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An Analogue of the Dichotomy Conjecture on Monoidally Distributive Posets

by

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Abstract

Hovey [in: The Čech centennial (Boston, MA, 1993), 1995, 225–250] proposed the dichotomy conjecture on the stable homotopy category of spectra. Hovey and Palmieri [in: Homotopy invariant algebraic structures (Baltimore, MD, 1998), 1999, 175–196] proved many interesting facts around the dichotomy conjecture from the viewpoint of the Bousfield lattice. The author, Shimomura and Tatehara [Publ. Res. Inst. Math. Sci. 50 (2014), 497–513] defined the notion of monoidally distributive posets as a generalization of the Bousfield lattice. In this paper we consider an analogue of the dichotomy conjecture on monoidally distributive posets, and prove several results around the analogue.

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§1. Introduction

Let p be a prime number and $\mathcal{S}_p$ the stable homotopy category of p-local spectra. For a spectrum $X \in \mathcal{S}_p$, a spectrum W is *X-acyclic* if $X \wedge W = 0$, and a spectrum V is *X-local* if any morphism $W \to V$ is trivial for any X-acyclic spectrum W. A spectrum X *has a finite acyclic* (resp. *a finite local*) if there exists a finite and X-acyclic (resp. X-local) spectrum. Hovey [1] proposed the following conjecture.

Conjecture 1.1 (Dichotomy conjecture [1, Conj. 3.10], [2, Conj. 7.5]). *Every spectrum has either a finite acyclic or a finite local.*

For a spectrum $X \in \mathcal{S}_p$, the Bousfield class $\langle X \rangle$ is defined to be the class consisting of X-acyclic spectra. Ohkawa showed that the collection $\mathbb{B}$ of Bousfield

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classes is a set [4]. This set has a lattice structure given by $\langle X\rangle \leq \langle Y\rangle \Leftrightarrow \langle X\rangle \supset \langle Y\rangle$. This lattice is called the *Bousfield lattice* (of $\mathcal{S}_p$). The author, Shimomura and Tatehara [3] defined *monoidally distributive posets* as a generalization of the Bousfield lattice $\mathbb{B}$ (see Section 2). Our aim in this paper is to consider an analogue of Conjecture 1.1 (and some related topics) on monoidally distributive posets.

Let B be a monoidally distributive poset. Then B is a lattice and also a commutative monoid with 0. For any $x \in B$, we have an operation $a\colon B \to B$ given by $a(x) = \bigvee\{w \in B\colon wx = 0\}$ (see (2.4)), and we define

$$\begin{aligned} BA &= \{x \in B\colon x \vee a(x) = 1\}, \\ \mathfrak{M} &= \{x \in B\colon x \text{ is minimal}\}, \\ M &= \mathfrak{M} \cap BA. \end{aligned}$$

Hereafter, we denote by $\vee$ and $\wedge$ the join and the meet, respectively. The subset $M^\vee$ is defined by

$$M^\vee = \left\{\textstyle\bigvee_{m\in S} m : S \text{ is a finite subset of } M\right\}.$$

We remark that if $S = \varnothing$, then $\bigvee_{m\in S} m = 0$. A nonzero element $d \in B$ is a *dichotomizer* if

$$B = M^\vee \cup \uparrow d,$$

where $\uparrow d = \{x \in B\colon x \geq d\}$ (see Section 4).

Problem 1.2 (Dichotomy problem). *What is a condition on B to which B has a dichotomizer?*

Remark 1.3. In Section 6, we consider the monoidally distributive poset $\mathbb{B}$, the Bousfield lattice of the stable homotopy category. From the viewpoint of this paper, Hovey and Palmieri [2] claim that the Bousfield class $\langle I\rangle$ of the Brown–Comenetz dual of the sphere spectrum is a candidate of a dichotomizer (Proposition 6.8). In particular, we may consider that Problem 1.2 is an analogue of Conjecture 1.1 on monoidally distributive posets. It is not difficult to see that $M^\vee \cup a(M^\vee)$ is a proper subset of $\mathbb{B}$ (see the proof of Theorem 6.9), where $a(M^\vee) = \{a(x)\colon x \in M^\vee\}$. Furthermore, we know that $\langle I\rangle$ satisfies $\langle I\rangle^2 = 0$ [2, Lem. 7.1(c)]. Besides, if Conjecture 1.1 holds, then $\langle I\rangle$ is minimal [2, Lem. 7.8]. With this background, Theorem 1.6 below supports Conjecture 1.1.

In this paper we prove the following results.

Theorem 1.4. *If B has a dichotomizer, then $BA = M^\vee \cup a(M^\vee)$. Furthermore, if M is an infinite subset of B, then $M^\vee \cap a(M^\vee) = \varnothing$.*

This is an analogue of [2, Cor. 7.11] on monoidally distributive posets.

Remark 1.5. In general, $M^\vee \cap a(M^\vee) \neq \varnothing$. For example, in the case for $B = \{0,1\}$, we have $M = \{1\}$ and so $M^\vee = \{0,1\}$. Hence, by (2.5), we have $a(M^\vee) = \{0,1\} = M^\vee$.

We put $\mathfrak{S} = \{x \in B : x^2 = 0\}$.

Theorem 1.6. *Suppose that $M^\vee \cup a(M^\vee)$ is a proper subset of B. If B has a dichotomizer d, then $M^\vee \cap {\uparrow} d = \varnothing$ and $\mathfrak{M} \cap \mathfrak{S} = \{d\}$. In particular, if a dichotomizer exists, then it is unique.*

We prove Theorems 1.4 and 1.6 in Section 4.

As a typical example of monoidally distributive posets, we have $\beta(\mathbb{Z}/n)$ for $n \geq 2$ (see Section 5). In Section 5, we prove the following.

