Publ. RIMS Kyoto Univ. **61** (2025), 153–232
DOI 10.4171/PRIMS/61-2-1

Étale Endomorphisms of 3-Folds. IV

by

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Abstract

We classify a smooth projective 3-fold X with $\kappa(X) = -\infty$ of type $(\mathrm{C}_{-\infty})$ admitting a nonisomorphic étale endomorphism up to a finite étale covering.

Mathematics Subject Classification 2020: 14J15 (primary); 14J25, 14J30, 14J60, 32J17 (secondary).
Keywords: endomorphism, étale sequence, extremal ray, semistable vector bundle, Atiyah surface.

Contents

§1. Introduction

This is the final part of a series of articles which provide proofs of results announced in [5]. We work over the field $\mathbb{C}$ of complex numbers and use the same notation as in our previous articles [5, 6, 8]. The main purpose of this paper is to study the structure of a smooth projective 3-fold X with negative Kodaira dimension which admits a nonisomorphic étale endomorphism. The following Theorem 1.1 announced in [5] is our main result.

Communicated by Y. Namikawa. Received December 8, 2022. Revised June 3, 2023; July 8, 2023.

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Theorem 1.1. *Let X be a smooth projective 3-fold with the Kodaira dimension $\kappa(X) = -\infty$ admitting a nonisomorphic étale endomorphism. Then up to a finite étale covering, X satisfies one of the following six conditions: (More precisely, replacing f by its suitable power f^k $(k > 0)$, there exist a finite étale Galois covering $\widetilde{X} \to X$ of X and a nonisomorphic étale endomorphism $\tilde{f}\colon \widetilde{X} \to \widetilde{X}$ which is a lift of $f\colon X \to X$. If we replace $f\colon X \to X$ by $\tilde{f}\colon \widetilde{X} \to \widetilde{X}$, then X satisfies one of the following.)*

(1) *$X \simeq S \times E$ for an elliptic curve E and a uniruled algebraic surface S.*

(2) *X is a $\mathbb{P}^1$-bundle over an abelian surface.*

(3) *X is obtained by a succession of blow-ups along elliptic curves from a $\mathbb{P}^1$-bundle Y over the product $S := C \times E$, where C is a curve of genus $g(C) \geq 1$ and E is an elliptic curve.*

(4) *X is a $\mathbb{P}^2$-bundle over an elliptic curve.*

(5) *X is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over an elliptic curve.*

(6) *By the Albanese map $\alpha_X\colon X \to C$, X is an analytic fiber bundle over an elliptic curve C whose fiber is birational to a Hirzebruch surface.*

We note that these six classes are not necessarily mutually disjoint from each other. For example, the product of the Hirzebruch surface $\mathbb{F}_1$ and an elliptic curve satisfies both conditions (1) and (6). Compared with classification results in the case of $\kappa(X) \geq 0$ (cf. [4, 9]), our Theorem 1.1 is not so simple and not strong enough for a complete classification of 3-folds with negative Kodaira dimension admitting nonisomorphic étale endomorphisms. We mainly give a necessary condition for the existence of such varieties and state a sufficient condition in some special cases.

Let us review the background briefly. By an *étale sequence of constant Picard number* (ESP for short) $W_\bullet = (v_n\colon W_n \to W_{n+1})_{n\in\mathbb{Z}}$ of smooth projective k-folds W_n, we mean that for any $n \in \mathbb{Z}$,

- v_n is a nonisomorphic finite étale covering, and
- the Picard number $\rho(W_n)$ is constant.

In [5], we applied the minimal model program (MMP for short) to the constant ESP $X_\bullet = (X, f)$ induced from a nonisomorphic étale endomorphism $f\colon X \to X$ of a smooth projective 3-fold X with $\kappa(X) = -\infty$ and constructed an FESP $Y_\bullet$ from $X_\bullet$ by a sequence of blow-downs of an ESP (cf. [5, Cor. 1.2]). Here, "FESP" means that there exist extremal rays $R_\bullet$ of fiber type on $\overline{\mathrm{NE}}(Y_\bullet)$ (cf. [5, Def. 3.6(3)]).

Compared with the case of $\kappa(X) \geq 0$ (cf. [4, 9]), one of the difficulties in the case of $\kappa(X) = -\infty$ is that there may exist infinitely many extremal rays of the Mori cone $\overline{\mathrm{NE}}(X)$. Here, by an *extremal ray* R, we always mean a K_X*-negative* extremal ray of $\overline{\mathrm{NE}}(X)$. Furthermore, it is not clear that we can find an extremal ray R of $\overline{\mathrm{NE}}(X)$ which is preserved by a suitable power f^k $(k > 0)$ of f. Hence the MMP does not necessarily work compatibly with étale endomorphisms. Thus we adapt a method to first study the rough structure of the original endomorphism $f\colon X \to X$ through the FESP $Y_\bullet$ constructed from X by a sequence of blow-downs of an ESP (cf. [5, Def. 3.7]). In particular, we shall focus our attention on the finiteness of extremal rays, which is equivalent to the finiteness of extremal rays of *divisorial type* (cf. [5, Thm. 8.6]). Once we know the finiteness of the set of extremal rays, then replacing f by its suitable power f^k $(k > 0)$, we can again run the MMP compatibly with étale endomorphisms and obtain another constant FESP (Y, g) induced from a nonisomorphic étale endomorphism $g\colon Y \to Y$. Throughout our articles, we have considered this problem in several stages. In [6] (resp. [8]), we classified a 3-fold X admitting a nonisomorphic étale endomorphism in the case where there exists an FESP $(Y, R_\bullet)$ of type (C_1), (C_0) (resp. of type (D)). Here, "type (C_1)" (resp. "type (C_0)") means that there exists the set $R_\bullet$ of extremal rays of fiber type on $\overline{\mathrm{NE}}\,(Y_\bullet)$ such that the contraction morphism $\mathrm{Cont}_{R_\bullet}\colon Y_\bullet \to S_\bullet$ is a conic bundle over an ESP $S_\bullet$ of smooth algebraic surfaces of Kodaira dimension 1 (resp. 0) (cf. [5, Defs 3.6(2) and 7.6]). Furthermore, "type (D)" means that there exists the set $R_\bullet$ of extremal rays of fiber type on $\overline{\mathrm{NE}}\,(Y_\bullet)$ such that the contraction morphism $\mathrm{Cont}_{R_\bullet}\colon Y_\bullet \to C_\bullet$ is a del Pezzo fibration over an ESP $C_\bullet$ of elliptic curves (cf. [5, Defs 3.6(2)]).

In this part IV article, we shall consider the only remaining case, i.e., we shall classify such varieties X in the case where there exists an FESP $(Y_\bullet, R_\bullet)$ of type $(\mathrm{C}_{-\infty})$ up to a finite étale covering (cf. [5, Defs 3.6(2) and 7.6]). Here, "type $(\mathrm{C}_{-\infty})$" means that there exists the set $R_\bullet$ of extremal rays of fiber type on $\overline{\mathrm{NE}}\,(Y_\bullet)$ such that the contraction morphism $\mathrm{Cont}_{R_\bullet}\colon Y_\bullet \to S_\bullet$ is a conic bundle over $S_\bullet$, where $S_\bullet$ is an ESP of elliptic ruled surfaces. As a by-product of our classifications, it will turn out that we can apply the MMP compatibly with étale endomorphisms in almost all cases (cf. Theorem 6.1). The only exception corresponds to case (1) in Theorem 1.1, where a suitable finite étale covering $\widetilde{X}$ of X is isomorphic to the product of a uniruled surface and an elliptic curve. Now we shall state the outline of the proof of Theorem 1.1. The proof is done according to the strategies stated in our previous article [5, Introduction].

(1) (Construction of an FESP): Using [5, Cor. 1.2], we shall first construct an FESP $Y_\bullet$ of a nonisomorphic étale endomorphism $f\colon X \to X$ by a sequence of blow-downs of an ESP.

(2) (Classifications of an FESP): Using Theorems 2.11, 2.13, Corollary 2.14, Propositions 4.3, 5.1 and the theory of vector bundles on an elliptic curve due to Atiyah (cf. [1]), we shall study the structure of the FESP $Y_\bullet$ explicitly.

(3) (Finiteness of extremal rays of $\overline{\mathrm{NE}}(X)$): Applying Theorems 2.11 and 4.7 to the FESP $(Y_\bullet, R_\bullet)$, we can show the finiteness of extremal rays or can find an f^k-invariant extremal ray R of $\overline{\mathrm{NE}}(X)$ for some $k > 0$ in all cases except the case where a suitable finite étale covering $\widetilde{X}$ is isomorphic to the product of a rational surface and an elliptic curve (cf. Theorems 4.12, 4.20 and Proposition 5.9).

(4) (Construction of a constant FESP): If (3) holds true, then we shall again run the MMP compatibly with étale endomorphisms and obtain another constant FESP (Z, v) induced from a nonisomorphic étale endomorphism $v\colon Z \to Z$.

(5) (Study the structure of a constant FESP): We shall again study the structure of the constant FESP (Z, v) (cf. Propositions 5.9 and 5.11).

(6) (Blow-ups of an FESP): Using [5, Lem. 3.3], we shall find a v-invariant elliptic curve E on Z and perform equivariant blow-ups along E to recover the original endomorphism $f\colon X \to X$. The structure of X can be studied in great detail. For example, we can show that the Albanese map of a suitable finite étale covering $\widetilde{X}$ of X is an analytic fiber bundle over an elliptic curve (cf. Proposition 4.25, Theorems 4.27 and 5.14).

Now we state our results in more detail. Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists an FESP $(Y'_\bullet, R'_\bullet)$ of type $(\mathrm{C}_{-\infty})$ constructed from $X_\bullet := (X, f)$ by a sequence of blow-downs of an ESP (cf. [5, Defs 3.6, 3.7 and 7.6]). Then by [5, Thm. 9.6], replacing X by a suitable finite étale covering $\widetilde{X}$ of X, we are reduced to the following situation: There exist Cartesian morphisms of ESPs

$$\begin{aligned} X_\bullet = (X, f) &\xrightarrow{\pi_\bullet} Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n \\ &\xrightarrow{\varphi_\bullet} S_\bullet = (u_n\colon S_n \to S_{n+1})_n \xrightarrow{\alpha_\bullet} C_\bullet = (C, h) \end{aligned}$$

such that the following conditions are satisfied:

- $Y_\bullet$ is an FESP constructed from $X_\bullet$ by a sequence of blow-downs of an ESP, i.e., $\pi_\bullet = (\pi_n)_n$ is a sequence of blow-ups along elliptic curves.
- There exists an extremal ray $R_\bullet := (R_n)_n$ $(\subset \overline{\mathrm{NE}}(Y_\bullet))$ of fiber type such that $\varphi_\bullet = (\varphi_n)_n$ is the contraction morphism associated to $R_\bullet$ and is a $\mathbb{P}^1$-bundle.
- $\alpha_\bullet = (\alpha_n)_n$ is a $\mathbb{P}^1$-bundle over the Albanese elliptic curve C of X.
- $h\colon C \to C$ is a nonisomorphic group homomorphism of C.
- The composite map $(\alpha_X)_\bullet := \alpha_\bullet \circ \varphi_\bullet \circ \pi_\bullet \colon X \to C$ coincides with the Albanese map $\alpha_X\colon X \to C$ of X, where $C_\bullet := (C, h)$.

Hereafter, we say that Cartesian morphisms of ESPs

$$X_\bullet \xrightarrow{\pi_\bullet} Y_\bullet \xrightarrow{\varphi_\bullet} S_\bullet \xrightarrow{\alpha_\bullet} C_\bullet$$

satisfy Condition ($\mathrm{P}_{-\infty}$) if all the above assumptions are satisfied; here, the letter "P" (resp. "$-\infty$") means that $\varphi_\bullet$ is a $\mathbb{P}^1$-bundle (resp. $S_\bullet$ is an ESP of elliptic ruled surfaces). Furthermore, when $R_\bullet$ is not relevant, we shall drop it and not mention it. Combined with [5, Prop. 5.10, Thm. 10.1] and taking a further finite étale covering, the following are all the possibilities for the elliptic ruled surface S_n over C:

(1) Any S_n is isomorphic to the Atiyah surface $\mathbb{S} := \mathbb{P}_C(\mathcal{F}_2)$ (cf. Definition 2.10).
(2) Any S_n is isomorphic to the elliptic ruled surface $\mathbb{P}_C(\mathcal{O}_C \oplus \ell_n)$ for some torsion line bundle ℓ_n on C.

The following is one of our main results.

Proposition 1.2. *Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Let $Y'_\bullet$ be an FESP constructed from $X_\bullet := (X, f)$ by a sequence of blow-downs of an ESP. Suppose that there exists an extremal ray $R'_\bullet$ of fiber type on $\overline{\mathrm{NE}}(Y'_\bullet)$ such that $(Y'_\bullet, R'_\bullet)$ is of type ($\mathrm{C}_{-\infty}$) (cf. [5, Defs 3.6(2) and 7.6]). Then, for a suitable multiplication mapping $\mu_k\colon C \to C$ by a positive integer k, if we replace X by a finite étale Galois covering $\widetilde{X} := X \times_{C,\mu_k} C$ of X and f by its lift $\tilde{f}\colon \widetilde{X} \to \widetilde{X}$, there exist Cartesian morphisms of ESPs,*

$$X_\bullet \xrightarrow{\pi_\bullet} Y_\bullet \xrightarrow{\varphi_\bullet} S_\bullet \xrightarrow{\alpha_\bullet} C_\bullet,$$

which satisfy Condition ($\mathrm{P}_{-\infty}$).

Furthermore, either of the following two cases can occur:

(1) *Any S_n is isomorphic to the Atiyah surface, i.e., $S_\bullet \simeq \mathbb{S}_\bullet$*
(2) *Any S_n is isomorphic to the elliptic ruled surface $\mathbb{P}_C(\mathcal{O}_C \oplus \ell_n)$ for a torsion line bundle ℓ_n on C. In particular, $\ell_0 \simeq \mathcal{O}_C$ and $S_0 \simeq C \times \mathbb{P}^1$ for $n = 0$.*

Definition 1.3. Let

$$X_\bullet \xrightarrow{\pi_\bullet} Y_\bullet \xrightarrow{\varphi_\bullet} S_\bullet \xrightarrow{\alpha_\bullet} C_\bullet$$

be Cartesian morphisms of ESPs which satisfy the conclusion as in Proposition 1.2. Furthermore, suppose that there exists an integer $a > 1$ such that each fiber of $Y_\bullet \to C_\bullet$ is a Hirzebruch surface $\mathbb{F}_a$. Then

- the FESP $Y_\bullet$ is of *Atiyah type* if any S_n is isomorphic to the Atiyah surface $\mathbb{S}$, that is, $S_\bullet \simeq \mathbb{S}_\bullet$.

- The FESP $Y_\bullet$ is of *torsion type* if any S_n is isomorphic to $\mathbb{P}_C(\mathcal{O}_C \oplus \ell_n)$ for a torsion line bundle ℓ_n on C, $\ell_0 \simeq \mathcal{O}_C$ and $S_0 \simeq C \times \mathbb{P}^1$.

The following theorems show the structure of a nonisomorphic étale endomorphism $f\colon X \to X$ of a smooth projective 3-fold X with $\kappa(X) = -\infty$ in the case where there exists an FESP $Y_\bullet$ satisfying Condition $(\mathrm{P}_{-\infty})$.

Theorem 1.4. *Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists an FESP $(Y'_\bullet, R'_\bullet)$ of type $(\mathrm{C}_{-\infty})$ constructed from $X_\bullet = (X, f)$ by a sequence of blow-downs of an ESP (cf.* [5, Def. 7.6]*). Furthermore, suppose that there exist Cartesian morphisms of ESPs,*

$$X_\bullet \xrightarrow{\pi'_\bullet} Y'_\bullet \xrightarrow{\varphi'_\bullet} \mathbb{S}_\bullet \xrightarrow{\alpha_\bullet} C_\bullet,$$

which satisfy Condition $(\mathrm{P}_{-\infty})$ and are of Atiyah type (cf. Definition 1.3). Then the following hold:

(a) *There exist at most finitely many extremal rays of $\overline{\mathrm{NE}}(X)$.*

(b) *The Albanese map $\alpha_X\colon X \to C$ is an analytic fiber bundle whose fiber is birational to the Hirzebruch surface $\mathbb{F}_a$.*

(c) *Replacing f by its suitable power f^k $(k > 0)$, we can obtain further Cartesian morphisms of constant ESPs satisfying Condition $(\mathrm{P}_{-\infty})$:*

$$X_\bullet \xrightarrow{\pi} (Y, g) \xrightarrow{\varphi} (\mathbb{S}, u) \xrightarrow{\alpha} C_\bullet.$$

(d) *There exists an exact sequence of vector bundles on the Atiyah surface $\mathbb{S}$,*

$$0 \longrightarrow \mathcal{O}_{\mathbb{S}}(a\,s_\infty) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{S}} \longrightarrow 0, \tag{1.1}$$

such that $Y \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{E})$ for a positive integer $a > 1$.

Theorem 1.5. *Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists an FESP $(Y'_\bullet, R'_\bullet)$ of type $(\mathrm{C}_{-\infty})$ constructed from $X_\bullet = (X, f)$ by a sequence of blow-downs of an ESP (cf.* [5, Def. 7.6]*). Furthermore, suppose that there exist Cartesian morphisms of ESPs,*

$$X_\bullet \xrightarrow{\pi'_\bullet} Y'_\bullet \xrightarrow{\varphi_\bullet} S_\bullet \xrightarrow{\alpha_\bullet} C_\bullet,$$

which satisfy Condition $(\mathrm{P}_{-\infty})$ and are of torsion type (cf. Definition 1.3). Then the following hold:

(a) *There exists an exact sequence of vector bundles on $S_0 \simeq C \times \mathbb{P}^1$*

$$0 \longrightarrow p_2^*\mathcal{O}_{\mathbb{P}^1}(a) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{S_0} \longrightarrow 0 \tag{1.2}$$

such that $Y_0 \simeq \mathbb{P}_{S_0}(\mathcal{E})$ for the second projection $p_2\colon S_0 \to \mathbb{P}^1$ and an integer $a > 1$.

(b) *The Albanese map* $\alpha_X : X \to C$ *is an analytic fiber bundle whose fiber is birational to the Hirzebruch surface* $\mathbb{F}_a$*.*

(c) *We set* $\Gamma := \{t \in \mathbb{P}^1 : Y_t \simeq \mathbb{S}\}$*, where* Y_t *is the fiber of* $p_2 \circ \varphi_0 : Y_0 \to \mathbb{P}^1$ *over a point* $t \in \mathbb{P}^1$*:*

   (c-1) *If the exact sequence* (1.2) *splits, then* $\Gamma = \emptyset$ *and* X *is isomorphic to the product* $T \times C$*, where* T *is birational to the Hirzebruch surface* $\mathbb{F}_a$*.*

   (c-2) *If the exact sequence* (1.2) *unsplits, then* $\emptyset \neq \Gamma \subsetneq \mathbb{P}^1$ *is a Zariski open subset such that*

   - *replacing* f *by its suitable power* f^k $(k > 0)$*, we can obtain further Cartesian morphisms of constant ESPs,*

     $$X_\bullet = (X, f) \xrightarrow{\pi_\bullet} Y_\bullet = (Y, g) \xrightarrow{\varphi_\bullet} S_\bullet = (C \times \mathbb{P}^1, v) \xrightarrow{\alpha_\bullet} C_\bullet = (C, h),$$

     such that $v = h \times u$ *for* $u \in \mathrm{Aut}(\mathbb{P}^1)$*;*
   - *the composite map* $\psi := p_2 \circ \varphi \circ \pi : X \to \mathbb{P}^1$ *is a smooth morphism which is not a fiber bundle and a jumping phenomenon occurs;*
   - *there exist at most finitely many* K_X*-negative extremal rays of* $\overline{\mathrm{NE}}(X)$*.*

Organization of this article. In Section 2 we show that any FESP $Y_\bullet$ of type $(\mathrm{C}_{-\infty})$ satisfying Condition $(\mathrm{P}_{-\infty})$ is a $\mathbb{P}^1$- bundle associated to a vector bundle of rank two on the Atiyah surface $\mathbb{S}$ or $C \times \mathbb{P}^1$ (cf. Theorem 2.13). In Definition 1.3, the type of an FESP $Y_\bullet$ is defined. Lemma 2.24 shows that an FESP $Y_\bullet$ can be described by an exact sequence of vector bundles on an elliptic ruled surface which is called the *fundamental exact sequence of bundles* (FES for short) (cf. Definition 2.25).

In Section 3 we review the theory of *elementary transformations of vector bundles* due to Maruyama [17], which will be used in Sections 4 and 5 to show the finiteness of extremal rays in certain cases.

In Section 4 we study a nonisomorphic étale endomorphism $f : X \to X$ admitting an FESP $Y_\bullet$ of Atiyah type (cf. Definition 1.3). Proposition 4.3 enables one to describe the structure of an FESP $Y_\bullet$ in terms of an FES on the Atiyah surface $\mathbb{S}$. In Definition 4.5, the FESP $Y_\bullet$ of Atiyah type will be classified into two cases according to whether the FES splits or unsplits. Then we shall apply the MMP compatibly with étale endomorphisms and show the existence of a constant FESP $Y_\bullet$ of Atiyah type so as to apply Theorem 4.7(2) or Corollary 4.8(2). Theorems 4.12 and 4.20 show the finiteness of K_X-negative extremal rays of $\overline{\mathrm{NE}}(X)$. Theorem 4.27 shows that in the case where the FESP $Y_\bullet$ is of Atiyah type, the Albanese map $\alpha_X : X \to C$ gives the analytic fiber bundle over the elliptic curve C. The proof of Theorem 1.4 will be given.

In Section 5 we study a nonisomorphic étale endomorphism $f\colon X \to X$ admitting an FESP $Y_\bullet$ of torsion type (cf. Definition 1.3). Proposition 5.1 enables one to describe the structure of an FESP $Y_\bullet$ in terms of an FES on $C \times \mathbb{P}^1$. In this case, Proposition 5.9 shows that the MMP works compatibly with étale endomorphisms except for the case where a suitable finite étale covering $\widetilde{X}$ of X is isomorphic to the product of a rational surface and an elliptic curve. In Definition 5.5, the FESP $Y_\bullet$ will be classified into two cases in terms of an FES. Theorem 5.6 and Proposition 5.11 describe the structure of the FESP $Y_\bullet$ in terms of an FES on $C \times \mathbb{P}^1$. Theorem 5.14 describes the structure of X in great detail. Finally, the proof of Theorem 1.5 will be given.

In Section 6 we prove Theorems 1.1 and 6.1, which are the main results of our series of articles (cf. [5, 6, 8] and this paper) and describe the structure of a smooth projective 3-fold X with $\kappa(X) = -\infty$ admitting a nonisomorphic étale endomorphism. As a by-product of our classifications, Theorem 6.1 shows that for a nonisomorphic étale endomorphisms of 3-folds $f\colon X \to X$, we can apply the MMP compatibly with étale endomorphisms and obtain a constant FESP $Y_\bullet = (Y, g)$ except for the case where a suitable finite étale covering $\widetilde{X}$ of X is isomorphic to the product of a rational surface and an elliptic curve.

In Section 7, Theorem 7.1 shows the uniqueness of the type of an FESP $Y_\bullet$ (cf. Definitions 1.3 and 5.5). If the FESP $Y_\bullet$ is of type (Torsion.A), then Theorem 7.8 shows that any FESP $Z_\bullet$ of $X_\bullet$ is of type (Torsion.A). Furthermore, Proposition 7.3 and Corollary 7.5 show the uniqueness of the fiber space structure of X over $\mathbb{P}^1$ up to isomorphism and also the finiteness of K_X-negative extremal rays of $\overline{\mathrm{NE}}\,(X)$.

In the appendix, Theorems A.1 and A.2 show the existence of certain 3-folds whose nonisomorphic surjective endomorphisms are necessarily étale, which is related to the endomorphisms of the Atiyah surface $\mathbb{S}$. Proposition A.5 shows the existence of a smooth projective 3-fold X with non-nef anti-canonical bundle $-K_X$ which admits a nonisomorphic étale endomorphism.

Section dependency. The contents of each section are related in the following diagram:

$$\begin{array}{ccccccc} & & 2 & \longrightarrow & 5 & & \\ & & \downarrow & & \downarrow & & \\ 3 & \longrightarrow & 4 & \longrightarrow & 6 & \longrightarrow & 7 \\ & & & & \downarrow & & \\ & & & & \text{Appendix.} & & \end{array}$$

§2. Set-up

Notation. Throughout this paper, we follow the notation in [5, Notation 2.1].

We first recall the following facts in [5] which play an important role in this paper.

Lemma 2.1 (Cf. [5, Lem. 2.6]). *Let $\lambda\colon V \to S$ and $\nu\colon W \to T$ be fiber spaces between normal varieties, i.e., λ and ν are proper surjective morphisms with connected fibers. Furthermore, suppose that $g\colon V \to W$ and $u\colon S \to T$ are finite surjective morphisms with the following commutative diagram:*

$$\begin{array}{ccc} V & \xrightarrow{g} & W \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\nu} \\ S & \xrightarrow[u]{} & T. \end{array}$$

Suppose that $g^{-1}(B) = A$ for irreducible subvarieties A $(\subset V)$ and B $(\subset W)$. If the above commutative diagram is Cartesian, then $\lambda(A) = u^{-1}(\nu(B))$.

Proposition 2.2 (Cf. [5, Prop. 3.1]). *Let $f\colon Y \to X$ be a finite surjective morphism between smooth projective n-folds with $\rho(X) = \rho(Y)$. Then the following assertions hold:*

(1) *The push-forward map $f_*\colon \mathrm{N}_1(Y) \to \mathrm{N}_1(X)$ is an isomorphism and $f_*\,\overline{\mathrm{NE}}(Y) = \overline{\mathrm{NE}}(X)$.*
(2) *Let $f_*\colon \mathrm{N}^1(Y) \to \mathrm{N}^1(X)$ be the map induced from the push-forward map $D \mapsto f_*D$ of divisors D. Then the dual map $f^*\colon \mathrm{N}_1(X) \to \mathrm{N}_1(Y)$ (called the pullback map) is an isomorphism and $f^*\,\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(Y)$.*
(3) *If f is étale and the canonical divisor K_X is not nef, then there is a one-to-one correspondence between the set of extremal rays of X and the set of extremal rays of Y under the isomorphisms f_* and f^*.*
(4) *Under the same assumption as in (3), let $\phi\colon X \to X'$ be the contraction morphism Cont_R associated to an extremal ray $R \subset \overline{\mathrm{NE}}(X)$ and let $\psi\colon Y \to Y'$ be the contraction morphism associated to the extremal ray f^*R. Then there exists a finite surjective morphism $f'\colon Y' \to X'$ and the Cartesian diagram below such that $f^{-1}(\mathrm{Exc}(\phi)) = \mathrm{Exc}(\psi)$ and $f'^{-1}(\phi(\mathrm{Exc}(\phi))) = \psi(\mathrm{Exc}(\psi))$:*

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\phi} \\ Y' & \xrightarrow{f'} & X'. \end{array}$$

Moreover, ϕ is a birational (resp. divisorial) contraction if and only if ψ is a birational (resp. divisorial) contraction.

Proposition 2.3 (Cf. [5, Prop. 4.8]). *Let $g\colon S \to S$ be a nonisomorphic étale endomorphism of a relatively minimal elliptic ruled surface S. Suppose that $S = \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L})$ for an invertible sheaf $\mathcal{L}$ on an elliptic curve C. Then $\mathcal{L} \in \mathrm{Pic}(C)$ is of finite order.*

Proposition 2.4 (Cf. [5, Prop. 5.2]). *Let $\pi\colon \mathbb{S} \to C$ be the Atiyah surface. Then the following hold:*

(1) *The canonical section s_∞ (cf. Definition 2.10) is a unique irreducible curve on $\mathbb{S}$ such that its self-intersection number is zero and is not contained in any fiber of π.*

(2) $\mathrm{h}^0(\mathbb{S}, \mathcal{O}_{\mathbb{S}}(a\, s_\infty)) = 1$ *for any positive integer a.*

Proposition 2.5 (Cf. [5, Prop. 5.5]). *Let $\pi\colon \mathbb{S} \to C$ be the Atiyah surface. Then every surjective endomorphism $\varphi\colon \mathbb{S} \to \mathbb{S}$ is a finite étale covering satisfying $\varphi^*(s_\infty) = s_\infty$.*

Proposition 2.6 (Cf. [5, Prop. 5.6]). *There exist no surjective morphisms from one to another among the following three $\mathbb{P}^1$-bundles S_i over an elliptic curve C_i $(1 \le i \le 3)$:*

(1) $\pi_1\colon S_1 = \mathbb{P}_{C_1}(\mathcal{O}_{C_1} \oplus \ell) \to C_1$ *for a line bundle ℓ of degree zero on C_1, where $\ell \in \mathrm{Pic}^0(C_1)$ is torsion.*

(2) $\pi_2\colon S_2 = \mathbb{P}_{C_2}(\mathcal{O}_{C_2} \oplus \mathcal{L}) \to C_2$ *for a line bundle $\mathcal{L}$ of degree zero on C_2, where $\mathcal{L} \in \mathrm{Pic}^0(C_2)$ is of infinite order.*

(3) $\pi_3\colon S_3 = \mathbb{S} \to C_3$ *is the Atiyah surface.*

Proposition 2.7 (Cf. [5, Prop. 5.10]). *Suppose that there exists an ESP $S_\bullet = (g_n\colon S_n \to S_{n+1})_n$ of elliptic ruled surfaces S_n. Then, one of the following cases occurs:*

(1) *There exists an integer $k \le \infty$ such that*

  (a) *For any $i \le k$, $S_i \simeq \mathbb{P}_{C_i}(\mathcal{O}_{C_i} \oplus \ell_i)$ for some torsion line bundle $\ell_i \in \mathrm{Pic}^0(C_i)$ on an elliptic curve C_i.*

  (b) *For any $i > k$, $S_i \simeq \mathbb{P}_{C_i}(\mathcal{E}_i)$, where $\mathcal{E}_i$ is a stable vector bundle of rank two and degree one on an elliptic curve C_i.*

(2) *For any i, $S_i \simeq \mathbb{P}_{C_i}(\mathcal{O}_{C_i} \oplus \mathcal{L}_i)$ for some nontorsion line bundle $\mathcal{L}_i$ of degree 0 on an elliptic curve C_i.*

(3) *S_i is isomorphic to the Atiyah surface $\mathbb{S}$ for any i.*

Proposition 2.8 (Cf. [5, Cor. 7.9]). *Let $X_\bullet = (f_n\colon X_n \to X_{n+1})_n$ be an ESP of smooth projective 3-folds X_n with $\kappa(X_n) = -\infty$. Let $(Y_\bullet, R_\bullet)$ be an FESP constructed from $X_\bullet$ by a sequence of blow-downs of an ESP and set $Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n$. Let $C_n^{(i)}$ be the elliptic curve which is the center of the blow-up $\pi_n^{(i-1)}\colon X_n^{(i-1)} \to X_n^{(i)}$. If we set $X_n^{(0)} := X_n$, $X_n^{(k)} := Y_n$, then $\gamma_n^{(i)} := \pi_n^{(k-1)} \circ \cdots \circ \pi_n^{(i)}(C_n^{(i)})$ is an elliptic curve on Y_n such that $\gamma_\bullet^{(i)} = (g_n\colon \gamma_n^{(i)} \to \gamma_{n+1}^{(i)})_n$ is an ESP of elliptic curves. Furthermore, the following hold:*

(1) *If $(Y_\bullet, R_\bullet)$ is of type* (D)*, then $\varphi_n(\gamma_n^{(i)}) = C_n$.*

(2) *If $(Y_\bullet, R_\bullet)$ is of type* (C_0) *or* (C_1)*, then $\Delta_n^{(i)} := \varphi_n(\gamma_n^{(i)})$ is some fiber of $\alpha_n\colon S_n \to C_n$ such that $\Delta_\bullet = (h_n\colon \Delta_n \to \Delta_{n+1})_n$ is an ESP of elliptic curves.*

(3) *If $(Y_\bullet, R_\bullet)$ is of type* ($\mathrm{C}_{-\infty}$)*, then $\Delta_n^{(i)} := \varphi_n(\gamma_n^{(i)})$ is an elliptic curve on S_n with self-intersection number* 0 *and it dominates C_n. Furthermore, $\Delta_\bullet = (h_n\colon \Delta_n \to \Delta_{n+1})_n$ is an ESP of elliptic curves.*

Hereafter, we shall use the following terminology.

Definition 2.9 (Cf. [5, Defs 2.4 and 3.6]). Let $X_\bullet = (f_n\colon X_n \to X_{n+1})_n$ be an ESP of smooth projective varieties:

(1) If there exists a smooth projective variety X such that $X_n = X$ for any n, then we say that $X_\bullet$ is a *stable ESP*.

(2) Furthermore, if $X_n = X$ and $f_n = f$ for any n for a nonisomorphic étale endomorphism $f\colon X \to X$ of a smooth projective variety X, then we say that $X_\bullet$ is a *constant ESP* induced by $f\colon X \to X$ and denote it by $X_\bullet = (X, f)$.

(3) Let $Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n$ be an ESP of smooth projective varieties with negative Kodaira dimension. Then $Y_\bullet$ is called an *ESP of fiber type* (FESP for short) if there exists an extremal ray R_n of fiber type on any $\overline{\mathrm{NE}}(Y_n)$, that is, the contraction morphism $\mathrm{Cont}_{R_n}\colon Y_n \to W_n$ associated to R_n is a Mori fiber space, i.e., $\dim W_n < \dim Y_n$ for any n.

Definition 2.10. Let $\mathcal{F}_2$ be an indecomposable semistable locally free sheaf of rank two and degree zero on an elliptic curve C (cf. [1]). Let $\pi\colon \mathbb{S} := \mathbb{P}_C(\mathcal{F}_2) \to C$ be the $\mathbb{P}^1$-bundle associated with $\mathcal{F}_2$ and s_∞ the section of π corresponding to a surjection $\mathcal{F}_2 \twoheadrightarrow \mathcal{O}_C$. Then we call $\mathbb{S}$ an *Atiyah surface* (over C) and s_∞ the *canonical section* of π.

The following *torsion theorem* proved in [5] will play a crucial role throughout this article.

Theorem 2.11 (Cf. [5, Thm. 10.1]). *Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Let $Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n$ be an FESP and $\pi_\bullet = (\pi_n)_n\colon X_\bullet \coloneqq (X, f) \to Y_\bullet$ a succession of Cartesian blow-ups along elliptic curves. Suppose that there exists an ESP $S_\bullet = (u_n\colon S_n \to S_{n+1})_n$ of smooth algebraic surfaces S_n and a Cartesian morphism $\varphi_\bullet = (\varphi_n)_n\colon Y_\bullet \to S_\bullet$ such that the following hold:*

(1) *$\varphi_n\colon Y_n \to S_n$ is a $\mathbb{P}^1$-bundle for any n.*
(2) *Any S_n is isomorphic to a $\mathbb{P}^1$-bundle $\mathbb{P}_C(\mathcal{O}_C \oplus \ell_n)$ for a line bundle ℓ_n of degree zero on the Albanese elliptic curve C of X.*

Then $\ell_n \in \mathrm{Pic}^0(C)$ is of finite order for any n.

Essentially, Proposition 1.2 has already been proved in [5]. We shall give a proof for convenience. We begin with an easy lemma.

Lemma 2.12 (Cf. [5, Prop. 4.4]). *Let $\mathcal{E}$ be a stable vector bundle of rank two and of odd degree on an elliptic curve C. We set $S \coloneqq \mathbb{P}_C(\mathcal{E})$. Let $\mu_2\colon C \to C$ be a multiplication mapping by 2 and consider the fiber product $\widetilde{S} \coloneqq S \times_{\mu_2, C} C$. Then there exists an isomorphism $\widetilde{S} \simeq C \times \mathbb{P}^1$ over C.*

Proof. By [1], S is isomorphic over C to the symmetric product $\mathrm{Sym}^2 C$ and the Albanese map $\alpha_S\colon S \to C$ is induced by $C \times C \to C$, $(x, y) \mapsto x + y$ for $x, y \in C$. Let $\mu_2\colon C \to C$, $x \mapsto 2x$ be a multiplication mapping by 2. Let $\widetilde{S} \coloneqq S \times_{\mu_2, C} C$ be the fiber product. Then the diagonal map $\Delta\colon C \to \mathrm{Sym}^2 C$ defined by $x \mapsto (x, x)$ induces a section of $\widetilde{S} \to C$. Hence $\widetilde{S} \simeq C \times \mathbb{P}^1$. □

Proof of Proposition 1.2. [5, Thm. 9.6] shows the existence of Cartesian morphisms of ESPs,

$$X_\bullet \xrightarrow{\pi_\bullet} Y_\bullet \xrightarrow{\varphi_\bullet} S_\bullet \xrightarrow{\alpha_\bullet} C_\bullet,$$

which satisfies Condition ($\mathrm{P}_{-\infty}$). Applying Proposition 2.7, Theorem 2.11, [5, Cor. 10.6] and Lemma 2.12, we see immediately that either case (1) or case (2) can occur. Suppose that we are in case (2). Then any S_n is isomorphic to an elliptic ruled surface $\mathbb{P}_C(\mathcal{O}_C \oplus \ell'_n)$ for some torsion line bundle $\ell'_n \in \mathrm{Pic}^0(C)$. If we set $d \coloneqq \mathrm{ord}(\ell'_0)$, then $\mu_d^* \ell'_0 \simeq \mathcal{O}_C$ for a multiplication by d mapping $\mu_d\colon C \to C$. If we consider the pullback $\widetilde{X} \coloneqq X \times_{C, \mu_d} C$, then there exists a nonisomorphic étale covering $\tilde{f}\colon \widetilde{X} \to \widetilde{X}$ which is a lift of f. Then we replace (X, f) by the pullback $(\widetilde{X}, \tilde{f})$. Applying an MMP to $X_\bullet \coloneqq (X, f)$ again, there exists the following Cartesian morphism of ESPs:

$$\begin{aligned} X_\bullet = (X, f) &\xrightarrow{\pi_\bullet} Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n \\ &\xrightarrow{\varphi_\bullet} S_\bullet = (u_n\colon S_n \to S_{n+1})_n \xrightarrow{\alpha_\bullet} C_\bullet = (C, h), \end{aligned}$$

such that any S_n is isomorphic to $\mathbb{P}_C(\mathcal{O}_C \oplus \ell_n)$ for a torsion line bundle $\ell_n \in \mathrm{Pic}^0(C)$ and $\ell_0 \simeq \mathcal{O}_C, S_0 \simeq C \times \mathbb{P}^1$. □

In the following, we show that each Y_n is isomorphic over S_n to a $\mathbb{P}^1$-bundle $\mathbb{P}_{S_n}(\mathcal{E}_n)$ associated to a locally free sheaf $\mathcal{E}_n$ of rank two on S_n.

