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Plurisubharmonic Functions on Affine Line Bundles over Compact Kähler Manifolds

Dedicated to Takeo Ohsawa on his seventieth birthday

by

Takayuki Koike and Tetsuo Ueda

Abstract

We investigate function-theoretic properties of holomorphic affine line bundles over compact Kähler manifolds. We discuss existence of (strictly) plurisubharmonic functions on the total space of such a bundle. Further, we give a precise restriction from below on the growth of such functions. This gives refinements of some previous results due to one of the present authors. In the proof, we construct a plurisubharmonic exhaustion function satisfying the Monge–Ampère equation and look at the foliation induced by this function.

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§1. Introduction and the statement of results

Let $\overline{X}$ be a compact complex manifold and Y a complex hypersurface in $\overline{X}$. The complement $X = \overline{X} \setminus Y$ sometimes exhibits function-theoretically interesting properties. It is known that there are examples of X that have the following properties simultaneously: (i) X is Stein, hence it admits a wealth of holomorphic functions; (ii) every non-constant holomorphic function on X has Y as essential singularity, hence X is not affine algebraic. These properties can be derived from statements about plurisubharmonic functions on X. Property (i) follows from the existence of strictly plurisubharmonic exhaustion function on X, and property (ii) comes from showing that any plurisubharmonic function on X increases rapidly near Y. (See [8] and also the remark at the end of this section.)

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T. Koike: Department of Mathematics, Osaka Metropolitan University, 3-3-138 Sugimoto, Sumiyoshi-ku Osaka 558-8585, Japan;
e-mail: tkoike@omu.ac.jp
T. Ueda: Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan;
e-mail: ueda.tetsuo.46c@st.kyoto-u.ac.jp

In the present paper we will examine the following concrete example: Let $\pi\colon \overline{X} \to M$ be a projective line bundle over a compact Kähler manifold M, that has a global section Y whose normal bundle is topologically trivial and X is the complement of Y. We will give a concrete form of (strictly) plurisubharmonic functions and a refined form of restriction on the growth of plurisubharmonic functions.

Let M be a compact complex manifold and $\pi\colon X \to M$ be an affine line bundle over M, i.e., a holomorphic fiber bundle that has the complex line $\mathbb{C}$ as typical fiber and affine automorphisms of $\mathbb{C}$ as structure group. Choosing an open covering $\{U_i\}_{i\in I}$ of M we can describe the structure of X as follows:

(i) For each i, there is a trivialization

$$\pi^{-1}(U_i) \cong U_i \times \mathbb{C}$$

so that a point $x \in \pi^{-1}(U_i)$ can be identified with $(p, w_i) \in U_i \times \mathbb{C}$, where $p = \pi(x)$ and w_i is the fiber coordinate.

(ii) For i, j with $U_i \cap U_j \neq \emptyset$, fiber coordinates are transformed as

$$w_i = a_{ij}(p)w_j + b_{ij}(p)$$

on $\pi^{-1}(U_i \cap U_j)$, where $a_{ij}(p)$ is a non-vanishing holomorphic function on $U_i \cap U_j$ and $b_{ij}(p)$ is a holomorphic function on $U_i \cap U_j$.

We can compactify each fiber $\pi^{-1}(p)$, $p \in M$ by adding a point at infinity, and we obtain a $\mathbb{P}^1$-bundle $\pi\colon \overline{X} \to M$ with the infinity section Y so that $X = \overline{X} \setminus Y$, i.e., $\pi\colon \overline{X} \to M$ is the $\mathbb{P}^1$-bundle obtained by the fiberwise projectivization of the rank-2 vector bundle over M defined by

$$\begin{pmatrix} a_{ij} & b_{ij} \\ 0 & 1 \end{pmatrix},$$

which actually defines a rank-2 vector bundle since one can check the 1-cocycle condition by using equations (1.1) and (1.2) below.

Now, since (a_{ij}) satisfies the cocycle condition

$$a_{ij}a_{jk} = a_{ik}, \tag{1.1}$$

this defines an element in $H^1(M, \mathcal{O}^*)$. We denote by E_X the complex line bundle defined by (a_{ij}). Further, we denote by $c_1(E_X)$ the first Chern class of E_X and by $c_{1,\mathbb{R}}(E_X)$ the real first Chern class, i.e., the image of $c_1(E_X)$ under the homomorphism $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$. Now, since we have

$$b_{ij} + a_{ij}b_{jk} = b_{ik}, \tag{1.2}$$

we can regard (b_{ij}) as a 1-cocycle of holomorphic sections of E_X and hence an element in $H^1(M, \mathcal{O}(E_X))$.

