,0}^+)_{\bar{0}}$. $\square$

§4.5.

Let $Z(U(\mathfrak{g}))$ denote the center of $U(\mathfrak{g})$, and write $Z(U(\mathfrak{g}, e))$ for the center of $U(\mathfrak{g}, e)$. It is immediate from the definition of $U(\mathfrak{g}, e)$ that the restriction of the linear map $\text{Pr}: U(\mathfrak{g}) \rightarrow U(\mathfrak{g})/I_\chi$ defines a natural algebra homomorphism $\text{Pr}: Z(U(\mathfrak{g})) \rightarrow Z(U(\mathfrak{g}, e))$. Denote the representation of $U(\mathfrak{g})$ in $\text{End}(Q_\chi)$ by ρ_χ . Thanks to [33, Cor. 13.2.2], the Harish-Chandra homomorphism $Z(U(\mathfrak{g})) \rightarrow U(\mathfrak{h})$ is injective, so is the restriction of $\rho_\chi: U(\mathfrak{g}) \rightarrow \text{End}(Q_\chi)$ to $Z(U(\mathfrak{g}))$, i.e., the map $\text{Pr}: Z(U(\mathfrak{g})) \rightarrow Z(U(\mathfrak{g}, e))$ is injective (the discussion here comes by the same strategy as in its non-super case [37, §6.2], and one can also refer to the proof of [43, Lem. 3.2] for more details). Moreover, when $\mathfrak{g}$ is a complex semi-simple Lie algebra, as explained in the footnote to [38, Que. 5.1] (see also [35, Rem. 2.1]), the map Pr is actually an algebra isomorphism. But it seems more complicated in the super case. We try to raise the following conjecture.

Conjecture 4.12. The center of a finite W -superalgebra $U(\mathfrak{g}, e)$ coincides with the image of $Z(U(\mathfrak{g}))$ in $U(\mathfrak{g}, e)$ under the projection $\text{Pr}: Z(U(\mathfrak{g})) \rightarrow Z(U(\mathfrak{g}, e))$.

We can give a positive answer to Conjecture 4.12 in some special cases. First, as an immediate consequence of Proposition 4.5, the conjecture is true for $\mathfrak{g} = \mathfrak{osp}(1|2)$ with e being regular nilpotent (which can also be considered as a root vector e_θ with $-\theta$ being a minimal root). Second, if $\mathfrak{g} = \mathfrak{gl}(m|n)$ with e being regular nilpotent, [11, Thm. 3.21] ensures the conjecture. Third, the conjecture for the Lie superalgebra $\mathfrak{q}(n)$ with e being regular nilpotent is confirmed in [36, Cor. 5.10]. However, whether this conclusion holds for the general situation is still an open problem.

Let us turn back to the minimal case. For further discussion, we will compute the action of the Casimir element $C \in Z(U(\mathfrak{g}, e))$ on the highest weight module $M_e(\lambda)$.

Lemma 4.13. *Retain the notation in Sections 2.2.2 and 3.2.3. For all $\lambda \in (\mathfrak{h}^e)^*$, C acts on the highest weight module $M_e(\lambda)$ as the scalar*

$$c_0 + (\lambda, \lambda + 2\bar{\rho}) + \sum_{i=1}^{k-1} (2\rho_{e,0}\delta + 3\delta^2)(h_i).$$

Proof. Since all positive root vectors annihilate v_λ , and $C'_\theta(v_\lambda) = -\frac{1}{8}v_\lambda$ by Proposition 4.3(i), a similar discussion to (3.20)–(3.24) entails that

$$\begin{aligned} C \cdot v_\lambda &= \pi_\epsilon(C)(v_\lambda) = (C + \epsilon)(v_\lambda) \\ &= \left(C'_\theta + (\lambda, \lambda + 2\bar{\rho}) + c_0 + \frac{1}{8} + \sum_{i=1}^{k-1} (2\rho_{e,0}\delta + 3\delta^2)(h_i) \right) (v_\lambda) \\ &= \left(c_0 + (\lambda, \lambda + 2\bar{\rho}) + \sum_{i=1}^{k-1} (2\rho_{e,0}\delta + 3\delta^2)(h_i) \right) (v_\lambda). \end{aligned}$$

Since $M_e(\lambda)$ is a cyclic module generated by v_λ , the conclusion follows. $\square$

Similarly to (3.29), we also have another description of Lemma 4.13. Taking Proposition 4.3 into consideration, by a similar discussion to (4.12) we obtain

$$\begin{aligned} C \cdot v_\lambda &= \pi_\epsilon(C)(v_\lambda) = (C + \epsilon)(v_\lambda) \\ &= \left(C'_\theta + \sum_{i=1}^{k-1} h_i^2 + \sum_{i=1}^{k-1} \left(\sum_{j=1}^w \beta_{0j} - \sum_{j=1}^\ell \beta_{1j} - 2 \sum_{j=1}^{\frac{s}{2}} \gamma_{0j} + 2 \sum_{j=1}^{\frac{r-1}{2}} \gamma_{1j} \right) (h_i) h_i \right. \\ &\quad \left. + c_0 + \frac{1}{8} + \sum_{i=1}^{k-1} (2\rho_{e,0}\delta + 3\delta^2)(h_i) \right) (v_\lambda) \\ (4.18) \quad &= \left(c_0 + \sum_{i=1}^{k-1} \left(\lambda(h_i)^2 + (2\rho_{e,0} + 4\delta)(h_i)\lambda(h_i) + (2\rho_{e,0}\delta + 3\delta^2)(h_i) \right) \right) (v_\lambda). \end{aligned}$$

This conclusion will be used in the formulation of Theorem 4.14.

A $U(\mathfrak{g}, e)$ -module V is of central character $\psi: Z(U(\mathfrak{g}, e)) \rightarrow \mathbb{C}$ if $z(v) = \psi(z)v$ for all $z \in Z(U(\mathfrak{g}, e))$ and $v \in V$. For the highest weight $U(\mathfrak{g}, e)$ -module $M_e(\lambda)$ with highest weight $\lambda \in (\mathfrak{h}^e)^*$, let $\psi^\lambda: Z(U(\mathfrak{g}, e)) \rightarrow \mathbb{C}$ be the corresponding central character. Under the above settings, we have the following theorem.

Theorem 4.14. *The number of isomorphism classes of irreducible highest weight modules for $U(\mathfrak{g}, e)$ with prescribed central character $\psi: Z(U(\mathfrak{g}, e)) \rightarrow \mathbb{C}$ is finite, i.e., the set $\{\lambda \in (\mathfrak{h}^e)^* \mid \psi^\lambda = \psi\}$ is finite.*

Proof. Given any irreducible highest weight $U(\mathfrak{g}, e)$ -module $L_e(\lambda)$ with prescribed central character $\psi: Z(U(\mathfrak{g}, e)) \rightarrow \mathbb{C}$, we can write $\psi = \psi^\lambda$. Let $\mu \in (\mathfrak{h}^e)^*$ be such

that $\psi^\mu = \psi^\lambda$. Then the action of $C \in Z(U(\mathfrak{g}, e))$ on v_μ of $L_e(\mu) = U(\mathfrak{g}, e)v_\mu$ coincides with that on v_λ of $L_e(\lambda) = U(\mathfrak{g}, e)v_\lambda$, i.e., $\psi^\mu(C) = \psi^\lambda(C)$. Thanks to (4.18), we have

$$\begin{aligned}
 0 &= \psi^\mu(C) - \psi^\lambda(C) \\
 &= \sum_{i=1}^{k-1} (\mu(h_i)^2 + (2\rho_{e,0} + 4\delta)(h_i)\mu(h_i)) \\
 &\quad - \sum_{i=1}^{k-1} (\lambda(h_i)^2 + (2\rho_{e,0} + 4\delta)(h_i)\lambda(h_i)) \\
 (4.19) \quad &= \sum_{i=1}^{k-1} (h_i^2 + (2\rho_{e,0} + 4\delta)(h_i)h_i)(\mu) - \sum_{i=1}^{k-1} (\lambda(h_i)^2 + (2\rho_{e,0} + 4\delta)(h_i)\lambda(h_i)),
 \end{aligned}$$

where the last equation of (4.19) is considered as a polynomial in indeterminates $h_1, \dots, h_{k-1}$ at μ . From linear algebra we know that the solution to (4.19) is finite, i.e., the set $\{\lambda \in (\mathfrak{h}^e)^* \mid \psi^\lambda = \psi\}$ is finite. $\square$

For the simple root system $\Delta = \{\alpha_1, \dots, \alpha_k\}$ for Φ , write $\beta_i := \alpha_i|_{(\mathfrak{h}^e)^*}$ for $1 \leq i \leq k-1$. Then $\{\beta_1, \dots, \beta_{k-1}\}$ is a system of restricted simple roots for $(\Phi'_e)^+$ by our earlier discussion. Denote $(Q^e)^+ := \sum_{i=1}^{k-1} \mathbb{Z}_+ \beta_i$. The following is a corollary of Theorem 4.14.

Corollary 4.15. *For $\lambda \in (\mathfrak{h}^e)^*$, the highest weight module $M_e(\lambda)$ has composition series.*

Proof. The corollary can be proved by imitating the standard argument in the classical case from [21, §7.6.1]. Let us put it explicitly.

Let N and N' be sub- $U(\mathfrak{g}, e)$ -modules of $M_e(\lambda)$ such that $N' \subseteq N$ and N/N' is simple. Since $M_e(\lambda) = \bigoplus_{\mu' \in (\mathfrak{h}^e)^*} M_e(\lambda)_{\mu'}$, we have $N/N' = \bigoplus_{\mu' \in (\mathfrak{h}^e)^*} (N/N')_{\mu'}$, and every μ' belongs to $\lambda - (Q^e)^+$. So there exists a weight μ of N/N' such that, for all $\alpha \in (\Phi'_e)^+$, $\mu + \alpha$ is not a weight of N/N' . If v is a non-zero element of $(N/N')_\mu$, we have $U(\mathfrak{g}, e)_\#(v) = 0$, and hence N/N' is isomorphic to $L_e(\mu)$ (up to parity switch) by Theorem 4.11. The central characters of N/N' and $M_e(\lambda)$ are equal, whence μ is in the finite set $\{\lambda \in (\mathfrak{h}^e)^* \mid \psi^\lambda = \psi\}$ by Theorem 4.14.

Recall that $U(\mathfrak{g}, e)$ is Noetherian by Proposition 1.12. Since $M_e(\lambda)$ is a cycle $U(\mathfrak{g}, e)$ -module, then $M_e(\lambda)$ is a Noetherian $U(\mathfrak{g}, e)$ -module. Every non-zero sub- $U(\mathfrak{g}, e)$ -module N of $M_e(\lambda)$ hence contains a sub- $U(\mathfrak{g}, e)$ -module N' such that N/N' is simple. If $M_e(\lambda)$ had no composition series, there would exist an infinite decreasing sequence $N_0 \supset N_1 \supset \dots$ of sub- $U(\mathfrak{g}, e)$ -modules of $M_e(\lambda)$ such that every N_i/N_{i+1} was simple. From the previous paragraph, infinitely many of these

quotients would be isomorphic to each other. Then one of the $\mathfrak{h}^e$ -weights of $M_e(\lambda)$ would have infinite multiplicity, contrary to Theorem 4.11(1). $\square$

§4.6.

Now we introduce an analogue of the BGG (short for Bernstein–Gelfand–Gelfand) category $\mathcal{O}$. Let $\mathcal{O}(e) = \mathcal{O}(e; \mathfrak{h}, \mathfrak{q})$ denote the category of all finitely generated $U(\mathfrak{g}, e)$ -modules V , that are semi-simple over $\mathfrak{h}^e$ with finite-dimensional $\mathfrak{h}^e$ -weight spaces, such that the set $\{\lambda \in (\mathfrak{h}^e)^* \mid V_\lambda \neq \{0\}\}$ is contained in a finite union of sets of the form $\{\nu \in (\mathfrak{h}^e)^* \mid \nu \leq \mu\}$ for $\mu \in (\mathfrak{h}^e)^*$.

In virtue of the results we obtained in this section, we introduce the proof of Theorem 0.3, which can be checked routinely as a counterpart for the ordinary BGG category $\mathcal{O}$ (see, e.g., [24, 33]). Let us put it explicitly.

Proof of Theorem 0.3. First note that $U(\mathfrak{g}, e)$ is Noetherian by Proposition 1.12. Then $\mathcal{O}(e)$ is closed under the operations of taking submodules, quotients, and finite direct sums.

(1) Let L be a simple object in $\mathcal{O}(e)$; we will show that $L \cong L_e(\lambda)$ for some $\lambda \in (\mathfrak{h}^e)^*$.