Theorem 2.13. *Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Let $X_\bullet = (X, f) \xrightarrow{\pi_\bullet} Y_\bullet := (g_n\colon Y_n \to Y_{n+1})_n \xrightarrow{\varphi_\bullet} S_\bullet := (u_n : S_n \to S_{n+1})_n \xrightarrow{\alpha_\bullet} C_\bullet = (C, h)$ be Cartesian morphisms of ESPs satisfying Condition $(\mathrm{P}_{-\infty})$ (cf. Section 1). Suppose that*

- *for a composite $\psi_\bullet := \alpha_\bullet \circ \varphi_\bullet$, some fiber of ψ_n is isomorphic to a Hirzebruch surface $\mathbb{F}_a$ $(a > 0)$ and*
- *$Y_\bullet$ is an FESP which contains no extremal rays of divisorial type.*

Then we have $a > 1$ and the following hold:

- *Any fiber of $\psi_\bullet$ is isomorphic to $\mathbb{F}_a$.*
- *For any $n \in \mathbb{Z}$, there exists a section Δ_n $(\subset Y_n)$ of $\varphi_n\colon Y_n \to S_n$ such that $\Delta_n \cap \psi_n^{-1}(t)$ $(t \in C)$ is a negative section of $\mathbb{F}_a$ $(\simeq \psi_n^{-1}(t))$ and forms a sub-ESP $\Delta_\bullet := (g_n|_{\Delta_n}\colon \Delta_n \to \Delta_{n+1})_n$ of $Y_\bullet$.*

Corollary 2.14. *Under the same assumption as in Theorem 2.13, there exist a rank-two vector bundle $\mathcal{E}_n$ and an invertible sheaf ℓ_n on S_n such that for any n,*

- *there exists an isomorphism $Y_n \simeq \mathbb{P}(\mathcal{E}_n)$, and*
- *the section Δ_n of $\varphi_n\colon Y_n \to S_n$ corresponds to a surjection $\mathcal{E}_n \twoheadrightarrow \ell_n$.*

Corollary 2.15. *Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Let $X_\bullet = (X, f) \xrightarrow{\pi_\bullet} Y_\bullet := (g_n\colon Y_n \to Y_{n+1})_n \xrightarrow{\varphi_\bullet} S_\bullet := (u_n\colon S_n \to S_{n+1})_n \xrightarrow{\alpha_\bullet} C_\bullet = (C, h)$ be Cartesian morphisms of ESPs satisfying Condition $(\mathrm{P}_{-\infty})$. For a composite $\psi_\bullet := \alpha_\bullet \circ \varphi_\bullet$, suppose that some fiber of ψ_n is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Then Y_n is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over C for any n.*

Remark 2.16. The case where $a = 0$ which is stated in Corollary 2.15 and the case where $a = 1$ (cf. Remark 5.16) have already been studied in our Part III article [8]. In this Part IV article, we shall study the case where there exists an integer $a > 1$ such that each fiber of $Y_\bullet \to C_\bullet$ is a Hirzebruch surface $\mathbb{F}_a$.

For the proof, we prepare some lemmas.

Lemma 2.17. *If some fiber of ψ_0 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, then any fiber of ψ_0 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. We take a point $x \in C$ such that $\psi_0^{-1}(x) \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Suppose to the contrary that there exists some point $p \in C$ such that $\psi_0^{-1}(p) \simeq \mathbb{F}_a$ for some $a > 0$. Since $H^1(\mathbb{P}^1 \times \mathbb{P}^1, \Theta) = 0$ for the tangent bundle Θ, we see that $\mathbb{P}^1 \times \mathbb{P}^1$ is rigid. Hence there exists a small open neighborhood U of x such that $\psi_0^{-1}(y) \simeq \mathbb{P}^1 \times \mathbb{P}^1$ for any $y \in U$. Let $h\colon C \to C$ be the endomorphism induced from f (cf. the notation of Theorem 2.13). Note that C is an elliptic curve and hence any nonisomorphic endomorphism of C is étale so that p cannot be h^{-1}-periodic. Hence, if we set

$$T := \bigcup_{n=0}^{\infty} (h^n)^{-1} \circ h^n(p),$$

then T is a dense subset of C containing p. Hence $T \cap U \neq \emptyset$. For any $q \in T \cap U$, we have $\psi_0^{-1}(q) \simeq \psi_0^{-1}(p) \simeq \mathbb{F}_a$ $(a > 0)$. Thus a contradiction is derived, since $q \in T$ and $\psi_0^{-1}(q) \simeq \mathbb{P}^1 \times \mathbb{P}^1$. □

Lemma 2.18. *Any fiber of ψ_0 contains no (-1)-curves.*

Proof. Suppose that $\psi_0^{-1}(p) \simeq \mathbb{F}_1$ for some $p \in C$. Then we first show that $\psi_0\colon Y_0 \to C$ is an $\mathbb{F}_1$-bundle. Suppose that there exists $q \in C$ such that $\psi_0^{-1}(q) \simeq \mathbb{F}_a$ $(a \neq 1)$ and we shall derive a contradiction. If we set

$$\Omega := \bigcup_{n=0}^{\infty} (h^n)^{-1} \circ h^n(q)$$

for the induced endomorphism $h\colon C \to C$ from f, then Ω is a dense subset of C, since $\deg(h) > 1$. By the "stability of (-1)-curves", there exists a small open neighborhood U of p such that $\psi_0^{-1}(y)$ contains (-1)-curves for any $y \in U$. Here, "stability of (-1)-curves" means that there exists a relative divisor D of X over U such that D_y $(\subset X_y)$ is a (-1) curve for any $y \in U$ (cf. [13, Def. 2 and Thm. 5] and [18, Sect. 11. Deformation of extremal rays]). Then $\Omega \cap U \neq \emptyset$. Thus a contradiction is derived, since for any $y \in \Omega \cap U \neq \emptyset$, $\psi_0^{-1}(y) \simeq \mathbb{F}_a$ $(a \neq 1)$ contains no (-1)-curves. Hence $\psi_0^{-1}(q) \simeq \mathbb{F}_1$ for any $q \in C$ and ψ_0 is an $\mathbb{F}_1$-bundle.

For a fixed point $o \in C$, let $\ell \in Y_o := \psi_0^{-1}(o)$ be a (-1)-curve on Y_o. Hereafter, we set $Y := Y_0$. Then we have the following exact sequence:

$$0 \longrightarrow N_{\ell/Y_o} \longrightarrow N_{\ell/Y} \longrightarrow N_{Y_o/Y}|_\ell \longrightarrow 0,$$

where $N_{\ell/Y_o} \cong \mathcal{O}_\ell(-1)$ and $N_{Y_o/Y}|_\ell \cong \mathcal{O}_\ell$.

Since $\operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-1)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$, the above exact sequence splits and $N_{\ell/Y} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Hence $h^0(\ell, N_{\ell/Y}) = 1$ and $h^1(\ell, N_{\ell/Y}) = 0$. Then, by the theory of Hilbert schemes, there exist a smooth curve B $(\subset \operatorname{Hilb}(Y/C))$ dominating C, a subspace $M \subset Y \times_C B$ and a proper flat morphism

$p_B|_M\colon M \to B$ (where $p_B\colon Y \times_C B \to B$ is the second projection) such that $M \cap p_B^{-1}(o') = \ell$ for a point $o' \in B$ lying over $o \in C$ under the natural projection $q\colon B \to C$. We put $\ell_t := M \cap p_B^{-1}(t)$ for $t \in B$, where $p_B^{-1}(t)$ is naturally identified with a fiber of $\varphi\colon Y \to C$ over $q(t)$. Then we have $(-K_Y, \ell_t) = (-K_Y, \ell) = 1$. Since $-K_Y$ is φ-ample, ℓ_t is irreducible and reduced, and hence isomorphic to $\mathbb{P}^1$. Since $T_{[\ell_t]} \operatorname{Hilb}(Y/C) \cong H^0(\ell_t, N_{\ell_t/Y_t}) = 0$, we have $\Omega^1_{B/C} \otimes \mathbb{C}([\ell_t]) = 0$. Hence $\Omega^1_{B/C} = 0$ around $[\ell_t]$ and the natural projection $p\colon B \to C$ is unramified. It is also proper and flat by the stability of (-1)-curves. Hence $p\colon B \to C$ is a finite étale covering. We note that ℓ is the unique (-1)-curve on Y_o, since $Y_o \simeq \mathbb{F}_1$. Thus we have an isomorphism $p\colon B \simeq C$. Hence there exists an irreducible and reduced divisor Z on Y such that $Z \cap \psi_0^{-1}(o)$ consists of only one (-1)-curve ℓ on $\psi_0^{-1}(o)$.

Next we show that $R := \mathbb{R}_{\geq 0}[\ell]$ is an extremal ray of type (E1) in the sense of [18]. Since $-K_Y + Z$ is ψ-nef and

$$\overline{\mathrm{NE}}(Y/C) \cap (-K_Y + Z)^{\perp} = R,$$

R is an extremal ray. Since $-K_Y$ is relatively semiample over C, $-K_Y + Z$ is semiample by the base-point-free theorem. Hence $\Phi_{|-K_Y+Z|}$ gives a divisorial contraction which is an extremal contraction $v := \mathrm{Cont}_R\colon Y \to Y'$ associated to R. This derives a contradiction, since by construction, $Y_\bullet$ is an FESP constructed from $X_\bullet$ by a sequence of blow-downs and there exist no extremal rays of divisorial type on $Y_\bullet$ (cf. [5, Cor. 1.2]). □

Lemma 2.19. *Under the same assumption as in Theorem 2.13, let $\mathcal{E}_n$ be a locally free sheaf on Y_n such that $\mathcal{E}_n \simeq g_n^*\mathcal{E}_{n+1}$ for any n. Let*

$$\psi_n^*\psi_{n*}\mathcal{E}_n \longrightarrow \mathcal{E}_n$$

be a canonical homomorphism of an $\mathcal{O}_{Y_n}$-module. For $t \in C$, let

$$\psi_{n*}\mathcal{E}_n \otimes_{\mathbb{C}} \mathbb{C}(t) \longrightarrow H^0(Y_{n,t}, \mathcal{E}_n|_{Y_{n,t}})$$

be an induced linear map. By $\tau_n\colon C \to \mathbb{Z}$ for $n \in \mathbb{Z}$, we denote the $\mathbb{Z}$-valued function on C defined by

$$t \longmapsto \dim_{\mathbb{C}} H^0(Y_{n,t}, \mathcal{E}_n|_{Y_{n,t}}).$$

Then τ_0 is a constant function on C.

Proof. Suppose the contrary. Then by the upper semicontinuity of dimension of cohomologies, for any $n \in \mathbb{Z}$, τ_n is constant on a Zariski open subset Γ_n of C and $\tau_n(x_n) > \tau_n(t_n)$ for $x_n \in C \setminus \Gamma_n$ and any $t_n \in \Gamma_n$ so that $\emptyset \neq \Gamma_n \subsetneq C$. Since $\mathcal{E}_n \simeq g_n^*\mathcal{E}_{n+1}$ and by assumption both g_n and h are étale, it follows that

$h^{-1}(\Gamma_{n+1}) = \Gamma_n$ for any n. We choose a point $x_0 \in C \setminus \Gamma_0$ arbitrarily and define a nonempty subset T of C by

$$T := \bigcup_{n>0} (h^n)^{-1} h^n(x_0).$$

Since $\deg(h) > 1$, T is a dense subset of C. Hence $T \cap \Gamma_0 \neq \emptyset$. From now on, we define τ_0 as the infimum of the function τ_0 by abuse of notation. Then for any $t' \in T \cap \Gamma_0$, we have $\tau_0(t') = \tau_0(x_0) > \tau_0$, since $x_0 \notin \Gamma_0$. On the other hand, we have $\tau_0(t') = \tau_0$, since $t' \in \Gamma_0$. Thus a contradiction is derived. □

Lemma 2.20. *For a positive integer n, let $\mathcal{E} := \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$ be a rank-two vector bundle on $T := \mathbb{P}^1$. Let $p\colon S := \mathbb{P}_T(\mathcal{E}) \to T := \mathbb{P}^1$ be a $\mathbb{P}^1$-bundle associated to $\mathcal{E}$. Then we have $h^0(S, \Theta_{S/T}) = n + 2$.*

Proof. Let

$$0 \longrightarrow \Omega^1_{S/T} \otimes \mathcal{O}_S(1) \longrightarrow p^*\mathcal{E} \longrightarrow \mathcal{O}_S(1) \longrightarrow 0$$

be the Euler sequence. After tensoring the dual of the above exact sequence with $\mathcal{O}_S(1)$, we have the following exact sequence of vector bundles on S:

$$0 \longrightarrow \mathcal{O}_S \longrightarrow p^*\mathcal{E}^\vee \otimes_{\mathcal{O}_S} \mathcal{O}_S(1) \longrightarrow \Theta_{S/T} \longrightarrow 0.$$

Since $R^1 p_* \mathcal{O}_S = 0$, taking direct images, we have the following exact sequence of vector bundles on T:

$$(\bigstar)\colon 0 \longrightarrow \mathcal{O}_T \longrightarrow \mathcal{E}^\vee \otimes_{\mathcal{O}_T} \mathcal{E} \longrightarrow p_*\Theta_{S/T} \longrightarrow 0.$$

Then the natural homomorphism $\mathcal{E}^\vee \otimes_{\mathcal{O}_T} \mathcal{E} \longrightarrow \mathcal{O}_T$ gives a splitting of $(\bigstar)$. Hence there exists an isomorphism

$$\mathcal{E}^\vee \otimes_{\mathcal{O}_T} \mathcal{E} \simeq \mathcal{O}_T \oplus p_*\Theta_{S/T}.$$

Since $\mathcal{E} := \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$, we have $\mathcal{E}^\vee \otimes_{\mathcal{O}_T} \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1}$. Hence there exists an isomorphism $p_*\Theta_{S/T} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$ and $h^0(S, \Theta_{S/T}) = h^0(T, p_*\theta_{S/T}) = n + 2$. □

Lemma 2.21. *For an integer $a \geq 2$, let S denote the Hirzebruch surface $\mathbb{F}_a$. Then*

$$h^0(S, \mathcal{O}_S(-K_S)) = \begin{cases} 9, & a = 2, \\ a + 6, & a \geq 3. \end{cases}$$

Proof. Let F be the fiber of the ruling $\mathbb{F}_a \to \mathbb{P}^1$. Since $K_S \sim -2s_\infty - (a+2)F$ and $\mathcal{O}_S(1) \sim s_\infty + aF$, we have $-K_S \sim \mathcal{O}_S(2) \otimes \mathcal{O}_S((2-a)F)$. Hence,

$$\begin{aligned} p_*\mathcal{O}_S(-K_S) &\simeq p_*\mathcal{O}_S(2) \otimes \mathcal{O}_{\mathbb{P}^1}(2-a) \\ &\simeq \mathrm{Sym}^2 p_*\mathcal{O}_S(1) \otimes \mathcal{O}_{\mathbb{P}^1}(2-a) \\ &\simeq \mathrm{Sym}^2(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a)) \otimes \mathcal{O}_{\mathbb{P}^1}(2-a) \\ &\simeq (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(2a)) \otimes \mathcal{O}_{\mathbb{P}^1}(2-a) \\ &\simeq \mathcal{O}_{\mathbb{P}^1}(2-a) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(a+2). \end{aligned}$$

Since $h^0(S, \mathcal{O}_S(K_S)) = h^0(\mathbb{P}^1, p_*\mathcal{O}_S(-K_S))$, the claim follows immediately. □

Lemma 2.22. *For $a \geq 2$, let S denote the Hirzebruch surface $\mathbb{F}_a$ and γ (resp. F) its negative section (resp. the fiber of the ruling $\mathbb{F}_a \to \mathbb{P}^1$). Then*

- *for $a \geq 3$, the linear system $|-K_S - \gamma|$ is very ample and $\mathrm{Bs}|-K_S| = \gamma$;*
- *for $a = 2$, the linear system $|-K_S|$ is semiample and $(-K_S, D) = 0$ for an irreducible curve D on S if and only if $D = \gamma$.*

Proof. We have $-K_S \sim 2\gamma + (a+2)F$.

(1) First suppose that $a \geq 3$. Since $-\gamma^2 = a < a+2$, by [11, p. 379, Thm. 2.17], we see that $-K_S - \gamma \sim \gamma + (a+2)F$ is very ample. Furthermore, since $(-K_S, \gamma) = 2(\gamma, \gamma) + (a+2)(F, \gamma) = 2 - a < 0$, we have $\mathrm{Bs}|-K_S| = \gamma$.

(2) Next, suppose that $a = 2$. Then $-K_S \sim 2\gamma + 4F$ is nef and $(-K_S, D) = 0$ for an irreducible curve D on S if and only if $D = \gamma$. Since $-K_S$ is big, $-2K_S = -K_S - K_S$ is nef and big. Hence $|-K_S|$ is semiample by the base-point-free theorem. □

Remark 2.23. If $a \geq 3$, then the linear equivalence relation

$$-K_{\mathbb{F}_a/\mathbb{P}^1} \sim (\gamma + aF) + \gamma$$

gives the Zariski decomposition (cf. [20]) of $-K_{\mathbb{F}_a/\mathbb{P}^1}$, where $\gamma + aF$ (resp. γ) is the positive (resp. negative) part of $-K_{\mathbb{F}_a/\mathbb{P}^1}$.

Proof of Theorem 2.13. Since g_n is étale, we see that $g_n^*\Theta_{Y_{n+1}/C} \simeq \Theta_{Y_n/C}$ for any n. Hence, applying Lemma 2.19 to $\mathcal{E}_n = \Theta_{Y_n/C}$ and combining Lemmas 2.17 and 2.20, we see immediately that any fiber of $\psi_n : Y_n \to C$ is isomorphic to $\mathbb{F}_a$ ($a > 0$). Thus $\psi_n : Y_n \to C$ is an $\mathbb{F}_a$-bundle for any n.

Lemma 2.18 shows that $a \geq 2$. First, suppose that $a \geq 3$. If we set

$$\mathcal{G}_n := \mathrm{Coker}(\psi_n^*(\psi_n)_*(-K_{Y_n/C}) \to -K_{Y_n/C}),$$

then $g_n^* \mathcal{G}_{n+1} = \mathcal{G}_n$ for any n. Furthermore, if we set $\Delta_n := \operatorname{Supp} \mathcal{G}_n$ for each n, then Lemma 2.22 shows that Δ_n is an effective divisor on Y_n and the restriction $\Delta_n|_{\psi_n^{-1}(t)}$ is the negative section of $\mathbb{F}_a$ for any $t \in C$. Hence Δ_n is a section of $\varphi_n \colon Y_n \to S_n$ for any n. Since $\Delta_n = g_n^{-1}(\Delta_{n+1})$ for any n, $\Delta_\bullet := (g_n|_{\Delta_n} \colon \Delta_n \to \Delta_{n+1})_n$ is a sub-ESP of $Y_\bullet$.

Next we consider the case where $a = 2$. Since $-K_{Y_n/C}$ is ψ_n-nef and $-K_{Y_n/C} - K_{Y_n/C} = -2K_{Y_n/C}$ is ψ_n-nef and ψ_n-big, $|-K_{Y_n/C}|$ is ψ_n-semiample by the base-point-free theorem. Let $\Phi_n := \Phi_{|-K_{Y_n/C}|} \colon Y_n \to Z_n$ be the relative birational morphism over C and $\Delta_n := \operatorname{Exc}(\Phi_n)$ be the Φ_n-exceptional divisor. Then Lemma 2.22 shows that the restriction $\Delta_n|_{\psi_n^{-1}(t)}$ is the negative section of $\mathbb{F}_2$ for any $t \in \mathbb{P}^1$. Hence Δ_n is a section of $\varphi_n \colon Y_n \to S_n$ for any n. Since $-K_{Y_n/C} \sim g_n^*(-K_{Y_{n+1}/C})$ for any n, we have $\Delta_n = g_n^{-1}(\Delta_{n+1})$. Hence $\Delta_\bullet := (g_n|_{\Delta_n} \colon \Delta_n \to \Delta_{n+1})_n$ is a sub-ESP of $Y_\bullet$. □

From now till the end of this section, we follow the notation of Theorem 2.13. Let $\mathcal{E}_n \twoheadrightarrow \ell_n$ be a surjective homomorphism from $\mathcal{E}_n$ to an invertible sheaf ℓ_n on S_n which corresponds to the section Δ_n of $\varphi_n \colon Y_n \to S_n$ as in Theorem 2.13. If we replace $\mathcal{E}_n$ by $\mathcal{E}_n \otimes \ell_n^{\otimes -1}$, then the isomorphism class of $Y_n := \mathbb{P}_{S_n}(\mathcal{E}_n)$ is invariant. Hence we may assume from the beginning that $\ell_n \simeq \mathcal{O}_{S_n}$. Thus there exists the following exact sequence of locally free sheaves:

$$0 \longrightarrow \mathcal{L}_n \longrightarrow \mathcal{E}_n \longrightarrow \mathcal{O}_{S_n} \longrightarrow 0,$$

such that for any n,

- $Y_n \simeq \mathbb{P}_{S_n}(\mathcal{E}_n)$, and
- $\mathcal{L}_n$ is a line bundle on S_n which is of degree a (> 1) when restricted to a fiber of $\alpha_n \colon S_n \to C$.

The following lemma is derived from a simple fact concerning surjective morphisms between Hirzebruch surfaces.

Lemma 2.24 (Cf. the notation in Theorem 2.13). *The following properties hold:*

(1) *Let Δ_n be the section of φ_n corresponding to a surjection $\mathcal{E}_n \twoheadrightarrow \mathcal{O}_{S_n}$. Then $g_n^{-1}(\Delta_{n+1}) = \Delta_n$ for all n.*

(2) *$\mathcal{E}_n \simeq u_n^* \mathcal{E}_{n+1}$ and $\mathcal{L}_n \simeq u_n^* \mathcal{L}_{n+1}$ for all n.*

Proof. For each n, let $\psi_n := \alpha_n \circ \varphi_n \colon Y_n \to C$ be a composite map. Then for any $t \in C$, the fiber $Y_{n,t} := \psi_n^{-1}(t)$ is isomorphic to a Hirzebruch surface $\mathbb{F}_a$ with $N_{n,t} := \Delta_n \cap Y_{n,t}$ as its negative section. Since g_n is finite étale, the restriction of g_n to each fiber of ψ_n gives an isomorphism $g_n(t) := g_n|_{Y_{n,t}} \colon Y_{n,t} \to Y_{n+1,h(t)}$. Since

$g_n(t)^{-1}(N_{n+1,h(t)})$ is a negative curve of $Y_{n,t}$, it coincides with the negative section of $\mathbb{F}_a$. Hence $g_n^{-1}(\Delta_{n+1}) = \Delta_n$ for all n. Thus assertion (1) has been proved.

Since $Y_n \simeq Y_{n+1} \times_{S_{n+1}} S_n \simeq \mathbb{P}(u_n^*\mathcal{E}_{n+1})$, there exists an isomorphism $\mathcal{E}_n \simeq u_n^*\mathcal{E}_{n+1} \otimes \varphi_n^*\mathcal{W}_n$ for some $\mathcal{W}_n \in \mathrm{Pic}(S_n)$. By construction, $\mathcal{O}_{Y_n}(1)|_{\Delta_n} \simeq \mathcal{O}_{\Delta_n}$ for any n. Since $\mathcal{O}_{Y_n}(1) \simeq g_n^*\mathcal{O}_{Y_{n+1}}(1) \otimes \varphi_n^*\mathcal{W}_n$, we have

$$\mathcal{O}_{Y_n}(1)|_{\Delta_n} \simeq g_n^*\mathcal{O}_{Y_{n+1}}(1)|_{\Delta_{n+1}} \otimes \mathcal{W}_n.$$

Hence $\mathcal{W}_n \simeq \mathcal{O}_{S_n}$. Thus $\mathcal{E}_n \simeq u_n^*\mathcal{E}_{n+1}$ for any n. Assertion (2) is derived from the following commutative diagram, where the second and the third vertical maps are isomorphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & u_n^*\mathcal{L}_{n+1} & \longrightarrow & u_n^*\mathcal{E}_{n+1} & \longrightarrow & \mathcal{O}_{S_n} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{L}_n & \longrightarrow & \mathcal{E}_n & \longrightarrow & \mathcal{O}_{S_n} & \longrightarrow & 0. \end{array}$$

□

Definition 2.25. We call the exact sequence of vector bundles on S_n

$$(\blacklozenge)_n\colon 0 \longrightarrow \mathcal{L}_n \longrightarrow \mathcal{E}_n \longrightarrow \mathcal{O}_{S_n} \longrightarrow 0,$$

satisfying the properties in Lemma 2.24, the fundamental exact sequence of vector bundles (FES for short) associated to the FESP $Y_\bullet$ of type $(\mathrm{P}_{-\infty})$. We abbreviate it as

$$(\blacklozenge)_\bullet\colon 0 \longrightarrow \mathcal{L}_\bullet \longrightarrow \mathcal{E}_\bullet \longrightarrow \mathcal{O}_{S_\bullet} \longrightarrow 0.$$

Remark 2.26 (Cf. [20, p. 69, 5.18, Example]). In general, a jumping phenomenon of ruled surfaces can occur, s the next example shows. On$\mathbb{P}^1$, there exists the following exact sequence of vector bundles:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0,$$

with group of extensions $\mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-2)) \simeq \mathbb{C}$. If the extension class is nontrivial, then $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Hence there exists a family of ruled surfaces $\pi\colon X \to \mathbb{P}^1 \times \mathbb{C} \to \mathbb{C}$ such that

- $X_t := \pi^{-1}(t)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ for $t \neq 0$;
- $X_0 \simeq \mathbb{F}_2$.

In Theorem 2.13, if we regard X as a fiber space over an elliptic curve C, then such a jumping phenomenon of ruled surfaces cannot occur because of the existence of the nonisomorphic étale endomorphism $f\colon X \to X$ of X.

§3. Elementary transformations

For a positive integer a, we consider the following exact sequence of locally free sheaves on the Atiyah surface $\mathbb{S}$:

$$(\spadesuit)\colon 0 \longrightarrow \mathcal{O}_{\mathbb{S}}(a\,s_\infty) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{S}} \longrightarrow 0.$$

Restricting the above exact sequence to the canonical section s_∞ ($\subset \mathbb{S}$), we obtain another exact sequence of sheaves:

$$(\diamondsuit)\colon 0 \longrightarrow \mathcal{O}_{s_\infty} \longrightarrow \mathcal{E}|_{s_\infty} \longrightarrow \mathcal{O}_{s_\infty} \longrightarrow 0.$$

Let D be a section of $\varphi\colon Y := \mathbb{P}_{\mathbb{S}}(\mathcal{E}) \to \mathbb{S}$ corresponding to the surjection $\mathcal{E} \twoheadrightarrow \mathcal{O}_{\mathbb{S}}$, $T := \varphi^{-1}(s_\infty)$ an elliptic ruled surface over s_∞ and $\Gamma := T \cap D$ the complete intersection curve. By construction, we infer that $T \simeq \mathbb{P}_{s_\infty}(\mathcal{E}|_\infty)$. The following lemma is useful to know the structure of T.

Lemma 3.1. *The extension class of* $(\spadesuit)$ *can be described as follows:*

(1) $\dim \operatorname{Ext}^1(\mathcal{O}_{\mathbb{S}}, \mathcal{O}_{\mathbb{S}}(a\,s_\infty)) = 1$.

(2) *The exact sequence* $(\spadesuit)$ *splits if and only if the exact sequence* $(\diamondsuit)$ *splits.*

Proof. Since $\mathcal{N}_{s_\infty/\mathbb{S}} \simeq \mathcal{O}_{s_\infty}$, for all $k > 0$, there exists the following exact sequence of sheaves:

$$0 \longrightarrow \mathcal{O}_{\mathbb{S}}((k-1)s_\infty) \longrightarrow \mathcal{O}_{\mathbb{S}}(k\,s_\infty) \longrightarrow \mathcal{O}_{s_\infty} \longrightarrow 0.$$

Then we obtain the following exact sequence of cohomologies:

$$\begin{aligned} 0 &\longrightarrow H^0(\mathbb{S}, \mathcal{O}_{\mathbb{S}}((k-1)s_\infty)) \longrightarrow H^0(\mathbb{S}, \mathcal{O}_{\mathbb{S}}(k\,s_\infty)) \longrightarrow H^0(s_\infty, \mathcal{O}_{s_\infty}) \\ &\longrightarrow H^1(\mathbb{S}, \mathcal{O}_{\mathbb{S}}((k-1)s_\infty)) \longrightarrow H^1(\mathbb{S}, \mathcal{O}_{\mathbb{S}}(k\,s_\infty)) \longrightarrow H^1(s_\infty, \mathcal{O}_\infty) \\ &\longrightarrow H^2(\mathbb{S}, \mathcal{O}_{\mathbb{S}}((k-1)s_\infty)). \end{aligned}$$

By Proposition 2.4, we see that $h^0(\mathbb{S}, \mathcal{O}_S(k\,s_\infty)) = 1$ for all $k > 0$. Thus there exists the following exact sequence:

$$\begin{aligned} 0 &\longrightarrow H^0(\mathbb{S}, \mathcal{O}_{s_\infty}) \longrightarrow H^1(\mathbb{S}, \mathcal{O}_{\mathbb{S}}((k-1)s_\infty)) \longrightarrow H^1(\mathbb{S}, \mathcal{O}_{\mathbb{S}}(k\,s_\infty)) \\ &\longrightarrow H^1(s_\infty, \mathcal{O}_{s_\infty}) \longrightarrow 0. \end{aligned}$$

Noting that $h^0(\mathbb{S}, \mathcal{O}_{\mathbb{S}}(k\,s_\infty)) = 1$ for all $k > 0$ by Proposition 2.4, the following exact sequence is derived:

$$\begin{aligned} 0 &\longrightarrow H^0(s_\infty, \mathcal{O}_{s_\infty}) \longrightarrow H^1(\mathbb{S}, \mathcal{O}_{\mathbb{S}}((k-1)s_\infty)) \longrightarrow H^1(\mathbb{S}, \mathcal{O}_{\mathbb{S}}(k\,s_\infty)) \\ &\xrightarrow{\text{res}} H^1(s_\infty, \mathcal{O}_{s_\infty}) \longrightarrow 0. \end{aligned}$$

Since $s_\infty^2 = 0$ and $K_{\mathbb{S}} \sim -2s_\infty$, we infer that $h^1(\mathbb{S}, \mathcal{O}_{\mathbb{S}}(k s_\infty)) = 1$ for all $k > 0$ by the Riemann–Roch formula. Hence the restriction map res is an isomorphism. Since $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{S}}, \mathcal{O}_{\mathbb{S}}(a s_\infty)) \simeq H^1(\mathbb{S}, \mathcal{O}_{\mathbb{S}}(a s_\infty))$, assertion (1) is derived and the restriction map res: $H^1(\mathbb{S}, \mathcal{O}_{\mathbb{S}}(k s_\infty)) \to H^1(s_\infty, \mathcal{O}_\infty)$ is an isomorphism. There exists the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Ext}^1(\mathcal{O}_{\mathbb{S}}, \mathcal{O}_{\mathbb{S}}(a s_\infty)) & \xrightarrow{\text{res}} & \mathrm{Ext}^1(\mathcal{O}_{s_\infty}, \mathcal{O}_{s_\infty}) \\ \downarrow & & \downarrow \\ H^1(\mathbb{S}, \mathcal{O}_{\mathbb{S}}(a s_\infty)) & \xrightarrow{\text{res}} & H^1(s_\infty, \mathcal{O}_\infty). \end{array}$$

Since the two vertical maps are both isomorphisms, the horizontal maps are both isomorphisms and assertion (2) is thus derived. □

Now we recall the theory of *elementary transformations* of vector bundles due to Maruyama [17]. Let S be a locally Noetherian scheme and T a subscheme of S whose defining ideal I_T is a Cartier divisor on S. For a vector bundle $\mathcal{E}$ of rank $N+1$ on S, assume that there is a surjective homomorphism $\delta\colon \mathcal{E} \twoheadrightarrow \mathcal{F}$ to a vector bundle $\mathcal{F}$ on T with $\mathrm{rank}(\mathcal{F}) < \mathrm{rank}(\mathcal{E})$. We set $X := \mathbb{P}(\mathcal{E})$ and $Y := \mathbb{P}(\mathcal{F})$. Then Y is a projective subbundle of $\mathbb{P}(\mathcal{E})|_T$ and a subscheme of $\mathbb{P}(\mathcal{E})$. Let $\pi\colon \widetilde{X} := \mathrm{Bl}_Y X \to X$ be the blow-up of X along Y. Set $D := \pi^{-1}(X_T)$, which is the proper transform of $X_T := \mathbb{P}(\mathcal{E}|_T)$ and let $G := \mathrm{Exc}(\pi)$ be the π-exceptional divisor. Our situation can be displayed in the following exact and commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \uparrow & & \uparrow & & & & \\ 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{E}|_T & \xrightarrow{\mu} & \mathcal{F} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{E} & \xrightarrow{\delta} & \mathcal{F} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & & & \\ & & \mathcal{E}(-T) & \xrightarrow{\mathrm{id}} & \mathcal{E}(-T) & & & & \\ & & \uparrow & & \uparrow & & & & \\ & & 0 & & 0. & & & & \end{array} \tag{3.1}$$

The following fundamental theorem is due to Maruyama [17, Thms 1.1 and 1.3].

Theorem 3.2. *Under the assumption above, $\mathcal{E}' := \mathrm{Ker}(\delta)$ is a vector bundle. Let $p' \colon X' := \mathbb{P}(\mathcal{E}') \to S$ be the projective bundle associated with a vector bundle $\mathcal{E}'$ on S. Then there exists a birational S-morphism $g \colon \widetilde{X} \to X'$ such that the following conditions are satisfied:*

(1) *$\mathcal{E}' = p_*(\mathcal{I}_Y \otimes \mathcal{O}_X(1))$, where $p \colon X = \mathbb{P}(\mathcal{E}) \to S$ is the projection, $\mathcal{I}_Y$ is the defining ideal of Y in X and $\mathcal{O}_X(1)$ is the tautological line bundle on X.*
(2) *The closed subscheme Y' defined by $g_*(\mathcal{I}_D)$ ($\subset g_*(\mathcal{O}_{\widetilde{X}}) \simeq \mathcal{O}_{X'}$) is a projective subbundle $\mathbb{P}(\mathcal{F}')$ of $\mathbb{P}(\mathcal{E}')|_T$, where $\mathcal{I}_D$ is the defining ideal of D in X.*
(3) *g is the blow-up of X' along Y'.*
(4) *$g^*(\mathcal{O}_{X'}(1)) \simeq \pi^*(\mathcal{O}_X(1)) \otimes \mathcal{O}_{\widetilde{X}}(-G)$, where $\mathcal{O}_{X'}(1)$ is the tautological line bundle of $\mathcal{E}'$.*

Definition 3.3. The birational map $g\pi^{-1} \colon X \cdots \to X'$ over S is called the elementary transformation along Y and denoted by $X' = \mathrm{elm}_Y(X)$.

Now we shall apply "elementary transformations" to two cases depending on whether the short exact sequence (♠) splits or not.

Definition 3.4. We say that the short exact sequence (♠) defined at the beginning of Section 3 is of *type* (A) (resp. of *type* (B)) if (♠) splits (resp. unsplits).

Case 1. First, we consider the case where the exact sequence (♠) is of type (A) (cf. Definition 3.4). Then Lemma 3.1 shows that the extension class $0 \neq \eta \in \mathrm{Ext}^1(\mathcal{O}_{\mathbb{S}}, \mathcal{O}_{\mathbb{S}})$ is uniquely determined up to a constant.

Lemma 3.5. *Let $Y := \mathbb{P}_{\mathbb{S}}(\mathcal{E})$ be the $\mathbb{P}^1$-bundle over $\mathbb{S}$ associated to the vector bundle $\mathcal{E}$ of rank two on $\mathbb{S}$. Then Y admits a nonisomorphic étale endomorphism.*

Proof. Let $\mu_k \colon C \to C$ be a multiplication mapping by an integer $k > 1$. Then [5, Prop. 4.13] shows the existence of a nonisomorphic étale endomorphism $\varphi \colon \mathbb{S} \to \mathbb{S}$ such that $\alpha_{\mathbb{S}} \circ \varphi = \mu_k \circ \alpha_{\mathbb{S}}$ for the Albanese map $\alpha_{\mathbb{S}} \colon \mathbb{S} \to C$. Pulling back (♠) by φ, we obtain the following exact sequence:

$$(\spadesuit') \colon 0 \longrightarrow \varphi^*\mathcal{O}_{\mathbb{S}}(a s_\infty) \longrightarrow \varphi^*\mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{S}} \longrightarrow 0.$$

Thanks to the projection formula and the finiteness of φ, we infer that

$$\begin{aligned} \mathrm{Ext}^1(\mathcal{O}_{\mathbb{S}}, \varphi^*\mathcal{O}_{\mathbb{S}}(a s_\infty)) \simeq H^1(\mathbb{S}, \varphi^*\mathcal{O}_{\mathbb{S}}(a s_\infty)) &\simeq H^1(\mathbb{S}, \varphi_*\varphi^*\mathcal{O}_{\mathbb{S}}(a s_\infty)) \\ &\simeq H^1(\mathbb{S}, \mathcal{O}_{\mathbb{S}}(a s_\infty) \otimes \varphi_*\mathcal{O}_{\mathbb{S}}). \end{aligned}$$

On the other hand, the trace map $\mathrm{Tr}_{\mathbb{S}/\mathbb{S}} \colon \varphi_*\mathcal{O}_{\mathbb{S}} \to \mathcal{O}_{\mathbb{S}}$ gives rise to a splitting of the natural inclusion $\mathcal{O}_{\mathbb{S}} \hookrightarrow \varphi_*\mathcal{O}_{\mathbb{S}}$. Hence $\mathcal{O}_{\mathbb{S}}(a s_\infty) \to \mathcal{O}_{\mathbb{S}}(a s_\infty) \otimes \varphi_*\mathcal{O}_{\mathbb{S}}$ likewise splits.

