Publ. RIMS Kyoto Univ. **61** (2025), 233–275
DOI 10.4171/PRIMS/61-2-2

A Characterization of Minimal Extended Affine Root Systems (Relations to Elliptic Lie Algebras)

by

Saeid Azam, Fatemeh Parishani and Shaobin Tan

Abstract

Extended affine root systems appear as the root systems of extended affine Lie algebras. A subclass of extended affine root systems, whose elements are called “minimal”, turns out to be of special interest, mostly because of the geometric properties of their Weyl groups; they possess the so-called *presentation by conjugation*. In this work, we characterize minimal extended affine root systems in terms of “minimal reflectable bases”, which resembles the concept of the “base” for finite and affine root systems. As an application, we construct elliptic Lie algebras by means of Serre-type generators and relations.

Mathematics Subject Classification 2020: 17B67 (primary); 20F55, 17B22, 17B65 (secondary).
Keywords: extended affine root systems, extended affine Weyl groups, reflectable bases, elliptic Lie algebras, presentations of Lie algebras.

§0. Introduction

Motivated by applications in “the construction of a flat structure for the base space of the universal deformation of a simple elliptic singularity”, Saito [Sa] introduced the concept of an extended affine root system. Considering [Sl], he predicted the “existence of Lie algebras corresponding to extended affine root systems which

Communicated by T. Arakawa. Received April 7, 2022. Revised May 10, 2023.

S. Azam: Department of Pure Mathematics, Faculty of Mathematics and Statistics, University of Isfahan, P.O. Box 81746-73441 Isfahan; School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746 Tehran, Iran;
e-mail: azam@ipm.ir
F. Parishani: Department of Pure Mathematics, Faculty of Mathematics and Statistics, University of Isfahan, P.O. Box 81746-73441 Isfahan; School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746 Tehran, Iran;
e-mail: f.parishani93@sci.ui.ac.ir, f.parishani93@yahoo.com
S. Tan: School of Mathematical Sciences, Xiamen University, 361005 Xiamen, P. R. China;
e-mail: tans@xmu.edu.cn

would describe the universal deformation of the simple elliptic singularity". The prediction was partially answered in [H-KT, P]. Later, in 1997, in a systematic study, a class of algebras associated with extended affine root systems was introduced via a set of axioms [AABGP]. Each element of this class is called an *extended affine Lie algebra*. Since then, the theory of extended affine root systems (Lie algebras) (see Definitions 1.1 and 6.1) and related topics have been under intensive investigation. It turns out that despite the striking similarities with the finite, and affine Lie theory, there are enough interesting differences to make the study of extended affine Lie theory a demanding subject. In what follows, we explain the motivation of the present work at two levels: the level of the root system and Weyl group, and the level of Lie algebra.

Level of root system and Weyl group. One of the astonishing privileges of finite and affine Weyl groups is that they are Coxeter groups with respect to certain appropriate generating sets. In finite types, these generating sets consist of reflections based on elements of a fundamental system of the corresponding root system [C, Hu2, MP]; thus by [B, Chap. IV, §8, Cor. 3], these generating sets are "minimal". In contrast, it is known that the Weyl group of an extended affine root system R is not, in general, a Coxeter group. In fact, one knows that an extended affine Weyl group is a Coxeter group if and only if the corresponding root system is either finite or affine [Ho, Thm. 3.6]. Therefore, finding a suitable presentation for an extended affine Weyl group has always been a demanding problem; see for example [Kr2, Kr1, A1, A2, AS1, AS2, AS4, AS3, AS5, SaT, AN]. A possible approach to achieve such a presentation is as follows. In the theory of root datum, it is known that the Coxeter presentation implies the so-called presentation by conjugation [MP, Chap. 5,§3]; see Definition 2.6. Therefore, it is natural to ask whether an extended affine Weyl group possesses the presentation by conjugation. This question was raised first by Krylyuk and was affirmatively answered in [Kr1] for simply laced extended affine Weyl groups of rank > 1. Later, in a series of works, the question was considered for general types [A2, Ho, AS4, AS3, AS5]. In brief, it is revealed that only elements of a subclass of extended affine Weyl groups enjoy the presentation by conjugation. The root system corresponding to an element of this class is called a "minimal" extended affine root system (see Definition 5.1). For general extended affine Weyl groups, one gives a weaker presentation, called the *generalized presentation by conjugation* [A2], which coincides with the presentation by conjugation if the corresponding root system is minimal. This explains our motivation for the study of minimal extended affine root systems.

The main objective of this work is to give a characterization of minimal extended affine root systems of reduced types in terms of minimal reflectable bases; see Theorem 5.6. We achieve this as a by-product of a study of interrelations between three classes of certain subsets of extended affine root systems which we denote by $\mathcal{M}_r$, $\mathcal{M}_m$ and $\mathcal{M}_c$, and we explain them and their related terminologies below. Let R be a reduced extended affine root system with the set of non-isotropic roots $R^\times$, and the Weyl group $\mathcal{W}$. For a subset $\mathcal{P}$ of $R^\times$, we denote the subgroup of $\mathcal{W}$ generated by reflections based on $\mathcal{P}$ by $\mathcal{W}_\mathcal{P}$. The set $\mathcal{P}$ is called a *reflectable base* for R if $\mathcal{W}_\mathcal{P}\mathcal{P} = R^\times$ and no proper subset of $\mathcal{P}$ has this property. The class of all such sets is denoted by $\mathcal{M}_r$. The class of all sets $\mathcal{P} \subseteq R^\times$ which are minimal with respect to the property that $\mathcal{W}_\mathcal{P} = \mathcal{W}$ is denoted by $\mathcal{M}_m$, and the class of all sets $\mathcal{P} \subseteq R^\times$ which have the minimal cardinality with respect to the property $\mathcal{W}_\mathcal{P} = \mathcal{W}$ is denoted by $\mathcal{M}_c$. Let $\mathcal{M}_{rm} = \mathcal{M}_r \cap \mathcal{M}_m$. In the finite theory, one concludes that $\mathcal{M}_r = \mathcal{M}_m = \mathcal{M}_c$; see Proposition 3.8.

Level of Lie algebra. Saito [Sa] conjectured that starting from any extended affine root system R there should be a Lie algebra with root system R. This led to many interesting investigations concerning the construction and classification of such Lie algebras; see for example [H-KT, AABGP, Neh]. An extended affine root system of nullity 2 (see Definition 1.1) is called an *elliptic root system*, and the corresponding Lie algebra is called an *elliptic Lie algebra*. In [SaY, Ya], the authors construct certain elliptic Lie algebras of rank > 1 through Serre-type generators and relations, using a concept of "base" which resembles the usual notion of base for finite and affine root systems. The bases used by [SaY, Ya] are not in general reflectable bases; they involve more roots than reflectable bases. One of our motivations for this work is to examine the concept of a reflectable base for constructing elliptic Lie algebras by a Serre-type presentation. We investigate this in Section 6, in particular, we construct elliptic Lie algebras of rank 1, the missing case in the works of [SaY, Ya]. We now explain the core materials appearing in each section.

In Section 1, we recall from [AABGP, Def. II.2.1] the definition of an extended affine root system and its internal structure in terms of a finite root system and certain subsets of the radical of the form called semilattices, where for our purposes in this work, we have modified the approach given in [AABGP, Chap. II, §2]; see Proposition 1.10. In Section 2, the Weyl group of an extended affine root system called an *extended affine Weyl group* is considered and a presentation for it called the *generalized presentation by conjugation* is explained. Some immediate related results are derived; see Corollary 2.8 and Lemma 2.9.

In Section 3, we introduce the classes $\mathcal{M}_c$, $\mathcal{M}_r$ and $\mathcal{M}_m$ for an extended affine root system R. It is shown that each element $\mathcal{P}$ of $\mathcal{M}_r$, $\mathcal{M}_c$ or $\mathcal{M}_m$ is a connected generating set for the root lattice; see Proposition 3.5. Moreover, associated to each connected subset $\mathcal{P} \subseteq R^\times$, a method of assigning an extended affine root subsystem $R_\mathcal{P}$ is provided such that if $\mathcal{P}$ belongs to either $\mathcal{M}_m$ or $\mathcal{M}_c$, then $R_\mathcal{P}$ has the same rank, nullity and type of R and if $\mathcal{P} \in \mathcal{M}_r$, $R_\mathcal{P} = R$; see Proposition 3.6. Finally, it is derived that for the finite case, we have $\mathcal{M}_r = \mathcal{M}_m = \mathcal{M}_c$ and that this equality fails for general extended affine root systems; see Proposition 3.8 and Example 3.9.

In Section 4, a characterization theorem for reflectable bases associated with reduced extended affine root systems is recorded from [AYY, Sect. 3] and [ASTY]. Then the cardinality of each element $\mathcal{P}$ in $\mathcal{M}_r$, $\mathcal{M}_m$ or $\mathcal{M}_c$ is investigated. It is shown that the cardinality of $\mathcal{P}$ is finite and, when R is of type A_1 or one of the non-simply-laced types, all reflectable bases (the elements of $\mathcal{M}_r$) have the same cardinality which can be described precisely in terms of the rank, nullity, twist number and indices of the involved semilattices. When R is of one of the simply laced types with rank > 1, $\mathcal{M}_m = \mathcal{M}_r$, and reflectable bases may have different cardinalities; see Proposition 4.4, Theorem 4.6, Proposition 4.10 and Example 4.9.

In Section 5, the concept of a minimal extended affine root system is recalled from the literature. Starting from a reflectable base $\mathcal{P}$, a presented group $\widehat{\mathcal{W}}$ defined by generators $\widehat{w}_\alpha$, $\alpha \in \mathcal{P}$, and certain conjugation relations, see Definition 2.6, is associated to R. It is shown that $\{\widehat{w}_\alpha \mid \alpha \in \mathcal{P}\}$ is a minimal generating set for $\widehat{\mathcal{W}}$; see Proposition 5.3. This utilizes the proof of the main result of the paper: a reduced extended affine root system R is minimal if and only if $\mathcal{M}_r = \mathcal{M}_{rm}$, if and only if the corresponding Weyl group has the presentation by conjugation. For details see Theorem 5.4. The section is concluded with Table 3, which illustrates the connections between the results obtained in Sections 1–5.

We now explain the concluding section, Section 6, in which some applications of the concept of a reflectable base are examined at the level of Lie algebra. It is declared that the core $\mathcal{L}_c$ of an extended affine Lie algebra $\mathcal{L}$ is finitely generated by showing that it is generated as a Lie algebra by the 1-dimensional root spaces $\mathcal{L}_{\pm\alpha}$, $\alpha \in \mathcal{P}$, $\mathcal{P}$ being a reflectable base for the ground root system R. Moreover, if R has index zero (see **1.13**(1.7)), then $\mathcal{P}$ satisfies a minimality condition concerning this property, Lemma 6.6. The rest of the section deals with constructing an elliptic Lie algebra utilizing a Serre-type presentation based on a reflectable base; see Proposition 6.13. As a by-product of this construction, we provide an elliptic Lie algebra of rank 1 whose non-isotropic root spaces are 1-dimensional, the missing case in [SaY, Ya]. We conclude the introduction by mentioning that in a recent work, see [AFI], reflectable bases have been used effectively in equipping the core of an extended affine Lie algebra of rank > 1 with an integral structure, a priority in establishing a modular theory for extended affine Lie algebras.

§1. Preliminaries

In this section we provide some preliminaries on extended affine root systems, and in particular, we describe the structure of a reduced extended affine root system in terms of a finite root system and certain subsets called *semilattices*.

All vector spaces are finite-dimensional and considered over the field $\mathbb{R}$. For a vector space $\mathcal{U}$, we denote the dual space of $\mathcal{U}$ by $\mathcal{U}^*$. For a subset S of $\mathcal{U}$, we denote by $\langle S\rangle$, the additive subgroup of $\mathcal{U}$ generated by S. The symbol $\uplus$ denotes the disjoint union of sets and $|\mathcal{P}|$ shows the cardinality of a set $\mathcal{P}$.

For a vector space $\mathcal{U}$ equipped with a symmetric bilinear form $(\cdot,\cdot)$, we denote the radical of the form by $\mathcal{U}^0$. For a subset R of $\mathcal{U}$, we set $R^0 := R\cap\mathcal{U}^0$ and $R^\times := R\setminus R^0$. A subset $\mathcal{P}$ of $\mathcal{U}^\times$ is called *connected* (with respect to the form), if $\mathcal{P}$ cannot be decomposed as $\mathcal{P}_1\uplus\mathcal{P}_2$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are non-empty subsets of $\mathcal{P}$ satisfying $(\mathcal{P}_1,\mathcal{P}_2)=\{0\}$.

§1.1. Extended affine root systems

Definition 1.1. Assume that $\mathcal{V}$ is a finite-dimensional real vector space equipped with a positive semidefinite symmetric bilinear form $(\cdot,\cdot)$. Suppose R is a subset of $\mathcal{V}$. Following [AABGP, Def. 2.1], we say R is an *extended affine root system* if

(1) $0\in R$;
(2) $R=-R$;
(3) R spans $\mathcal{V}$;
(4) if $\alpha\in R^\times$, then $2\alpha\notin R$;
(5) R is discrete in $\mathcal{V}$, with respect to the natural topology of $\mathcal{V}$ when identified with $\mathbb{R}^n$, $n=\dim\mathcal{V}$;
(6) if $\alpha\in R^\times$ and $\beta\in R$, then there exist non-negative integers d, u satisfying $\{\beta+n\alpha\mid n\in\mathbb{Z}\}\cap R=\{\beta-d\alpha,\dots,\beta+u\alpha\}$ and $d-u=\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$;
(7) $R^\times$ is connected;
(8) if $\sigma\in R^0$, then there exists $\alpha\in R^\times$ such that $\alpha+\sigma\in R$.

Elements of $R^\times$ are called *non-isotropic roots*, and elements of R^0 are called *isotropic roots*. The dimension ν of $\mathcal{V}^0$ is called the *nullity* of R. Since the form is positive semidefinite, one can easily check that

$$R^0=\{\alpha\in R\mid(\alpha,\alpha)=0\}\quad\text{and}\quad R^\times=\{\alpha\in R\mid(\alpha,\alpha)\neq0\}.$$

As announced in [AF], the definition of an extended affine root system is equivalent to the following.

Definition 1.2. Let $\mathcal{V}$ be a finite-dimensional real vector space equipped with a positive semidefinite symmetric bilinear form $(\cdot,\cdot)$, and R be a subset of $\mathcal{V}$. Then R is called an *extended affine root system* if the following axioms hold:

(R1) $\langle R\rangle$ is a full lattice in $\mathcal{V}$;

(R2) $(\beta,\alpha^\vee)\in\mathbb{Z}$, $\alpha,\beta\in R^\times$;

(R3) $w_\alpha(\beta)\in R$ for $\alpha\in R^\times$, $\beta\in R$;

(R4) $R^0=\mathcal{V}^0\cap(R^\times-R^\times)$;

(R5) $\alpha\in R^\times\Rightarrow 2\alpha\notin R$;

(R6) $R^\times$ is connected.

Remark 1.3. Concerning Definition 1.2, we emphasize the following points:

(i) Axiom (R1) means that the natural map $\langle R\rangle\otimes_{\mathbb{Z}}\mathbb{R}\to\mathcal{V}$ is a vector space isomorphism. In other words $\langle R\rangle$ is the $\mathbb{Z}$-span of an $\mathbb{R}$-basis of $\mathcal{V}$.

(ii) From (R4), we conclude that $R^\times\neq\emptyset$.

(iii) The relations between extended affine root systems and the root systems defined by Saito [Sa] are investigated in [A3].

Let R be an extended affine root system in a vector space $\mathcal{V}$ and let $^-\colon\mathcal{V}\to\overline{\mathcal{V}}:=\mathcal{V}/\mathcal{V}^0$ be the canonical map. It is known that $\bar{R}$, the image of R under the map $^-$, is an irreducible finite root system in $\bar{\mathcal{V}}$; see [AABGP, Prop. II.2.9(d)].

Definition 1.4. The *type* and the *rank* of extended affine root system R are defined as the type and the rank of $\bar{R}$. In this work, we always assume that R is of *reduced type*, that is, $\bar{R}$ has one of the types A, B, C, D, $E_{6,7,8}$, F_4 or G_2.

Definition 1.5. For a real vector space $\mathcal{U}$ equipped with a symmetric bilinear form $(\cdot,\cdot)$, and $\alpha\in\mathcal{U}$ with $(\alpha,\alpha)\neq 0$, the *reflection based on* α is defined by $\beta\mapsto\beta-\frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$, $\beta\in\mathcal{U}$. By convention, if $(\alpha,\alpha)=0$, we interpret w_α as the identity map. For a subset $\mathcal{P}\subseteq\mathcal{U}$, we denote by $\mathcal{W}_{\mathcal{P}}$ the subgroup of $\mathrm{GL}(\mathcal{U})$ generated by w_α, $\alpha\in\mathcal{P}$.

§1.2. Semilattices

Let R be an extended affine root system in $\mathcal{V}$ with corresponding form $(\cdot,\cdot)$. As was mentioned before, we denote the radical of $(\cdot,\cdot)$ by $\mathcal{V}^0$. Certain subsets of $\mathcal{V}^0$ called *semilattices* play an important role in describing the internal structure of extended affine root systems.

Definition 1.6. A *semilattice* in $\mathcal{V}^0$ is a spanning subset S of $\mathcal{V}^0$ which is discrete, contains 0 and satisfies $S\pm 2S\subseteq S$.

Here we record some facts about semilattices needed in the sequel.

1.7. Let S be a semilattice in $\mathcal{V}^0$ and set $\Lambda := \langle S\rangle$.

(i) Λ is a lattice in $\mathcal{V}^0$; that is, Λ is the $\mathbb{Z}$-span of a basis of $\mathcal{V}^0$.

(ii) $2\Lambda + S \subseteq S \subseteq \Lambda$ and so $S = \bigcup_{i=0}^{m}(\tau_i + 2\Lambda)$, for some $\tau_i \in S$, with $\tau_0 = 0$.

(iii) Λ admits a $\mathbb{Z}$-basis $B = \{\sigma_1, \ldots, \sigma_\nu\}$ consisting of elements of S.

Definition 1.8. Assume that S is a semilattice with $\Lambda = \langle S\rangle$.

(i) The *index* of S, denoted $\mathrm{ind}(S)$, is by definition the least positive integer m such that 1.7(ii) holds.

(ii) Assume that $\mathrm{ind}(S) = m$, and the τ_i and B are as in 1.7(iii). For $\sigma = \sum_{i=1}^{\nu} n_i\sigma_i \in \Lambda$, we define $\mathrm{supp}(\sigma) := \{i \mid n_i \in 2\mathbb{Z}+1\}$. Then $\sigma = \sum_{i\in\mathrm{supp}(\sigma)} \sigma_i$ (mod 2Λ). The collection $\mathrm{supp}(S) = \{\mathrm{supp}(\tau_i) \mid 0 \le i \le m\}$, is called the *supporting class* of S with respect to the basis B.

The supporting class determines S uniquely. In fact, we have

$$S = \biguplus_{J\in\mathrm{supp}(S)} (\tau_J + 2\Lambda), \tag{1.1}$$

where $\tau_J := \sum_{r\in J} \sigma_r$.

§1.3. Internal structure of extended affine root systems

In [AABGP, Chap. II, §2], the integral structure of an extended affine root system in terms of a finite root system and certain semilattices is studied. Based on our purposes in this work, we need to make some modifications to [AABGP]'s approach. As before, we assume that R is a reduced extended affine root system in $\mathcal{V}$ equipped with the positive semidefinite form $(\cdot,\cdot)$.

We start by recording the following lemma whose proof for simply laced types is given in [ASTY, Prop. 2.8], and for non-simply-laced types concludes from [AYY, Props 2.6, 2.9 and Rem. 2.15].

Lemma 1.9. *Let Φ be a reduced irreducible finite root system in a vector space $\mathcal{U}$, and $\mathcal{P}$ be a minimal subset (with respect to inclusion) of $\Phi^\times$ such that $\mathcal{W}_\mathcal{P}\mathcal{P} = \Phi^\times$. Then $|\mathcal{P}| = \mathrm{rank}\,\Phi$.*

Proposition 1.10. *Let R be an extended affine root system and $\mathcal{P}$ be a subset of $R^\times$ satisfying $\mathcal{W}_\mathcal{P}\mathcal{P} = R^\times$. Then $\mathcal{P}$ contains a subset $\dot\Pi$ such that the following hold:*

(i) *$\dot R := (\mathcal{W}_{\dot\Pi}\dot\Pi) \cup \{0\}$ is a finite root system in $\dot{\mathcal{V}} := \mathrm{span}_\mathbb{R}\,\dot R$;*

(ii) *$\dot{R}$ is isomorphic to $\bar{R}$;*

(iii) *$\mathcal{V} = \dot{\mathcal{V}} \oplus \mathcal{V}^0$;*

(iv) *for a fixed $\alpha \in R^\times$, there exists a finite root system $\dot{R}_\alpha \subseteq R$ such that $\alpha \in \dot{R}_\alpha$ and* (ii) *and* (iii) *hold with $\dot{R}_\alpha$ and $\dot{\mathcal{V}}_\alpha := \mathrm{span}_{\mathbb{R}}\, \dot{R}_\alpha$ in place of $\dot{R}$ and $\dot{\mathcal{V}}$.*

Proof. Since $R^\times = \mathcal{W}_{\mathcal{P}}\mathcal{P}$, we have $\bar{R}^\times = \mathcal{W}_{\overline{\mathcal{P}}}\overline{\mathcal{P}}$. Let $\overline{\Pi}$ be a subset of $\overline{\mathcal{P}}$ such that $\bar{R}^\times = \mathcal{W}_{\overline{\Pi}}\overline{\Pi}$ and no proper subset of $\overline{\Pi}$ satisfies this property. From Lemma 1.9, we have $|\overline{\Pi}| = \mathrm{rank}\, \bar{R}$ and so $\overline{\Pi}$ is a basis of the vector space $\overline{\mathcal{V}}$. Let $\overline{\Pi} = \{\bar{\alpha}_1, \ldots, \bar{\alpha}_\ell\}$. We fix a preimage $\dot{\alpha}_i \in \mathcal{P}$ for $\bar{\alpha}_i$. Set $\dot{\Pi} := \{\dot{\alpha}_1, \ldots, \dot{\alpha}_\ell\}$, $\dot{\mathcal{V}} := \mathrm{span}_{\mathbb{R}}\, \dot{\Pi}$ and $\dot{R} := (\mathcal{W}_{\dot{\Pi}}\dot{\Pi}) \cup \{0\}$. Then the epimorphism $^-\colon \mathcal{V} \to \overline{\mathcal{V}}$ induces an isometry $\dot{\mathcal{V}} \to \overline{\mathcal{V}}$ which maps $\dot{R}$ isometrically onto $\bar{R}$. It follows that $\mathcal{V} = \dot{\mathcal{V}} \oplus \overline{\mathcal{V}}$. It completes the proof of (i)–(iii).