Since L is $\mathbb{Z}_2$ -graded, there exists $\mu \in (\mathfrak{h}^e)^*$ such that L_μ contains a non-zero homogeneous element v . Since $N = U(\mathfrak{g}, e)_\#(v)$ is finite-dimensional by the definition of $\mathcal{O}(e)$, there exists $\lambda \in (\mathfrak{h}^e)^*$ such that $N_\lambda \neq 0$, but $N_{\lambda+\alpha} = 0$ for all restricted positive roots $\alpha \in (\mathfrak{h}^e)^*$. Since N_λ is finite-dimensional, we can further assume that $\Theta_{[v_{\frac{r+1}{2}}, e]} \cdot N_\lambda = 0$. Now N_λ is a finite-dimensional $\mathbb{Z}_2$ -graded $U(\mathfrak{g}, e)_0/U(\mathfrak{g}, e)_0, \# \cong \bar{U}(\mathfrak{g}_0, e)$ -module, so N_λ contains a $U(\mathfrak{g}_0, e)$ -submodule isomorphic to V_λ by Theorem 4.6. Since $U(\mathfrak{g}, e)V_\lambda$ is a $U(\mathfrak{g}, e)$ -submodule of L , it equals L by simplicity. Then $L \cong L_e(\lambda)$ follows from Theorem 4.11(3)–(4).

(2) Let M be any non-zero module in $\mathcal{O}(e)$. We claim that M has a finite filtration $0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$ with non-zero quotients each of which is a highest weight module. Then statement (2) follows from Corollary 4.15.

In fact, since M can be generated by finitely many weight vectors, and for each $m \in M$, the subspace $U(\mathfrak{g}, e)_\#(m)$ is finite-dimensional by definition, then the $U(\mathfrak{g}, e)_\#$ -submodule V generated by such a generating family of weight vectors is finite-dimensional. If $\dim V = 1$, it is clear that M itself is a highest weight module. Otherwise proceed by induction on $\dim V$.

Start with a non-zero weight vector $v \in V$ of weight $\lambda \in (\mathfrak{h}^e)^*$ which is maximal among all weights of V and is therefore a maximal vector in M . It generates a submodule M_1 , while the quotient $\bar{M} := M/M_1$ again lies in $\mathcal{O}(e)$ and is generated by the image $\bar{V}$ of V . Since $\dim \bar{V} < \dim V$, the induction hypothesis can be

applied to M , yielding a chain of highest weight submodules whose pre-images in M are the desired $M_2, \dots, M_n$.

(3) Define a subspace of M for each fixed ψ^λ by

$$M^{\psi^\lambda} := \{m \in M \mid (z - \psi^\lambda(z))^n(m) = 0 \text{ for some } n > 0 \\ \text{depending on } z \in Z(U(\mathfrak{g}, e))\}.$$

It is clear that M^{ψ^λ} is a $U(\mathfrak{g}, e)$ -submodule of M , while the subspaces M^{ψ^λ} are independent.

Now $Z(U(\mathfrak{g}, e))$ stabilizes each weight space M_μ , as $Z(U(\mathfrak{g}, e))$ and $U(\mathfrak{h}^e)$ commute. Then a standard result from linear algebra on families of commuting operators implies that $M_\mu = \bigoplus_\mu (M_\mu \cap M^{\psi^\lambda})$. Because M is generated by finitely many weight vectors, it must therefore be the direct sum of finitely many non-zero submodules M^{ψ^λ} . Thanks to statement (1), each central character ψ occurring in this way must be of the form ψ^λ for some $\lambda \in (\mathfrak{h}^e)^*$.

Denote by $\mathcal{O}_{\psi^\lambda}(e)$ the (full) subcategory of $\mathcal{O}(e)$ whose objects are the modules M for which $M = M^{\psi^\lambda}$. Then statement (3) follows. $\square$

At the extreme, if $\mathfrak{g} = \mathfrak{g}_0 \cong \mathfrak{h}^e \oplus \mathfrak{osp}(1|2)$, then e is a distinguished nilpotent element of $\mathfrak{g}$, i.e., the only semi-simple elements of $\mathfrak{g}$ that centralize e belong to the center of $\mathfrak{g}$. In this case, $\mathcal{O}(e)$ is the category of all finite-dimensional $U(\mathfrak{g}, e)$ -modules that are semi-simple over the Lie algebra center of $\mathfrak{g}$.

Remark 4.16. Let $U(\mathfrak{g}, e)$ be the finite W -algebra associated with a complex semi-simple Lie algebra $\mathfrak{g}$ and its nilpotent element e . In [30], by the method of quantized symplectic actions, Losev established an equivalence of the BGG category $\mathcal{O}$ for $U(\mathfrak{g}, e)$ in [12] with a certain category of $\mathfrak{g}$ -modules. In the case when e is of principal Levi type, the category of $\mathfrak{g}$ -modules of interest is the category of generalized Whittaker modules. Recall that in Section 3 we achieved this goal for Verma modules of minimal finite W -superalgebras. It will be an interesting topic to generalize all these to finite W -superalgebras associated with arbitrary even nilpotent elements.

Remark 4.17. For the minimal finite W -superalgebra $U(\mathfrak{g}, e)$ of type odd, it can be checked that both the Verma module $Z_{U(\mathfrak{g}, e)}(\lambda, c)$ introduced in Section 2.2 and the highest weight module $M_e(\lambda)$ constructed in this section, are essentially identical. Actually, the action of $\Theta_{v_{\frac{r+1}{2}}}, \Theta_{h_i}$ for $1 \leq i \leq k-1$, C , and $\Theta_{[v_{\frac{r+1}{2}}, e]}$ on $Z_{U(\mathfrak{g}, e)}(\lambda, c)$ is translated into the action of $U(\mathfrak{g}_0, e)$ on $M_e(\lambda)$. And the restriction (2.8) for $Z_{U(\mathfrak{g}, e)}(\lambda, c)$ is converted into the one in Lemma 4.13 (or (4.18)) for $M_e(\lambda)$.

However, each of the two definitions has its own advantages. On one hand, the $U(\mathfrak{g}, e)$ -module $Z_{U(\mathfrak{g}, e)}(\lambda, c)$ is easy to construct, and it is much like the highest weight theory for $U(\mathfrak{g})$. Moreover, it is more convenient to establish a link between $Z_{U(\mathfrak{g}, e)}(\lambda, c)$ and the $\mathfrak{g}$ -modules obtained by parabolic induction from Whittaker modules for $\mathfrak{osp}(1|2)$ (i.e., the standard Whittaker modules) as in Section 3.2. On the other hand, one can observe that the related theory for $U(\mathfrak{g}, e)$ -module $M_e(\lambda)$ is more fruitful, not only allowing us to define the corresponding BGG category $\mathcal{O}$, but also providing a method that may be generalized to finite W -superalgebras associated with other nilpotent elements.

The above discussion also apply for the minimal refined W -superalgebra W'_χ of both types. See Appendix A for more details.

Appendix A. On the category $\mathcal{O}$ for minimal refined W -superalgebras

As a counterpart of Section 4, we will consider the abstract universal highest weight modules for minimal refined W -superalgebras W'_χ of both types, and then consider the corresponding BGG category $\mathcal{O}$ in this appendix. Since the discussions for them are similar to Section 4, we will just sketch them, omitting the proofs.

Appendix A.1.

We first consider the minimal refined W -superalgebra W'_χ of type odd.

Keep the notation in Sections 3.2 and 4. Recall that for the restricted root decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi'_e} \mathfrak{g}_\alpha$ with respect to $\mathfrak{h}^e$, we have $\mathfrak{g}_0 \cong \mathfrak{h}^e \oplus \mathfrak{osp}(1|2)$ as in Section 3.2.2. Moreover, $(\Phi'_e)^+ = (\Phi^+ \setminus \{\frac{\theta}{2}, \theta\})|_{(\mathfrak{h}^e)^*}$ is a system of positive roots in the restricted root system Φ'_e , and $(\Phi'_e)^- = -(\Phi'_e)^+$. Define the minimal refined W -superalgebra $(W_0)'_\chi$ associated to $e \in \mathfrak{g}_0$ by $(W_0)'_\chi := (Q_0)_{\chi}^{\text{ad } \mathfrak{m}'_0}$, where $\mathfrak{m}'_0$ is the *extended χ -admissible subalgebra* introduced in Section 4.2. By the same discussion as in Section 4.2, we have the following proposition.

Proposition A.1. *The following statements hold:*

(1) *The minimal refined W -superalgebra $(W_0)'_\chi$ is generated by*

- (i) $\Theta'_{h_i} = h_i \otimes 1_\chi$ for $1 \leq i \leq k-1$;
- (ii) $\Theta'_E = (E + \frac{1}{2}Fh - \frac{3}{4}F) \otimes 1_\chi$;
- (iii) $C'_\theta = (2e + \frac{1}{2}h^2 - \frac{3}{2}h + FE) \otimes 1_\chi$;

subject to the relation $[\Theta'_E, \Theta'_E] = C'_\theta + \frac{1}{8} \otimes 1_\chi$, and the commutators between other generators are all zero.

(2) *The center $Z((W_0)'_\chi)$ of $(W_0)'_\chi$ is generated by Θ'_{h_i} for $1 \leq i \leq k-1$ and C'_θ .*

- (3) The map $\text{Pr}_0: U(\mathfrak{g}_0) \twoheadrightarrow U(\mathfrak{g}_0)/(I_0)_\chi$ sends $Z(U(\mathfrak{g}_0))$ isomorphically onto the center of $(W_0)'_\chi$.
- (4) For $\lambda \in (\mathfrak{h}^e)^*$, let $V_\lambda := \mathbb{C}v_\lambda$ be a vector space spanned by $v_\lambda \in (V_\lambda)_{\bar{0}}$ satisfying $\Theta'_E(v_\lambda) = 0$, $C'_\theta(v_\lambda) = -\frac{1}{8}v_\lambda$, and $\Theta'_t(v_\lambda) = \lambda(t)v_\lambda$ for all $t \in \mathfrak{h}^e$. Then the set $\{V_\lambda \mid \lambda \in (\mathfrak{h}^e)^*\}$ forms a complete set of pairwise inequivalent finite-dimensional irreducible $(W_0)'_\chi$ -modules, all of which are of type M .

We have the following PBW basis for W'_χ :

$$(A.1) \quad \prod_{i=1}^w \Theta_{x_i}^{a_i} \cdot \prod_{i=1}^{\ell} \Theta_{y_i}^{c_i} \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i}^{m_i} \cdot \prod_{i=1}^{\frac{r-1}{2}} \Theta_{g_i}^{p_i} \cdot \prod_{i=1}^{k-1} \Theta_{h_i}^{t_i} \cdot C^{t_k} \\ \cdot \Theta_{[v_{\frac{r+1}{2}}, e]}^{\varepsilon} \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i^*}^{n_i} \cdot \prod_{i=1}^{\frac{r-1}{2}} \Theta_{g_i^*}^{q_i} \cdot \prod_{i=1}^w \Theta_{x_i^*}^{b_i} \cdot \prod_{i=1}^{\ell} \Theta_{y_i^*}^{d_i},$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_+^w$, $\mathbf{c}, \mathbf{d} \in \Lambda_\ell$, $\mathbf{m}, \mathbf{n} \in \mathbb{Z}_+^{\frac{s}{2}}$, $\mathbf{p}, \mathbf{q} \in \Lambda_{\frac{r-1}{2}}$, $\varepsilon \in \Lambda_1$, $\mathbf{t} \in \mathbb{Z}_+^k$. Let v be any element in (4.7) excluding e . Since $\theta(\mathfrak{h}^e) = 0$ by definition, from the explicit description of Θ_v in Theorem 2.2, we see that v and Θ_v have the same $\mathfrak{h}^e$ -weight. Also note that the $\mathfrak{h}^e$ -weight of C is zero. As $\mathfrak{h}^e$ -root spaces, W'_χ can be decomposed into $W'_\chi = \bigoplus_{\alpha \in \mathbb{Z}\Phi'_e} (W'_\chi)_\alpha$. The restricted root space $(W'_\chi)_\alpha$ has a basis given by all the PBW monomials as in (A.1) such that

$$\sum_i (-a_i + b_i) \beta_{\bar{0}i} + \sum_i (-c_i + d_i) \beta_{\bar{1}i} + \sum_i (m_i - n_i) \gamma_{\bar{0}i} + \sum_i (p_i - q_i) \gamma_{\bar{1}i} = \alpha.$$

In particular, $(W'_\chi)_0$ is the zero weight space of W'_χ . Set $(W'_\chi)_\#$ (resp. $(W'_\chi)_b$) to be the left (resp. right) ideal of W'_χ generated by all $(W'_\chi)_\alpha$ for $\alpha \in (\Phi'_e)^+$ (resp. $\alpha \in (\Phi'_e)^-$), and denote $(W'_\chi)_{0,\#} := (W'_\chi)_0 \cap (W'_\chi)_\#$, $(W'_\chi)_{b,0} := (W'_\chi)_b \cap (W'_\chi)_0$. By the PBW theorem we have $(W'_\chi)_{0,\#} = (W'_\chi)_{b,0}$, hence it is a two-sided ideal of $(W'_\chi)_0$.

Let S_ϵ be a shift on W'_χ by keeping all other generators as in Proposition A.1(1) invariant and sending C'_θ to $C'_\theta + \epsilon$ with ϵ having the same meaning as in (3.4). By a similar discussion to the proof of Proposition 4.8, we obtain the following proposition.