Thus $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{S}}, \mathcal{O}_{\mathbb{S}}(a\,s_\infty))$ embeds as a direct summand of $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{S}}, \varphi^*\mathcal{O}_{\mathbb{S}}(a\,s_\infty))$. Hence the natural homomorphism $\varphi^*\colon \mathrm{Ext}^1(\mathcal{O}_{\mathbb{S}}, \mathcal{O}_{\mathbb{S}}(a\,s_\infty)) \to \mathrm{Ext}^1(\mathcal{O}_{\mathbb{S}}, \varphi^*\mathcal{O}_{\mathbb{S}}(a\,s_\infty))$ is injective. On the other hand, Proposition 2.5 shows that $\varphi^*\mathcal{O}_{\mathbb{S}}(a\,s_\infty) \simeq \mathcal{O}_{\mathbb{S}}(a\,s_\infty)$. Hence, by Lemma 3.1, φ^* is an isomorphism. Thus $\varphi^*\mathcal{E} \simeq \mathcal{E}$. If we set $\widetilde{Y} := Y \times_{\varphi,\mathbb{S}} \mathbb{S}$, then there exists an isomorphism $\widetilde{Y} \simeq \mathbb{P}_{\mathbb{S}}(\varphi^*\mathcal{E}) \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{E}) =: Y$. Thus the natural projection $\widetilde{Y} \to Y$ gives a nonisomorphic étale endomorphism $g\colon Y \to Y$ of Y. $\square$

Now we shall apply Theorem 3.2 to the $\mathbb{P}^1$-bundle $Y := \mathbb{P}_{\mathbb{S}}(\mathcal{E})$ over the Atiyah surface $\mathbb{S}$. We recall the following unsplit exact sequence of vector bundles on $\mathbb{S}$:

$$(\spadesuit) : 0 \longrightarrow \mathcal{O}_{\mathbb{S}}(a\,s_\infty) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{S}} \longrightarrow 0.$$

Let D be a section of $\varphi\colon Y := \mathbb{P}_{\mathbb{S}}(\mathcal{E}) \to \mathbb{S}$ corresponding to the surjection $\mathcal{E} \twoheadrightarrow \mathcal{O}_{\mathbb{S}}$. Then Lemma 3.1 shows that $T := \varphi^{-1}(s_\infty) \simeq \mathbb{S}$ and $\Gamma := T \cap D$ is the canonical section of D ($\simeq \mathbb{S}$). Let $\delta\colon \mathcal{E} \twoheadrightarrow \mathcal{O}_\Gamma$ be the composite of $\mathcal{E} \twoheadrightarrow \mathcal{O}_{\mathbb{S}}$ and the canonical homomorphism $\mathcal{O}_{\mathbb{S}} \twoheadrightarrow \mathcal{O}_\Gamma$. We set $\mathcal{E}' := \mathrm{Ker}(\delta)$. Then $\mathcal{E}'$ is a vector bundle of rank two on $\mathbb{S}$ and we have the following exact and commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{\mathbb{S}}(a\,s_\infty) & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{O}_{\mathbb{S}} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{S}}(a\,s_\infty) & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{O}_{\mathbb{S}}(-s_\infty) & \longrightarrow & 0.
\end{array}$$

We set $Y' := \mathbb{P}_{\mathbb{S}}(\mathcal{E}')$. Then by Theorem 3.2, we see that Y' is obtained from Y by performing elementary transformations along Γ, i.e., $Y' = \mathrm{elm}_\Gamma(Y)$. By construction, there exists the following exact sequence of vector bundles on $\mathbb{S}$:

$$(\spadesuit\spadesuit)\colon 0 \longrightarrow \mathcal{O}_{\mathbb{S}}((a+1)s_\infty) \longrightarrow \mathcal{E}' \otimes \mathcal{O}_{\mathbb{S}}(s_\infty) \longrightarrow \mathcal{O}_{\mathbb{S}} \longrightarrow 0,$$

such that $Y' \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{E}' \otimes \mathcal{O}_{\mathbb{S}}(s_\infty))$.

Lemma 3.6. *The above exact sequence* $(\spadesuit\spadesuit)$ *splits; in particular,*

$$Y' \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{O}_{\mathbb{S}}((a+1)s_\infty) \oplus \mathcal{O}_{\mathbb{S}}).$$

Proof. We shall give two proofs.

The first proof. We use the same notation as explained just after Definition 3.3. Let $\pi\colon \widetilde{Y} := \mathrm{Bl}_\Gamma(Y) \to Y$ be the blow-up of Y along $\Gamma := T \cap D$ ($\simeq C$). Since Γ is the intersection of two effective Cartier divisors, the normal bundle of Γ in Y is trivial. Hence the π-exceptional divisor E is isomorphic to $C \times \mathbb{P}^1$. Let $T' \simeq \mathbb{S}$ be the strict transform of T in $\widetilde{X}$. Then Y' is obtained from $\widetilde{Y}$ by blow-down of T' to an elliptic curve. If we set $\varphi'\colon Y' \to \mathbb{S}$, then the image of E in Y' is the elliptic ruled surface $\varphi'^{-1}(s_\infty)$ over s_∞ ($\subset \mathbb{S}$) and is isomorphic to $C \times \mathbb{P}^1$. Then Lemma 3.1(2) shows that $(\spadesuit\spadesuit)$ splits.

The second proof. By the same argument as in the proof of Lemma 3.1, there exists the following exact sequence of cohomologies:

$$\begin{aligned}0 &\longrightarrow H^0(\mathbb{S}, \mathcal{O}_{\mathbb{S}}(a\,s_\infty)) \longrightarrow H^0(\mathbb{S}, \mathcal{O}_{\mathbb{S}}((a+1)s_\infty)) \longrightarrow H^0(s_\infty, \mathcal{O}_{s_\infty})\\ &\xrightarrow{c} H^1(\mathbb{S}, \mathcal{O}_{\mathbb{S}}(a\,s_\infty)) \xrightarrow{i} H^1(\mathbb{S}, \mathcal{O}_{\mathbb{S}}((a+1)s_\infty)) \longrightarrow H^1(s_\infty, \mathcal{O}_{s_\infty})\\ &\longrightarrow H^2(\mathbb{S}, \mathcal{O}_{\mathbb{S}}(a\,s_\infty)) = 0.\end{aligned}$$

Since $h^0(\mathbb{S}, \mathcal{O}_{\mathbb{S}}(a\,s_\infty)) = h^1(\mathbb{S}, \mathcal{O}_{\mathbb{S}}(a\,s_\infty)) = 1$ for any $a > 0$ by Lemma 3.1, c is an isomorphism and i is a zero map. Since $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{S}}, \mathcal{O}_{\mathbb{S}}(a\,s_\infty)) \simeq H^1(\mathbb{S}, \mathcal{O}_{\mathbb{S}}(a\,s_\infty))$, $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{S}}, \mathcal{O}_{\mathbb{S}}(a\,s_\infty)) \to \mathrm{Ext}^1(\mathcal{O}_{\mathbb{S}}, \mathcal{O}_{\mathbb{S}}((a+1)s_\infty))$ is a zero map. Thus the exact sequence (♠♠) splits. □

With the same notation as above, we have the following lemma.

Lemma 3.7. $\dim|D + aT| = 0$, $\dim|D + (a+1)T| = 1$.

Proof. Since $D|_D \sim -a\gamma$ and $T|_D = \gamma$, we have $D + aT|_D \sim \mathcal{O}_D$. Hence there exists the following exact sequence of vector bundles on Y:

$$0 \longrightarrow \mathcal{O}_Y(aT) \longrightarrow \mathcal{O}_Y(D + aT) \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

Then we have the following exact sequence:

$$0 \longrightarrow H^0(Y, \mathcal{O}_Y(aT)) \longrightarrow H^0(Y, \mathcal{O}_Y(D + aT)) \longrightarrow H^0(D, \mathcal{O}_D) \longrightarrow \cdots.$$

Since $H^0(Y, \mathcal{O}_Y(aT)) \simeq H^0(\mathbb{S}, \mathcal{O}_{\mathbb{S}}(a\,s_\infty)) \simeq \mathbb{C}$ and $H^0(D, \mathcal{O}_D) \simeq \mathbb{C}$, we have $1 \leq h^0(Y, \mathcal{O}_Y(D + aT)) \leq 2$. Suppose that $h^0(Y, \mathcal{O}_Y(D + aT)) = 2$. Then there exists an effective divisor $\Delta \in |D + aT|$ which is a section of $Y \to \mathbb{S}$ and is disjoint from D. This contradicts the assumption that (♠) is an unsplit exact sequence. Hence $h^0(Y, \mathcal{O}_Y(D + aT)) = 1$.

We have $\mathcal{O}_Y(1) \sim D + aT$. There exists the following exact sequence of sheaves on $\mathbb{S}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{O}_{\mathbb{S}}((a+1)s_\infty) & \longrightarrow & \mathcal{E} \otimes \mathcal{O}_{\mathbb{S}}(s_\infty) & \longrightarrow & \mathcal{O}_{\mathbb{S}}(s_\infty) & \longrightarrow & 0 \\ & & \uparrow \mathrm{id} & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathrm{O}_{\mathbb{S}}((a+1)s_\infty) & \longrightarrow & \mathcal{E}' \otimes \mathcal{O}_{\mathbb{S}}(s_\infty) & \longrightarrow & \mathcal{O}_{\mathbb{S}} & \longrightarrow & 0. \end{array}$$

where the exact sequence on the top row is obtained from (♠) by tensoring with $\mathcal{O}_{\mathbb{S}}(s_\infty)$. Since $h^0(\mathbb{S}, \mathrm{O}_{\mathbb{S}}((a+1)s_\infty)) = 1$, we have $h^0(\mathbb{S}, \mathcal{E} \otimes \mathcal{O}_{\mathbb{S}}(s_\infty)) \leq 2$ from the exact sequence on the top row. Furthermore, since the exact sequence on the second row splits by Lemma 3.6, we have $h^0(\mathbb{S}, \mathcal{E}') = 2$. Then we have

$h^0(\mathbb{S}, \mathcal{E} \otimes \mathcal{O}_{\mathbb{S}}(s_\infty)) \geq h^0(\mathbb{S}, \mathcal{E}') = 2$. Thus we have $h^0(\mathbb{S}, \mathcal{E} \otimes \mathcal{O}_{\mathbb{S}}(s_\infty)) = 2$. Since $D + (a+1)T \sim \mathcal{O}_Y(1) \otimes \varphi^*\mathcal{O}_{\mathbb{S}}(s_\infty)$, the projection formula shows that

$$h^0(Y, \mathcal{O}(D+(a+1)T)) = h^0(\mathbb{S}, \varphi_*\mathcal{O}_Y(1) \otimes \mathcal{O}_{\mathbb{S}}(s_\infty)) = h^0(\mathbb{S}, \mathcal{E} \otimes \mathcal{O}_{\mathbb{S}}(s_\infty)) = 2. \quad \square$$

Lemma 3.8. *Let $f\colon X \to X$ be a nonisomorphic étale endomorphism. Let L be the complete linear system on X defined by $L := |D + (a+1)T|$. Then for any member $\Lambda \in L$, we have $f^*\Lambda \in L$, i.e., L is preserved by f.*

Proof. Since $f^*D \sim D$ and $f^*T \sim T$, we have $f^*\Lambda \in L$ for any $\Lambda \in L$. □

Case 2. Next we consider the case where the exact sequence (♠) is of type (B) (cf. Definition 3.4), that is, $Y \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{O}_{\mathbb{S}}(a\,s_\infty) \oplus \mathcal{O}_{\mathbb{S}})$.

Lemma 3.9. *The variety Y admits a nonisomorphic étale endomorphism.*

Proof. Let $\mu_k\colon C \to C$ be a multiplication mapping by an integer $k > 1$. Then [5, Prop. 4.13] shows the existence of a nonisomorphic étale endomorphism $\nu\colon \mathbb{S} \to \mathbb{S}$ such that $\alpha_{\mathbb{S}} \circ \nu = \mu_k \circ \alpha_{\mathbb{S}}$ for the Albanese map $\alpha_{\mathbb{S}}\colon \mathbb{S} \to C$. Then applying Proposition 2.5 to $\mathcal{E} := \mathcal{O}_{\mathbb{S}}(a\,s_\infty) \oplus \mathcal{O}_{\mathbb{S}}$, we see that $\nu^*\mathcal{E} \simeq \mathcal{E}$. Hence there exists an isomorphism $\widetilde{Y} := Y \times_{\nu,\mathbb{S}} \mathbb{S} \simeq \mathbb{P}_{\mathbb{S}}(\nu^*\mathcal{E}) \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{E}) =: Y$ and the first projection $\widetilde{Y} \to Y$ gives a nonisomorphic étale endomorphism of Y. □

For $\mathcal{E} := \mathcal{O}_{\mathbb{S}}(a\,s_\infty) \oplus \mathcal{O}_{\mathbb{S}}$, let D_∞ (resp. D_0) be the section of $\varphi\colon Y \to \mathbb{S}$ corresponding to a surjection $\mathcal{E} \twoheadrightarrow \mathcal{O}_{\mathbb{S}}$ (resp. $\mathcal{E} \twoheadrightarrow \mathcal{O}_{\mathbb{S}}(a\,s_\infty)$). If we set $T := \varphi^{-1}(s_\infty)$, then $T \simeq C \times \mathbb{P}^1$. We set $\Gamma_\infty := T \cap D_\infty$, $\Gamma_0 := T \cap D_0$. Let $q\colon T \to \mathbb{P}^1$ be the second projection such that $\Gamma_\infty = q^{-1}(\infty)$ and $\Gamma_0 = q^{-1}(0)$. We set $\Gamma_t := q^{-1}(t)$ for $t \in \mathbb{P}^1$. The following lemma will play a key role in the proof of Theorem 4.12.

Lemma 3.10. *If $t \neq 0, \infty$, then the normal bundle $N_{\Gamma_t/Y}$ of Γ_t in Y is isomorphic to $\mathcal{F}_2$.*

Proof. First we consider the case where $a = 1$. For a 3-fold $Y_1 := \mathbb{P}_{\mathbb{S}}(\mathcal{O}_{\mathbb{S}}(s_\infty) \oplus \mathcal{O}_{\mathbb{S}})$, the assertion follows immediately from [8, Lem. 6.20]. Here we assume that $a > 1$. Let $i\colon Y_1 := \mathbb{P}_{\mathbb{S}}(\mathcal{O}_{\mathbb{S}}(s_\infty) \oplus \mathcal{O}_{\mathbb{S}}) \hookrightarrow \mathbb{P}_{\mathbb{S}}(\mathrm{Sym}^a(\mathcal{O}_{\mathbb{S}}(s_\infty) \oplus \mathcal{O}_{\mathbb{S}}))$ be the a-th Veronese embedding. Then the a-th symmetric product $\mathrm{Sym}^a(\mathcal{O}_{\mathbb{S}}(s_\infty) \oplus \mathcal{O}_{\mathbb{S}})$ contains $\mathcal{E}$ as a direct summand. Let $p\colon \mathbb{P}_{\mathbb{S}}(\mathrm{Sym}^a(\mathcal{O}_{\mathbb{S}}(s_\infty) \oplus \mathcal{O}_{\mathbb{S}})) \cdots \to Y := \mathbb{P}_{\mathbb{S}}(\mathcal{E})$ be the induced projection. Then the composite $p \circ i\colon Y_1 \to Y$ is a finite covering of degree a ramified over Γ_0 and Γ_∞ and is unramified over $Y \setminus (\Gamma_0 \cup \Gamma_\infty)$. Thus the assertion follows immediately. □

With the same notation as above, we have the following lemma.

Lemma 3.11. *The dimension of the complete linear system $|D_0|$ is as follows:*

(1) $\dim_{\mathbb{C}}|D_\infty + (a-1)T| = 0$.

(2) $\dim_{\mathbb{C}}|D_\infty + aT| = 1$ *and* $D_0 \in |D_\infty + aT|$.

Proof. Since $D_\infty|_{D_\infty} \sim -a\Gamma_\infty$ and $T|_{D_\infty} = \Gamma_\infty$, we have $(D_\infty + (a-1)T)|_{D_\infty} \sim -\Gamma_\infty$. Hence D_∞ is contained in the base locus of $|D_\infty + (a-1)T|$. Hence $\dim|D_\infty + (a-1)T| = \dim|(a-1)T| = 0$. Since D_0 and D_∞ are disjoint sections of $\varphi\colon Y \to \mathbb{S}$, we have $D_0 - D_\infty \sim \varphi^*\Delta$ for some divisor Δ on $\mathbb{S}$. Since $(D_0 - D_\infty)|_{D_\infty} \sim a\Gamma_\infty$, we have $a s_\infty \sim \Delta$ on $\mathbb{S}$. Hence $D_0 \in |D_\infty + aT|$. Since $(D_\infty + aT)|_{D_\infty} \sim 0$, we have the following exact sequences of vector bundles:

$$0 \longrightarrow \mathcal{O}_Y(aT) \longrightarrow \mathcal{O}_Y(D_\infty + aT) \longrightarrow \mathcal{O}_{D_\infty} \longrightarrow 0.$$

Then, taking the long exact sequence of cohomologies, we have

$$h^0(Y, \mathcal{O}_Y(D_\infty + aT)) \leq h^0(Y, \mathcal{O}_Y(aT)) + h^0(D_\infty, \mathcal{O}_{D_\infty}) = 2.$$

Since $D_0 \in |D_\infty + aT|$, we have $\dim|D_\infty + aT| = 1$. □

Lemma 3.12. *Let Y' (resp. Y'') be a 3-fold obtained from Y by performing elementary transformations along an elliptic curve Γ_∞ (resp. Γ_0), i.e., $Y' := \mathrm{elm}_{\Gamma_\infty}(Y)$ (resp. $Y'' := \mathrm{elm}_{\Gamma_0}(Y)$). Then $Y' \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{O}_{\mathbb{S}}((a+1)s_\infty) \oplus \mathcal{O}_{\mathbb{S}})$ (resp. $Y'' \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{O}_{\mathbb{S}}((a-1)s_\infty) \oplus \mathcal{O}_{\mathbb{S}})$).*

Proof. Let $\pi\colon \widehat{Y} \to Y$ (resp. $\pi'\colon \widehat{Y}' \to Y$) be the blow-up of Y along Γ_∞ (resp. Γ_0). Since $N_{\Gamma_\infty/Y}$ (resp. $N_{\Gamma_0/Y}$) $\simeq \mathcal{O}_C \oplus \mathcal{O}_C$, the π (resp. π')-exceptional divisor E (resp. E') is isomorphic to $C \times \mathbb{P}^1$. Let $g\colon \widehat{Y} \to Y'$ (resp. $g'\colon \widehat{Y}' \to Y''$) be the blow-down of the proper transform $\overline{T}$ (resp. $\overline{T}'$) of T by π (resp. π'). Then the birational map $g \circ \pi^{-1}\colon Y \cdots\longrightarrow Y'$ (resp. $g' \circ {\pi'}^{-1}\colon Y \cdots\longrightarrow Y''$) coincides with the elementary transformation $\mathrm{elm}_{\Gamma_\infty}\colon Y \cdots\longrightarrow Y'$ (resp. $\mathrm{elm}_{\Gamma_0}\colon Y \cdots\longrightarrow Y''$) along Γ_∞ (resp. Γ_0). The proper transforms of D_∞ and D_0 on Y' (resp. Y'') are two disjoint sections of the $\mathbb{P}^1$-bundle $\varphi'\colon Y' \to \mathbb{S}$ (resp. $\varphi''\colon Y'' \to \mathbb{S}$) and Y' (resp. Y'') is an $\mathbb{F}_{a+1}$ (resp. $\mathbb{F}_{a-1}$)-bundle over C. Furthermore, ${\varphi'}^{-1}(s_\infty)$ (resp. ${\varphi''}^{-1}(s_\infty)$) is the proper transform E' (resp. E'') of E on Y' (resp. Y'') and is isomorphic to $C \times \mathbb{P}^1$. By Lemma 3.1, we see that the short exact sequence (♠) splits if and only if their restriction to s_∞ splits. Hence there exists an isomorphism $Y' \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{O}_{\mathbb{S}}((a+1)s_\infty) \oplus \mathcal{O}_{\mathbb{S}})$ (resp. $Y'' \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{O}_{\mathbb{S}}((a-1)s_\infty) \oplus \mathcal{O}_{\mathbb{S}})$). □

Lemma 3.13. *Let Z be a 3-fold obtained from Y by performing elementary transformations along an elliptic curve Γ_t ($t \neq 0, \infty$) as in Lemma 3.10, i.e., $Z = \mathrm{elm}_{\Gamma_t}(Y)$. Then Z is independent of the choice of t and there exists an isomorphism $Z \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{E})$, where $\mathcal{E}$ is a vector bundle of rank two on $\mathbb{S}$ satisfying the* *following unsplit exact sequence of sheaves:*

$$(*)\colon 0 \longrightarrow \mathcal{O}_{\mathbb{S}}((a-1)s_\infty) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{S}} \longrightarrow 0.$$

Proof. Let $\pi\colon \widehat{Y} \to Y$ be the blow-up of Y along Γ_t ($t \neq 0, \infty$). Since $N_{\Gamma_t/Y} \simeq \mathcal{F}_2$ by Lemma 3.10, the π-exceptional divisor E is isomorphic to $\mathbb{S}$. Let $g\colon \widehat{Y} \to Z$ be the blow-down of the proper transform $\overline{T}$ of T by π. Then $g \circ \pi^{-1}\colon Y \cdots \longrightarrow Z$ coincides with the elementary transformations $\mathrm{elm}_{\Gamma_t}\colon Y \cdots \longrightarrow Z$. We set $\varphi'\colon Z \to \mathbb{S}$. Then $\varphi'^{-1}(s_\infty)$ is the proper transform of E on Z and is isomorphic to $\mathbb{S}$. Furthermore, Z is an $\mathbb{F}_{a-1}$-bundle over C and the proper transform $\overline{D_\infty}$ of D_∞ on Z is a section of φ' which is the negative section of $\mathbb{F}_{a-1}$ when restricted to each fiber of $Z \to C$. Hence there exists an exact sequence $(*)$ of vector bundles on $\mathbb{S}$ such that

- $Z \simeq \mathbb{P}(\mathcal{E})$;
- the section $\overline{D_\infty}$ of φ' corresponds to a surjection $\mathcal{E} \twoheadrightarrow \mathcal{O}_{\mathbb{S}}$.

By construction, $\varphi'^{-1}(s_\infty)$ is the proper transform of E on Z and is isomorphic to $\mathbb{S}$. Hence Lemma 3.1 shows that the exact sequence $(*)$ does not split and Z is independent of t. □

Remark 3.14. Since $D_0 \sim D_\infty + aT$, we have $\pi^* D_0 \sim \pi^* D_\infty + a\overline{T} + aE$. Hence the push-forward g_* gives the relation $D'_0 \sim D'_\infty + aE'$, where D'_0 (resp. D'_∞ and E') is the proper transform of D_0 (resp. D_∞ and E) on Z. Lemmas 3.7 and 3.13 show that $\dim|D'_\infty + aE'| = 1$ and $D'_0 \in |D'_\infty + aE'|$.

§4. Atiyah case

Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Let

$$X_\bullet \xrightarrow{\pi_\bullet} Y_\bullet \xrightarrow{\varphi_\bullet} S_\bullet \xrightarrow{\alpha_\bullet} C_\bullet$$

be Cartesian morphisms of ESPs which satisfy Condition ($\mathrm{P}_{-\infty}$). In this section, we consider the case where the FESP $Y_\bullet$ is of *Atiyah type*, i.e., $S_\bullet \simeq \mathbb{S}_\bullet$, in other words, $S_n \simeq \mathbb{S}$ for any n (cf. Definition 1.3). Then using Proposition 4.3, we show that the FES associated to the FESP $Y_\bullet \xrightarrow{\varphi_\bullet} \mathbb{S}_\bullet$ (cf. Definition 2.25) can be reduced to the following simple form:

$$(\blacklozenge)_\bullet\colon 0 \longrightarrow \mathcal{O}_{\mathbb{S},\bullet}(a\, s_\infty) \longrightarrow \mathcal{E}_\bullet \longrightarrow \mathcal{O}_{\mathbb{S}_\bullet} \longrightarrow 0.$$

In Definition 4.4, the FESP of *Atiyah type* will be classified into two cases according to whether the above ESP $(\blacklozenge)_\bullet$ splits or unsplits. We can apply the MMP compatibly with étale endomorphisms and show the existence of a constant FESP of *Atiyah type* so as to apply Theorem 4.7(2) or Corollary 4.7(2). Our main results in the section are Theorems 4.12 and 4.20 which show the finiteness of extremal rays. Another main result is Theorem 4.27 which shows that the Albanese map $\alpha_X : X \to C$ of X is an analytic fiber bundle over an elliptic curve C. Combining these results, the proof of Theorem 1.4 will be given.

Now we recall the notion of the *shift* of an ESP.

Definition 4.1 ([8, Def. 3.13]). Let $f: X \to X$ be a nonisomorphic étale endomorphism of a smooth projective variety X and $X_\bullet = (X, f)$ the induced constant ESP. Let $D_\bullet \hookrightarrow X_\bullet$ be a sub-ESP. Then for any $m \in \mathbb{Z}$, we define a new sub-ESP $D_\bullet[m] \hookrightarrow X_\bullet$ by $D_k[m] := D_{k+m}$ for any $k \in \mathbb{Z}$ and $D_\bullet[m] := (f|_{D_{k+m}} : D_{k+m} \to D_{k+m+1})_k$. We call $D_\bullet[m]$ the *shift* of $D_\bullet$ by m.

Proposition 4.2 (Cf. [5, Prop. 7.10], [8, Prop. 6.10]). *Let $X_\bullet = (f_n : X_n \to X_{n+1})_n$ be an ESP of smooth projective* 3*-folds X_n with $\kappa(X_n) = -\infty$. Suppose that there exists an FESP $(Y_\bullet, R'_\bullet)$ of type* (C) *(cf.* [5, Def. 3.6]*) constructed from $X_\bullet$ by a sequence of blow-downs of an ESP $\pi_\bullet : X_\bullet \to Y_\bullet$. Suppose that $\pi_\bullet$ is not an isomorphism. Suppose furthermore that for $Y_\bullet = (g_n : Y_n \to Y_{n+1})_n$, the contraction morphism $\varphi_\bullet := \mathrm{Cont}_{R'_\bullet} : Y_\bullet \to S_\bullet := (u_n : S_n \to S_{n+1})_n$ is a $\mathbb{P}^1$-bundle. Let Δ_n be the discriminant locus of $\psi_n := \varphi_n \circ \pi_n : X_n \to S_n$. Then the following hold:*

- *ψ_n is an equidimensional $\mathbb{P}^1$-fiber space, that is, each fiber of ψ_n is connected and general fiber of ψ_n is isomorphic to $\mathbb{P}^1$.*
- *Δ_n is nonsingular and any of its irreducible component $\Delta_{n,i}$ is an elliptic curve disjoint with each other.*
- *For any i, $\psi_n^{-1}(\Delta_{n,i})$ is a simple normal crossing divisor and any of its irreducible components is a $\mathbb{P}^1$-bundle over an elliptic curve associated to a semistable vector bundle of rank two.*

Using Propositions 2.6 and 4.2, we can show that any $\mathcal{M}_n \in \mathrm{Pic}(C)$ is a torsion line bundle whose order is uniformly bounded above by a constant depending only on an endomorphism $f: X \to X$.

Proposition 4.3. *For any n, $\mathcal{M}_n \in \mathrm{Pic}(C)$ is of finite order and its order $\mathrm{ord}\,\mathcal{M}_n$ is bounded above by a constant which is determined by an endomorphism $f: X \to X$ and is independent of n.*

Proof. We first show that any $\mathcal{M}_n$ is of finite order. Lemma 2.24 shows that $\mathcal{L}_n \sim u_n^* \mathcal{L}_{n+1}$. Since $u_n^* s_{\infty,n+1} \sim s_{\infty,n}$ by Proposition 2.5, we have $u_n^* \mathcal{M}_{n+1} \sim \mathcal{M}_n$

for any n. Hence, by taking a truncated sequence, it is sufficient to show that $\mathcal{M}_0 \in \operatorname{Pic}(C)$ is of finite order without loss of generality. Let $s_{\infty,n}$ denote the canonical section of $S_n \simeq \mathbb{S}$ and we set $T_n := \varphi_n^{-1}(s_{\infty,n})$. By Proposition 2.5, $u_n^{-1}(s_{\infty,n+1}) = s_{\infty,n}$ for any n and $s_{\infty,\bullet} := (u_n|_{s_{\infty,n}} \colon s_{\infty,n} \to s_{\infty,n+1})_n$ forms a sub-ESP of $\mathbb{S}_\bullet$. Since $\varphi_\bullet \colon Y_\bullet \to \mathbb{S}_\bullet$ is a Cartesian morphism of ESPs, we see that $g_n^{-1}(T_{n+1}) = T_n$ for any n. Thus there is induced an ESP $T_\bullet = (g_n|_{T_n} \colon T_n \to T_{n+1})_n$ of elliptic ruled surfaces such that the inclusion morphism $T_\bullet \hookrightarrow Y_\bullet$ is Cartesian. Since $T_n = \mathbb{P}_{s_\infty}(\mathcal{E}_n|_{s_\infty})$, the vector bundle $\mathcal{E}_n|_{s_\infty}$ over s_∞ is semistable by [5, Prop. 4.1]. By construction, $\mathcal{L}_n|_{s_\infty} \simeq \mathcal{M}_n$ and there exists the following exact sequence of sheaves:

$$0 \longrightarrow \mathcal{M}_n \longrightarrow \mathcal{E}_n|_{s_\infty} \longrightarrow \mathcal{O}_{s_\infty} \longrightarrow 0.$$

Hence $\deg \mathcal{M}_n \leq 0$ for all n. Since $\mathcal{L}_n \simeq u_n^* \mathcal{L}_{n+1}$, we see that $\mathcal{M}_n \simeq h^* \mathcal{M}_{n+1}$ and $\mathcal{M}_0 \simeq (h^n)^* \mathcal{M}_n$ for all n. Suppose to the contrary that $\deg \mathcal{M}_0 \neq 0$. Then $0 < |\deg \mathcal{M}_n| = (\deg h)^{-n} |\deg \mathcal{M}_0| < 1$ for a sufficiently positive integer n. This contradicts the fact that $\deg \mathcal{M}_n \in \mathbb{Z}$. Hence $\deg \mathcal{M}_0 = 0$. Now we show that $\mathcal{M}_0 \in \operatorname{Pic}^0 C$ is of finite order. The proof is by contradiction. Suppose that $\mathcal{M}_0 \in \operatorname{Pic}^0 C$ is of infinite order. Let $\overline{T_n}$ be the proper transform of T_n by the blow-up $\pi_n \colon X \to Y_n$ for $n \in \mathbb{Z}$. We note that by [5, Cor. 7.9], $\pi_\bullet$ is a succession of blow-ups along elliptic curves contained in $\pi_\bullet^{-1}(s_{\infty,\bullet})$ and hence is an isomorphism outside $\pi_\bullet^{-1}(s_{\infty,\bullet})$. Then Proposition 4.2 shows the existence of an isomorphism $\overline{T_n} \simeq T_n \simeq \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{M}_n)$, where $\mathcal{M}_n \in \operatorname{Pic}^0(C)$ is of infinite order. Then $\overline{T}_\bullet := (f|_{\overline{T_n}} \colon \overline{T_n} \to \overline{T_{n+1}})_n$ is a sub-ESP of $X_\bullet$. For any $k \in \mathbb{Z}$, let $\overline{T}_\bullet[k]$ be the shift of $\overline{T}_\bullet$ (cf. Definition 4.1). Then there exist the following Cartesian morphisms of ESPs:

$$\begin{array}{ccc} \overline{T}_\bullet[k] & \xrightarrow{i_\bullet[k]} & X_\bullet \\ & & \downarrow \pi_\bullet \\ & & Y_\bullet \\ & & \downarrow \varphi_\bullet \\ & & \mathbb{S}_\bullet. \end{array}$$

By Proposition 2.6, there exists no surjective morphism from $\overline{T_n}$ onto the Atiyah surface $\mathbb{S}$. Hence, by Lemma 2.1 and Proposition 4.11(4), we see that $\varphi_\bullet \circ \pi_\bullet(\overline{T}_\bullet[k])$ is a sub-ESP of elliptic curves on $\mathbb{S}$ and equals $s_{\infty,\bullet}$ for the canonical section s_∞ ($\subset \mathbb{S}$). Thus any $\overline{T_k}$ is contained in $(\varphi \circ \pi_0)^{-1}(s_\infty)$, which has a finite number of irreducible components. Hence $\overline{T_p} = \overline{T_q}$ for some integers $p < q$. Then the restriction of $f^{q-p} \colon X \to X$ to the surface $\overline{T_p}$ induces a nonisomorphic finite étale endomorphism of $\overline{T_p}$. Hence Proposition 2.3 shows that $\mathcal{M}_n \in \mathrm{Pic}(C)$ is of finite order. Thus a contradiction is derived.

Next we show that $\mathrm{ord}\,\mathcal{M}_n$ $(< \infty)$ is independent of n. If we set $\psi_0 := \varphi_0 \circ \pi_0 \colon X \to \mathbb{S}$, then Proposition 4.2 shows that all the irreducible components of $\psi_0^{-1}(s_\infty)$ are elliptic ruled surfaces crossing normally with each other. Suppose that some $\mathcal{M}_n \in \mathrm{Pic}^0(C)$ is not trivial but torsion, then there exists an isomorphism $\overline{T_n} \simeq \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{M}_n)$. By Proposition 2.6, there exist no surjective morphisms from $\overline{T_n}$ to $\mathbb{S}$. Hence, by the same method as above, we see that for any $k \geq 0$, $f^{\pm k}(\overline{T_n})$ is contained $\psi_0^{-1}(s_\infty)$. Let Ω denote the finite set of all the irreducible components of $\psi_0^{-1}(s_\infty)$ and $p > 0$ its cardinality. Then both f and f^{-1} induce a permutation of Ω. Hence there exists some positive integer r $(\leq p!)$ such that $(f^r)^{-1}(\overline{T_n}) = \overline{T_n}$ for any $\overline{T_n} \simeq \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{M}_n)$. Thus there is induced a nonisomorphic étale endomorphism $f^r|_{\overline{T_n}} \colon \overline{T_n} \to \overline{T_n}$. Then [5, Cor. 4.9] shows that $\mathrm{ord}\,\mathcal{M}_n$ is bounded above by a constant which is determined by $f \colon X \to X$ and is independent of n. □

With the aid of Proposition 4.3, we see that there exists an integer $k > 0$ such that $\mu_k^* \mathcal{M}_n \simeq \mathcal{O}_C$ for any n, where $\mu_k \colon C \to C$ denotes the multiplication mapping by $k > 0$. Hence, replacing X (resp. Y_n and S_n) by the pullback $\widetilde{X} := X \times_{C,\mu_k} C$ (resp. $\widetilde{Y}_n := Y_n \times_{C,\mu_k} C$ and $\widetilde{S}_n := S_n \times_{C,\mu_k} C$), there exists a nonisomorphic étale endomorphism $\tilde{f} \colon \widetilde{X} \to \widetilde{X}$ (resp. nonisomorphic finite étale coverings $\tilde{g}_n \colon \widetilde{Y}_n \to \widetilde{Y}_{n+1}$ and $\tilde{u}_n \colon \widetilde{S}_n \to \widetilde{S}_{n+1}$) which is a lift of f (resp. g_n and u_n). Therefore, we may assume from the beginning that $Y_n = \mathbb{P}_{\mathbb{S}}(\mathcal{E}_n)$ and $\mathcal{L}_n \simeq \mathcal{O}_{\mathbb{S}}(a\, s_\infty)$ with $a > 1$, for any n. Furthermore, [5, Prop. 1.1 and Cor. 1.2] show that each $\pi_n \colon X \to Y_n$ is a succession of blow-ups along elliptic curves and thus $\pi_\bullet \colon X_\bullet \to Y_\bullet$ is a Cartesian morphism of ESPs. Furthermore, the FES associated to the FESP $Y_\bullet$ (cf. Definition 2.25) can be reduced to the following form:

$$(\blacklozenge)_\bullet \colon 0 \longrightarrow \mathcal{O}_{\mathbb{S},\bullet}(a\, s_\infty) \longrightarrow \mathcal{E}_\bullet \longrightarrow \mathcal{O}_{\mathbb{S}_\bullet} \longrightarrow 0.$$

Lemma 4.4. *The FESP $Y_\bullet$ enjoys the following properties:*

(1) *The FESP $Y_\bullet$ is a stable FESP (cf. Definition 2.9).*

(2) *The FES $(\blacklozenge)_\bullet$ splits (resp. does not split) if and only if $T_\bullet \simeq C_\bullet \times \mathbb{P}^1$ (resp. $T_\bullet \simeq \mathbb{S}_\bullet$).*

Proof. Since $s_{\infty,\bullet} \hookrightarrow \mathbb{S}_\bullet$ is a sub-ESP, $T_\bullet := \varphi_\bullet^{-1}(s_\bullet)$ is a sub-ESP of $Y_\bullet$. Restriction of $(\blacklozenge)_\bullet$ to $s_{\infty,\bullet}$ gives the following exact sequence for any n:

$$(\blacklozenge)_n|_{s_\infty} \colon 0 \longrightarrow \mathcal{O}_{\mathbb{S}} \longrightarrow \mathcal{E}_n|_{s_\infty} \longrightarrow \mathcal{O}_{\mathbb{S}} \longrightarrow 0.$$

Since $T_n \simeq \mathbb{P}(\mathcal{E}_n|_{s_\infty})$, we have $T_n \simeq \mathbb{S}$ (resp. $T_n \simeq C \times \mathbb{P}^1$) if and only if the exact sequence $(\blacklozenge)_n|_{s_\infty}$ unsplits (resp. splits). Furthermore, Proposition 2.6 shows that there exist no surjective morphisms between $\mathbb{S}$ and $C \times \mathbb{P}^1$. Hence assertion (2) follows. Assertion (1) is a direct consequence of (2). □

By Lemma 4.4, the possibilities for the FESP $Y_\bullet$ of *Atiyah type* are divided into the following two cases.

Definition 4.5. Let $F_\bullet$ be an FESP of *Atiyah type*. Then

- $F_\bullet$ is of type (Atiyah.A) if the exact sequence (♠) defined at the beginning of Section 3 does not split, equivalently, $T_\bullet \simeq \mathbb{S}_\bullet$;
- $F_\bullet$ is of type (Atiyah.B) if the exact sequence (♠) defined at the beginning of Section 3 splits, equivalently, $T_\bullet \simeq C_\bullet \times \mathbb{P}^1$.

