Mathematisches Forschungsinstitut Oberwolfach

Report No. 45/2024

DOI: 10.4171/OWR/2024/45

Anabelian Geometry and Representations of Fundamental Groups

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29 September – 4 October 2024

Abstract. The focus of the workshop was on recent developments in local systems in arithmetic geometry in the broad sense, including anabelian geometry. The talks covered results in complex variations of Hodge structures, ℓ-adic local systems, p-adic Hodge theory with p-adic local systems, and new insights into absolute Galois groups, with applications to rational points.

Mathematics Subject Classification (2020): 11-XX, 14-XX, 18-XX.

Introduction by the Organizers

The workshop *Anabelian Geometry and Representations of Fundamental Groups* was attended by 42 participants mainly from Japan, Europe and North America, including a number of young participants, with further participants following the talks virtually. We had 16 talks of 60 minutes each and two talks of 30 minutes.

Investigating the connections between étale and Tannakian fundamental groups and arithmetic/geometric data is a central goal of several branches of modern arithmetic geometry. Such relationships have been intensively studied from different points of view, including anabelian geometry, the yoga of motives and local systems, and Hodge-theoretic methods (both complex-analytic and p-adic). The increased interaction between mathematicians working from these different points of view towards similar goals was a focal point of the workshop.

Anabelian geometry was represented by the talks of Tamagawa who reported on work on analogues of the Neukirch-Uchida theorem for m-step solvable Galois groups, and by Bresciani who defined the toric fundamental group and applied it to Grothendieck's section conjecture. Tim Holzschuh used the étale homotopy type, for instance, for simply connected and possibly singular varieties over the reals to prove an analogue of the real section conjecture. The homotopical point of view was also present in Mair's talk about condensed shapes. Furthermore, Lüdtke presented work connecting the section conjecture to the Chabauty-Kim approach to rational points, Kwon discussed the number theoretic Shafarevich conjecture, and Gropper talked about techniques from mapping class groups in the study of outer automorphism groups of the pro-p Galois groups of p-adic fields.

In the talk by Kerz we learned about an inductive approach to Deligne's monodromy-weight conjecture based on an arithmetic Kashiwara conjecture. Saito discussed how to expand singular supports to mixed characteristic.

Complex analytic spaces and the role of fundamental groups in their structure was the main focus of the talk by Brunebarbe circling around the Shafarevich conjecture, this time in the context of complex geometry. Krämer explained how a Tannakian approach to sheaves on complex abelian varieties and convolution combine with big monodromy to give arithmetic finiteness results, in relation with Shafarevich's finiteness conjectures for varieties with good reduction outside a fixed set of places of a number field. The talk by Olsson explained how to categorically reconstruct point objects in the bounded derived category of coherent sheaves on torsors under abelian varieties. Litt reported on formal solutions of algebraic differential equations, discussing a criterion for the formal solution to be algebraic which is related to the p-curvature conjecture.

The p-adic side of local systems featured in the talk by Shimizu, which described a pointwise criterion for a $\mathbb{Z}_p$-local system to be semistable, and purity for prismatic F-crystals. Petrov explained the theory of characteristic classes for p-adic étale local systems. Then we had two talks about Poincaré duality in p-adic cohomology: Nizioł spoke about the intricacies of Poincaré duality in proétale cohomology of partially proper rigid analytic varieties, while Reinecke discussed relative mod-n Poincaré duality with respect to a smooth, proper morphism of locally noetherian analytic adic spaces. The notion of a tame fundamental group for rigid analytic spaces was the topic of Achinger's talk who connected finiteness properties to those for fundamental groups of complex analytic spaces via a chain of reductions, while passing through logarithmic fundamental groups in positive characteristic.

The exceptional environment at the Mathematisches Forschungsinstitut Oberwolfach stimulated inspiring discussions among all the participants, who immensely enjoyed the workshop and are very grateful for the institute's hospitality.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-2230648, "US Junior Oberwolfach Fellows".