Proposition A.2. *The projection $\pi: U(\mathfrak{g})_0 \twoheadrightarrow U(\mathfrak{g}_0)$ as in (4.6) induces a surjective homomorphism $\pi: (W'_\chi)_0 \twoheadrightarrow (W_0)'_\chi$ with $\ker \pi = (W'_\chi)_{0,\#}$, and there exists an algebra isomorphism*

$$\pi_\epsilon := S_\epsilon \circ \pi: (W'_\chi)_0 / (W'_\chi)_{0,\#} \cong (W_0)'_\chi.$$

For a W'_χ -module M and $\alpha \in (\mathfrak{h}^e)^*$, we define the α -weight space

$$M_\alpha := \{m \in M \mid (\Theta_t - \delta(t))(m) = \alpha(t)m \text{ for all } t \in \mathfrak{h}^e\}.$$

Since the right action of $(W'_\chi)_0$ factors through the map π_e from Proposition A.2 to make $W'_\chi/(W'_\chi)_\#$ into a $(W'_\chi, (W_0)'_\chi)$ -bimodule, then the highest weight W'_χ -module with highest weight λ can be defined as

$$M_e(\lambda) := (W'_\chi/(W'_\chi)_\#) \otimes_{(W_0)'_\chi} V_\lambda,$$

where the $(W_0)'_\chi$ -module V_λ is one-dimensional, defined as in Proposition A.1(4). Moreover, we have the following theorem.

Theorem A.3. *For $\lambda \in (\mathfrak{h}^e)^*$, let v_λ be a basis for the $(W_0)'_\chi$ -module V_λ with $\mathfrak{h}^e$ -weight λ as in Proposition A.1(4).*

- (1) *The vectors $\{\prod_{i=1}^w \Theta_{x_i}^{a_i} \cdot \prod_{i=1}^\ell \Theta_{y_i}^{b_i} \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i}^{c_i} \cdot \prod_{i=1}^{\frac{r-1}{2}} \Theta_{g_i}^{d_i}(v_\lambda) \mid \mathbf{a} \in \mathbb{Z}_+^w, \mathbf{b} \in \Lambda_\ell, \mathbf{c} \in \mathbb{Z}_+^{\frac{s}{2}}, \mathbf{d} \in \Lambda_{\frac{r-1}{2}}\}$ form a basis of $M_e(\lambda)$.*
- (2) *The weight λ is the unique maximal weight of $M_e(\lambda)$ in the dominance ordering, $M_e(\lambda)$ is generated by the maximal weight space $M_e(\lambda)_\lambda$, and $M_e(\lambda)_\lambda \cong V_\lambda$ as $(W_0)'_\chi$ -modules.*
- (3) *The module $M_e(\lambda)$ is a universal highest weight module with highest weight λ , i.e., if M is another highest weight module generated by a maximal weight space M_μ and $f: V_\lambda \xrightarrow{\sim} M_\mu$ is a $(W_0)'_\chi$ -module isomorphism, then there is a unique W'_χ -module homomorphism $\tilde{f}: M_e(\lambda) \twoheadrightarrow M$ extending f .*
- (4) *There is a unique maximal proper submodule $M_e^{\max}(\lambda)$ in $M_e(\lambda)$,*

$$(A.2) \quad L_e(\lambda) := M_e(\lambda)/M_e^{\max}(\lambda)$$

is an irreducible module type M , and $\{L_e(\lambda) \mid \lambda \in (\mathfrak{h}^e)^\}$ is a complete set of pairwise inequivalent irreducible highest weight modules over W'_χ . Moreover, any finite-dimensional simple W'_χ -module is isomorphic to one of the modules $L_e(\lambda)$ for $\lambda \in \Lambda_0^+ = \{\lambda \in (\mathfrak{h}^e)^* \mid \lambda(h_\alpha) \in \mathbb{Z}_+ \text{ for } \alpha \in (\Phi_{e,0}^+)_{\bar{0}}\}$.*

We say that a W'_χ -module V is of central character $\psi: Z(W'_\chi) \rightarrow \mathbb{C}$ if $z(v) = \psi(z)v$ for all $z \in Z(W'_\chi)$ and $v \in V$. For the highest weight W'_χ -module $M_e(\lambda)$ with highest weight $\lambda \in (\mathfrak{h}^e)^*$, let $\psi^\lambda: Z(W'_\chi) \rightarrow \mathbb{C}$ be the corresponding central character. Repeating verbatim the proofs of Theorem 4.14 and Corollary 4.15, we obtain the following theorem.

Theorem A.4. *The number of isomorphism classes of irreducible highest weight modules for W'_χ with prescribed central character $\psi: Z(W'_\chi) \rightarrow \mathbb{C}$ is finite, i.e., the set $\{\lambda \in (\mathfrak{h}^e)^* \mid \psi^\lambda = \psi\}$ is finite.*

Corollary A.5. *For each $\lambda \in (\mathfrak{h}^e)^*$, the highest weight module $M_e(\lambda)$ has composition series.*

Now an analogue of the BGG category $\mathcal{O}$ can also be introduced. Denote $\mathfrak{q} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in (\Phi'_e)^+} \mathfrak{g}_\alpha$, and let $\mathcal{O}(e) = \mathcal{O}(e; \mathfrak{h}, \mathfrak{q})$ denote the category of all finitely generated W'_χ -modules V that are semi-simple over $\mathfrak{h}^e$ with finite-dimensional $\mathfrak{h}^e$ -weight spaces, such that the set $\{\lambda \in (\mathfrak{h}^e)^* \mid V_\lambda \neq \{0\}\}$ is contained in a finite union of sets of the form $\{\nu \in (\mathfrak{h}^e)^* \mid \nu \leq \mu\}$ for $\mu \in (\mathfrak{h}^e)^*$. Then we have the following theorem.

Theorem A.6. *For the category $\mathcal{O}(e)$, the following statements hold:*

- (1) *There is a complete set of isomorphism classes of simple objects which is $\{L_e(\lambda) \mid \lambda \in (\mathfrak{h}^e)^*\}$ as in (A.2).*
- (2) *The category $\mathcal{O}(e)$ is Artinian. In particular, every object has finite length of composition series.*
- (3) *The category $\mathcal{O}(e)$ has a block decomposition as $\mathcal{O}(e) = \bigoplus_{\psi^\lambda} \mathcal{O}_{\psi^\lambda}(e)$, where the direct sum is over all central characters $\psi^\lambda: Z(U(\mathfrak{g}, e)) \rightarrow \mathbb{C}$, and $\mathcal{O}_{\psi^\lambda}(e)$ denotes the Serre subcategory of $\mathcal{O}(e)$ generated by the irreducible modules $\{L_e(\mu) \mid \mu \in (\mathfrak{h}^e)^* \text{ such that } \psi^\mu = \psi^\lambda\}$.*

Appendix A.2.

It remains to deal with the minimal refined (equivalently, finite) W -superalgebra W'_χ of type even. In this case, it is much like the situation of finite W -algebras as in [12, §4], and we still just give a brief description.

Keep the notation in Sections 3.3 and 4.1. The adjoint action of $\mathfrak{h}^e$ on $\mathfrak{g}$ induces the restricted root decompositions $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi'_e} \mathfrak{g}_\alpha$, and we have $\mathfrak{g}_0 = \mathfrak{h}^e \oplus \mathfrak{s}_\theta \cong \mathfrak{h}^e \oplus \mathfrak{sl}(2)$ as defined in Section 3.3. Let $(\Phi'_e)^+$ be a system of positive roots in the restricted root system Φ'_e , and $(\Phi'_e)^- = -(\Phi'_e)^+$. Let $\mathfrak{m}_0 := \mathbb{C}f$ be the χ -admissible subalgebra of $\mathfrak{g}_0$, $\mathfrak{p}_0 := \mathfrak{h}^e \oplus \mathbb{C}h \oplus \mathbb{C}e$ a parabolic subalgebra of $\mathfrak{g}_0$ with nilradical $\mathbb{C}e$ and Levi subalgebra $\mathfrak{h}$. Define $(Q_0)_\chi := U(\mathfrak{g}_0) \otimes_{U(\mathfrak{m}_0)} \mathbb{C}_\chi$, where $\mathbb{C}_\chi = \mathbb{C}1_\chi$ is a one-dimensional $\mathfrak{m}_0$ -module such that $x \cdot 1_\chi = \chi(x)1_\chi$ for all $x \in \mathfrak{m}_0$. Let $(I_0)_\chi$ denote the $\mathbb{Z}_2$ -graded left ideal in $U(\mathfrak{g}_0)$ generated by all $x - \chi(x)$ with $x \in \mathfrak{m}_0$, and write $\text{Pr}_0: U(\mathfrak{g}_0) \twoheadrightarrow U(\mathfrak{g}_0)/(I_0)_\chi$ for the canonical homomorphism. Recall that the minimal refined W -superalgebra $(W_0)_\chi'$ associated to e is defined by

$$(W_0)_\chi' := (\text{End}_{\mathfrak{g}_0}(Q_0)_\chi)^{\text{op}} \cong (Q_0)_\chi^{\text{ad } \mathfrak{m}_0}.$$

So $(W_0)_\chi' \cong \{u \in U(\mathfrak{p}_0) \mid \text{Pr}_0([x, u]) = 0 \text{ for all } x \in \mathfrak{m}_0\}$, which is a subalgebra of $U(\mathfrak{p}_0)$. By a similar discussion to Section 4.2, we have the following proposition.

Proposition A.7. *The following statements hold:*

(1) *The minimal refined W -superalgebra $(W_0)'_\chi$ is generated by*

- (i) $\Theta'_{h_i} = h_i \otimes 1_\chi$ for $1 \leq i \leq k-1$;
- (ii) $C'_\theta = (2e + \frac{1}{2}h^2 - h) \otimes 1_\chi$,

and the commutators between the generators are all zero.

(2) *The algebra $(W_0)'_\chi$ is commutative.*

(3) *The map Pr_0 sends $Z(U(\mathfrak{g}_0))$ isomorphically onto $(W_0)'_\chi$.*

(4) *For $\lambda \in (\mathfrak{h}^e)^*$ and $c \in \mathbb{C}$, let $V_{\lambda,c} := \mathbb{C}v_{\lambda,c}$ be a vector space spanned by $v_{\lambda,c} \in (V_{\lambda,c})_{\bar{0}}$ satisfying $C'_\theta(v_{\lambda,c}) = cv_{\lambda,c}$, and $\Theta'_t(v_{\lambda,c}) = \lambda(t)v_{\lambda,c}$ for all $t \in \mathfrak{h}^e$. Then the set $\{V_{\lambda,c} \mid \lambda \in (\mathfrak{h}^e)^*, c \in \mathbb{C}\}$ forms a complete set of pairwise inequivalent finite-dimensional irreducible $(W_0)'_\chi$ -modules, all of which are of type M .*

The adjoint action of $\mathfrak{h}^e$ on the universal enveloping algebra $U(\mathfrak{g})$ induces decomposition $U(\mathfrak{g}) = \bigoplus_{\alpha \in \mathbb{Z}\Phi'_e} U(\mathfrak{g})_\alpha$. Then $U(\mathfrak{g})_0$, the zero weight space of $U(\mathfrak{g})$ with respect to the adjoint action, is a subalgebra of $U(\mathfrak{g})$. Let $U(\mathfrak{g})_\#$ (resp. $U(\mathfrak{g})_\flat$) denote the left (resp. right) ideal of $U(\mathfrak{g})$ generated by the root spaces $\mathfrak{g}_\alpha$ for $\alpha \in (\Phi'_e)^+$ (resp. $\alpha \in (\Phi'_e)^-$). Let

$$U(\mathfrak{g})_{0,\#} := U(\mathfrak{g})_0 \cap U(\mathfrak{g})_\#, \quad U(\mathfrak{g})_{\flat,0} := U(\mathfrak{g})_\flat \cap U(\mathfrak{g})_0,$$

which are left and right ideals of $U(\mathfrak{g})_0$, respectively. By the PBW theorem, $U(\mathfrak{g})_{0,\#}$ is a two-sided ideal of $U(\mathfrak{g})_0$, and $U(\mathfrak{g})_0 = U(\mathfrak{g}_0) \oplus U(\mathfrak{g})_{0,\#}$. The projection along this decomposition defines a surjective algebra homomorphism

$$(A.3) \quad \pi: U(\mathfrak{g})_0 \twoheadrightarrow U(\mathfrak{g}_0)$$

with $\ker \pi = U(\mathfrak{g})_{0,\#}$. Hence $U(\mathfrak{g})_0/U(\mathfrak{g})_{0,\#} \cong U(\mathfrak{g}_0)$ as $\mathbb{C}$ -algebras.