Proposition 4.6. *Suppose that there exists the following unsplit exact sequence of sheaves on* $\mathbb{S}$*:*

$$0 \longrightarrow \mathcal{O}_{\mathbb{S}}(a\,s_\infty) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{S}} \longrightarrow 0,$$

where $a > 1$*. If we set* $Y := \mathbb{P}_{\mathbb{S}}(\mathcal{E})$ *and* $Z := \mathbb{P}_{\mathbb{S}}(\mathcal{O}_S \oplus \mathcal{O}_{\mathbb{S}}([a\,s_\infty]))$*, then there exists no surjective morphism from* Y *to* Z*, and vice versa from* Z *to* Y*.*

Proof. The proof is by contradiction. Let $\pi_Y\colon Y \to \mathbb{S}$ and $\pi_Z\colon Z \to \mathbb{S}$ be the projections.

(1) Suppose that there exists a surjective morphism $\varphi\colon Z \twoheadrightarrow Y$. Assume that some fiber of the projection $T := \pi_Z^{-1}(s_\infty) \simeq C \times \mathbb{P}^1 \to C$ is mapped onto a curve on $\mathbb{S}$ by the morphism $\pi_Y \circ \varphi\colon Z \to \mathbb{S}$. Then by the rigidity lemma (cf. [14, Lem. 1.6]), any fiber of π_Z is mapped onto a curve on $\mathbb{S}$ by $\pi_Y \circ \varphi$. Hence, by φ, there is induced a surjective morphism $T \simeq C \times \mathbb{P}^1 \xrightarrow{\varphi} \mathbb{S}$, which contradicts Proposition 2.6. Hence any fiber of π_Z is mapped to a point by the morphism $\pi_Z \circ \varphi\colon Z \to Y$ and there exists a surjective morphism $g\colon \mathbb{S} \to \mathbb{S}$ with $\pi_Y \circ \varphi = g \circ \pi_Z$. On the other hand, Proposition 2.5 shows that $g^{-1}(s_\infty) = s_\infty$. Hence, by φ, there is induced a surjective morphism $\pi_Z^{-1}(s_\infty) \simeq C \times \mathbb{P}^1 \twoheadrightarrow \pi_Y^{-1}(s_\infty) \simeq \mathbb{S}$, which again contradicts Proposition 2.6.

(2) Suppose that there exists a surjective morphism $\psi\colon Y \twoheadrightarrow Z$. Assume that some fiber of $\pi_Y : \pi_Y^{-1}(s_\infty) \simeq \mathbb{S} \to C$ is mapped to a point on $\mathbb{S}$ by the morphism $\pi_Z \circ \psi\colon Y \to \mathbb{S}$. Then by the rigidity lemma (cf. [14, Lem. 1.6]), there exists a surjective morphism $u\colon \mathbb{S} \twoheadrightarrow \mathbb{S}$ with $\pi_Z \circ \psi = u \circ \pi_Y$. With the aid of Proposition 2.5, we infer that $u^{-1}(s_\infty) = s_\infty$. Thus, by ψ, there is induced a surjective morphism $\pi_Y^{-1}(s_\infty) \simeq \mathbb{S} \twoheadrightarrow \pi_Z^{-1}(s_\infty) = s_\infty \times \mathbb{P}^1$. Since the restriction $\alpha_{\mathbb{S}}|_{s_\infty} : s_\infty \to C$ of the Albanese map $\alpha_{\mathbb{S}} : \mathbb{S} \to C$ is an isomorphism, this contradicts Proposition 2.6. Hence, by the rigidity lemma (cf. [14, Lem. 1.6]), any fiber of π_Y is mapped to a curve on $\mathbb{S}$ by the morphism $\pi_Z \circ \psi\colon Y \to \mathbb{S}$. Let $\alpha_Z : Z \to C$ (resp. $\alpha_Y : Y \to C$)

be the Albanese map of Z (resp. Y). Then, by the universality of the Albanese map, there exists a finite morphism $h\colon C \to C$ such that $\alpha_Z \circ \psi = h \circ \alpha_Y$. Hence there is induced a finite morphism $\alpha_Y^{-1}(o) \simeq \mathbb{F}_a \xrightarrow{\psi} \alpha_Z^{-1}(h(o)) \simeq \mathbb{F}_a$. The pullback map ψ^* induces an isomorphism of the Kleiman–Mori cone $\overline{\mathrm{NE}}(\mathbb{F}_a)$, which is a 2-dimensional closed polyhedral cone generated by the numerical equivalence class $[F]$ and $[\Delta]$, where F is the fiber of the ruling which is contracted by π_Y or π_Z and Δ is a negative section. Thus there is a one-to-one correspondence between the extremal rays of $\overline{\mathrm{NE}}(\mathbb{F}_a)$ of the source variety and that of the target variety. By the above remark, we see that $\psi^*\Delta \equiv \alpha F$ for some $\alpha \in \mathbb{R}$. Taking the self-intersection number of both sides, we infer that $\deg \psi = 0$, which derives a contradiction. □

We recall the following fact which will be essentially used to show the finiteness of extremal rays.

Theorem 4.7 (Cf. [8, Thm. 3.9]). *Let $X_\bullet = (f_n\colon X_n \to X_{n+1})_n$ be an ESP of smooth projective 3-fold X_n's with $\kappa(X_n) = -\infty$. Let $Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n$ and $Y'_\bullet = (g'_n\colon Y'_n \to Y_{n+1})_n$ be two FESPs constructed from $X_\bullet$ by a sequence of blow-downs of an ESP (cf.* [5, Def. 3.7]*). Suppose that there exists an extremal ray $R_\bullet$ (resp. $R'_\bullet$) of fiber type on $\overline{\mathrm{NE}}(Y_\bullet)$ (resp. $\overline{\mathrm{NE}}(Y'_\bullet)$) such that both $(Y_\bullet, R_\bullet)$ and $(Y'_\bullet, R'_\bullet)$ are of type* $(\mathrm{C}_{-\infty})$ *and satisfy the following conditions:*

- *There exist Cartesian morphisms $\pi_\bullet = (\pi_n)_n\colon X_\bullet \to Y_\bullet$, $\varphi_\bullet = (\varphi_n)_n\colon Y_\bullet \to S_\bullet = (u_n\colon S_n \to S_{n+1})_n$, and $\alpha_\bullet = (\alpha_n)_n\colon S_\bullet \to C_\bullet := (h_n\colon C_n \to C_{n+1})_n$, where $\pi_\bullet$ is a sequence of blow-downs of an ESP, $C_\bullet$ is an ESP of elliptic curves, any S_n is isomorphic to the Atiyah surface $\mathbb{S}$ over C_n, and $\varphi_n\colon Y_n \to S_n$ is a $\mathbb{P}^1$-bundle over $\mathbb{S}$.*
- *There exist further Cartesian morphisms $\pi'_\bullet = (\pi'_n)_n\colon X_\bullet \to Y'_\bullet, \varphi'_\bullet = (\varphi'_n)_n\colon Y'_\bullet \to T_\bullet = (T_n \to T_{n+1})_n$ and $\alpha'_\bullet = (\alpha'_n)_n\colon T_\bullet \to C_\bullet$, where $\pi'_\bullet$ is a sequence of blow-downs of an ESP and $\varphi'_n\colon Y'_n \to T_n$ is a $\mathbb{P}^1$-bundle over an elliptic ruled surface $T_n := \mathbb{P}_C(\mathcal{O}_C \oplus \ell_n)$. Here, any ℓ_n is a torsion line bundle on C_n and $\ell_0 \simeq \mathcal{O}_C$ for $n = 0$.*

Then the following hold:

1. *The natural morphism $X_0 \to S_0 \times_C T_0$ induced by $(\varphi_0 \circ \pi_0, \varphi'_0 \circ \pi'_0)\colon X_0 \to S_0 \times T_0$ is birational and a succession of blow-ups along elliptic curves.*
2. *The number of extremal rays of divisorial type on $\overline{\mathrm{NE}}(X_n)$ is finite for any n.*

Corollary 4.8 (Cf. [8, Cor. 3.12]). *Under the same assumption as in Theorem 4.7, in addition, suppose that $X_\bullet$ is a constant ESP (X, f) induced from a nonisomorphic étale endomorphism $f\colon X \to X$ of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Then the following hold:*

(1) *The natural morphism* $X \to S \times_C T \simeq S \times \mathbb{P}^1$ *which is induced by the morphism* $(\varphi \circ \pi, \varphi' \circ \pi')\colon X \to S \times T$ *is birational and a succession of blow-ups along elliptic curves.*

(2) *The number of extremal rays on* $\overline{\mathrm{NE}}(X)$ *is finite.*

(3) *Replacing* f *by its suitable power* f^k $(k > 0)$*, there exist two Cartesian morphisms of constant ESPs below:*
   - $(X, f) \xrightarrow{\pi} (Y, g) \xrightarrow{\varphi} (S, u)$;
   - $(X, f) \xrightarrow{\pi'} (Y', g') \xrightarrow{\varphi'} (T, v)$.

Theorem 4.9 (Cf. [8, Thm. 1.4]). *Let* X *be a smooth projective* 3*-fold* X *which admits a nonisomorphic étale endomorphism. Let* R *be an arbitrary extremal ray of divisorial type on* $\overline{\mathrm{NE}}(X)$ *and* $\varphi := \mathrm{Cont}_R\colon X \to X'$ *the contraction morphism associated to* R*. Then the following hold:*

(1) *The target* X' *is nonsingular and* φ *is (the inverse of) the blow-up of* X' *along an elliptic curve* C*.*

(2) *The exceptional divisor* $D := \mathrm{Exc}(\varphi)$ *of* φ *is isomorphic to either the Atiyah surface* $\mathbb{S}$ *or* $\mathbb{P}_C(\mathcal{O}_C \oplus \ell)$ *for a torsion line bundle* ℓ *on an elliptic curve* C*.*

Theorem 4.10 (Cf. [8, Thm. 7.3]). *Let* $f\colon X \to X$ *be a nonisomorphic étale endomorphism of a smooth projective* 3*-fold. Let* $Y_\bullet$ *be an arbitrary FESP constructed from the constant ESP* $X_\bullet := (X, f)$ *by a sequence of blow-downs of an ESP,*

$$X_0 := X \xrightarrow{\pi_n^{(0)}} \cdots \longrightarrow X_n^{(i)} \xrightarrow{\pi_n^{(i)}} X_n^{(i+1)} \longrightarrow \cdots \longrightarrow Y_n := X_n^{(k)}.$$

For any n *and* $0 \le i \le k-1$*, let* $E_n^{(i)}$ *be the* $\pi_n^{(i)}$*-exceptional divisor. Then a suitable finite étale covering of* $E_n^{(i)}$ *is isomorphic to either the Atiyah surface* $\mathbb{S}$ *or the product of* $\mathbb{P}^1$ *and an elliptic curve.*

Proposition 4.11 (Cf. [5, Prop. 6.9]). *Let* $S_\bullet = (g_n\colon S_n \to S_{n+1})_n$ *be an ESP of smooth algebraic surfaces* S_n*. Suppose that there exists an ESP* $\gamma_\bullet = (g_n|_{\gamma_n}\colon \gamma_n \to \gamma_{n+1})_n$ *of irreducible curves* γ_n *on* S_n*. Then every* γ_n *is an elliptic curve satisfying* $(\gamma_n)^2 = 0$ *and the following hold:*

(1) *If* $\kappa(S_n) = 1$ *for any* n*, then* $\varphi_n(\gamma_n)$ *is a point for the Iitaka fibration* $\varphi_n\colon S_n \to C_n$ *of* S_n*.*

(2) *If any* S_n *is an abelian surface, then there exists an elliptic fiber bundle structure* $\varphi_n\colon S_n \to C_n$ *over an elliptic curve* C_n *such that*
   - $\varphi_n(\gamma_n)$ *is a point on* C_n*, and*
   - $\varphi_{n+1} \circ g_n = u_n \circ \varphi_n$ *for an isomorphism* $u_n\colon C_n \cong C_{n+1}$*.*

(3) *If S_n is a hyperelliptic surface, then one of the following cases holds:*

(3.1) *The Albanese map $\alpha_n\colon S_n \to C_n$ is an elliptic fiber bundle such that*

- $\alpha_n(\gamma_n)$ *is a point, and*
- $\alpha_{n+1} \circ g_n = u_n \circ \alpha_n$ *for an isomorphism* $u_n\colon C_n \cong C_{n+1}$.

(3.2) $\mathrm{Aut}^0(S_n)$ *is an elliptic curve and the natural projection* $p_n\colon S_n \to \Gamma_n \coloneqq S_n/\mathrm{Aut}^0(S_n) \cong \mathbb{P}^1$ *is a Seifert elliptic surface such that*

- *there exists a finite morphism* $v_n\colon \Gamma_n \to \Gamma_{n+1}$ *such that* $p_{n+1} \circ g_n = v_n \circ p_n$ *and* $p_n(\gamma_n)$ *is a point, and*
- v_n *is an isomorphism or* $p_n(\gamma_n)$ *is a ramification point of* v_n.

(3.3)
- *The Albanese map* $\alpha_n\colon S_n \to C_n$ *is an elliptic fiber bundle such that* $\alpha_n(\gamma_n)$ *is a point,*
- $p_{n+1}\colon S_{n+1} \to \Gamma_{n+1} \simeq \mathbb{P}^1$ *is a Seifert elliptic surface such that* $\alpha_{n+1}(\gamma_{n+1})$ *is a point, and*
- *there exists a finite morphism* $w_n\colon C_n \to \Gamma_{n+1}$ *such that* $p_{n+1} \circ g_n = w_n \circ \alpha_n$.

(4) *If $\kappa(S_n) = -\infty$ for any n, then the Albanese map $\alpha_n\colon S_n \to C_n$ is a $\mathbb{P}^1$-bundle over an elliptic curve C_n associated to a semistable vector bundle $\mathcal{E}_n$ of rank two such that*

- $\alpha_n(\gamma_n) = C_n$ *and*
- *there exists a Cartesian morphism* $\alpha_\bullet\colon \gamma_\bullet \to C_\bullet = (h_n\colon C_n \to C_{n+1})_n$.

Furthermore, one of the following hold.

(4.1) *For any n, $S_n \simeq \mathbb{P}_{C_n}(\mathcal{E}_n)$ for a stable vector bundle $\mathcal{E}_n$ on C_n and γ_n is a multi-section of α_n.*

(4.2) *For any n, $S_n \simeq \mathbb{S}$ and γ_n is the canonical section of α_n.*

(4.3) *For any n,*

- *there exists an isomorphism* $S_n \simeq \mathbb{P}_{C_n}(\mathcal{O}_{C_n} \oplus \mathcal{L}_n)$ *for a line bundle* $\mathcal{L}_n \in \mathrm{Pic}^0(C_n)$ *of infinite order, and*
- *the elliptic curve* γ_n *coincides with either of the two sections of* α_n *corresponding to the first projection* $\mathcal{O}_{C_n} \oplus \mathcal{L}_n \twoheadrightarrow \mathcal{O}_{C_n}$ *or the second projection* $\mathcal{O}_{C_n} \oplus \mathcal{L}_n \twoheadrightarrow \mathcal{L}_n$.

(4.4) *For any n, there exists an isomorphism $S_n \simeq \mathbb{P}_{C_n}(\mathcal{O}_{C_n} \oplus \mathcal{L}_n)$ for a line bundle $\mathcal{L}_n \in \mathrm{Pic}^0(C_n)$ which is of finite order.*

§4.1. Subcase (Atiyah.B)

Now we consider the case where there exists an FESP $Y_\bullet$ of type (Atiyah.B), i.e., $\mathcal{E}_n \simeq \mathcal{O}_{\mathbb{S}}(a s_\infty) \oplus \mathcal{O}_{\mathbb{S}}$ for $a > 1$ and $Y_n \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{E}_n)$ for any n (cf. Theorem 2.13 and Definition 4.5). Let $D_{\infty,n}$ (resp. $D_{0,n}$) be the section of φ_n corresponding to a surjection $\mathcal{E}_n \twoheadrightarrow \mathcal{O}_{\mathbb{S}}$ (resp. $\mathcal{E} \twoheadrightarrow \mathcal{O}_{\mathbb{S}}(a s_\infty)$). If we set $T_n := \varphi_n^{-1}(s_\infty)$ for all n, then $T_n \simeq C \times \mathbb{P}^1$. Since the canonical section s_∞ ($\subset \mathbb{S}$) forms a sub-ESP $s_{\infty,\bullet}$ ($\subset \mathbb{S}_\bullet$), the T_n also form a sub-ESP $T_\bullet := (g_n|_{T_n} : T_n \to T_{n+1})_n$ of $Y_\bullet$. Let $\Gamma_{\infty,n} := T_n \cap D_{\infty,n}$ (resp. $\Gamma_0 := T \cap D_0$), the complete intersection curve on Y_n (resp. Y_0).

Theorem 4.12. *Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists an FESP $Y_\bullet$ of type (Atiyah.B). Then there exist at most finitely many extremal rays of $\overline{\mathrm{NE}}(X)$.*

Proof. The proof is done by using the theory of elementary transformations (cf. Section 3) and applying Theorem 4.7.

Step 1. With the aid of [5, Rem. 8.7(1)], it is sufficient to show the finiteness of extremal rays of *divisorial type* on $\overline{\mathrm{NE}}\,(X)$. Hereafter, by R ($\subset \overline{\mathrm{NE}}\,(X)$) (resp. E_R), we always denote an extremal ray of *divisorial* type (resp. the exceptional divisor of the contraction morphism $\mathrm{Cont}_R\colon X \to X'$ associated to R). We set $R^n := (f^n)_* R$ for $n \geq 0$, $R^n := (f^{-n})^* R$ for $n < 0$ and $R_\bullet := \{R_n\}_n$. If we set $E_n := E_{R_n}$, then $E_{R_\bullet} = (f|_{E_n} : E_n \to E_{n+1})_n$ is a sub-ESP of $X_\bullet = (X, f)$ by Proposition 2.2. Since $\psi_\bullet := \varphi_\bullet \circ \pi_\bullet \colon X_\bullet \to \mathbb{S}_\bullet$ is a Cartesian morphism of ESPs, the image $\psi_\bullet(E_{R_\bullet})$ is also a sub-ESP of $\mathbb{S}_\bullet$ by Lemma 2.1.

Suppose that $\psi_\bullet(E_{R_\bullet})$ is an ESP of elliptic curves. Then for any $k \in \mathbb{Z}$, its shift $\psi_\bullet(E_{R_\bullet})[k] = \psi_\bullet(E_{R_\bullet}[k])$ is also an ESP of elliptic curves and equals $s_{\infty,\bullet}$ by Proposition 4.11(4.2). Hence any E_k is contained in $\psi_0^{-1}(s_\infty)$, which has a finite number of irreducible components. Hence, by [7, Thm. 1.1], we see that the number of such extremal rays $R (\subset \overline{\mathrm{NE}}(X))$ is finite.

Suppose that there exists no extremal ray R of divisorial type such that E_R dominates $\mathbb{S}$. Then the finiteness of extremal rays R follows from the argument above. Hence, to show the finiteness of extremal rays of $\overline{\mathrm{NE}}\,(X)$, it is sufficient to assume from the beginning that there exists some extremal ray R ($\subset \overline{\mathrm{NE}}\,(X)$) of divisorial type such that $\psi_\bullet(E_{R_\bullet}) = \mathbb{S}_\bullet$. Then by Propositions 2.5, 2.6 and Theorem 4.9, $E_k \simeq \mathbb{S}$ for any k and $\psi_k|_{E_k} : E_k \to \mathbb{S}$ is a finite étale covering. Then $\psi\colon E \to \mathbb{S}$ for $\psi := \psi_0|_E$ and $E := E_0 = E_R$ induces a surjective morphism $u\colon\ \mathrm{Alb}(E) \to C$ of Albanese elliptic curves. We take some positive integer k so that u is factored by a multiplication mapping $\mu_k : C \to C$ by k. We set $\widetilde{X} := X \times_{C,\mu_k} C$.

We note that $\deg h = \deg f > 1$, since f is étale and the general fiber of the Albanese map $\alpha_X\colon X \to C$ is simply connected and hence the fiber degree of α_X has to be 1. Since C is an elliptic curve, we see that $h\colon C \to C$ has a fixed point $o \in C$ by using the Lefschetz fixed point theorem. Hence, if we endow C with a group structure so that $o \in C$ is the zero element, then we may assume that h is a group homomorphism of C. Since $h \circ \mu_k = \mu_k \circ h$, there exists a nonisomorphic étale endomorphism $\tilde{f}\colon \widetilde{X} \to \widetilde{X}$ which is a lift of $f\colon X \to X$. Let $\rho\colon \widetilde{X} \to X$ be the natural projection. Then the Galois group $\mathrm{Gal}(\widetilde{X}/X)$ acts on the normalization of $E \times_{C,\mu_k} E$ and hence $\rho^{-1}(E)$ is divided into a disjoint union $\rho^{-1}(E) = \amalg_i \widetilde{E}^{(i)}$ of sections of $\widetilde{X} \to \mathbb{S}$ which is a copy of $\mathbb{S}$ and one of its connected component is induced from the diagonal section $\boldsymbol{\Delta}$ $(\subset E \times_{C,\mu_k} E)$ with $\boldsymbol{\Delta} \simeq \mathbb{S}$. For each i, there exists an extremal ray $\widetilde{R}_i$ of $\overline{\mathrm{NE}}(\widetilde{X})$ such that $E_{\widetilde{R}_i} = \widetilde{E}^{(i)}$ and $\rho_*(\widetilde{R}_i) = R$. Hence, replacing $f\colon X \to X$ by its lift $\tilde{f}\colon \widetilde{X} \to \widetilde{X}$, we may assume from the beginning that E_R is a section of the $\mathbb{P}^1$-fiber space $\varphi_0 \circ \pi_0\colon X \to \mathbb{S}_0$. Thus $D_R \coloneqq \pi_0(E_R)$ is also a section of $\varphi_0\colon Y_0 \to \mathbb{S}_0$.

Step 2. By construction, we have the following Cartesian morphisms of ESPs:

$$\begin{array}{ccc} T_\bullet & \xrightarrow{i_\bullet} & Y_\bullet \\ \big\downarrow & & \big\downarrow{\scriptstyle \varphi_\bullet} \\ s_{\infty,\bullet} & \longrightarrow & \mathbb{S}_\bullet . \end{array}$$

Furthermore, we have further Cartesian morphisms of ESPs:

$$\begin{array}{ccc} E_{R,\bullet} & \xrightarrow{i_\bullet} & X_\bullet \\ & & \big\downarrow{\scriptstyle \pi_\bullet} \\ & & Y_\bullet \\ & & \big\downarrow{\scriptstyle \varphi_\bullet} \\ & & \mathbb{S}_\bullet . \end{array}$$

By Lemma 2.1, we see that $\Delta_{R,\bullet} \coloneqq \pi_\bullet(E_{R,\bullet})$ is a sub-ESP of $Y_\bullet$. By assumption, the composite $\varphi_\bullet \circ \pi_\bullet \circ i_\bullet\colon E_{R,\bullet} \to \mathbb{S}_\bullet$ is an isomorphism of ESPs consisting of Atiyah surfaces. Hence $\Gamma_\bullet \coloneqq \Delta_{R,\bullet} \cap T_\bullet \hookrightarrow Y_\bullet$ is a sub-ESP of elliptic curves. Thus there exist the following Cartesian morphisms of ESPs:

$$\begin{array}{ccc} \Gamma_\bullet & \longrightarrow & Y_\bullet \\ \big\downarrow & & \big\downarrow{\scriptstyle \varphi_\bullet} \\ s_{\infty,\bullet} & \longrightarrow & \mathbb{S}_\bullet . \end{array}$$

Since the inclusion morphism $\Gamma_\bullet \hookrightarrow T_\bullet \simeq (C \times \mathbb{P}^1)_\bullet$ is Cartesian, we see that $\Gamma_\bullet$ is a fiber of the second projection $T_\bullet \simeq (C \times \mathbb{P}^1)_\bullet \to \mathbb{P}^1$ by Proposition 4.11(4). By construction, there exist two sub-ESPs $\Delta_{R,\bullet} \hookrightarrow Y_\bullet$ and $D_{\infty,\bullet} \hookrightarrow Y_\bullet$.

Lemma 4.13. *We have* $\Delta_{R,\bullet} \neq D_{\infty,\bullet}$.

Proof. Suppose that $\Delta_{R,\bullet} = D_{\infty,\bullet}$. Let F be a general fiber of $\alpha_X : X \to C$. Then $(\Delta_{R,n}|_{\pi(F)})^2 = -a < -1$. Since $E_{R,\bullet}$ is the strict transform of $D_{\infty,\bullet}$ on $X_\bullet$, we have $(E_{R,n}|_F)^2 \leq (\Delta_{R,n}|_{\pi(F)})^2 = -a < -1$. On the other hand, $(E_{R,n}|_F)^2 = -1$, since R is of type (E1) in the sense of [18] (cf. Theorem 4.9). Thus a contradiction is derived. □

We set $\Gamma_{\infty,\bullet} = D_{\infty,\bullet} \cap T_\bullet$. The following lemma is crucial.

Lemma 4.14. *We have either* $\Delta_{R,\bullet} \cap D_{\infty,\bullet} = \Gamma_\bullet = \Gamma_{\infty,\bullet}$, *or* $\Delta_{R,\bullet} \cap D_{\infty,\bullet} = \emptyset$ *and* $\Gamma_n = D_{0,n} \cap T_n$ *for any* n.

Proof. If $\Delta_{R,\bullet} \cap D_{\infty,\bullet} \neq \emptyset$, then $\Delta_{R,\bullet} \cap D_{\infty,\bullet} \hookrightarrow D_{\infty,\bullet}$ is a sub-ESP of $D_{\infty,\bullet} \simeq \mathbb{S}_\bullet$ and equals the canonical section $s_{\infty,\bullet}$. Hence $\Delta_{R,\bullet} \cap D_{\infty,\bullet} = \Gamma_{\infty,\bullet}$.

Since $\Gamma_\bullet$ is a sub-ESP of $T_\bullet$ which is isomorphic to a stable ESP $(C \times \mathbb{P}^1)_\bullet$, $\Gamma_\bullet$ equals some fiber of the second projection $p\colon \Gamma_\bullet \simeq (C \times \mathbb{P}^1)_\bullet \to \mathbb{P}^1$. We set $\Gamma_t := p^{-1}(t)$ for $t \in \mathbb{P}^1$ such that $\Gamma_0 = p^{-1}(0)$ and $\Gamma_\infty = p^{-1}(\infty)$. Then Lemma 3.10 shows that for any $t \neq 0, \infty$, we have $\mathrm{N}_{\Gamma_t/Y} \simeq \mathcal{F}_2$. Suppose that $\Gamma = \Gamma_t$ for some $t \neq 0, \infty$. Since $\mathrm{N}_{\Gamma/Y} \simeq \mathcal{O}_C \oplus \mathcal{O}_C$ is decomposable, this contradicts the fact that $\mathcal{F}_2$ is indecomposable. Hence either $\Gamma_n = \Gamma_{0,n} \cap T_n$ for any n or $\Gamma_\bullet = \Gamma_{\infty,\bullet}$. Thus the proof has been done. □

Step 3. First we consider the case where $\Delta_{R,\bullet} \cap D_{\infty,\bullet} = \emptyset$ and $\Gamma_n = D_{0,n} \cap T_n$ for any n. We have the following lemma.

Lemma 4.15. *If we set* $\Delta_R := \Delta_{R,0}$, $D_\infty := D_{\infty,0}$, $D_0 := D_{0,0}$ *and* $T := T_0$, *then we have* $\Delta_R \sim D_\infty + aT$ *and* $\Delta_R \sim D_0$.

Proof. We have $\Delta_R - D_\infty \sim \varphi_0^*(p s_\infty + \alpha_0^* \mathcal{W})$ for some $p \in \mathbb{Z}$ and a divisor $\mathcal{W}$ on C. Restriction to the surface T gives the relation $\Delta_R|_T - D_\infty|_T \sim \varphi_0^* \alpha_0^* \mathcal{W}|_T$. Since $\Delta_R|_T \sim D_\infty|_T \sim \Gamma_0$, we see that $\mathcal{W} \sim 0$. Restriction to the surface D_∞ shows that $p = a$, since $\Delta_R \cap D_\infty = \emptyset$ and $D_\infty|_{D_\infty} \sim -a\Gamma_\infty$. □

Step 3-1. Since $\Delta_{R,\bullet} \cap D_{\infty,\bullet} = \emptyset$ and $Y_\bullet \simeq \mathbb{P}_\mathbb{S}(\mathcal{O}_\mathbb{S}(a s_\infty) \oplus \mathcal{O}_\mathbb{S})_\bullet$, there exist isomorphisms $X_\bullet \times_\mathbb{S} \Delta_{R,\bullet} \simeq X_\bullet$ and $Y_\bullet \times_\mathbb{S} \Delta_{R,\bullet} \simeq Y_\bullet \simeq \mathbb{P}_\mathbb{S}(\mathcal{O}_\mathbb{S}(a s_\infty) \oplus \mathcal{O}_\mathbb{S})_\bullet$. Hence, replacing $X_\bullet$ (resp. $Y_\bullet$) by $X_\bullet \times_\mathbb{S} \Delta_{R,\bullet}$ (resp. $Y_\bullet \times_\mathbb{S} \Delta_{R,\bullet}$), we may assume from the beginning that $\Delta_{0,n} = D_{0,n}$ for any n. Thus $\Delta_{R,\bullet} = D_{0,\bullet}$ and $\Gamma_{0,\bullet} := D_{0,\bullet} \cap T_\bullet$ form a sub-ESP of $Y_\bullet$. Then Lemma 4.15 shows that $(\Delta_R^2, F) = a \geq 2$ for a general fiber F of $\alpha \circ \varphi_0 \colon Y_0 \to C$, and hence $\pi_\bullet \colon X_\bullet \to Y_\bullet$ is not an isomorphism, since it should be of type (E1) in the sense of [18]. Let

$$X_\bullet := X_\bullet^{(0)} \longrightarrow \cdots \longrightarrow X_\bullet^{(k-2)} \xrightarrow{\pi_\bullet^{(k-2)}} X_\bullet^{(k-1)} \xrightarrow{\pi_\bullet^{(k-1)}} Y_\bullet := X_\bullet^{(k)}$$

be a sequence of blow-ups of an ESP along elliptic curves. Let $\gamma_n^{(i)}$ be the center of the blow-up $\pi_n^{(i-1)} \colon X_n^{(i-1)} \to X_n^{(i)}$ and let $\gamma_n := \gamma_n^{(k)}$ ($\subset Y_n$) be the center of the first blow-up $\pi_n^{(k-1)}$ from the bottom. Then $\gamma_\bullet^{(i)} = (\gamma_n^{(i)})_n$ is a sub-ESP of $X_\bullet^{(i)}$ and there exist the following Cartesian morphisms of ESPs:

$$\begin{array}{ccc} \gamma_\bullet^{(i)} & \longrightarrow & X_\bullet^{(i)} \\ & & \downarrow{\scriptstyle \pi_\bullet^{(k-1)} \circ \cdots \circ \pi_\bullet^{(i)}} \\ & & Y_\bullet \\ & & \downarrow{\scriptstyle \varphi_\bullet} \\ & & \mathbb{S}_\bullet . \end{array}$$

Step 3-2. We begin with a remark.

Remark 4.16. In this Step 3-2, the following argument works under the assumption that $a > 0$. The assumption that $a > 1$ is only used in Lemma 4.13 to show that $\Delta_{R,\bullet} \neq D_{\infty,\bullet}$. In the case where $a = 1$ (i.e., $Y_\bullet \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{O}_{\mathbb{S}}(s_\infty) \oplus \mathcal{O}_{\mathbb{S}})$) we may *assume* that $\Delta_{R,\bullet} \neq D_{\infty,\bullet}$ to show the finiteness of extremal rays of $\overline{\mathrm{NE}}\,(X)$ (cf. [8, Thm. 8.27 and Rem. 8.33]).

In this step, we use the assumption that $a > 0$. By Lemma 2.1, Proposition 4.11 and Corollary 2.8, we see that $\varphi_\bullet \circ \pi_\bullet^{(k-1)} \circ \cdots \circ \pi_\bullet^{(i)}(\gamma_\bullet^{(i)})$ is a sub-ESP of $\mathbb{S}_\bullet$ and equals $s_{\infty,\bullet}$. Hence $G_\bullet^{(i)} := \pi_\bullet^{(k-1)} \circ \cdots \circ \pi_\bullet^{(i)}(\gamma_\bullet^{(i)})$ is a sub-ESP of the ESP $T_\bullet = \varphi_\bullet^{-1}(s_{\infty,\bullet}) \simeq (C \times \mathbb{P}^1)_\bullet$ and thus is contained in a fiber of the second projection $p_2 \colon T_\bullet^{(k)} \to \mathbb{P}^1$. Hence we have either $G_\bullet^{(i)} = \Gamma_{0,\bullet}$ or $G_\bullet^{(i)} \cap \Gamma_{0,\bullet} = \emptyset$ (cf. Lemma 4.14). Suppose that $G_\bullet^{(i)} \cap \Gamma_{0,\bullet} = \emptyset$ for any $0 \le i \le k-1$. Since $E_{R,\bullet}$ is the proper transform of $\Delta_{R,\bullet}$ by $\pi_\bullet$, we have an isomorphism $E_{R,\bullet} \simeq D_{0,\bullet}$ and thus $(E_R^2, V) = ({D_0}^2, V) = a > 0$ for a general fiber V of $\alpha_0 \circ \varphi_0 \circ \pi_0 \colon X \to C$. On the other hand, since the extremal ray R is of type (E1) in the sense of [18], we have $(E_R^2, V) = -1$. Thus a contradiction is derived. Hence $G_\bullet^{(i)} = \Gamma_{0,\bullet}$ for some $0 \le i \le k-1$ and at any rate, we have to blow up along the elliptic curve $\Gamma_{0,\bullet}$ and $\pi_\bullet \colon X_\bullet \to Y_\bullet$ factors through the blow-up $\mathrm{Bl}_{\Gamma_{0,\bullet}} Y_\bullet$ of $Y_\bullet$ along $\Gamma_{0,\bullet}$. Thus, if we change the ordering of a sequence of blow-ups of an ESP if necessary, then we may assume that the center of the first blow-up $\pi_\bullet^{(k-1)} \colon X_\bullet^{(k-1)} \to X_\bullet^{(k)} = Y_\bullet$ from the bottom coincides with $\Gamma_{0,\bullet}$. Then we can perform an elementary transformation to $Y_\bullet$ along $\Gamma_{0,\bullet}$ (cf. Definition 3.3). Let $D_{0,\bullet}^{(k-1)} \simeq \mathbb{S}_\bullet$ (resp. $T_\bullet^{(k-1)} \simeq \mathbb{S}_\bullet$) be the proper transform of $D_{0,\bullet}$ (resp. $T_\bullet$) in $X_\bullet^{(k-1)}$ and $E_\bullet^{(k-1)} := \mathrm{Exc}(\pi_\bullet^{(k-1)})$ ($\simeq (C \times \mathbb{P}^1)_\bullet$) be the exceptional divisor of $\pi_n^{(k-1)} \colon X_n^{(k-1)} \to Y$. There exists a birational morphism $\nu_n^{(k-1)} \colon X_n^{(k-1)} \to Z_n^{(k)}$ which contracts the divisor $T_n^{(k-1)} \simeq C \times \mathbb{P}^1$ to an elliptic curve. Since $E_n^{(k-1)} \simeq C \times \mathbb{P}^1$, there exists an isomorphism $Z_n^{(k)} \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{O}_{\mathbb{S}}((a-1)s_\infty) \oplus \mathcal{O}_{\mathbb{S}})$ by Lemma 3.12 and $E_n^{(k)} := \nu_n^{(k-1)}(E_n^{(k-1)})$ is isomorphic to $\mathbb{P}^1 \times C$. Since $\Gamma_{0,\bullet}$ and $T_\bullet^{(k-1)}$ are sub-ESPs of $Y_\bullet$, $Z_n^{(k)}$ ($n \in \mathbb{Z}$) forms an ESP $Z_\bullet^{(k)}$ (cf. [5, Prop. 3.1 and Lem. 3.3]). Thus we have obtained further Cartesian morphisms of ESPs,

$$X_\bullet = (X, f) \xrightarrow{\mu_\bullet} X_\bullet^{(k-1)} \xrightarrow{\nu_\bullet^{(k-1)}} Z_\bullet^{(k)} = (g'_n \colon Z_n^{(k)} \to Z_{n+1}^{(k)})_n,$$

such that the following conditions are satisfied:

- $\nu_\bullet^{(k-1)} \circ \mu_\bullet$ is a sequence of blow-ups of an ESP along elliptic curves.
- $Z_\bullet^{(k)}$ is another FESP of $X_\bullet$ obtained from $Y_\bullet$ by performing elementary transformations $\mathrm{elm}_{\Gamma_{0,\bullet}}$ along $\Gamma_{0,\bullet}$ and is isomorphic to the stable ESP $\mathbb{P}_{\mathbb{S}}(\mathcal{O}_{\mathbb{S}}((a-1)s_\infty) \oplus \mathcal{O}_{\mathbb{S}})_\bullet$.
- The sub-ESP $E_\bullet^{(k)} := (g'_n|_{E_n^{(k)}} \colon E_n^{(k)} \to E_{n+1}^{(k)})_n$ of $Z_\bullet^{(k)}$ is isomorphic to a stable ESP $(\mathbb{P}^1 \times C)_\bullet$.
- The sub-ESP $D'^{(k)}_{0,\bullet} := \nu_\bullet^{(k-1)}(D_{0,\bullet}^{(k-1)})$ is a section of $Z_\bullet^{(k)} \to \mathbb{S}_\bullet$ which corresponds to a surjection $\mathcal{O}_{\mathbb{S}}((a-1)s_\infty) \oplus \mathcal{O}_{\mathbb{S}} \to \mathcal{O}_{\mathbb{S}}((a-1)s_\infty)$.