Next, let $\alpha \in R^\times$. Then $\bar{\alpha} \in \bar{R} = \mathcal{W}_{\overline{\Pi}}\overline{\Pi}$, and so $\bar{\alpha} = w_{\bar{\alpha}_{i_1}} \cdots w_{\bar{\alpha}_{i_k}}(\bar{\alpha}_{i_{k+1}})$ for some $\bar{\alpha}_{i_1}, \ldots, \bar{\alpha}_{i_k}, \bar{\alpha}_{i_{k+1}} \in \overline{\Pi}$. Set

$$\overline{\Pi}_\alpha := w_{\bar{\alpha}_{i_1}} \cdots w_{\bar{\alpha}_{i_k}}(\overline{\Pi}).$$

Now $\overline{\Pi}_\alpha$ is a basis of $\overline{\mathcal{V}}$ such that $\bar{\alpha} \in \overline{\Pi}_\alpha$ and $\mathcal{W}_{\overline{\Pi}_\alpha}\overline{\Pi}_\alpha = \bar{R}^\times$. Therefore, in the proof of part (i), we may work with $\overline{\Pi}_\alpha$ in place of $\overline{\Pi}$. Then as above we fix a preimage $\dot{\Pi}_\alpha$ of $\overline{\Pi}_\alpha$, where we pick α as the preimage of $\bar{\alpha}$. Finally, we construct $\dot{R}_\alpha$ in the same manner as $\dot{R}$. Then $\dot{R}_\alpha$ and $\dot{\mathcal{V}}_\alpha := \mathrm{span}_{\mathbb{R}}\, \dot{R}_\alpha$ satisfy (ii)–(iii) in place of $\dot{R}$ and $\dot{\mathcal{V}}$, and the proof of (iv) is completed. $\square$

Let $\dot{R}$ and $\dot{\mathcal{V}}$ be as in Proposition 1.10. We write $\dot{R}^\times = \dot{R}_{\mathrm{sh}} \cup \dot{R}_{\mathrm{lg}}$, where $\dot{R}_{\mathrm{sh}}$ and $\dot{R}_{\mathrm{lg}}$ are the sets of short and long roots of $\dot{R}$. By convention, we assume that all non-zero roots of $\dot{R}$ are short if there is only one root length in $\dot{R}$.

1.11. It is shown in [AABGP, Chap. II] that using the finite root system $\dot{R}$, one can obtain a description of R in the form

$$\begin{aligned} R &= (S+S) \cup (\dot{R}_{\mathrm{sh}} + S) \cup (\dot{R}_{\mathrm{lg}} + L), \\ \text{where}\quad S &= \left\{\sigma \in \mathcal{V}^0 \;\middle|\; \sigma + \alpha \in R \text{ for some } \alpha \in \dot{R}_{\mathrm{sh}}\right\}, \\ L &= \left\{\sigma \in \mathcal{V}^0 \;\middle|\; \sigma + \alpha \in R \text{ for some } \alpha \in \dot{R}_{\mathrm{lg}}\right\}, \end{aligned} \tag{1.2}$$

where S and L are *semilattices* in $\mathcal{V}^0$. If $\dot{R}_{\mathrm{lg}} = \emptyset$, we interpret the terms L and $\dot{R}_{\mathrm{lg}} + L$ as empty sets. Semilattices S and L satisfy some further properties which we record here.

1.12. If R is simply laced of rank > 1, or is of type C_ℓ, $\ell \geq 3$, F_4 or G_2, then S is a lattice. If R is of type B_ℓ, $\ell \geq 3$, F_4 or G_2, then L is a lattice. Furthermore, for non-simply-laced types, we have

$$L + S = S \quad\text{and}\quad kS + L = L, \tag{1.3}$$

with

$$k = \begin{cases} 3 & \text{if } X = G_2, \\ 2 & \text{if } X \neq G_2. \end{cases} \tag{1.4}$$

Now (1.3) implies that

$$k\langle S\rangle \subseteq \langle L\rangle \subseteq S, \tag{1.5}$$

and so $\langle S\rangle/\langle L\rangle$ is a finite-dimensional vector space over the finite field $\mathbb{Z}_k$. The integer $0 \leq t \leq \nu$ satisfying $|\langle S\rangle/\langle L\rangle| = k^t$ is called the *twist number* of R. We make the convention that if R is of simply laced type, then $t = 0$ and $k = 1$.

We proceed with a fixed description of R in the form (1.2). We assume that R is of type X, has rank ℓ, nullity ν and twist number t.

1.13. From [AABGP, II, §4(b)], we see that there exist subspaces $\mathcal{V}_1^0$ and $\mathcal{V}_2^0$ of vector space $\mathcal{V}^0$ of dimensions t and $\nu - t$ respectively such that $\mathcal{V}^0 = \mathcal{V}_1^0 \oplus \mathcal{V}_2^0$, and there are semilattices S_1 and S_2 in $\mathcal{V}_1^0$ and $\mathcal{V}_2^0$, respectively such that

$$S = S_1 \oplus \langle S_2\rangle \quad \text{and} \quad L = k\langle S_1\rangle \oplus S_2. \tag{1.6}$$

Following [A1, Sect. 4], we define the *index* of R, denoted $\operatorname{ind}(R)$, by

$$\operatorname{ind}(R) := \begin{cases} 0, & X = A_\ell(\ell \geq 2),\ D_\ell(\ell \geq 4), \\ & \qquad E_{6,7,8},\ F_4,\ G_2, \\ \operatorname{ind}(S) - \nu, & X = A_1, \\ \operatorname{ind}(S_1) + \operatorname{ind}(S_2) - \nu, & X = B_2, \\ \operatorname{ind}(S_1) - t, & X = B_\ell(\ell \geq 3), \\ \operatorname{ind}(S_2) - (\nu - t), & X = C_\ell(\ell \geq 3). \end{cases} \tag{1.7}$$

We write S_1 and S_2 in the form **1.7**(ii), namely

$$S_1 = \biguplus_{i=0}^{\operatorname{ind}(S_1)} (\gamma_i + 2\langle S_1\rangle) \quad \text{and} \quad S_2 = \biguplus_{i=0}^{\operatorname{ind}(S_2)} (\eta_i + 2\langle S_2\rangle). \tag{1.8}$$

1.14. We fix a fundamental system $\dot{\Pi} = \{\alpha_1, \ldots, \alpha_\ell\}$ for $\dot{R}$. By **1.7**(iii), $\Lambda = \langle S\rangle$ admits a $\mathbb{Z}$-basis $\{\sigma_1, \ldots, \sigma_\nu\} \subseteq S$. Let θ be any root in $\dot{R}$ if $\dot{R}$ is of simply laced type, and θ_s and θ_l be any short and any long root in $\dot{R}$, respectively if $\dot{R}$ is of non-simply-laced type. For later use, for each type X, we introduce a subset $\mathcal{P}(X)$ of $R^\times$ as given in Table 1.

Type	$\mathcal{P}(X)$
A_1	$\{\alpha_1, \tau_1 - \alpha_1, \dots, \tau_\rho - \alpha_1\}$ ($\rho = \mathrm{ind}(S)$)
A_ℓ ($\ell > 1$)	$\{\alpha_1, \dots, \alpha_\ell, \sigma_1 - \theta, \dots, \sigma_\nu - \theta\}$
D_ℓ	$\{\alpha_1, \dots, \alpha_\ell, \sigma_1 - \theta, \dots, \sigma_\nu - \theta\}$
$E_{6,7,8}$	$\{\alpha_1, \dots, \alpha_\ell, \sigma_1 - \theta, \dots, \sigma_\nu - \theta\}$ ($\ell = 6, 7, 8$)
F_4	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \sigma_1 - \theta_s, \dots, \sigma_t - \theta_s, \sigma_{t+1} - \theta_l, \dots, \sigma_\nu - \theta_l\}$
G_2	$\{\alpha_1, \alpha_2, \sigma_1 - \theta_s, \dots, \sigma_t - \theta_s, \sigma_{t+1} - \theta_l, \dots, \sigma_\nu - \theta_l\}$
B_2	$\{\alpha_1, \alpha_2, \gamma_1 - \theta_s, \dots, \gamma_{\rho_1} - \theta_s, \eta_1 - \theta_l, \dots, \eta_{\rho_2} - \theta_l\}$ ($\rho_i = \mathrm{ind}(S_i)$, $i = 1, 2$)
B_ℓ ($\ell > 2$)	$\{\alpha_1, \dots, \alpha_\ell, \gamma_1 - \theta_s, \dots, \gamma_{\rho_1} - \theta_s, \sigma_{t+1} - \theta_l, \dots, \sigma_\nu - \theta_l\}$ ($\rho_1 = \mathrm{ind}(S_1)$)
C_ℓ ($\ell > 2$)	$\{\alpha_1, \dots, \alpha_\ell, \sigma_1 - \theta_s, \dots, \sigma_t - \theta_s, \eta_1 - \theta_l, \dots, \eta_{\rho_2} - \theta_l\}$ ($\rho_2 = \mathrm{ind}(S_2)$)

Table 1. The sets $\mathcal{P}(X)$

§2. Extended affine Weyl groups

We proceed with the same notation as in Section 1, in particular we assume that R is a reduced extended affine root system in $\mathcal{V}$ of type X, rank ℓ and nullity ν. As before, $\mathcal{V}^0$ is the radical of the form.

§2.1. The setting

2.1. We fix a description of R in the form $R = (S + S) \cup (\dot{R}_{\mathrm{sh}} + S) \cup (\dot{R}_{\mathrm{lg}} + L)$ as in **1.11**. We recall that $\dot{\mathcal{V}} := \mathrm{span}_{\mathbb{R}} \dot{R}$ and $\mathcal{V} = \dot{\mathcal{V}} \oplus \mathcal{V}^0$. We consider the $(\ell + 2\nu)$-dimensional real vector space $\widetilde{\mathcal{V}} := \dot{\mathcal{V}} \oplus \mathcal{V}^0 \oplus (\mathcal{V}^0)^*$. We normalize the form on $\dot{\mathcal{V}}$ such that

$$(\dot{\alpha}, \dot{\alpha}) := 2 \quad \text{for } \dot{\alpha} \in \dot{R}_{\mathrm{sh}}.$$

We extend the form $(\cdot,\cdot)$ on $\mathcal{V}$ to a symmetric form on $\widetilde{\mathcal{V}}$, denoted again by $(\cdot,\cdot)$, as follows:

$$(\dot{\mathcal{V}}, (\mathcal{V}^0)^*) = ((\mathcal{V}^0)^*, (\mathcal{V}^0)^*) := \{0\},$$
$$(\sigma, \lambda) = \lambda(\sigma), \quad \text{for } \sigma \in \mathcal{V}^0,\ \lambda \in (V^0)^\star.$$

This forces the form on $\widetilde{\mathcal{V}}$ to be non-degenerate. For $\alpha \in \mathcal{V}$ with $(\alpha, \alpha) \neq 0$ set $\alpha^\vee := 2\alpha/(\alpha, \alpha)$.

Definition 2.2. The *extended affine Weyl group* $\mathcal{W}$ of R (or the Weyl group of extended affine root system R) is defined to be the subgroup of $\mathrm{GL}(\widetilde{\mathcal{V}})$ generated by reflections $w_\alpha \colon \beta \mapsto \beta - (\beta, \alpha^\vee)\alpha$, $\alpha \in R^\times$. Since $0 \in L \subseteq S$, we have $\dot{R} \subseteq R$, and so we may identify $\dot{\mathcal{W}}$, the Weyl group of $\dot{R}$, as a subgroup of $\mathcal{W}$. We denote by $\mathcal{Z}(\mathcal{W})$, the center of $\mathcal{W}$.

§2.2. Presentation by conjugation

It is known that an extended affine Weyl group $\mathcal{W}$ associated with R is not, in general, a Coxeter group. It is proved that $\mathcal{W}$ is a Coxeter group if and only if R has nullity ≤ 1 (see [Ho, Thm. 3.6]). Nevertheless, extended affine Weyl groups enjoy another interesting presentation called *generalized presentation by conjugation*. When R has some "minimality condition", then $\mathcal{W}$ admits an even more delightful presentation, called the *presentation by conjugation*. We record these definitions here, as they play a crucial role in the sequel.

2.3. We begin by recalling some notation from [A2, Sects 1, 2]. We fix the $\mathbb{Z}$-basis $\{\sigma_1,\ldots,\sigma_\nu\}\subseteq S$ of $\Lambda=\langle S\rangle$ and the integer k given in **1.7**(iii) and **1.12**. For $1\leq i\leq\nu$ and $\alpha\in\langle R\rangle$, define the linear map $t_\alpha^{(i)}:\widetilde{\mathcal{V}}\to\widetilde{\mathcal{V}}$ by

$$t_\alpha^{(i)}(\lambda):=\lambda-\Big(\lambda,\frac{1}{k}\sigma_i\Big)\alpha+\Big((\lambda,\alpha)-\frac{1}{2}(\alpha,\alpha)\Big(\lambda,\frac{1}{k}\Big)\sigma_i\Big)\frac{1}{k}\sigma_i.$$

The map $t_\alpha^{(i)}$ is invertible with the inverse $t_{-\alpha}^{(i)}$, so $t_\alpha^{(i)}\in\mathrm{Aut}(\widetilde{\mathcal{V}})$. For $1\leq i,j\leq\nu$ set

$$c_{ij}:=t_{-\sigma_j}^{(i)}.$$

One checks that $(t_\alpha^{(i)},t_\beta^{(j)})=c_{ij}^{\frac{1}{k}(\alpha,\beta)}$, for $\alpha,\beta\in R$, where $(x,y):=xyx^{-1}y^{-1}$ denotes the group commutator.

For $\alpha\in R^\times$ and $\sigma=\sum_{i=1}^{\nu}m_i\sigma_i\in R^0$ with $\alpha+\sigma\in R$, we set

$$c_{(\alpha,\sigma)}:=(w_{\alpha+\sigma}w_\alpha)(w_\alpha w_{\alpha+\sigma_1})^{m_1}\ldots(w_\alpha w_{\alpha+\sigma_\nu})^{m_\nu}. \tag{2.1}$$

By [A2, Sect. 2], we have

$$c_{(\alpha,\sigma)}=\prod_{i<j}c_{ij}^{k(\alpha)m_im_j}, \tag{2.2}$$

where the integers $k(\alpha)$ are defined as follows:

$$k(\alpha):=\begin{cases}1 & \text{if } \alpha\in R_{\mathrm{lg}},\\ k & \text{if } \alpha\in R_{\mathrm{sh}}.\end{cases}$$

(If R is simply laced, then $k(\alpha)=1$ for all $\alpha\in R^\times$.)

2.4. Assume that θ_s and θ_l are the highest short and highest long roots of $\dot{R}$ respectively (for simply laced cases $\theta:=\theta_s=\theta_\ell$), and for $1\leq p\leq\nu$ consider a triple

$$\Big(\varepsilon_p,\alpha_p,\eta_p=\sum_{i=1}^{m}m_{ip}\sigma_i\Big)\in\Big(\{\pm1\}\times\{\theta_s\}\times\sum_{i=1}^{\nu}\mathbb{Z}\sigma_i\Big)\cup\Big(\{\pm1\}\times\{\theta_l\}\times\sum_{i=t+1}^{\nu}\mathbb{Z}\sigma_i\Big).$$

The collection $\{(\varepsilon_p, \alpha_p, \eta_p)\}_{p=1}^m$ is called a *reduced collection* if

$$\sum_{p=1}^m k(\alpha_p)\varepsilon_p m_{ip} m_{jp} = 0 \quad \text{for all } 1 \leq i < j \leq \nu.$$

If $\{(\varepsilon_p, \alpha_p, \eta_p)\}_{p=1}^m$ is a reduced collection, then $c_{(\alpha_p,\eta_p)}$ is an element of $\mathcal{W}$ for all $1 \leq p \leq m$ (see [A2, Sect. 1]).

The following theorem is due to [A2, Thm. 2.7], [Kr2, Thm. 1.18], [AS4, Thm. 5.19(ii)] and [AS5, Cors 2.34, 2.36(b)].

Theorem 2.5. *Let R be an extended affine root system with Weyl group $\mathcal{W}$ and $\widehat{\mathcal{W}}$ be the group defined by generators $\widehat{w}_\alpha$, $\alpha \in R^\times$, subject to the relations*

$$\widehat{w}_\alpha^2 = 1 \qquad \text{for every } \alpha \in R^\times, \tag{2.3}$$

$$\widehat{w}_\alpha \widehat{w}_\beta \widehat{w}_\alpha = \widehat{w}_{w_\alpha(\beta)} \qquad \text{for every } \alpha, \beta \in R^\times, \tag{2.4}$$

$$\prod_{p=1}^m \hat{c}_{(\alpha_p,\eta_p)}^{\varepsilon_p} = 1 \qquad \text{for any reduced collection } \{(\varepsilon_p, \alpha_p, \eta_p)\}_{p=1}^m, \tag{2.5}$$

where $\hat{c}_{(\alpha_p,\eta_p)}$ is the element in $\widehat{\mathcal{W}}$ corresponding to $c_{(\alpha_p,\eta_p)}$, under the assignment $w_\alpha \mapsto \widehat{w}_\alpha$.

(i) *The extended affine Weyl group $\mathcal{W}$ is isomorphic to $\widehat{\mathcal{W}}$.*

(ii) *If R is simply laced of rank > 1, or is of type F_4 or G_2, then relations of the form (2.5) are consequences of relations of the forms (2.3) and (2.4); in the sense of Definition 2.6 below, $\mathcal{W}$ has the presentation by conjugation.*

(iii) *If $\nu \leq 2$, then $\mathcal{W}$ has the presentation by conjugation (Definition 2.6).*

Definition 2.6. The given presentation in Theorem 2.5 is called the *generalized presentation by conjugation*. If $\mathcal{W}$ is isomorphic to the presented group defined by generators $\widehat{w}_\alpha$, $\alpha \in R^\times$ and relations (2.3) and (2.4), then $\mathcal{W}$ is said to have the *presentation by conjugation*.

2.7. As a consequence of [AS4, Thms 4.23, 5.16] and [AS5, Thm. 2.33], the extended affine Weyl group $\mathcal{W}$ has the presentation by conjugation if and only if the epimorphism

$$\begin{aligned} \psi\colon \widehat{\mathcal{W}} &\to \mathcal{W}, \\ \widehat{w}_\alpha &\mapsto w_\alpha, \end{aligned} \tag{2.6}$$

is an isomorphism.

For $\alpha \in R^\times$, we denote the orbit of α under the action of $\mathcal{W}$ on $R^\times$ by $\mathcal{W}\alpha$, namely $\mathcal{W}\alpha = \{w\alpha \mid w \in \mathcal{W}\}$.

Corollary 2.8. *Assume that $\mathcal{W}$ has the presentation by conjugation and fix $\beta \in R^\times$. Then the assignments*

$$w_\alpha \overset{\Phi_\beta}{\longmapsto} \begin{cases} 1 & \text{if } \alpha \in \mathcal{W}\beta, \\ 0 & \text{if } \alpha \in R^\times \setminus \mathcal{W}\beta, \end{cases} \qquad w_\alpha \overset{\Psi}{\longmapsto} \begin{cases} 0 & \text{if } \alpha \in R_{\mathrm{lg}}, \\ 1 & \text{if } \alpha \in R_{\mathrm{sh}}. \end{cases}$$

extend to epimorphisms Φ_β and Ψ from $\mathcal{W}$ onto $\mathbb{Z}_2$, respectively.

Proof. First we consider Φ_β. Since for $\alpha, \gamma \in R^\times$, we have $\gamma \in \mathcal{W}\beta$ if and only if $w_\alpha(\gamma) \in \mathcal{W}\beta$, it easily follows that the defining relations of the form $w_\alpha w_\gamma w_\alpha w_{w_\alpha(\gamma)}$ are mapped to zero under the assignment Φ_β. The argument for defining relations of the form $w_\alpha w_\alpha$ is obvious.