Theorem 1.7. *Suppose that $M^\vee \cup a(M^\vee)$ is a proper subset of $\beta(\mathbb{Z}/n)$. Then an element $d \in \beta(\mathbb{Z}/n)$ is a dichotomizer if and only if $\mathfrak{M} \cap \mathfrak{S} = \{d\}$.*

Conjecture 1.8. *Suppose that $M^\vee \cup a(M^\vee)$ is a proper subset of a monoidally distributive poset B. Then an element $d \in B$ is a dichotomizer if and only if $\mathfrak{M} \cap \mathfrak{S} = \{d\}$.*

§2. Monoidally distributive posets

First we recall the definition of monoidal posets.

Definition 2.1 ([3, Def. 2.4]). A *monoidal poset* B consists of the following data:

(1) $(B, \leq)$ is a poset.

(2) $(B, \cdot, 1, 0)$ is a commutative monoid with 0.

(3) For any x and y in B, the following are equivalent:

- $x \leq y$;
- $wy = 0$ for $w \in B$ implies $wx = 0$.

Hereafter, we denote $xy = x \cdot y$. We also denote by $\vee$ and $\wedge$ the join and the meet, respectively.

It is easy to see that 0 (resp. 1) is the minimum (resp. maximum) element. Furthermore, we note that

$$x \leq y \text{ implies } wx \leq wy \text{ for any } w \in B. \tag{2.2}$$

Indeed, for any $c \in B$, $x \leq y$ implies $cwy = 0 \Rightarrow cwx = 0$. In particular, for any x and y, we have $x \geq xy$ and $y \geq xy$, and so $x \wedge y \geq xy$.

Definition 2.3 ([3, Def. 3.6]). A monoidal poset B is a *monoidally distributive poset* if the following hold:

(1) B is a complete lattice.
(2) For any $x \in B$ and $\{y_\lambda\}_\lambda \subset B$, we have $x(\bigvee_\lambda y_\lambda) = \bigvee_\lambda (xy_\lambda)$.

Hereafter, we assume that B is a monoidally distributive poset. We define

$$a\colon B \to B; \quad x \mapsto \bigvee\{w \in B\colon wx = 0\}. \tag{2.4}$$

It is easy to see that

$$a(0) = 1, \quad a(1) = 0. \tag{2.5}$$

We also have

$$xa(x) = 0 \quad \text{for any } x \in B. \tag{2.6}$$

Indeed, $xa(x) = x(\bigvee\{w \in B\colon wx = 0\}) = \bigvee\{wx\colon wx = 0\} = 0$.

Proposition 2.7 ([3, Prop. 3.8]). *For any x and y in B, the following hold:*

(1) *$x \leq y$ implies $a(x) \geq a(y)$.*
(2) *$xy = 0$ if and only if $x \leq a(y)$.*
(3) *$a^2(x) = x$.*

Proof.

(1) We note that $x \leq y$ implies $\{w \in B\colon wx = 0\} \supset \{w \in B\colon wy = 0\}$. Therefore, $a(x) = \bigvee\{w \in B\colon wx = 0\} \geq \bigvee\{w \in B\colon wy = 0\} = a(y)$.
(2) If $xy = 0$, then $x \in \{w \in B\colon wy = 0\}$. Hence $x \leq \bigvee\{w \in B\colon wy = 0\} = a(y)$. Conversely, if $x \leq a(y)$, then $xy \leq a(y)y = 0$ by (2.2) and (2.6).
(3) By (2.6) and (2), we have $x \leq a^2(x)$. From (1), we obtain $a(x) \geq a^3(x)$. On the other hand, $a(x)a^2(x) = 0$ and (2) imply $a(x) \leq a^3(x)$. Therefore, $a(x) = a^3(x)$. Thus, we have

$$\begin{aligned} wx = 0 &\Leftrightarrow w \leq a(x) && \text{by (2)} \\ &\Leftrightarrow w \leq a^3(x) = aa^2(x) \\ &\Leftrightarrow wa^2(x) = 0 && \text{by (2)}, \end{aligned}$$

and therefore $x = a^2(x)$. □

Lemma 2.8. *For any x and y in B, we have $a(x \wedge y) = a(x) \vee a(y)$ and $a(x \vee y) = a(x) \wedge a(y)$.*

Proof. First we show that $a(x \wedge y) = a(x) \vee a(y)$, that is, $a(x \wedge y)$ is the least upper bound of $\{a(x), a(y)\}$. By Proposition 2.7(1), $a(x \wedge y)$ is an upper bound of $\{a(x), a(y)\}$. If z is an upper bound of $\{a(x), a(y)\}$, then $z \geq a(x)$ and $z \geq a(y)$. By (1) and (3) of Proposition 2.7, we have $a(z) \leq a^2(x) = x$ and $a(z) \leq a^2(y) = y$, and so $a(z) \leq x \wedge y$. Thus $z = a^2(z) \geq a(x \wedge y)$, and we see $a(x \wedge y) = a(x) \vee a(y)$.

The second claim is given by $a(x \vee y) = a(a^2(x) \vee a^2(y)) = a^2(a(x) \wedge a(y)) = a(x) \wedge a(y)$. □

§3. The subset $M^{\vee}$

We consider a subset

$$BA = \{x \in B \colon x \vee a(x) = 1\}.$$

By (2.5), the elements 0 and 1 are in BA. We also note that

$$x \in BA \iff a(x) \in BA. \tag{3.1}$$

Lemma 3.2. *If x is in BA, then $x^2 = x$.*

Proof. By (2.6), we have $x^2 = x^2 \vee xa(x) = x(x \vee a(x)) = x \cdot 1 = x$. □

Lemma 3.3. *The following are equivalent:*

(1) *BA is a proper subset of B.*

(2) *There exists a nonzero element y such that $y^2 = 0$.*

Proof. If there exists $x \in B \setminus BA$, then $x \vee a(x) \neq 1$. Thus, we have a nonzero element y such that $y(x \vee a(x)) = 0$. This implies $yx = 0$ and $ya(x) = 0$. Hence $y \leq a(x)$ and $y \leq x$ by Proposition 2.7. We then have $y^2 = yy \leq xa(x) = 0$ by (2.2). Conversely, if $y \neq 0$ and $y^2 = 0$, then $y \notin BA$ by Lemma 3.2. □

Lemma 3.4 (Cf. [2, (e) and (f) of Lem. 4.3]). *If x and y are in BA, then xy and $x \vee y$ are in BA.*