Workshop: Anabelian Geometry and Representations of Fundamental Groups

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Abstracts

m-step solvable anabelian geometry of finitely generated fields

Akio Tamagawa
(joint work with Mohamed Saïdi)

Anabelian geometry of fields (sometimes called "birational anabelian geometry") has much longer history than anabelian geometry of schemes intiated by Grothendieck around 1980 [1][2]. Roughly speaking, it asks whether from the absolute Galois group of certain types of fields — e.g. fields finitely generated over the prime field — one can reconstruct the original field purely group-theoretically. The main results of anabelian geometry of fields were satisfactorily established by mid-1990s, by Neukirch-Uchida [3][7] (for number fields), Uchida [8] (for global function fields), and Pop [4][5] (for finitely generated fields).

In this talk, we discuss m-step solvable anabelian geometry of finitely generated fields, which is a variant of anabelian geometry of finitely generated fields and where the absolute Galois group is replaced by its maximal m-step solvable quotient for some m (≥ 2). Among others, we present the speaker's recent joint work with Saïdi on m-step solvable versions of theorems of Neukirch-Uchida ($m \geq 2$; see [6] for $m \geq 3$), Uchida ($m \geq 2$), and Pop ($m \geq 9$). We also discuss the common structure — local theory and global theory — of proofs of these results and more details on proofs for number fields and global function fields.

References

[1] A. Grothendieck, *Esquisse d'un programme*, Geometric Galois actions, 1, pp. 5–48, London Math. Soc. Lecture Note Ser., **242**, Cambridge University Press, 1997.
[2] A. Grothendieck, *Brief an G. Faltings*, Geometric Galois actions, 1, pp. 49–58, London Math. Soc. Lecture Note Ser., **242** Cambridge University Press, 1997.
[3] J. Neukirch, *Kennzeichnung der p-adischen und der endlichen algebraischen Zahlkörper*, Invent. Math. **6** (1969), 296–314.
[4] F. Pop, *On Grothendieck's conjecture of birational anabelian geometry*, Ann. of Math. (2) **139** (1994), 145–182.
[5] F. Pop, *Alterations and birational anabelian geometry*, Resolution of singularities (Obergurgl, 1997), pp. 519–532, Progr. Math., **181**, Birkhäuser Verlag, 2000.
[6] M. Saïdi, A. Tamagawa, *The m-step solvable anabelian geometry of number fields*, J. Reine Angew. Math. **789** (2022), 153–186.
[7] K. Uchida, *Isomorphisms of Galois groups*, J. Math. Soc. Japan **28** (1976), 617–620.
[8] K. Uchida, *Isomorphisms of Galois groups of algebraic function fields*, Ann. of Math. (2) **106** (1977), 589–598.

Pure ℓ-adic local systems over local fields

Moritz Kerz

(joint work with Hélène Esnault)

We define and study pure $\overline{\mathbb{Q}}_\ell$-adic étale local systems on a smooth variety X over a p-adic local field K with $p \neq \ell$, in analogy to [1]. The two most basic open problems about such local systems are:

(1) Whether they degenerate to a limiting mixed structure at a ramification point, and
(2) Whether their cohomology is pure.

Definition 1.

(i) A continuous Galois representation

$$\rho\colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_{\overline{\mathbb{Q}}_\ell}(V)$$

has weight $w \in \mathbb{Z}$ if a lift of the geometric Frobenius Fr_q, acting on the monodromy graded piece $\mathrm{gr}_a^{\mathsf{M}} V$, has eigenvalues with complex absolute values $q^{\frac{a+w}{2}}$.

(ii) A $\overline{\mathbb{Q}}_\ell$-adic local system on X is pure of weight $w \in \mathbb{Z}$ if, for all closed points $x \in X$, the local Galois representation $\mathbb{L}_{\overline{x}}$ is pure of weight w, and if the underlying geometric local system $\mathbb{L}_{\overline{K}}$ is semi-simple and arithmetic.

Here, 'arithmetic' means that the geometric local system descends to a local system over a model of X over a finitely generated field.

Conjecture 2 (Meta Conjecture)**.** *All known results about variations of (integral) pure Hodge structure have an analog for the pure local systems of Definition 1.*

For example, let us make this precise for X a smooth curve with smooth compactification $j\colon X \to \overline{X}$, and a pure local system $\mathbb{L}$ on X of weight w.

Conjecture 3.

(1) For a closed point $x \in \overline{X} \setminus X$, the geometric monodromy graded piece $\mathrm{gr}_a^{\mathsf{M}^{\mathrm{geo}}} \mathbb{L}$ is pure of weight $a + w$ for all $a \in \mathbb{Z}$.