Now we assume the following conditions:

(a) The first Chern class $c_1(E_X)$ of E_X is torsion, i.e., $c_{1,\mathbb{R}}(E_X) = 0$ (note that this condition implies that E_X is topologically trivial when M is a compact Riemann surface).

(b) The affine line bundle X is not isomorphic to the line bundle E_X.

We note that condition (b) is equivalent to saying that $(b_{ij}) \neq 0$ as an element in $H^1(M, \mathcal{O}(E_X))$, or that the affine line bundle X does not admit a holomorphic section (see Abe [1]).

The following three theorems are the main results of the present paper.

Theorem 1.1. *Let M be a compact Kähler manifold and $\pi\colon X \to M$ an affine line bundle satisfying conditions* (a) *and* (b)*. Then there exists a plurisubharmonic exhaustion function $\Psi(x)$ on X such that*

$$\Psi(x) \sim 1/\operatorname{dist}(x, Y) \quad \textit{as } x \to Y.$$

Here, $\operatorname{dist}(x, Y)$ *denotes the distance of x from the infinity section Y with respect to a Riemannian metric on $\overline{X}$.*

Theorem 1.2. *Let $\pi\colon X \to M$ be as in Theorem 1.1.*

(1) *If* $\dim M = 1$*, i.e., if M is a compact Riemann surface, then there exists a strictly plurisubharmonic exhaustion function $\Phi(x)$ on X such that*

$$\Phi(x) \sim 1/\operatorname{dist}(x, Y) \quad \textit{as } x \to Y.$$

In particular, X is a Stein manifold.

(2) *If* $\dim M \geq 2$*, then X does not admit a strictly plurisubharmonic function. In particular, X is not a Stein manifold.*

Theorem 1.3. *Under the same situation as above, let V be an open set with $Y \subset V \subset \overline{X}$, and let $\varphi(x)$ be a plurisubharmonic function on $V \setminus Y$. Suppose that*

$$\varphi(x) = o(1/\operatorname{dist}(x, Y)) \quad \textit{as } x \to Y.$$

Then there is an open set V_0 with $Y \subset V_0 \subset V$ such that $\varphi(x)$ is a constant on $V_0 \setminus Y$.

In the proof of Theorem 1.1 we give a concrete form of the function Ψ. We note that Ohsawa [6, 7] introduced this function Ψ and showed that its square Ψ^2 is plurisubharmonic.

This function Ψ satisfies the Monge–Ampère equation $(\partial\bar{\partial}\Psi)^2 = 0$ and hence it is nowhere *strictly* plurisubharmonic. In the case $\dim M = 1$, we can produce a strictly plurisubharmonic function by modifying Ψ (Theorem 1.2(1)). As for the case $\dim M \geq 2$, we note that a more general result is obtained by Ohsawa [5].

The $(1,1)$ form $\partial\bar{\partial}\Psi$ defines a foliation (Monge–Ampère foliation) $\mathcal{F}$. The function Ψ is pluriharmonic when restricted to each leaf of $\mathcal{F}$. Whereas the leaves of $\mathcal{F}$ are holomorphically immersed submanifolds, it is not a holomorphic foliation. Theorem 1.3 is proved with the aid of this foliation.

Remark. Here we discuss the results of [8] as the background to the present paper. Let C be a compact complex curve embedded in a 2-dimensional complex manifold (not necessarily compact nor Kähler) with topologically trivial normal bundle. The complex normal bundle N_C of C is defined to be the restriction $[C]|_C$ of the line bundle $[C]$ to C. If N_C is topologically trivial, N_C can be given a structure of unitary flat line bundle, and N_C can be extended to a unitary flat line bundle F over a neighborhood of the curve C. The type n ($n = 1, 2, \dots$, or ∞) of the embedded curve C is defined to be the order of coincidence of $[C]$ and F around C. In the case where C is of type $n < \infty$, the following facts are proved in [8]:

(1) For any $\varepsilon > 0$, there exist an open set V containing C and a strictly plurisubharmonic function $\Phi(x)$ on $V \setminus C$ such that $\Phi(x) \sim 1/\operatorname{dist}(x, C)^{n+\varepsilon}$ as $x \to C$.