Proof. Assume that $x, y \in BA$. We then have

$$\begin{aligned}
xy \vee a(xy) &\geq xy \vee a(x \wedge y) && \text{by (2.2) and Proposition 2.7(1)} \\
&= xy \vee a(x) \vee a(y) && \text{by Lemma 2.8} \\
&\geq xy \vee a(x)y \vee a(y) && \text{by (2.2)} \\
&= (x \vee a(x))y \vee a(y) = y \vee a(y) = 1
\end{aligned}$$

and

$$
\begin{aligned}
x \vee y \vee a(x \vee y) &= x \vee y \vee (a(x) \wedge a(y)) && \text{by Lemma 2.8} \\
&\geq x \vee a(x)y \vee a(x)a(y) && \text{by (2.2)} \\
&= x \vee a(x)(y \vee a(y)) = x \vee a(x) = 1.
\end{aligned}
$$

Therefore, xy and $x \vee y$ are in BA. $\square$

We define

$$\mathfrak{M} = \{x \in B \colon x \text{ is minimal}\}, \quad M = \mathfrak{M} \cap BA \tag{3.5}$$

and

$$M^{\vee} = \left\{\bigvee_{m \in S} m : S \text{ is a finite subset of } M\right\}. \tag{3.6}$$

We remark that $S = \varnothing$ implies $\bigvee_{m \in S} m = 0$. We also define $a(M^{\vee}) = \{a(x) \colon x \in M^{\vee}\}$. From (3.1) and Lemma 3.4, we obtain the following corollary.

Corollary 3.7. *$M^{\vee} \cup a(M^{\vee}) \subset BA$.*

Lemma 3.8. *For any $m \in \mathfrak{M}$ and $x \in B$, the product mx is either 0 or m.*

Proof. By (2.2), $mx \leq m$. Since m is minimal, $mx = 0$ or $mx = m$. $\square$

Lemma 3.9. *For any m and m' in M,*

$$mm' = \begin{cases} 0, & m \neq m', \\ m \ (= m'), & m = m'. \end{cases}$$

Proof. We take m and m' in $M = \mathfrak{M} \cap BA$. First we consider the case for $m \neq m'$. Note that $mm' \leq m$ and $mm' \leq m'$. Since m and m' are minimal, if $mm' \neq 0$, then $m = mm' = m'$, which is a contradiction. Therefore $mm' = 0$. In the case for $m = m'$, since $m \in BA$, we have $mm' = m^2 = m$ by Lemma 3.2. $\square$

Corollary 3.10. *If m is minimal and c is nilpotent, then $mc = 0$.*

Proof. By Lemma 3.8, we have $mc = 0$ or m. We note that $c^n = 0$ for some $n \geq 1$. If $mc = m$, then $m = mc = (mc)c = \cdots = mc^n = 0$, which is a contradiction. $\square$

For a subset T of B, we define

$$
\begin{aligned}
\downarrow T &= \{x \in B \colon x \leq t \text{ for some } t \in T\}, \\
\uparrow T &= \{x \in B \colon x \geq t \text{ for some } t \in T\}
\end{aligned}
$$

and, for $x \in B$,

$$\downarrow x = \downarrow \{x\} \quad \text{and} \quad \uparrow x = \uparrow \{x\}.$$

Lemma 3.11. *If $x \in BA \cap \uparrow y$, then $xy = y$.*

Proof. Since $x \in BA \cap \uparrow y$, we have $x \vee a(x) = 1$ and $ya(x) = 0$ by Proposition 2.7(2). Hence $y = y(x \vee a(x)) = xy \vee ya(x) = xy$. □

Remark 3.12. Lemma 3.2 is a corollary of Lemma 3.11. Indeed, if $x \in BA$, then $x \in BA \cap \uparrow x$. This and Lemma 3.11 imply $x^2 = xx = x$.

For the sake of simplicity, we denote

$$\bigvee T = \bigvee_{t \in T} t.$$

Lemma 3.13. *We have $\downarrow M^{\vee} = M^{\vee}$ and $\uparrow a(M^{\vee}) = a(M^{\vee})$.*

Proof. First we prove that $\downarrow M^{\vee} = M^{\vee}$. It is easy to see that $\downarrow M^{\vee} \supset M^{\vee}$. If $x \in \downarrow M^{\vee}$, then $x \leq \bigvee S$ for a finite subset S of M. Then we have

$$\begin{aligned} x &= x\Big(\bigvee S \vee a\Big(\bigvee S\Big)\Big) && \text{by Corollary 3.7} \\ &= x\Big(\bigvee S\Big) \vee xa\Big(\bigvee S\Big) = x\Big(\bigvee S\Big) && \text{by Proposition 2.7(2)} \\ &= \bigvee_{m \in S} xm = \bigvee_{xm = m \in S} m && \text{by Lemma 3.8,} \end{aligned}$$

and so $x \in M^{\vee}$. Therefore $\downarrow M^{\vee} \subset M^{\vee}$.

Next turn to $\uparrow a(M^{\vee}) = a(M^{\vee})$. It suffices to show that $\uparrow a(M^{\vee}) \subset a(M^{\vee})$. If $x \in \uparrow a(M^{\vee})$, then $x \geq a(z)$ for some $z \in M^{\vee}$. By Proposition 2.7, we have $a(x) \leq z$, and so $a(x) \in \downarrow M^{\vee}$. Since $\downarrow M^{\vee} = M^{\vee}$, we have $a(x) \in M^{\vee}$, which implies $x \in a(M^{\vee})$. Therefore $\uparrow a(M^{\vee}) \subset a(M^{\vee})$. □

Lemma 3.14. *If two subsets S and T of M satisfy $\bigvee S \leq \bigvee T$, then $S \subset T$.*

Proof. If $S \not\subset T$, then there exists m_0 such that $m_0 \in S$ and $m_0 \notin T$. By Lemma 3.9, we have $m_0(\bigvee S) = \bigvee_{m \in S} m_0 m = m_0 \neq 0$ and $m_0(\bigvee T) = \bigvee_{m \in T} m_0 m = 0$. Therefore $\bigvee S \not\leq \bigvee T$. □