(2) $H^i(X_{\overline{K}}, j_*\mathbb{L})$ is pure of weight $w + i$ for all $i \in \mathbb{Z}$.

A standard Lefschetz pencil argument shows that Conjecture 3(2) would imply Deligne's monodromy-weight conjecture.

A potential way to approach Conjecture 3 is to reduce it to an arithmetic version of Kashiwara's conjecture [2]. In some cases, we can prove this arithmetic Kashiwara conjecture by using an idea of Grothendieck [4, I.5] of "tilting" the problem to the usual known Kashiwara conjecture in equal characteristic zero.

Theorem 4. *If $\mathbb{L}$ is moreover tame, and if, for a complex embedding $K \hookrightarrow \mathbb{C}$, the isomorphism class $[\mathbb{L}_{\overline{K}}]$ has a finite orbit under the action of the mapping class group (see* [3]*), then Conjecture 3 holds.*

References

[1] P. Deligne, *Théorie de Hodge I*, Actes, Congrès intern. math. 1970, Tome 1, 425–430.
[2] M. Kashiwara, *Semisimple holonomic D-modules*, in Topological field theory, primitive forms and related topics (M. Kashiwara and al. editors), Progress in Mathematics **160** (1998), 267–271.
[3] A. Landesman, D. Litt, *Canonical representations of surface groups*, Ann. of Math. (2) **199** (2024), no.2, 823–897.
[4] A. Grothendieck and al., *Groupes de monodromie en Géométrie Algébrique*, SGA 7 I, Springer Lecture Notes **288** (1972), 1–533.

On the generalised real Section Conjecture

Tim Holzschuh

Utilising the so-called *Sullivan Conjecture* from equivariant unstable homotopy theory, we derive the *generalised (pro-2) real Section Conjecture* for a large class of varieties.

1. The generalised Section Conjecture

Before introducing the *generalised* Section Conjecture, we quickly recall the ordinary Section Conjecture. To this end, we fix the following situation:

Setup. *Let k be a field with separable closure $\bar{k} \supset k$, and let X be a geometrically connected quasi-compact and quasi-separated scheme over k. Write* $\mathrm{Gal}_k := \mathrm{Gal}(\bar{k}/k)$ *for the absolute Galois group of k with respect to $\bar{k}$, $X_{\bar{k}}$ for the base change of X to $\bar{k}$, and let $\bar{x} \in X_{\bar{k}}$ be a geometric point.*

Recall that in the above situation, one has a short exact sequence of profinite groups

$$1 \longrightarrow \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \longrightarrow \pi_1^{\text{ét}}(X, \bar{x}) \longrightarrow \mathrm{Gal}_k \longrightarrow 1, \tag{$\pi_1^{\text{ét}}(X/k)$}$$

the so-called *fundamental exact sequence* of étale fundamental groups.

Notation. *We write* $\Gamma(\pi_1^{\text{ét}}(X/k)) := \mathrm{Hom}^{\mathrm{out}}_{\mathrm{Gal}_k}(\mathrm{Gal}_k, \pi_1^{\text{ét}}(X))$ *for the set of sections of the above fundamental exact sequence.*

With these notations in place, we can now state Grothendieck's Section Conjecture:

Section Conjecture. *Let X be a proper hyperbolic curve over a finitely generated field extension k of $\mathbb{Q}$. Then the canonical map*

$$X(k) \to \Gamma(\pi_1^{\text{ét}}(X/k))$$

is a bijection.

Note that, in the language Artin and Mazurs (profinite) *étale homotopy type* $\Pi^{\text{ét}}(X)$ of [1], hyperbolic curves X are étale $\mathrm{K}(\pi, 1)$-varieties , i.e. all the higher étale homotopy groups $\pi_{\mathrm{n}}^{\text{ét}}(X)$ vanish for $n > 1$. Many higher-dimensional varieties do not share this property. It is thus unreasonable to expect the Section Conjecture to hold for such varieties.