(2) If V is an open set containing C and $\varphi(x)$ is a plurisubharmonic function on $V \setminus C$ such that $\varphi(x) = o(1/\operatorname{dist}(x, C)^{n-\varepsilon})$, then there is an open set V_0 with $C \subset V_0 \subset V$ such that $\varphi(x)$ is constant on V_0.

In the situation of the present paper, the infinity section Y is embedded in $\overline{X}$ with topologically trivial normal bundle by condition (a), and it is of type 1 by condition (b). Therefore, our present results are considered to be refinements of the results in [8] in concrete examples of ruled surfaces.

§2. Construction of a plurisubharmonic exhaustion function

Let $\pi\colon X \to M$ be an affine line bundle over a complex manifold and let $\{w_i\}$ be a collection of fiber coordinates for an open covering $\{U_i\}$. We will say that $\{w_i\}$ is a collection of *admissible* fiber coordinates if the following conditions are satisfied:

(i) a_{ij} are constants with $|a_{ij}| = 1$.

(ii) b_{ij} are constants.

(iii) There exist anti-holomorphic functions $\overline{g_i}$ on U_i such that

$$\overline{g_i} = a_{ij}\overline{g_j} + b_{ij} \quad \text{on } U_i \cap U_j.$$

Proposition 2.1. *Let $\pi\colon X \to M$ be a holomorphic affine line bundle over a compact Kähler manifold M. If the Chern class $c_1(E_X)$ of the associated line bundle E_X is torsion, then there is a collection $\{w_i\}$ of admissible fiber coordinates for an open covering $\{U_i\}$ of M.*

Proof. By [8, Prop. 1], the line bundle E_X has the structure of flat line bundle, Hence there are non-vanishing holomorphic functions a_i on U_i such that $a_{ij}a_ia_j^{-1}$ are constants of modulus 1 on $U_i \cap U_j$. By replacing w_i with a_iw_i on $\pi^{-1}(U_i)$, we may assume that condition (i) is satisfied.

Now E_X is a flat line bundle and the notions of constant, holomorphic, anti-holomorphic and pluriharmonic sections of E_X are well defined. We denote by $\mathcal{O}(E_X)$ [resp. $\mathcal{H}(E_X)$] the sheaves of holomorphic [resp. pluriharmonic] sections of E_X.

Since $\{b_{ij}\}$ satisfies the condition $b_{ij} + a_{ij}b_{ij} = b_{ik}$, it is regarded as an element in $H^1(M, \mathcal{O}(E_X))$. By [8, Prop. 2], the homomorphism $H^1(M, \mathcal{O}(E_X)) \to H^1(M, \mathcal{H}(E_X))$ is a zero map. Hence there are pluriharmonic functions h_i on U_i such that

$$h_i = a_{ij}h_j + b_{ij}.$$

We write $h_i = f_i + \overline{g_i}$ with holomorphic f_i and anti-holomorphic $\overline{g_i}$ on U_i. Then

$$b_{ij} - (a_{ij}f_j - f_i) = a_{ij}\overline{g_j} - \overline{g_i} \quad \text{on } U_i \cap U_j.$$

Both sides are holomorphic and anti-holomorphic at the same time, hence a constant, which we will denote by $\hat{b}_{ij}$. We define new fiber coordinates $\widehat{w}_i$ on $\pi^{-1}(U_i)$ by

$$\widehat{w}_i = w_i - f_i(p).$$

Then we have

$$\widehat{w}_i = a_{ij}\widehat{w}_j + \hat{b}_{ij}.$$

Further, $\widehat{w}_i = \overline{g_i(p)}$ defines a global anti-holomorphic section over M, since $a_{ij}\overline{g_j(p)} - \overline{g_i(p)} = \hat{b}_{ij}$. Thus the proposition is proved. □

Remark. The section σ is unique if E is not analytically trivial. Additionally, it is non-constant if we assume the condition (b).

Proof of Theorem 1.1. In what follows we assume that $\{w_i\}$ is a collection of admissible fiber coordinates.

We define the function $\Psi(x)$ as the distance of x from the anti-holomorphic section $\sigma(M)$ on each fiber, namely,

$$\Psi(x) := |w_i - \overline{g_i(p)}| \quad \text{on } \pi^{-1}(U_i) \cong U_i \times \mathbb{C}.$$

This function $\Psi(x)$ is well defined on X, since

$$w_i - \overline{g_i(p)} = a_{ij}(w_i - \overline{g_i(p)}) \quad \text{on } \pi^{-1}(U_i \cap U_j)$$

with $|a_{ij}| = 1$. It is clear that $\Psi(x)$ is an exhaustion function on X and that it is real analytic on $X \setminus \sigma(M)$.