Proposition 3.15. *If M is an infinite subset of B, then $M^{\vee} \cap a(M^{\vee}) = \varnothing$.*

Proof. Assume that $M^{\vee} \cap a(M^{\vee}) \neq \varnothing$. Then we have an element $x \in M^{\vee} \cap a(M^{\vee})$, that is, $\bigvee S = x = a(\bigvee T)$ for some finite subsets S and T of M. If $m \in M \setminus S$, then $m(\bigvee S) = 0$ by Lemma 3.9. We then have m $(= \bigvee\{m\}) \leq a(\bigvee S) = \bigvee T$ by Proposition 2.7. This implies $m \in T$ by Lemma 3.14, and therefore we have $M \setminus S \subset T$. However, since $M \setminus S$ is an infinite subset and T is a finite subset, this is a contradiction. □

§4. Dichotomizer

First we define the notion of *dichotomizer*.

Definition 4.1. A nonzero element $d \in B$ is a *dichotomizer* if

$$B = M^\vee \cup \uparrow d.$$

Remark 4.2. If $B = M^\vee$, then any nonzero element is a dichotomizer.

We consider a subset

$$\mathfrak{S} = \{x \in B \colon x^2 = 0\}.$$

Lemma 4.3. *If d is a dichotomizer, then d belongs to $\mathfrak{S} \cup a(M^\vee)$.*

Proof. Assume that d is a dichotomizer and $d \notin \mathfrak{S}$. Then we have $d^2 \neq 0$, and so $d \not\leq a(d)$ by Proposition 2.7(2). Hence $a(d) \notin \uparrow d$. Since $B = M^\vee \cup \uparrow d$, we have $a(d) \in M^\vee$. Therefore, $d \in a(M^\vee)$. □

Lemma 4.4. *If $M^\vee \cup a(M^\vee)$ is a proper subset of B, then any $e \in a(M^\vee)$ is not a dichotomizer.*

Proof. For any $e \in a(M^\vee)$, we have $\uparrow e \subset \uparrow a(M^\vee) = a(M^\vee)$ by Lemma 3.13. This implies $M^\vee \cup \uparrow e \subset M^\vee \cup a(M^\vee)$. Since $M^\vee \cup a(M^\vee)$ is a proper subset of B, we have $M^\vee \cup \uparrow e \neq B$, and therefore e is not a dichotomizer. □

From Lemmas 4.3 and 4.4, we obtain the following.

Corollary 4.5. *If d is a dichotomizer and $M^\vee \cup a(M^\vee)$ is a proper subset of B, then $d^2 = 0$.*

Proposition 4.6. *If $M^\vee \cup a(M^\vee)$ is a proper subset of B, then every dichotomizer is minimal.*

Proof. Assume that d is a dichotomizer and $x < d$. This implies $x \notin \uparrow d$. Since $B = M^\vee \cup \uparrow d$, we have $x \in M^\vee$, and so $x = \bigvee S$ for a finite subset S of M. If $S \neq \varnothing$, then there exists $m \in S$. Hence $m \leq \bigvee S = x < d$. On the other hand, we have $m^2 = m \neq 0$ by Lemma 3.2, and $md = 0$ by Corollaries 3.10 and 4.5. This contradicts $m < d$. Hence $S = \varnothing$, and so $x = \bigvee S = \bigvee \varnothing = 0$. □

Proof of Theorem 1.4. Assume that d is a dichotomizer, that is, $B = M^\vee \cup \uparrow d$. By Corollary 3.7, it suffices to show that $BA \subset M^\vee \cup a(M^\vee)$. Take an element $x \in BA$. If $x \notin M^\vee$, then $x \in \uparrow d$. Hence $dx = d \neq 0$ by Lemma 3.11. This implies $a(x) \notin \uparrow d$ by Proposition 2.7(2), and so $a(x) \in M^\vee$ ($\Leftrightarrow x \in a(M^\vee)$). Therefore, $BA \subset M^\vee \cup a(M^\vee)$. The second claim is Proposition 3.15. □

Proof of Theorem 1.6. First we prove that $M^\vee \cap \uparrow d = \varnothing$. If there exists $x \in M^\vee \cap \uparrow d$, then $d \le x = \bigvee S$ for a finite subset S of M. Since $x \in BA$ by Corollary 3.7, we have

$$\begin{aligned} d &= dx && \text{by Lemma 3.11} \\ &= d\Big(\bigvee S\Big) = \bigvee_{m \in S} dm = 0 && \text{by Corollaries 3.10 and 4.5,} \end{aligned}$$

which is a contradiction. Therefore, $M^\vee \cap \uparrow d = \varnothing$.

Next we turn to the assertion that $\mathfrak{M} \cap \mathfrak{S} = \{d\}$. From Corollary 4.5 and Proposition 4.6, we obtain $d \in \mathfrak{M} \cap \mathfrak{S}$. If there exists an element $e \in \mathfrak{M} \cap \mathfrak{S}$ other than d, then $e \notin \uparrow d$. (Indeed, if $e \ge d (\neq 0)$, then $e = d$ since e is minimal.) Therefore, $e \in M^\vee$. This implies that $e \in BA$ by Corollary 3.7, and we have $e^2 = e \neq 0$ by Lemma 3.2. However, this contradicts $e \in \mathfrak{S}$. Therefore, $\mathfrak{M} \cap \mathfrak{S} = \{d\}$. □

§5. The case for $B = \beta(\mathbb{Z}/n)$

Let R be a commutative monoid with 0. From R we obtain a typical example $\beta(R)$ of monoidal posets as follows (see [3, §2]): For $x \in R$, we define

$$\langle x \rangle = \{c \in R : xc = 0\}.$$

We denote by $\beta(R)$ the set

$$\beta(R) = \{\langle x \rangle : x \in R\}.$$

Then $\beta(R)$ is a monoidal poset, whose structure is given by

- $\langle x \rangle \langle y \rangle = \langle xy \rangle$,
- $\langle x \rangle \le \langle y \rangle \;\Leftrightarrow\; \langle x \rangle \supset \langle y \rangle$.