Recall that in Sections 1.2 and 2.1.1 we have a basis consisting of $\mathfrak{h}^e$ -weight vectors

$$(A.4) \quad \begin{aligned} & x_1, \dots, x_w, y_1, \dots, y_\ell, f_1, \dots, f_{\frac{r}{2}}, g_1, \dots, g_{\frac{r}{2}}, h_1, \dots, h_{k-1}, \\ & e, f_1^*, \dots, f_{\frac{r}{2}}^*, g_1^*, \dots, g_{\frac{r}{2}}^*, x_1^*, \dots, x_w^*, y_1^*, \dots, y_\ell^* \end{aligned}$$

of $\mathfrak{g}^e$ so that the weights of x_i, y_j, f_k, g_l are respectively $-\beta_{\bar{0}i}, -\beta_{\bar{1}j}, \theta + \gamma_{\bar{0}k}, \theta + \gamma_{\bar{1}l} \in (\Phi'_e)^-$, and the weights of $f_k^*, g_l^*, x_i^*, y_j^*$ are respectively $\theta + \gamma_{\bar{0}k}^*, \theta + \gamma_{\bar{1}l}^*, \beta_{\bar{0}i}, \beta_{\bar{1}j} \in (\Phi'_e)^+$, while $h_i, e \in \mathfrak{g}_0^e$. Moreover, we have the following PBW basis

for W'_χ :

$$(A.5) \quad \prod_{i=1}^w \Theta_{x_i}^{a_i} \cdot \prod_{i=1}^\ell \Theta_{y_i}^{c_i} \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i}^{m_i} \cdot \prod_{i=1}^{\frac{r}{2}} \Theta_{g_i}^{p_i} \cdot \prod_{i=1}^{k-1} \Theta_{h_i}^{t_i} \\ \cdot C^{t_k} \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i^*}^{n_i} \cdot \prod_{i=1}^{\frac{r}{2}} \Theta_{g_i^*}^{q_i} \cdot \prod_{i=1}^w \Theta_{x_i^*}^{b_i} \cdot \prod_{i=1}^\ell \Theta_{y_i^*}^{d_i},$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_+^w$, $\mathbf{c}, \mathbf{d} \in \Lambda_\ell$, $\mathbf{m}, \mathbf{n} \in \mathbb{Z}_+^{\frac{s}{2}}$, $\mathbf{p}, \mathbf{q} \in \Lambda_{\frac{r}{2}}$, $\mathbf{t} \in \mathbb{Z}_+^k$. Let v be one of the elements in (A.4) excluding e . Since $\theta(\mathfrak{h}^e) = 0$ by definition, from the explicit description of Θ_v in Theorem 2.2, we see that v and Θ_v have the same $\mathfrak{h}^e$ -weight. Also note that the $\mathfrak{h}^e$ -weight of C is zero. As $\mathfrak{h}^e$ -root spaces, W'_χ can be decomposed as $W'_\chi = \bigoplus_{\alpha \in \mathbb{Z}\Phi'_e} (W'_\chi)_\alpha$. The restricted root space $(W'_\chi)_\alpha$ has a basis given by all the PBW monomials as in (A.5) such that

$$\sum_i (-a_i + b_i) \beta_{\bar{0}i} + \sum_i (-c_i + d_i) \beta_{\bar{1}i} + \sum_i (m_i - n_i) \gamma_{\bar{0}i} + \sum_i (p_i - q_i) \gamma_{\bar{1}i} = \alpha,$$

and $(W'_\chi)_0$ is the zero weight space of W'_χ . Set $(W'_\chi)_\#$ (resp. $(W'_\chi)_b$) to be the left (resp. right) ideal of W'_χ generated by all $(W'_\chi)_\alpha$ for $\alpha \in (\Phi'_e)^+$ (resp. $\alpha \in (\Phi'_e)^-$), and denote $(W'_\chi)_{0,\#} := (W'_\chi)_0 \cap (W'_\chi)_\#$, $(W'_\chi)_{b,0} := (W'_\chi)_b \cap (W'_\chi)_0$. We have $(W'_\chi)_{0,\#} = (W'_\chi)_{b,0}$; then it is a two-sided ideal of $(W'_\chi)_0$.

Retain the notation δ in (3.32). By a similar discussion to the proof of Proposition 4.8, we have the following proposition.

Proposition A.8. *The projection $\pi: U(\mathfrak{g})_0 \rightarrow U(\mathfrak{g}_0)$ as in (A.3) induces a surjective homomorphism $\pi: (W'_\chi)_0 \rightarrow (W_0)_\chi'$ with $\ker \pi = (W'_\chi)_{0,\#}$. Hence there exists an algebra isomorphism*

$$\pi: (W'_\chi)_0 / (W'_\chi)_{0,\#} \cong (W_0)_\chi'.$$

Remark A.9. Comparing Proposition A.8 with Proposition 4.8, one can observe a significant difference in the absence of the shift S_ϵ . This is because the element $\Theta_{[v_{\frac{r+1}{2}}, e]}$ does not exist for the present case. So we need not consider (4.14), which directly leads to the emergence of S_ϵ in Proposition 4.8.

For a W'_χ -module M and $\alpha \in (\mathfrak{h}^e)^*$, we define the α -weight space

$$M_\alpha := \{m \in M \mid (\Theta_t - \delta(t))(m) = \alpha(t)m \text{ for all } t \in \mathfrak{h}^e\}.$$

By the same consideration as in Section 4.4, we get the action of $\mathfrak{h}^e$ on M_α via

$$t(m) = \Theta'_t(m) = \alpha(t)m$$

for all $t \in \mathfrak{h}^e$, which explains why the additional shift by $-\delta$ in the definition of the α -weight space of a W'_χ -module is necessary.

Associated with the $(W_0)'_\chi$ -module $V_{\lambda,c}$ as in Proposition A.7(4), we can define the highest weight W'_χ -module of type (λ, c) as

$$M_e(\lambda, c) := (W'_\chi / (W'_\chi)_\#) \otimes_{(W_0)'_\chi} V_{\lambda,c}.$$

Theorem A.10. *For $\lambda \in (\mathfrak{h}^e)^*$ and $c \in \mathbb{C}$, let $v_{\lambda,c}$ be a basis for a $(W_0)'_\chi$ -module $V_{\lambda,c}$ of level c with $\mathfrak{h}^e$ -weight λ as in Proposition A.7(4).*

- (1) *The vectors $\{\prod_{i=1}^w \Theta_{x_i}^{a_i} \cdot \prod_{i=1}^\ell \Theta_{y_i}^{b_i} \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i}^{c_i} \cdot \prod_{i=1}^{\frac{t}{2}} \Theta_{g_i}^{d_i}(v_\lambda) \mid \mathbf{a} \in \mathbb{Z}_+^w, \mathbf{b} \in \Lambda_\ell, \mathbf{c} \in \mathbb{Z}_+^{\frac{s}{2}}, \mathbf{d} \in \Lambda_{\frac{t}{2}}\}$ form a basis of $M_e(\lambda, c)$.*
- (2) *The weight λ is the unique maximal weight of $M_e(\lambda, c)$ in the dominance ordering, $M_e(\lambda, c)$ is generated by the maximal weight space $M_e(\lambda, c)_\lambda$, and $M_e(\lambda, c)_\lambda \cong V_{\lambda,c}$ as $(W_0)'_\chi$ -modules.*
- (3) *The module $M_e(\lambda, c)$ is a universal highest weight module of type (λ, c) , i.e., if M is another highest weight module generated by a maximal weight space $M_{\mu,c}$ and $f: V_{\lambda,c} \xrightarrow{\sim} M_{\mu,c}$ is a $(W_0)'_\chi$ -module isomorphism, then there is a unique W'_χ -module homomorphism $f: M_e(\lambda, c) \rightarrow M$ extending f .*
- (4) *There is a unique maximal proper submodule $M_e^{\max}(\lambda, c)$ in $M_e(\lambda, c)$,*

$$(A.6) \quad L_e(\lambda, c) := M_e(\lambda, c) / M_e^{\max}(\lambda, c)$$

is an irreducible module type M , and $\{L_e(\lambda, c) \mid \lambda \in (\mathfrak{h}^e)^, c \in \mathbb{C}\}$ is a complete set of pairwise inequivalent irreducible highest weight modules over W'_χ . Moreover, any finite-dimensional simple W'_χ -module is isomorphic to one of the modules $L_e(\lambda, c)$ for $\lambda \in \Lambda_0^+ = \{\lambda \in (\mathfrak{h}^e)^* \mid \lambda(h_\alpha) \in \mathbb{Z}_+ \text{ for } \alpha \in (\Phi_{e,0}^+)_{\bar{0}}\}$ and $c \in \mathbb{C}$. We further have that c is a rational number in the case when $\mathfrak{g}$ is a simple Lie algebra except type $A(m)$, or when $\mathfrak{g} = \mathfrak{psl}(2|2)$, or when $\mathfrak{spo}(2|m)$ with m being even such that $\mathfrak{g}^e(0) = \mathfrak{so}(m)$, or when $\mathfrak{osp}(4|2m)$ with $\mathfrak{g}^e(0) = \mathfrak{sl}(2) \oplus \mathfrak{sp}(2m)$, or when $\mathfrak{g} = \mathfrak{osp}(5|2m)$ with $\mathfrak{g}^e(0) = \mathfrak{osp}(1|2m) \oplus \mathfrak{sl}(2)$, or when $\mathfrak{g} = D(2, 1; \alpha)$ with $\alpha \in \overline{\mathbb{Q}}$, or when $\mathfrak{g} = F(4)$ with $\mathfrak{g}^e(0) = \mathfrak{so}(7)$.*

We say that a W'_χ -module V is of central character $\psi: Z(W'_\chi) \rightarrow \mathbb{C}$ if $z(v) = \psi(z)v$ for all $z \in Z(W'_\chi)$ and $v \in V$. For the highest weight W'_χ -module $M_e(\lambda, c)$ of type $(\lambda, c) \in (\mathfrak{h}^e)^* \times \mathbb{C}$, let $\psi^{\lambda,c}: Z(W'_\chi) \rightarrow \mathbb{C}$ be the corresponding central character. By a similar discussion to Theorem 4.14 and Corollary 4.15, we obtain the following theorem.

Theorem A.11. *The number of isomorphism classes of irreducible highest weight modules for W'_χ with prescribed central character $\psi: Z(W'_\chi) \rightarrow \mathbb{C}$ is finite, i.e., the set $\{(\lambda, c) \in (\mathfrak{h}^e)^* \times \mathbb{C} \mid \psi^{\lambda,c} = \psi\}$, is finite.*

Corollary A.12. *For each $(\lambda, c) \in (\mathfrak{h}^e)^* \times \mathbb{C}$, the highest weight module $M_e(\lambda, c)$ has composition series.*

Now an analogue of the BGG category $\mathcal{O}$ can also be introduced. Denote $\mathfrak{q} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in (\Phi'_e)^+} \mathfrak{g}_\alpha$, and let $\mathcal{O}(e) = \mathcal{O}(e; \mathfrak{h}, \mathfrak{q})$ denote the category of all finitely generated W'_χ -modules V , that are semi-simple over $\mathfrak{h}^e$ with finite-dimensional $\mathfrak{h}^e$ -weight spaces, such that the set $\{\lambda \in (\mathfrak{h}^e)^* \mid V_\lambda \neq \{0\}\}$ is contained in a finite union of sets of the form $\{\nu \in (\mathfrak{h}^e)^* \mid \nu \leq \mu\}$ for $\mu \in (\mathfrak{h}^e)^*$. Then we have the following theorem.

Theorem A.13. *For the category $\mathcal{O}(e)$, the following statements hold:*

- (1) *There is a complete set of isomorphism classes of simple objects which is $\{L_e(\lambda, c) \mid \lambda \in (\mathfrak{h}^e)^*, c \in \mathbb{C}\}$ as in (A.6).*
- (2) *The category $\mathcal{O}(e)$ is Artinian. In particular, every object has finite length of composition series.*
- (3) *The category $\mathcal{O}(e)$ has a block decomposition as $\mathcal{O}(e) = \bigoplus_{\psi^{\lambda, c}} \mathcal{O}_{\psi^{\lambda, c}}(e)$, where the direct sum is over all central characters $\psi^{\lambda, c}: Z(W'_\chi) \rightarrow \mathbb{C}$, and $\mathcal{O}_{\psi^{\lambda, c}}(e)$ denotes the Serre subcategory of $\mathcal{O}(e)$ generated by the irreducible modules $\{L_e(\mu, c) \mid \mu \in (\mathfrak{h}^e)^* \text{ such that } \psi^{\mu, c} = \psi^{\lambda, c}\}$.*

Remark A.14. It is worth mentioning that there are also some other methods to interpret the representation theory of finite W -superalgebras, which are only suitable for some basic Lie superalgebras but associated with arbitrary even nilpotents. To be explicit, given a basic Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of type I and arbitrary nilpotent element $e \in \mathfrak{g}_0$, denote by $\mathcal{W}$ the corresponding finite W -superalgebras. Xiao [47] introduced an extended W -superalgebra $\widetilde{\mathcal{W}}$ which contains $\mathcal{W}$ as a subalgebra, and then established a bijection between the isomorphism classes of finite-dimensional irreducible $\mathcal{W}$ -modules and $\widetilde{\mathcal{W}}$ -modules. With aid of this bijection, the *Verma modules* (which are much like the Kac modules formulated for basic Lie superalgebras) for $\widetilde{\mathcal{W}}$ were also defined there, and the finite-dimensional simple $\mathcal{W}$ -modules with integral central character were classified. Furthermore, an algorithm for computing their characters was given. If $e = e_\theta$ with $-\theta$ being a minimal root, it corresponds to the cases when $\mathfrak{g}$ is a simple Lie algebra, or when $\mathfrak{sl}(m|n)$ with $m \neq n$, $m \geq 2$, $\mathfrak{psl}(m|m)$ with $m \geq 2$, or when $\mathfrak{spo}(2m|2)$. All these correspond to a subclass of the minimal refined W -superalgebra W'_χ of type even in Table 2, which we have already studied directly; see the beginning of Section 3.3 for more details.