Instead, one ought to formulate a *generalised* Section Conjecture that replaces the set $\Gamma(\pi_1^{\text{ét}}(X/k))$ with an appropriate variant $\Gamma^{\text{ét}}(X/k)$ that incorporates all the higher étale homotopy-theoretic information of X/k. In joint work with Peter. J. Haine and Sebastian Wolf [2], we prove a higher-dimensional generalisation of the fundamental exact sequence, the so-called *fundamental fibre sequence*, which states that the canonical square

$$\begin{array}{ccc} \Pi^{\text{ét}}(X_{\bar{k}}) & \longrightarrow & \Pi^{\text{ét}}(X) \\ \downarrow & & \downarrow \\ \Pi^{\text{ét}}(\bar{k}) \simeq * & \longrightarrow & \Pi^{\text{ét}}(k) \end{array}$$

is a homotopy pullback square.[1] Such a fibre sequence always induces a long exact sequence on homotopy groups, which in this case, using that k is also an étale $\mathrm{K}(\pi, 1)$, recovers the fundamental exact sequence together with isomorphisms $\pi_{\mathrm{n}}^{\text{ét}}(X_{\bar{k}}) \to \pi_{\mathrm{n}}^{\text{ét}}(X)$ for any $n > 1$.

It is now very natural to replace the set $\Gamma(\pi_1^{\text{ét}}(X/k))$ of "π_1-sections" with the set of sections of the fibration $\Pi^{\text{ét}}(X) \to \Pi^{\text{ét}}(k)$ *up to homotopy*:

Definition. *We write* $\Gamma^{\text{ét}}(X/k) := \pi_0 \operatorname{map}_{\Pi^{\text{ét}}(k)}(\Pi^{\text{ét}}(k), \Pi^{\text{ét}}(X))$ *for the set of* étale sections *of* X/k*.*

The generalised Section Conjecture thus states:

Generalised Section Conjecture. *The canonical map*

$$X(k) \to \Gamma^{\text{ét}}(X/k), \quad a \mapsto [a_*]$$

is a bijection.

Remark.

(1) *Of course, one still has to determine for which varieties X/k the above conjecture should hold, as it clearly does not hold in general (e.g. $X = \mathbb{A}_k^1$).*
(2) *One can check that if X is aspherical, i.e. if $\pi_{\mathrm{n}}^{\text{ét}}(X) = 0$ for $n > 1$, then $\Gamma^{\text{ét}}(X/k) = \Gamma(\pi_1^{\text{ét}}(X/k))$ and the above conjecture precisely recovers the ordinary Section Conjecture.*
(3) *Using a homotopy-theoretic analogue of the group-theoretic pro-p completion, one can also formulate an appropriate* generalised pro-p Section Conjecture *for any choice of prime p. We denote the set of generalised pro-p sections by $\Gamma_p^{\text{ét}}(X/k)$.*

2. SC = π_0(SC)

Note that $X(k) = X(\bar{k})^{\mathrm{Gal}_k}$ is the set of fixed points under the natural Galois action on $X(\bar{k})$. Similarly, one can show that the set $\Gamma^{\text{ét}}(X/k)$ is given as the set of (homotopy classes of) *homotopy fixed points* under the natural action of Gal_k on $\Pi^{\text{ét}}(X_{\bar{k}})$ induced by its geometric action on $X_{\bar{k}}$:

[1] Here, and in the following, we exclusively work with Lurie's ∞-categorical incarnation of the étale homotopy type via the (profinite) *shape* of the étale ∞-topos of X, as introduced in [3, Appendix E].

Observation. *There is a canonical identification*

$$\Gamma^{\text{ét}}(X/k) = \pi_0(\Pi^{\text{ét}}(X_{\bar{k}})^{\text{hGal}_k}).$$

So from this perspective the Section Conjecture is asking for a comparison

$$X(k) = X(\bar{k})^{\text{Gal}_k} \leftrightsquigarrow \pi_0\,\Pi^{\text{ét}}(X_{\bar{k}})^{\text{hGal}_k} = \Gamma^{\text{ét}}(X/k)$$

of *fixed points* with *homotopy fixed points.*

A deep conjecture, first proposed in [4], (now a theorem by work of Miller [5], Carlsson [6] and Lannes [7]) in equivariant unstable homotopy theory, the so-called *Sullivan Conjecture*, is precisely about such comparisons:

Sullivan Conjecture. *Let G be a finite p-group and K a finite-dimensional G-CW-complex. Then the composition*

$$(K^G)^\wedge_p \to (K^{\text{h}G})^\wedge_p \to (K^\wedge_p)^{\text{h}G}$$

is an equivalence of p-profinite homotopy types. Here, $(-)^\wedge_p$ denotes p-profinite completion *of homotopy types.*

Since $(-)^\wedge_p$ preserves connected components, we arrive at the following:

Slogan. SC $= \pi_0(\text{SC})$*, i.e. the (generalised) Section Conjecture is the π_0-portion of the Sullivan Conjecture.*

In [8], we turn the above slogan into a theorem when working over $k = \mathbb{R}$:

Theorem. *Let $X/\mathbb{R}$ be any equivariantly triangulable*[2] *scheme. Then:*

(1) X satisfies the generalised pro-2 Section Conjecture*, i.e. the map*

$$X(\mathbb{R}) \to \Gamma_2^{\text{ét}}(X/\mathbb{R})$$

is a bijection.

(2) If, in addition, X is geometrically étale nilpotent[3]*, then X satisfies the* generalised Section Conjecture*, i.e. the map*

$$X(\mathbb{R}) \to \Gamma^{\text{ét}}(X/\mathbb{R})$$

is a bijection.

Remark.

(1) This is the first result in anabelian geometry for non $\mathrm{K}(\pi,1)$*-varieties.*

(2) Also note that the above result is in some sense orthogonal *to all other existing results in anabelian geometry, since one e.g. assumes that $\pi_1^{\text{ét}}(X_{\bar{k}})$ vanishes.*

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[2]By definition, this just means that $X(\mathbb{C})$ carries the structure of a finite-dimensional $\text{Gal}_{\mathbb{R}}$-CW-complex and is required in order to apply the Sullivan Conjecture. This is for example satisfied if

(1) $X/\mathbb{R}$ is affine and of finite type, or
(2) $X/\mathbb{R}$ is projective, or
(3) $X/\mathbb{R}$ is smooth.

[3]This is for example satisfied if $X_{\mathbb{C}}$ is étale simply connected, i.e. if $\pi_1^{\text{ét}}(X_{\mathbb{C}}) = 1$.

(3) *In accordance with our proposed slogan, the above Theorem is actually the* π_0*-portion of another theorem comparing* $X(\mathbb{R})^\wedge_2$ *with* $(\Pi^{\text{ét}}(X_{\mathbb{C}})^\wedge_2)^{\mathrm{hGal}_{\mathbb{R}}}$ *or (in the nilpotent case) with* $(\Pi^{\text{ét}}(X_{\mathbb{C}})^{\mathrm{hGal}_{\mathbb{R}}})^\wedge_2$.

(4) *The first part of the above Theorem generalises the results obtained independently by Mochizuki* [9]*, Stix* [10]*, Vistoli–Bresciani* [11]*, Wickelgren* [12]*, and Pál* [13] *in the case of hyperbolic curves over* $\mathbb{R}$.

Let us conclude with a conjecture:

Conjecture. *Any separated scheme of finite type* $X/\mathbb{R}$ *is equivariantly triangulable.*

References

[1] M. Artin & B. Mazur, *Étale Homotopy*, Lecture Notes in Mathematics (1969)
[2] P. Haine, T. Holzschuh, S. Wolf, *The fundamental fiber sequence in étale homotopy theory*, IMRN, Volume 2024, Issue 1, 175–196
[3] J. Lurie, *Spectral Algebraic Geometry*, `https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf`
[4] D.P. Sullivan, *Geometric topology: localization, periodicity and Galois symmetry*, The 1970 MIT Notes
[5] H. Miller, *The Sullivan Conjecture on Maps from Classifying Space*, Annals of Mathematics (1985)
[6] G. Carlsson, *Equivariant stable homotopy and Sullivan's conjecture*, Inventiones mathematicae (1991)
[7] J. Lannes, *Sur les espaces fonctionnels dont la source est le classifiant d'un p-groupe abélien élémentaire*, Inst. Hautes Études Sci. Publ. Math. (1992)
[8] T. Holzschuh, *On the generalised real Section Conjecture*, PhD thesis, `https://archiv.ub.uni-heidelberg.de/volltextserver/35190/`
[9] S. Mochizuki, *Topics surrounding the anabelian geometry of hyperbolic curves*, Math. Sci. Res. Inst. Publ. Vol. 41 (2003), pp. 119–165.
[10] J. Stix, *On the period-index problem in light of the section conjecture*, Amer. J. Math. (2010)
[11] A. Vistoli & G. Bresciani, *An elementary approach to Stix's proof of the real section conjecture*, (2020)
[12] K. Wickelgren, 2*-nilpotent real Section Conjecture*, Math. Ann., vol. 358, no. 1-2, pp. 361–387, (2014)
[13] A. Pal, *The real section conjecture and Smith's fixed-point theorem for pro-spaces*, J. Lond. Math. Soc. (2011)