Let P be a principal ideal domain, and (q) a nontrivial ideal of P. In [3, §4], the authors consider the monoidal poset $\beta(P/(q))$. By [3, Cor. 4.3], we see that

$$\beta(P/(q)) = \beta(\mathbb{Z}/n) \quad \text{for some } n \ge 2.$$

In this section we consider Problem 1.2 on $\beta(\mathbb{Z}/n)$ for $n \ge 2$.

We denote by $[x] \in \mathbb{Z}/n$ the class represented by an integer x. For $[x] \in \mathbb{Z}/n$, we denote

$$\langle x \rangle = \langle [x] \rangle \in \beta(\mathbb{Z}/n).$$

Proposition 5.1 (Cf. [3, Thm. 4.1]). *Let n be an integer ≥ 2.*

(1) $\beta(\mathbb{Z}/n) = \{\langle x\rangle : x \mid n\}$ *as sets. In particular,* $\langle n\rangle = \langle 0\rangle$.

(2) $\langle x\rangle \geq \langle y\rangle$ *in* $\beta(\mathbb{Z}/n)$ *if and only if* $x \mid y$.

Proof. First we prove (1). If a nonzero integer x is prime to n, then $ax + bn = 1$ for some $a, b \in \mathbb{Z}$. In particular, $[ax] = [1] \in \mathbb{Z}/n$. Take $[y] \in \langle x\rangle$. Then $[x][y] = [0] \in \mathbb{Z}/n$, and so $[y] = [ax][y] = [a][x][y] = [0]$. Hence $\langle x\rangle = \{[0]\} = \langle 1\rangle$. We put

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}, \tag{5.2}$$

where the p_i are different prime numbers and $e_i > 0$ for every i. In the case that x is not prime to n, we put

$$x = p_1^{f_1} \cdots p_k^{f_k} p_{k+1}^{f_{k+1}} \cdots p_l^{f_l},$$

where the p_i are different prime numbers, $f_i \geq 0$ for $1 \leq i \leq k$ and $f_i \geq 1$ for $k < i \leq l$. Here, the p_i for $1 \leq i \leq k$ are in (5.2). We remark that $p_{k+1}^{f_{k+1}} \cdots p_l^{f_l}$ is prime to n. We put $m_i = \min\{e_i, f_i\}$ for $1 \leq i \leq k$; then

$$\begin{aligned}\langle x\rangle &= \langle p_1^{f_1} \cdots p_k^{f_k} p_{k+1}^{f_{k+1}} \cdots p_l^{f_l}\rangle \\ &= \langle p_1^{f_1} \cdots p_k^{f_k}\rangle\langle p_{k+1}^{f_{k+1}} \cdots p_l^{f_l}\rangle = \langle p_1^{f_1} \cdots p_k^{f_k}\rangle = \langle p_1^{m_1} \cdots p_k^{m_k}\rangle,\end{aligned}$$

and $p_1^{m_1} \cdots p_k^{m_k}$ divides n. Therefore, we have $\beta(\mathbb{Z}/n) = \{\langle x\rangle : x \mid n\}$.

Next turn to (2). By (1) and (5.2), for any $\langle x\rangle$ and $\langle y\rangle$ in $\beta(\mathbb{Z}/n)$, we may consider

$$x = p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k} \quad \text{and} \quad y = p_1^{y_1} p_2^{y_2} \cdots p_k^{y_k},$$

where $0 \leq x_i \leq e_i$ and $0 \leq y_i \leq e_i$ for any i. We assume that $x \mid y$; then $x_i \leq y_i$ for any i. If $[z] \in \langle x\rangle$, then $p_1^{e_1 - x_1} \cdots p_k^{e_k - x_k} \mid z$, and so $p_1^{e_1 - y_1} \cdots p_k^{e_k - y_k} \mid z$. This implies $[z] \in \langle y\rangle$. Hence $\langle x\rangle \geq \langle y\rangle$. Conversely, we assume that $\langle x\rangle \geq \langle y\rangle$. Then $[xz] = [0]$ implies $[yz] = [0]$. Thus $p_1^{e_1 - x_1} \cdots p_k^{e_k - x_k} \mid z$ implies $p_1^{e_1 - y_1} \cdots p_k^{e_k - y_k} \mid z$. Hence $x_i \leq y_i$ for any i, and so $x \mid y$. ☐

Proposition 5.3 (Cf. [3, Cor. 4.4]). *The set* $\beta(\mathbb{Z}/n)$ *is a monoidally distributive poset.*

Proof. We use the notation in (5.2). By Proposition 5.1, any element of $\beta(\mathbb{Z}/n)$ is of the form $\langle p_1^{f_1} p_2^{f_2} \cdots p_k^{f_k}\rangle$, where $0 \leq f_i \leq e_i$ for $1 \leq i \leq k$. It is easy to see that

$$\langle p_1^{f_1} p_2^{f_2} \cdots p_k^{f_k}\rangle \vee \langle p_1^{g_1} p_2^{g_2} \cdots p_k^{g_k}\rangle = \langle p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k}\rangle, \tag{5.4}$$

where $l_i = \min\{f_i, g_i\}$. Then we immediately see that $\beta(\mathbb{Z}/n)$ is monoidally distributive. ☐

By the above argument, for $\beta(\mathbb{Z}/n)$, it is easy to see that

$$\mathfrak{M} = \{\langle n/p_i\rangle : 1 \le i \le k\} \tag{5.5}$$

under the notation of (5.2).