Appendix B. Proof of Lemma 2.7

This appendix is devoted to the proof of Lemma 2.7. We mainly follow Premet's strategy on finite W -algebras as in [38, Lem. 7.1], with a few modifications. Compared with the non-super situation, one can observe significant differences for the emergence of the restriction (2.8).

Appendix B.1.

For $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_+^w$, $\mathbf{c}, \mathbf{d} \in \Lambda_\ell$, $\mathbf{m}, \mathbf{n} \in \mathbb{Z}_+^{\frac{s}{2}}$, $\mathbf{p}, \mathbf{q} \in \Lambda_{\frac{r-1}{2}}^k$, $\mathbf{t} \in \mathbb{Z}_+^k$, $\iota, \varepsilon \in \Lambda_1$, set

$$\begin{aligned} \Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q}, \mathbf{t}, \iota, \varepsilon) \\ := \prod_{i=1}^w \Theta_{x_i}^{a_i} \cdot \prod_{i=1}^\ell \Theta_{y_i}^{c_i} \cdot \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i}^{m_i} \cdot \prod_{i=1}^{\frac{r-1}{2}} \Theta_{g_i}^{p_i} \cdot \Theta_{v_{\frac{r+1}{2}}}^{\iota} \cdot \prod_{i=1}^{k-1} \Theta_{h_i}^{t_i} \\ \cdot C^{t_k} \cdot \Theta_{[v_{\frac{r+1}{2}}, e]}^{\varepsilon} \prod_{i=1}^{\frac{s}{2}} \Theta_{f_i^*}^{n_i} \cdot \prod_{i=1}^{\frac{r-1}{2}} \Theta_{g_i^*}^{q_i} \cdot \prod_{i=1}^w \Theta_{x_i^*}^{b_i} \cdot \prod_{i=1}^\ell \Theta_{y_i^*}^{d_i}. \end{aligned}$$

By Theorem 1.4, the PBW monomials $\Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q}, \mathbf{t}, \iota, \varepsilon)$ form a $\mathbb{C}$ -basis of $U(\mathfrak{g}, e)$. Note that

$$\begin{aligned} \deg_e(\Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q}, \mathbf{t}, \iota, \varepsilon)) \\ = 4t_k + 3(|\mathbf{m}| + |\mathbf{n}| + |\mathbf{p}| + |\mathbf{q}| + \varepsilon) \\ + 2(|\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}| + |\mathbf{d}|) + 2 \sum_{i=1}^{k-1} t_i + \iota. \end{aligned}$$

Recall that $[v_{\frac{r+1}{2}}, e]$ is a root vector corresponding to an odd simple root $\frac{\theta}{2} \in \Phi_{e,1}^+$, and $[\Theta_{v_{\frac{r+1}{2}}}, \Theta_v] = [\Theta_{v_{\frac{r+1}{2}}}, \Theta_w] = [\Theta_{v_{\frac{r+1}{2}}}, C] = 0$ for all $v \in \mathfrak{g}^e(0)$ and $w \in \mathfrak{g}^e(1)$ by Theorem 2.4.

Appendix B.2.

As $C - c$ is in the center of finite W -superalgebra $U(\mathfrak{g}, e)$, we have

$$\Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q}, \mathbf{t}, \iota, \varepsilon)(C - c) \in I_{\lambda, c}.$$

On the other hand, it follows from Theorem 2.4 (more precisely, Theorem 2.3(1)–(2)) that $\Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q}, \mathbf{t}, \iota, \varepsilon) \cdot (\Theta_{h_i} - \lambda(h_i)) \in I_{\lambda, c}$ for $1 \leq i \leq k-1$. Moreover, since $\Theta_{\mathfrak{n}^+(0)}$ is a Lie subalgebra of $\Theta_{\mathfrak{g}^e(0)}$, Theorem 2.4 entails that $\Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q}, \mathbf{t}, \iota, \varepsilon) \cdot \Theta_{e_\alpha} \in I_{\lambda, c}$ for all $\alpha \in \Phi_{e,0}^+$.

Appendix B.3.

It remains to show that $\Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q}, \mathbf{t}, \iota, \varepsilon) \cdot \Theta_H \in I_{\lambda, c}$ with $H \in \{f_1^*, \dots, f_{\frac{r}{2}}^*, g_1^*, \dots, g_{\frac{r-1}{2}}^*, [v_{\frac{r+1}{2}}, e]\}$. To prove this, we will use induction on $\deg_e(\Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q}, \mathbf{t}, \iota, \varepsilon))$. Obviously it is true for the case with $\deg_e(\Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q}, \mathbf{t}, \iota, \varepsilon)) = 0$. From now on we assume that $\deg_e(\Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q}, \mathbf{t}, \iota, \varepsilon)) = N$ and $F_k U(\mathbf{g}, e) \cdot \Theta_H \in I_{\lambda, c}$ for all $k < N$.

B.3.1. Note that the span of $f_1^*, \dots, f_{\frac{r}{2}}^*, g_1^*, \dots, g_{\frac{r-1}{2}}^*, [v_{\frac{r+1}{2}}, e]$ equals $\mathfrak{n}^+(1)$, hence is stable under the adjoint action of $\mathfrak{n}^+(0)$. As we have already established that $U(\mathbf{g}, e) \cdot \Theta_{e_\alpha} \in I_{\lambda, c}$ for all $\alpha \in \Phi_{e, 0}^+$, it follows from Theorem 2.4 (more precisely, Theorem 2.3(2)) that

$$(B.1) \quad \begin{aligned} & \Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q}, \mathbf{t}, \iota, \varepsilon) \cdot \Theta_H \\ & \in \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q}, \mathbf{t}, \iota, \varepsilon) \cdot \Theta_{\mathfrak{n}^+(1)} + I_{\lambda, c}. \end{aligned}$$

Thus we can assume that $\mathbf{b} = \mathbf{d} = \mathbf{0}$.

B.3.2. (i) If $q_j = 0$ for all $j \geq i$, it is immediate that

$$(B.2) \quad \begin{aligned} & \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q}, \mathbf{t}, \iota, \varepsilon) \cdot \Theta_{g_i^*} \\ & = \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q} + \mathbf{e}_i, \mathbf{t}, \iota, \varepsilon) \in I_{\lambda, c}. \end{aligned}$$

So we just need to consider the case $\mathbf{q} = (q_1, \dots, q_k, 0, \dots, 0)$ for some $q_k = 1$ and $k \geq i$. In virtue of [49, Thm. 3.7(3)] and our induction assumption we have

$$(B.3) \quad \begin{aligned} & \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q}, \mathbf{t}, \iota, \varepsilon) \cdot \Theta_{g_i^*} \\ & \in \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q} - \mathbf{e}_k, \mathbf{t}, \iota, \varepsilon) [\Theta_{g_k^*}, \Theta_{g_i^*}] \\ & \quad + (-1)^{q_{i+1} + \dots + q_k} \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q} + \mathbf{e}_i, \mathbf{t}, \iota, \varepsilon) \\ & \quad + F_{N-2} U(\mathbf{g}, e) \cdot \Theta_{g_k^*} \\ & \in \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q} - \mathbf{e}_k, \mathbf{t}, \iota, \varepsilon) [\Theta_{g_k^*}, \Theta_{g_i^*}] \\ & \quad + (-1)^{q_{i+1} + \dots + q_k} \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q} + \mathbf{e}_i, \mathbf{t}, \iota, \varepsilon) \\ & \quad + I_{\lambda, c}. \end{aligned}$$

Now we consider the first term in the last equation of (B.3). Since $\frac{\theta}{2}$ is an odd simple root by Conventions 2.1, then for $1 \leq i, k \leq \frac{r-1}{2}$, we have

$$[z_\alpha^*, g_i^*]^\sharp, [[g_i^*, z_\alpha^*]^\sharp, [z_\alpha^*, g_k^*]^\sharp] \in \bigcup_{\beta \in \Phi_{e, 0}^+} \mathbb{C} e_\beta$$

for $1 \leq \alpha \leq \frac{s}{2}$ and $s+1 \leq \alpha \leq s + \frac{r+1}{2}$. As

$$\begin{aligned} ([g_k^*, g_i^*], f) &= ([e, v_k^*], [e, v_i^*], f) = ([e, v_k^*], [[e, v_i^*], f]) \\ &= ([e, v_k^*], [e, [v_i^*, f]]) + ([e, v_k^*], [[e, f], v_i^*]) \\ &= -([e, v_k^*], v_i^*) = -(e, [v_k^*, v_i^*]) = 0, \end{aligned}$$

one can conclude from Theorem 2.4 and the discussion in Appendix B.2 that

$$\begin{aligned} [\Theta_{g_k^*}, \Theta_{g_i^*}] &= -\frac{1}{2} \sum_{\alpha \in S(-1)} (\Theta_{[g_k^*, z_\alpha]^\#} \Theta_{[z_\alpha^*, g_i^*]^\#} + \Theta_{[g_i^*, z_\alpha]^\#} \Theta_{[z_\alpha^*, g_k^*]^\#}) \\ (B.4) \quad &\in \sum_{\alpha \in \Phi_{e,0}^+} U(\mathfrak{g}, e) \cdot \Theta_{e_\alpha} \in I_{\lambda, c}. \end{aligned}$$

In particular, if $i = k$, the right-hand side of (B.3) equals $\frac{1}{2} \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q} - \mathbf{e}_i, \mathbf{t}, \iota, \varepsilon) \cdot [\Theta_{g_i^*}, \Theta_{g_i^*}] + I_{\lambda, c}$ by definition, which is contained in $I_{\lambda, c}$.

Let us consider the second term in the last equation of (B.3). For $1 \leq i \leq \frac{r-1}{2}$, if $q_i + e_i = 1$, then

$$(B.5) \quad \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q} + \mathbf{e}_i, \mathbf{t}, \iota, \varepsilon) \in I_{\lambda, c}$$

by definition. If $q_i + e_i = 2$, we have

$$\Theta_{g_i^*}^2 = \frac{1}{2} [\Theta_{g_i^*}, \Theta_{g_i^*}] = -\frac{1}{2} \sum_{\alpha \in S(-1)} \Theta_{[g_i^*, z_\alpha]^\#} \Theta_{[z_\alpha^*, g_i^*]^\#}.$$

For $1 \leq i < k \leq \frac{r-1}{2}$, since

$$[z_\alpha^*, g_i^*]^\#, [[z_\alpha^*, g_i^*]^\#, g_k^*], [[g_i^*, z_\alpha^*]^\#, [z_\alpha^*, g_i^*]^\#], [[[g_i^*, z_\alpha^*]^\#, [z_\alpha^*, g_i^*]^\#], g_k^*] \in \bigcup_{\beta \in \Phi_{e,1}^+} \mathbb{C} e_\beta$$

for $1 \leq \alpha \leq \frac{s}{2}$ and $s+1 \leq \alpha \leq s + \frac{r+1}{2}$, then it follows from (B.4), Theorem 2.4, the discussion in Appendix B.2, and our induction assumption that

$$\begin{aligned} &\Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q} + \mathbf{e}_i, \mathbf{t}, \iota, \varepsilon) \\ &\in \sum_{\alpha \in \Phi_{e,0}^+} U(\mathfrak{g}, e) \cdot \Theta_{e_\alpha} + F_{N-4} U(\mathfrak{g}, e) \cdot \Theta_{g_k^*} \\ &\quad + \sum_{\substack{1 \leq \alpha \leq \frac{s}{2}, \\ s+1 \leq \alpha \leq s + \frac{r+1}{2}}} F_{N-4} U(\mathfrak{g}, e) \cdot \Theta_{[[z_\alpha^*, g_i^*]^\#, g_k^*]} \\ (B.6) \quad &+ \sum_{\substack{1 \leq \alpha \leq \frac{s}{2}, \\ s+1 \leq \alpha \leq s + \frac{r-1}{2}}} F_{N-6} U(\mathfrak{g}, e) \cdot \Theta_{[[[g_i^*, z_\alpha^*]^\#, [z_\alpha^*, g_i^*]^\#], g_k^*]} \in I_{\lambda, c}. \end{aligned}$$

Therefore, (B.2)–(B.6) show that

$$(B.7) \quad \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q}, \mathbf{t}, \iota, \varepsilon) \cdot \Theta_{g_i^*} \in I_{\lambda, c}$$

for all $1 \leq i \leq \frac{r-1}{2}$.