Step 3-3. If we take a self-intersection number within a general fiber of $Z_0^{(k)} \to C$, then we have $(D'^{(k)}_0)^2_{Z_0^{(k)} \to C} = a - 1 \geq 0$. In the case where $a = 1$, we stop here. From now on, we shall use the assumption that $a > 1$, Then $(D'^{(k)}_0)^2_{Z_0^{(k)} \to C} = a - 1 > 0$. Hence, applying the same argument as before, we see that the blow-up of an ESP $\nu_\bullet^{(k-1)} \circ \pi_\bullet^{(k-2)} \circ \cdots \circ \pi_\bullet^{(0)} \colon X_\bullet \to Z_\bullet^{(k)}$ factors through the blow-up of $Z_\bullet^{(k)}$ along $E_\bullet^{(k)} \cap D'^{(k)}_{0,\bullet}$. Hence we can apply elementary transformations to $Z_\bullet^{(k-1)}$ along $E_\bullet^{(k)} \cap D'^{(k)}_{0,\bullet}$. By changing the order of a sequence of blow-ups of an ESP, we obtain the following Cartesian morphisms of ESPs:

$$X_\bullet \longrightarrow Z_\bullet^{(k-1)} \longrightarrow \mathbb{S}_\bullet,$$

where

- $X_\bullet \to Z_\bullet^{(k-1)}$ is a blow-up sequence of an ESP, and
- $Z_\bullet^{(k-1)} \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{O}_{\mathbb{S}}((a-2)s_\infty) \oplus \mathcal{O}_{\mathbb{S}})_\bullet$ is a stable FESP of $X_\bullet$ (cf. Definition 2.9).

We shall continue these procedures and apply successions of elementary transformations along an elliptic curve a times (cf. Definition 3.3). Changing the ordering of a sequence of blow-ups of an ESP, we decrease a one by one and finally reach $a = 0$. Thus we eventually obtain the following morphisms of ESPs:

$$X_\bullet \longrightarrow Z_\bullet^{(k-a+1)} \longrightarrow \mathbb{S}_\bullet,$$

where

- $X_\bullet \to Z_\bullet^{(k-a+1)}$ is a blow-up sequence of an ESP, and
- $Z_\bullet^{(k-a+1)} \simeq \mathbb{S} \times \mathbb{P}^1$ is another stable FESP (cf. Definition 2.9) of $X_\bullet$.
- $(D_0^{(k-a+1)}, D_0^{(k-a+1)}, F^{(k-a+1)}) = 0$ for the proper transform $D_0^{(k-a+1)}$ of D_0 on $Z^{(k-a+1)}$ and the general fiber $F^{(k-a+1)}$ of $Z^{(k-a+1)} \to C$.

Then [8, Prop. 8.10] shows the existence of the $\mathbb{P}^1$-bundle $Z_\bullet \to (C \times \mathbb{P}^1)_\bullet$ which is a Cartesian morphism of stable ESPs. Thus there exist the following Cartesian morphisms of ESPs:

$$\begin{array}{ccc} X_\bullet & = & X_\bullet \\ {\scriptstyle \pi_\bullet}\downarrow & & \downarrow \\ Y_\bullet & & Z_\bullet^{(k-a+1)} \\ {\scriptstyle \varphi_\bullet}\downarrow & & \downarrow \\ \mathbb{S}_\bullet & & (C \times \mathbb{P}^1)_\bullet. \end{array}$$

Hence, by Theorem 4.7, there exist only finitely many extremal rays of divisorial type on X.

Step 4. Next we consider the case where $\Delta_{R,\bullet} \cap D_{\infty,\bullet} = \Gamma_\bullet = \Gamma_{\infty,\bullet}$.

Lemma 4.17. *Suppose that there exists another section D ($\neq D_\infty$) of $\varphi_0 \colon Y_0 \to S_0$ such that $D \cap D_\infty = \Gamma$ and $D \cap T = \Gamma$. Set $D|_{D_\infty} \sim m\Gamma$ for $m > 0$. Then $D \sim D_\infty + (m + a)T$.*

Proof. We see that $D - D_\infty \sim \varphi_0^*(p s_\infty + \alpha_0^* \mathcal{W})$ for some $p \in \mathbb{Z}$ and a divisor $\mathcal{W}$ of C. Restriction to the surface T gives a relation $D|_T - D_\infty|_T \sim \varphi_0^*(\alpha_0^* \mathcal{W})|_T$. Since $D|_T \sim D_\infty|_T \sim \Gamma$, we see that $\mathcal{W} \sim 0$. Hence $D - D_\infty \sim pT$. Restricting to the surface D_∞, we infer that $D|_{D_\infty} - D_\infty|_{D_\infty} \sim p\Gamma$. Since $D|_{D_\infty} \sim m\Gamma$ and $D_\infty|_{D_\infty} \sim -a\Gamma$, we infer that $p = m + a$. □

Since $a > 1$, we see easily that $\Delta_{R,\bullet} \neq D_{\infty,\bullet}$ by the same method as in the proof of Lemma 4.13. Setting $\Delta_R|_{D_\infty} \sim m\Gamma_\infty$ ($m > 0$) and applying Lemma 4.17, we infer that $\Delta_R \sim D_\infty + (a + m)T$. Since $(\Delta_R^2, F) = 2m + a > 2$ for a general fiber F of $\alpha \circ \varphi_0 \colon Y_0 \to C$ (cf. Lemma 4.17), $\pi_\bullet \colon X_\bullet \to Y_\bullet$ is not an isomorphism.

Let

$$X_\bullet := X_\bullet^{(0)} \longrightarrow \cdots \longrightarrow X_\bullet^{(k-2)} \xrightarrow{\pi_\bullet^{(k-2)}} X_\bullet^{(k-1)} \xrightarrow{\pi_\bullet^{(k-1)}} Y_\bullet := X_\bullet^{(k)}$$

be a sequence of blow-ups of an ESP along elliptic curves. Let $\gamma_\bullet^{(i)}$ be the center of the blow-up $\pi_\bullet^{(i-1)}\colon X_\bullet^{(i-1)} \to X_\bullet^{(i)}$ and let $\gamma_\bullet := \gamma_\bullet^{(k)}$ ($\subset Y_\bullet$) be the center of the first blow-up $\pi_\bullet^{(k-1)}$ from the bottom. Then $\gamma_\bullet^{(i)} = (\gamma_n^{(i)})_n$ is a sub-ESP of $X_\bullet^{(i)}$ and there exist the following Cartesian morphisms of ESPs:

$$\begin{array}{ccc} \gamma_\bullet^{(i)} & \longrightarrow & X_\bullet^{(i)} \\ & & \Big\downarrow {\scriptstyle \pi_\bullet^{(k-1)}\circ\cdots\circ\pi_\bullet^{(i)}} \\ & & Y_\bullet. \end{array}$$

Then by Lemma 2.1, Proposition 2.8 and the same argument as in Step 3, we may assume that $\pi_\bullet^{(k-1)}\circ\cdots\circ\pi_\bullet^{(i)}(\gamma_\bullet^{(i)})$ is a sub-ESP of $Y_\bullet$ and equals $\Gamma_{\infty,\bullet}$. Hence $X_\bullet^{(k-1)}$ is isomorphic to $\mathrm{Bl}_{\Gamma_{\infty,\bullet}}(Y_\bullet)$, which is the blow-up of $Y_\bullet$ along the elliptic curve $\Gamma_{\infty,\bullet}$. Now we shall perform elementary transformations of an ESP to $Y_\bullet$ along elliptic curves $\Gamma_{\infty,\bullet}$ (cf. Definition 3.3). Let $D_{\infty,\bullet}^{(k-1)} \simeq \mathbb{S}_\bullet$ (resp. $\Delta_{R,\bullet}^{(k-1)} \simeq \mathbb{S}_\bullet$ and $T_\bullet^{(k-1)} \simeq (C\times\mathbb{P}^1)_\bullet$) be the proper transform of $D_{\infty,\bullet}$ (resp. $\Delta_{R,\bullet}$ and $T_\bullet$) in $X_\bullet^{(k-1)}$ and $E_\bullet^{(k-1)} := \mathrm{Exc}(\pi_\bullet^{(k-1)})$ ($\simeq (C\times\mathbb{P}^1)_\bullet$) be the exceptional divisor of the first blow-up $\pi_\bullet^{(k-1)}\colon X_\bullet^{(k-1)} \to Y_\bullet$. There exists a birational morphism $\nu_n\colon X_n^{(k-1)} \to W_n^{(k)}$ which contracts the divisor $T_n^{(k-1)} \simeq C\times\mathbb{P}^1$ to an elliptic curve. Since ${E'}_n^{(k)} := \nu_n(E_n^{(k-1)})$ is isomorphic to $C\times\mathbb{P}^1$, there exists an isomorphism $W_n^{(k-1)} \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{O}_{\mathbb{S}}((a+1)s_\infty)\oplus\mathcal{O}_{\mathbb{S}})$ by Lemma 3.12. Thus we have obtained further Cartesian morphisms of ESPs,

$$X_\bullet = (X,f) \xrightarrow{\mu_\bullet} X_\bullet^{(k-1)} \xrightarrow{\nu_\bullet} W_\bullet^{(k)} = (g_n\colon Z_n^{(k)} \to Z_{n+1}^{(k)})_n,$$

such that the following conditions are satisfied:

- $\nu_\bullet\circ\mu_\bullet$ is a sequence of blow-ups of an ESP along elliptic curves.
- $W_\bullet^{(k)}$ is another FESP of $X_\bullet$ obtained from $Y_\bullet$ by performing elementary transformations $\mathrm{elm}_{\Gamma_{\infty,\bullet}}$ along $\Gamma_{\infty,\bullet}$ and is isomorphic to the stable ESP $\mathbb{P}_{\mathbb{S}}(\mathcal{O}_{\mathbb{S}}((a+1)s_\infty)\oplus\mathcal{O}_{\mathbb{S}})_\bullet$.
- ${E'}_\bullet^{(k)} := (g_n|_{{E'}_n^{(k)}}\colon {E'}_n^{(k)} \to {E'}_{n+1}^{(k)})_n$ is a sub-ESP of $Z_\bullet^{(k)}$ and is isomorphic to a stable ESP $(\mathbb{P}^1\times C)_\bullet$.
- The sub-ESP ${D'}_{\infty,\bullet}^{(k)} := \nu_\bullet(D_{\infty,\bullet}^{(k-1)})$ is a section of $Z_\bullet^{(k)} \to \mathbb{S}_\bullet$ which corresponds to a surjection $\mathcal{O}_{\mathbb{S}}((a+1)s_\infty)\oplus\mathcal{O}_{\mathbb{S}} \twoheadrightarrow \mathcal{O}_{\mathbb{S}}$.

Then we have $(\Delta_R^{(k-1)}, \Delta_R^{(k-1)}, F^{(k-1)}) = 2m+a-1$ and $(\Delta_R^{(k-1)}, D_\infty^{(k-1)}, F^{(k-1)})$ $= m-1$ for a general fiber $F^{(k-1)}$ of $X^{(k-1)} \to C$. If $m=1$, then we have

$\Delta_R^{(k-1)} \cap D_\infty^{(k-1)} = \emptyset$ and thus we are reduced to the case as in Step 3. If $m > 1$, then by the same method as in Step 3, we can apply successions of elementary transformations along an elliptic curve m times and changing the ordering of a sequence of blow-ups of an ESP, we eventually obtain the following morphisms of ESPs:

$$X_\bullet \longrightarrow W_\bullet^{(k-m)} \longrightarrow \mathbb{S}_\bullet,$$

where

- $X_\bullet \to W_\bullet^{(k-m)}$ is a blow-up sequence of an ESP, and
- $W_\bullet^{(k-m)} \simeq \mathbb{P}(\mathcal{O}_\mathbb{S}((a+m)s_\infty) \oplus \mathcal{O}_\mathbb{S})_\bullet$ is a stable FESP (cf. Definition 2.9) of $X_\bullet$.
- $\Delta_{R,\bullet}^{(k-m)}$, which is the proper transform of $\Delta_{R,\bullet}$ on $W_\bullet^{(k-m)}$, is disjoint from the section $D_{\infty,\bullet}'^{(k-m)}$ corresponding to a surjection $\mathcal{O}_\mathbb{S}((a+m)s_\infty) \oplus \mathcal{O}_\mathbb{S} \twoheadrightarrow \mathcal{O}_\mathbb{S}$.

Thus we are reduced to the situation in Step 3. Then the finiteness of extremal rays of $\overline{\mathrm{NE}}(X)$ follows from Step 3. □

Remark 4.18. The same conclusion as in Theorem 4.12 also holds true in the case of $a = 1$ (cf. [8, Thm. 8.27]). In Steps 2 and 3 of the proof of Theorem 4.12, the case where $\Delta_{R,\bullet} = D_{\infty,\bullet}$ can occur, which means that $E_{R,\bullet} = D_{\infty,\bullet}$. To show the finiteness of extremal rays, we may assume that $\Delta_{R,\bullet} \neq D_{\infty,\bullet}$. Then completely the same proof as above works (cf. Remark 4.16). Furthermore, in the case of $a = 1$, there exists an extremal ray of divisorial type in the FESP $Y_\bullet$. Using this, in [8, Prop. 8.34] we studied the structure of a certain nonisomorphic étale endomorphism $f\colon X \to X$ admitting an FESP $Y_\bullet$ of type (D) (cf. [5, Def. 3.6]).

§4.2. Subcase (Atiyah.A)

Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. We consider the case where there exists an FESP $Y_\bullet$ of type (Atiyah.A) (cf. Definition 4.5). Let

$$X_\bullet \xrightarrow{\pi_\bullet^{(0)}} \cdots \longrightarrow X_\bullet^{(k-2)} \xrightarrow{\pi_\bullet^{(k-2)}} X_\bullet^{(k-1)} \xrightarrow{\pi_\bullet^{(k-1)}} X_\bullet^{(k)} =: Y_\bullet$$

be a sequence of blow-ups of an ESP along an elliptic curve $\gamma_\bullet^{(j)}$ on $X_\bullet^{(j)}$ such that $\pi_\bullet = \pi_\bullet^{(k-1)} \circ \cdots \circ \pi_\bullet^{(0)}$ (cf. [5, Cor. 1.2, Fig. 1, Def. 3.7]). First, we show that the center $\gamma_\bullet^{(k)}$ of the first blow-up $\pi_\bullet^{(k-1)}$ from the bottom is uniquely determined.

Lemma 4.19. *We have $\gamma_\bullet^{(k)} = \Gamma_\bullet$, where Γ is the intersection of $T \simeq \mathbb{S}$ (cf. the explanations just before Lemma 3.1) and D which is the section of $\varphi\colon Y \to \mathbb{S}$ corresponding to the surjection $\mathcal{E} \twoheadrightarrow \mathcal{O}_\mathbb{S}$.*

Proof. By construction, there exists a Cartesian morphism of ESPs $\varphi_\bullet \colon Y_\bullet \to \mathbb{S}_\bullet$. Furthermore, the center $\gamma_\bullet^{(k)}$ of the first blow-up of $\pi_\bullet \colon X_\bullet \to Y_\bullet$ from the bottom is a sub-ESP of $Y_\bullet$ and the inclusion $\gamma_\bullet^{(k)} \hookrightarrow Y_\bullet$ is Cartesian. Applying Lemma 2.1 and Proposition 2.8, we see that $\varphi_\bullet(\gamma_\bullet^{(k)})$ is a sub-ESP of $\mathbb{S}_\bullet$ and equals $s_{\infty,\bullet}$. There is induced an ESP $T_\bullet = (g_n|_{T_n} \colon T_n \to T_{n+1})_n$ of Atiyah surfaces $T_n := \varphi_n^{-1}(s_\infty) \simeq \mathbb{S}$ and the inclusion $\gamma_\bullet^{(k)} \hookrightarrow T_\bullet$ is Cartesian. Then Proposition 4.11 shows that $\gamma_\bullet^{(k)} = D_\bullet \cap T_\bullet \simeq s_{\infty,\bullet}$ and $\gamma_0^{(k)} = \Gamma$. □

Now we are ready to prove the following fundamental theorem.

Theorem 4.20. *Let $f \colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists an FESP $Y_\bullet$ of type (Atiyah.A) (cf. Definition 4.5). Then there exist at most finitely many extremal rays of $\overline{\mathrm{NE}}(X)$.*

Proof. We shall use almost the same notation as in the proof of Theorem 4.12, except that $D_\bullet$ denotes the ESP of only one canonical section with respect to the surjection $\mathcal{E} \twoheadrightarrow \mathcal{O}_{\mathbb{S}}$.

Step 1. By the same argument as in the proof of Theorem 4.12, we may assume the existence of an extremal ray R ($\subset \overline{\mathrm{NE}}(X)$) of divisorial type such that the composite

$$E_{R,\bullet} \hookrightarrow X_\bullet \xrightarrow{\pi_\bullet} Y_\bullet \xrightarrow{\varphi_\bullet} \mathbb{S}_\bullet$$

is an isomorphism.

Step 2. Applying the same argument as in the proof of Lemma 4.13, we easily see that $\Delta_{R,\bullet} \neq D_\bullet$. Furthermore, $\Delta_{R,\bullet} \cap D_\bullet \neq \emptyset$, since the exact sequence (♠) stated at the beginning of Section 3 does not split. We set $T_n := \varphi_n^{-1}(s_\infty)$. Let $\Gamma := T_0 \cap D$ be the complete intersection curve of T_0 and D. Then Γ is the canonical section of $T_0 \simeq \mathbb{S}$ and $D \simeq \mathbb{S}$.

By construction, we have the following Cartesian morphisms of ESPs:

$$\begin{array}{ccc} E_{R,\bullet} & \xrightarrow{i_\bullet} & X_\bullet \\ & & \downarrow \pi_\bullet \\ & & Y_\bullet \\ & & \downarrow \varphi_\bullet \\ & & \mathbb{S}_\bullet. \end{array}$$

By Lemma 2.1, we see that $\Delta_{R,\bullet} := \pi_\bullet(E_{R,\bullet})$ is a sub-ESP of $Y_\bullet$. Now we show that $\Delta_R \cap D = \Gamma$. By assumption, the composite $\varphi_\bullet \circ \pi_\bullet \circ i_\bullet \colon E_{R,\bullet} \to \mathbb{S}_\bullet$ is an isomorphism of ESPs consisting of Atiyah surfaces. By Lemma 2.24, the section of $\varphi_\bullet\colon Y_\bullet \to \mathbb{S}_\bullet$ forms another ESP $D_\bullet$ of Atiyah surfaces. Hence $\Delta_{R,\bullet} \cap D_\bullet$ is a sub-ESP of elliptic curves and thus equals an ESP $\Gamma_\bullet := T_\bullet \cap D_\bullet$ which is isomorphic to the ESP $s_{\infty,\bullet}$ of canonical sections (cf. Proposition 4.11).

We have $\Delta_R|_T \sim \Gamma$, since $\Delta_R := \pi_0(E_0)$ is a section of φ. Setting $\Delta_R|_D \sim m\Gamma$ ($m > 0$) and applying Lemma 4.17, we infer that $\Delta_R \sim D + (a+m)T$. Since we have $(\Delta_R^2, F) = 2m + a \geq 2$ for a general fiber F of $\alpha \circ \varphi_0\colon Y_0 \to C$, we see that $\pi_\bullet\colon X_\bullet \to Y_\bullet$ is not an isomorphism. Let

$$X_\bullet := X_\bullet^{(0)} \longrightarrow \cdots \longrightarrow X_\bullet^{(k-2)} \xrightarrow{\pi_\bullet^{(k-2)}} X_\bullet^{(k-1)} \xrightarrow{\pi_\bullet^{(k-1)}} Y_\bullet := X_\bullet^{(k)}$$

be a sequence of blow-ups of an ESP along elliptic curves. Let $\gamma_n^{(i)}$ be the center of the blow-up $\pi_n^{(i-1)}\colon X_n^{(i-1)} \to X_n^{(i)}$ and $\gamma_n := \gamma_n^{(k)}$ ($\subset Y_n$) the center of the first blow-up $\pi_n^{(k-1)}$ from the bottom. Then $\gamma_\bullet^{(i)} = (\gamma_n^{(i)})_n$ is a sub-ESP of $X_\bullet^{(i)}$ and there exist the following Cartesian morphisms of ESPs:

$$\begin{array}{ccc} \gamma_\bullet^{(i)} & \longrightarrow & X_\bullet^{(i)} \\ & & \Big\downarrow{\scriptstyle \pi_\bullet^{(k-1)} \circ \cdots \circ \pi_\bullet^{(i)}} \\ & & Y_\bullet. \end{array}$$

Then by Lemma 2.1 and Proposition 2.8, we see that $\pi_\bullet^{(k-1)} \circ \cdots \circ \pi_\bullet^{(i)}(\gamma_\bullet^{(i)})$ is a sub-ESP of $Y_\bullet$ and equals $\Gamma_\bullet$. In particular, by Lemma 4.19, we see that $X_\bullet^{(k-1)}$ is isomorphic to $\mathrm{Bl}_{\Gamma_\bullet}(Y_\bullet)$ which is the blow-up of $Y_\bullet$ along the elliptic curve $\Gamma_\bullet$ and $X_\bullet^{(k-1)} = (f_n^{(k-1)}\colon X_n^{(k-1)} \to X_{n+1}^{(k-1)})_n$ is a stable ESP isomorphic to $\mathrm{Bl}_{\Gamma_\bullet}(Y_\bullet)$. Now we shall perform elementary transformations on $Y_\bullet$ along elliptic curves $\Gamma_\bullet$ (cf. Definition 3.3). Let $D_n^{(k-1)} \simeq \mathbb{S}$ (resp. $\Delta_{R,n}^{(k-1)} \simeq \mathbb{S}$ and $T_n^{(k-1)} \simeq \mathbb{S}$) be the proper transform of D_n (resp. $\Delta_{R,n}$ and T_n) in $X_n^{(k-1)}$ and $E_n^{(k-1)} := \mathrm{Exc}(\pi_n^{(k-1)})$ ($\simeq C \times \mathbb{P}^1$) be the exceptional divisor of the first blow-up $\pi_n^{(k-1)}\colon X_n^{(k-1)} \to Y$ from the bottom of $\pi_\bullet\colon X_\bullet \to Y_\bullet$. Then, by construction, all of them form sub-ESPs of $X_\bullet^{(k-1)}$. We set

$$\begin{aligned} D_\bullet^{(k-1)} &:= (f_n^{(k-1)}\colon D_n^{(k-1)} \longrightarrow D_{n+1}^{(k-1)})_n, \\ \Delta_\bullet^{(k-1)} &:= (f_n^{(k-1)}\colon \Delta_n^{(k-1)} \longrightarrow \Delta_{n+1}^{(k-1)})_n, \\ T_\bullet^{(k-1)} &:= (f_n^{(k-1)}\colon T_n^{(k-1)} \longrightarrow T_{n+1}^{(k-1)})_n, \\ E_\bullet^{(k-1)} &:= (f_n^{(k-1)}|_{E_n^{(k-1)}}\colon E_n^{(k-1)} \longrightarrow E_{n+1}^{(k-1)})_n. \end{aligned}$$

There exists a birational morphism $\nu_n\colon X_n^{(k-1)} \to Z_n^{(k)}$ which contracts the divisor $T_n^{(k-1)} \simeq \mathbb{S}$ to an elliptic curve. Since ${E'}_n^{(k)} := \nu_n(E_n^{(k-1)}) \simeq C \times \mathbb{P}^1$, there exists an isomorphism $Z_n^{(k-1)} \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{O}_{\mathbb{S}}((a+1)s_\infty) \oplus \mathcal{O}_{\mathbb{S}})$ by Lemma 3.1. Thus we have obtained further Cartesian morphisms of ESPs,

$$X_\bullet = (X, f) \xrightarrow{\mu_\bullet} X_\bullet^{(k-1)} \xrightarrow{\nu_\bullet} Z_\bullet^{(k)} = (g_n : Z_n^{(k)} \to Z_{n+1}^{(k)})_n,$$

such that the following conditions are satisfied:

- $\mu_\bullet = (\mu_n)_n$ is a sequence of blow-ups of an ESP along elliptic curves.
- $Z_\bullet^{(k)}$ is another FESP of $X_\bullet$.
- $Z_\bullet^{(k)}$ is a stable ESP and is isomorphic to a $\mathbb{P}^1$-bundle $\mathbb{P}_{\mathbb{S}}(\mathcal{O}_{\mathbb{S}}((a+1)s_\infty) \oplus \mathcal{O}_{\mathbb{S}})_\bullet$ over $\mathbb{S}_\bullet$.
- $E'^{(k)}_\bullet := (g_n|_{E'^{(k)}_n} : E'^{(k)}_n \to E'^{(k)}_{n+1})_n$ is a sub-ESP of $Z_\bullet^{(k)}$ and is isomorphic to a stable ESP $(\mathbb{P}^1 \times C)_\bullet$.
- The sub-ESP $D'^{(k)}_\bullet := \nu_\bullet(D_\bullet^{(k-1)})$ is a section of $Z_\bullet^{(k)} \to \mathbb{S}_\bullet$ which corresponds to a surjection $\mathcal{O}_{\mathbb{S}}((a+1)s_\infty) \oplus \mathcal{O}_{\mathbb{S}} \to \mathcal{O}_{\mathbb{S}}$.

Hence, by Theorem 4.12, there exist at most finitely many extremal rays of $\overline{\mathrm{NE}}(X)$. □

Remark 4.21. In the proofs of Theorems 4.12 and 4.20, the *existence* of an extremal ray R ($\subset \overline{\mathrm{NE}}(X)$) of divisorial type such that $E_R := \mathrm{Exc}(\mathrm{Cont}_R)$ dominates $\mathbb{S}$ is not established. To show the finiteness of extremal rays R ($\subset \overline{\mathrm{NE}}(X)$), it is sufficient to show the finiteness of an extremal ray R of divisorial type such that E_R dominates $\mathbb{S}$ (cf. Step 1 in the proof of Theorems 4.12 and 4.20). Thus only the *finiteness* of such extremal rays is established. The proofs are done by changing the blow-down process of ESPs with the aid of *elementary transformations* to obtain another FESP $\pi'_\bullet : X_\bullet \to (C \times \mathbb{P}^1)_\bullet$, so as to apply Theorem 4.7. Thus it is not so easy to show the existence of an extremal ray R ($\subset \overline{\mathrm{NE}}(X)$) such that E_R dominates $\mathbb{S}$. On the other hand, in our previous article [8, Thm. 8.13], the finiteness of extremal rays is proved without using *elementary transformations* and the *existence* of such an extremal ray R of divisorial type so that E_R dominates $\mathbb{S}$ is also shown in certain cases (cf. [8, Thm. 10.2]). We shall briefly describe its outline.

Let $f : X \to X$ be a nonisomorphic étale endomorphism such that there exists an FESP $Y_\bullet \simeq \mathbb{S} \times_C \mathbb{S}$ for the Albanese map $\alpha_{\mathbb{S}} : \mathbb{S} \to C$. In other words, there exists an *unsplit* exact sequence of vector bundles on $\mathbb{S}$,

$$0 \longrightarrow \mathcal{O}_{\mathbb{S}} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{S}} \longrightarrow 0,$$

such that $Y \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{E})$. Let

$$X_\bullet \xrightarrow{\pi_\bullet} Y_\bullet \xrightarrow{p_{i,\bullet}} \mathbb{S}_\bullet$$

be the Cartesian morphisms of ESPs, where $\pi_\bullet$ is a succession of blow-downs to an FESP and $p_{i,\bullet}$ $(i = 1, 2)$ is a $\mathbb{P}^1$-bundle. Let $\mathbf{\Delta}$ $(\subset Y := \mathbb{S} \times_C \mathbb{S})$ be the diagonal divisor of Y. Then we have $\mathbf{\Delta} \simeq \mathbb{S}, \dim|\mathbf{\Delta}| = 1$ and the base locus $\mathrm{Bs}|\mathbf{\Delta}|$ of $|\mathbf{\Delta}|$ equals the complete intersection curve $p_1^{-1}(s_\infty) \cap p_2^{-1}(s_\infty)$ for $p_i^{-1}(s_\infty) \simeq \mathbb{S}$. Furthermore, all the members of $|\mathbf{\Delta}|$ can be separated after twice blowing up $\pi' \colon X^{(2)} \to Y$. Suppose that there exists an extremal ray R $(\subset \overline{\mathrm{NE}}(X))$ of divisorial type such that the exceptional divisor $E_R := \mathrm{Exc}(\mathrm{Cont}_R)$ is mapped onto $\mathbb{S}$ isomorphically by the morphism $X \xrightarrow{\pi} Y \xrightarrow{p_i} \mathbb{S}$ $(i = 1, 2)$. Then $\pi_\bullet \colon X_\bullet \to Y_\bullet$ is a succession of blow-ups along elliptic curves and the centers of the first and the second blow-up from the bottom are uniquely determined. If we set $G_{R,\bullet} := \pi_\bullet(E_{R,\bullet})$, then $G_{R,\bullet}$ is a sub-ESP of $Y_\bullet$ and the natural projection $G_{R,\bullet} \to \mathbb{S}_i$ $(i = 1, 2)$ is an isomorphism and G_R is a member of $|\mathbf{\Delta}|$. We note that E_R is the strict transform of G_R by $\pi \colon X \to Y$. Since $(G_R^2, \text{general fiber of } \alpha_Y \colon Y \to C) = 2$ and $(E_R^2, \text{general fiber of } \alpha_X \colon X \to C) = -1$, $\pi_\bullet \colon X_\bullet \to Y_\bullet$ factors through the blow-ups $\pi_{1,\bullet} \circ \pi_{2,\bullet} \colon X_\bullet^{(2)} \to Y_\bullet$ such that $\pi' = \pi_1 \circ \pi_2$. Thus the indeterminacy of the rational mapping $\Phi_{|\mathbf{\Delta}|} \colon Y \cdots \to \mathbb{P}^1$ is eliminated after twice blowing up $Y_\bullet$ to have a morphism $\Phi' \colon X^{(2)} \to \mathbb{P}^1$. Hence there exists another blow-down of an ESP $X_\bullet \to Y'_\bullet = (\mathbb{S} \times \mathbb{P}^1)_\bullet \to (C \times \mathbb{P}^1)_\bullet$. Then applying Theorem 4.7, the finiteness of extremal rays of $\overline{\mathrm{NE}}(X)$ is derived.

Conversely, let $\varphi \colon \mathbb{S} \to \mathbb{S}$ be a nonisomorphic étale endomorphism of the Atiyah surface $\mathbb{S}$. If we set $g := \varphi \times_C \varphi \colon Y \to Y$, then g is a nonisomorphic étale endomorphism of Y such that $g^{-1}(\mathbf{\Delta}) = \mathbf{\Delta}$ for the diagonal divisor $\mathbf{\Delta}$ $(\subset \mathbb{S} \times_C \mathbb{S})$. Furthermore, there exists a lift $g^{(2)} \colon X^{(2)} \to X^{(2)}$ of $g \colon Y \to Y$. Let $\mathbf{\Delta}^{(2)}$ $(\subset X^{(2)})$ be the proper transform of $\mathbf{\Delta}$. Then $(\mathbf{\Delta}^{(2)}, g^{(2)}|_{\mathbf{\Delta}^{(2)}})$ is a sub-ESP of $(X^{(2)}, g^{(2)})$ and $(g^{(2)})^{-1}(s_\infty) = s_\infty$ for the canonical section s_∞ $(\subset \mathbf{\Delta}^{(2)} \simeq \mathbb{S})$. Let $\pi^{(3)} \colon X^{(3)} := \mathrm{Bl}_{s_\infty}(X^{(2)})$ be the blow-up of $X^{(2)}$ along s_∞ $(\subset \mathbf{\Delta}^{(2)})$ and E_3 the $\pi^{(3)}$-exceptional divisor. Then there exists an extremal ray R $(\subset \overline{\mathrm{NE}}(X^{(3)}))$ such that

- $E_R = E_3$ and
- E_R is a section of $X^{(3)} \to Y \xrightarrow{p_i} \mathbb{S}$ for $i = 1, 2$.

4.2.1. Blow-up process in the Atiyah case. We summarize all the information we have obtained about the construction of an FESP in the *Atiyah case* (cf. Definition 1.3). With the aid of Theorems 4.12, 4.20 and [5, Prop. 3.8], the MMP works compatibly with étale endomorphisms. Replacing $f \colon X \to X$ by its suitable power f^k $(k > 0)$, we obtain Cartesian morphisms of constant ESPs

$$X_\bullet = (X, f) \xrightarrow{\pi} Y_\bullet = (Y, g) \xrightarrow{\varphi} \mathbb{S}_\bullet = (\mathbb{S}, u) \xrightarrow{\alpha} C_\bullet = (C, h)$$

which satisfy the following:

- $Y_\bullet$ is a constant FESP (cf. Definition 2.9) constructed from $X_\bullet$ by a sequence of equivariant blow-downs $\pi_\bullet$.
- There exists the exact sequence of vector bundles called "FES" (cf. Definition 2.25)

$$(\spadesuit)\colon 0 \longrightarrow \mathcal{O}_{\mathbb{S}}(a s_\infty) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{S}} \longrightarrow 0 \quad (a > 1)$$

  on the Atiyah surface $\mathbb{S}$ so that $\varphi\colon Y \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{E}) \to \mathbb{S}$ is a $\mathbb{P}^1$-bundle associated to $\mathcal{E}$.
- The composite map $\alpha \circ \varphi\colon Y \to C$ is an $\mathbb{F}_a$-bundle over the Albanese elliptic curve C of X.

We begin with an easy lemma.

Lemma 4.22. *Suppose that we are in the case (Atiyah.A), that is, the above exact sequence* ($\spadesuit$) *does not split (cf. Definition 4.5). For* $\eta \in C$, *let* $T_\eta\colon C \simeq C$ *be an automorphism of* C *defined by* $\zeta \mapsto \zeta + \eta$ *under the group law of* C. *Let* $\alpha_Y : Y \to C$ *be the Albanese map of* Y. *Then for any* $\eta \in C$, *there exists an automorphism* $u_\eta\colon Y \simeq Y$ *such that* $\alpha_Y \circ u_\eta = T_\eta \circ \alpha_Y$.

Proof. It follows by [15, 16] that there exists an automorphism $v_\eta \in \mathrm{Aut}^0(\mathbb{S})$ such that $\alpha_{\mathbb{S}} \circ v_\eta = T_\eta \circ \alpha_{\mathbb{S}}$ for the Albanese map $\alpha_{\mathbb{S}}\colon \mathbb{S} \to C$. Pulling back ($\spadesuit$) by v_η, we obtain the following exact sequence:

$$(\spadesuit')\colon 0 \longrightarrow v_\eta^*\mathcal{O}_{\mathbb{S}}(a s_\infty) \longrightarrow v_\eta^*\mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{S}} \longrightarrow 0.$$

Since $v_\eta^*\mathcal{O}_{\mathbb{S}}(s_\infty) \simeq \mathcal{O}_{\mathbb{S}}(s_\infty)$ by Proposition 2.4, we infer that

$$\begin{aligned}\mathrm{Ext}^1(\mathcal{O}_{\mathbb{S}}, v_\eta^*\mathcal{O}_{\mathbb{S}}(a s_\infty)) \simeq H^1(\mathbb{S}, v_\eta^*\mathcal{O}_{\mathbb{S}}(a s_\infty)) &\simeq H^1(\mathbb{S}, \mathcal{O}_{\mathbb{S}}(a s_\infty)) \\ &\simeq \mathrm{Ext}^1(\mathcal{O}_{\mathbb{S}}, \mathcal{O}_{\mathbb{S}}(a s_\infty)).\end{aligned}$$

Thus $v_\eta^*\colon \mathrm{Ext}^1(\mathcal{O}_{\mathbb{S}}, \mathcal{O}_{\mathbb{S}}(a s_\infty)) \to \mathrm{Ext}^1(\mathcal{O}_{\mathbb{S}}, \mathcal{O}_{\mathbb{S}}(a s_\infty))$ is an isomorphism and $v_\eta^*\mathcal{E} \simeq \mathcal{E}$. If we set $\widetilde{Y} := Y \times_{v_\eta, \mathbb{S}} \mathbb{S}$, then there exists an isomorphism $\widetilde{Y} \simeq \mathbb{P}_{\mathbb{S}}(v_\eta^*\mathcal{E}) \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{E}) =: Y$. Thus the composite map $Y \simeq \widetilde{Y} \to Y$ gives an automorphism $u_\eta\colon Y \simeq Y$ such that $\alpha_Y \circ u_\eta = T_\eta \circ \alpha_Y$. □

Lemma 4.23. *Let* $V_\bullet \hookrightarrow Y_\bullet$ *be any sub-ESP of elliptic curves. Then* $V_\bullet = \Gamma_\bullet$, *where* $\Gamma_\bullet := D_\bullet \cap T_\bullet$ *(cf. the notation of Lemma 4.19). In particular,* $V_\bullet$ *is a constant sub-ESP of* $Y_\bullet$, *Proof.* Applying Lemma 2.1 to the Cartesian morphisms of ESPs

$$\begin{array}{ccc} V_\bullet & \xrightarrow{i_\bullet} & Y_\bullet \\ & & \downarrow{\scriptstyle \varphi_\bullet} \\ & & \mathbb{S}_\bullet, \end{array}$$

it follows that $\varphi_\bullet(V_\bullet)$ is an ESP of elliptic curves such that the inclusion $\varphi_\bullet(V_\bullet) \hookrightarrow \mathbb{S}_\bullet$ is Cartesian. Hence Proposition 4.11(4) shows that $\varphi_\bullet(V_\bullet) = s_{\infty,\bullet}$. Thus V_n is contained in $T := \varphi^{-1}(s_\infty) \simeq \mathbb{S}$. Hence, again by Proposition 4.11(4), we see that $V_n = \Gamma_n$ for any n. □

Now we recall a structure theorem of the automorphism groups of compact Kähler manifolds due to Fujiki [3].

Theorem 4.24 ([3, p. 251, Thm. 5.5]). *Let X be a compact Kähler manifold. Then we have the following:*

(1) *$G := \mathrm{Aut}^0(X)$ has a natural structure of a meromorphic group which acts biregularly and meromorphically on X.*

(2) *There exists an exact sequence of complex Lie groups*

$$1 \longrightarrow L(G) \longrightarrow G \xrightarrow{\alpha} T(G) \longrightarrow 1,$$

where $T(G)$ is the Albanese torus of the compactification G^* of G, $\alpha\colon G \to T(G)$ is the Albanese homomorphism of G and the kernel $L(G)$ of α (which is called the linear part of G) is meromorphically isomorphic to a linear algebraic group.