Lemma 5.6. *If $n = p_1p_2\cdots p_k$, where the p_i are different prime numbers and $k \ge 2$, then $\beta(\mathbb{Z}/n) = M^\vee \cup a(M^\vee)$.*

Proof. By Proposition 5.1, we have $\beta(\mathbb{Z}/n) = \{\langle p_1^{\varepsilon_1}p_2^{\varepsilon_2}\cdots p_k^{\varepsilon_k}\rangle : \varepsilon_i \in \{0,1\}$ for $1 \le i \le k\}$. Note that $\mathfrak{M} = \{\langle n/p_i\rangle : 1 \le i \le k\}$ by (5.5). From (5.4), we obtain

$$\langle n/p_i\rangle \vee a(\langle n/p_i\rangle) = \langle n/p_i\rangle \vee \langle p_i\rangle = \langle 1\rangle \quad \text{for } 1 \le i \le k.$$

Hence we have $M = \mathfrak{M} \cap BA = \mathfrak{M}$. This and (5.4) imply that

$$\langle p_1^{\varepsilon_1}p_2^{\varepsilon_2}\cdots p_k^{\varepsilon_k}\rangle = \bigvee_{\varepsilon_i=0} \langle n/p_i\rangle \in M^\vee$$

for any $\langle p_1^{\varepsilon_1}p_2^{\varepsilon_2}\cdots p_k^{\varepsilon_k}\rangle \in \beta(\mathbb{Z}/n)$, and therefore we have $\beta(\mathbb{Z}/n) = M^\vee$. This implies $\beta(\mathbb{Z}/n) = M^\vee \cup a(M^\vee)$. □

Lemma 5.7. *If $n = p_1^e p_2\cdots p_k$, where the p_i are different prime numbers and $e \ge 2$, then $\beta(\mathbb{Z}/n)$ has a dichotomizer $\langle p_1^{e-1}p_2\cdots p_k\rangle$.*

Proof. By Proposition 5.1 we have

$$\beta(\mathbb{Z}/n) = \left\{\langle p_1^{e_1}p_2^{\varepsilon_2}\cdots p_k^{\varepsilon_k}\rangle : 0 \le e_1 \le e \text{ and } \varepsilon_i \in \{0,1\} \text{ for } 2 \le i \le k\right\}.$$

Note that $\mathfrak{M} = \{\langle n/p_i\rangle : 1 \le i \le k\}$ by (5.5). Any $\langle n/p_i\rangle \in \mathfrak{M}$ satisfies

$$\langle n/p_i\rangle \vee a(\langle n/p_i\rangle) = \langle n/p_i\rangle \vee \langle p_i\rangle = \begin{cases} \langle p_1\rangle, & i = 1, \\ \langle 1\rangle, & 2 \le i \le k, \end{cases}$$

by $e \ge 2$ and (5.4). Therefore, we have $M = \mathfrak{M} \cap BA = \{\langle n/p_i\rangle : 2 \le i \le k\}$. This and (5.4) imply

$$M^\vee = \left\{\langle p_1^e p_2^{\varepsilon_2}\cdots p_k^{\varepsilon_k}\rangle : \varepsilon_i \in \{0,1\} \text{ for } 2 \le i \le k\right\}.$$

By Proposition 5.1, if $\langle p_1^{e_1}p_2^{\varepsilon_2}\cdots p_k^{\varepsilon_k}\rangle \notin \uparrow \langle p_1^{e-1}p_2\cdots p_k\rangle$, then $p_1^{e_1}p_2^{\varepsilon_2}\cdots p_k^{\varepsilon_k}$ does not divide $p_1^{e-1}p_2\cdots p_k$. Hence $e_1 = e$, and so $\langle p_1^{e_1}p_2^{\varepsilon_2}\cdots p_k^{\varepsilon_k}\rangle = \langle p_1^e p_2^{\varepsilon_2}\cdots p_k^{\varepsilon_k}\rangle \in M^\vee$. Therefore, $\beta(\mathbb{Z}/n) = M^\vee \cup \uparrow \langle p_1^{e-1}p_2\cdots p_k\rangle$. □

Lemma 5.8. *If $p_1^2p_2^2 \mid n$ where p_1 and p_2 are different prime numbers, then $\beta(\mathbb{Z}/n)$ has no dichotomizer.*

Proof. Note that $\beta(\mathbb{Z}/n)$ contains $\langle n/p_1\rangle$ and $\langle n/p_2\rangle$. It is easy to see that $\langle n/p_i\rangle^2 = \langle 0\rangle$ for $i \in \{1,2\}$. Furthermore, $\langle n/p_i\rangle$ is minimal for $i \in \{1,2\}$. Hence $\mathfrak{M}\cap\mathfrak{S}$ contains different two elements $\langle n/p_1\rangle$ and $\langle n/p_2\rangle$, and so $\beta(\mathbb{Z}/n)$ has no dichotomizer by Theorem 1.6. □

Proof of Theorem 1.7. Note that any $n \geq 2$ is of the form

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k},$$

where the p_i are different prime numbers and

$$e_1 \geq e_2 \geq \cdots \geq e_k \geq 0.$$

In the case for $(e_1, e_2) = (1, 0)$, that is, $n = p_1$, we have $\beta(\mathbb{Z}/n) = \{\langle 1\rangle, \langle p_1\rangle\} = \{\langle 0\rangle, \langle 1\rangle\}$ by Proposition 5.1. In this case, we have $M = \{1\}$, and so $\beta(\mathbb{Z}/n) = M^\vee \cup a(M^\vee)$.

In the case for $(e_1, e_2) = (1, 1)$, that is, $n = p_1 p_2 \cdots p_\ell$ with $\ell \geq 2$, we have $\beta(\mathbb{Z}/n) = M^\vee \cup a(M^\vee)$ by Lemma 5.6.

In the case for $e_1 \geq 2 > e_2$, that is, $n = p_1^e p_2 \cdots p_\ell$ with $e \geq 2$, $\beta(\mathbb{Z}/n)$ has a dichotomizer $\langle p_1^{e-1} p_2 \cdots p_k\rangle$ by Lemma 5.7. We note that $\mathfrak{S} = \{\langle p_1^f p_2 \cdots p_k\rangle \colon 2/e \leq f \leq e\}$ by (5.4). From this and (5.5), we obtain $\mathfrak{M} \cap \mathfrak{S} = \{\langle p_1^{e-1} p_2 \cdots p_k\rangle\}$.