Now we show that the function $\Psi(x)$ is plurisubharmonic but nowhere strictly plurisubharmonic on X.

In the following calculations, we will work on $\pi^{-1}(U_i)$ and suppress the suffix i. The anti-holomorphic section σ is expressed as $w = \overline{g(p)}$, where $g(p)$ is a holomorphic function on U_i. We write

$$u \coloneqq w - \overline{g(p)}.$$

Then

$$\Psi(x) = (u\bar{u})^{1/2}$$

and

$$\begin{aligned}
\partial\bar{\partial}\Psi &= \frac{1}{4}(u\bar{u})^{-3/2}\bigl\{2u\bar{u}\,\partial\bar{\partial}(u\bar{u}) - \partial(u\bar{u}) \wedge \bar{\partial}(u\bar{u})\bigr\} \\
&= \frac{1}{4}(u\bar{u})^{-3/2}\bigl\{2u\bar{u}(\partial u \wedge \bar{\partial}\bar{u} + \partial\bar{u} \wedge \bar{\partial}u) - (\bar{u}\,\partial u + u\,\partial\bar{u}) \wedge (u\,\bar{\partial}\bar{u} + \bar{u}\,\bar{\partial}u)\bigr\} \\
&= \frac{1}{4}(u\bar{u})^{-3/2}(\bar{u}\,\partial u - u\,\partial\bar{u}) \wedge (u\,\bar{\partial}\bar{u} - \bar{u}\,\bar{\partial}u) \\
&= \frac{1}{4}\Psi(x)^{-3}(\bar{u}\,dw + u\,dg) \wedge (\overline{\bar{u}\,dw + u\,dg}).
\end{aligned}$$

The Levi form L_Ψ of Ψ is the quadratic form associated to the $(1,1)$ form $\sqrt{-1}\partial\bar{\partial}\Psi$, and it is expressed, with respect to the coordinates $(z,w) = (z_1, z_2, \ldots, z_n, w)$, as

$$L_\Psi(x; dz_1, \ldots, dz_n, dw) = \frac{1}{4}\Psi(x)^{-3}|u\,dg(z) + \bar{u}\,dw|^2.$$

This is semipositive and degenerates on the subspace of the tangent space at x which is given by the condition $u\,dg + \bar{u}\,dw = 0$ for any point x in $\pi^{-1}(U_i) \setminus \sigma(U_i)$.

This shows that Ψ is plurisubharmonic on $X \setminus \sigma(M)$. Obviously, Ψ is plurisubharmonic also at the points on $\sigma(M)$. $\square$

Remark. It has been shown in Ohsawa [6, 7] that Ψ^2 is a plurisubharmonic exhaustion function on X.

§3. Existence of strictly plurisubharmonic exhaustion functions

In this section we will give a proof of Theorem 1.2.

First we consider the case $\dim M = 1$. We begin with constructing an exhaustion function that is strictly plurisubharmonic almost everywhere on X by modifying the function Ψ which we constructed in the previous section. We define

$$\Phi(x) := (1 + \Psi(x)^2)^{1/2}.$$

We calculate $\partial\bar\partial\Phi$ on $\pi^{-1}(U_i)$, with the same notation as in the calculation of Ψ. We have $\Phi(x) = (1 + u\bar u)^{1/2}$ and

$$\begin{aligned}\partial\bar\partial\Phi &= \frac{1}{4}(1+u\bar u)^{-3/2}\big\{2(1+u\bar u)\partial\bar\partial(u\bar u) - \partial(u\bar u)\wedge\bar\partial(u\bar u)\big\}\\ &= \frac{1}{4}(1+u\bar u)^{-3/2}\big\{2\partial\bar\partial(u\bar u) + (\bar u\,\partial u - u\,\partial\bar u)\wedge(u\,\bar\partial\bar u - \bar u\,\partial\bar u)\big\}\\ &= \frac{1}{4}\Phi^{-3}\big\{2(dw\wedge d\overline{w} + dg\wedge d\bar g) + (\bar u\,dw + u\,dg)\wedge(\overline{\bar u\,dw + u\,dg})\big\}.\end{aligned}$$

Hence the Levi form of Φ is given by

$$L_\Phi(x; dz, dw) = \frac{1}{4}\Phi^{-3}\big\{2(|dw|^2 + |dg(z)|^2) + |\bar u\,dw + u\,dg(z)|^2\big\},$$

where z denotes a coordinate on U_i.