(3) *Let $T(X) = \mathrm{Alb}(\mathrm{Aut}^0(X))$ be the Albanese torus of $\mathrm{Aut}^0(X)$ and $\phi\colon X \to \mathrm{Alb}(X)$ the Albanese map of X. Then the natural homomorphism $\phi_*\colon G \to A(X) := \mathrm{Aut}^0(\mathrm{Alb}(X))$ is meromorphic and factors through the Albanese homomorphism α, i.e., there exists a unique homomorphism $h_*\colon T(X) \to A(X)$ such that $\phi_* = h_* \circ \alpha$. Furthermore, the kernel of h_* is a finite group.*

Using Proposition 4.25 which is an application of Theorem 4.24, we shall prove that the Albanese map $\alpha_X\colon X \to C$ is an analytic fiber bundle in the case where there exists an FESP $Y_\bullet$ of *Atiyah type* (cf. Definition 1.3).

Proposition 4.25 ([8, Prop. 8.19]). *Let $\alpha_X\colon X \to A := \mathrm{Alb}(X)$ be the Albanese map of a smooth projective variety X and set $G := \mathrm{Aut}^0(X)$. Suppose the following conditions:*

- *α_X is a fiber space.*

- *The induced group homomorphism* $(\alpha_X)_*\colon G \to \mathrm{Aut}^0(A) \simeq A$ *is surjective.*

Then α_X *is an analytic fiber bundle. Furthermore, let* D *be a* G*-stable subvariety (which may be singular) of* X *admitting a surjective morphism* $p\colon D \to V$ *with connected fibers onto a lower-dimensional manifold* V *with* $b_1(V) = 0$ *and* $\dim V > 0$*. If* Γ *is a smooth irreducible fiber of* p*, then the following hold:*

(1) $\alpha_X(\Gamma) = A$.

(2) *Let* $\pi\colon Z := \mathrm{Bl}_\Gamma(X) \to X$ *be the blow-up of* X *along* Γ*. Then the algebraic group homomorphism* $(\alpha_Z)_*\colon \mathrm{Aut}^0(Z) \to \mathrm{Aut}^0(A) \simeq A$ *induced by the Albanese map* $\alpha_Z\colon Z \to A$ *of* Z *is also surjective. In particular,* α_Z *is an analytic fiber bundle.*

Now we recall the following result concerning automorphism groups of the Atiyah surface $\mathbb{S}$.

Lemma 4.26 (Cf. [16]). *Let* $\mathbb{S}$ *be the Atiyah surface and* s_∞ *its canonical section. We set* $G := \mathrm{Aut}^0(\mathbb{S})$*. Then we have the following:*

- *There exists an unsplit exact sequence of algebraic groups,*

$$1 \longrightarrow \mathbb{G}_a \longrightarrow G \xrightarrow{\varphi} C \longrightarrow 1,$$

where $\varphi\colon G \to \mathrm{Aut}^0(C) \simeq C$ *is a group homomorphism induced by the universality of the Albanese map of* $\mathbb{S}$ *(cf. Theorem 4.24). That is,* G *is a nontrivial extension of an elliptic curve* C *by the additive group* $\mathbb{G}_a \simeq \mathbb{C}$*. Moreover,* G *is commutative.*

- G *preserves* s_∞ *and acts transitively on its complement* $\mathbb{S}^0 := \mathbb{S} \setminus s_\infty$*. In particular,* $\mathbb{S}$ *is an almost homogeneous variety.*

Theorem 4.27. *Let* $f\colon X \to X$ *be a nonisomorphic étale endomorphism of a smooth projective* 3*-fold* X *with* $\kappa(X) = -\infty$*. Let* $(Y_\bullet, R_\bullet)$ *be an FESP of type* $(\mathrm{C}_{-\infty})$ *constructed from* $X_\bullet = (X, f)$ *by a sequence of blow-downs of an ESP. Suppose that* $Y_\bullet$ *is of* Atiyah type *(cf. Definition 1.3). Then the Albanese map* $\alpha_X\colon X \to C$ *is an analytic fiber bundle over the Albanese elliptic curve* C *whose fiber is a blow-up of the Hirzebruch surface* $\mathbb{F}_a$*.*

Proof. Hereafter, we shall give a proof in the case where there exists an FESP $Y_\bullet$ of type (Atiyah.A) (cf. Definition 4.5).

Step 1 (Some reduction). By Theorem 4.20, we can construct a constant FESP $Y_\bullet = (Y, g)$ of $X_\bullet$. There exists a sequence of blow-downs of an ESP

$$X_\bullet = (X, f) \longrightarrow \cdots \longrightarrow X_\bullet^{(i-1)} \xrightarrow{\pi^{(i-1)}} X_\bullet^{(i)} \longrightarrow \cdots \longrightarrow Y_\bullet,$$

such that the following conditions are satisfied for each i:

- $X_\bullet^{(i)} = (X^{(i)}, f^{(i)})$ is a constant ESP induced from a nonisomorphic étale endomorphism $f^{(i)}\colon X^{(i)} \to X^{(i)}$.
- $\pi^{(i-1)}\colon X_\bullet^{(i-1)} \to X_\bullet^{(i)}$ is the blow-up along an elliptic curve $C^{(i)}$ on $X^{(i)}$, where we set $X^{(0)} := X$ and $X^{(k)} := Y$.
- ${f^{(i)}}^{-1}(C^{(i)}) = C^{(i)}$.

Then applying Lemma 2.1 to the Cartesian morphism

$$C_\bullet^{(i)} \hookrightarrow X_\bullet^{(i)} \xrightarrow{\pi_\bullet^{(k-1)}\circ\cdots\circ\pi_\bullet^{(i)}} X_\bullet^{(k)} =: Y_\bullet,$$

we see that the image $\gamma_\bullet^{(i)}$ of $C_\bullet^{(i)}$ in $Y_\bullet$ is an ESP of elliptic curves such that the inclusion $\gamma_\bullet^{(i)} \hookrightarrow Y_\bullet$ is Cartesian. Hence Lemma 4.23 shows that $\gamma_\bullet^{(i)} = \Gamma_\bullet$ and any $C^{(i)}$ dominates $\Gamma \simeq C$. The exceptional divisor $E^{(i-1)} := \mathrm{Exc}(\pi^{(i-1)})$ of $\pi^{(i-1)}$ is isomorphic to either $\mathbb{S}$ or $\mathbb{P}_{C_i}(\mathcal{O}\oplus\ell^{(i)})$, where $\ell^{(i)} \in \mathrm{Pic}^0(C^{(i)})$ is a torsion line bundle by Theorems 4.9 and 4.10. Let $\alpha^{(i)}\colon X^{(i)} \to C$ be the Albanese map of $X^{(i)}$. Then by the universality of the Albanese map, there exists an endomorphism $h\colon C \to C$ such that $\alpha_X \circ f = h \circ \alpha_X$. By the same reason as in the proof of Theorem 4.12(Step 1), we have $\deg h = \deg f > 1$ and h has a fixed point $o \in C$. Hence, if C is endowed with the group structure so that $o \in C$ is the zero element, then we may assume that h is a group homomorphism. We can choose a multiplication map $\mu_n\colon C \to C$ by an integer $n > 0$ such that μ_n is factored through $\alpha^{(i)}\colon X^{(i)} \to C$ for any i. Then $\mu_n \circ h = \mu_n \circ h$. Hence, if we set $\widetilde{X} := X \times_{C,h} C$, then $f\colon X \to X$ can be lifted to a nonisomorphic étale endomorphism $\tilde{f}\colon \widetilde{X} \to \widetilde{X}$. Since the property that "the Albanese map α_X gives an analytic fiber bundle structure" is preserved by étale base change, without loss of generality we may assume the following as in the proof of Theorem 4.12:

(1) Each elliptic curve $C^{(i)}$ is a section of the Albanese map $\alpha^{(i)} := \alpha_Y \circ \pi^{(k-1)} \circ \cdots \circ \pi^{(i)}\colon X^{(i)} \to C$.

(2) Each $E^{(i)} := \mathrm{Exc}(\pi^{(i)})$ is isomorphic to either $\mathbb{S}$ or $C \times \mathbb{P}^1$.

Step 2. Let $(\alpha^{(i)})_*\colon \mathrm{Aut}^0(X^{(i)}) \to \mathrm{Aut}^0(C) \simeq C$ be the induced algebraic group homomorphism. Note that the elliptic curve C acts on itself transitively by translations. Now we recall the following lemma which gives a sufficient condition for a fiber space to be an analytic fiber bundle whose proof is done with the aid of [2].

Lemma 4.28 (Cf. [8, Lem. 4.8]). *Let $\varphi\colon X \to Y$ be a fiber space of smooth projective varieties X and Y with $0 < \dim Y < \dim X$. Suppose that Y is a homogeneous variety and the induced algebraic group homomorphism $\varphi_*\colon \mathrm{Aut}^0(X) \to \mathrm{Aut}^0(Y)$ is surjective. Then φ is an analytic fiber bundle.*

Hence, applying Lemma 4.28, it is sufficient to show the surjectivity of the group homomorphism $(\alpha_X)_*\colon \mathrm{Aut}^0(X) \to \mathrm{Aut}^0(C) \simeq C$. This follows immediately from the following assertions for the case of $i = 0$.

Lemma 4.29. *For the induced algebraic group homomorphism* $(\alpha^{(i)})_*\colon \mathrm{Aut}^0(X^{(i)}) \to C$, *we have the following:* $(\mathbf{Surj})_i\colon (\alpha^{(i)})_*$ *is surjective.*

Proof. The proof of the assertion is by a descending induction on i. For the case of $i = k$, the assertion is trivial by Lemma 4.22. Next suppose that the assertion holds true for i and set $u_i := \pi^{(k-1)} \circ \cdots \circ \pi^{(i)}\colon X^{(i)} \to Y$. Note that by the construction of FESP, the following hold (cf. Proposition 4.2 and Theorem 4.9):

- The exceptional locus $\mathrm{Exc}(u_i)$ is a simple normal crossing divisor.
- Any irreducible component of $\mathrm{Exc}(u_i)$ is isomorphic to either $\mathbb{S}$ or $C \times \mathbb{P}^1$ and is $\mathrm{Aut}^0(X^{(i)})$-stable.
- The blow-up center $C^{(i)}$ of $\pi^{(i-1)}$ is contained in some irreducible component Δ_i of $\mathrm{Exc}(u_i)$ for $i < k$, and $C^{(k)}$, which is the first blow-up center from the bottom, equals Γ $(:= T \cap D)$ (cf. Lemma 4.19).

If $\Delta_i \simeq \mathbb{S}$, then $C^{(i)} = s_\infty$. Applying Proposition 2.5, we infer that $C^{(i)}$ is $\mathrm{Aut}^0(X^{(i)})$-invariant. Hence, by [8, Prop. 4.7], the induced injective algebraic group homomorphism $\pi_*^{(i-1)}\colon \mathrm{Aut}^0(X^{(i-1)}) \to \mathrm{Aut}^0(X^{(i)})$ is an isomorphism. Thus the composite map $(\alpha^{(i-1)})_* = (\alpha^{(i)})_* \circ \pi_*^{(i-1)}\colon \mathrm{Aut}^0(X^{(i-1)}) \to C$ is also surjective and the assertion holds for $i - 1$.

Suppose that $\Delta_i \simeq C \times \mathbb{P}^1$. Then $C^{(i)}$ is some fiber of the second projection $p_2\colon C \times \mathbb{P}^1 \to \mathbb{P}^1$. Then by Proposition 4.25, the induced algebraic group homomorphism $\alpha_*^{(i-1)}$ is also surjective and the assertion holds for $i - 1$. □

The proof in the case where $Y_\bullet$ is of type (Atiyah.B) (cf. Definition 4.5) is completely the same, so we omit it. □

Remark 4.30. We note that the linear part $L(G)$ of $G := \mathrm{Aut}^0(Y)$ is nontrivial. The morphism $\varphi\colon Y \to \mathbb{S}$ induces a group homomorphism $\psi_*\colon G \to \mathrm{Aut}^0(\mathbb{S})$. Since s_∞ $(\subset \mathbb{S})$ is stabilized by $\psi_*(G)$, ψ_* induces an automorphism group of $T := \psi^{-1}(s_\infty) \simeq \mathbb{S}$. Suppose that $L(G)$ is trivial. Then by Theorem 4.24, we see that G is an elliptic curve. Thus there is induced a nontrivial group homomorphism $\psi_*\colon G \to \mathrm{Aut}^0(T) \simeq \mathrm{Aut}^0(\mathbb{S})$, which contradicts Lemma 4.26.

Proof of Theorem 1.4. It follows immediately from [5, Prop. 3.8], Proposition 4.3 and Theorems 4.12, 4.20 and 4.27. □

§5. Torsion case

Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists an FESP $F_\bullet$ of *torsion type* (cf. Definition 1.3). There exist the following Cartesian morphisms of ESPs:

$$X_\bullet = (X, f) \xrightarrow{\pi_\bullet} Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n \xrightarrow{\varphi_\bullet} S_\bullet = (u_n\colon S_n \to S_{n+1})_n \xrightarrow{\alpha_\bullet} C_\bullet = (C, h)$$

such that any S_n is isomorphic to $\mathbb{P}_C(\mathcal{O}_C \oplus \ell_n)$ for a torsion line bundle $\ell_n \in \mathrm{Pic}^0(C)$ and $S_0 \simeq C \times \mathbb{P}^1$. Let $p_1\colon S_0 \to C$ (resp. $p_2\colon S_0 \to \mathbb{P}^1$) be the first (resp. the second) projection and s a general fiber of p_2. Then there exist a divisor D on C and the following exact sequence of sheaves on S_0:

$$(\diamondsuit)\colon 0 \longrightarrow p_2^*\mathcal{O}_{\mathbb{P}^1}(a) \otimes p_1^*\mathcal{O}_C(D) \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{O}_{S_0} \longrightarrow 0$$

such that $Y_0 = \mathbb{P}_{S_0}(\mathcal{E}_0)$ for a rank-two vector bundle $\mathcal{E}_0$ on S_0 (cf. Theorem 2.13). The following proposition is crucial.

Proposition 5.1. *The line bundle* $[D] \in \mathrm{Pic}(C)$ *is of finite order.*

Proof. Note that by Proposition 2.8, the fibration $p_2 \circ \varphi_0 \circ \pi_0\colon X \to \mathbb{P}^1$ is smooth over a nonempty Zariski open subset T of $\mathbb{P}^1$ and its smooth fiber X_t over a point $t \in T$ is isomorphic to a $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{E}_0|_{\{t\}\times C})$ over C.

Suppose to the contrary that $[D] \in \mathrm{Pic}(C)$ is of infinite order. Then $X_t \simeq \mathbb{P}_C(\mathcal{O}_C(D) \oplus \mathcal{O}_C)$ for $t \in T$. If $\deg(D) = 0$ and $[D]$ is of infinite order, then by Proposition 2.6, the surface X_t is mapped to an irreducible curve on S_0 by the morphism $\varphi_0 \circ \pi_0 \circ f\colon X \to S_0$. If $\deg(D) \neq 0$, then there exists no surjective morphism from $\mathbb{P}_C(\mathcal{O}_C(D) \oplus \mathcal{O}_C)$ to S_0 by considering the induced Mori cone isomorphism similar to the argument in the proof (2) of Proposition 4.6. Thus, applying the rigidity lemma (cf. [14, Lem. 1.6]), we see that any fiber of $\varphi_0 \circ \pi_0\colon X \to S_0$ is mapped to a point by the morphism $\varphi_0 \circ \pi_0 \circ f\colon X \to S_0$. Hence there exists a surjective endomorphism $v\colon S_0 \to S_0$ so that the following commutative diagram holds:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle \varphi_0\circ\pi_0}\downarrow & & \downarrow{\scriptstyle \varphi_0\circ\pi_0} \\ S_0 & \xrightarrow{v} & S_0 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ C & \xrightarrow[h]{} & C. \end{array}$$

Then we have $\deg f = \deg v$ (resp. $\deg f = \deg h$), since f is étale and each fiber of $\varphi_0 \circ \pi_0$ (resp. $p_1 \circ \varphi_0 \circ \pi_0$) is simply connected and hence has fiber degree one. Thus $\deg v = \deg h$ and v has degree one when restricted to a fiber of p_1 so that $v|_{p_1^{-1}(t)}$ is an isomorphism for any $t \in C$. Since C is complete and $\mathrm{Aut}(\mathbb{P}^1)$ is affine, the morphism $\Phi\colon C \to \mathrm{Aut}(\mathbb{P}^1)$, $t \mapsto v|_{p_1^{-1}(t)}$ is a constant map. Thus $v = h \times u$ for a unique automorphism $u\colon \mathbb{P}^1 \simeq \mathbb{P}^1$. Hence, for a point $t \in T$, the restriction of $f\colon X \to X$ to X_t induces a nonisomorphic finite étale covering $f|_{X_t}\colon X_t \to X_{u(t)}$. Since there exists another isomorphism $X_t \simeq X_{u(t)} \simeq \mathbb{P}_C(\mathcal{O}_C(D) \oplus \mathcal{O}_C)$, we can regard $f|_{X_t}$ as a nonisomorphic étale endomorphism of X_t. Thus, by Proposition 2.3, we see that $[D] \in \mathrm{Pic}(C)$ is of finite order, which contradicts the assumption. □

Then Proposition 5.1 shows that $e := \mathrm{ord}([D]) < \infty$ and let $\mu_e\colon C \to C$ be a multiplication mapping by a positive integer e. Further replacing X by its finite étale Galois covering $\widetilde{X} := X \times_{C,\mu_e} C$ of X and $f\colon X \to X$ by its lift $\tilde{f}\colon \widetilde{X} \to \widetilde{X}$, we may assume from the beginning that $D = 0$. Hence the exact sequence ($\diamondsuit$) just before Proposition 5.1 can be reduced to the following exact sequence of sheaves on $S_0 \simeq C \times \mathbb{P}^1$:

$$(\bigstar)\colon 0 \longrightarrow p_2^*\mathcal{O}_{\mathbb{P}^1}(a) \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{O}_{S_0} \longrightarrow 0.$$

Hereafter, for simplicity, we set $Y := Y_0$ and $S := S_0$. By the composite map $\psi\colon Y \xrightarrow{\varphi_0} S \xrightarrow{p_2} \mathbb{P}^1$, we regard $Y = \mathbb{P}_S(\mathcal{E}_0)$ as a fiber space over $\mathbb{P}^1$. For each $t \in \mathbb{P}^1$, let Y_t be the fiber of ψ over t. Then by [5, Prop. 7.8] and Proposition 2.8, we see that X is constructed from Y by a succession of blow-ups along an elliptic curve which belongs to Y_t for some $t \in \mathbb{P}^1$ and dominates C. Hence there exists a Zariski open subset $(\mathbb{P}^1)^0$ of $\mathbb{P}^1$ such that $\pi_0|_{X_t}\colon X_t \simeq Y_t$ is an isomorphism for all $t \in (\mathbb{P}^1)^0$.

Lemma 5.2. *For any $t \in \mathbb{P}^1$, Y_t is isomorphic to either an Atiyah surface $\mathbb{S}$ or the product $C \times \mathbb{P}^1$.*

Proof. The restriction of ($\bigstar$) to $(p_2)^{-1}(t)$ gives the following exact sequence of sheaves:

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E}_0|_{(p_2)^{-1}(t)} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

such that $Y_t \simeq \mathbb{P}(\mathcal{E}_0|_{(p_2)^{-1}(t)})$. Hence, if the above exact sequence unsplits (resp. splits), then $Y_t \simeq \mathbb{S}$ (resp. $Y_t \simeq C \times \mathbb{P}^1$). □

By Ω ($\subset \mathbb{P}^1$), we denote the set of points $t \in \mathbb{P}^1$ such that Y_t is isomorphic to the Atiyah surface $\mathbb{S}$. By construction, if $t \in \mathbb{P}^1 \setminus \Omega$, then $Y_t \simeq C \times \mathbb{P}^1$. Applying the same method as in the proof of [6, Lem. 2.3], we can show the following.

Lemma 5.3. *The set Ω is a Zariski open subset of $\mathbb{P}^1$.*

Proposition 5.4. *Under the assumption in* (★), *suppose that* $\Omega \neq \emptyset$. *Then* $S_n \simeq C \times \mathbb{P}^1$ *for any* n, *that is,* $S_\bullet$ *is a stable ESP.*

Proof. For any $t \in (\mathbb{P}^1)^0$, there exists an isomorphism $\pi|_{X_t} \colon X_t \simeq Y_t \simeq \mathbb{S}$. Since $S_n \simeq \mathbb{P}_C(\mathcal{O}_C \oplus \ell_n)$ for $\ell_n \in \mathrm{Pic}^0(C)$, Proposition 2.6 shows that the image of X_t by the composite map $\varphi_n \circ \pi_n \colon X \to S_n$ is a curve on S_n. Hence some fiber of $X_t \to C$ is mapped to a point by $\varphi_n \circ \pi_n \colon X \to S_n$. Applying the "rigidity lemma" (cf. [14, Lem. 1.6]) to the equidimensional morphism $\varphi_0 \circ \pi_0 \colon X \to S$, we see that any fiber of $\varphi_0 \circ \pi_0$ is mapped to a point by $\varphi_n \circ \pi_n \colon X \to S_n$. Hence there exists a surjective morphism $v_n \colon S_0 \to S_n$ such that $v_n \circ \varphi_0 \circ \pi_0 = \varphi_n \circ \pi_n$. Since $\rho(S_0) = \rho(S_n) = 2$, v_n is a finite morphism. Furthermore, since both fibers of $\varphi_0 \circ \pi_0$ and $\varphi_n \circ \pi_n$ are connected, we have $\deg(v_n) = 1$. Thus v_n is an isomorphism by Zariski's main theorem. Hence $S_n \simeq C \times \mathbb{P}^1$ for any n. □

Hence our situation is divided into two cases.

- Case (Torsion.A): If $\Omega \neq \emptyset$, then $Y \to \mathbb{P}^1$ is a smooth morphism whose general fiber Y_t is isomorphic to $\mathbb{S}$. We can show that $\Omega \subsetneq \mathbb{P}^1$ (cf. Proposition 5.11).
- Case (Torsion.B): If $\Omega = \emptyset$, then $Y \to \mathbb{P}^1$ is a fiber bundle whose fiber is isomorphic to $C \times \mathbb{P}^1$.

Definition 5.5. Let $f \colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists an FESP $Y_\bullet$ of torsion type (cf. Definition 1.3). We set $Y := Y_0$ and let Y_t be a fiber of a smooth morphism $Y \to \mathbb{P}^1$ over a point $t \in \mathbb{P}^1$. Let Ω denote the set of points $t \in \mathbb{P}^1$ such that Y_t is isomorphic to the Atiyah surface $\mathbb{S}$. Then we say that

- $Y_\bullet$ is of type (Torsion.A) if $\Omega \neq \emptyset$.
- $Y_\bullet$ is of type (Torsion.B) if $\Omega = \emptyset$.

§5.1. Classifications in the case (Torsion.B)

In this section, we shall give the structure theorem of a nonisomorphic étale endomorphism $f \colon X \to X$ which has an FESP $Y_\bullet$ of type (Torsion.B).

Theorem 5.6. *Let* $f \colon X \to X$ *be a nonisomorphic étale endomorphism of a smooth projective* 3*-fold* X *with* $\kappa(X) = -\infty$. *Let* $Y_\bullet$ *be an FESP constructed from* $X_\bullet = (X, f)$ *by a sequence of blow-downs of an ESP. Suppose that* $Y_\bullet = (g_n \colon Y_n \to Y_{n+1})_n$ *is of type (Torsion.B) (cf. Definition 5.5). Then the following hold:*

(1) Y_0 *is isomorphic to the product* $T \times C$ *of an elliptic curve* C *and a Hirzebruch surface* $T \simeq \mathbb{F}_a$ $(a \geq 0)$.

(2) *There exists an isomorphism* $X \simeq W \times C$*, where* $W \to T$ *is a birational morphism.*

(3) $f = u \times h$ *for an automorphism* $u\colon W \simeq W$ *and an isomorphic étale endomorphism* $h\colon C \to C$*.*

Theorem 5.6 can be proved by a similar method to the proof of [6, Prop. 3.1] and [8, Thm. 6.8], though it needs slight modifications. It requires new arguments to seek the candidate for the blow-up centers of $\pi_\bullet\colon X_\bullet \to Y_\bullet$. The trouble is that in the proof of Theorem 5.6, it is not at all clear whether $\operatorname{ord} \ell_n$ ($\in \operatorname{Pic}(C)$) is bounded above by a positive constant which is independent of n.

Lemma 5.7. *Let* $f\colon X \to X$ *be a nonisomorphic étale endomorphism of a smooth projective 3-fold* X *with* $\kappa(X) = -\infty$*. Let* $Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n$ *be an FESP constructed from* $X_\bullet := (X, f)$ *by a sequence of blow-downs of an ESP. Suppose that* $Y_\bullet$ *is of type (Torsion.B). Then there exists an isomorphism* $Y_0 \simeq T \times C$ *of an elliptic curve* C *and a Hirzebruch surface* $T \simeq \mathbb{F}_a$*.*

Proof. We set $Y := Y_0$. Since all the fibers Y_t of ψ over $t \in \mathbb{P}^1$ are isomorphic to $C \times \mathbb{P}^1$, $\psi\colon Y \to \mathbb{P}^1$ is a holomorphic fiber bundle over $\mathbb{P}^1$ by the theorem of Fischer–Grauert [2]. Since the relative anti-canonical bundle $-K_{Y/\mathbb{P}^1}$ is ψ-free, there is induced an elliptic fibration

$$\alpha\colon Y \twoheadrightarrow T \subset \mathbb{P}_{\mathbb{P}^1}(\psi_*(-K_{Y/\mathbb{P}^1}))$$

over a $\mathbb{P}^1$-bundle $q\colon T \to \mathbb{P}^1$ so that $\psi = q \circ \alpha$. This elliptic fibration α is an elliptic fiber bundle, since for all $t \in \mathbb{P}^1$, $\alpha|_{Y_t}\colon Y_t \to \mathbb{P}^1$ is a trivial elliptic bundle over $\mathbb{P}^1$. For the composite map $\varphi'\colon Y \xrightarrow{\varphi} S = C \times \mathbb{P}^1 \xrightarrow{p_1} C$, let us consider the canonical morphism $\Psi := \alpha \times \varphi'\colon Y \to T \times C$. Then by construction, Ψ is an isomorphism, since Ψ is of degree one when restricted to each fiber of ψ. Furthermore, since the composite map $Y \xrightarrow{\Psi} T \times C \xrightarrow{p_2} C$ is the Albanese map of Y, we see that $T \simeq \mathbb{F}_a$. Thus the proof is finished. □

Proof of Theorem 5.6. In the case where $a = 0$ (that is, $Y \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times C$), the assertion has been proved in our previous article [8, Thm. 8.11]. Hence, hereafter we assume that $a > 1$. By construction, there exists the following Cartesian morphism of ESPs:

$$X_\bullet = (X, f) \xrightarrow{\pi_\bullet} Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n \xrightarrow{\varphi_\bullet} S_\bullet = (u_n\colon S_n \to S_{n+1})_n$$
$$\xrightarrow{\alpha_\bullet} C_\bullet = (C, h)$$

such that the following conditions are satisfied:

- $\pi_\bullet$ is a sequence of blow-ups along elliptic curves.

- Any S_n is isomorphic to $\mathbb{P}_C(\mathcal{O}_C \oplus \ell_n)$ for a torsion line bundle $\ell_n \in \mathrm{Pic}^0(C)$ and $S_0 \simeq C \times \mathbb{P}^1$.

We shall describe the centers of a sequence of blow-ups $\pi_\bullet$. There exist an ESP $Y_\bullet^{(j)} = (g_n^{(j)} : Y_n^{(j)} \to Y_{n+1}^{(j)})_n$ for $0 \le j \le k$, Cartesian morphisms $\nu_\bullet^{(j)} = (\nu_n^{(j)})_n : Y_\bullet^{(j-1)} \to Y_\bullet^{(j)}$ for $1 \le j \le k$ and the Cartesian diagram of ESPs,

$$\begin{array}{ccccccccc} X_\bullet & \xrightarrow{\nu_\bullet^{(1)}} & Y_\bullet^{(1)} & \longrightarrow \cdots & Y_\bullet^{(j-1)} & \xrightarrow{\nu_\bullet^{(j)}} & Y_\bullet^{(j)} & \cdots \longrightarrow & Y_\bullet \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ E_\bullet^{(0)} & \longrightarrow & E_\bullet^{(1)} & \longrightarrow \cdots & E_\bullet^{(j-1)} & \longrightarrow & E_\bullet^{(j)} & \cdots \longrightarrow & E_\bullet^{(k)}, \end{array}$$

such that the following conditions are satisfied:

- $(g_n^{(j)})^{-1}(E_{n+1}^{(j)}) = E_n^{(j)}$, where $g_n^{(k)} := g$, $g_n^{(0)} = f$, that is, $E_\bullet^{(j)} = (g_n^{(j)}|_{E_n^{(j)}} : E_n^{(j)} \to E_{n+1}^{(j)})_n$ is a sub-ESP of elliptic curves on $Y_\bullet^{(j)}$.
- $\nu_\bullet^{(j)} : Y_\bullet^{(j-1)} \to Y_\bullet^{(j)}$ is (the inverse of) the blow-up along $E_\bullet^{(j)}$ on $Y_\bullet^{(j)}$, where $Y_\bullet^{(0)} := X_\bullet$ and $Y_\bullet^{(k)} := Y_\bullet$.

In what follows, for simplicity, we omit the subscript "0" and set $Y^{(i)} := Y_0^{(i)}, \nu^{(i)} := \nu_0^{(i)}$, $S^{(i)} := S_0^{(i)}$ and $E^{(i)} := E_0^{(i)}$. We shall prove the following assertions.

Assertions. For any $0 \le i \le k$,

$(1)_i$: $Y^{(i)}$ is isomorphic to the direct product $T^{(i)} \times C$ over $T^{(i)}$, where $T^{(i)} \to \mathbb{F}_a$ is a birational morphism;

$(2)_i$: for any sub-ESP $\Delta_\bullet^{(i)}$ $(\hookrightarrow Y_\bullet^{(i)})$ of elliptic curves, $\Delta^{(i)} := \Delta_0^{(i)}$ is some fiber of the first projection $p_i : Y^{(i)} \to T^{(i)}$;

$(2)'_i$: the blow-up center $E^{(i)}$ of $\nu^{(i)} : Y^{(i-1)} \to Y^{(i)}$ is some fiber of the first projection $p_i : Y^{(i)} \to T^{(i)}$.

Proof. We note that $(2)'_i$ follows from $(2)_i$, since $E_\bullet^{(i)}$ is an ESP of elliptic curves. The proof is done by a descending induction on i. For $i = k$, $(1)_k$ follows immediately from Lemma 5.7. Next we shall prove the $(2)_k$. Applying Lemma 2.1 to the Cartesian morphisms of ESPs,

$$\begin{array}{ccc} \Delta_\bullet^{(k)} & \xrightarrow{i_\bullet} & Y_\bullet \\ & & \downarrow{\scriptstyle \varphi_\bullet} \\ & & S_\bullet^{(k)}, \end{array}$$

it follows that $\varphi_\bullet(\Delta_\bullet^{(k)})$ is an ESP of elliptic curves such that the inclusion $\varphi_\bullet(\Delta_\bullet^{(k)}) \hookrightarrow S_\bullet^{(k)}$ is Cartesian. Applying Proposition 4.11, we see that for any n, $\varphi_n(\Delta_n^{(k)})$ is an elliptic curve on $S_n^{(k)}$ whose self-intersection number equals zero. Since $S_0^{(k)} \simeq C \times \mathbb{P}^1$, $\varphi_0(\Delta_0^{(k)})$ is some fiber of the second projection $p_2\colon S_0^{(k)} \simeq C \times \mathbb{P}^1 \to \mathbb{P}^1$. Hence the proof of Lemma 5.7 shows assertion $(2)_k$.

Next suppose that the assertions hold true for $i+1$. Assertion $(1)_i$ follows immediately from $(1)_{i+1}$ and $(2)'_{i+1}$. Applying Lemma 2.1 to the Cartesian morphisms of ESPs

$$\Delta_\bullet^{(i)} \hookrightarrow Y_\bullet^{(i)} \xrightarrow{\nu_\bullet^{(i+1)}} Y_\bullet^{(i+1)},$$

we see that $\nu_\bullet^{(i+1)}(\Delta_\bullet^{(i)})$ is an ESP of elliptic curves such that the inclusion $\nu_\bullet^{(i+1)}(\Delta_\bullet^{(i)}) \hookrightarrow Y_\bullet^{(i+1)}$ is Cartesian. Hence, by $(2)_{i+1}$, $\nu_0^{(i+1)}(\Delta_0^{(i)})$ is some fiber of the first projection $p_{i+1}\colon Y_0^{(i+1)} \to T_0^{(j+1)}$. By $(2)'_{i+1}$, we have either of the following two cases:

(a) $\nu^{(i+1)}(\Delta^{(i)}) \cap E^{(i+1)} = \emptyset$, or
(b) $\nu^{(i+1)}(\Delta^{(i)}) = E^{(i+1)}$.

In case (a), $(2)_i$ is clear by construction. In case (b), $\Delta_\bullet^{(i)}$ is a sub-ESP of $D_\bullet^{(i)} := \operatorname{Exc}(\nu_\bullet^{(i+1)})$ which is an ESP of elliptic ruled surfaces. Since $D^{(i)} := D_0^{(i)} \simeq C \times \mathbb{P}^1$ and the self-intersection number $(\Delta^{(i)})^2$ in $D^{(i)}$ equals 0 (cf. [5, Prop. 6.9]), $\Delta^{(i)}$ is contained in some fiber of the second projection $D^{(i)} \to \mathbb{P}^1$. Thus assertion $(2)_i$ has been proved. □

Theorem 5.6 follows immediately from assertion $(1)_0$. □

Remark 5.8. Let $u\colon S \to \mathbb{P}^1$ be a relatively minimal rational elliptic surface with a section whose Mordell–Weil rank is positive. Then S contains infinitely many (-1)-curves. If we consider S as an elliptic curve over the function field $\mathbb{C}(\mathbb{P}^1)$ of $\mathbb{P}^1$, then there exists a relative automorphism g over $\mathbb{P}^1$ with infinite order which is induced by a holomorphic section of u with infinite order. If we set $X := S \times C$ for an elliptic curve C, then there exist infinitely many K_X-negative extremal rays. Since S is obtained from $\mathbb{P}^2$ by blow-up 9-points, S is an 8-points blow-up of $\mathbb{F}_2$. Let $\mu_n\colon C \to C$ be a multiplication mapping by a positive integer $n > 1$ and $f := g \times \mu_n$ a nonisomorphic étale endomorphism of X. Hence, if we set $Y_0 := C \times \mathbb{F}_2 \to C \times \mathbb{P}^1$, then there is induced an FESP $Y_\bullet$ of $X_\bullet := (X, f)$ which is of type (Torsion.B).

§5.2. Classifications in the case (Torsion.A)

Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Here we shall study the structure of X in the case where there exists an FESP $Y_\bullet$ of type (Torsion.A) (cf. Definition 5.5). In this case, we can show that the MMP works compatibly with étale endomorphisms and we can obtain a constant FESP (cf. Definition 2.9) of $X_\bullet = (X, f)$.

Proposition 5.9. *Suppose that $\pi_0 \colon X \to Y_0$ is not an isomorphism. Let X_s be a fiber of $p_2 \circ \varphi \circ \pi_0 \colon X \to \mathbb{P}^1$ over a point $s \in \mathbb{P}^1$. Suppose that $X_s \simeq \mathbb{S}$ for some $s \in (\mathbb{P}^1)^0$. Then there exists an extremal ray R ($\subset \overline{\mathrm{NE}}(X)$) of divisorial type such that $(f^k)_* R = R$ for some integer $k > 0$. Furthermore, replacing f by a suitable power f^k ($k > 0$), there exist Cartesian morphisms of constant ESPs*

$$X_\bullet = (X, f) \xrightarrow{\pi} Y_\bullet = (Y, g) \xrightarrow{\varphi} S_\bullet = (S, v)$$

such that the following hold:

- *$Y_\bullet$ is a constant FESP (cf. Definition 2.9) and $X_\bullet$ is constructed from $Y_\bullet$ by a sequence of equivariant blow-ups $\pi_\bullet = \pi \colon X \to Y$ along elliptic curves.*
- *$v = h \times u$ for some automorphism u of $\mathbb{P}^1$ and a nonisomorphic group homomorphism $h \colon C \to C$ of an elliptic curve C.*

Proof. By Proposition 2.6, we see that $\varphi_0 \circ \pi_0 \circ f(X_s)$ is a curve on S_0. Hence, by the rigidity lemma (cf. [14, Lem. 1.6]), some fiber of $\varphi_0 \circ \pi_0 \colon X_s \to C$ is mapped to a point on S_0 by the morphism $\varphi_0 \circ \pi_0 \circ f \colon X_s \to C$. Since $\varphi_0 \circ \pi_0 \colon X \to S_0$ is equidimensional, again by the rigidity lemma (cf. [14, Lem. 1.6]), any fiber of $\varphi_0 \circ \pi_0$ is mapped to a point by $\varphi_0 \circ \pi_0 \circ f \colon X \to S_0$. Hence there exists a surjective endomorphism $v \colon S_0 \to S_0$ such that $\varphi_0 \circ \pi_0 \circ f = v \circ \varphi_0 \circ \pi_0$. Thus we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle \varphi_0 \circ \pi_0}\downarrow & & \downarrow{\scriptstyle \varphi_0 \circ \pi_0} \\ S_0 & \xrightarrow{v} & S_0 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ C & \xrightarrow[h]{} & C. \end{array}$$

Then, since f is étale, we have $\deg f = \deg v = \deg h$ and v can be expressed as $v = h \times u$ for some automorphism u of $\mathbb{P}^1$ as in the proof of Proposition 5.1.

By construction, the $\mathbb{P}^1$-fiber space $\varphi_0 \colon \pi_0 \colon X \to S_0$ is smooth over S_0 outside $B := p_2^{-1}(\mathbb{P}^1 \setminus (\mathbb{P}^1)^0)$ consisting of finitely many fibers of the second projection $p_2 \colon S_0 \to \mathbb{P}^1$ and the blow-up centers of π lie over B (cf. Proposition 2.8(3)). Then an étale endomorphism $f \colon X \to X$ induces a permutation between the finite set M consisting of all the irreducible components of $(\varphi_0 \circ \pi_0)^{-1}(B)$. Hence, replacing

f by a suitable power f^k ($k > 0$), we may assume that $f^{-1}(\sigma) = \sigma$ for all $\sigma \in M$. Let R ($\subset \overline{\mathrm{NE}}(X)$) be an extremal ray of divisorial type such that $\varphi_0 \circ \pi_0(\mathrm{Exc}(\pi_R))$ is contained in B. Since $\mathrm{Exc}(\pi_R) \in M$, we have $f_*R = R$. Then applying the same argument as in the proof of [5, Prop. 3.8], the MMP works compatibly with étale endomorphisms and we obtain Cartesian morphisms of *constant* ESPs which satisfies the desired property. □

Remark 5.10. In Corollary 7.5, using Proposition 5.9, we can strengthen our statement: "There exist only finitely many extremal rays R ($\subset \overline{\mathrm{NE}}(X)$) of divisorial type. In particular, there exists some positive integer k such that $(f^k)_*R = R$ for any extremal ray R." (Cf. Remark 7.7.)