For the assignment Ψ, clearly, the defining relations are mapped to zero under Ψ, as reflections preserve the length. $\square$

Lemma 2.9. *Let R be of non-simply-laced type. Let $\{w_\alpha \mid \alpha \in \mathcal{P}\}$ be a set of generators for $\mathcal{W}$. Then $\mathcal{P} \cap R_{\mathrm{sh}} \neq \emptyset$ and $\mathcal{P} \cap R_{\mathrm{lg}} \neq \emptyset$.*

Proof. First, assume that $\mathcal{W}$ has the presentation by conjugation and consider the epimorphism $\Psi\colon \mathcal{W} \to \mathbb{Z}_2$ of Corollary 2.8. Now if $\mathcal{P} \subseteq R_{\mathrm{lg}}$, then for $\alpha \in R_{\mathrm{sh}}$, we have $w_\alpha = w_{\alpha_1} \cdots w_{\alpha_m}$ for some $\alpha_1, \ldots, \alpha_m \in R_{\mathrm{lg}}$. Then $1 = \Psi(w_\alpha) = \Psi(w_{\alpha_1}) + \cdots + \Psi(w_{\alpha_m}) = 0$ which is absurd. An analogous argument works for the case $\mathcal{P} \subseteq R_{\mathrm{sh}}$, replacing the roles of short and long roots. Thus the result holds if $\mathcal{W}$ has the presentation by conjugation.

For the general case, we consider the epimorphism $\mathcal{W} \to \mathcal{W}_{\bar{R}}$, $w_\alpha \mapsto w_{\bar{\alpha}}$, where $\mathcal{W}_{\bar{R}}$ denotes the Weyl group of the finite root system $\bar{R}$. Now, since $\{w_\alpha \mid \alpha \in \mathcal{P}\}$ generates $\mathcal{W}$, the set $\{w_{\bar{\alpha}} \mid \alpha \in \mathcal{P}\}$ generates $\mathcal{W}_{\bar{R}}$. Since $\mathcal{W}_{\bar{R}}$ has the presentation by conjugation (see [MP, Prop. 5.3.3]), we get from the first part of the proof that $\bar{R}_{\mathrm{sh}} \cap \overline{\mathcal{P}} \neq \emptyset$ and $\bar{R}_{\mathrm{lg}} \cap \overline{\mathcal{P}} \neq \emptyset$. Since for $\alpha \in R$, α and $\bar{\alpha}$ have the same length, we are done. $\square$

We return to the discussion about the existence of presentation by conjugation for extended affine Weyl groups in future sections.

§3. The classes $\mathcal{M}_r$, $\mathcal{M}_c$ and $\mathcal{M}_m$

We proceed by assuming that R is a reduced extended affine root system in $\mathcal{V}$ of rank ℓ and nullity $\nu \geq 0$. Let $\mathcal{W}$ be the Weyl group of R. We begin by introducing some notation. For a subset $\mathcal{P}$ of $R^\times$, we set

$$\begin{aligned}\mathcal{P}_{\mathrm{sh}} &:= \mathcal{P} \cap R_{\mathrm{sh}},\\ \mathcal{P}_{\mathrm{lg}} &:= \mathcal{P} \cap R_{\mathrm{lg}},\\ S_{\mathcal{P}} &= \{w_\alpha \mid \alpha \in \mathcal{P}\},\\ \mathcal{W}_{\mathcal{P}} &:= \langle S_{\mathcal{P}} \rangle = \langle w_\alpha \mid \alpha \in \mathcal{P}\rangle.\end{aligned}$$

§3.1. Inter-relations between $\mathcal{M}_r$, $\mathcal{M}_c$ and $\mathcal{M}_m$

As we will see in the sequel (Proposition 4.4), the extended affine Weyl group $\mathcal{W}$ is finitely generated.

Definition 3.1. Let c denote the least number of reflections generating $\mathcal{W}$ and let $\mathcal{P} \subseteq R^\times$. We say $\mathcal{P}$ is a *c-minimal* set in $R^\times$ if $|\mathcal{P}| = c$ and $S_{\mathcal{P}}$ generates $\mathcal{W}$.

Definition 3.2 ([AYY, Def. 1.19]). A subset $\mathcal{P} \subseteq R^\times$ is called a *reflectable set* if $\mathcal{W}_{\mathcal{P}}\mathcal{P} = R^\times$. A subset $\mathcal{P} \subseteq R^\times$ is called a *reflectable base* if

(i) $\mathcal{P}$ is a reflectable set, $\mathcal{W}_{\mathcal{P}}\mathcal{P} = R^\times$;
(ii) no proper subset of $\mathcal{P}$ is a reflectable set.

Definition 3.3. Let $\mathcal{P} \subseteq R^\times$; $\mathcal{P}$ is called an *S-minimal* set (or simply a *minimal* set) in $R^\times$ if

(i) $S_{\mathcal{P}}$ generates $\mathcal{W}$;
(ii) no proper subset of $S_{\mathcal{P}}$ generates $\mathcal{W}$.

3.4. We denote the class of reflectable bases, c-minimal sets, and S-minimal sets of $R^\times$ by $\mathcal{M}_r$, $\mathcal{M}_c$ and $\mathcal{M}_m$, respectively. We also set

$$\mathcal{M}_{rc} := \mathcal{M}_r \cap \mathcal{M}_c \quad \text{and} \quad \mathcal{M}_{rm} := \mathcal{M}_r \cap \mathcal{M}_m.$$

Clearly, we have

$$\mathcal{M}_c \subseteq \mathcal{M}_m \quad \text{and} \quad \mathcal{M}_{rc} \subseteq \mathcal{M}_{rm}. \tag{3.1}$$

Note that if $\mathcal{P} \in \mathcal{M}_r \cup \mathcal{M}_m$, then as $\mathcal{W} = \mathcal{W}_{\mathcal{P}}$, we have $\mathcal{V} = \mathrm{span}_{\mathbb{R}} \mathcal{P}$, and then

$$\dim(\mathcal{V}) = \ell + \nu \leq c \leq |\mathcal{P}|. \tag{3.2}$$

Proposition 3.5. *If $\mathcal{P} \subseteq R^\times$ satisfies $\mathcal{W} = \mathcal{W}_{\mathcal{P}}$, then $\langle \mathcal{P} \rangle = \langle R \rangle$ and $\mathcal{P}$ is connected. In particular, if $\mathcal{P}$ belongs to either of $\mathcal{M}_c$, $\mathcal{M}_m$ and $\mathcal{M}_r$, then $\langle \mathcal{P} \rangle = \langle R \rangle$ and $\mathcal{P}$ is connected.*

Proof. For $\alpha \in R^\times$, we have $w_\alpha = w_{\alpha_1} \cdots w_{\alpha_n}$ for some $\alpha_1, \ldots, \alpha_n \in \mathcal{P}$. If there exists $\beta \in R^\times$ with $(\beta, \alpha^\vee) = -1$, then

$$\alpha + \beta = w_\alpha(\beta) = w_{\alpha_1} \cdots w_{\alpha_n}(\beta) = \beta + k_1\alpha_1 + \cdots + k_n\alpha_n$$

for some $k_1, \ldots, k_n \in \mathbb{Z}$. Thus $\alpha \in \langle \mathcal{P} \rangle$ and so $\langle \mathcal{P} \rangle = \langle R \rangle$. In particular, this holds for the types $X \neq A_1, B_\ell, C_\ell$. Assume now that R is of type A_1. Then

$$-\alpha = w_\alpha(\alpha) = w_{\alpha_1} \cdots w_{\alpha_n}(\alpha) = \alpha + 2k_1\alpha_1 + \cdots + 2k_n\alpha_n$$

for some $k_i \in \mathbb{Z}$, which again gives $\alpha \in \langle \mathcal{P} \rangle$, as required.

Next assume that R is of type B_ℓ. If $\alpha \in R_{\mathrm{lg}}$, then there exists $\beta \in R_{\mathrm{sh}}$, such that $(\beta, \alpha^\vee) = -1$, and so we have $\alpha \in \langle \mathcal{P} \rangle$ by the preceding paragraph. Thus $\langle R_{\mathrm{lg}} \rangle \subseteq \langle \mathcal{P} \rangle$. It remains to show that $R_{\mathrm{sh}} \subseteq \langle \mathcal{P} \rangle$. By **1.13**(1.6), $R_{\mathrm{sh}} = \dot{R}_{\mathrm{sh}} + S_1 \oplus \langle S_2 \rangle$. Since $\langle S_2 \rangle \subseteq \langle R_{\mathrm{lg}} \rangle$, it only remains to show that $\dot{R}_{\mathrm{sh}} + S_1 \subseteq \langle \mathcal{P} \rangle$. But as $\langle S_1 \rangle = \sum_{i=1}^{t} \mathbb{Z}\sigma_t$, we are done if we show that $\dot{R}_{\mathrm{sh}} \subseteq \langle \mathcal{P} \rangle$ and $\sigma_1, \ldots, \sigma_t \in \langle \mathcal{P} \rangle$.

By Lemma 2.9, $\mathcal{P}$ contains a short root α. By Proposition 1.10(iv), we may assume that $\alpha \in \dot{R}_{\mathrm{sh}}$. Since $\dot{R}_{\mathrm{lg}} \subseteq \langle \mathcal{P} \rangle$, this implies that $\dot{R}_{\mathrm{sh}} \subseteq \langle \mathcal{P} \rangle$ and so $\dot{R} \subseteq \langle \mathcal{P} \rangle$. To conclude the proof for type B_ℓ, it remains to show that $\sigma_1, \ldots, \sigma_t \in \langle \mathcal{P} \rangle$.

For this, we fix $1 \leq i \leq t$ and for $\alpha \in R^\times$ we consider the assignment

$$w_\alpha \overset{\psi_i}{\longmapsto} \begin{cases} 1 & \text{if } \alpha \in \dot{R}_{\mathrm{sh}} + \sigma_i + \langle L \rangle, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that this assignment can be lifted to an epimorphism $\mathcal{W} \to \mathbb{Z}_2$. By Theorem 2.5, we need to show that the defining relations of the generalized presentation by conjugation vanish in $\mathbb{Z}_2$ under this assignment. The relations of the form $w_\alpha w_\alpha$ are clear. To check that $\psi_i(w_\alpha)\psi_i(w_\beta)\psi_i(w_\alpha)\psi_i(w_{w_\alpha(\beta)}) = 0$, we only need to verify that $\beta \in \dot{R}_{\mathrm{sh}} + \sigma_i + \langle L \rangle$ if and only if $w_\alpha(\beta) \in \dot{R}_{\mathrm{sh}} + \sigma_i + \langle L \rangle$. So suppose $\beta = \dot{\beta} + \sigma_i + \lambda$ for some $\dot{\beta} \in \dot{R}_{\mathrm{sh}}$ and $\lambda \in \langle L \rangle$. Then

$$w_\alpha(\beta) = w_\alpha(\dot{\beta}) + \sigma_i + \lambda \in \begin{cases} \pm\dot{\beta} + 2\Lambda + \sigma_i + \lambda \subseteq \dot{R}_{\mathrm{sh}} + \sigma_i + \langle L \rangle & \text{if } \alpha \in R_{\mathrm{sh}}, \\ \dot{R}_{\mathrm{sh}} + L + \sigma_i + \lambda \subseteq \dot{R}_{\mathrm{sh}} + \sigma_i + \langle L \rangle & \text{if } \alpha \in R_{\mathrm{lg}}, \end{cases}$$

as required.

Finally, we consider relations of the form **2.4**(2.5). For this, we must show that for

$$\left(\alpha, \sigma = \sum_{j=1}^{\nu} m_j\sigma_j\right) \in (\{\theta_s\} \times \Lambda) \cup \left(\{\theta_\ell\} \times \sum_{i=t+1}^{\nu} \mathbb{Z}\sigma_i\right),$$

we have

$$(\psi_i(w_{\alpha+\sigma})\psi_i(w_\alpha))(\psi_i(w_{\alpha+\sigma_1})\psi_i(w_\alpha))^{m_1} \cdots (\psi_i(w_{\alpha+\sigma_\nu})\psi_i(w_\alpha))^{m_\nu} = 0.$$

Clearly, this holds if $\alpha = \theta_\ell$. Suppose now that $\alpha = \theta_s$. Since $\psi_i(w_{\theta_s}) = 0$ and $\psi_i(w_{\theta_s+\sigma_j}) = 0$ for all $j \neq i$, we must show $\psi_i(w_{\theta_s+\sigma}) + m_i\psi_i(w_{\theta_s+\sigma_i}) = 0$. Now if $\sigma \in \sigma_i + \langle L\rangle$, then as $1 \leq i \leq t$, we have $m_i \in 2\mathbb{Z}+1$. Then $\psi_i(w_{\theta_s+\sigma}) = 1$ and $m_i\psi_i(w_{\theta_s+\sigma_i}) = 1$ and we are done. Finally, we consider the case $\sigma \notin \sigma_i + \langle L\rangle$. This forces $m_i \in 2\mathbb{Z}$. Then we have $\psi_i(w_{\theta_s+\sigma}) = 0$ and $m_i\psi_i(w_{\theta_s+\sigma_i}) = 0$ which again gives the result. Thus, as was claimed, the assignment ψ_i induces an epimorphism $\psi_i \colon \mathcal{W} \to \mathbb{Z}_2$.

Now let $\alpha \in \dot{R}_{\mathrm{sh}}$ and fix $1 \leq i \leq t$. Then $\alpha+\sigma_i \in R$ and so $w_{\alpha+\sigma_i} = w_{\gamma_1} \cdots w_{\gamma_n}$ for some $\gamma_i \in \mathcal{P}$. Then $\psi_i(w_{\alpha+\sigma_i}) = \psi_i(w_{\gamma_1} \cdots w_{\gamma_n})$, and so

$$1 = \psi_i(w_{\alpha+\sigma_i}) = \sum_{j=1}^{k} \psi_i(w_{\gamma_{i_j}}),$$

where $\gamma_{i_j} \in \mathcal{P} \cap R_{\mathrm{sh}}$. Thus, at least for one j, we get $\psi(w_{\gamma_{i_j}}) = 1$, implying that $\gamma_{i_j} \in (\dot{R}_{\mathrm{sh}} + \sigma_i + \langle L\rangle) \cap \mathcal{P}$. Since $\langle L\rangle \subseteq \langle \mathcal{P}\rangle$, this implies $\dot{\gamma} + \sigma_i \in \langle \mathcal{P}\rangle$ for some $\dot{\gamma} \in \dot{R}_{\mathrm{sh}}$. Since $\dot{R}_{\mathrm{sh}} \subseteq \langle \mathcal{P}\rangle$, we get $\sigma_i \in \langle \mathcal{P}\rangle$ and the proof for type B_ℓ is completed. The proof for type C_ℓ is analogous, replacing the roles of short and long roots.

Finally, we show that $\mathcal{P}$ is connected. If not, then $\mathcal{P} = \mathcal{P}_1 \uplus \mathcal{P}_2$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are non-empty sets and $(\mathcal{P}_1, \mathcal{P}_2) = \{0\}$. Now let $\alpha \in R^\times$. Since $\mathcal{P}$ spans $\mathcal{V}$, there exists $\beta \in \mathcal{P}$ with $(\alpha, \beta) \neq 0$. Assume without loss of generality that $\beta \in \mathcal{P}_1$. We have $w_\alpha = w_{\alpha_1} \cdots w_{\alpha_m}$ for some $\alpha_1, \ldots, \alpha_m \in \mathcal{P}$. Since $\mathcal{P}_1$ and $\mathcal{P}_2$ are orthogonal, it follows that $\beta - (\beta, \alpha^\vee)\alpha = w_\alpha(\beta) = w_{\alpha_{j_1}} \cdots w_{\alpha_{j_k}}(\beta)$, where $\alpha_{j_1}, \ldots, \alpha_{j_k}$ are in $\mathcal{P}_1$. Since $(\alpha, \beta) \neq 0$, this forces $\alpha \in \operatorname{span}_{\mathbb{R}} \mathcal{P}_1$. This argument gives $R^\times \subseteq (R^\times \cap \operatorname{span}_{\mathbb{R}} \mathcal{P}_1) \uplus (R^\times \cap \operatorname{span}_{\mathbb{R}} \mathcal{P}_2)$ which is absurd as R is connected. $\square$

To each subset $\mathcal{P}$ of $R^\times$, we associate two subsets of R as follows:

$$R^n_{\mathcal{P}} \coloneqq \mathcal{W}_{\mathcal{P}}\mathcal{P} \quad \text{and} \quad R^i_{\mathcal{P}} \coloneqq (R^n_{\mathcal{P}} - R^n_{\mathcal{P}}) \cap R^0. \tag{3.3}$$

Proposition 3.6. *Let $\emptyset \neq \mathcal{P} \subseteq R^\times$ be connected.*

- (i) *The set $R_{\mathcal{P}} = R^n_{\mathcal{P}} \cup R^i_{\mathcal{P}}$ is an extended affine root system in $\mathcal{V}_{\mathcal{P}} \coloneqq \operatorname{span}_{\mathbb{R}} R_{\mathcal{P}}$ satisfying $R^\times_{\mathcal{P}} = R^n_{\mathcal{P}}$, $R^0_{\mathcal{P}} = R^i_{\mathcal{P}}$ and $\mathcal{W}_{\mathcal{P}} = \mathcal{W}_{R_{\mathcal{P}}}$.*
- (ii) *If $\mathcal{W} = \mathcal{W}_{\mathcal{P}}$, then $R_{\mathcal{P}}$ has the same rank, nullity and type of R. In particular, if $\mathcal{P}$ belongs to either of the classes $\mathcal{M}_c$ or $\mathcal{M}_m$, then $R_{\mathcal{P}}$ has the same rank, nullity and type of R.*
- (iii) *If $\mathcal{P}$ is a reflectable set, then $R_{\mathcal{P}} = R$.*

Proof. First, we note from the way $R_{\mathcal{P}}$ is defined that $R^0_{\mathcal{P}} = R^i_{\mathcal{P}}$ and $R^\times_{\mathcal{P}} = R^n_{\mathcal{P}}$.

(i) From the definition of $R_{\mathcal{P}}$ it is clear that axioms (R1)–(R3) and (R5) of an extended affine root system (Definition 1.2) hold for $R_{\mathcal{P}}$. Since $\mathcal{P}$ is connected it follows easily that $R_{\mathcal{P}}$ is connected, so (R6) also holds. For (R4), since $\mathcal{V}^0_{\mathcal{P}} = \mathcal{V}_{\mathcal{P}} \cap \mathcal{V}^0$, we have

$$\begin{aligned} R^0_{\mathcal{P}} = R^i_{\mathcal{P}} &= (R^n_{\mathcal{P}} - R^n_{\mathcal{P}}) \cap R^0 \\ &= (R^{\times}_{\mathcal{P}} - R^{\times}_{\mathcal{P}}) \cap ((R^{\times} - R^{\times}) \cap \mathcal{V}^0) \\ &= (R^{\times}_{\mathcal{P}} - R^{\times}_{\mathcal{P}}) \cap \mathcal{V}^0_{\mathcal{P}}. \end{aligned}$$

(ii) If $\mathcal{W} = \mathcal{W}_{\mathcal{P}}$, then $\operatorname{span}_{\mathbb{R}} \mathcal{P} = \mathcal{V}$ and $\mathcal{V}_{\mathcal{P}} = \mathcal{V}$, so $\operatorname{span}_{\mathbb{R}} R^{\times}_{\mathcal{P}} = \operatorname{span}_{\mathbb{R}} \mathcal{P} = \mathcal{V}$ and $\operatorname{span}_{\mathbb{R}} R^0_{\mathcal{P}} = \mathcal{V}^0$. Thus $R_{\mathcal{P}}$ has the same nullity as R. This gives $\operatorname{rank} R_{\mathcal{P}} = \dim \mathcal{V}_{\mathcal{P}} - \dim \mathcal{V}^0_{\mathcal{P}} = \dim \mathcal{V} - \dim \mathcal{V}^0 = \operatorname{rank} R$. Also by Lemma 2.9, R and $R_{\mathcal{P}}$ have the same number of root lengths. Thus both $\bar{R}$ and $\bar{R}_{\mathcal{P}}$ have the same rank and the same number of root lengths, with $\bar{R}_{\mathcal{P}} \subseteq \bar{R}$. Now consulting the root data of finite root systems (see [Hu1, Table 12.2.1]), we conclude that $\bar{R}_{\mathcal{P}} = \bar{R}$.

(iii) Since $R^{\times}_{\mathcal{P}} = R^n_{\mathcal{P}} = R^{\times}$ and $R^0_{\mathcal{P}} = (R^n_{\mathcal{P}} - R^n_{\mathcal{P}}) \cap R^0 = (R^{\times} - R^{\times}) \cap \mathcal{V}^0 = R^0$, we are done. □

§3.2. The finite case

We show that $\mathcal{M}_r = \mathcal{M}_c = \mathcal{M}_m$ if R is finite.

Lemma 3.7. *Let R be a reduced irreducible finite root system of rank ℓ; then $c = \ell$.*

Proof. Since the rank of a finite root system is equal to the minimal number of generators for the corresponding Weyl group, we get $\ell = c$. □

Proposition 3.8. *Let R be a reduced irreducible finite root system; then $\mathcal{M}_c = \mathcal{M}_m = \mathcal{M}_r$.*

Proof. Let $\mathcal{P}$ be a reflectable base for R. By Lemma 1.9, $|\mathcal{P}| = \operatorname{rank}(R)$. Thus by Lemma 3.7, $\mathcal{M}_r \subseteq \mathcal{M}_c$. Now let $\mathcal{P} \in \mathcal{M}_c$; since $\mathcal{W} = \mathcal{W}_{\mathcal{P}}$, we get from Proposition 3.6 that $R_{\mathcal{P}}$ is a finite subsystem of R of the same type and rank as R. Thus $R = R_{\mathcal{P}}$ and so $\mathcal{P}$ is a reflectable set for R. This proves that $\mathcal{M}_c \subseteq \mathcal{M}_r$, and so $\mathcal{M}_c = \mathcal{M}_r$.