(ii) By parity considerations we know that $([g_k^*, f_i^*], f) = 0$ for all $1 \leq k \leq \frac{r-1}{2}$ and $1 \leq i \leq \frac{s}{2}$, and it is obvious that

$$[z_\alpha^*, f_i^*]^\sharp, [z_\alpha^*, g_k^*]^\sharp, [[g_k^*, z_\alpha^*]^\sharp, [z_\alpha^*, f_i^*]^\sharp], [[f_i^*, z_\alpha^*]^\sharp, [z_\alpha^*, g_k^*]^\sharp] \in \bigcup_{\beta \in \Phi_{e,0}^+} \mathbb{C}e_\beta$$

for $1 \leq \alpha \leq \frac{s}{2}$ and $s+1 \leq \alpha \leq s + \frac{r+1}{2}$. Then Theorem 2.4 yields

$$(B.8) \quad \begin{aligned} [\Theta_{g_k^*}, \Theta_{f_i^*}] &= -\frac{1}{2} \sum_{\alpha \in S(-1)} (\Theta_{[g_k^*, z_\alpha]^\sharp} \Theta_{[z_\alpha^*, f_i^*]^\sharp} - \Theta_{[f_i^*, z_\alpha]^\sharp} \Theta_{[z_\alpha^*, g_k^*]^\sharp}) \\ &\in \sum_{\alpha \in \Phi_{e,0}^+} U(\mathbf{g}, e) \cdot \Theta_{e_\alpha}. \end{aligned}$$

As $U(\mathbf{g}, e) \cdot \Theta_{e_\alpha} \in I_{\lambda, c}$ for all $\alpha \in \Phi_{e,0}^+$, if there exists $q_l \neq 0$ for $1 \leq l \leq \frac{r-1}{2}$, then (B.7) and (B.8) entail that

$$(B.9) \quad \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q}, \mathbf{t}, \iota, \varepsilon) \cdot \Theta_{f_i^*} \in I_{\lambda, c}.$$

(iii) By weight considerations we know that $([g_i^*, [v_{\frac{r+1}{2}}, e]], f) = 0$ for $1 \leq i \leq \frac{r-1}{2}$. Since

$$\begin{aligned} &[z_\alpha^*, g_i^*]^\sharp, [z_\alpha^*, [v_{\frac{r+1}{2}}, e]]^\sharp, [[g_i^*, z_\alpha^*]^\sharp, [z_\alpha^*, [v_{\frac{r+1}{2}}, e]]^\sharp], [[[v_{\frac{r+1}{2}}, e], z_\alpha^*]^\sharp, [z_\alpha^*, g_i^*]^\sharp] \\ &\in \bigcup_{\beta \in \Phi_{e,0}^+} \mathbb{C}e_\beta \end{aligned}$$

for $1 \leq \alpha \leq \frac{s}{2}$ and $s+1 \leq \alpha \leq s + \frac{r+1}{2}$, applying Theorem 2.4 again we have

$$(B.10) \quad \begin{aligned} [\Theta_{g_i^*}, \Theta_{[v_{\frac{r+1}{2}}, e]}] &= -\frac{1}{2} \sum_{\alpha \in S(-1)} (\Theta_{[g_i^*, z_\alpha]^\sharp} \Theta_{[z_\alpha^*, [v_{\frac{r+1}{2}}, e]]^\sharp} + \Theta_{[[v_{\frac{r+1}{2}}, e], z_\alpha]^\sharp} \Theta_{[z_\alpha^*, g_i^*]^\sharp}) \\ &\in \sum_{\alpha \in \Phi_{e,0}^+} U(\mathbf{g}, e) \cdot \Theta_{e_\alpha} \in I_{\lambda, c}. \end{aligned}$$

If there exists $q_l \neq 0$ for $1 \leq l \leq \frac{r-1}{2}$, then (B.7) and (B.10) yield

$$(B.11) \quad \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q}, \mathbf{t}, \iota, \varepsilon) \cdot \Theta_{[v_{\frac{r+1}{2}}, e]} \in I_{\lambda, c}.$$

In virtue of (B.1), (B.7), (B.9) and (B.11), we may further assume that $\mathbf{b} = \mathbf{d} = \mathbf{q} = \mathbf{0}$.

B.3.3. Repeating verbatim the discussions in Appendix B.3 but substituting $\Theta_{g_i^*}$, $\Theta_{g_j^*}$ with $\Theta_{f_i^*}$, $\Theta_{f_j^*}$, Theorem 2.4 shows that

$$(B.12) \quad \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \ell, \varepsilon) \cdot \Theta_H \in I_{\lambda, c}$$

for $H \in \{f_1^*, \dots, f_{\frac{s}{2}}^*, [v_{\frac{r+1}{2}}, e]\}$. Then by (B.12) we may assume that $\mathbf{b} = \mathbf{d} = \mathbf{n} = \mathbf{q} = \mathbf{0}$.

B.3.4. Thanks to (B.12), it remains to show that

$$(B.13) \quad \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \ell, \varepsilon) \cdot \Theta_{[v_{\frac{r+1}{2}}, e]} \in I_{\lambda, c}.$$

(i) First note that

$$\begin{aligned} [[v_{\frac{r+1}{2}}, e], [v_{\frac{r+1}{2}}, e]] &= [[[v_{\frac{r+1}{2}}, e], v_{\frac{r+1}{2}}], e] - [v_{\frac{r+1}{2}}, [[v_{\frac{r+1}{2}}, e], e]] \\ &= [[[v_{\frac{r+1}{2}}, e], v_{\frac{r+1}{2}}], e] \\ &= [[[v_{\frac{r+1}{2}}, v_{\frac{r+1}{2}}], e], e] + [[v_{\frac{r+1}{2}}, [e, v_{\frac{r+1}{2}}]], e] \\ &= [[f, e], e] + [[[e, v_{\frac{r+1}{2}}], v_{\frac{r+1}{2}}], e] \\ &= -2e - [[[v_{\frac{r+1}{2}}, e], v_{\frac{r+1}{2}}], e]. \end{aligned}$$

As a result,

$$(B.14) \quad [[v_{\frac{r+1}{2}}, e], [v_{\frac{r+1}{2}}, e]] = [[[v_{\frac{r+1}{2}}, e], v_{\frac{r+1}{2}}], e] = -e.$$

In virtue of Theorem 2.4 (more precisely, Theorem 2.3(3)) and (B.14), we have

$$\begin{aligned} \Theta_{[v_{\frac{r+1}{2}}, e]}^2 &= \frac{1}{2} [\Theta_{[v_{\frac{r+1}{2}}, e]}, \Theta_{[v_{\frac{r+1}{2}}, e]}] \\ &= \frac{1}{4} ([[v_{\frac{r+1}{2}}, e], [v_{\frac{r+1}{2}}, e]], f)(C - \Theta_{\text{Cas}} - c_0) \\ &\quad - \frac{1}{2} \sum_{\alpha \in S(-1)} (\Theta_{[[v_{\frac{r+1}{2}}, e], z_\alpha]}^\# \Theta_{[z_\alpha^*, [v_{\frac{r+1}{2}}, e]]}^\#) \\ (B.15) \quad &= -\frac{1}{4}(C - c_0) + \frac{1}{4}\Theta_{\text{Cas}} - \frac{1}{2} \sum_{\alpha \in S(-1)} (\Theta_{[[v_{\frac{r+1}{2}}, e], z_\alpha]}^\# \Theta_{[z_\alpha^*, [v_{\frac{r+1}{2}}, e]]}^\#). \end{aligned}$$

Second, by the definition of Θ_{Cas} in (2.5), Theorem 2.4 (more precisely, Theorem 2.3(1)) implies that

$$\Theta_{\text{Cas}} = \sum_{i=1}^{k-1} \Theta_{h_i}^2 + \sum_{i=1}^w \Theta_{x_i} \Theta_{x_i^*} + \sum_{i=1}^w \Theta_{x_i^*} \Theta_{x_i} + \sum_{i=1}^\ell \Theta_{y_i} \Theta_{y_i^*} - \sum_{i=1}^\ell \Theta_{y_i^*} \Theta_{y_i}$$

$$\begin{aligned}
&= \sum_{i=1}^{k-1} \Theta_{h_i}^2 + 2 \sum_{i=1}^w \Theta_{x_i} \Theta_{x_i^*} + \sum_{i=1}^w [\Theta_{x_i^*}, \Theta_{x_i}] + 2 \sum_{i=1}^{\ell} \Theta_{y_i} \Theta_{y_i^*} - \sum_{i=1}^{\ell} [\Theta_{y_i^*}, \Theta_{y_i}] \\
\text{(B.16)} \quad &= \sum_{i=1}^{k-1} \Theta_{h_i}^2 + 2 \sum_{i=1}^w \Theta_{x_i} \Theta_{x_i^*} + \sum_{i=1}^w \Theta_{[x_i^*, x_i]} + 2 \sum_{i=1}^{\ell} \Theta_{y_i} \Theta_{y_i^*} - \sum_{i=1}^{\ell} \Theta_{[y_i^*, y_i]}.
\end{aligned}$$

Note that x_i^*, x_i, y_j^*, y_j are in the Lie algebra $\mathfrak{g}^e(0)$ for all i, j . Then both $[x_i^*, x_i]$ and $[y_j^*, y_j]$ are linear combinations of $h_1, \dots, h_{k-1}$ by weight considerations. Moreover, if we write $[x_i^*, x_i] = \sum_{j=1}^{k-1} l_j h_j$, then for any $1 \leq r \leq k-1$ we have

$$l_r = \sum_{j=1}^{k-1} l_j (h_r, h_j) = (h_r, [x_i^*, x_i]) = ([h_r, x_i^*], x_i) = \beta_{\bar{0}i}(h_r)(x_i^*, x_i) = \beta_{\bar{0}i}(h_r),$$

which shows that

$$\text{(B.17)} \quad [x_i^*, x_i] = \sum_{j=1}^{k-1} \beta_{\bar{0}i}(h_j) h_j.$$

As a result,

$$\text{(B.18)} \quad \Theta_{[x_i^*, x_i]} = \sum_{j=1}^{k-1} \beta_{\bar{0}i}(h_j) \Theta_{h_j}.$$

By the same discussion as above, we can also obtain

$$\text{(B.19)} \quad [y_i^*, y_i] = \sum_{j=1}^{k-1} \beta_{\bar{1}i}(h_j) h_j$$

and

$$\text{(B.20)} \quad \Theta_{[y_i^*, y_i]} = \sum_{j=1}^{k-1} \beta_{\bar{1}i}(h_j) \Theta_{h_j}.$$

Taking (B.18) and (B.20) into consideration, (B.16) shows that

$$\begin{aligned}
\Theta_{\text{Cas}} &= \sum_{i=1}^{k-1} \Theta_{h_i}^2 + 2 \sum_{i=1}^w \Theta_{x_i} \Theta_{x_i^*} + \sum_{i=1}^{k-1} \sum_{j=1}^w \beta_{\bar{0}j}(h_i) \Theta_{h_i} \\
\text{(B.21)} \quad &- \sum_{i=1}^{k-1} \sum_{j=1}^{\ell} \beta_{\bar{1}j}(h_i) \Theta_{h_i} + 2 \sum_{i=1}^{\ell} \Theta_{y_i} \Theta_{y_i^*}.
\end{aligned}$$

For the last term in the final equation of (B.15), as

$$[[v_{\frac{r+1}{2}}, e], v_{\frac{r+1}{2}}] = [v_{\frac{r+1}{2}}, [e, v_{\frac{r+1}{2}}]] + [[v_{\frac{r+1}{2}}, v_{\frac{r+1}{2}}], e] = -[[v_{\frac{r+1}{2}}, e], v_{\frac{r+1}{2}}] - h,$$

we have

$$(B.22) \quad [[v_{\frac{r+1}{2}}, e], v_{\frac{r+1}{2}}] = -\frac{h}{2},$$

and then

$$(B.23) \quad \Theta_{[[v_{\frac{r+1}{2}}, e], v_{\frac{r+1}{2}}]}^\# \Theta_{[v_{\frac{r+1}{2}}^*, [v_{\frac{r+1}{2}}, e]]}^\# = \Theta_{(-\frac{h}{2})}^\# \Theta_{(-\frac{h}{2})}^\# = 0.$$

Since $[[v_{\frac{r+1}{2}}, e], u_i^*]^\#$, $[u_i, [v_{\frac{r+1}{2}}, e]]^\# \in \mathfrak{g}^e(0)$ for all $1 \leq i \leq \frac{s}{2}$, then $[[[v_{\frac{r+1}{2}}, e], u_i^*]^\#, [u_i, [v_{\frac{r+1}{2}}, e]]^\#]$ is a linear combination of $h_1, \dots, h_{k-1}$ by weight considerations. For any $t \in \mathfrak{h}^e$, by definition we have $\theta(t) = 0$. Taking (B.22) into consideration, if we write

$$[[[v_{\frac{r+1}{2}}, e], u_i^*]^\#, [u_i, [v_{\frac{r+1}{2}}, e]]^\#] = \sum_{j=1}^{k-1} l_j h_j,$$

then for any $1 \leq r \leq k-1$ and $1 \leq i \leq \frac{s}{2}$, we have

$$\begin{aligned} l_r &= \sum_{j=1}^{k-1} l_j (h_r, h_j) = (h_r, [[[v_{\frac{r+1}{2}}, e], u_i^*]^\#, [u_i, [v_{\frac{r+1}{2}}, e]]^\#]) \\ &= (h_r, [[[[v_{\frac{r+1}{2}}, e], u_i^*], [u_i, [v_{\frac{r+1}{2}}, e]]]]) \\ &= ([h_r, [[v_{\frac{r+1}{2}}, e], u_i^*]], [u_i, [v_{\frac{r+1}{2}}, e]]) \\ &= \left(\frac{\theta}{2} + \gamma_{0i}^*\right) (h_r) ([v_{\frac{r+1}{2}}, e], u_i^*, [u_i, [v_{\frac{r+1}{2}}, e]]) \\ &= -\gamma_{\bar{0}i}(h_r) ([[[[v_{\frac{r+1}{2}}, e], u_i^*], u_i], v_{\frac{r+1}{2}}], e) \\ &= -\gamma_{\bar{0}i}(h_r) ([[[[v_{\frac{r+1}{2}}, e], v_{\frac{r+1}{2}}], u_i^*], u_i], e) \\ &= \gamma_{\bar{0}i}(h_r) \left([\frac{h}{2}, u_i^*], u_i, e\right) \\ (B.24) \quad &= -\frac{\gamma_{\bar{0}i}}{2}(h_r) ([u_i^*, u_i], e) = -\frac{\gamma_{\bar{0}i}}{2}(h_r). \end{aligned}$$