Now we set

$$Z := \bigcup_i \big\{p \in U_i \bigm| dg_i(p) = 0\big\}.$$

Since g_i are non-constant, the set Z consists of a finite number of points. The calculation above shows that $L_\Phi(x; dz, dw)$ is positive definite except at the points on $\pi^{-1}(Z)$.

We will modify Φ on a neighborhood of $\pi^{-1}(p)$ for each $p \in Z$ to obtain a strictly plurisubharmonic function on all of X.

Let $p_0 \in Z$ and choose a coordinate neighborhood $\Delta \cong \{z \in \mathbb{C} \mid |z| < 1\}$ with center p_0. We assume that Δ is sufficiently small so that $\bar\Delta \cap Z = \{p_0\}$.

We take a real-valued C^∞ function $\rho(z)$ on Δ with compact support such that

$$\rho(z) = |z|^2 \quad \text{on } \Delta_{1/2} = \{|z| < 1/2\},$$

and set

$$\eta(x) := \Phi(x)^{-3}\rho(z),$$

where $x \in \pi^{-1}(\Delta)$ and $z = \pi(x)$.

Lemma 3.1. *We have the following estimates of the Levi form of η:*

(1) *There exists a number A such that*

$$|L_\eta(x; dz, dw)| \le A\Phi^{-3}(|dz|^2 + |dw|^2)$$

when $x \in \pi^{-1}(\Delta)$.

(2) *There exists a number* r *with* $0 < r < 1/2$ *such that*

$$L_\eta(x; dz, dw) \geq \frac{1}{2}\Phi^{-3}(|dz|^2 - |dw|^2)$$

when $x \in \pi^{-1}(\Delta_r)$*, where* $\Delta_r = \{|z| < r\}$*.*

Proof. On $\pi^{-1}(\Delta)$ we have

$$\partial\bar{\partial}\eta = \Phi^{-3}\,\partial\bar{\partial}\rho + \partial\Phi^{-3} \wedge \bar{\partial}\rho + \partial\rho \wedge \bar{\partial}\Phi^{-3} + \rho\,\partial\bar{\partial}\Phi^{-3}$$

and assertion (1) follows easily.

Now, on $\pi^{-1}(\Delta_{1/2})$, we have

$$\partial\bar{\partial}\eta = \Phi^{-3} dz \wedge d\bar{z} + \alpha,$$

where we have set

$$\alpha := z\,\partial\Phi^{-3} \wedge d\bar{z} + \bar{z}\,dz \wedge \bar{\partial}\Phi^{-3} + |z|^2\,\partial\bar{\partial}\Phi^{-3}.$$

If we denote by $Q(x; dz, dw)$ the quadratic form associated to the (1,1) form $\sqrt{-1}\alpha$, we have

$$|Q(x; dz, dw)| \leq B|z|\Phi^{-3}(|dz|^2 + |dw|^2)$$

for some constant B. We choose r so that $0 < r < \min\{\frac{1}{2}, \frac{1}{2B}\}$. Then

$$|Q(x; dz, dw)| \leq \frac{1}{2}\Phi^{-3}(|dz|^2 + |dw|^2)$$

on $\pi^{-1}(\Delta_r)$.

Thus we have

$$\begin{aligned} L_\eta(x; dz, dw) &= \Phi^{-3}|dz|^2 + Q(x; dz, dw) \\ &\geq \frac{1}{2}\Phi^{-3}(|dz|^2 - |dw|^2) \end{aligned}$$

on $\pi^{-1}(\Delta_r)$. Thus Lemma 3.1 is proved. □

Now we return to the proof of Theorem 1.1 and consider the function

$$\Phi_\varepsilon(x) = \Phi(x) + \varepsilon\eta(x).$$

In what follows, we let r be a positive number which satisfies the condition in Lemma 3.1(2). On $\pi^{-1}(\Delta_r)$ we have

$$\begin{aligned} L_{\Phi_\varepsilon}(x; dz, dw) &= L_\Phi(x; dz, dw) + \varepsilon L_\eta(x; dz, dw) \\ &\geq \frac{1}{2\Phi^3}(|dw|^2 + \varepsilon(|dz|^2 - |dw|^2)) \\ &= \frac{1}{2\Phi^3}(\varepsilon|dz|^2 + (1-\varepsilon)|dw|^2). \end{aligned}$$

This shows that L_{Φ_ε} is strictly positive at every point in $\pi^{-1}(\Delta_r)$ whenever $0 < \varepsilon < 1$.