In the case for $e_2 \geq 2$, that is, $p_1^2 p_2^2 \mid n$ and $p_1 \neq p_2$, $\beta(\mathbb{Z}/n)$ has no dichotomizer by Lemma 5.8. By the proof of Lemma 5.8, the subset $\mathfrak{M} \cap \mathfrak{S}$ is not of the form $\{d\}$. □

§6. The case for $B = \mathbb{B}$

Let p be a prime number. In this section we consider the case for $B = \mathbb{B}$, the Bousfield lattice of the stable homotopy category $\mathcal{S}_p$ of p-local spectra. Recall that $\mathbb{B} = \{\langle X\rangle \colon X \in \mathcal{S}_p\}$, where $\langle X\rangle = \{W \in \mathcal{S}_p \colon X \wedge W = 0\}$. This is a monoidally distributive poset, whose lattice structure is given by

$$\langle X\rangle \leq \langle Y\rangle \;\;\Leftrightarrow\;\; \langle X\rangle \supset \langle Y\rangle.$$

We also have $\langle X\rangle\langle Y\rangle = \langle X \wedge Y\rangle$, $\langle X\rangle \vee \langle Y\rangle = \langle X \vee Y\rangle$, $0 = \langle 0\rangle$ and $1 = \langle S^0\rangle$. Here, S^0 is the p-local sphere spectrum.

Let $F(n)$ be a finite spectrum of type n, and $T(n)$ the telescope of a v_n-self-map on $F(n)$. By [1, Lems 1.2 and 1.3], the Bousfield classes $\langle F(n)\rangle$ and $\langle T(n)\rangle$ depend on only n. We also note that the Bousfield class $\langle K(n)\rangle$ of the nth Morava K-theory spectrum is minimal for any $n \geq 0$ [1, Cor. 1.7].

Conjecture 6.1 (Telescope conjecture [5, 10.5]). *We have $\langle K(n)\rangle = \langle T(n)\rangle$ for any $n \geq 0$.*

For a spectrum E, we have the Bousfield localization functor $L_E\colon \mathcal{S}_p \to \mathcal{S}_p$ with respect to E. We define the spectrum $A(n)$ by the cofiber sequence

$$F(n) \to L_{K(n)}F(n) \to A(n) \to \Sigma F(n).$$

By [1, Prop. 1.6], we know that

- $\langle A(n)\rangle$ depends on only n,
- $\langle T(n)\rangle = \langle K(n)\rangle \vee \langle A(n)\rangle$ for any $n \geq 0$,
- $A(n) \wedge K(m) = 0$ for all m.

Furthermore, $\langle A(n)\rangle$ belongs to BA, and $A(n) \wedge A(n) = A(n)$ for any $n \geq 0$ (see [2, §5]).

Lemma 6.2. *The following are equivalent:*

(1) $\langle K(n)\rangle = \langle T(n)\rangle$.

(2) $A(n) = 0$.

Proof. If $A(n) = 0$, then $\langle T(n)\rangle = \langle K(n)\rangle \vee \langle A(n)\rangle = \langle K(n)\rangle$. Conversely, if $\langle K(n)\rangle = \langle T(n)\rangle$, then we have $\langle K(n)\rangle \vee \langle A(n)\rangle = \langle T(n)\rangle = \langle K(n)\rangle$. This implies $\langle A(n)\rangle \leq \langle K(n)\rangle$. Since $A(n) \wedge K(n) = 0$, we have $A(n) = A(n) \wedge A(n) = 0$. □

Hovey and Palmieri modified Conjecture 6.1 as follows.

Conjecture 6.3 ([2, Conj. 5.1]). *For any $n \geq 0$, the Bousfield class $\langle A(n)\rangle$ is 0 or minimal.*

Remark 6.4. In 2023, a disproof of Conjecture 6.1 was announced by Burkland, Hahn, Levy and Schlank. It has not yet been published.

For a spectrum E, a spectrum X is *E-acyclic* if $E \wedge X = 0$, and a spectrum Y is *E-local* if any morphism $X \to Y$ is trivial for any E-acyclic spectrum X. Furthermore, a spectrum E *has a finite acyclic* (resp. *a finite local*) if there exists a nonzero finite spectrum which is E-acyclic (resp. E-local). We denote $L_n^f = L_{T(0)\vee T(1)\vee\cdots\vee T(n)}$. Then $\langle L_n^f S^0\rangle = \bigvee_{i=0}^{n}\langle T(i)\rangle$. Furthermore, $\langle L_n^f S^0\rangle$ is in BA, and $a(\langle L_n^f S^0\rangle) = \langle F(n+1)\rangle$ (see [2, §5]).

Hereafter, for the sake of simplicity, we denote

$$f_n = \langle F(n)\rangle, \quad t_n = \langle T(n)\rangle, \quad k_n = \langle K(n)\rangle, \quad a_n = \langle A(n)\rangle \quad \text{and} \quad \ell_n^f = \langle L_n^f S^0\rangle.$$

We put

$$\mathfrak{A} = \{n \geq 0\colon a_n \neq 0\}.$$

We recall the subset M in (3.5). If Conjecture 6.3 is true, then we have

$$KA := \{k_n, a_m\colon n \geq 0, m \in \mathfrak{A}\} \subset M.$$

Proposition 6.5 ([2, Conj. 5.1]). *Assume that Conjecture 6.3 is true. Then, for any spectrum X which has a finite acyclic, the Bousfield class $x = \langle X\rangle$ belongs to*

$$KA^{\vee} = \left\{\bigvee S : S \text{ is a finite subset of } KA\right\}.$$

Proof. If a nonzero spectrum X has a finite acyclic, then $x = \langle X\rangle$ satisfies that $xf_{n+1} = 0$ for some $n \geq 0$. Hence we have

$$\begin{aligned} x &= x \cdot 1 = x(\ell_n^f \vee f_{n+1}) = x\ell_n^f \vee xf_{n+1} \\ &= x\ell_n^f = x\left(\bigvee_{i=0}^{n} t_i\right) = \bigvee_{i=0}^{n} xt_i = \bigvee_{i=0}^{n} x(k_i \vee a_i) \\ &= \bigvee_{i\in K(x)} k_i \vee \bigvee_{i\in T(x)} a_i \qquad \text{by the assumption and Lemma 3.8,} \end{aligned}$$

where $K(x) = \{i\colon 0 \leq i \leq n,\ xk_i = k_i\}$ and $T(x) = \{i \in \mathfrak{A}\colon 0 \leq i \leq n,\ xa_i = a_i\}$. Therefore $x \in KA^{\vee}$. ☐

We consider the Bousfield class

$$i = \langle I\rangle$$

where I is the Brown–Comenetz dual of the sphere spectrum.