Next, let $\mathcal{P} \in \mathcal{M}_m$. As we saw in the previous paragraph, $\mathcal{P}$ is a reflectable set. If $|\mathcal{P}| > \operatorname{rank} R = c$, then $\mathcal{P}$ contains a proper subset $\mathcal{P}'$ which is a reflectable base, and so $\mathcal{W}_{\mathcal{P}'} = \mathcal{W}$, contradicting that $\mathcal{P} \in \mathcal{M}_m$. Thus

$$\mathcal{M}_c \subseteq \mathcal{M}_m \subseteq \mathcal{M}_r \subseteq \mathcal{M}_c,$$

completing the proof that $\mathcal{M}_c = \mathcal{M}_m = \mathcal{M}_r$. □

The following example shows that Proposition 3.8 may fail if the extended affine root system R is not finite.

Example 3.9. Consider the extended affine root system $R = \Lambda \cup (\pm\epsilon + \Lambda)$ of type A_1, where $\{0, \pm\epsilon\}$ is a finite root system of type A_1 and $\Lambda = \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2 \oplus \mathbb{Z}\sigma_3$. Set

$$\mathcal{P} := \{\epsilon, \sigma_1 - \epsilon, \sigma_2 - \epsilon, \sigma_3 - \epsilon, \sigma_1 + \sigma_2 - \epsilon, \sigma_1 + \sigma_3 - \epsilon, \sigma_2 + \sigma_3 - \epsilon, \sigma_1 + \sigma_2 + \sigma_3 - \epsilon\}.$$

By Proposition 4.4(i), $\mathcal{P}$ is a reflectable base for R, so $\mathcal{P} \in \mathcal{M}_r$. A simple computation shows that

$$w_{\sigma_1+\sigma_2+\sigma_3-\epsilon} = w_{\sigma_1+\sigma_3-\epsilon} w_{\sigma_1-\epsilon} w_{\sigma_1+\sigma_3-\epsilon} w_\epsilon w_{\sigma_3-\epsilon} w_\epsilon w_{\sigma_2-\epsilon} w_{\sigma_2+\sigma_3-\epsilon} w_\epsilon,$$

so $w_{\sigma_1+\sigma_2+\sigma_3-\epsilon} \in \langle w_\alpha \mid \alpha \in \mathcal{P} \setminus \{w_{\sigma_1+\sigma_2+\sigma_3-\epsilon}\}\rangle$; thus $\mathcal{P} \notin \mathcal{M}_m$ and then by (3.1), $\mathcal{P} \notin \mathcal{M}_c$. That is, $\mathcal{P} \notin \mathcal{M}_{rc}$.

§4. Reflectable bases, the class $\mathcal{M}_r$

In this section we consider various aspects of reflectable bases; in particular, we investigate their existences and their possible cardinalities.

§4.1. Characterization theorem for reflectable bases

We recall two recognition theorems for reflectable bases associated with reduced extended affine root systems given in [AYY, Thms 3.1, 3.14, 3.24, 3.26, 3.27] and [ASTY, Thm. 3.2].

Theorem 4.1 (Recognition theorem for simply laced types). *Let R be simply laced of type X and $\Pi \subseteq R^\times$ satisfy $\langle \Pi \rangle = \langle R \rangle$. Then Π is a reflectable base for R if and only if*

(i) $R^\times = \biguplus_{\alpha\in\Pi}((\alpha + 2\langle R \rangle) \cap R)$ *if* $X = A_1$*;*

(ii) Π *is a minimal generating set for the free abelian group* $\langle R \rangle$*, if* X *is simply laced of rank* > 1.

Theorem 4.2 (Recognition theorem for non-simply-laced types). *Let R be non-simply-laced of type X and let $\Pi \subseteq R^\times$ satisfy $\langle \Pi \rangle = \langle R \rangle$. Then Π is a reflectable base for R if and only if*

(i) $R_{\mathrm{sh}} = \biguplus_{\alpha\in\Pi_{\mathrm{sh}}}((\alpha + \langle R_{\mathrm{lg}} \rangle) \cap R_{\mathrm{sh}})$ *and* $R_{\mathrm{lg}} = \biguplus_{\alpha\in\Pi_{\mathrm{lg}}}((\alpha + 2\langle R_{\mathrm{sh}} \rangle) \cap R_{\mathrm{lg}})$ *if* $X = B_2$*;*

(ii) $R_{\mathrm{sh}} = \biguplus_{\alpha\in\Pi_{\mathrm{sh}}}((\alpha + \langle R_{\mathrm{lg}} \rangle) \cap R_{\mathrm{sh}})$*, and* Π_{lg} *is a minimal set with respect to the property that* $\{\alpha + 2\langle R_{\mathrm{sh}} \rangle \mid \alpha \in \Pi_{\mathrm{lg}}\}$ *is a basis for the* $\mathbb{Z}_2$*-vector space* $\langle R_{\mathrm{lg}} \rangle / 2\langle R_{\mathrm{sh}} \rangle$*, if* $X = B_\ell$*,* $\ell \geq 3$*;*

(iii) $R_{\rm lg} = \biguplus_{\alpha\in\Pi_{\rm lg}}((\alpha + 2\langle R_{\rm sh}\rangle) \cap R_{\rm lg})$, *and* $\Pi_{\rm sh}$ *is minimal with respect to the property that the set* $\{\alpha + \langle R_{\rm lg}\rangle \mid \alpha \in \Pi_{\rm sh}\}$ *is a basis for the* $\mathbb{Z}_2$*-vector space* $\langle R_{\rm sh}\rangle/\langle R_{\rm lg}\rangle$, *if* $X = C_\ell$, $\ell \geq 3$;

(iv) $\Pi_{\rm sh}$ *is minimal with respect to the property that the set* $\{\alpha+\langle R_{\rm lg}\rangle \mid \alpha \in \Pi_{\rm sh}\}$ *is a basis for the* $\mathbb{Z}_2$*-vector space* $\langle R_{\rm sh}\rangle/\langle R_{\rm lg}\rangle$, *and* $\Pi_{\rm lg}$ *is minimal with respect to the property that the set* $\{\alpha+k\langle R_{\rm sh}\rangle \mid \alpha \in \Pi_{\rm lg}\}$ *is a basis for the* $\mathbb{Z}_2$*-vector space* $\langle R_{\rm lg}\rangle/k\langle R_{\rm sh}\rangle$, *if* $X = F_4, G_2$, *where* k *is as in* **1.12**.

Remark 4.3. If in Theorems 4.1 and 4.2, we drop the terms "minimal" and "basis" and change the symbol "$\biguplus$" to "$\bigcup$", then the conditions there characterize the reflectable sets in R.

§4.2. Concrete family of reflectable bases

We show that set $\mathcal{P}(X)$ given in Table 1 is a reflectable base for the extended affine root system of type X.

Proposition 4.4. *Let* R *be a reduced extended affine root system of type* X, *nullity* ν *and twist number* t. *Then the following hold:*

(i) *the set* $\mathcal{P}(X)$ *given in Table 1 is a reflectable base for* R *with* $|\mathcal{P}(X)| = \mathrm{ind}(R) + \ell + \nu$;

(ii) *if* $\mathrm{ind}(R) = 0$, *then* $\mathcal{P}(X) \in \mathcal{M}_c$.

Proof. We follow the same notation as in **1.12**.

(i) Suppose first that R is simply laced of rank > 1, or is of type F_4 or G_2. Since $|\mathcal{P}(X)| = \ell+\nu$, we get from Theorem 4.1(ii) and Theorem 4.2(iv) that $\mathcal{P}(X)$ is a reflectable base.

Next suppose $X = B_\ell$ with $\ell \geq 3$. We have

$$\mathcal{P} := \mathcal{P}(B_\ell) = \{\alpha_1, \ldots, \alpha_\ell, \gamma_1 - \theta_s, \ldots, \gamma_{\rho_1} - \theta_s, \delta_{t+1} - \theta_l, \ldots, \delta_\nu - \theta_l\},$$

where $\rho_1 = \mathrm{ind}(S_1)$. We assume that the only short root of $\dot{\Pi}$ is α_ℓ. We have

$$\mathcal{P}_{\rm sh} = \{\alpha_\ell, \gamma_1 - \theta_s, \ldots, \gamma_{\rho_1} - \theta_s\},$$
$$\mathcal{P}_{\rm lg} = \{\alpha_1, \ldots, \alpha_{\ell-1}, \delta_{t+1} - \theta_l, \ldots, \delta_\nu - \theta_l\}.$$

By Proposition 4.4, $\mathcal{P}$ is a reflectable set for R and so by Remark 4.3,

(1) $R_{\rm sh} = \bigcup_{\alpha\in\mathcal{P}_{\rm sh}}(\alpha + \langle R_{\rm lg}\rangle) \cap R_{\rm sh}$;

(2) $\{\alpha + 2\langle R_{\rm sh}\rangle \mid \alpha \in \mathcal{P}_{\rm lg}\}$ spans the $\mathbb{Z}_2$-vector space $\mathcal{U} = \langle R_{\rm lg}\rangle/2\langle R_{\rm sh}\rangle$.

Thus, to show that $\mathcal{P}$ is a reflectable base, it is enough to show (by Theorem 4.2(ii)) that the union in (1) is disjoint and that the set in (2) is a basis for $\mathcal{U}$. We start by proving that the union in (1) is disjoint.

If $\alpha_\ell \in \bigcup_{\alpha\in\mathcal{P}_{\rm sh}\setminus\{\alpha_\ell\}}(\alpha+\langle R_{\rm lg}\rangle)\cap R_{\rm sh}$, then there is a $1\leq i\leq \mathrm{ind}(S_1)$ such that $\alpha_\ell\in\gamma_i-\theta_s+\langle R_{\rm lg}\rangle$ and this implies that $\gamma_i\in\langle L\rangle$, which contradicts **1.13**(1.6). Next suppose there exists a $1\leq i\leq \mathrm{ind}(S_1)$ such that $\gamma_i-\theta_s\in\bigcup_{\alpha\in\mathcal{P}_{\rm sh}}\setminus\{\gamma_i-\theta_s\}(\alpha+\langle R_{\rm lg}\rangle)\cap R_{\rm sh}$. Now if $\gamma_i-\theta_s\in\alpha_\ell+\langle R_{\rm lg}\rangle$, then $\gamma_i\in\langle L\rangle$ which again contradicts **1.13**(1.6). If $\gamma_i-\theta_s\in\gamma_j-\theta_s+\langle R_{\rm lg}\rangle$ for some $1\leq j\neq i\leq \mathrm{ind}(S_1)$, we get $\gamma_i-\gamma_j\in 2\langle S_1\rangle$, which contradicts **1.13**(1.8). This completes the proof for (1).

Next, we consider (2). As the set $\{\alpha+2\langle R_{\rm sh}\rangle\mid\alpha\in\mathcal{P}_{\rm lg}\}$ spans $\mathcal{U}$ and

$$\dim_{\mathbb{Z}_2}\mathcal{U}=(\ell-1)+(\nu-t)=|\mathcal{P}_{\rm lg}|=(\ell-1)+(\nu-t),$$

this set is a basis for $\mathcal{U}$. Thus $\mathcal{P}$ is a reflectable base for R and $|\mathcal{P}|=\mathrm{ind}(R)+\ell+\nu$.

The proof for type C_ℓ $(\ell\geq 3)$ is analogous to type B_ℓ, replacing the roles of short and long roots in the proof, and using the characterization of reflectable bases of type C_ℓ given in Theorem 4.2(iii). The proofs for types A_1 and B_2 can also be carried out in a similar manner, using recognition theorems, Theorems 4.1(i) and 4.2(i).

(ii) Let $\mathcal{P}=\mathcal{P}(X)$. If $\mathrm{ind}(R)=0$, then $|\mathcal{P}|=\ell+\nu$ and as $\mathcal{W}=\mathcal{W}_{\mathcal{P}}$, we have $|\mathcal{P}|=\ell+\nu=\dim(\mathcal{V})=\dim(\mathrm{span}_{\mathbb{R}}\mathcal{P})$; thus $\mathcal{P}$ is a basis for $\mathcal{V}$. Since $\dim(\mathcal{V})\leq c\leq|\mathcal{P}|$, we have $c=|\mathcal{P}|=\ell+\nu$, that is, $\mathcal{P}\in\mathcal{M}_c$. □

For the proof of the following corollary, we recall the elementary fact from group theory that if a group G is finitely generated, then any generating set for G contains a finite generating set.

Corollary 4.5. *Let $\mathcal{P}$ be in either $\mathcal{M}_c$, $\mathcal{M}_m$ or $\mathcal{M}_r$. Then $\mathcal{P}$ is finite.*

Proof. By **3.4**(3.1), we only need to consider $\mathcal{P}\in\mathcal{M}_m$ or $\mathcal{P}\in\mathcal{M}_r$. By Proposition 4.4, $\mathcal{W}$ is generated by reflections based on the finite set $\mathcal{P}(X)$, and so is finitely generated. Thus any generating set for $\mathcal{W}$ contains a finite generating set. In particular, if $\mathcal{P}\in\mathcal{M}_m$ then $\mathcal{P}$ contains a finite subset $\mathcal{P}'$ such that $S_{\mathcal{P}'}$ generates $\mathcal{W}$. But the minimality of $\mathcal{P}$ gives $\mathcal{P}=\mathcal{P}'$, and we are done.

Next assume $\mathcal{P}\in\mathcal{M}_r$. We have $\mathcal{W}=\mathcal{W}_{\mathcal{P}}$, and so as explained above, $\mathcal{W}$ is generated by reflections based on a finite subset $\mathcal{P}'$ of $\mathcal{P}$. Again let $\mathcal{P}(X)$ be the reflectable base given in Proposition 4.4. Then $\mathcal{P}(X)\subseteq R^\times=\mathcal{W}_{\mathcal{P}}\mathcal{P}=\mathcal{W}_{\mathcal{P}'}\mathcal{P}$. As $\mathcal{P}(X)$ is finite, we have $\mathcal{P}(X)\subseteq\mathcal{W}_{\mathcal{P}'}\mathcal{P}''$ for some finite subset $\mathcal{P}''$ of $\mathcal{P}$. Now setting $\widehat{\mathcal{P}}:=\mathcal{P}'\cup\mathcal{P}''$, we get

$$R^\times=\mathcal{W}\mathcal{P}(X)=\mathcal{W}_{\widehat{\mathcal{P}}}\mathcal{P}(X)\subseteq\mathcal{W}_{\widehat{\mathcal{P}}}\mathcal{W}_{\mathcal{P}'}\mathcal{P}''\subseteq\mathcal{W}_{\widehat{\mathcal{P}}}\widehat{\mathcal{P}}.$$

Now the minimality of $\mathcal{P}$ gives $\widehat{\mathcal{P}}=\mathcal{P}$, and we are done. □

§4.3. On cardinality of reflectable bases; types A_1, B_ℓ, C_ℓ, F_4, G_2

In Corollary 4.5, we just saw that reflectable bases have finite cardinalities. Do any two reflectable bases for an extended affine root system have the same cardinality? As we will see in the sequel, the answer is not positive in general, but we will discuss situations in which the response is affirmative.

Theorem 4.6. *Let R be an extended affine root system of rank ℓ, nullity ν and twist number t, and let $\mathcal{P}$ and $\mathcal{P}'$ be two reflectable bases for R. Then for the types given in Table 2, $|\mathcal{P}| = |\mathcal{P}'|$. Moreover, if $\mathcal{P} \in \mathcal{M}_r$, then $|\mathcal{P}|$, $|\mathcal{P}_{\rm sh}|$ and $|\mathcal{P}_{\rm lg}|$ are given by Table 2, where S, S_1 and S_2 are as in* **1.13**(1.6).

Type	$\lvert\mathcal{P}\rvert$	$\lvert\mathcal{P}_{\rm sh}\rvert$	$\lvert\mathcal{P}_{\rm lg}\rvert$
A_1	$1+\mathrm{ind}(S)$	—	—
B_2	$2+\mathrm{ind}(S_1)+\mathrm{ind}(S_2)$	$1+\mathrm{ind}(S_1)$	$1+\mathrm{ind}(S_2)$
B_ℓ $(\ell\geq 3)$	$\ell+\mathrm{ind}(S_1)+(\nu-t)$	$1+\mathrm{ind}(S_1)$	$(\ell-1)+(\nu-t)$
C_ℓ $(\ell\geq 3)$	$\ell+t+\mathrm{ind}(S_2)$	$(\ell-1)+t$	$1+\mathrm{ind}(S_2)$
F_4	$4+\nu$	$2+t$	$2+(\nu-t)$
G_2	$2+\nu$	$1+t$	$1+(\nu-t)$

Table 2. Cardinality of reflectable bases

Proof. If R is of type A_1, then we have $|\mathcal{P}| = 1+\mathrm{ind}(S)$, by [AP, Lem. 2.3], and we are done in this case.

Next suppose R is of type B_2 and $\mathcal{P}$ and $\mathcal{P}'$ are two reflectable bases for R. By Theorem 4.2(i), we have the following coset description of $R_{\rm sh}$ and $R_{\rm lg}$:

$$R_{\rm sh} = \biguplus_{\alpha\in\mathcal{P}_{\rm sh}} (\alpha+\langle R_{\rm lg}\rangle)\cap R_{\rm sh} = \biguplus_{\alpha'\in\mathcal{P}'_{\rm sh}} (\alpha'+\langle R_{\rm lg}\rangle)\cap R_{\rm sh}$$

and

$$R_{\rm lg} = \biguplus_{\alpha\in\mathcal{P}_{\rm lg}} (\alpha+2\langle R_{\rm sh}\rangle)\cap R_{\rm lg} = \biguplus_{\alpha'\in\mathcal{P}'_{\rm lg}} (\alpha'+2\langle R_{\rm sh}\rangle)\cap R_{\rm lg}.$$

It follows that $|\mathcal{P}_{\rm sh}| = |\mathcal{P}'_{\rm sh}|$ and $|\mathcal{P}_{\rm lg}| = |\mathcal{P}'_{\rm lg}|$. Thus $|\mathcal{P}| = |\mathcal{P}'|$.

Assume next that R is of type B_ℓ $(\ell\geq 3)$ or C_ℓ $(\ell\geq 3)$. We give the proof for type B_ℓ; the proof for type C_ℓ is analogous, replacing the roles of short and long roots. Assume $\mathcal{P}$ and $\mathcal{P}'$ are two reflectable bases for R. According to

Theorem 4.2(ii), we have

$$R_{\mathrm{sh}} = \biguplus_{\alpha\in\mathcal{P}_{\mathrm{sh}}} (\alpha + \langle R_{\mathrm{lg}}\rangle) \cap R_{\mathrm{sh}} = \biguplus_{\alpha'\in\mathcal{P}'_{\mathrm{sh}}} (\alpha' + \langle R_{\mathrm{lg}}\rangle) \cap R_{\mathrm{sh}}$$

and

$$|\mathcal{P}_{\mathrm{lg}}| = \dim_{\mathbb{Z}_2}(\langle R_{\mathrm{lg}}\rangle/2\langle R_{\mathrm{sh}}\rangle) = |\mathcal{P}'_{\mathrm{lg}}|.$$

Thus $|\mathcal{P}_{\mathrm{sh}}| = |\mathcal{P}'_{\mathrm{sh}}|$ and $|\mathcal{P}_{\mathrm{lg}}| = |\mathcal{P}'_{\mathrm{lg}}|$, as required.

Next assume R is of type F_4 or G_2. By Theorem 4.2(iv), if $\mathcal{P}$ and $\mathcal{P}'$ are two reflectable bases for R, then

$$|\mathcal{P}_{\mathrm{lg}}| = \dim_{\mathbb{Z}_2}(\langle R_{\mathrm{lg}}\rangle/2\langle R_{\mathrm{sh}}\rangle) = |\mathcal{P}'_{\mathrm{lg}}| \quad\text{and}\quad |\mathcal{P}_{\mathrm{sh}}| = \dim_{\mathbb{Z}_2}(\langle R_{\mathrm{sh}}\rangle/\langle R_{\mathrm{lg}}\rangle) = |\mathcal{P}'_{\mathrm{sh}}|.$$

Finally, we consider the last assertion in the statement. Let $\mathcal{P}$ be a reflectable base of type $X = A_1, B_\ell, C_\ell, F_4$ or G_2. Since now we know that any two reflectable bases for R have the same cardinality, it is enough to assume that $\mathcal{P} = \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the reflectable base given in Proposition 4.4. The result can now be seen from the information given in Table 1. □

Corollary 4.7. *The cardinality of a reflectable base is an isomorphism invariant of an extended affine root system of types $X = A_1, B_\ell, C_\ell, F_4, G_2$.*

Proof. One notes that the index of semilattices S, S_1 and S_2 appearing in the structure of an extended affine root system, as well as the nullity, the rank and the twist number are isomorphism invariants for an extended affine root system; see [AABGP, Chap. II]. Now the result is immediate from Theorem 4.6. □

Lemma 4.8. *Suppose $\mathcal{P}$ is a reflectable base for R such that $|\mathcal{P}| = \ell + \nu$; then $\mathcal{P} \in \mathcal{M}_{rm}$ (that is, $\mathcal{P}$ is a minimal reflectable base).*

Proof. Since $\mathcal{P}$ is a reflectable base, the reflections based on $\mathcal{P}$ generate $\mathcal{W}$, so it is enough to show that no proper subset of $\mathcal{P}$ has this property. But this is clear as $|\mathcal{P}| = \dim \mathcal{V}$. □

§4.4. On cardinalities of reflectable bases; simply laced types of rank > 1

Let R be a simply laced extended affine root system of rank $\ell > 1$ and nullity $\nu > 1$. We start with the following example which shows that Theorem 4.6 does not hold in general for R, namely reflectable bases for R might have different cardinalities, even when $\nu = 1$ which is the affine case. The example also shows that $c = \ell + \nu$ and that $\mathcal{M}_{rc}$ is a proper subclass of $\mathcal{M}_{rm}$. We then show that $\mathcal{M}_r = \mathcal{M}_{rm}$. The subsection is concluded with a technical result that shows that each reflectable base for R contains a subset of cardinality $\ell+\nu$ which is a reflectable base for a subsystem of R of the same type, rank and nullity as R.