For $\frac{s}{2}+1 \leq i \leq s$, it follows from (B.24) and Theorem 2.4 (more precisely, Theorem 2.3(1)) that

$$\begin{aligned} &\Theta_{[[v_{\frac{r+1}{2}}, e], u_i]}^\# \Theta_{[u_i^*, [v_{\frac{r+1}{2}}, e]]}^\# \\ &= -\Theta_{[[v_{\frac{r+1}{2}}, e], u_{s+1-i}^*]}^\# \Theta_{[u_{s+1-i}, [v_{\frac{r+1}{2}}, e]]}^\# \\ &= \Theta_{[u_{s+1-i}, [v_{\frac{r+1}{2}}, e]]}^\# \Theta_{[[v_{\frac{r+1}{2}}, e], u_{s+1-i}^*]}^\# \\ &\quad - \Theta_{[[v_{\frac{r+1}{2}}, e], u_{s+1-i}^*]}^\# [u_{s+1-i}, [v_{\frac{r+1}{2}}, e]]^\# \end{aligned}$$

$$\begin{aligned}
&= \Theta_{[u_{s+1-i}, [v_{\frac{r+1}{2}}, e]]^\#} \Theta_{[[v_{\frac{r+1}{2}}, e], u_{s+1-i}^\ast]^\#} + \frac{1}{2} \sum_{j=1}^{k-1} \gamma_{\bar{0}s+1-i}(h_j) \Theta_{h_j} \\
(B.25) \quad &= \Theta_{[[v_{\frac{r+1}{2}}, e], u_{s+1-i}]^\#} \Theta_{[u_{s+1-i}^\ast, [v_{\frac{r+1}{2}}, e]]^\#} + \frac{1}{2} \sum_{j=1}^{k-1} \gamma_{\bar{0}s+1-i}(h_j) \Theta_{h_j}.
\end{aligned}$$

By the same discussion as in (B.25), we have

$$\begin{aligned}
&\Theta_{[[v_{\frac{r+1}{2}}, e], v_i]^\#} \Theta_{[v_i^\ast, [v_{\frac{r+1}{2}}, e]]^\#} = \Theta_{[[v_{\frac{r+1}{2}}, e], v_{r+1-i}]^\#} \Theta_{[v_{r+1-i}^\ast, [v_{\frac{r+1}{2}}, e]]^\#} \\
(B.26) \quad &\quad - \frac{1}{2} \sum_{j=1}^{k-1} \gamma_{\bar{1}r+1-i}(h_j) \Theta_{h_j}
\end{aligned}$$

for $\frac{r+3}{2} + 1 \leq i \leq r$.

Now combining (B.15), (B.21), (B.23), (B.25) with (B.26), we obtain

$$\begin{aligned}
\Theta_{[v_{\frac{r+1}{2}}, e]}^2 &= -\frac{1}{4}C + \frac{1}{4}c_0 + \frac{1}{4} \sum_{i=1}^{k-1} \Theta_{h_i}^2 + \frac{1}{2} \sum_{i=1}^w \Theta_{x_i} \Theta_{x_i^\ast} + \frac{1}{4} \sum_{i=1}^{k-1} \sum_{j=1}^w \beta_{\bar{0}j}(h_i) \Theta_{h_i} \\
&\quad - \frac{1}{4} \sum_{i=1}^{k-1} \sum_{j=1}^\ell \beta_{\bar{1}j}(h_i) \Theta_{h_i} + \frac{1}{2} \sum_{i=1}^\ell \Theta_{y_i} \Theta_{y_i^\ast} \\
&\quad - \sum_{i=1}^{\frac{s}{2}} \Theta_{[[v_{\frac{r+1}{2}}, e], u_i]^\#} \Theta_{[u_i^\ast, [v_{\frac{r+1}{2}}, e]]^\#} - \frac{1}{4} \sum_{i=1}^{k-1} \sum_{j=1}^{\frac{s}{2}} \gamma_{\bar{0}j}(h_i) \Theta_{h_i} \\
(B.27) \quad &\quad - \sum_{i=1}^{\frac{r-1}{2}} \Theta_{[[v_{\frac{r+1}{2}}, e], v_i]^\#} \Theta_{[v_i^\ast, [v_{\frac{r+1}{2}}, e]]^\#} + \frac{1}{4} \sum_{i=1}^{k-1} \sum_{j=1}^{\frac{r-1}{2}} \gamma_{\bar{1}j}(h_i) \Theta_{h_i}.
\end{aligned}$$

(ii) Now we introduce the proof of (B.13). If $\varepsilon = 0$, then (B.13) follows by definition. Now we will consider the case with $\varepsilon = 1$. Since $C - c$ is in the center of $U(\mathfrak{g}, e)$, and $[v_{\frac{r+1}{2}}^\ast, [v_{\frac{r+1}{2}}, e]]^\# = [v_{\frac{r+1}{2}}, [v_{\frac{r+1}{2}}, e]]^\# = (-\frac{h}{2})^\# = 0$ by (B.22), then by (B.27) we have

$$\begin{aligned}
&\Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \ell, 1) \cdot \Theta_{[v_{\frac{r+1}{2}}, e]} \\
&= \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \ell, 0) \cdot \Theta_{[v_{\frac{r+1}{2}}, e]}^2 \\
&= -\frac{1}{4} \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \ell, 0) (C - c) \\
&\quad - \frac{1}{4} (c - c_0) \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \ell, 0)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \left(\sum_{i=1}^{k-1} \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \iota, 0) \cdot (\Theta_{h_i} - \lambda(h_i))^2 \right. \\
& \quad + 2 \sum_{i=1}^{k-1} \lambda(h_i) \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \iota, 0) \cdot (\Theta_{h_i} - \lambda(h_i)) \\
& \quad + \sum_{i=1}^{k-1} (\lambda(h_i))^2 \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \iota, 0) \\
& \quad + 2 \sum_{i=1}^w \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \iota, 0) \cdot \Theta_{x_i} \Theta_{x_i^*} \\
& \quad + \sum_{i=1}^{k-1} \sum_{j=1}^w \beta_{\bar{0}j}(h_i) \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \iota, 0) \cdot (\Theta_{h_i} - \lambda(h_i)) \\
& \quad + \sum_{i=1}^{k-1} \sum_{j=1}^w \lambda(h_i) \beta_{\bar{0}j}(h_i) \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \iota, 0) \\
& \quad + 2 \sum_{i=1}^{\ell} \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \iota, 0) \cdot \Theta_{y_i} \Theta_{y_i^*} \\
& \quad - \sum_{i=1}^{k-1} \sum_{j=1}^{\ell} \beta_{\bar{1}j}(h_i) \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \iota, 0) \cdot (\Theta_{h_i} - \lambda(h_i)) \\
& \quad \left. - \sum_{i=1}^{k-1} \sum_{j=1}^{\ell} \lambda(h_i) \beta_{\bar{1}j}(h_i) \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \iota, 0) \right) \\
& + \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \iota, 0) \\
& \cdot \left(- \sum_{i=1}^{\frac{s}{2}} \Theta_{[[v_{\frac{r+1}{2}}, e], u_i]^{\#}} \Theta_{[u_i^*, [v_{\frac{r+1}{2}}, e]]^{\#}} - \frac{1}{4} \sum_{i=1}^{k-1} \sum_{j=1}^{\frac{s}{2}} \gamma_{\bar{0}j}(h_i) (\Theta_{h_i} - \lambda(h_i)) \right. \\
& \quad - \frac{1}{4} \sum_{i=1}^{k-1} \sum_{j=1}^{\frac{s}{2}} \gamma_{\bar{0}j}(h_i) \lambda(h_i) - \sum_{i=1}^{\frac{r-1}{2}} \Theta_{[[v_{\frac{r+1}{2}}, e], v_i]^{\#}} \Theta_{[v_i^*, [v_{\frac{r+1}{2}}, e]]^{\#}} \\
& \quad \left. + \frac{1}{4} \sum_{i=1}^{k-1} \sum_{j=1}^{\frac{r-1}{2}} \gamma_{\bar{1}j}(h_i) (\Theta_{h_i} - \lambda(h_i)) + \frac{1}{4} \sum_{i=1}^{k-1} \sum_{j=1}^{\frac{r-1}{2}} \gamma_{\bar{1}j}(h_i) \lambda(h_i) \right) \\
& = -\frac{1}{4} \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t} + \mathbf{e}_k, \iota, 0) \\
& \quad + \frac{1}{4} \left(\sum_{i=1}^{k-1} \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t} + 2\mathbf{e}_i, \iota, 0) \right.
\end{aligned}
\tag{B.28}$$

$$\begin{aligned}
& + 2 \sum_{i=1}^w \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \ell, 0) \cdot \Theta_{x_i} \Theta_{x_i^*} \\
& + 2 \sum_{i=1}^{\ell} \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \ell, 0) \cdot \Theta_{y_i} \Theta_{y_i^*} \\
& + \left(\sum_{i=1}^{k-1} \left(2\lambda + \sum_{j=1}^w \beta_{\bar{0}j} - \sum_{j=1}^{\ell} \beta_{\bar{1}j} - \sum_{j=1}^{\frac{s}{2}} \gamma_{\bar{0}j} + \sum_{j=1}^{\frac{r-1}{2}} \gamma_{\bar{1}j} \right) (h_i) \right) \\
& \quad \cdot \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t} + \mathbf{e}_i, \ell, 0) \\
& + \left(\sum_{i=1}^{k-1} \left(\lambda^2 + \lambda \left(\sum_{j=1}^w \beta_{\bar{0}j} - \sum_{j=1}^{\ell} \beta_{\bar{1}j} - \sum_{j=1}^{\frac{s}{2}} \gamma_{\bar{0}j} + \sum_{j=1}^{\frac{r-1}{2}} \gamma_{\bar{1}j} \right) \right) (h_i) \right. \\
& \quad \left. - c + c_0 \right) \cdot \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \ell, 0) \\
& - \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \ell, 0) \\
& \cdot \left(\sum_{i=1}^{\frac{s}{2}} \Theta_{[[v_{\frac{r+1}{2}}, e], u_i]}^{\#} \Theta_{[u_i^*, [v_{\frac{r+1}{2}}, e]]}^{\#} + \sum_{i=1}^{\frac{r-1}{2}} \Theta_{[[v_{\frac{r+1}{2}}, e], v_i]}^{\#} \Theta_{[v_i^*, [v_{\frac{r+1}{2}}, e]]}^{\#} \right)
\end{aligned}
\tag{B.29}$$

Now we discuss the terms in (B.29). By definition we have

$$\Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t} + \mathbf{e}_k, \ell, 0) \in I_{\lambda, c},$$

and also

$$\Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t} + \mathbf{e}_i, \ell, 0), \quad \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t} + 2\mathbf{e}_i, \ell, 0) \in I_{\lambda, c}$$

for $1 \leq i \leq k-1$. For all i, j , by definition we have $x_i^* \in \mathfrak{n}_0^+(0)$ and $y_j^* \in \mathfrak{n}_1^+(0)$ respectively, thus x_i^*, y_j^* are all in the span of e_α with $\alpha \in \Phi_{e,0}^+$. Therefore,

$$\Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \ell, 0) \cdot \Theta_{x_i} \Theta_{x_i^*} \quad \text{and} \quad \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \ell, 0) \cdot \Theta_{y_i} \Theta_{y_i^*}$$

are in $\sum_{\alpha \in \Phi_{e,0}^+} U(\mathfrak{g}, e) \cdot \Theta_{e_\alpha}$ of $I_{\lambda, c}$. Moreover, for $1 \leq i \leq \frac{s}{2}$ and $1 \leq j \leq \frac{r-1}{2}$, since $[u_i^*, [v_{\frac{r+1}{2}}, e]]^\#, [v_j^*, [v_{\frac{r+1}{2}}, e]]^\# \in \bigcup_{\beta \in \Phi_{e,0}^+} \mathbb{C}e_\beta$, then both

$$\Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \ell, 0) \cdot \Theta_{[[v_{\frac{r+1}{2}}, e], u_i]}^{\#} \Theta_{[u_i^*, [v_{\frac{r+1}{2}}, e]]}^{\#}$$

and

$$\Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \ell, 0) \cdot \Theta_{[[v_{\frac{r+1}{2}}, e], v_j]}^{\#} \Theta_{[v_j^*, [v_{\frac{r+1}{2}}, e]]}^{\#}$$

are also in $\sum_{\alpha \in \Phi_{e,0}^+} U(\mathfrak{g}, e) \cdot \Theta_{e_\alpha}$ of $I_{\lambda,c}$. Apart from all the terms mentioned above, what remains in (B.29) are

$$(B.30) \quad \frac{1}{4} \left(\sum_{i=1}^{k-1} \left(\lambda^2 + \lambda \cdot \left(\sum_{j=1}^w \beta_{\bar{0}j} - \sum_{j=1}^\ell \beta_{\bar{1}j} - \sum_{j=1}^{\frac{s}{2}} \gamma_{\bar{0}j} + \sum_{j=1}^{\frac{r-1}{2}} \gamma_{\bar{1}j} \right) \right) (h_i) - c + c_0 \right) \cdot \Theta(\mathbf{a}, \mathbf{0}, \mathbf{c}, \mathbf{0}, \mathbf{m}, \mathbf{0}, \mathbf{p}, \mathbf{0}, \mathbf{t}, \mathbf{0}).$$

Thanks to our assumption in (2.8), (B.30) must be zero.