On $\pi^{-1}(\Delta \setminus \Delta_r)$, we have

$$L_\Phi(x; dz, dw) \geq \frac{1}{2}\Phi^{-3}(\delta|dz|^2 + |dw|^2),$$

where

$$\delta = \min\bigl\{|g'(z)| \bigm| z \in \overline{\Delta \setminus \Delta_r}\bigr\} > 0.$$

Hence

$$L_{\Phi_\varepsilon}(x; dz, dw) \geq \frac{1}{2}\Phi^{-3}(\delta|dz|^2 + |dw|^2) - \varepsilon A\Phi^{-3}(|dz|^2 + |dw|^2),$$

which is positive definite when ε is sufficiently small.

Thus $\Phi_\varepsilon(x)$ is strictly plurisubharmonic at any point in $\pi^{-1}(\Delta)$ when ε is sufficiently small. By performing the same procedure for every point in Z, we obtain a strictly plurisubharmonic function on X with the desired properties. This completes the proof of Theorem 1.2(1).

Now we deal with the case $\dim M \geq 2$. Suppose that $\varphi(x)$ is a plurisubharmonic function on X. Let r be a non-negative number and put

$$\Sigma_r = \bigl\{x \in X \bigm| \Psi(x) = r\bigr\}.$$

Then $\varphi(x)$ takes its maximum on the compact set Σ_r at a point $x^* \in \Sigma_r$. We suppose $x^* \in \pi^{-1}(U_i)$. Identifying $\pi^{-1}(U_i)$ with its trivialization $U_i \times \mathbb{C}$, we denote $x^* = (p^*, w_i^*)$. Let

$$Z = \bigl\{x = (p, w_i) \in \pi^{-1}(U_i) \bigm| g_i(p) = g_i(p^*),\ w_i = w_i^*\bigr\}.$$

Then Z is an analytic subset of positive dimension in $\pi^{-1}(U_i)$, containing x^*. Further, Z is contained in Σ_r, since

$$\Psi(x) = |w_i - \overline{g_i(p)}| = |w_i^* - \overline{g_i(p^*)}| = \Psi(x^*).$$

Then $\varphi(x)$ is constant on the set Z by the maximum principle. This shows that $\varphi(x)$ is not strictly plurisubharmonic and Theorem 1.2(2) is proved. □

§4. Growth condition on plurisubharmonic functions

We set $X_0 = X \setminus \sigma(M)$. As we noticed in the proof of Theorem 1.1, the plurisubharmonic function $\Psi(x)$ is real-analytic on X_0 and its Levi form L_Ψ degenerates on the subspace of the tangent space at $x \in X_0$ given by the condition $\bar{u}\,\partial u - u\,\partial\bar{u} = 0$, where $u = w - \overline{g(p)}$. In other words, the tangent bundle $T_{\mathcal{F}} \subset T_{X_0}$ of the Monge–Ampère foliation $\mathcal{F}$ on X_0 of $\partial\bar{\partial}\Psi$ satisfies

$$(T_{\mathcal{F}})_x = \{v \in (T_{X_0})_x \mid \langle \bar{u}\, \partial u - u\, \partial \bar{u}, v\rangle = 0\}$$

for each $x \in X_0$, where T_{X_0} is the holomorphic part $T^{1,0}X_0$ of the decomposition $T^{\mathbb{C}}X_0 = T^{1,0}X_0 \oplus T^{0,1}X_0$ of the complexified bundle $T^{\mathbb{C}}X_0$ of the tangent bundle of the C^∞ manifold X_0. Here, *the Monge–Ampère foliation induced by* $\partial\bar{\partial}\Psi$ is the foliation such that the tangent vectors of leaves are eigenvectors of L_Ψ belonging to the eigenvalue 0. Generally, it is known that such a foliation exists when Ψ is plurisubharmonic and the dimension of the 0-eigenspace of its complex Hessian is constant, and that each leaf of it is a holomorphically immersed submanifold (see [2] and [4, §3.2.2] for example).