Example 4.9. Let R be a simply laced extended affine root system of type X and rank >1. Then $R=\Lambda\cup(\dot{R}+\Lambda)$, where $\dot{R}$ is a finite root system of type X and $\Lambda=\mathbb{Z}\sigma_1\oplus\cdots\oplus\mathbb{Z}\sigma_\nu$; see **1.11** and **1.12**. Let $\dot{\Pi}=\{\alpha_1,\ldots,\alpha_\ell\}$ be a fundamental system for $\dot{R}$. Set

$$\Pi=\{\alpha_1,\ldots,\alpha_\ell,\alpha_1+m_1\sigma_1,\alpha_2+m_2\sigma_1,\alpha_2+\sigma_2,\ldots,\alpha_2+\sigma_\nu\},$$

where $m_1,m_2>1$ are relatively prime integers. We claim that Π is a reflectable base for R. Since $(m_1,m_2)=1$, we have $\langle\Pi\rangle=\langle\dot{R}\rangle\oplus\Lambda=\langle R\rangle$, and so we get from Theorem 4.1(ii) and Remark 4.3 that Π is a reflectable set for R. Therefore, using Theorem 4.1(ii) again, it is enough to show that for each $\alpha\in\Pi$, $\Pi'_\alpha:=\Pi\setminus\{\alpha\}$ does not generate $\langle R\rangle$. Clearly, we only need to check this for $\alpha\in\{\alpha_1,\alpha_2,\alpha_1+m_1\sigma_1,\alpha_2+m_2\sigma_1\}$. Since $m_1,m_2>1$, we have $\sigma_1\notin\langle\Pi'_{\alpha_1+m_1\sigma_1}\rangle$ and similarly $\sigma_1\notin\langle\Pi'_{\alpha_2+m_2\sigma_1}\rangle$. If $\sigma_1\in\Pi'_{\alpha_1}$, then $\sigma_1=k\alpha_2+k'(\alpha_1+m_1\sigma_1)+k''(\alpha_2+m_2\sigma_1)$ for some $k,k',k''\in\mathbb{Z}$. This gives $k'=0$ and so $k''m_2=1$ which is impossible. The same reasoning gives $\sigma_1\notin\langle\Pi'_{\alpha_2}\rangle$. Thus Π is a reflectable base. Note that $|\Pi|=\ell+\nu+1>|\mathcal{P}(X)|=\ell+\nu$, where $\mathcal{P}(X)$ is the reflectable base given in Table 1. We conclude that $\mathcal{M}_{rc}$ is a proper subclass of $\mathcal{M}_{rm}$.

Proposition 4.10. *Let R be a simply laced extended affine root system of rank >1. Then $\mathcal{M}_r=\mathcal{M}_{rm}$.*

Proof. We must show $\mathcal{M}_r\subseteq\mathcal{M}_m$. Assume that $\mathcal{P}$ is a reflectable base for R; by Theorem 4.1(ii), $\mathcal{P}$ is minimal with respect to the property that $\langle\mathcal{P}\rangle=\langle R\rangle$. If $\mathcal{P}\notin\mathcal{M}_m$, then there is at least one $\alpha_0\in\mathcal{P}$ such that $w_{\alpha_0}\in\langle w_\gamma\mid\gamma\in\mathcal{P}\setminus\{\alpha_0\}\rangle$. But then Proposition 3.5 gives $\langle\mathcal{P}\setminus\{\alpha_0\}\rangle=\langle R\rangle$, which contradicts the minimality of $\mathcal{P}$. □

Proposition 4.11. *Let R be a simply laced extended affine root system of rank $\ell>1$ and nullity ν. Let Π be a reflectable base for R. Then Π contains a subset Π' of cardinality $\ell+\nu$ such that Π' is a reflectable base for a root subsystem R' of R, where R' has the same type, rank and nullity as R.*

Proof. Let Π be a reflectable base for R; then $|\Pi|\geq\ell+\nu$. If $|\Pi|=\ell+\nu$, there is nothing to prove. Now assume that $|\Pi|>\ell+\nu$. Recall from Section 1.1 that $^-$ is the canonical map from $\mathcal{V}$ onto $\overline{\mathcal{V}}:=\mathcal{V}\setminus\mathcal{V}^0$ and $\bar{R}$ is the image of R under $^-$. Since $\mathcal{W}_\Pi\Pi=R^\times$, we get from Proposition 1.10 that there exists a subset $\dot{\Pi}$ of Π, such that $\dot{R}=(\mathcal{W}_{\dot{\Pi}}\dot{\Pi})\cup\{0\}\subseteq R$ is an irreducible finite root system isomorphic to $\bar{R}$, and we get the description $R=\Lambda\cup(\dot{R}+\Lambda)$ for R where Λ is a lattice; see **1.11**.

We fix a reflectable base $\{\alpha_1,\ldots,\alpha_\ell\}\subseteq\dot\Pi$ of $\dot R$; see **3.2**. Then $\dot{\mathcal V}=\mathrm{span}_{\mathbb R}\dot R$. We extend $\{\alpha_1,\ldots,\alpha_\ell\}$ to a basis $\Pi'=\{\alpha_1,\ldots,\alpha_\ell,\alpha_{\ell+1},\ldots,\alpha_{\ell+\nu}\}$ of $\mathcal V$, such that $\Pi'\subseteq\Pi$. Since Π' spans $\mathcal V$, it is connected.

We now consider the extended affine root system $R':=R_{\Pi'}$ defined in Proposition 3.6. By the same argument as in the proof of Proposition 3.6(ii), R' has the same type, rank and nullity as R. Since $|\Pi'|=\ell+\nu=\dim\mathcal V$, it is a reflectable base for R'. □

§5. A characterization of minimal root systems

As in the previous sections, we let R be an extended affine root system of reduced type X with corresponding Weyl group $\mathcal W$. It is known that $\mathcal W$ has the presentation by conjugation, see Definition 2.6, if and only if R is "minimal" in the sense of Definition 5.1 below. Therefore, the minimality condition reflects the geometric aspects of extended affine root systems. In this section we characterize minimal root systems in terms of minimal reflectable bases.

§5.1. Minimal extended affine root systems

Definition 5.1. Following [Ho], we say the extended affine root system R is *minimal* if for each $\alpha\in R^\times$, $\mathcal W_{\mathcal P}\subsetneq\mathcal W$, where $\mathcal P=R^\times\setminus\mathcal W\alpha$.

Lemma 5.2. *Let $\mathcal P\subseteq R^\times$.*

(i) *$\mathcal W\mathcal P=\mathcal P$ if and only if $S_{\mathcal P}=S_{\mathcal W\mathcal P}$ and $\mathcal P=-\mathcal P$.*
(ii) *The extended affine root system R is minimal if and only if whenever $\mathcal P\subseteq R^\times$ with $\mathcal W_{\mathcal P}=\mathcal W$ and $\mathcal W\mathcal P=\mathcal P$, then $\mathcal P=R^\times$.*

Proof. (i) The "if" part is clear. Now assume that $S_{\mathcal P}=S_{\mathcal W\mathcal P}$ and $\mathcal P=-\mathcal P$. Then for $w'\in\mathcal W$ and $\alpha\in\mathcal P$ we have $w_{w'(\alpha)}\in S_{\mathcal P}$; thus there exists $\beta\in\mathcal P$ such that $w_{w'(\alpha)}=w_\beta$ and then we have $w'(\alpha)=\pm\beta$. Now as $\mathcal P=-\mathcal P$, we have $w'(\alpha)\in\mathcal P$ and therefore $\mathcal W\mathcal P=\mathcal P$.

(ii) Assume first that R is minimal and $\mathcal P$ is a subset of $R^\times$ such that $\mathcal W=\mathcal W_{\mathcal P}$ and $\mathcal W\mathcal P=\mathcal P$. We must show $\mathcal P=R^\times$. If not, we pick $\alpha\in R^\times\setminus\mathcal P$. Since $\mathcal W\mathcal P=\mathcal P$, we get $\mathcal P\subset R^\times\setminus\mathcal W\alpha$. But now the assumption $\mathcal W=\mathcal W_{\mathcal P}$ contradicts minimality of R.

Next assume that the "only if" part holds, and that R is not minimal. So there exists $\alpha\in R^\times$ such that $\mathcal W_{\mathcal P}=\mathcal W$ for $\mathcal P:=R^\times\setminus\mathcal W\alpha$. Now since $\mathcal W\mathcal P=\mathcal P$, we get by the assumption that $R^\times=\mathcal P$, which is absurd. □

§5.2. Minimality and presentation by conjugation

Let R be a reduced extended affine root system with Weyl group $\mathcal{W}$. Let $\widehat{\mathcal{W}}$ be the group defined by generators $\widehat{w}_\alpha$, $\alpha \in R^\times$ and relations

- $\widehat{w}^2 = 1$, $\alpha \in R^\times$;
- $\widehat{w}_\alpha \widehat{w}_\beta \widehat{w}_\alpha = \widehat{w}_{w_\alpha \beta}$, $\alpha, \beta \in R^\times$.

We recall from Definition 2.6 that if $\mathcal{W} \cong \widehat{\mathcal{W}}$, then $\mathcal{W}$ is said to have the presentation by conjugation. We refer to this group as the *conjugation presented group*, associated with R.

Proposition 5.3. *Given any reflectable base $\mathcal{P}$ for R, the set $\{\widehat{w}_\alpha \mid \alpha \in \mathcal{P}\}$ is a minimal generating set for the conjugation presented group $\widehat{\mathcal{W}}$ associated to R.*

Proof. If $\nu = 0$ then R is a finite root system and so $\mathcal{W} \cong \widehat{\mathcal{W}}$. Then we are done by Proposition 3.8.

Assume next that $\nu \geq 1$. If R is simply laced of rank > 1 or has one of types F_4 or G_2, then by Lemma 4.8, Theorem 4.6 and Proposition 4.10, $\{w_\alpha \mid \alpha \in \mathcal{P}\}$ is a minimal generating set for $\mathcal{W}$. But by Theorem 2.5, for the types under consideration $\mathcal{W} \cong \widehat{\mathcal{W}}$ and so we are done.

Next, suppose that R is of type A_1; then by Theorem 4.1(i), we have

$$R^\times = \biguplus_{\alpha \in \mathcal{P}} (\alpha + 2\langle R \rangle) \cap R^\times. \tag{5.1}$$

On the other hand, since $R^\times = \bigcup_{\alpha \in \mathcal{P}} \mathcal{W}\alpha$, and for $\alpha \in R^\times$ we have $\mathcal{W}\alpha \subseteq \alpha + 2\langle R \rangle$, we get from (5.1),

$$R^\times = \biguplus_{\alpha \in \mathcal{P}} \mathcal{W}\alpha. \tag{5.2}$$

If $\{\widehat{w}_\alpha \mid \alpha \in \mathcal{P}\}$ is not a minimal generating set for $\widehat{\mathcal{W}}$, then there is at least one $\beta \in \mathcal{P}$ such that $\widehat{w}_\beta = \widehat{w}_{\gamma_1} \cdots \widehat{w}_{\gamma_n}$ for some $\gamma_i \in \mathcal{P} \setminus \{\beta\}$, $1 \leq i \leq n$. By Corollary 2.8, the assignment

$$w_\alpha \overset{\Phi_\beta}{\longmapsto} \begin{cases} 1 & \text{if } \alpha \in \mathcal{W}\beta, \\ 0 & \text{if } \alpha \in R^\times \setminus \mathcal{W}\beta, \end{cases}$$

induces an epimorphism $\Phi_\beta \colon \mathcal{W} \to \mathbb{Z}_2$. Now, by (5.2), $\mathcal{P} \setminus \{\beta\} \subseteq R^\times \setminus \mathcal{W}\beta$, and so we get $\Phi_\beta(\widehat{w}_\beta) = 1$ and $\Phi_\beta(\widehat{w}_{\gamma_1} \ldots \widehat{w}_{\gamma_n}) = 0$ which is a contradiction.

Next we consider type B_2. In this case, it is easy to see that

$$\mathcal{W}\alpha \subseteq \begin{cases} \alpha + \langle R_{\mathrm{lg}} \rangle & \text{if } \alpha \in R_{\mathrm{sh}}, \\ \alpha + 2\langle R_{\mathrm{sh}} \rangle & \text{if } \alpha \in R_{\mathrm{lg}}. \end{cases} \tag{5.3}$$

By Theorem 4.1(iii), we have

$$R_{\mathrm{sh}} = \biguplus_{\alpha\in\mathcal{P}_{\mathrm{sh}}} ((\alpha + \langle R_{\mathrm{lg}}\rangle) \cap R_{\mathrm{sh}}) \quad \text{and} \quad R_{\mathrm{lg}} = \biguplus_{\alpha\in\mathcal{P}_{\mathrm{lg}}} ((\alpha + 2\langle R_{\mathrm{sh}}\rangle) \cap R_{\mathrm{lg}}).$$

Using these equations together with (5.3), we get that the union $R^\times = \bigcup_{\alpha\in\mathcal{P}} \mathcal{W}\alpha$ is disjoint. Then using the same argument as in type A_1, we achieve the desired result.

Finally, suppose that R is of type B_ℓ ($\ell > 2$), and assume to the contrary that $\{\widehat{w}_\alpha \mid \alpha \in \mathcal{P}\}$ is not a minimal generating set for $\widehat{\mathcal{W}}$; then there is at least one $\beta \in \mathcal{P}$ such that $\widehat{w}_\beta \in \langle \widehat{w}_\alpha \mid \alpha \in \mathcal{P} \setminus \{\beta\}\rangle$. We proceed with the proof by considering the following two cases.

$\beta \in R_{\mathrm{sh}}$**.** Similar to the case B_2, using Theorem 4.1(iv), we see that $R_{\mathrm{sh}} = \biguplus_{\alpha\in\mathcal{P}_{\mathrm{sh}}} \mathcal{W}\alpha$; then $\mathcal{P} \setminus \{\beta\} \subseteq R^\times \setminus \mathcal{W}\beta$. Therefore, repeating the same argument as in type B_2 leads to a contradiction.

$\beta \in R_{\mathrm{lg}}$**.** We have $\widehat{w}_\beta = \widehat{w}_{\gamma_1} \cdots \widehat{w}_{\gamma_n}$ for some $\gamma_i \in \mathcal{P} \setminus \{\beta\}$. Applying the epimorphism ψ given in **2.7**(2.6), we get $w_\beta = w_{\gamma_1} \dots w_{\gamma_n}$. Since $\ell \geq 3$, there exists $\alpha \in R_{\mathrm{lg}}$ such that $(\alpha, \beta^\vee) = -1$. If $\{\gamma_{i_1}, \dots, \gamma_{i_m}\} = \{\gamma_1, \dots, \gamma_n\} \cap R_{\mathrm{lg}}$ then we have

$$\alpha + \beta = w_\beta(\alpha) = w_{\gamma_1} \dots w_{\gamma_n}(\alpha) \in w_{\gamma_{i_1}} \dots w_{\gamma_{i_m}}(\alpha) + 2\langle R_{\mathrm{sh}}\rangle;$$

thus

$$\beta \in \langle \gamma_{i_1}, \dots, \gamma_{i_m}\rangle + 2\langle R_{\mathrm{sh}}\rangle, \quad \gamma_{i_j} \in \mathcal{P}_{\mathrm{lg}} \setminus \{\beta\}, \tag{5.4}$$

which is a contradiction as, by Theorem 4.1(iv), $\mathcal{P}_{\mathrm{lg}}$ is minimal with respect to the property that the set $\{\alpha + 2\langle R_{\mathrm{sh}}\rangle \mid \alpha \in \mathcal{P}_{\mathrm{lg}}\}$ is a basis for the $\mathbb{Z}_2$ vector space $\langle R_{\mathrm{lg}}\rangle / 2\langle R_{\mathrm{sh}}\rangle$.

As usual, the argument for the case C_ℓ is analogous to B_ℓ, replacing the roles of short and long roots. □

Theorem 5.4. *The following are equivalent:*

(i) *R is minimal;*

(ii) *$\mathcal{W}$ has the presentation by conjugation;*

(iii) *$\mathcal{M}_r = \mathcal{M}_{rm}$.*

Proof. Conditions (i) and (ii) are equivalent by [Ho, Thm. 5.8]. Next assume (ii) holds, namely $\mathcal{W} \cong \widehat{\mathcal{W}}$ under the isomorphism induced by the assignment $\psi\colon \widehat{\mathcal{W}} \to \mathcal{W}$, $\widehat{w}_\alpha \mapsto w_\alpha$, $\alpha \in R^\times$. Let $\Pi \in \mathcal{M}_r$. By Proposition 5.3, the set $\{\widehat{w}_\alpha \mid \alpha \in \Pi\}$ is a minimal generating set for $\widehat{\mathcal{W}}$, and so we are done. This gives (iii).

Finally, assume that (iii) holds. Consider the set $\mathcal{P}(X)$ given in Table 1. By Proposition 4.4, $\mathcal{P}(X) \in \mathcal{M}_r$ and so $\mathcal{P}(X) \in \mathcal{M}_m$. Then R is minimal by Theorem 5.6. This gives (i) and the proof is completed. $\square$

Corollary 5.5. *Let R be a reduced extended affine root system of type X, rank ℓ and nullity ν. Then under either of the two conditions*

(i) $\nu \leq 2$,

(ii) $X = A_\ell$ $(\ell > 1)$, D_ℓ, $E_{6,7,8}$, F_4 *or* G_2,

we have $\mathcal{M}_r = \mathcal{M}_{rm}$. Moreover, if $\operatorname{ind}(R) = 0$ *and $X = A_1, B_\ell$ $(\ell \geq 2)$ or C_ℓ $(\ell \geq 3)$, then $\mathcal{M}_r = \mathcal{M}_{rc} = \mathcal{M}_{rm}$.*

Proof. For the first part of the statement, we only need to show by Theorem 5.4 that under either of the conditions (i) and (ii), $\mathcal{W}$ has the presentation by conjugation. But this is immediate from Theorem 2.5(ii)–(iii).

Next we assume $\operatorname{ind}(R) = 0$, where $X = A_1, B_\ell, C_\ell$. By **3.4**(3.1), it is enough to show that $\mathcal{M}_r \subseteq \mathcal{M}_{rc}$. Let $\mathcal{P} \in \mathcal{M}_r$. By Theorem 4.6, we have $|\mathcal{P}| = \ell + \nu$. Now as $\mathcal{W} = \mathcal{W}_\mathcal{P}$, we get $\mathcal{V} = \operatorname{span}_\mathbb{R} \mathcal{P}$, and since by **3.4**(3.2), $\dim(\mathcal{V}) \leq c \leq |\mathcal{P}|$, we get $c = \ell + \nu$. Thus $\mathcal{P} \in \mathcal{M}_{rc}$. $\square$

Theorem 5.6. *Let R be a reduced extended affine root system of type X and $\mathcal{P}(X)$ be as in Table 1. Then $\mathcal{P}(X) \in \mathcal{M}_m$ if and only if R is a minimal root system.*

Proof. First, suppose that R is of type A_1 or one of the non-simply-laced types; then by [AS4, Thm. 5.16] and [AS5, Thm. 2.33], $\mathcal{P}(X) \in \mathcal{M}_m$ if and only if R is a minimal root system. Next assume that R is simply laced of rank > 1 and $\mathcal{P} \in \mathcal{M}_m$. If R is not minimal, then by [Ho, Thm. 4.5], $\mathcal{W}$ does not have the presentation by conjugation, contradicting Theorem 2.5(iii).

Conversely, assume that R is a minimal root system. As $|\mathcal{P}(X)| = \ell + \nu = \dim(\mathcal{V})$, we have $\mathcal{P}(X) \in \mathcal{M}_c \subseteq \mathcal{M}_m$. $\square$

§5.3. Gallery of results

Table 3 concludes and illustrates our investigation concerning the interrelations between classes $\mathcal{M}_r$, $\mathcal{M}_m$ and $\mathcal{M}_c$. Here, R is an irreducible, reduced, extended, affine root system of type X, rank ℓ and nullity ν.

§6. Relations to extended affine Lie algebras

We begin with a brief introduction to the definition of an extended affine Lie algebra and some of its basic properties. For more details on the subject and further terminology, we refer the reader to [AABGP, Chap. I].