Next we shall describe the structure of the constant FESP $Y_\bullet$ in terms of *FES* (cf. Definition 2.25).

Proposition 5.11. *Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists an FESP $Y'_\bullet$ of type (Torsion.A) (cf. Definition 5.5). Let*

$$X_\bullet = (X, f) \xrightarrow{\pi} Y_\bullet = (Y, g) \xrightarrow{\varphi} S_\bullet = (S, v)$$

be Cartesian morphisms of constant ESPs obtained after replacing f by its suitable power f^k ($k > 0$) as in Proposition 5.9. Let $p_2\colon S = C \times \mathbb{P}^1 \to \mathbb{P}^1$ be the second projection. Then there exist an integer $a > 1$ and an unsplit exact sequence of vector bundles called "FES" (cf. Definition 2.25) on S,

$$0 \longrightarrow p_2^*\mathcal{O}_{\mathbb{P}^1}(a) \longrightarrow \mathcal{E} \xrightarrow{q} \mathcal{O}_S \longrightarrow 0 \tag{5.1}$$

which satisfies the following properties:

(i) *The surjection q corresponds to the section D of φ.*
(ii) *There exists an isomorphism $Y \simeq \mathbb{P}_S(\mathcal{E})$ over $S = C \times \mathbb{P}^1$.*
(iii) *$v^*\mathcal{E} \simeq \mathcal{E}$ and the extension class $\eta \in \mathrm{Ext}^1(\mathcal{O}_S, p_2^*\mathcal{O}_{\mathbb{P}^1}(a))$ of (5.1) is preserved by $v\colon S \to S$ up to a scalar.*
(iv) *$p_2 \circ \varphi\colon Y \to \mathbb{P}^1$ is a smooth morphism but is not a fiber bundle, and a jumping phenomenon occurs: There exists a nonempty Zariski open subset M ($\subsetneq \mathbb{P}^1$) of $\mathbb{P}^1$ such that*

$$Y_t \simeq \begin{cases} \mathbb{S}, & t \in M, \\ C \times \mathbb{P}^1, & t \notin M, \end{cases}$$

where $Y_t := (p_2 \circ \varphi)^{-1}(t)$ is a fiber over a point $t \in \mathbb{P}^1$.

Proof. First we note that $\deg h = \deg f > 1$, since f is étale and each fiber of $p_1 \circ \varphi \circ \pi\colon X \to C$ is simply connected so that the fiber degree of f equals one. Since C is an elliptic curve, h has a fixed point $o \in C$. If C is endowed with a group structure so that $o \in C$ is the zero element, then we may assume that $h\colon C \to C$ is a group homomorphism. Since u is a nonisomorphic étale endomorphism of $S = C \times \mathbb{P}^1$, we have $u = h \times u$ for some $u \in \operatorname{Aut} \mathbb{P}^1$.

By assumption, assertions (i) and (ii) hold and Proposition 5.1 shows the existence of the above exact sequence (5.1). If we set $S_t := p_2^{-1}(t)$ for $t \in C$, then the restriction of (5.1) to $S_t \simeq C$ gives the following exact sequence:

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E}|_{S_t} \longrightarrow \mathcal{O}_C \longrightarrow 0. \tag{5.2}$$

Since $Y_t \simeq \mathbb{P}_C(\mathcal{E}|_{S_t})$, we see that

$$\mathcal{E}|_{S_t} \simeq \begin{cases} \mathcal{F}_2 & \text{if (5.2) unsplits,} \\ \mathcal{O}_C \oplus \mathcal{O}_C & \text{if (5.2) splits.} \end{cases}$$

If (5.1) splits, then $Y_t \simeq \mathbb{P}^1 \times C$ for any $t \in \mathbb{P}^1$, which contradicts our assumption. Hence (5.1) unsplits. Let M $(\subset \mathbb{P}^1)$ be a nonempty Zariski open subset such that S_t is isomorphic to the Atiyah surface $\mathbb{S}$ for any $t \in M$. Then $s_{\infty,t} := S_t \cap D$ $(\subset D \simeq \mathbb{S})$ is the canonical section of $\mathbb{S}$. By Proposition 2.5, we see that $g^{-1}(s_{\infty,u(t)}) = s_{\infty,t}$ for any $t \in M$. Hence we have $g^{-1}(D) = D$, since $\overline{\bigcup_{t\in M} s_{\infty,t}} = D$. The former assertion in (iii) follows immediately from this fact.

By the Künneth formula, there exist isomorphisms

$$\operatorname{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_S, p_2^*\mathcal{O}_{\mathbb{P}^1}(a)) \simeq H^1(S, p_2^*\mathcal{O}_{\mathbb{P}^1}(a)) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)) \otimes H^1(C, \mathcal{O}_C), \tag{5.3}$$

since $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)) = 0$ for $a > 1$. Hence $\operatorname{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_S, p_2^*\mathcal{O}_{\mathbb{P}^1}(a)) \neq 0$ and let $0 \neq \eta \in \operatorname{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_S, p_2^*\mathcal{O}_{\mathbb{P}^1}(a))$ be the nonzero extension class of (5.1). Then we have $\eta = \eta_1 \otimes \eta_2$ for some nonzero $\eta_1 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ and $\eta_2 \in H^1(C, \mathcal{O}_C)$. Note that the group homomorphism $h\colon C \to C$ acts on the vector space $H^1(C, \mathcal{O}_C)$ as a multiplication by a nonzero constant $\mu_2 \in \mathbb{C}^\times$. Hence $u^*\eta_2 = \mu_2\eta_2$. Furthermore, since the extension class η is preserved by $v = h \times u$ $(u \in \operatorname{Aut}(\mathbb{P}^1))$, we have $h^*\eta_1 = \mu_1\eta_1$ for some nonzero constant μ_1. Thus the latter assertion in (iii) has been proved.

Let Λ $(\neq \emptyset)$ be the finite set of $\mathbb{P}^1$ on which η_1 vanishes and set $M := \mathbb{P}^1 \setminus \Lambda$. Then

$$Y_t \simeq \begin{cases} \mathbb{S}, & t \in M, \\ C \times \mathbb{P}^1, & t \notin M. \end{cases}$$

Thus assertion (iv) has been proved and we are done. □

The automorphism group $\operatorname{Aut}^0(Y)$ is not linear, as the next lemma shows.

Lemma 5.12. *We pose the same assumption as in Proposition 5.11. For $\eta \in C$, let $T_\eta\colon C \simeq C$ be an automorphism of C defined by $\zeta \mapsto \zeta + \eta$ under the group law of C. Let $\alpha_Y : Y \to C$ be the Albanese map of Y. Then for any $\eta \in C$, there exists an automorphism $u_\eta\colon Y \simeq Y$ such that $\alpha_Y \circ u_\eta = T_\eta \circ \alpha_Y$.*

Proof. The proof is done by a similar method to the proof of [6, Cor. 4.3]. Pulling back the exact sequence (4) by $v_\eta := T_\eta \times \mathrm{id}_{\mathbb{P}^1}$, we obtain the following exact sequence:

$$0 \longrightarrow p_2^*\mathcal{O}_{\mathbb{P}}(a) \longrightarrow v_\eta^*\mathcal{E} \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

Let $0 \neq \mu \in \mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_S, p_2^*\mathcal{O}_{\mathbb{P}^1}(a))$ be the nonzero extension class of (5.1). Since there exist isomorphisms

$$\text{(5.4)}\quad \mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_S, p_2^*\mathcal{O}_{\mathbb{P}^1}(a)) \simeq H^1(S, p_2^*\mathcal{O}_{\mathbb{P}^1}(a)) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)) \otimes H^1(C, \mathcal{O}_C),$$

we have $\mu = \mu_1 \otimes \mu_2$ for some nonzero $\mu_1 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ and $\mu_2 \in H^1(C, \mathcal{O}_C)$. Note that the translation $T_\eta\colon C \to C$ acts trivially on the vector space $H^1(C, \mathcal{O}_C)$. Hence $v_\eta^*\mu = \mu$. Hence $v_\eta^*\mathcal{E} \simeq \mathcal{E}$. If we set $\widetilde{Y} := Y \times_{v_\eta, S} S$, then there exists an isomorphism $\widetilde{Y} \simeq \mathbb{P}_S(v_\eta^*\mathcal{E}) \simeq \mathbb{P}_S(\mathcal{E}) =: Y$. Thus the composite map $Y \simeq \widetilde{Y} \to Y$ gives an automorphism $u_\eta\colon Y \simeq Y$ such that $\alpha_Y \circ u_\eta = T_\eta \circ \alpha_Y$. □

The next lemma describes the blow-up centers of $\pi_\bullet\colon X_\bullet \to Y_\bullet$.

Lemma 5.13. *Let $V_\bullet \hookrightarrow Y_\bullet$ be any sub-ESP of elliptic curves. Then any V_n is contained in Y_t for some $t \in \mathbb{P}^1$ and dominates C isomorphically.*

Proof. Applying Lemma 2.1 to the Cartesian morphisms of ESPs,

$$\begin{array}{ccc} V_\bullet & \xrightarrow{i_\bullet} & Y_\bullet \\ & & \downarrow{\scriptstyle \varphi_\bullet} \\ & & (C \times \mathbb{P}^1)_\bullet, \end{array}$$

it follows that $\varphi_\bullet(V_\bullet)$ is an ESP of elliptic curves such that the inclusion $\varphi_\bullet(V_\bullet) \hookrightarrow (C \times \mathbb{P}^1)_\bullet$ is Cartesian. Hence Proposition 4.11 shows that $\varphi_\bullet(V_\bullet)$ is a fiber of the second projection $p_2\colon (C \times \mathbb{P}^1)_\bullet \to \mathbb{P}^1$. Thus V_n is contained in Y_t, where Y_t is a fiber of $Y \to \mathbb{P}^1$ over $t \in \mathbb{P}^1$. Furthermore, Proposition 5.11 shows that any Y_t is isomorphic to either $\mathbb{S}$ or $\mathbb{P}^1 \times C$. Hence, applying Proposition 4.11, if $Y_t \simeq \mathbb{S}$ (resp. $Y_t \simeq \mathbb{P}^1 \times C$), then V_n is isomorphic to the canonical section s_∞ and $V_n = D_\infty \cap Y_t$ (resp. V_n is a fiber of the projection $Y_t \to \mathbb{P}^1$). In both cases, any V_n dominates C isomorphically. □

Now we shall state one of our main results.

Theorem 5.14. *Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Let $Y'_\bullet$ be an FESP constructed from $X_\bullet = (X, f)$ by a sequence of blow-downs of an ESP. Suppose that $Y'_\bullet$ is of type (Torsion.A). Then replacing $f\colon X \to X$ by its suitable power f^k ($k > 0$), we obtain further Cartesian morphisms of constant ESPs:*

$$X_\bullet \xrightarrow{\pi} Y_\bullet = (Y, g) \xrightarrow{\varphi} S_\bullet = (S, v) \xrightarrow{p_1} C_\bullet = (C, h)$$

which satisfy the following:

(1) *$Y_\bullet$ is a constant FESP (cf. Definition 2.9) constructed from $X_\bullet$ by a sequence of equivariant blow-downs π.*
(2) *$S \simeq C \times \mathbb{P}^1$ and $v = h \times u$ for an automorphism u of $\mathbb{P}^1$.*
(3) *There exists an unsplit exact sequence of vector bundles on S,*

$$0 \longrightarrow p_2^*\mathcal{O}_{\mathbb{P}^1}(a) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_S \longrightarrow 0,$$

for an integer $a > 1$ such that $Y = \mathbb{P}_S(\mathcal{E})$.
(4) *$p_2 \circ \varphi\colon Y \to \mathbb{P}^1$ is a smooth morphism but is not a fiber bundle, and a jumping phenomenon occurs: There exists a nonempty Zariski open subset M ($\subsetneq \mathbb{P}^1$) of $\mathbb{P}^1$ such that*

$$Y_t \simeq \begin{cases} \mathbb{S}, & t \in M, \\ C \times \mathbb{P}^1, & t \notin M, \end{cases}$$

where $Y_t := (p_2 \circ \varphi)^{-1}(t)$ is a fiber over a point $t \in \mathbb{P}^1$.
(5) *By the Albanese map $\alpha_X = p_1 \circ \varphi \circ \pi\colon X \to C$, X is a fiber bundle over C whose fiber is birational to the Hirzebruch surface $\mathbb{F}_a$.*

Proof. Assertions (1), (2), (3) and (4) follow immediately from Propositions 5.9 and 5.11. Applying Lemmas 5.12, 5.13 and a similar argument to Step 2 in the proof of Theorem 4.27, we see that the group homomorphism $(\alpha_X)_*\colon \mathrm{Aut}^0(X) \to \mathrm{Aut}^0(C) \simeq C$ induced by the Albanese map $\alpha_X\colon X \to C$ is surjective. Hence Lemma 4.28 shows that $\alpha_X\colon X \to C$ is an analytic fiber bundle. Thus assertion (5) is derived. □

Applying elementary transformations (cf. Definition 3.3) to $\mathbb{S} \times \mathbb{P}^1 \to \mathbb{P}^1$ successively, we can construct an example stated as in Theorem 5.14(3).

Remark 5.15. Now we shall apply Theorem 3.2 to the $\mathbb{P}^1$-bundle over $S := C \times \mathbb{P}^1$. We take the following *unsplit* exact sequence of vector bundles on S:

$$(\mathbf{E}_a)\colon 0 \longrightarrow p_2^*\mathcal{O}_{\mathbb{P}^1}(a) \longrightarrow \mathcal{E}_a \longrightarrow \mathcal{O}_S \longrightarrow 0,$$

where $a \geq 0$. Let D_a be a section of $\varphi_a\colon Y_a := \mathbb{P}_S(\mathcal{E}_a) \to S$ corresponding to the surjection $\mathcal{E}_a \twoheadrightarrow \mathcal{O}_S$. Let $\mu_n\colon C \to C$ be a multiplication mapping by an integer $n > 1$. Then $\rho := \mu_n \times \mathrm{id}_{\mathbb{P}^1}$ is a nonisomorphic étale endomorphism of S. Since $\rho^*\mathcal{E}_a \simeq \mathcal{E}_a$, there exists a nonisomorphic étale endomorphism $g_a\colon Y_a \to Y_a$. We choose a general point $t \in \mathbb{P}^1$ such that $(Y_a)_t \simeq \mathbb{S}$ and let $(\Gamma_a)_t := (Y_a)_t \cap D_a$ be the canonical section of $(Y_a)_t \simeq \mathbb{S}$. Let $\delta_a\colon \mathcal{E}_a \twoheadrightarrow \mathcal{O}_{(\Gamma_a)_t}$ be the composite of $\mathcal{E}_a \twoheadrightarrow \mathcal{O}_S$ and the canonical homomorphism $\mathcal{O}_S \twoheadrightarrow \mathcal{O}_{(\Gamma_a)_t}$. We set $\mathcal{E}'_a := \mathrm{Ker}(\delta_a)$. Then $\mathcal{E}'_a$ is a vector bundle of rank two on S and we have the following exact and commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & p_2^*\mathcal{O}_{\mathbb{P}^1}(a) & \longrightarrow & \mathcal{E}_a & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0 \\
 & & \mathrm{id}\big\uparrow & & \big\uparrow & & \big\uparrow & & \\
0 & \longrightarrow & p_2^*\mathcal{O}_{\mathbb{P}^1}(a) & \longrightarrow & \mathcal{E}'_a & \longrightarrow & \mathcal{O}_S(-(\Gamma_a)_t) & \longrightarrow & 0.
\end{array}$$

We set $Y_{a+1} := \mathbb{P}_S(\mathcal{E}'_a)$. Then by Theorem 3.2, we see that Y_{a+1} is obtained from Y_a by performing elementary transformations along $(\Gamma_a)_t$, i.e., $Y_{a+1} = \mathrm{elm}_{(\Gamma_a)_t}(Y_a)$. Since $g_a^{-1}((\Gamma_a)_t) = (\Gamma_a)_t$, there is induced a nonisomorphic étale endomorphism $g_{a+1}\colon Y_{a+1} \to Y_{a+1}$. If we set $\mathcal{E}_{a+1} := \mathcal{E}'_a \otimes \mathcal{O}_S((\Gamma_a)_t)$, then there exists the following unsplit exact sequence of vector bundles on S:

$$(\mathbf{E}_{a+1})\colon 0 \longrightarrow p_2^*\mathcal{O}_{\mathbb{P}^1}(a+1) \longrightarrow \mathcal{E}_{a+1} \longrightarrow \mathcal{O}_S \longrightarrow 0,$$

such that $Y_{a+1} \simeq \mathbb{P}_S(\mathcal{E}_{a+1})$. First, if we start from $a = 0$, then there exists the unsplit exact sequence of vector bundles on S

$$(\mathbf{E}_0)\colon 0 \longrightarrow p_2^*\mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{O}_S \longrightarrow 0,$$

such that $Y_0 := \mathbb{P}(\mathcal{E}_0) \simeq \mathbb{S} \times \mathbb{P}^1$ and $(Y_0)_t \simeq \mathbb{S}$ for any $t \in \mathbb{P}^1$. Now, for an integer $k > 0$, we choose distinct k points $t_i \in \mathbb{P}^1$ $(1 \leq i \leq k)$ arbitrarily and perform elementary transformations successively on Y_0 along the canonical section $(s_\infty)_i$ $(\subset (Y_0)_{t_i})$ as explained above. Then we obtain the following *unsplit* exact sequence of vector bundles on S:

$$(\mathbf{E}_k)\colon 0 \longrightarrow p_2^*\mathcal{O}_{\mathbb{P}^1}(k) \longrightarrow \mathcal{E}_k \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

If we set $Y_k := \mathbb{P}_S(\mathcal{E}_k)$, then the composite $\varphi_k\colon Y_k \to S \simeq C \times \mathbb{P}^1 \to \mathbb{P}^1$ is a smooth morphism but not a fiber bundle. By construction, we have

$$(Y_k)_t \simeq \begin{cases} \mathbb{S}, & t \notin \mathbb{P}^1 \setminus \{t_1, \ldots, t_k\}, \\ C \times \mathbb{P}^1, & t = t_i, \end{cases}$$

for $(Y_k)_t := \varphi_k^{-1}(t)$, $t \in \mathbb{P}^1$. The case where $k = 1$ was considered in [5, Rem. 8.3].

Remark 5.16. In our previous article (cf. [8, Thms 1.2 and 1.3]), we classified a smooth projective 3-fold X admitting a nonisomorphic étale endomorphism $f\colon X \to X$ in the case where there exists an FESP of type (D) (cf. [5, Def. 3.6(2)]). In this case, after taking a finite étale covering and performing divisorial contractions, we can reduce to the case where an FESP $Y_\bullet$ is a $\mathbb{P}^2$-bundle or $\mathbb{P}^1\times\mathbb{P}^1$-bundle over an elliptic curve C. Hereafter we assume that $\pi_\bullet\colon X_\bullet \to Y_\bullet$ is not an isomorphism and take a sequence of blow-downs to an FESP $Y_\bullet$:

$$X_\bullet := (X, f) = X_\bullet^{(0)} \xrightarrow{\pi_{0,\bullet}} X_\bullet^{(1)} \longrightarrow \cdots \longrightarrow X_\bullet^{(k-1)} \xrightarrow{\pi_{k-1,\bullet}} X_\bullet^{(k)} := Y_\bullet.$$

(1) In the case where $Y_\bullet \to C_\bullet$ is a $\mathbb{P}^2$-bundle, then the FESP $Y_\bullet$ can be classified into three types:

- If $Y \simeq C \times \mathbb{P}^1 \times \mathbb{P}^1$, then X is isomorphic to the product of a rational surface and an elliptic curve C.
- If $Y_\bullet \to C_\bullet$ is a $\mathbb{P}^2$-bundle such that $Y_\bullet \simeq \mathbb{P}_C(\mathcal{F}_2 \oplus \mathcal{O}_C)$, or $Y_\bullet \simeq \mathbb{P}_C(\mathcal{F}_3)$, then by the first blow-up from the bottom $\pi_{k-1,\bullet}\colon X_\bullet^{(k-1)} \to Y_\bullet$, $X_\bullet^{(k-1)}$ is a $\mathbb{P}^1$-bundle over the ESP of Atiyah surfaces $\mathbb{S}_\bullet$ or $(C \times \mathbb{P}^1)_\bullet$. Furthermore, $X_\bullet^{(k-1)}$ can be described by a rank-two vector bundle on $\mathbb{S}$ (i.e., of *Atiyah type*) or $C \times \mathbb{P}^1$ (i.e., of *torsion type*).
  - If $Y_\bullet$ is of Atiyah type, then there exists the following exact sequence of vector bundles on the Atiyah surface $\mathbb{S}$:

    $$0 \longrightarrow \mathcal{O}_{\mathbb{S}}(s_\infty) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{S}} \longrightarrow 0$$

    such that $X^{(k-1)} \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{E})$. This corresponds to the case with $a = 1$ in the exact sequence (♠) defined at the beginning of Section 3.
  - If $Y_\bullet$ is of torsion type, then there exists the following exact sequence of vector bundles on $C \times \mathbb{P}^1$:

    $$0 \longrightarrow p_2^*\mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow \mathcal{E}' \longrightarrow \mathcal{O}_{C\times\mathbb{P}^1} \longrightarrow 0$$

    such that $X^{(k-1)} \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{E})$. This corresponds to the case with $a = 1$ in the exact sequence (3) defined in Proposition 5.14.

(2) If $Y_\bullet \to C_\bullet$ is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle, then after taking a finite étale covering, we can reduce to the case where Y is isomorphic to one of the following: $C \times \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{S} \times C$ or $\mathbb{S} \times_C \mathbb{S}$.

- If $Y \simeq C \times \mathbb{P}^1 \times \mathbb{P}^1$, then X is isomorphic to the direct product of a rational surface and an elliptic curve C.

- If $Y \simeq \mathbb{S} \times_C \mathbb{S}$, then Y can be described by rank-two vector bundles on $\mathbb{S}$. There exists the following *unsplit* exact sequence of sheaves:

$$0 \longrightarrow \mathcal{O}_{\mathbb{S}} \longrightarrow \mathcal{E}'' \longrightarrow \mathcal{O}_{\mathbb{S}} \longrightarrow 0$$

  such that $Y \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{E}'')$. This corresponds to the case with $a = 0$ in the exact sequence (♠) defined at the beginning of Section 3.
- If $Y \simeq \mathbb{S} \times \mathbb{P}^1$, then Y can be described by rank-two vector bundles on $C \times \mathbb{P}^1$. There exists the following exact *unsplit* sequence of sheaves:

$$0 \longrightarrow p_2^* \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{E}''' \longrightarrow \mathcal{O}_{C \times \mathbb{P}^1} \longrightarrow 0$$

  such that $Y \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{E}''')$. This corresponds to the case with $a = 0$ in the exact sequence (3) defined in Proposition 5.14.

In this article, the classifications of a smooth projective 3-fold X admitting a nonisomorphic étale endomorphism and whose FESP is of type ($\mathrm{C}_{-\infty}$) are done under the assumption that $a > 1$

- in the exact sequence (♠) defined at the beginning of Section 3, and
- in the exact sequence (3) defined in Proposition 5.14.

Proof of Theorem 1.5. This follows immediately from Theorems 5.6, 5.14 and Corollary 7.5. □

§6. Main theorems

In this section, combining all the preceding results, we shall prove Theorem 1.1.

Proof of Theorem 1.1. Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Since K_X is not nef, there exists an extremal ray. Suppose that there exists some extremal ray of divisorial type. Then applying [5, Cor. 1.2], we can construct an FESP $Y_\bullet$ from the constant ESP $X_\bullet = (X, f)$ by a sequence of blow-downs of an ESP $\pi_\bullet\colon X_\bullet \to Y_\bullet$. Take an extremal ray $R_\bullet$ of fiber type on $\overline{\mathrm{NE}}(Y_\bullet)$ arbitrarily. We shall prove Theorem 1.1 according to each type of the FESP $(Y_\bullet, R_\bullet)$. Theorem 1.1 has been proved in the case where $(Y_\bullet, R_\bullet)$ is of type (C_0) or of type (C_1) (resp. of type (D)) in our previous article [6] (resp. [8]). Hence, hereafter we may assume that the FESP $(Y_\bullet, R_\bullet)$ is of type ($\mathrm{C}_{-\infty}$). Then, using [5, Thm. 9.6 and Cor. 10.6] and replacing X by its suitable finite étale covering, we may assume the following for the FESP $Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n$:

- There exists an ESP $S_\bullet = (h_n\colon S_n \to S_{n+1})_n$ of elliptic ruled surfaces S_n. Furthermore, there exists either of the following two cases:

- Case (A): Any S_n is isomorphic to the Atiyah surface $\mathbb{S}$.
- Case (T): For any n, we have $\mathbb{P}_C(\mathcal{O}_C \oplus \ell_n)$ for a torsion line bundle $\ell_n \in \mathrm{Pic}(C)$ and in particular, $S_0 \simeq C \times \mathbb{P}^1$.

- $\varphi_\bullet \colon Y_\bullet \to S_\bullet$ is a Cartesian morphism of ESPs such that each Y_n is a $\mathbb{P}^1$-bundle over S_n. Furthermore, there exists a vector bundle $\mathcal{E}$ of rank two on S_0 satisfying the exact sequences below such that $Y_0 \simeq \mathbb{P}_{S_0}(\mathcal{E})$.
  - In Case (A),
$$(\spadesuit)\colon 0 \longrightarrow \mathcal{O}_{\mathbb{S}}(a s_\infty) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{S}} \longrightarrow 0,$$
    for a nonnegative integer a, and
  - in Case (T),
$$(\bigstar)\colon 0 \longrightarrow p_2^*\mathcal{O}_{\mathbb{P}^1}(a) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{S_0} \longrightarrow 0,$$
    for the second projection $p_2 \colon S_0 \to \mathbb{P}^1$ and a nonnegative integer a.

First suppose that $a = 0$ (cf. Remark 5.16). In Case (A), we have $Y_n \simeq \mathbb{S} \times_C \mathbb{S}$ (resp. $Y_0 \simeq \mathbb{S} \times \mathbb{P}^1$) if the above exact sequence ($\spadesuit$) unsplits (resp. splits). In all these cases, the classifications of such varieties have been done in our previous article [8, Thms 1.1, 1.2 and 1.3]. Similarly, if $a = 0$, then in Case (B) we have $Y_0 \simeq \mathbb{S} \times \mathbb{P}^1$ (resp. $C \times \mathbb{P}^1 \times \mathbb{P}^1$) if the above exact sequence ($\bigstar$) unsplits (resp. splits). In all these cases, classifications of such varieties also have been done in our previous article [8, Thms 1.1, 1.2 and 1.3].

Hence we next consider the case where $a > 0$. Then by Proposition 1.2, Theorems 1.4 and 1.5, we see that X satisfies condition (1) or (6) in Theorem 1.1.

Finally, we consider the case where there exists an extremal ray R which is not of divisorial type. Then (X, R) itself is already an FESP and we may only consider the case where (X, R) is of type ($\mathrm{C}_{-\infty}$). By [5, Thm. 3.10] and [10], $(f^k)_*R = R$ for some $k > 0$. Hence, by replacing f by f^k, we may assume from the beginning that $f_*R = R$. Let $\varphi \coloneqq \mathrm{Cont}_R \colon X \to W$ be the contraction morphism associated to R. Then by [5, Props 7.3 and 7.5], we see that φ is a conic bundle over a smooth surface W which is a $\mathbb{P}^1$-bundle over an elliptic curve. Furthermore, there is induced a nonisomorphic étale endomorphism $g \colon W \to W$ such that $\varphi \circ f = g \circ \varphi$. Suppose that φ is not smooth. Then we can apply the same argument as in the case where the extremal ray R is of divisorial type. After a finite étale base and divisorial contractions, we can reduce to the case where $\varphi \colon X \to W$ is a $\mathbb{P}^1$-bundle. Consequently, a suitable finite étale covering $\widetilde{X}$ of X is of type (1), (5) or (6) in Theorem 1.1. Thus we have finished the proof of Theorem 1.1. □

As a by-product of our series of articles (cf. [5, 6, 8] and this article) concerning classifications of 3-folds with nonisomorphic étale endomorphisms, we have obtained the following.

Theorem 6.1. *Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Then applying the MMP working compatibly with étale endomorphisms, there exists a constant FESP $Y_\bullet = (Y, g)$ constructed from $X_\bullet = (X, f)$ by equivariant blow-downs except the case where a suitable finite étale covering $\widetilde{X}$ of X is isomorphic to the product of a rational surface and an elliptic curve.*

Proof. Let $(Y_\bullet, R_\bullet)$ be an FESP obtained from $X_\bullet := (X, f)$ by a sequence of blow-downs of an ESP. If $(Y_\bullet, R_\bullet)$ is of type (C_1) or (C_0) (resp. of type (D)), then the assertion has been proved in [6] (resp. [8]). If $(Y_\bullet, R_\bullet)$ is of type $(\mathrm{C}_{-\infty})$, then the assertion follows immediately from [5, Prop. 3.8], Theorems 4.12, 4.20, Proposition 5.9 and Corollary 7.5. □

§7. Uniqueness of FESPs

In this section we shall show the uniqueness of an FESP in certain cases.

Theorem 7.1. *Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Then the case where there exist both an FESP of type (Torsion.A) and another FESP of type (Atiyah) does not occur (cf. Definitions 1.3 and 5.5).*

Proof. We note that Theorems 1.4 and 1.5 show the existence of constant ESPs.

Case 1. First, suppose to the contrary that there exist both an FESP $Y_\bullet$ of type (Torsion.A) and another FESP $Y'_\bullet$ of type (Atiyah.A) and we shall derive a contradiction.

Let

$$X_\bullet \xrightarrow{\pi} Y_\bullet = (Y, g) \xrightarrow{\varphi} S_\bullet = (S, v) \xrightarrow{p_1} C_\bullet = (C, h)$$

be Cartesian morphisms of constant ESPs such that

- $Y_\bullet$ is a constant FESP (cf. Definition 2.9) of type (Torsion.A), and
- $S \simeq C \times \mathbb{P}^1$ and $v = h \times u$ for some $u \in \mathrm{Aut}(\mathbb{P}^1)$.

We set $\psi := p_2 \circ \varphi \circ \pi\colon X \to \mathbb{P}^1$ and $X_t = \psi^{-1}(t)$ for $t \in \mathbb{P}^1$. By construction, $X_t \simeq \mathbb{S}$ for a general point $t \in \mathbb{P}^1$. Let $X_{t,n} := X_{u^n(t)}$ for $n \in \mathbb{Z}$. Then there is induced a sub-ESP $X_{t,\bullet} := (f|_{X_{t,n}} : X_{t,n} \to X_{t,n+1})_n$ of $X_\bullet$ which is isomorphic to an ESP $\mathbb{S}_\bullet$ of Atiyah surfaces $\mathbb{S}$. Furthermore, let

$$X_\bullet \xrightarrow{\pi'} Y'_\bullet = (Y', g') \xrightarrow{\varphi'} \mathbb{S}_\bullet = (\mathbb{S}, u) \xrightarrow{\alpha} C_\bullet = (C, h)$$

be Cartesian morphisms of constant ESPs such that $Y'_\bullet$ is a constant FESP (cf. Definition 2.9) of type (Atiyah.A). Then we have the following Cartesian morphisms of ESPs:

$$\begin{array}{ccc} X_{t,\bullet} & \xrightarrow{i_\bullet} & X_\bullet \\ & & \downarrow{\scriptstyle \pi'} \\ & & Y'_\bullet \\ & & \downarrow{\scriptstyle \varphi'} \\ & & \mathbb{S}_\bullet. \end{array}$$

Then Lemma 2.1 shows that $\varphi' \circ \pi' \circ i_\bullet(X_{t,\bullet})$ is a sub-ESP of $\mathbb{S}_\bullet$. Suppose that $\varphi' \circ \pi' \circ i_\bullet(X_{t,\bullet})$ is a sub-ESP of elliptic curves. Then, since the $\mathbb{P}^1$-fiber space (i.e., a fiber space whose general fiber is isomorphic to $\mathbb{P}^1$) $\varphi \circ \pi \colon X \to S$ is equidimensional, any fiber of $\varphi \circ \pi$ is mapped to a point by $\varphi' \circ \pi' \colon X \to \mathbb{S}$ by the rigidity lemma (cf. [14, Lem. 1.6]). Thus there exists a surjective morphism $w \colon S \simeq C \times \mathbb{P}^1 \to \mathbb{S}$ such that $w \circ \varphi \circ \pi = \varphi' \circ \pi'$, which contradicts Proposition 2.6. Hence $\varphi' \circ \pi' \circ i_\bullet(X_{t,\bullet}) = \mathbb{S}_\bullet$. By Proposition 2.5, the morphism $\varphi' \circ \pi' \circ i_\bullet \colon X_{t,\bullet} \to \mathbb{S}_\bullet$ is finite étale, and hence an isomorphism, since $p_1 \circ \varphi \circ \pi|_{X_t} \colon X_t \to C$ is an isomorphism and the composite map $X_t \xrightarrow{\varphi' \circ \pi'} \mathbb{S} \xrightarrow{\alpha} C$ is the restriction of the Albanese map $\alpha_X \colon X \to C$ to X_t and equals $p_1 \circ \varphi \circ \pi|_{X_t}$. Hence $D_{t,\bullet} := \pi' \circ i_\bullet(X_{t,\bullet})$ is a sub-ESP of $Y'_\bullet$ consisting of Atiyah surfaces $\mathbb{S}$ and $\varphi'|_{D_{t,\bullet}} \colon D_{t,\bullet} \to \mathbb{S}_\bullet$ is an isomorphism of ESPs.

Since $\varphi' \circ \pi' \colon X_\bullet \to \mathbb{S}_\bullet$ is a Cartesian morphism of ESPs and the canonical section $s_{\infty,\bullet}$ ($\subset \mathbb{S}_\bullet$) forms a sub-ESP of $\mathbb{S}_\bullet$, $T_\bullet := (\varphi')^{-1}(s_{\infty,\bullet})$ is a sub-ESP of $Y'_\bullet$ and an ESP of Atiyah surfaces. Let $D_\bullet := (D, g'|_D)$ be a sub-ESP of $Y'_\bullet$ consisting of canonical sections of $\varphi' \colon Y'_\bullet \to \mathbb{S}_\bullet$. Then the complete intersection $\gamma_\bullet := D_\bullet \cap T_\bullet$ is also an ESP of elliptic curves and the image $\gamma_\bullet \hookrightarrow D_\bullet \simeq \mathbb{S}_\bullet$ (resp. $\gamma_\bullet \hookrightarrow T_\bullet \simeq \mathbb{S}_\bullet$) equals the canonical section of $\mathbb{S}_\bullet$. Then we have $D_{t,\bullet} \cap D_\bullet \neq \emptyset$, since $Y'_\bullet$ is of type (Atiyah.A) so that $\mathcal{E}'$ does not split and hence there do not exist two disjoint sections of φ'. Hence, for general $t \in \mathbb{P}^1$, $D_{t,\bullet} \cap D_\bullet \hookrightarrow D_\bullet$ is a sub-ESP of elliptic curves and equals $\gamma_\bullet$. Furthermore, since $D_{t,\bullet} \cap T_\bullet \hookrightarrow T_\bullet \simeq \mathbb{S}_\bullet$ is a sub-ESP of elliptic curves, we see that $D_{t,\bullet} \cap T_\bullet = \gamma_\bullet$. We set $(D_t, D) = m\gamma$ ($m > 0$). Since $(D_t, T) = \gamma$, we have the following linear equivalence relation of divisors: $D_t \sim D + (a+m)T$ by Lemma 4.15. Let L be the linear pencil generated by $\{D_t\}_t$ and $\Phi := \Phi_{|L|} \colon Y \cdots \to \mathbb{P}^1$ the rational map associated to $|L|$. Lemma 4.19 shows that $\pi'\colon X \to Y'$ is an equivariant blow-up along elliptic curves and the center of the first blow-up of Y' equals γ. Let F be the general fiber of $\alpha\circ\varphi'\colon Y' \to C$. Then by Lemma 4.17 we see that $(D_t, D_t)_F = a+2m$. As in the proofs of Theorems 4.12 and 4.20, by performing equivariant blow-up along elliptic curves $(a+2m)$ times on Y' successively, all the general members of D_t $(t \in \mathbb{P}^1)$ can be separated and the strict transform of $D_{t,\bullet}$ on X equals X_t. Thus we have recovered the original fiber space $\psi\colon X \to \mathbb{P}^1$. On the other hand, the proofs of Theorems 4.12 and 4.20 show that $Y \simeq \mathbb{S} \times \mathbb{P}^1$. Hence any fiber of the Albanese map $\alpha_Y\colon Y \to C$ of Y is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. This contradicts the assumption that $Y_\bullet$ is of type (Torsion.A) so that any fiber of α_Y is isomorphic to $\mathbb{F}_a$ $(a > 1)$. Thus the proof is finished.

Case 2. Next we show that the case where there exist both an FESP of type (Torsion.A) and another FESP of type (Atiyah.B) does not occur. Since the argument in Case 1 also works with minor changes, we mainly state some modifications. We shall use the same notation as in Case 1. Let

$$X_\bullet \xrightarrow{\pi} Y_\bullet = (Y, g) \xrightarrow{\varphi} S_\bullet \xrightarrow{p_1} C_\bullet = (C, h)$$

be Cartesian morphisms of constant ESPs such that

- $Y_\bullet$ is a constant FESP (cf. Definition 2.9) of type (Torsion.A), and
- $S \simeq C \times \mathbb{P}^1$ and $v = h \times u$ for some $u \in \mathrm{Aut}(\mathbb{P}^1)$.