First let us give an explicit description of the leaves of $\mathcal{F}$. Since complex conjugation of the equation above gives $\bar{u}\,\bar{\partial} u - u\,\bar{\partial}\bar{u} = 0$, this condition is equivalent to

$$\bar{u}\, du - u\, d\bar{u} = 0$$

or

$$d \log(u/\bar{u}) = 0.$$

Hence the integral manifolds of this equation are given by

$$\arg(w - \overline{g(p)}) = \theta,$$

where $\theta \in \mathbb{R}$ are arbitrary constants. Setting $\lambda = e^{i\theta}$, this equation can be put in the form

$$\bar{\lambda}(w - \overline{g(p)}) = \lambda(\overline{w} - g(p)),$$

or

$$\bar{\lambda} w + \lambda g(p) = \lambda \overline{w} + \bar{\lambda}\overline{g(p)}.$$

From this we know that, for any real number r, the Levi form L_Ψ degenerates on the complex curve defined by

$$\bar{\lambda} w + \lambda g(p) = r.$$

Thus we have that leaves of the foliation $\mathcal{F}$ are locally given by the equations above.

To prove Theorem 1.3, let us describe this foliation $\mathcal{F}$ and its leaves more closely. For that purpose, we define a family of holomorphic sections $\gamma_{[\lambda,r]}$ of X over U_i by

$$w_i = \lambda r - \lambda^2 g_i(p),$$

or

$$\bar{\lambda} w_i + \lambda g_i(p) = r$$

with $|\lambda| = 1$ and $r \in (2\,\mathrm{Re}(\lambda g_i(p)), \infty) \subset \mathbb{R}$.

Lemma 4.1. *We have the following for the family of holomorphic sections $\gamma_{[\lambda,r]}$:*

(1) *For any point $x = (p, w) \in \pi^{-1}(U_i)$, there exists a unique section $\gamma_{[\lambda,r]}$ passing though x, which is given by $q \mapsto (q, \lambda r - \lambda^2 g(q))$ with*

$$(\lambda, r) = \left(\frac{w - \overline{g(p)}}{|w - \overline{g(p)}|}, \frac{|w|^2 - |g(p)|^2}{|w - \overline{g(p)}|} \right).$$

(2) *The sections $\gamma_{[\lambda,r]}$ satisfy the differential equation*

$$(\overline{w} - g(p))\, dw - (w - \overline{g(p)})\, dg = 0.$$

Proof. (1) This follows from $\bar{\lambda} w + \lambda g = r$ and its conjugate $\lambda \overline{w} + \bar{\lambda} \bar{g} = r$.

(2) By differentiating $\bar{\lambda} w + \lambda g = r$ one has $\bar{\lambda}\, dw + \lambda\, dg = 0$. Since $\lambda = (w - \overline{g(p)})/|w - \overline{g(p)}|$, the assertion holds. □

Lemma 4.2. *The restriction of Ψ to a leaf of the foliation $\mathcal{F}$ is a non-constant pluriharmonic function.*

Proof. This is clear since the differential equation above defines in the tangent space the direction on which the Levi form degenerates. We can also directly verify this: We have

$$\begin{aligned} \Psi \circ \gamma_{\lambda,r}(p) &= |\lambda r - \lambda^2 g(p) - \overline{g(p)}| \\ &= |r - \lambda g(p) - \overline{\lambda g(p)}| \\ &= |r - 2\,\mathrm{Re}(\lambda g(p))| \\ &= r - 2\,\mathrm{Re}(\lambda g(p)), \end{aligned}$$

which is harmonic except at the points where Ψ vanishes. □

Remark. Suppose that $U_i \cap U_j \neq \emptyset$. Consider a section given by

$$\bar{\lambda} w_i + \lambda g_i = r.$$

Then we have

$$\bar{\lambda}(a_{ij} w_j - b_{ij}) + \lambda(\overline{a_{ij}} g_j - \overline{b_{ij}}) = r.$$

Thus in the fiber coordinate w_j,

$$\overline{\lambda'} w_j + \lambda' g_j = r'$$

holds for $\lambda' = \lambda\overline{a_{ij}}$ and $r' = r + 2\,\mathrm{Re}(\bar{\lambda}b_{ij})$, from which one can concretely check that sections $\gamma_{[\lambda,r]}$ glue together to give a global foliation (note that this fact itself is nothing but a simple conclusion from Lemma 4.1(2) and the definition of the foliation $\mathcal{F}$).