$R =$ extended affine root system	$\xRightarrow{\text{Proposition 4.4}}$	$\mathcal{P}(X)$ is a reflectable base for R
$\mathcal{P}(X) \in \mathcal{M}_m$	$\xLeftrightarrow{\text{Theorem 5.6}}$	R is minimal
$\mathcal{W}_{\mathcal{P}} = \mathcal{W}$	$\xRightarrow{\text{Proposition 3.5}}$	$\langle \mathcal{P} \rangle = \langle R \rangle$, $\mathcal{P}$ is connected
R finite	$\xRightarrow{\text{Proposition 3.8}}$	$\mathcal{M}_c = \mathcal{M}_m = \mathcal{M}_r$, $c = \text{rank}(R)$
$\mathcal{P} \in \mathcal{M}_c$, $\mathcal{M}_m$ or $\mathcal{M}_r$	$\xRightarrow{\text{Corollary 4.5}}$	$\mathcal{P}$ is finite
$X = A_1, B_\ell, C_\ell, F_4, G_2$	$\xRightarrow{\text{Theorem 4.6}}$	All elements of $\mathcal{M}_r$ have the same cardinality
$X =$ simply laced of rank > 1	$\xRightarrow[\text{Example 4.9}]{\text{Proposition 4.10}}$	$\mathcal{M}_r = \mathcal{M}_{rm}$ and $\mathcal{M}_{rc} \subsetneq \mathcal{M}_r$
$\mathcal{P} \in \mathcal{M}_r$	$\xRightarrow{\text{Proposition 5.3}}$	$\mathcal{P}$ is a minimal generating set for the conjugation presented group $\widehat{\mathcal{W}}$
$\mathcal{M}_r = \mathcal{M}_{rm}$	$\xLeftrightarrow{\text{Theorem 5.4}}$	R is minimal
$\nu \leq 2$, or $X =$ simply laced of rank > 1, $X = F_4, G_2$, $X = A_1, B_\ell, C_\ell$, and $\text{ind}(R) = 0$	$\xRightarrow{\text{Corollary 5.5}}$	$\mathcal{M}_r = \mathcal{M}_{rm}$
$X = F_4, G_2$	$\xRightarrow{\text{Corollary 5.5}}$	$\mathcal{M}_r = \mathcal{M}_{rc} = \mathcal{M}_{rm}$
$\text{ind}(R) = 0$, $X = A_1, B_\ell, C_\ell$	$\xRightarrow{\text{Corollary 5.5}}$	$\mathcal{M}_r = \mathcal{M}_{rc} = \mathcal{M}_{rm}$

Table 3. Gallery of results on the level of root system and Weyl group

§6.1. Extended affine Lie algebras

Definition 6.1. A triple $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$ in which $\mathcal{L}$ is a complex Lie algebra, $\mathcal{H}$ is a subalgebra and $(\cdot,\cdot)$ is a bilinear form on $\mathcal{L}$ is called an *extended affine Lie algebra* if the following five axioms hold:

(E1) The form $(\cdot,\cdot)$ is symmetric, invariant and non-degenerate.

(E2) $\mathcal{H}$ is non-trivial, finite-dimensional and self-centralizing such that

$$\mathcal{L} = \sum_{\alpha \in R} \mathcal{L}_\alpha,$$

where

$$\mathcal{L}_\alpha = \big\{ x \in \mathcal{L} \;\big|\; [h, x] = \alpha(h)(x) \text{ for all } h \in \mathcal{H} \big\}$$

and

$$R = \big\{ \alpha \in \mathcal{H}^\star \;\big|\; \mathcal{L}_\alpha \neq \{0\} \big\}.$$

We call R the *root system* of $\mathcal{L}$ with respect to $\mathcal{H}$. The form on $\mathcal{L}$ restricted to $\mathcal{H}$ is non-degenerate so the form can be transferred to $\mathcal{H}^\star$ in a natural way. We denote by t_α the unique element in $\mathcal{H}$ which represents $\alpha \in \mathcal{H}^\star$ via the form, namely

$$\alpha(h) = (t_\alpha, h) \quad (h \in \mathcal{H}).$$

Set

$$R^0 := \{\alpha \in R \mid (\alpha, \alpha) = 0\} \quad \text{and} \quad R^\times := R \setminus R^0.$$

(E3) $\mathrm{ad}(x)$ is locally nilpotent for $x \in \mathcal{L}_\alpha$, $\alpha \in R^\times$.

(E4) R is a discrete subset of $\mathcal{H}^\star$.

(E5) R is irreducible, meaning that $R^\times$ is connected, and for $\sigma \in R^0$, there exists $\alpha \in R^\times$ with $\alpha + \sigma \in R$.

6.2. We assume from now on that $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$ is an extended affine Lie algebra with root system R. One knows that R is an extended affine root system in $\mathcal{V} := \mathrm{span}_{\mathbb{R}} R$, all non-isotropic root spaces are 1-dimensional, and that for $\alpha \in R^\times$, one may choose $e_{\pm\alpha} \in \mathcal{L}_{\pm\alpha}$ such that $(e_\alpha, h_\alpha := [e_\alpha, e_{-\alpha}], e_{-\alpha})$ is an $\mathfrak{sl}_2$-triple. From now on, *we assume that R is reduced*. For $\alpha \in R^\times$ and $x_{\pm\alpha} \in \mathcal{L}_{\pm\alpha}$, we define $\Phi_\alpha \in \mathrm{Aut}(\mathcal{L})$ by

$$\Phi_\alpha := \exp(\mathrm{ad}\, x_\alpha) \exp(-\mathrm{ad}\, x_{-\alpha}) \exp(\mathrm{ad}\, x_\alpha).$$

Then $\Phi_\alpha(\mathcal{L}_\beta) = \mathcal{L}_{w_\alpha(\beta)}$ for $\beta \in R$; see [AABGP, Prop. I.1.27]. As in Definition 2.2, we denote by $\mathcal{W}$ the extended affine Weyl group of R. We also denote by $\mathcal{W}_{\mathcal{L}}$ the Weyl group of $\mathcal{L}$, the subgroup of $\mathrm{GL}(\mathcal{H}^\star)$ generated by reflections $\beta \mapsto \beta - (\beta, \alpha^\vee)\alpha$, $\alpha \in R^\times$. Then Φ_α restricted to $\mathcal{H} \equiv \mathcal{H}^\star$ coincides with the reflection $w_\alpha \in \mathcal{W}$, identifying $\mathcal{W}_{\mathcal{L}}$ with the extended affine Weyl group $\mathcal{W}$ of R.

We recall that the core $\mathcal{L}_c$ of $\mathcal{L}$ is the subalgebra of $\mathcal{L}$ generated by the non-isotropic root spaces $\mathcal{L}_\alpha$, $\alpha \in R^\times$.

§6.2. Reflectable bases and the core

We begin with a lemma that shows, using the concept of a reflectable set, that the root spaces associated with a reflectable set (up to a plus-minus sign) generate the core, implying that the core of an extended affine Lie algebra is finitely generated (compare with [Neh, Sect. 6.12] and [A5, Cor. 2.14]). We start with terminology.

Definition 6.3. We call a subset $\mathcal{P}$ of $R^\times$ a *root-generating set* for $\mathcal{L}$ if the root spaces $\mathcal{L}_{\pm\alpha}$, $\alpha \in \mathcal{P}$ generate $\mathcal{L}_c$. We call a root-generating set *minimal* if no proper subset of $\mathcal{P}$ is a root-generating set.

Lemma 6.4. *Any reflectable set for R is a root-generating set for $\mathcal{L}$. In particular, $\mathcal{L}_c$ is finitely generated.*

Proof. Let $\mathcal{P}$ be a reflectable set and let $\alpha \in R^{\times}$. Then $\alpha = w_{\alpha_1} \cdots w_{\alpha_t}(\alpha_{t+1})$ for some $\alpha_i \in \mathcal{P}$. Therefore,

$$\Phi_{\alpha_1} \cdots \Phi_{\alpha_t}(\mathcal{L}_{\alpha_{t+1}}) = \mathcal{L}_{w_{\alpha_1} \cdots w_{\alpha_t}(\alpha_{t+1})} = \mathcal{L}_{\alpha}.$$

Thus $\mathcal{L}_\alpha$ is contain in the subalgebra of $\mathcal{L}$ generated by $\mathcal{L}_\beta$, $\beta \in \mathcal{P}^{\pm}$. The second statement now follows from Corollary 4.5. □

Lemma 6.5. *Let R be a simply laced extended affine root system of type X and rank > 1. Let $\mathcal{P}$ be a reflectable base for R. Then $\mathcal{P}$ is a minimal root-generating set for $\mathcal{L}$. In particular, the set $\mathcal{P}(X)$ given in Table 1 is a minimal root-generating set for $\mathcal{L}$.*

Proof. Assume to the contrary that $\mathcal{P}$ contains a proper root-generating set $\mathcal{P}'$. If $\alpha \in \mathcal{P}$, then $\mathcal{L}_\alpha$ is generated by some root spaces $\mathcal{L}_{\alpha_1}, \ldots, \mathcal{L}_{\alpha_t}$ for some $\alpha_1, \ldots, \alpha_t \in \mathcal{P}'$. Thus $\alpha \in \langle \mathcal{P}' \rangle$, and so $\langle \mathcal{P}' \rangle = \langle \mathcal{P} \rangle = \langle R \rangle$. Then by [ASTY, Thm. 3.2], $\mathcal{P}'$ is a reflectable set for R. But this contradicts the minimality of $\mathcal{P}$ as a reflectable base. The second assertion in the statement follows from Proposition 4.4. □

Lemma 6.6. *If $\mathrm{ind}(R) = 0$, then any reflectable base $\mathcal{P}$ is a minimal root-generating set for $\mathcal{L}$. In particular, if R is simply laced of rank > 1, or is of type F_4 or G_2, then any reflectable base is a minimal root-generating set for $\mathcal{L}$.* □

Proof. If R is simply laced of rank > 1, then we are done by Lemma 6.5. For the remaining types, we have from Theorem 4.6 that all reflectable bases have the same cardinality. Now, if $\mathrm{ind}(R) = 0$ then we see from **1.13** that $|\mathcal{P}| = |\mathcal{P}(X)| = \ell + \nu$, which is equal to the rank of the free abelian group $\langle R \rangle$. Since the roots corresponding to any generating set of root vectors for $\mathcal{L}_c$ must generate $\langle R \rangle$, the first assertion in the statement holds. The second assertion is now clear as $\mathrm{ind}(R) = 0$ for the given types. □

The following example shows that if $\mathrm{ind}(R) \neq 0$ then Lemma 6.6 may fail. Also, it shows that an extended affine Lie algebra might have minimal root-generating sets with different cardinalities.

Example 6.7. Suppose $\mathcal{L}$ is an extended affine Lie algebra with root system R.

(i) Let R be of type A_1 and nullity 2 with $\mathrm{ind}(R) > 0$. We know that (see **1.13**) up to isomorphism $R = \Lambda + (\pm\alpha + \Lambda)$, where $\Lambda = \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2$. We consider the reflectable base $\mathcal{P}(X) = \{\alpha, \sigma_1 - \alpha, \sigma_2 - \alpha, \sigma_1 + \sigma_2 - \alpha\}$ for R; see Table 1. From [A4, Rem. 1.5(ii)], we know that

$$[[\mathcal{L}_{\sigma_1 - \alpha}, \mathcal{L}_\alpha], \mathcal{L}_{\sigma_2 - \alpha}] \neq \{0\}.$$

Thus the 1-dimensional space $\mathcal{L}_{\sigma_1+\sigma_2-\alpha}$ is generated by root spaces corresponding to roots $\{\alpha, \sigma_1-\alpha, \sigma_2-\alpha\}$. Therefore, by Lemma 6.4, $\mathcal{L}_c$ is generated by root spaces $\mathcal{L}_{\pm\beta}$, $\beta \in \{\alpha, \sigma_1-\alpha, \sigma_2-\alpha\}$. Thus $\mathcal{P}(X)$ is not a minimal root-generating set for $\mathcal{L}_c$.

(ii) Let R be of type A_2. Then up to isomorphism $R = \Lambda \cup (\dot{R} + \Lambda)$, where $\Lambda = \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2$ and $\dot{R}$ is a finite root system of type A_2 with a base $\{\alpha_1, \alpha_2\}$. Then $\mathcal{P} = \{\alpha_1, \alpha_2, \alpha_1+\sigma_1, \alpha_2+\sigma_2\}$ and $\mathcal{P}' = \{\alpha_1, \alpha_2, \alpha_1+2\sigma_1, \alpha_2+3\sigma_1, \alpha_2+\sigma_2\}$ are reflectable bases for R; see Example 4.9. By Lemma 6.6 both $\mathcal{P}$ and $\mathcal{P}'$ are minimal root-generating sets with $|\mathcal{P}| \neq |\mathcal{P}'|$ for the corresponding Lie algebra $\mathcal{L}$.

§6.3. Elliptic Lie algebras versus 2-extended affine Lie algebras

We begin with a definition. By a 2-*extended affine Lie algebra* (*root system*) we mean an extended affine Lie algebra (root system) of nullity 2. Throughout this section, R is a reduced 2-extended affine root system.

Definition 6.8. Below we define the concepts of an elliptic root system and an elliptic Lie algebra:

(i) In the literature, a 2-extended affine root system is referred to as an *elliptic root system.*

(ii) Let $\mathcal{L}$ be a Lie algebra containing a subalgebra $\mathcal{H}$ with a decomposition $\mathcal{L} = \sum_{\alpha\in\mathcal{H}^\star} \mathcal{L}_\alpha$, where

$$\mathcal{L}_\alpha = \big\{x \in \mathcal{L} \bigm| [h,x] = \alpha(h)x \text{ for all } h \in \mathcal{H}\big\}.$$

The set $R \coloneqq \{\alpha \in \mathcal{H}^\star \mid \mathcal{L}_\alpha \neq \{0\}\}$ is called the *root system* of $\mathcal{L}$ with respect to $\mathcal{H}$. We call $(\mathcal{L}, \mathcal{H})$, or simply $\mathcal{L}$, an *elliptic Lie algebra* if R is an elliptic root system, with respect to a symmetric positive semidefinite form on the real span of R.

Corollary 6.9. *Let $\mathcal{L}$ be a 2-extended affine Lie algebra of type X, rank ℓ and twist number t. Let S be as in* **1.11**. *Suppose any of the following hold:*

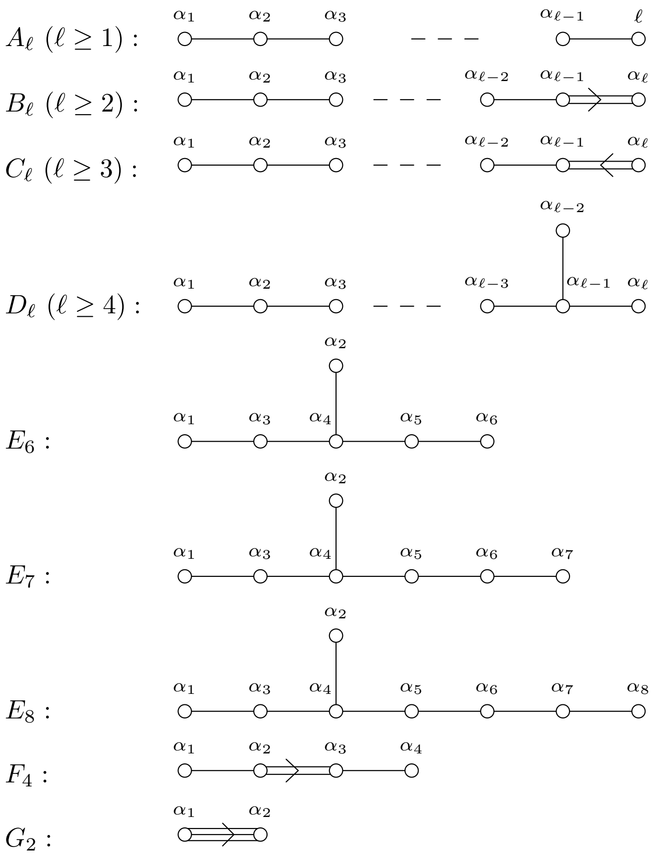
- *X is simply laced of rank > 1;*
- *$X = A_1$ and $\mathrm{ind}(S) = 2$;*
- *$X = F_4$, G_2;*
- *$X = B_2$ with $t = 1$; or $t = 0, 2$ and $\mathrm{ind}(S) = 2$;*
- *$X = B_\ell$ ($\ell > 2$) with $t = 0, 1$; or $t = 2$ and $\mathrm{ind}(S) = 2$;*
- *$X = C_\ell$ with $t = 1, 2$; or $t = 0$ and $\mathrm{ind}(S) = 2$.*

Then any reflectable base is a minimal root-generating set for $\mathcal{L}$.

Proof. For all the types given in the statement we have $\mathrm{ind}(R) = 0$ and so by Lemma 6.6 we are done. $\square$

§6.4. A prototype presentation

Let $R = R^0 \cup (\dot{R}_{\mathrm{sh}} + S) \cup (\dot{R}_{\mathrm{lg}} + L)$ be an elliptic root system, where $\dot{R} = \dot{R}_{\mathrm{sh}} \cup \dot{R}_{\mathrm{lg}} \cup \{0\}$ and S, L are defined in **1.11**. Let $\dot{\mathcal{P}} = \{\alpha_1, \ldots, \alpha_\ell\}$ be a fundamental system for $\dot{R}$ and consider the following Dynkin diagrams:



As in **2.1**, we set $\mathcal{V} = \mathrm{span}_{\mathbb{R}} R$, $\widetilde{\mathcal{V}} = \mathcal{V} \oplus (\mathcal{V}^0)^\star$ and we let $(\cdot, \cdot)$ be the non-degenerate form on $\widetilde{\mathcal{V}}$. We recall that $\mathcal{V} = \dot{\mathcal{V}} \oplus \mathcal{V}^0$, and that $\mathcal{V}^0 = \mathbb{R}\sigma_1 \oplus \mathbb{R}\sigma_2$, where $\sigma_1, \sigma_2 \in R^0$ and $\langle R^0 \rangle = \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2$. We fix a basis $\{\lambda_1, \lambda_2\}$ for $(\mathcal{V}^0)^\star$ by

$$\lambda_i(\sigma_j) = \delta_{ij} \quad (1 \leq i, j \leq 2). \tag{6.1}$$

We consider λ_1, λ_2 as elements of $\mathcal{V}^\star$ by requiring $\lambda_1(\dot{\mathcal{V}}) = \lambda_2(\dot{\mathcal{V}}) = \{0\}$.

Assume that $\mathcal{P} = \mathcal{P}(X)$ is the reflectable base for R given in Table 4. We recall from Proposition 3.6 and axiom (R4) of Definition 1.2 that

$$R = \mathcal{W}_{\mathcal{P}}\mathcal{P} \cup ((\mathcal{W}_{\mathcal{P}}\mathcal{P} - \mathcal{W}_{\mathcal{P}}\mathcal{P}) \cap \mathcal{V}^0),$$

so R is uniquely determined by $\mathcal{P}$.

For $\alpha, \beta \in R^\times$, we define

$$n_{\alpha,\beta} := \min\{n \in \mathbb{Z}_{>0} \mid n\alpha + \beta \notin R\}.$$

In what follows, by $\mathcal{P}^\pm$ we mean $\mathcal{P} \cup (-\mathcal{P})$. Also, a term $[x_1, \ldots, x_n]$ in a Lie algebra means $[x_1, [x_2, [x_3, \ldots, [x_{n-1}, x_n] \cdots]]]$.

Let $\widehat{\mathcal{L}}$ be the Lie algebra defined by generators

$$\widehat{X}_\alpha,\ \widehat{H}_\alpha,\ \hat{d}_1,\ \hat{d}_2 \quad (\alpha \in \mathcal{P}^\pm),$$

subject to the following relations:

$$\begin{aligned}
&\text{(I)}\ \ \textstyle\sum_{i=1}^k \widehat{H}_{\beta_i} = \widehat{H}_{\sum_{i=1}^k \beta_i} \text{ if } \beta_1, \ldots, \beta_k,\ \sum_{i=1}^k \beta_i \in \mathcal{P};\\
&\text{(II)}\ \ [\widehat{H}_\alpha, \widehat{H}_\beta] = 0,\ \alpha, \beta \in \mathcal{P}^\pm;\\
\text{(6.2)}\quad &\text{(III)}\ \ [\widehat{H}_\alpha, \widehat{X}_\beta] = (\beta, \alpha^\vee)\widehat{X}_\beta,\ \alpha, \beta \in \mathcal{P}^\pm;\\
&\text{(IV)}\ \ [\widehat{X}_\alpha, \widehat{X}_{-\alpha}] = \widehat{H}_\alpha,\ \alpha \in \mathcal{P}^\pm;\\
&\text{(V)}\ \ [\widehat{X}_{\alpha_1}, \ldots, \widehat{X}_{\alpha_n}] = 0,\ \alpha_i \in \mathcal{P}^\pm,\ \textstyle\sum_{i=1}^n \alpha_i \notin R;\\
&\text{(VI)}\ \ [\hat{d}_i, \widehat{X}_\alpha] = \lambda_i(\alpha)\widehat{X}_\alpha,\ [\hat{d}_i, \hat{d}_j] = [\hat{d}_i, \widehat{H}_\alpha] = 0,\ \alpha \in \mathcal{P}^\pm,\ 1 \le i, j \le 2.
\end{aligned}$$

We see from defining relations of the form (V) that

$$\text{(V}'\text{)}\quad \operatorname{ad} \widehat{X}_\alpha^{n_{\alpha,\beta}}(\widehat{X}_\beta) = 0,\ \alpha, \beta \in \mathcal{P}^\pm.$$

Remark 6.10. Let Φ be an irreducible finite or affine root system, and $\mathcal{P}$ be a fundamental system for Φ. Let $\widehat{\mathcal{L}}$ be the Lie algebra defined by generators $\widehat{X}_\alpha$, $\widehat{H}_\alpha$, $\alpha \in \mathcal{P}^\pm$, and relations (I)–(IV), (V$'$), (VI) (while $\hat{d}_i$ is dropped). Then $\widehat{\mathcal{L}}$ is either a finite-dimensional simple Lie algebra or the derived subalgebra of an affine Lie algebra with root system Φ; see [Kac]. Since relations of the form (V) hold in $\widehat{\mathcal{L}}$, it follows that (V) and (V$'$) are equivalent in this case.