Furthermore, let

$$X_\bullet \xrightarrow{\pi'} Y'_\bullet = (Y', g') \xrightarrow{\varphi'} \mathbb{S}_\bullet = (\mathbb{S}, u) \xrightarrow{\alpha} C_\bullet = (C, h)$$

be Cartesian morphisms of constant ESPs such that $Y'_\bullet$ is a constant FESP (cf. Definition 2.9) of type (Atiyah.B). Then by the same argument as in Case 1, $D_{t,\bullet} := \pi'\circ i_\bullet(X_{t,\bullet})$ is a sub-ESP of $Y'_\bullet$ consisting of Atiyah surfaces $\mathbb{S}$ and $\varphi'|_{D_{t,\bullet}}\colon D_{t,\bullet} \to \mathbb{S}_\bullet$ is an isomorphism of ESPs. Now we use the same notation as in the proof of Theorem 4.12. If we set $\Gamma_{\infty,\bullet} := D_\infty \cap T_\bullet$, then by the same argument as in the proof of Lemma 4.14, either of the following two cases can occur:

Case 2-1. $D_{t,\bullet} \cap D_{\infty,\bullet} = D_{t,\bullet} \cap T_\bullet = \Gamma_{\infty,\bullet}$.

Case 2-2. $D_{t,\bullet} \cap D_{\infty,\bullet} = \emptyset$ and $\Gamma_{t,n} = D_{0,n} \cap T_n$ for all n.

Let L be the linear pencil generated by $\{D_t\}_t$ and $\Phi := \Phi_{|L|}\colon Y \cdots \to \mathbb{P}^1$ the rational map associated to $|L|$. Then as explained in Case 1, all the members of L can be separated by performing equivariant blow-ups along elliptic curves $(a+2m)$ times (resp. a times) on Y' in Case 2-1 (resp. Case 2-2) (cf. Lemmas 4.15 and 4.17). Then the rest of the arguments are the same as in Case 1 and thus a contradiction is derived. □

Remark 7.2. There exists a nonisomorphic étale endomorphism $f\colon X \to X$ of a smooth projective 3-fold X with $\kappa(X) = -\infty$ admitting both an FESP of type (Atiyah.A) and an FESP of type (Atiyah.B). We shall give such an example. Let Y be a 3-fold admitting a nonisomorphic étale endomorphism $g\colon Y \to Y$ stated as in Lemma 3.5. Then for an integer $a > 1$, there exists an unsplit exact sequence of vector bundles

$$0 \longrightarrow \mathcal{O}_{\mathbb{S}}(a\, s_\infty) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{S}} \longrightarrow 0$$

such that $Y = \mathbb{P}_{\mathbb{S}}(\mathcal{E})$. Let D denote the section of φ corresponding to the surjection $\mathcal{E} \to \mathcal{O}_{\mathbb{S}}$ and set $\gamma := \varphi^{-1}(s_\infty) \cap D$ for $\varphi\colon Y \to \mathbb{S}$. Let $X := \mathrm{Bl}_\gamma(Y)$ be the blow-up of Y along γ. Since $g^{-1}(\gamma) = \gamma$, there exists a nonisomorphic étale endomorphism $f\colon X \to X$ which is a lift of $g\colon Y \to Y$. Let $Y' := \mathrm{elm}_\gamma(Y)$ be a 3-fold obtained from Y by performing an elementary transformation along γ. Then f descends to a nonisomorphic étale endomorphism $g'\colon Y' \to Y'$. There exists a Cartesian morphism of a constant ESP $\pi\colon (X, f) \to (Y, g)$ such that (Y, g) is an FESP of type (Atiyah.A). Furthermore, there exists another Cartesian morphism of a constant ESP $\pi'\colon (X, f) \to (Y', g')$ such that (Y', g') is an FESP of type (Atiyah.B), since $Y \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{O}_{\mathbb{S}}((a+1)s_\infty) \oplus \mathcal{O}_{\mathbb{S}})$ is a splitting projective bundle over $\mathbb{S}$ by Lemma 3.6.

The following proposition shows that in the case of (Torsion.A), the fiber space structure of X over $\mathbb{P}^1$ is unique up to isomorphism.

Proposition 7.3. *Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists an FESP $Y_\bullet$ of type (Torsion.A). Let*

$$X_\bullet := (X, f) \xrightarrow{\pi} Y_\bullet := (Y, g) \xrightarrow{\varphi} S_\bullet := (S, v) \xrightarrow{p_1} C_\bullet := (C, h)$$

be a Cartesian morphism of constant ESPs as in Theorem 5.14. We set $\psi := p_2 \circ \varphi \circ \pi\colon X \to \mathbb{P}^1$ for the second projection $p_2\colon S \to \mathbb{P}^1$. Then for any fiber space $\psi'\colon X \to \mathbb{P}^1$ (that is, any fiber of ψ' is connected), there exists an isomorphism $w\colon \mathbb{P}^1 \simeq \mathbb{P}^1$ such that $\psi' = w \circ \psi$:

$$\begin{array}{ccc} X & = & X \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi'} \\ \mathbb{P}^1 & \xrightarrow{w} & \mathbb{P}^1. \end{array}$$

For the proof, we begin with the following easy lemma.

Lemma 7.4. *Let $g\colon \mathbb{S} \twoheadrightarrow \mathbb{P}^1$ be a surjective morphism. Then there exists a finite surjective morphism $\tau\colon C \twoheadrightarrow \mathbb{P}^1$ such that $g = \tau \circ \alpha_{\mathbb{S}}$.*

Proof. Set $\Psi := (\alpha_{\mathbb{S}}, g)\colon \mathbb{S} \to C \times \mathbb{P}^1$. Then Proposition 2.6 shows that ψ is not surjective and $\Gamma := \Psi(\mathbb{S})$ is an irreducible curve on $C \times \mathbb{P}^1$. Let $\mathbb{S} \xrightarrow{\Psi'} \Gamma' \xrightarrow{u} \Gamma$ be the Stein factorization of $\Psi\colon \mathbb{S} \twoheadrightarrow \Gamma$. Then we have the following composite map: $\Gamma' \xrightarrow{u} \Gamma \hookrightarrow C \times \mathbb{P}^1 \xrightarrow{P_1} C$. Since Γ dominates C and $q(\mathbb{S}) = 1$, Γ' is an elliptic curve. Then by the universality of the Albanese map, we see that the composite map $\Gamma' \xrightarrow{u} \Gamma \to C$ is an isomorphism. Hence $\Gamma' \simeq \Gamma \simeq C$. Let $\tau\colon C \to \mathbb{P}^1$ be the composite map $C \simeq \Gamma \hookrightarrow C \times \mathbb{P}^1 \xrightarrow{p_2} \mathbb{P}^1$. Then by construction, we have $g = \tau \circ \alpha_{\mathbb{S}}$. □

Proof of Proposition 7.3. We use the same notation as in Proposition 5.11 and Theorem 5.14. Suppose that $\psi'(X_t)$ is not a point for some $t \in M$, where M denotes the set of points $t \in \mathbb{P}^1$ such that Y_t is isomorphic to the Atiyah surface $\mathbb{S}$. Since ψ_0 is equidimensional, the rigidity lemma (cf. [14, Lem. 1.6]) shows that $\psi'(X_t) = \mathbb{P}^1$ for any $t \in M$. Applying Lemma 7.4, we see that $\psi'_t := \psi'|_{X_t}\colon X_t(\simeq \mathbb{S}) \to \mathbb{P}^1$ factors through the Albanese map $\alpha_t\colon X_t \simeq \mathbb{S} \to \mathrm{Alb}(X_t) \simeq C$. Let $\varphi\colon X \to C\times\mathbb{P}^1$ be the projection. Since $X_t := \varphi^{-1}(C\times\{t\})$, ψ' maps each fiber of $\varphi_t\colon X_t \to C\times\{t\}$ to a point on $\mathbb{P}^1$. Applying the rigidity lemma again (cf. [14, Lem. 1.6]) to the equidimensional morphism $\varphi\colon X \to C \times \mathbb{P}^1$, we see that there exists a surjective morphism $v\colon C \times \mathbb{P}^1 \to \mathbb{P}^1$ such that $\psi' = v \circ \varphi$. Hence, for each $s \in C$, if we denote by v_s the restriction of v to $p_1^{-1}(s) \cong \mathbb{P}^1$, we obtain a holomorphic map $\rho\colon C \to \mathrm{End}(\mathbb{P}^1)$ defined by $\rho(s) := v_s$. Here, the set $\mathrm{End}(\mathbb{P}^1)$ consisting of all endomorphisms of $\mathbb{P}^1$ has a natural complex space structure. Then we show that v_s is surjective for any $s \in C$. Suppose the contrary. Then some fiber of $p_1\colon C\times\mathbb{P}^1 \to C$ is contracted to a point by v. Then the rigidity lemma shows that $v\colon C \times \mathbb{P}^1 \to \mathbb{P}^1$ factors through $C \to \mathbb{P}^1$. Hence $\psi'\colon X \to \mathbb{P}^1$ also factors through $C \to \mathbb{P}^1$. Since $\deg(C/\mathbb{P}^1) > 1$, this contradicts the assumption that any fiber of ψ' is connected. Since $\rho(C)$ is a compact subvariety of $\mathrm{End}_{\mathrm{surj}}(\mathbb{P}^1)$ consisting of all the surjective endomorphisms of $\mathbb{P}^1$, it follows from [12, Thm. 3.1] that $\rho(C)$ is contained in a left $\mathrm{Aut}(\mathbb{P}^1)$-orbit of v_0, where $0 \in C$ is the zero element. Hence, for each $s \in C$, there exists a unique automorphism $h_s \in \mathrm{Aut}(\mathbb{P}^1)$ such that $v_s = h_s \circ v_0$. Thus we obtain a holomorphic map $H\colon C \to \mathrm{Aut}(\mathbb{P}^1)$ defined by $H(s) := h_s$, $s \in C$. Since $\mathrm{Aut}(\mathbb{P}^1)$ is a linear algebraic group and C is compact, H is a constant map. Since $h_0 = \mathrm{id}_{\mathbb{P}^1}$, we have $h_s \equiv \mathrm{id}_{\mathbb{P}^1}$ and $v_s \equiv v_0$ for all $s \in C$. If we set $w := v_0\colon \mathbb{P}^1 \to \mathbb{P}^1$, then the composite

$$C \times \mathbb{P}^1 \xrightarrow{\mathrm{pr}_2} \mathbb{P}^1 \xrightarrow{w} \mathbb{P}^1$$

equals v. Hence ψ' maps any $X_t = \varphi^{-1}(C \times \{t\})$ to a point $w(t)$ on $\mathbb{P}^1$, which derives a contradiction. Hence $\psi'(X_t)$ is a point on $\mathbb{P}^1$ for any $t \in M$. Since ψ is equidimensional, the rigidity lemma (cf. [14, Lem. 1.6]) shows the existence of a surjective endomorphism $w\colon \mathbb{P}^1 \to \mathbb{P}^1$ such that $\psi' = w \circ \psi$. In summary, we have the following commutative diagram:

$$\begin{array}{ccc} X & = & X \\ {\scriptstyle \varphi\circ\pi}\downarrow & & \downarrow{\scriptstyle \psi'} \\ C \times \mathbb{P}^1 & \xrightarrow{v} & \mathbb{P}^1 \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathbb{P}^1 & \xrightarrow{w} & \mathbb{P}^1. \end{array}$$

Since each fiber of ψ' is connected, w is of degree one and hence an isomorphism. □

Using Proposition 7.3, we can show the finiteness of extremal rays in the case where there exists an FESP of type (Torsion.A).

Corollary 7.5. *Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists an FESP $Y_\bullet$ of type (Torsion.A) obtained from $X_\bullet = (X, f)$ by a sequence of blow-downs of an ESP. Then there exist at most finitely many K_X-negative extremal rays of $\overline{\mathrm{NE}}\,(X)$.*

Proof. Let

$$X_\bullet := (X, f) \xrightarrow{\pi} Y_\bullet := (Y, g) \xrightarrow{\varphi} S_\bullet := (S, v) \xrightarrow{p_1} C_\bullet := (C, h)$$

be Cartesian morphism of constant ESPs as in Theorem 5.14. With the aid of [5, Rem. 8.7(1)], it is sufficient to show the finiteness of extremal rays of *divisorial* type of $\overline{\mathrm{NE}}\,(X)$. We take an arbitrary extremal ray R of divisorial type on $\overline{\mathrm{NE}}\,(X)$ and construct an FESP $Z_\bullet$ from $X_\bullet = (X, f)$ by a sequence of blow-downs of an ESP. We note that the finiteness of the set of K_X-negative extremal rays of $\overline{\mathrm{NE}}\,(X)$ is invariant by taking a finite étale covering of X (cf. [5, Lem. 3.12]). Suppose that $Z_\bullet$ is of type (C_1) or (C_0) (cf. [5, Def. 3.6]). Then by [5, Cor. 8.1] and [6, Thms 3.2, 4.7 and 5.1], we see that by taking a finite étale covering, $Z_\bullet$ is a constant FESP and is isomorphic to a $\mathbb{P}^1$-bundle over an abelian surface or the product of a smooth curve of genus ≥ 2 and an elliptic curve. Hence we have $q(X) = q(Z) \geq 2$, which contradicts $q(X) = 1$. Hence $Z_\bullet$ is of type (D) or type ($\mathrm{C}_{-\infty}$). Suppose that $Z_\bullet$ is of type (D) (cf. [5, Def. 3.6]). Then $Z_\bullet$ is a $\mathbb{P}^2$-bundle over C. Theorem 5.14 shows that the Albanese $\alpha_X\colon X \to C$ is an analytic fiber bundle over C whose fiber is birational to the Hirzebruch surface $\mathbb{F}_a$ ($a \geq 2$). Hence, the birational contraction $X \to Z$ is not an isomorphism. Thus, applying the results of [8, Sects 6.3, 6.4 and

6.5], we can choose an FESP $Z_\bullet$ of type ($\mathrm{C}_{-\infty}$). Then Theorem 7.1 shows that by a suitable base change, we may assume that $Z_\bullet$ is of type (Torsion). Let

$$X_\bullet \xrightarrow{\pi'} Z_\bullet \xrightarrow{\psi} T_\bullet \xrightarrow{p_1} C_\bullet = (C, h)$$

be Cartesian morphisms of constant ESPs such that $T \coloneqq T_0 \simeq C \times \mathbb{P}^1$. Let E_R ($\subset X$) be the exceptional divisor of the contraction morphism Cont_R associated to R. Then the image of the sub-ESP $E_{R,\bullet}$ ($\subset X_\bullet$) by $\psi_\bullet \circ \pi'_\bullet \colon X_\bullet \to T_\bullet$ is a sub-ESP of $T_\bullet$ consisting of elliptic curves. Hence E_R is contracted by $\psi \circ \pi'$ to an elliptic curve on $T \simeq C \times \mathbb{P}^1$ which is a fiber of the second projection $p_2 \colon T \to \mathbb{P}^1$. Then E_R is contained in a fiber of $p_2 \circ \psi \circ \pi' \colon X \to \mathbb{P}^1$ (cf. Proposition 4.11). Applying Proposition 7.3, we see that there exists an automorphism $w \colon \mathbb{P}^1 \simeq \mathbb{P}^1$ such that the following commutative diagram is satisfied:

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle p_2 \circ \varphi \circ \pi} \downarrow & & \downarrow {\scriptstyle p_2 \circ \psi \circ \pi'} \\ \mathbb{P}^1 & \xrightarrow{w} & \mathbb{P}^1. \end{array}$$

Thus E_R is also contained in a fiber of $\psi \coloneqq p_2 \circ \varphi \circ \pi \colon X \to \mathbb{P}^1$. Let M ($\subset \mathbb{P}^1$) be a nonempty Zariski open subset over which ψ is smooth. Then E_R is contained in $\psi^{-1}(\mathbb{P}^1 \setminus M)$, which has a finite number of irreducible components. Then applying [7, Thm. 1.1], the proof is finished. □

Corollary 7.6. *Let $f \colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective* 3*-fold X with $\kappa(X) = -\infty$. Then the case where there exist both an FESP of type (Torsion.A) and another FESP of type (Torsion.B) does not occur (cf. Definitions* 5.5*).*

Proof. Suppose to the contrary that there exist both an FESP $Y_\bullet$ of type (Torsion.A) and another FESP $Y'_\bullet$ of type (Torsion.B) and we shall derive a contradiction. Since $Y_\bullet$ is of type (Torsion.A), X is a fiber space over $\mathbb{P}^1$ whose general fiber is the Atiyah surface $\mathbb{S}$. On the other hand, since $Y'_\bullet$ is of type (Torsion.B), Theorem 5.6 shows that X is isomorphic to the product $T \times C$ of an elliptic curve C and a smooth surface T birational to a Hirzebruch surface $\mathbb{F}_a$. Hence, by the composite $X \to T \to \mathbb{P}^1$, X is a fiber space over $\mathbb{P}^1$ whose general fiber is isomorphic to $\mathbb{P}^1 \times C$. This contradicts Proposition 7.3. □

Remark 7.7. In Propositions 5.9 and 5.11, to show the existence of constant FESPs, it is sufficient to show the existence of a K_X-negative extremal ray R ($\subset \overline{\mathrm{NE}}(X)$) such that $(f^k)_* R = R$ for some $k > 0$. Based on this, the structure of X is analyzed in Theorem 5.14 in the case where there exists an FESP of type (Torsion.A). The *finiteness of K_X-negative extremal rays* follows from Theorem 5.14 and Proposition 7.3.

Theorem 7.8. *Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists an FESP $Y_\bullet$ of type (Torsion.A) obtained from $X_\bullet = (X, f)$ by a sequence of blow-downs of an ESP:*

$$X_\bullet \xrightarrow{\pi_\bullet} Y_\bullet \xrightarrow{\varphi_\bullet} S_\bullet \xrightarrow{p_{1,\bullet}} C_\bullet.$$

Then any FESP $Z_\bullet$ of $X_\bullet$ which is of type $(\mathrm{C}_{-\infty})$ is of type (Torsion.A). Furthermore, after replacing f by a suitable power f^k ($k > 0$) and taking a finite étale base change of C, we have the commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{\mathrm{id}_X} & X \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\
Y & \cdots\xrightarrow{\tau} & Z \\
{\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\
S & \xrightarrow{\rho} & T \\
{\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_2} \\
\mathbb{P}^1 & \xrightarrow{w} & \mathbb{P}^1,
\end{array}$$

such that

(1) *π and π' are successions of equivariant blow-ups along elliptic curves,*
(2) *τ is a birational map which is an isomorphism over a nonempty Zariski open subset of $\mathbb{P}^1$,*
(3) *w is an automorphism of $\mathbb{P}^1$, and*
(4) *$\rho = (\mathrm{id}_C, w)\colon S \to T \simeq C \times \mathbb{P}^1$ is an isomorphism.*

Proof. Replacing X by its suitable finite étale covering, we may assume that there exist both an FESP $Y_\bullet$ of type (Torsion.A) and an FESP $Z_\bullet$ of type $(\mathrm{C}_{-\infty})$. Then Theorem 7.1 and Corollary 7.6 show that $Z_\bullet$ is of type (Torsion.A). Thus, applying Proposition 7.3, the theorem follows immediately. □

Remark 7.9. In Remark 5.15, using elementary transformations, we constructed an example of a nonisomorphic étale endomorphism $f\colon X \to X$ with two nonisomorphic constant FESPs of type (Torsion.A). There exists the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ Y & \cdots\xrightarrow{\mathrm{elm}_{\gamma_t}} & Y' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ S & \xrightarrow{\mathrm{id}_S} & S. \end{array}$$

Then both Y and Y' are constant FESPs (cf. Definition 2.9) of type (Torsion.A) which are not isomorphic over S.

Appendix. On 3-folds with negative Kodaira dimension whose surjective endomorphisms are necessarily étale

In our classifications of 3-folds admitting a nonisomorphic étale endomorphism, Propositions 2.4, 2.5 and 2.7 play an important role. In Theorems A.1 and A.2, we shall construct certain 3-folds whose nonisomorphic surjective endomorphisms are necessarily *étale*, which is related to the endomorphisms of the Atiyah surface $\mathbb{S}$. Proposition A.5 shows the existence of a smooth projective 3-fold X with *non-nef* anti-canonical bundle $-K_X$ which admits a nonisomorphic étale endomorphism.

Theorem A.1. *Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists a constant FESP $Y_\bullet := (Y, g)$ (cf. Definition 2.9) obtained from $X_\bullet := (X, f)$ by a sequence of equivariant blow-downs such that Y is isomorphic to $\mathbb{S} \times_C \mathbb{S}$ for the Atiyah surface $\mathbb{S}$ over an elliptic curve C. Then any nonisomorphic surjective endomorphism $F\colon X \to X$ of X is an étale endomorphism.*

Proof. Let $\alpha_X\colon X \to C$ be the Albanese map of X. Then by [8, Thm. 8.13(2)], α_X is an analytic fiber bundle over the elliptic curve C. By the universality of the Albanese map, there is induced a surjective morphism $u\colon C \to C$ such that $\alpha_X \circ F = u \circ \alpha_X$. Let S be a typical fiber of α_X which is birational to $\mathbb{P}^1 \times \mathbb{P}^1$. If we set $X_p := \alpha_X^{-1}(p)$ for $p \in C$, then $X_p \simeq S$ for any $p \in C$.

First, we show that for any $p \in C$, the induced morphism $F_p := F|_{X_p}\colon X_p \to X_{u(p)}$ is an isomorphism. Suppose that F_p is nonisomorphic for some $p \in C$ and we shall derive a contradiction. Let $T_a\colon C \to C$ be a translation mapping for some $a \in C$ such that $T_a \circ u(p) = p$. By [8, Thm. 8.13(2)] and its proof, the natural algebraic group homomorphism $(\alpha_X)_*\colon \mathrm{Aut}^0(X) \to \mathrm{Aut}^0(C)$ is surjective. Hence there exists $\widetilde{T}_a \in \mathrm{Aut}^0(X)$ such that $\alpha_X \circ \widetilde{T}_a = T_a \circ \alpha_X$. Thus, replacing F (resp. u) by the composite $\widetilde{T}_a \circ F$ (resp. $T_a \circ u$), we may assume from the beginning that

$u(p) = p$. Then $F_p\colon X_p \to X_p$ is a nonisomorphic endomorphism of $X_p \simeq S$. By [8, Thm. 8.13(1)], there exist at most finitely many extremal rays of $\overline{\mathrm{NE}}(X)$. Let

$$(X = X_{(0)}, f) \xrightarrow{\pi_0} (X_{(1)}, f_{(1)}) \xrightarrow{\pi_1} \cdots \longrightarrow (X_{(i-1)}, f_{(i-1)}) \xrightarrow{\pi_{i-1}} (X_{(i)}, f_{(i)}) \longrightarrow \cdots \longrightarrow (Y = X_{(k)}, f_{(k)})$$

be a sequence of equivariant blow-downs, where π_{i-1} is (the inverse of) the blow-up of X_i along an elliptic curve E_i of X_i such that $f_{(i)}^{-1}(E_i) = E_i$. Then for any i, the natural morphism $\psi_i\colon X_{(i)} \to C$ gives the Albanese map of $X_{(i)}$ and $\psi_i(E_i) = C$. We take a sufficiently large positive integer q such that the multiplication by q mapping $\mu_q\colon C \to C$ is factored through $\psi_i|_{E_i}\colon E_i \to C$ for any i. Replacing X by $\widetilde{X} = X \times_{C,\mu_q} C$ and f by its lift $\tilde{f}\colon \widetilde{X} \to \widetilde{X}$, we may assume that $\psi_i|_{E_i}\colon E_i \to C$ is an isomorphism for any i. Let Δ_i $(\subset X_i)$ be the exceptional divisor of π_i. Then $\gamma_0 := \Delta_0 \cap X_p$ $(\subset X_p \simeq S)$ is a (-1)-curve on S and spans an extremal ray R_0 of $\overline{\mathrm{NE}}(X)$. Let M be the set consisting of negative curves on S. Then by [19, Prop. 11], M is a finite set. Hence, by replacing F by its suitable power f^k $(k > 0)$, we may assume that F induces an identity permutation of M. Since $\gamma_0 \in M$, $F(\gamma_0) = \gamma_0$, which shows that $F_*R_0 = R_0$. Hence there exists a surjective endomorphism $F_1\colon X_{(1)} \to X_{(1)}$ such that $\pi_0 \circ F = F_1 \circ \pi_0$ for $\pi_0 = \mathrm{Cont}_{R_0}$. By repeating the same argument as above, there exists a surjective endomorphism $F_k\colon Y \to Y$ such that $\pi \circ F = F_k \circ \pi$, where $\pi := \pi_{k-1} \circ \cdots \circ \pi_0\colon X \to Y$ is the composite map. Then by [8, Prop. 8.8], F_k is an étale endomorphism of Y. Since $f_{(i)}^{-1}(E_i) = E_i$ for any i, $F\colon X \to X$ is an étale endomorphism. Since S is birational to $\mathbb{P}^1 \times \mathbb{P}^1$ and is simply connected, an étale endomorphism $F_p\colon S \to S$ is an isomorphism, which derives a contradiction. Thus, for any $p \in C$, F_p is an isomorphism.

Hence $F\colon X \to X$ is an endomorphism of $\deg(F) = \deg(u) > 1$. By construction, we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ {\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \alpha_X} \\ C & \xrightarrow{u} & C. \end{array}$$

Then $F\colon X \to X$ is factored through a finite étale covering $X \times_{C,u} C \to X$ of degree $\deg(u)$. Hence the natural morphism $X \to X \times_{C,u} C$ is an isomorphism by Zariski's main theorem and the fact that the above commutative diagram is Cartesian. Thus $F\colon X \to X$ is a nonisomorphic étale endomorphism of X. □

We have the following similar result.

Theorem A.2. *Let $f\colon X \to X$ be a nonisomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists a constant FESP (cf. Definition 2.9) $Y_\bullet = (Y, G)$ which is of type (Atiyah.A). Then any nonisomorphic surjective endomorphism $F\colon X \to X$ of X is an étale endomorphism.*

We insert the following lemma.

Lemma A.3. *Any surjective endomorphism $G\colon Y \to Y$ of Y is étale.*

Proof. Let

$$(\spadesuit)\colon 0 \longrightarrow \mathscr{O}_{\mathbb{S}}(a\, s_\infty) \longrightarrow \mathcal{E} \longrightarrow \mathscr{O}_{\mathbb{S}} \longrightarrow 0.$$

be the exact sequence of locally free sheaves on the Atiyah surface $\mathbb{S}$ defined at the beginning of Section 3. Then by Definition 4.5, we consider the case where $(\spadesuit)$ unsplits and $Y \simeq \mathbb{P}_{\mathbb{S}}(\mathcal{E})$. By the universality of the Albanese map $\alpha_Y\colon Y \to C$, there is induced a surjective morphism $u\colon C \to C$ such that $\alpha_Y \circ G = u \circ \alpha_Y$. Then any fiber $Y_t := \alpha_Y^{-1}(t)$ $(t \in C)$ of α_Y is isomorphic to a Hirzebruch surface $\mathbb{F}_a$. Let D be the unique section of $\varphi\colon Y \to \mathbb{S}$ corresponding to the surjection $\mathcal{E} \twoheadrightarrow \mathscr{O}_{\mathbb{S}}$. If we set $N_t := Y_t \cap D$, then N_t is a negative section of $\mathbb{F}_a$. Then we have $G^{-1}(D) = D$, since N_t is a unique negative curve on Y_t and its numerical class $[N_t]$ spans the extremal ray of $\overline{\mathrm{NE}}(Y_t)$ which is a 2-dimensional closed convex cone. Hence we have $G^*D \sim kD$ for an integer $k > 0$. Furthermore, if we set $T := \varphi^{-1}(s_\infty)$ for the canonical section s_∞ $(\subset \mathbb{S})$, then $T \simeq \mathbb{S}$ (cf. Lemma 3.1) and we see that $G^*T = T$, since $u^*s_\infty = s_\infty$ by Proposition 2.5. Restricting this relation to T, we have $(G|_T)^*\gamma \sim k\gamma$ for $\gamma := D \cap T$ in $T \simeq \mathbb{S}$. Then Proposition 2.5 shows that $G|_T$ is étale and $k = 1$. Thus $G^*D = D$. Since $K_Y \sim -2D - (a+2)T$ and $K_Y \sim G^*K_Y + R_G$ for the ramification divisor $R_G \geq 0$ of G, we have $R_G \sim 2(G^*D - D) = 0$. Hence G is étale. □

Remark A.4. The conclusion of Theorem A.1 does not necessarily hold true if we do not assume the existence of an FESP of type (Atiyah.A). If we set $Y := \mathbb{S} \times \mathbb{P}^1$, then Y admits a nonisomorphic surjective endomorphism which is not étale. Such an example can be easily constructed: We consider Y as a trivial $\mathbb{S}$-bundle over $\mathbb{P}^1$ and let $Y_t \simeq \mathbb{S}$ be a fiber over $t \in \mathbb{P}^1$. For an integer $n \geq 2$, let $u\colon \mathbb{P}^1 \to \mathbb{P}^1$, $z \mapsto z^n$ be a nonisomorphic endomorphism of $\mathbb{P}^1$. Then $F := \mathrm{id}_{\mathbb{S}} \times u\colon Y \to Y$ is a nonisomorphic surjective endomorphism of Y of degree n which ramifies along Y_0 and Y_∞ with ramification index n.

Let $\alpha_{\mathbb{S}}\colon \mathbb{S} \to C$ be the Albanese map of $\mathbb{S}$ and $\varphi := \alpha_{\mathbb{S}} \times \mathrm{id}_{\mathbb{P}^1}\colon Y \to S := C \times \mathbb{P}^1$ the $\mathbb{P}^1$-bundle. Let $p_2\colon Y \to \mathbb{P}^1$ be the second projection. Then $\gamma_0 := s_\infty \times \mathbb{P}^1 \cap p_2^{-1}(0)$ (resp. $\gamma_\infty := s_\infty \times \mathbb{P}^1 \cap p_2^{-1}(\infty)$) is an elliptic curve on Y with $F^{-1}(\gamma_0) = \gamma_0$ (resp. $F^{-1}(\gamma_\infty) = \gamma_\infty$). Now we shall perform elementary transformations along

γ_0 and γ_∞ (cf. Definition 3.3). We set $Z := \mathrm{elm}_{\gamma_0} \circ \mathrm{elm}_{\gamma_\infty}(Y)$. Then $\varphi' \colon Z \to S$ is a $\mathbb{P}^1$-bundle over S which is isomorphic to $\mathbb{P}_S(\mathcal{E})$, where $\mathcal{E}$ is a vector bundle of rank two on S which satisfies the following unsplit exact sequence (cf. Theorem 5.14):

$$0 \longrightarrow p_2^*\mathcal{O}_{\mathbb{P}^1}(2) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

By the composite map $\psi \colon Z \xrightarrow{\varphi'} C \times \mathbb{P}^1 \xrightarrow{p_2} \mathbb{P}^1$, Z is a smooth fiber space over $\mathbb{P}^1$. The general fiber of ψ is isomorphic to $\mathbb{S}$ and its two special fibers $Z_0 := \psi^{-1}(0)$ and $Z_\infty := \psi^{-1}(\infty)$ are both isomorphic to $C \times \mathbb{P}^1$. In other words, Z is an FESP of type (Torsion.A) and admits a nonisomorphic étale endomorphism. Furthermore, a nonisomorphic endomorphism F of Y induces a nonisomorphic endomorphism $G \colon Z \to Z$ of degree n which ramifies only along Z_0 and Z_∞ with ramification index n.

Proof of Theorem A.2. Theorem 4.20 shows the existence of equivariant blow-downs $\pi \colon (X, f) \to (Y, g)$ such that (Y, g) is an FESP of type (Atiyah.A). Furthermore, by Theorem 4.27, the Albanese map $\alpha_X \colon X \to C$ is an analytic fiber bundle over an elliptic curve C. Hence, combining with Lemma A.3, we can apply completely the same method as in the proof of Theorem A.1. □

We pose the following question:

Question. Let X be a smooth projective 3-fold with $\kappa(X) = -\infty$ admitting a nonisomorphic étale endomorphism. Then is it true that the anti-canonical divisor $-K_X$ is nef?

We shall give a negative answer to this question.

Proposition A.5. *There exists a smooth projective 3-fold X with the following properties:*

- *X admits a nonisomorphic étale endomorphism.*
- *The anti-canonical divisor $-K_X$ is not nef.*

Proof. For an integer $a > 2$, we take the following unsplit exact sequence of vector bundles on the Atiyah surface $\mathbb{S}$:

$$(\spadesuit) \colon 0 \longrightarrow \mathcal{O}_{\mathbb{S}}(a\, s_\infty) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{S}} \longrightarrow 0.$$

If we set $\varphi \colon Y := \mathbb{P}_{\mathbb{S}}(\mathcal{E}) \to \mathbb{S}$, then Lemma 3.5 shows the existence of a nonisomorphic étale endomorphism $g \colon Y \to Y$. Set $T := \varphi^{-1}(s_\infty)$ and let D be the section of φ corresponding to a surjection $\mathcal{E} \twoheadrightarrow \mathcal{O}_{\mathbb{S}}$. Then both T and D are isomorphic to $\mathbb{S}$ which are preserved by $g \colon Y \to Y$. The Albanese map $\alpha_Y \colon Y \to C$ is an $\mathbb{F}_a$-bundle over C. If we set $\zeta := \alpha_Y^{-1}(0) \cap D$ for a point $0 \in C$, then ζ is the negative section of $\alpha_Y^{-1}(0) \simeq \mathbb{F}_a$. We have the following relations:

$$-K_Y \sim 2D + (a+2)T, \quad (D, \zeta) = -a, \quad (T, \zeta) = 1.$$

Hence we have $(-K_Y, \zeta) = 2 - a < 0$ and thus $-K_Y$ is not nef.

We set $\gamma := D \cap T$ and let $\pi\colon X := \mathrm{Bl}_\gamma(Y) \to Y$ be the blow-up of Y along γ. Since $g^{-1}(\gamma) = \gamma$, there exists a nonisomorphic étale endomorphism $f\colon X \to X$ which is a lift of $g\colon Y \to Y$. If we let $E := \mathrm{Exc}(\pi)$ be the π-exceptional divisor, then $E \simeq C \times \mathbb{P}^1$. Let ζ' be the proper transform of ζ by π. Since ζ intersects T transversally at one point, we have $(E, \zeta') = 1$. Since $K_X \sim \pi^* K_Y + E$ and $(\pi^*(-K_Y), \zeta') = (-K_Y, \zeta) < 0$, we have $(-K_X, \zeta') = (\pi^*(-K_Y) - E, \zeta') = (-K_Y, \zeta) - 1 < 0$. Thus $-K_X$ is not nef. □

Acknowledgements

The author wishes to express sincere thanks to Professor Noboru Nakayama for many useful discussions. The author is grateful to the referee for the very careful reading of the manuscript, error corrections and many constructive suggestions to make this article more readable. This paper is dedicated to the memory of my late mother Junko Fujimoto for her deep love and constant encouragement throughout my life. The author is supported by the Grant-in-Aid for Scientific Research (C), no. 20K03518, Japan Society for the Promotion of Science.

References

[1] M. F. Atiyah, Vector bundles over an elliptic curve, Proc. London Math. Soc. (3) **7** (1957), 414–452. Zbl 0084.17305 MR 0131423

[2] W. Fischer and H. Grauert, Lokal-triviale Familien kompakter komplexer Mannigfaltigkeiten, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **1965** (1965), 89–94. Zbl 0135.12601 MR 0184258

[3] A. Fujiki, On automorphism groups of compact Kähler manifolds, Invent. Math. **44** (1978), 225–258. Zbl 0367.32004 MR 0481142

[4] Y. Fujimoto, Endomorphisms of smooth projective 3-folds with non-negative Kodaira dimension, Publ. Res. Inst. Math. Sci. **38** (2002), 33–92. Zbl 1053.14049 MR 1873169

[5] Y. Fujimoto, Étale endomorphisms of 3-folds. I, Osaka J. Math. **55** (2018), 195–257. Zbl 1394.14025 MR 3787744

[6] Y. Fujimoto, Étale endomorphisms of 3-folds. II, Osaka J. Math. **59** (2022), 1–14. Zbl 1483.14020 MR 4373482

[7] Y. Fujimoto, Some remarks on finiteness of extremal rays of divisorial type, Proc. Japan Acad. Ser. A Math. Sci. **98** (2022), 7–12. Zbl 1482.14014 MR 4388915

[8] Y. Fujimoto, Étale endomorphisms of 3-folds, III, Kyoto J. Math. **64** (2024), 95–204. Zbl 1540.14034 MR 4677749

[9] Y. Fujimoto and N. Nakayama, Endomorphisms of smooth projective 3-folds with nonnegative Kodaira dimension. II, J. Math. Kyoto Univ. **47** (2007), 79–114. Zbl 1138.14023 MR 2359102

[10] Y. Fujimoto and N. Nakayama, Termination of extremal rays of fibration type for the iteration of surjective endomorphisms, Appendix (pp. 164–168) to D.-Q. Zhang, Polarized endomorphisms of uniruled varieties, Compos. Math. **146** (2010), 145–168. Zbl 1198.14019 MR 2581245

[11] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics 52, Springer, New York-Heidelberg, 1977. Zbl 0367.14001 MR 0463157

[12] C. Horst, Compact varieties of surjective holomorphic endomorphisms, Math. Z. **190** (1985), 499–504. Zbl 0581.32013 MR 0808917

[13] K. Kodaira, On stability of compact submanifolds of complex manifolds, Amer. J. Math. **85** (1963), 79–94. Zbl 0173.33101 MR 0153033

[14] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics 134, Cambridge University Press, Cambridge, 1998. MR 1658959

[15] M. Maruyama, *On classification of ruled surfaces*, Lectures in Mathematics, Department of Mathematics, Kyoto University 3, Kinokuniya Book Store, Tokyo, 1970. Zbl 0214.20103 MR 0276243

[16] M. Maruyama, On automorphism groups of ruled surfaces, J. Math. Kyoto Univ. **11** (1971), 89–112. Zbl 0213.47803 MR 0280493

[17] M. Maruyama, On a family of algebraic vector bundles, in *Number theory, algebraic geometry and commutative algebra, in honor of Yasuo Akizuki*, Kinokuniya Book Store, Tokyo, 1973, 95–146. MR 0360587

[18] S. Mori, Threefolds whose canonical bundles are not numerically effective, Ann. of Math. (2) **116** (1982), 133–176. Zbl 0557.14021 MR 0662120

[19] N. Nakayama, Ruled surfaces with non-trivial surjective endomorphisms, Kyushu J. Math. **56** (2002), 433–446. Zbl 1049.14029 MR 1934136

[20] N. Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs 14, Mathematical Society of Japan, Tokyo, 2004. Zbl 1061.14018 MR 2104208