Taking all the above into consideration, we finally obtain (B.13).

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References

- [1] D. Adamović, V. G. Kac, P. Möseneder Frajria, P. Papi and O. Perše, [Conformal embeddings of affine vertex algebras in minimal \$W\$ -algebras II: Decompositions](#), Jpn. J. Math. **12** (2017), 261–315. [Zbl 1425.17037](#) [MR 3694933](#)
- [2] D. Adamović, V. G. Kac, P. Möseneder Frajria, P. Papi and O. Perše, [Conformal embeddings of affine vertex algebras in minimal \$W\$ -algebras I: Structural results](#), J. Algebra **500** (2018), 117–152. [Zbl 1420.17028](#) [MR 3765450](#)
- [3] T. Arakawa, T. Creutzig, K. Kawasetsu and A. R. Linshaw, [Orbifolds and cosets of minimal \$W\$ -algebras](#), Comm. Math. Phys. **355** (2017), 339–372. [Zbl 1407.17030](#) [MR 3670736](#)
- [4] T. Arakawa and A. Moreau, [Joseph ideals and lisse minimal \$W\$ -algebras](#), J. Inst. Math. Jussieu **17** (2018), 397–417. [Zbl 1416.17014](#) [MR 3773273](#)
- [5] I. Bagci, K. Christodouloupoulou and E. Wiesner, [Whittaker categories and Whittaker modules for Lie superalgebras](#), Comm. Algebra **42** (2014), 4932–4947. [Zbl 1364.17008](#) [MR 3210421](#)
- [6] H. Bao, [Kazhdan–Lusztig theory of super type D and quantum symmetric pairs](#), Represent. Theory **21** (2017), 247–276. [Zbl 1411.17015](#) [MR 3696376](#)
- [7] H. Bao and W. Wang, [A new approach to Kazhdan–Lusztig theory of type \$B\$ via quantum symmetric pairs](#), Astérisque **402** (2018), vii+134 pp. [Zbl 1411.17001](#) [MR 3864017](#)
- [8] J. Brundan, [Kazhdan–Lusztig polynomials and character formulae for the Lie superalgebra \$\mathfrak{gl}\(m|n\)\$](#) , J. Amer. Math. Soc. **16** (2003), 185–231. [Zbl 1050.17004](#) [MR 1937204](#)
- [9] J. Brundan, [Representations of the general linear Lie superalgebra in the BGG category \$\mathcal{O}\$](#) , in *Developments and retrospectives in Lie theory*, Developments in Mathematics 38, Springer, Cham, 2014, 71–98. [Zbl 1391.17003](#) [MR 3308778](#)
- [10] J. Brundan and S. M. Goodwin, [Good grading polytopes](#), Proc. Lond. Math. Soc. (3) **94** (2007), 155–180. [Zbl 1120.17007](#) [MR 2293468](#)
- [11] J. Brundan and S. M. Goodwin, [Whittaker coinvariants for \$\mathrm{GL}\(m|n\)\$](#) , Adv. Math. **347** (2019), 273–339. [Zbl 1472.17027](#) [MR 3916872](#)

- [12] J. Brundan, S. M. Goodwin and A. Kleshchev, [Highest weight theory for finite \$W\$ -algebras](#), *Int. Math. Res. Not. IMRN* **15** (2008), article no. rnn051, 53 pp. [Zbl 1211.17024](#) [MR 2438067](#)
- [13] R. W. Carter, *Finite groups of Lie type*, Pure and Applied Mathematics (New York), John Wiley & Sons, New York, 1985. [Zbl 0567.20023](#) [MR 0794307](#)
- [14] C.-W. Chen, [Whittaker modules for classical Lie superalgebras](#), *Comm. Math. Phys.* **388** (2021), 351–383. [Zbl 1500.17007](#) [MR 4328057](#)
- [15] C.-W. Chen and S.-J. Cheng, [Whittaker categories of quasi-reductive Lie superalgebras and quantum symmetric pairs](#), *Forum Math. Sigma* **12** (2024), article no. e37. [Zbl 07827375](#) [MR 4726500](#)
- [16] C.-W. Chen, S.-J. Cheng and V. Mazorchuk, [Whittaker categories, properly stratified categories and Fock space categorification for Lie superalgebras](#), *Comm. Math. Phys.* **401** (2023), 717–768. [Zbl 1523.17016](#) [MR 4604906](#)
- [17] S.-J. Cheng, N. Lam and W. Wang, [Super duality and irreducible characters of ortho-symplectic Lie superalgebras](#), *Invent. Math.* **183** (2011), 189–224. [Zbl 1246.17007](#) [MR 2755062](#)
- [18] S.-J. Cheng, N. Lam and W. Wang, [The Brundan–Kazhdan–Lusztig conjecture for general linear Lie superalgebras](#), *Duke Math. J.* **164** (2015), 617–695. [Zbl 1387.17013](#) [MR 3322307](#)
- [19] S.-J. Cheng, V. Mazorchuk and W. Wang, [Equivalence of blocks for the general linear Lie superalgebra](#), *Lett. Math. Phys.* **103** (2013), 1313–1327. [Zbl 1376.17010](#) [MR 3117249](#)
- [20] S.-J. Cheng and W. Wang, [Dualities and representations of Lie superalgebras](#), Graduate Studies in Mathematics 144, American Mathematical Society, Providence, RI, 2012. [Zbl 1271.17001](#) [MR 3012224](#)
- [21] J. Dixmier, *Enveloping algebras*, Graduate Studies in Mathematics 11, American Mathematical Society, Providence, RI, 1996. [Zbl 0867.17001](#) [MR 1393197](#)
- [22] R. Fiorese and F. Gavarini, [Chevalley supergroups](#), *Mem. Amer. Math. Soc.* **215** (2012), no. 1014, vi+64 pp. [Zbl 1276.14075](#) [MR 2918543](#)
- [23] F. Gavarini, [Chevalley supergroups of type \$D\(2, 1; a\)\$](#) , *Proc. Edinb. Math. Soc. (2)* **57** (2014), 465–491. [Zbl 1304.14069](#) [MR 3200319](#)
- [24] J. E. Humphreys, [Representations of semisimple Lie algebras in the BGG category \$\mathcal{O}\$](#) , Graduate Studies in Mathematics 94, American Mathematical Society, Providence, RI, 2008. [Zbl 1177.17001](#) [MR 2428237](#)
- [25] V. G. Kac, [Lie superalgebras](#), *Advances in Math.* **26** (1977), 8–96. [Zbl 0366.17012](#) [MR 0486011](#)
- [26] V. Kac, [Representations of classical Lie superalgebras](#), in *Differential geometrical methods in mathematical physics, II (Proc. Conf., Univ. Bonn, Bonn, 1977)*, Lecture Notes in Mathematics 676, Springer, Berlin, 1978, 597–626. [Zbl 0388.17002](#) [MR 0519631](#)
- [27] V. G. Kac, P. Möseneder Frajria and P. Papi, [Unitarity of minimal \$W\$ -algebras and their representations I](#), *Comm. Math. Phys.* **401** (2023), 79–145. [Zbl 1542.17051](#) [MR 4604893](#)
- [28] V. Kac, S.-S. Roan and M. Wakimoto, [Quantum reduction for affine superalgebras](#), *Comm. Math. Phys.* **241** (2003), 307–342. [Zbl 1106.17026](#) [MR 2013802](#)
- [29] V. G. Kac and M. Wakimoto, [Quantum reduction and representation theory of superconformal algebras](#), *Adv. Math.* **185** (2004), 400–458; [Corrigendum](#): *Adv. Math.* **193** (2005), 453–455. [Zbl 1049.17025](#) [MR 2060475](#) [MR 2137292](#)
- [30] I. Losev, [On the structure of the category \$\mathcal{O}\$ for \$W\$ -algebras](#), in *Geometric methods in representation theory. II*, Séminaires et Congrès 24, Société Mathématique de France, Paris, 2012, 353–370. [Zbl 1356.17010](#) [MR 3203033](#)

- [31] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, Pure and Applied Mathematics (New York), John Wiley & Sons, Chichester, 1987. [Zbl 0644.16008](#) [MR 0934572](#)
- [32] D. Milićić and W. Soergel, [The composition series of modules induced from Whittaker modules](#), *Comment. Math. Helv.* **72** (1997), 503–520. [Zbl 0956.17004](#) [MR 1600134](#)
- [33] I. M. Musson, [Lie superalgebras and enveloping algebras](#), Graduate Studies in Mathematics 131, American Mathematical Society, Providence, RI, 2012. [Zbl 1255.17001](#) [MR 2906817](#)
- [34] L. Pan and B. Shu, [Jantzen filtration and strong linkage principle for modular Lie superalgebras](#), *Forum Math.* **30** (2018), 1573–1598. [Zbl 1404.17027](#) [MR 3871461](#)
- [35] D. Panyushev, A. Premet and O. Yakimova, [On symmetric invariants of centralisers in reductive Lie algebras](#), *J. Algebra* **313** (2007), 343–391. [Zbl 1163.17012](#) [MR 2326150](#)
- [36] E. Poletaeva and V. Serganova, [On Kostant's theorem for the Lie superalgebra \$Q\(n\)\$](#) , *Adv. Math.* **300** (2016), 320–359. [Zbl 1402.17022](#) [MR 3534835](#)
- [37] A. Premet, [Special transverse slices and their enveloping algebras](#), *Adv. Math.* **170** (2002), 1–55. [Zbl 1005.17007](#) [MR 1929302](#)
- [38] A. Premet, [Enveloping algebras of Slodowy slices and the Joseph ideal](#), *J. Eur. Math. Soc. (JEMS)* **9** (2007), 487–543. [Zbl 1134.17307](#) [MR 2314105](#)
- [39] A. Premet, [Primitive ideals, non-restricted representations and finite \$W\$ -algebras](#), *Mosc. Math. J.* **7** (2007), no. 4, 743–762. [Zbl 1139.17005](#) [MR 2372212](#)
- [40] A. Premet, [Commutative quotients of finite \$W\$ -algebras](#), *Adv. Math.* **225** (2010), 269–306. [Zbl 1241.17015](#) [MR 2669353](#)
- [41] M. Scheunert, *The theory of Lie superalgebras*, Lecture Notes in Mathematics 716, Springer, Berlin, 1979. [Zbl 0407.17001](#) [MR 0537441](#)
- [42] B. Shu and H. Xiao, [Super formal Darboux–Weinstein theorem and finite \$W\$ -superalgebras](#), *J. Algebra* **550** (2020), 242–265. [Zbl 1473.17058](#) [MR 4056057](#)
- [43] B. Shu and Y. Zeng, [Centers and Azumaya loci for finite \$W\$ -algebras in positive characteristic](#), *Math. Proc. Cambridge Philos. Soc.* **173** (2022), 35–66. [Zbl 1510.17025](#) [MR 4438330](#)
- [44] U. R. Suh, [Classical affine \$W\$ -algebras associated to Lie superalgebras](#), *J. Math. Phys.* **57** (2016), article no. 021703, 34 pp. [Zbl 1345.81062](#) [MR 3453843](#)
- [45] W. Wang, Nilpotent orbits and finite W -algebras, in [Geometric representation theory and extended affine Lie algebras](#), Fields Institute Communications 59, American Mathematical Society, Providence, RI, 2011, 71–105. [Zbl 1278.17007](#) [MR 2777648](#)
- [46] W. Wang and L. Zhao, [Representations of Lie superalgebras in prime characteristic. I](#), *Proc. Lond. Math. Soc. (3)* **99** (2009), 145–167. [Zbl 1176.17013](#) [MR 2520353](#)
- [47] H. Xiao, [On finite-dimensional representations of finite \$W\$ -superalgebras](#), *Sci. China Math.* **66** (2023), 1737–1750. [Zbl 1536.17014](#) [MR 4621126](#)
- [48] Y. Zeng and B. Shu, [Finite \$W\$ -superalgebras for basic Lie superalgebras](#), *J. Algebra* **438** (2015), 188–234. [Zbl 1394.17033](#) [MR 3353030](#)
- [49] Y. Zeng and B. Shu, [Minimal \$W\$ -superalgebras and the modular representations of basic Lie superalgebras](#), *Publ. Res. Inst. Math. Sci.* **55** (2019), 123–188. [Zbl 1472.17068](#) [MR 3898326](#)