To proceed, we recall that in [AABGP, Chap. III], associated to each elliptic root system R, an extended affine Lie algebra $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$ with root system R is constructed such that $\mathcal{L} = \mathcal{L}_c \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$, where d_1, d_2 act as derivations on $\mathcal{L}_c$, and $[d_1, d_2] = 0$. Moreover, $\mathcal{H} = \sum_{\dot\alpha \in \dot R} \mathbb{C}t_{\dot\alpha} \oplus \mathbb{C}t_{\sigma_1} \oplus \mathbb{C}t_{\sigma_2}$, and $(d_i, t_{\sigma_j}) = \delta_{ij} = \lambda_i(\sigma_j)$, $i, j = 1, 2$. So we may identify $\lambda_i \in (\mathcal{V}^0)^\star \subseteq \mathcal{H}^{\star\star} \equiv \mathcal{H}$ with d_i. Then for

Type	$\mathcal{P}(X)$
A_1	$\{\alpha_1, \sigma_1 - \alpha_1, \sigma_2 - \alpha_1\}$ if $\mathrm{ind}(R) = 0$, $\{\alpha_1, \sigma_1 - \alpha_1, \sigma_2 - \alpha_1, \sigma_1 + \sigma_2 - \alpha_1\}$ if $\mathrm{ind}(R) = 1$
A_ℓ $(\ell > 1)$ D_ℓ $E_{6,7,8}$	$\{\alpha_1, \dots, \alpha_\ell, \sigma_1 - \alpha_1, \sigma_2 - \alpha_1\}$
F_4	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \sigma_1 - \alpha_1, \sigma_2 - \alpha_1\}$ if $t = 0$, $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \sigma_1 - \alpha_4, \sigma_2 - \alpha_1\}$ if $t = 1$, $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \sigma_1 - \alpha_4, \sigma_2 - \alpha_4\}$ if $t = 2$
G_2	$\{\alpha_1, \alpha_2, \sigma_1 - \alpha_1, \sigma_2 - \alpha_1\}$ if $t = 0$, $\{\alpha_1, \alpha_2, \sigma_1 - \alpha_2, \sigma_2 - \alpha_1\}$ if $t = 1$, $\{\alpha_1, \alpha_2, \sigma_1 - \alpha_2, \sigma_2 - \alpha_2\}$ if $t = 2$
B_2	$\{\alpha_1, \alpha_2, \sigma_1 - \alpha_1, \sigma_2 - \alpha_1\}$ if $t = 0$, $\mathrm{ind}(R) = 0$, $\{\alpha_1, \alpha_2, \sigma_1 - \alpha_1, \sigma_2 - \alpha_1, \sigma_1 + \sigma_2 - \alpha_1\}$ if $t = 0$, $\mathrm{ind}(R) = 1$, $\{\alpha_1, \alpha_2, \sigma_1 - \alpha_1, \sigma_2 - \alpha_2\}$ if $t = 1$, $\{\alpha_1, \alpha_2, \sigma_1 - \alpha_2, \sigma_2 - \alpha_2\}$ if $t = 2$, $\mathrm{ind}(R) = 0$, $\{\alpha_1, \alpha_2, \sigma_1 - \alpha_2, \sigma_2 - \alpha_2, \sigma_1 + \sigma_2 - \alpha_2\}$ if $t = 2$, $\mathrm{ind}(R) = 1$
B_ℓ $(\ell > 2)$	$\{\alpha_1, \dots, \alpha_\ell, \sigma_1 - \alpha_1, \sigma_2 - \alpha_1\}$ if $t = 0$, $\{\alpha_1, \dots, \alpha_\ell, \sigma_1 - \alpha_1, \sigma_2 - \alpha_\ell\}$ if $t = 1$, $\{\alpha_1, \dots, \alpha_\ell, \sigma_1 - \alpha_\ell, \sigma_2 - \alpha_\ell\}$ if $t = 2$, $\mathrm{ind}(R) = 0$, $\{\alpha_1, \dots, \alpha_\ell, \sigma_1 - \alpha_\ell, \sigma_2 - \alpha_\ell, \sigma_1 + \sigma_2 - \alpha_\ell\}$ if $t = 2$, $\mathrm{ind}(R) = 1$
C_ℓ $(\ell > 2)$	$\{\alpha_1, \dots, \alpha_\ell, \sigma_1 - \alpha_1, \sigma_2 - \alpha_1\}$ if $t = 2$, $\{\alpha_1, \dots, \alpha_\ell, \sigma_1 - \alpha_1, \sigma_2 - \alpha_\ell\}$ if $t = 1$, $\{\alpha_1, \dots, \alpha_\ell, \sigma_1 - \alpha_\ell, \sigma_2 - \alpha_\ell\}$ if $t = 0$, $\mathrm{ind}(R) = 0$, $\{\alpha_1, \dots, \alpha_\ell, \sigma_1 - \alpha_\ell, \sigma_2 - \alpha_\ell, \sigma_1 + \sigma_2 - \alpha_\ell\}$ if $t = 0$, $\mathrm{ind}(R) = 1$

Table 4. A reflectable base for the elliptic root system R

$\alpha \in R$, $\lambda_i(\alpha) = \alpha(d_i)$. We fix this Lie algebra $\mathcal{L}$ throughout the section and set $\mathcal{H}_c := \mathcal{H} \cap \mathcal{L}_c$.

For each $\alpha \in R^\times$, we fix an $\mathfrak{sl}_2$-triple $(e_\alpha, h_\alpha, e_{-\alpha})$, where $e_{\pm\alpha} \in \mathcal{L}_{\pm\alpha}$, $[e_\alpha, e_{-\alpha}] = h_\alpha$ and $[h_\alpha, e_{\pm\alpha}] = \pm 2e_{\pm\alpha}$; see **6.2**.

Since relations (6.2) hold in $\mathcal{L}$ for e_α, h_α, d_1, d_2, $\alpha \in \mathcal{P}^\pm$, in place of $\widehat{X}_\alpha$, $\widehat{H}_\alpha$, $\hat{d}_1$, $\hat{d}_2$, and since by Lemma 6.4 the elements e_α, h_α, d_1, d_2, $\alpha \in \mathcal{P}^\pm$ generate $\mathcal{L}$, we get an epimorphism

$$\begin{gathered} \Psi\colon \widehat{\mathcal{L}} \to \mathcal{L}, \\ \widehat{X}_\alpha \mapsto e_\alpha, \quad \widehat{H}_\alpha \mapsto h_\alpha \ (\alpha \in \mathcal{P}^\pm), \quad \hat{d}_i \mapsto d_i. \end{gathered} \tag{6.3}$$

We set

$$\widehat{\mathcal{H}} := \operatorname{span}_{\mathbb{C}}\{\widehat{H}_\alpha, \hat{d}_1, \hat{d}_2 \mid \alpha \in \mathcal{P}^\pm\} = \operatorname{span}_{\mathbb{C}}\{\widehat{H}_\alpha, \hat{d}_1, \hat{d}_2 \mid \alpha \in \mathcal{P}\}.$$

We note that the second equality holds here, since $\widehat{H}_{-\alpha} = -\widehat{H}_\alpha$, for $\alpha \in \mathcal{P}$, by relations of the form (IV) in (6.2).

Lemma 6.11. *The epimorphism Ψ restricts to an isomorphism $\widehat{\mathcal{H}} \to \mathcal{H}$. In particular, $\widehat{\mathcal{H}}$ is an abelian subalgebra of $\widehat{\mathcal{L}}$ with* $\dim \widehat{\mathcal{H}} = \dim \mathcal{H} = \ell + 4$.

Proof. Consider the epimorphism $\Psi\colon \widehat{\mathcal{L}} \to \mathcal{L}$ given by (6.3). From [AABGP, Chap. III], we know that $\dim \mathcal{H} = \ell + 4$ and $\mathcal{H} = \sum_{\alpha\in\mathcal{P}} \mathbb{C}h_\alpha \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$. So if $|\mathcal{P}| = |\mathcal{P}(X)| = \ell + 2$, then $\{h_\alpha, d_1, d_2 \mid \alpha \in \mathcal{P}\}$ is a linearly independent set and so its preimage $\{\widehat{H}_\alpha, \hat{d}_1, \hat{d}_2 \mid \alpha \in \mathcal{P}\}$ under Ψ is also a linearly independent set; therefore we are done in this case.

Assume next that $|\mathcal{P}| > \ell + 2$. By Table 4 we are in one of the cases $X = A_1$, $X = B_\ell, C_\ell$, with $\operatorname{ind}(R) = 1$. For type A_1, we have $\mathcal{P}(A_1) = \{\alpha_1, \sigma_1 - \alpha_1, \sigma_2 - \alpha_1, \sigma_1 + \sigma_2 - \alpha_1\}$. Now the elements $\widehat{H}_{\alpha_1}$, $\widehat{H}_{\sigma_1-\alpha_1}$ and $\widehat{H}_{\sigma_2-\alpha_1}$ are linearly independent in $\widehat{\mathcal{H}}$ as are their images in $\mathcal{H}$. Also from relations of the form (I) in (6.2), we have $\widehat{H}_{\sigma_1+\sigma_2-\alpha} = \widehat{H}_{\sigma_1-\alpha} + \widehat{H}_{\sigma_2-\alpha} + \widehat{H}_\alpha$. Thus $\dim \widehat{\mathcal{H}} = \dim \mathcal{H}$.

Assume next that $X = B_\ell$ ($\ell > 2$). From Table 4 we see that the only reflectable base $\mathcal{P}$ with $|\mathcal{P}| > \ell + 2$ is the one given in the fourth row, namely

$$\{\alpha_1, \dots, \alpha_\ell, \sigma_1 - \alpha_\ell, \sigma_2 - \alpha_\ell, \sigma_1 + \sigma_2 - \alpha_\ell\}.$$

We know that $\widehat{H}_{\alpha_1}, \dots, \widehat{H}_{\alpha_\ell}, \widehat{H}_{\sigma_1-\alpha_\ell}, \widehat{H}_{\sigma_2-\alpha_\ell}$ are linearly independent, as are their images in $\mathcal{H}$. Also, $\widehat{H}_{\sigma_1+\sigma_2-\alpha_\ell} = \widehat{H}_{\sigma_1-\alpha_\ell} + \widehat{H}_{\sigma_2-\alpha_\ell} + \widehat{H}_{\alpha_\ell}$. Thus $\dim \widehat{\mathcal{H}} = \dim \mathcal{H}$. The remaining types can be treated in a similar manner. $\square$

Lemma 6.12. *Let $\widehat{Y} = [\widehat{Y}_1, \dots, \widehat{Y}_n]$, where $n \geq 2$ and $\widehat{Y}_i \in \{\widehat{X}_\alpha, \widehat{H}_\alpha, \hat{d}_1, \hat{d}_2 \mid \alpha \in \mathcal{P}^\pm\}$. Then $\widehat{Y}$ is in the span of brackets of the form $[\widehat{X}_{\alpha_1}, \dots, \widehat{X}_{\alpha_m}]$, $\alpha_i \in \mathcal{P}^\pm$, $m \geq 1$.*

Proof. We use induction on n. If $n = 2$, then we are done, using defining relations (6.2). Assume now that $n > 2$. By induction steps, we may assume without loss of generality that $\widehat{Y} = [\widehat{Y}_1, \widehat{Y}']$, where $\widehat{Y}' = [\widehat{X}_{\alpha_1}, \dots, \widehat{X}_{\alpha_m}]$ for some $\alpha_i \in \mathcal{P}^{\pm}$. Now if $\widehat{Y}_1 = \widehat{X}_\alpha$ for some $\alpha \in \mathcal{P}^{\pm}$, then we are done. Otherwise, we have

$$\widehat{Y} = [\widehat{Y}_1, [\widehat{X}_{\alpha_1}, \widehat{X}']] = -[\widehat{X}', [\widehat{Y}_1, \widehat{X}_{\alpha_1}]] - [\widehat{X}_{\alpha_1}, [\widehat{X}', \widehat{Y}_1]],$$

with $\widehat{X}' = [\widehat{X}_{\alpha_2}, \dots, \widehat{X}_{\alpha_m}]$, and so we are done again by induction steps and by defining relations (6.2). $\square$

Using Lemma 6.11, we identify $\widehat{\mathcal{H}}$ with $\mathcal{H}$. For $\alpha \in \mathcal{H}^\star$, we set

$$\widehat{\mathcal{L}}_\alpha := \big\{\widehat{X} \in \widehat{\mathcal{L}} \mid [h, \widehat{X}] = \alpha(h)\widehat{X} \text{ for all } h \in \mathcal{H}\big\}.$$

Proposition 6.13. *The Lie algebra $(\widehat{\mathcal{L}}, \widehat{\mathcal{H}})$ is an elliptic Lie algebra with root system R.*

Proof. We first prove that $\widehat{\mathcal{L}} = \sum_{\alpha \in \mathcal{H}^\star} \widehat{\mathcal{L}}_\alpha$. For this we need to show that any bracket in $\widehat{\mathcal{L}}$ of the form $\widehat{Y} = [\widehat{Y}_1, \dots, \widehat{Y}_n]$, where the $\widehat{Y}_i$ belong to defining generators of $\widehat{\mathcal{L}}$, sits in $\widehat{\mathcal{L}}_\alpha$ for some $\alpha \in \mathcal{H}^\star$. By the defining relations of $\widehat{\mathcal{L}}$, we have $\widehat{X}_\alpha \in \widehat{\mathcal{L}}_\alpha$ and $\widehat{H}_\alpha, \hat{d}_1, \hat{d}_2 \in \widehat{\mathcal{L}}_0$, for $\alpha \in \mathcal{P}^{\pm}$. So we are done if $n = 1$. Assume now that $n \geq 2$. By Lemma 6.12, we may assume $\widehat{Y} = [\widehat{X}_{\alpha_1}, \dots, \widehat{X}_{\alpha_n}]$ for some $\alpha_i \in \mathcal{P}^{\pm}$. Now if $\beta \in \mathcal{P}^{\pm}$, then by the Jacobi identity and defining relations of $\widehat{\mathcal{L}}$, we get $[\widehat{H}_\beta, [\widehat{X}_{\alpha_1}, \widehat{X}_{\alpha_2}]] = (\alpha_1 + \alpha_2)(\widehat{H}_\beta)[\widehat{X}_{\alpha_1}, \widehat{X}_{\alpha_2}]$. Also,

$$[\hat{d}_i, [\widehat{X}_{\alpha_1}, \widehat{X}_{\alpha_2}]] = \lambda_i(\alpha_1 + \alpha_2)[\widehat{X}_{\alpha_1}, \widehat{X}_{\alpha_2}] = (\alpha_1 + \alpha_2)(\hat{d}_i)[\widehat{X}_{\alpha_1}, \widehat{X}_{\alpha_2}].$$

Thus $[\widehat{X}_{\alpha_1}, \widehat{X}_{\alpha_2}] \in \widehat{\mathcal{L}}_{\alpha_1+\alpha_2}$. Then an induction argument on $n \geq 2$ proves that $\widehat{Y} \in \widehat{\mathcal{L}}_{\alpha_1+\dots+\alpha_n}$. The above reasoning shows that $\widehat{\mathcal{L}} = \sum_{\alpha \in \mathcal{H}^\star} \widehat{\mathcal{L}}_\alpha$, and moreover, $\widehat{\mathcal{L}}_\alpha$ is spanned by brackets $[\widehat{X}_{\alpha_1}, \dots, \widehat{X}_{\alpha_n}]$ with $\alpha_1 + \dots + \alpha_n = \alpha$, where $\widehat{X}_{\alpha_i} \in \{\widehat{X}_\alpha, \widehat{H}_\alpha, \hat{d}_1, \hat{d}_t \mid \alpha \in \mathcal{P}^{\pm}\}$. This completes the proof that $\widehat{\mathcal{L}} = \sum_{\alpha \in H^\star} \widehat{\mathcal{L}}_\alpha$. Now, defining relations of the form (V) imply that $\widehat{\mathcal{L}} = \sum_{\alpha \in R} \widehat{\mathcal{L}}_\alpha$. Since $\Psi(\widehat{\mathcal{L}}_\alpha) = \mathcal{L}_\alpha$, we get $\widehat{\mathcal{L}}_\alpha \neq \{0\}$ for $\alpha \in R$. $\square$

§6.5. Type A_1

We specialize the presented Lie algebra $\widehat{\mathcal{L}}$ to the case $X = A_1$ as an example. Since $\nu = 2$, we have up to isomorphism two elliptic root systems of type A_1:

$$R = \begin{cases} (\mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2) \cup (\pm\alpha + \mathbb{Z}\sigma_1 + 2\mathbb{Z}\sigma_2) \cup (\pm\alpha + 2\mathbb{Z}\sigma_1 + \mathbb{Z}\sigma_2) & \text{if } \operatorname{ind}(R) = 0, \\ (\mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2) \cup (\pm\alpha + \mathbb{Z}\sigma_1 + \mathbb{Z}\sigma_2) & \text{if } \operatorname{ind}(R) = 1. \end{cases}$$

We fix the reflectable base

$$\mathcal{P} = \begin{cases} \{\alpha, \sigma_1 - \alpha, \sigma_2 - \alpha\} & \text{if } \operatorname{ind}(R) = 0, \\ \{\alpha, \sigma_1 - \alpha, \sigma_2 - \alpha, \sigma_1 + \sigma_2 - \alpha\} & \text{if } \operatorname{ind}(R) = 1, \end{cases} \tag{6.4}$$

for R.

For a non-isotropic root β, we define the automorphism $\Phi_\beta \in \operatorname{Aut}(\widehat{\mathcal{L}})$ by

$$\Phi_\beta := \exp(\operatorname{ad} \widehat{X}_\beta) \exp(-\operatorname{ad} \widehat{X}_{-\beta}) \exp(\operatorname{ad} \widehat{X}_\beta).$$

Lemma 6.14. *The automorphism $\Phi_{\sigma_1-\alpha}$ satisfies $\Phi_{\sigma_1-\alpha}(\widehat{X}_{\sigma_2-\alpha}) \in \widehat{\mathcal{L}}_{\sigma_2-2\sigma_1+\alpha}$ if and only if*

$$\frac{1}{2} \operatorname{ad} \widehat{X}_{\sigma_1-\alpha} \operatorname{ad} \widehat{X}_{\alpha-\sigma_1} [\widehat{X}_{\alpha-\sigma_1}, \widehat{X}_{\sigma_2-\alpha}] = [\widehat{X}_{\alpha-\sigma_1}, \widehat{X}_{\sigma_2-\alpha}]. \tag{6.5}$$

Proof. We have

$$\begin{aligned}
&\Phi_{\sigma_1-\alpha}(\widehat{X}_{\sigma_2-\alpha}) \\
&\quad = \exp \operatorname{ad}(\widehat{X}_{\sigma_1-\alpha})(\widehat{X}_{\sigma_2-\alpha} - [\widehat{X}_{\alpha-\sigma_1}, \widehat{X}_{\sigma_2-\alpha}] + \frac{1}{2}[\widehat{X}_{\alpha-\sigma_1}, [\widehat{X}_{\alpha-\sigma_1}, \widehat{X}_{\sigma_2-\alpha}]]) \\
&\quad = \widehat{X}_{\sigma_2-\alpha} - [\widehat{X}_{\alpha-\sigma_1}, \widehat{X}_{\sigma_2-\alpha}] - [\widehat{X}_{\sigma_1-\alpha}, [\widehat{X}_{\alpha-\sigma_1}, \widehat{X}_{\sigma_2-\alpha}]] \\
&\qquad + \frac{1}{2}[\widehat{X}_{\alpha-\sigma_1}, [\widehat{X}_{\alpha-\sigma_1}, \widehat{X}_{\sigma_2-\alpha}]] + \frac{1}{2}[\widehat{X}_{\sigma_1-\alpha}, [\widehat{X}_{\alpha-\sigma_1}, [\widehat{X}_{\alpha-\sigma_1}, \widehat{X}_{\sigma_2-\alpha}]]] \\
&\qquad + \frac{1}{4}[\widehat{X}_{\sigma_1-\alpha}, [\widehat{X}_{\sigma_1-\alpha}, [\widehat{X}_{\alpha-\sigma_1}, [\widehat{X}_{\alpha-\sigma_1}, \widehat{X}_{\sigma_2-\alpha}]]]].
\end{aligned}$$

We note that we have

$$\begin{aligned}
[\widehat{X}_{\sigma_1-\alpha}, [\widehat{X}_{\alpha-\sigma_1}, \widehat{X}_{\sigma_2-\alpha}]] &= -[\widehat{X}_{\sigma_2-\alpha}[\widehat{X}_{\sigma_1-\alpha}, \widehat{X}_{\alpha-\sigma_1}]] - [\widehat{X}_{\alpha-\sigma_1}, [\widehat{X}_{\sigma_2-\alpha}, \widehat{X}_{\sigma_1-\alpha}]] \\
&= -[\widehat{X}_{\sigma_2-\alpha}, H_{\sigma_1-\alpha}] = (\alpha, \alpha^\vee)\widehat{X}_{\sigma_2-\alpha} = 2\widehat{X}_{\sigma_2-\alpha}.
\end{aligned}$$