Lemma 4.3. *Let V be an open set such that $Y \subset V \subset X \setminus \sigma(M)$ and let L be a connected leaf of the restriction of the foliation $\mathcal{F}$ to V. Then the restriction of the function $\Psi(x)$ to L is unbounded from above.*

Proof. Suppose that $\Psi|L$ is bounded from above and put $B = \sup_{x\in L}\Psi(x)$. We choose a sequence $\{x_n\}$ of points in L such that $\Psi(x_n) \to B$ $(n \to \infty)$. By shifting to a subsequence we assume that $\{x_n\}$ converges to a point x_0 in $\overline{X}$. Then $x_0 \in V \setminus Y$ and $\Psi(x_0) = B$. We choose U_i such that $x_0 \in \pi^{-1}(U_i)$. For sufficiently large n, we have $x_n \in \pi^{-1}(U_i)$. We denote by γ_n the connected component of $L \cap \pi^{-1}(U_i)$ that passes through x_n. Then γ_n is expressed by $w_i = \lambda_n r_n - \lambda^2 g_i(p)$.

Since $\Psi(x) \leq B$ on γ_n, we have $\Psi(x) \leq B$ on γ_0. The restriction of $\Psi(x)$ to γ_0 is pluriharmonic and takes the value B at x_0. Hence, by the maximum principle $\Psi(x)$ is constant on γ_0, which contradicts Lemma 4.2. □

We set

$$X_R = \left\{x \in X \;\middle|\; \Psi(x) > R\right\}$$

and

$$\Sigma_R = \left\{x \in X \;\middle|\; \Psi(x) = R\right\}$$

for $R \geq 0$.

Lemma 4.4. *Any bounded plurisubharmonic function on X_R $(R \geq 0)$ is constant.*

Proof. By a theorem of Grauert and Remmert [3], any plurisubharmonic function φ on X_R is extended to a plurisubharmonic function on $X_R \cup Y$, which we denote by the same letter φ. We will prove that φ is constant on $X_{R'}$ for any $R' > R$. The function φ takes its maximum B on $X_{R'}$ at a point $x_0 \in \Sigma_{R'}$. Since $\Psi(x_0) = |w_0 - \overline{g(p_0)}| = R'$, we can write

$$w_0 - \overline{g(p_0)} = R'\lambda,$$

where λ is a complex number with $|\lambda| = 1$. We define a holomorphic section $s\colon \Delta \to \pi^{-1}(\Delta)$ by $s(p) = (p, h(p))$, where

$$h(p) = w_0 + \lambda^2(g(p) - g(p_0)).$$

Then

$$\begin{aligned}\Psi(p, h(p)) &= |w_0 + \lambda^2(g(p) - g(p_0)) - \overline{g(p)}| \\ &= |R'\lambda + \overline{g(p_0)} + \lambda^2(g(p) - g(p_0)) - \overline{g(p)}| \\ &= \bigl|R' + 2i \operatorname{Im}\bigl(\lambda(g(p) - g(p_0))\bigr)\bigr| \\ &\geq R'.\end{aligned}$$

This shows that $s(\Delta) \subset \overline{X_{R'}}$. The function $\varphi(s(p))$ is subharmonic and attains its maximum B at p_0, and so $\varphi(s(p)) = B$ identically on Δ. Further, since $g(p)$ is non-constant by the assumption, $s(\Delta)$ contains an interior point of $X_{R'}$. Thus φ takes its maximum at an interior point of $X_{R'}$ and hence is constant on $X_{R'}$. □

Proof of Theorem 1.3. Suppose that $\varphi(x)$ is a plurisubharmonic function on $V \setminus Y$. Take a sufficiently large real number R such that X_R is relatively compact in V. We set $V_0 = X_R \cup Y$.

By Lemma 4.4, it is sufficient to lead to the contradiction by assuming that φ is unbounded from above. We can assume that $\varphi(x) < 0$ on the boundary of ∂V_0 by subtracting a constant. If $\varphi(x) = o(1/\operatorname{dist}(x, Y))$, we have

$$\lim_{x \to Y} \frac{\varphi(x)}{\Psi(x)} = 0.$$

If $\varphi(x)$ is unbounded above, it takes a positive value at some point in V_0. Hence $\frac{\varphi(x)}{\Psi(x)}$ attains its maximum $B > 0$ at some interior point x_0 of V_0. Hence we have

$$\varphi(x) - B\Psi(x) \leq 0 \quad (x \in V_0)$$

and

$$\varphi(x_0) - B\Psi(x_0) = 0.$$

Let L_0 be the leaf of the foliation $\mathcal{F}$ that passes through x_0. The restriction of $\varphi(x) - B\Psi(x)$ to L_0 is subharmonic and attains its maximum at x_0. By the maximum principle, we have $\varphi(x) - B\Psi(x) = 0$ on L_0. This contradicts the assumption that $\varphi(x) = o(\Psi(x))$ $(x \to Y)$. Theorem 1.3 is thereby proved. □

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