Proposition 6.6 ([2, Prop. 7.2]). *A spectrum X has a finite local if and only if the Bousfield class $x = \langle X\rangle$ satisfies $x \geq i$.*

Proposition 6.7 ([2, Cor. 7.11]). *If Conjectures 1.1 and 6.3 are true, then*

$$BA = KA^{\vee} \cup a(KA^{\vee}),$$

where $a(KA^{\vee}) = \{a(x)\colon x \in KA^{\vee}\}$. Furthermore, $KA^{\vee} \cap a(KA^{\vee}) = \varnothing$.

Proof. Since $KA \subset M \subset BA$, we have $KA^{\vee} \cup a(KA^{\vee}) \subset BA$ by (3.1) and Lemma 3.4. Hence we prove that $BA \subset KA^{\vee} \cup a(K^{\vee})$. Take an element $x = \langle X\rangle \in BA$. If X has a finite acyclic, then $x \in KA^{\vee}$ by Proposition 6.5. If X has no finite acyclic, then, since we assume that Conjecture 1.1 is true, X has a finite local. Hence, by Proposition 6.6, we have $x \geq i$. From Lemma 3.11, we obtain $xi = i \neq 0$, which implies $a(x) \not\geq i$ by Proposition 2.7(2). Let aX be a spectrum such that $\langle aX\rangle = a(x)$. Then aX has no finite local by Proposition 6.6. This and Conjecture 1.1 imply that aX has a finite acyclic, and so $a(x) \in KA^{\vee}$ by Proposition 6.5. Therefore, $x \in a(KA^{\vee})$.

We prove the second claim $KA^{\vee} \cap a(KA^{\vee}) = \varnothing$. Since $KA \subset M$ and KA contains all the k_n the subset M is an infinite subset. Therefore, by Proposition 3.15, we have $KA^{\vee} \cap a(KA^{\vee}) \subset M^{\vee} \cap a(M^{\vee}) = \varnothing$. ☐

Proposition 6.8. *If Conjectures 1.1 and 6.3 are true, then i is a dichotomizer of $\mathbb{B}$. Furthermore, $\mathbb{B} = KA^\vee \cup \uparrow i = M^\vee \cup \uparrow i$ and $KA^\vee \cap \uparrow i = \varnothing$.*

Proof. Take an element $x = \langle X \rangle \in \mathbb{B}$. Since we assume that Conjecture 1.1 is true, if $x \notin \uparrow i$, then X has a finite acyclic by Proposition 6.6. Thus, $x \in KA^\vee$ by Proposition 6.5. Therefore, we have $\mathbb{B} \subset KA^\vee \cup \uparrow i \subset M^\vee \cup \uparrow i \subset \mathbb{B}$.

Next turn to $KA^\vee \cap \uparrow i = \varnothing$. If there exists $y \in KA^\vee \cap \uparrow i$, then $i \le y = \bigvee S$ for a finite subset S of KA. We put $\overline{S} = \{k_n \vee a_n \colon k_n \in S \text{ or } a_n \in S\} = \{t_n \colon k_n \in S \text{ or } a_n \in S\}$. Then $i \le y \le \bigvee \overline{S} \le \bigvee_{0 \le i \le N} t_i = \ell^f_N$ for some $N \ge 0$. Since $\ell^f_N f_{N+1} = 0$, we have $i f_{N+1} = 0$. On the other hand, we know that $i f_n = i$ for any $n \ge 0$ [2, Lem. 7.1(e)]. Therefore, $i f_{N+1} = 0$ is a contradiction, and so $KA^\vee \cap \uparrow i = \varnothing$. □

Theorem 6.9. *If $KA^\vee \cap \uparrow i \ne \varnothing$ or $\mathfrak{M} \cap \mathfrak{S} \ne \{i\}$, then at least one of Conjecture 1.1 and Conjecture 6.3 does not hold.*

Proof. By Proposition 6.8, if Conjectures 1.1 and 6.3 hold, then i is a dichotomizer. We note that $i^2 = 0$ [2, Lem. 7.1(c)]. This and Lemma 3.3 imply that BA is a proper subset of $\mathbb{B}$, and so $M^\vee \cup a(M^\vee)$ ($\subset BA$) is a proper subset. Hence, by Theorem 1.6, we have $KA^\vee \cap \uparrow i \subset M^\vee \cap \uparrow i = \varnothing$ and $\mathfrak{M} \cap \mathfrak{S} = \{i\}$. □

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References

[1] M. Hovey, Bousfield localization functors and Hopkins' chromatic splitting conjecture, in *The Čech centennial (Boston, MA, 1993)*, Contemporary Mathematics 181, American Mathematical Society, Providence, RI, 1995, 225–250. Zbl 0830.55004 MR 1320994

[2] M. Hovey and J. H. Palmieri, The structure of the Bousfield lattice, in *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, Contemporary Mathematics 239, American Mathematical Society, Providence, RI, 1999, 175–196. Zbl 0947.55014 MR 1718080

[3] R. Kato, K. Shimomura and Y. Tatehara, Generalized Bousfield lattices and a generalized retract conjecture, Publ. Res. Inst. Math. Sci. **50** (2014), 497–513. Zbl 1305.55006 MR 3262447

[4] T. Ohkawa, The injective hull of homotopy types with respect to generalized homology functors, Hiroshima Math. J. **19** (1989), 631–639. Zbl 0697.55008 MR 1035147

[5] D. C. Ravenel, Localization with respect to certain periodic homology theories, Amer. J. Math. **106** (1984), 351–414. Zbl 0586.55003 MR 0737778