Then from this and (6.5), we get

$$\begin{aligned}
&\widehat{X}_{\sigma_2-\alpha} - [\widehat{X}_{\sigma_1-\alpha}, [\widehat{X}_{\alpha-\sigma_1}, \widehat{X}_{\sigma_2-\alpha}]] + \frac{1}{4}[\widehat{X}_{\sigma_1-\alpha}, [\widehat{X}_{\sigma_1-\alpha}, [\widehat{X}_{\alpha-\sigma_1}, [\widehat{X}_{\alpha-\sigma_1}, \widehat{X}_{\sigma_2-\alpha}]]]] \\
&\quad = \widehat{X}_{\sigma_2-\alpha} - 2\widehat{X}_{\sigma_2-\alpha} + \widehat{X}_{\sigma_2-\alpha} = 0
\end{aligned}$$

and

$$\begin{aligned}
&- [\widehat{X}_{\alpha-\sigma_1}, \widehat{X}_{\sigma_2-\alpha}] + \frac{1}{2}[\widehat{X}_{\sigma_1-\alpha}, [\widehat{X}_{\alpha-\sigma_1}, [\widehat{X}_{\alpha-\sigma_1}, \widehat{X}_{\sigma_2-\alpha}]]] \\
&\quad = -[\widehat{X}_{\alpha-\sigma_1}, \widehat{X}_{\sigma_2-\alpha}] + [\widehat{X}_{\alpha-\sigma_1}, \widehat{X}_{\sigma_2-\alpha}] = 0.
\end{aligned}$$

□

Lemma 6.15. *Suppose R is an elliptic root system of type A_1 and $\mathcal{P}$ is the reflectable base given in (6.4) for R. Let $\widehat{\mathcal{L}}$ be the Lie algebra defined by the* *generators* $\widehat{X}_\alpha$, $\widehat{H}_\alpha$, $\hat{d}_1$, $\hat{d}_2$, $\alpha \in \mathcal{P}^\pm$, *and relations* (I)–(VI) *given in* (6.2) *plus relation* (6.5). *Then* $(\widehat{\mathcal{L}}, \widehat{\mathcal{H}})$ *is an elliptic Lie algebra satisfying*

$$\Phi_\gamma(\widehat{\mathcal{L}}_\beta) = \widehat{\mathcal{L}}_{w_\gamma(\beta)} \quad (\gamma \in \mathcal{P}, \beta \in R^\times).$$

Proof. We first note that (6.5) holds in $\mathcal{L}$ for $e_{\sigma_1-\alpha}$, $e_{\sigma_2-\alpha}$ in place of $\widehat{X}_{\sigma_1-\alpha}$ and $\widehat{X}_{\sigma_2-\alpha}$. Thus the epimorphism $\Psi\colon \widehat{\mathcal{L}} \to \mathcal{L}$ given in (6.3) holds in our case. Suppose γ, β are as in the statement. Let $\widehat{X} \in \widehat{\mathcal{L}}_\beta$. We may assume without loss of generality that $\widehat{X} = [\widehat{X}_{\beta_1}, [\widehat{X}_{\beta_2}, [\dots, [\widehat{X}_{\beta_{k-1}}, \widehat{X}_{\beta_k}] \dots]]]$ for some $\beta_i \in \mathcal{P}$ with $\beta = \beta_1 + \cdots + \beta_k$. Then for $\gamma \in \mathcal{P}$, we have

$$\Phi_\gamma(\widehat{X}) = [\Phi_\gamma(\widehat{X}_{\beta_1}), [\Phi_\gamma(\widehat{X}_{\beta_2}), [\dots, [\Phi_\gamma(\widehat{X}_{\beta_{k-1}}), \Phi_\gamma(\widehat{X}_{\beta_k})] \cdots]]].$$

Now, if $\{\gamma, \beta_i\} \neq \{\sigma_1 - \alpha, \sigma_2 - \alpha\}$, then the vectors $\widehat{X}_\gamma$, $\widehat{X}_{\beta_i}$ can be identified as root vectors of a subalgebra of $\widehat{\mathcal{L}}$ which is either finite-dimensional simple, or a derived subalgebra of an affine Lie algebra, implying that $\Phi_\gamma(\widehat{X}_{\beta_i}) \in \widehat{\mathcal{L}}_{w_\gamma(\beta_i)}$. If $\{\gamma, \beta_i\} = \{\sigma_1 - \alpha, \sigma_2 - \alpha\}$, then since (6.5) holds by assumption, we get from Lemma 6.14 that $\Phi_\gamma(\widehat{X}_{\beta_i}) \in \widehat{\mathcal{L}}_{w_{\gamma(\beta_i)}}$. Thus $\Phi_\gamma(\widehat{\mathcal{L}}_\beta) \subseteq \widehat{\mathcal{L}}_{w_\gamma(\beta)}$. This completes the proof. □

§6.6. Type A_1, ind$(R) = 0$

We now restrict our attention to the case when R is of type A_1 and $\mathrm{ind}(R) = 0$. We consider the corresponding reflectable base $\mathcal{P}$ given in (6.4). Let $\widehat{\mathcal{L}}$ be the corresponding presented Lie algebra.

We recall that for any elements $X_0, \dots X_m$ of a Lie algebra and for $1 \leq j < m$, we have

$$\begin{aligned}[X_0, \dots X_m] = {}& [[X_0, X_1], X_2, \dots, X_m] \\ &+ [X_1, [X_0, X_2], X_3, \dots, X_m] \\ &+ \cdots \\ &+ [X_1, X_2, \dots, X_{j-1}, [X_0, X_j], X_{j+1}, \dots X_m] \\ &+ [X_1, \dots, X_{j-1}, X_0, X_j, \dots, X_m]. \end{aligned} \tag{6.6}$$

Lemma 6.16. *Let* $\beta, \alpha_1, \dots, \alpha_m \in \mathcal{P}^\pm$ *with* $\alpha_1 + \cdots + \alpha_m = \beta$. *Then*

(i) m *is odd, and for each* $\gamma \in \mathcal{P}^\pm$ *with* $\gamma \neq \beta$, *there are an even number of* j*'s such that* $\alpha_j = \gamma$;

(ii) *if* $A = [\widehat{X}_{\alpha_1}, \widehat{X}_{\alpha_2}, \dots, \widehat{X}_{\alpha_m}] \in \widehat{\mathcal{L}}_\beta$, *then* A *can be written as a sum of expressions of the form* $[\widehat{X}_{\beta_1}, \dots, \widehat{X}_{\beta_k}]$, *where all the* β_i *belong to* $\pm\{\alpha, \sigma_1 - \alpha\}$ *or to* $\pm\{\alpha, \sigma_2 - \alpha\}$.

Proof. (i) is clear as α, σ_1, σ_2 are linearly independent.

(ii) We recall that $R = R^0 \cup R^\times$ with

$$R^0 = \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2 \quad \text{and} \quad R^\times = (\pm\alpha + 2\mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2) \cup (\pm\alpha + \mathbb{Z}\sigma_1 \oplus 2\mathbb{Z}\sigma_2).$$

In particular, we have

$$\epsilon\alpha + n_1\sigma_1 + n_2\sigma_2 \notin R \quad \text{for } \epsilon \in \{0, \pm 1\},\ n_1, n_2 \in 2\mathbb{Z}+1. \tag{6.7}$$

This fact will be used frequently in the proof without further mention. We also recall that $\mathcal{P} = \{\alpha, \sigma_1 - \alpha, \sigma_2 - \alpha\}$.

Without loss of generality, we may assume $\beta = \alpha$. If all the α_i belong to $\{\alpha, \sigma_i - \alpha\}^\pm$, $i = 1, 2$, we are done. So we may assume that

$$\alpha_i = \pm(\sigma_1 - \alpha) \quad \text{and} \quad \alpha_j = \pm(\sigma_2 - \alpha) \quad \text{for some } i,\ j. \tag{6.8}$$

If $m = 1$, there is nothing to prove. Now (6.7), (6.8) and relations of the form (V′) show that no non-zero expression of the form A exists for length $m = 3$. By part (i), for $m = 5$ the only possibility is $\{\alpha_1, \dots, \alpha_5\} = \{\alpha, \pm(\sigma_1 - \alpha), \pm(\sigma_2 - \alpha)\}$ and the only possibility for A, up to a permutation of $\mathcal{P}$, is $[\widehat{X}_{\alpha-\sigma_2}, \widehat{X}_{\sigma_2-\alpha}, \widehat{X}_{\alpha-\sigma_1}, \widehat{X}_{\sigma_1-\alpha}, \widehat{X}_\alpha]$, which reduces to the case $m = 3$, by using (6.6) and applying defining relations of the form (III). In general, by using (6.6), we write A as a sum of terms of length $m-2$, except for the last term $[\widehat{X}_1, \dots, \widehat{X}_{j-1}, \widehat{X}_0, \widehat{X}_j, \dots, \widehat{X}_m]$ which has length m. But then, using (6.7), we choose an appropriate j such that this term becomes zero, by applying a relation of the form (V′). $\square$

Theorem 6.17. *Let $R = \Lambda \cup (\pm\alpha + S)$ be an elliptic root system of type A_1 where $\Lambda = \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2$ and $S = (\sigma_1 + 2\Lambda) \cup (\sigma_2 + 2\Lambda)$. Let $(\widehat{\mathcal{L}}, \widehat{\mathcal{H}})$ be the Lie algebra defined by generators and relations (6.2). Then $\widehat{\mathcal{L}}$ is an elliptic Lie algebra with root system R. Moreover, $\dim \widehat{\mathcal{H}} = 5$ and $\dim \widehat{\mathcal{L}}_\beta = 1$ for $\beta \in R^\times$.*

Proof. By Proposition 6.13 and Lemma 6.11, $\widehat{\mathcal{L}}$ is an elliptic Lie algebra with root system R satisfying $\dim \widehat{\mathcal{H}} = 5$. Recall from Proposition 3.6 that if $\mathcal{P} = \{\alpha, \sigma_1 - \alpha\}$ or $\mathcal{P} = \{\alpha, \sigma_2 - \alpha\}$, then the root system $R_\mathcal{P} \subseteq R$ is an affine root system of type A_1. We now consider the Lie algebra $\widehat{\mathcal{L}}_\mathcal{P}$ defined by generators $\widehat{X}_\alpha$, $\widehat{H}_\alpha$, $\alpha \in \mathcal{P}^\pm$, subject to relations (6.2)(I)–(V). By Remark 6.10, we see that $\widehat{\mathcal{L}}_\mathcal{P}$ is isomorphic to the derived algebra of an affine Lie algebra of type A_1. Thus, identifying $\widehat{\mathcal{L}}_\mathcal{P}$ as a subalgebra of $\widehat{\mathcal{L}}$, and using Lemma 6.16, we conclude that $\dim \widehat{\mathcal{L}}_\beta = 1$ for $\beta \in \mathcal{P}$. We note that by (6.7) and (6.2)(V), both sides of equation (6.5) are zero and so this relation is a consequence of relations that hold in $\widehat{\mathcal{L}}$. Thus Lemma 6.15 implies that $\dim \widehat{\mathcal{L}}_\beta = 1$ for all $\beta \in R^\times$. $\square$

§6.7. Further considerations

In [SaY, Ya], the authors construct elliptic Lie algebras using Serre-type generators and relations. The defining generators of the given presentations are based on a "root base" assigned to the corresponding elliptic root system. This root base, which we call a *Saito root base* and denote by $\mathcal{P}_S$, is defined by Saito [Sa, Sect. (5.2)]. For a given type, in Table 5 the cardinality of a Saito root base is compared with the corresponding reflectable base of Table 4.

Let R be an elliptic root system of rank > 1. Let $\mathcal{P}$ be the reflectable base for R given in Table 4, and $\mathcal{P}_S$ be the Saito base for R. We have $|\mathcal{P}| \leq |\mathcal{P}_S|$; see Table 5. Considering some justifications, we may identify the reflectable base $\mathcal{P}$ with a subset of $\mathcal{P}_S$. Let $\widehat{\mathcal{L}}_S$ be the corresponding presented Lie algebra given in [SaY, Ya] associated to $\mathcal{P}_S$. Consider the map which assigns each generator of $\widehat{\mathcal{L}}$ to the corresponding generator in $\widehat{\mathcal{L}}_S$; see [SaY, Def. 2] and [Ya, Def. 3.1]. One sees from [SaY, Thm. 2] and [Ya, Def. 3.1 and Thm. 3.1] that the defining relations (I)–(IV) and (VI) of (6.2) hold in $\widehat{\mathcal{L}}_S$ for the corresponding generators. Moreover, since R is the set of roots of $\widehat{\mathcal{L}}_S$, it follows that relations of the form (V) also hold in $\widehat{\mathcal{L}}_S$. Therefore, we get a homomorphism $\Phi\colon \widehat{\mathcal{L}} \to \widehat{\mathcal{L}}_S$ which preserves root spaces. This suggests that to achieve a finite Serre-type presentation associated with $\mathcal{P}$, with 1-dimensional (non-isotropic) root spaces, one should investigate replacing relations of the form (V) with more appropriate relations such as (V′) and those given in [SaY] and [Ya].

Acknowledgements

This work is based upon research funded by the Iran National Science Foundation (INSF) under project no. 4001480, and by the NSF of China (nos. 12131018, 12161141001). The research was in part carried out at the IPM-Isfahan Branch.

References

[AABGP] B. N. Allison, S. Azam, S. Berman, Y. Gao and A. Pianzola, Extended affine Lie algebras and their root systems, Mem. Amer. Math. Soc. **126** (1997), x+122. Zbl 0879.17012 MR 1376741

[A1] S. Azam, Extended affine Weyl groups, J. Algebra **214** (1999), 571–624. Zbl 0927.17013 MR 1680585

[A2] S. Azam, A presentation for reduced extended affine Weyl groups, Comm. Algebra **28** (2000), 465–488. Zbl 0952.20028 MR 1739339

[A3] S. Azam, Extended affine root systems, J. Lie Theory **12** (2002), 515–527. Zbl 1014.17020 MR 1923782

[A4] S. Azam, Generalized reductive Lie algebras: Connections with extended affine Lie algebras and Lie tori, Canad. J. Math. **58** (2006), 225–248. Zbl 1105.17012 MR 2209277

Type X	$\lvert\mathcal{P}_S(X)\rvert$	$\lvert\mathcal{P}(X)\rvert$
A_1	3 if $\mathrm{ind}(R)=0$, 4 if $\mathrm{ind}(R)=1$	3 if $\mathrm{ind}(R)=0$, 4 if $\mathrm{ind}(R)=1$
A_ℓ $(\ell>1)$	$2\ell+2$	$\ell+2$
D_ℓ	$2\ell-2$	$\ell+2$
$E_{6,7,8}$	8,9,10	8,9,10
F_4	6 if $t=0$, 6 if $t=1$, 6 if $t=2$	6 if $t=0$, 6 if $t=1$, 6 if $t=2$
G_2	4 if $t=0$, 4 if $t=1$, 4 if $t=2$	4 if $t=0$, 4 if $t=1$, 4 if $t=2$
B_2	5 if $t=0$, $\mathrm{ind}(R)=0$, 6 if $t=0$, $\mathrm{ind}(R)=1$, 4 if $t=1$, 5 if $t=2$, $\mathrm{ind}(R)=0$, 6 if $t=2$, $\mathrm{ind}(R)=1$	4 if $t=0$, $\mathrm{ind}(R)=0$, 5 if $t=0$, $\mathrm{ind}(R)=1$, 4 if $t=1$, 4 if $t=2$, $\mathrm{ind}(R)=0$, 5 if $t=2$, $\mathrm{ind}(R)=1$
B_ℓ $(\ell>2)$	$2\ell-1$ if $t=0$, 2ℓ if $t=1$, $2\ell+1$ if $t=2$, $\mathrm{ind}(R)=0$, $2\ell+2$ if $t=2$, $\mathrm{ind}(R)=1$	$\ell+2$ if $t=0$, $\ell+2$ if $t=1$, $\ell+2$ if $t=2$, $\mathrm{ind}(R)=0$, $\ell+3$ if $t=2$, $\mathrm{ind}(R)=1$
C_ℓ $(\ell>2)$	$2\ell+2$ if $t=2$, 2ℓ if $t=1$, $2\ell+1$ if $t=0$, $\mathrm{ind}(R)=0$, $2\ell+2$ if $t=0$, $\mathrm{ind}(R)=1$	$\ell+2$ if $t=2$, $\ell+2$ if $t=1$, $\ell+2$ if $t=0$, $\mathrm{ind}(R)=0$, $\ell+3$ if $t=0$, $\mathrm{ind}(R)=1$

Table 5. Cardinals of Saito bases and reflectable bases

[A5] S. Azam, Derivations of multi-loop algebras, Forum Math. **19** (2007), 1029–1045. Zbl 1172.17016 MR 2367953

[AF] S. Azam and A. Farahmand Parsa, Groups of extended affine Lie type, Publ. Res. Inst. Math. Sci. **55** (2019), 627–649. Zbl 1471.17044 MR 3986531

[AFI] S. Azam, A. Farahmand Parsa and M. Izadi Farhadi, Integral structures in extended affine Lie algebras, J. Algebra **597** (2022), 116–161. Zbl 1512.17043 MR 4372130

[AN] S. Azam and M. Nikouei, Weyl groups associated with affine reflection systems of type A_1 (Coxeter type defining relations), Publ. Res. Inst. Math. Sci. **49** (2013), 123–153. Zbl 1272.20041 MR 3029998

[AP] S. Azam and F. Parishani, Cayley graph characterization of geometric reflections, J. Lie Theory **31** (2021), 413–438. Zbl 1543.20045 MR 4225036

[AS1] S. Azam and V. Shahsanaei, Simply laced extended affine Weyl groups (a finite presentation), Publ. Res. Inst. Math. Sci. **43** (2007), 403–424. Zbl 1133.20026 MR 2341017

[AS2] S. Azam and V. Shahsanaei, Extended affine Weyl groups of type A_1, J. Algebraic Combin. **28** (2008), 481–493. Zbl 1162.20026 MR 2443282

[AS3] S. Azam and V. Shahsanaei, On the presentations of extended affine Weyl groups, Publ. Res. Inst. Math. Sci. **44** (2008), 131–161. Zbl 1144.20020 MR 2405870

[AS4] S. Azam and V. Shahsanaei, Presentation by conjugation for A_1-type extended affine Weyl groups, J. Algebra **319** (2008), 1428–1449. Zbl 1143.20024 MR 2383054

[AS5] S. Azam and V. Shahsanaei, Extended affine Weyl groups: Presentation by conjugation via integral collection, Comm. Algebra **39** (2011), 730–749. Zbl 1257.20041 MR 2810615

[ASTY] S. Azam, M. B. Soltani, M. Tomie and Y. Yoshii, Characterization and enumeration of reflectable bases of types $E_{6,7,8}$, Comm. Algebra **50** (2022), 2694–2718. Zbl 1515.17036 MR 4413861

[AYY] S. Azam, H. Yamane and M. Yousofzadeh, Reflectable bases for affine reflection systems, J. Algebra **371** (2012), 63–93. Zbl 1305.17011 MR 2975388

[B] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), Springer, Berlin, 2002. Zbl 0983.17001 MR 1890629

[C] R. W. Carter, *Lie algebras of finite and affine type*, Cambridge Studies in Advanced Mathematics 96, Cambridge University Press, Cambridge, 2005. MR 2188930

[H-KT] R. Høegh-Krohn and B. Torrésani, Classification and construction of quasisimple Lie algebras, J. Funct. Anal. **89** (1990), 106–136. Zbl 0792.17019 MR 1040958

[Ho] G. W. Hofmann, Weyl groups with Coxeter presentation and presentation by conjugation, J. Lie Theory **17** (2007), 337–355. Zbl 1144.20021 MR 2325703

[Hu1] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics 9, Springer, New York-Berlin, 1978. Zbl 0447.17002 MR 0499562

[Hu2] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics 29, Cambridge University Press, Cambridge, 1990. Zbl 0725.20028 MR 1066460

[Kac] V. G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge University Press, Cambridge, 1990. Zbl 0716.17022 MR 1104219

[Kr1] Y. Krylyuk, On the structure of quasi-simple Lie algebras and their automorphism groups, PhD thesis, University of Saskatchewan, 1995. MR 2694404

[Kr2] Y. Krylyuk, On automorphisms and isomorphisms of quasi-simple Lie algebras, J. Math. Sci. (New York) **100** (2000), 1944–2002. Zbl 0988.17006 MR 1774363

[MP] R. V. Moody and A. Pianzola, *Lie algebras with triangular decompositions*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, New York, 1995. Zbl 0874.17026 MR 1323858

[Neh] E. Neher, Extended affine Lie algebras and other generalizations of affine Lie algebras—A survey, in *Developments and trends in infinite-dimensional Lie theory*, Progress in Mathematics 288, Birkhäuser Boston, Boston, MA, 2011, 53–126. Zbl 1261.17023 MR 2743761

[P] U. Pollmann, Realisation der biaffinen Wurzelsysteme von Saito in Lie Algebren, Hamburger Beiträgezur Mathematik aus dem Mathematischen Seminar, Heft 29,1994.

[Sa] K. Saito, Extended affine root systems. I. Coxeter transformations, Publ. Res. Inst. Math. Sci. **21** (1985), 75–179. Zbl 0573.17012 MR 0780892

[SaT] K. Saito and T. Takebayashi, Extended affine root systems. III. Elliptic Weyl groups, Publ. Res. Inst. Math. Sci. **33** (1997), 301–329. Zbl 0901.20016 MR 1442503

[SaY] K. Saito and D. Yoshii, Extended affine root system. IV. Simply-laced elliptic Lie algebras, Publ. Res. Inst. Math. Sci. **36** (2000), 385–421. Zbl 0987.17012 MR 1781435

[Sl] P. Slodowy, Another new class of Lie algebras, Preprint, 1982.

[Ya] H. Yamane, A Serre-type theorem for the elliptic Lie algebras with rank ≥ 2, Publ. Res. Inst. Math. Sci. **40** (2004), 441–469. Zbl 1142.17308 MR 2049642