Geometric duality for p-adic pro-étale cohomology of analytic varieties

Wiesława Nizioł

(joint work with Pierre Colmez, Sally Gilles)

Let p be a prime. Let K be a finite extension of $\mathbf{Q}_p$. Let $\overline{K}$ be an algebraic closure of K and let $C = \widehat{\overline{K}}$ be its p-adic completion; let $\mathscr{G}_K = \mathrm{Gal}(\overline{K}/K)$. Or analytic varieties are separated.

We are in the process of writing down a proof of the following result.

Theorem 1. (Poincaré Duality) *Let X be a smooth, partially proper rigid analytic variety over K, connected, of dimension d. Then, for $j \in \mathbf{Z}$, there is a natural quasi-isomorphism in* TVS

$$\mathrm{R}\Gamma_{\text{proét}}(X_C, \mathbf{Q}_p(j)) \xrightarrow{\sim} \mathrm{R}\operatorname{Hom}_{\mathrm{TVS}}(\mathrm{R}\Gamma_{\text{proét},c}(X_C, \mathbf{Q}_p(d-j))[2d], \mathbf{Q}_p).$$

Here TVS is the category of Topological Vector Spaces, i.e., v-sheaves of enriched (in condensed sets) solid $\underline{\mathbf{Q}}_p$-vector spaces on Perf_C and pro-étale cohomology is seen in TVS; the Hom is internal.

The proof passes to syntomic cohomology (via a geometric version of a comparison theorem), represents syntomic cohomology via a complex of solid quasi-coherent sheaves on the Fargues-Fontaine curve, proves a Poincaré duality for such complex, and then projects this duality down to the TVS category. The Poincaré duality on the FF curve reduces to Hyodo-Kato duality on the whole curve and the filtered $\mathbf{B}^+_{\mathrm{dR}}$-duality at infinity (both of which are known). The functional analytic problems can be solved because all the infinite data "come from the base" and can be "taken out" via a projection formula. The enriched structure is necessary for the computation of Ext-groups between Banach-Colmez spaces (and enters through the enriched Yoneda Lemma). See Proposition 3.

Theorem 1 yields the computation:

Corollary 2. *There is a natural exact sequence in* TVS

$$\begin{aligned} 0 &\to \operatorname{Ext}^1_{\mathrm{TVS}}(H^{2d-i+1}_{\text{proét},c}(X_C, \mathbf{Q}_p(d)), \mathbf{Q}_p) \\ &\to H^i_{\text{proét}}(X_C, \mathbf{Q}_p) \\ &\to \operatorname{Hom}_{\mathrm{TVS}}(H^{2d-i}_{\text{proét},c}(X_C, \mathbf{Q}_p(d)), \mathbf{Q}_p) \\ &\to 0 \end{aligned}$$

This is proved by a reduction to the following vanishing result (a topological version of a result of Anschütz-Le Bras)

Proposition 3. *Let $\mathscr{F}_1, \mathscr{F}_2$ be Banach-Colmez spaces. Then*

$$\operatorname{Ext}^a_{\mathrm{TVS}}(\mathscr{F}_1, \mathscr{F}_2) = 0, \quad a \geq 2.$$

Which, in turn, is proved by using Mac-Lane resolutions (those are naturally enriched in this setting) and reducing, via the enriched Yoneda Lemma, to the computation of cohomologies of complexes built from cohomologies of affine spaces with values in the sheaves $\mathbf{Q}_p$ and $\mathbb{G}_a$. These complexes can be represented by complexes of Fréchet spaces and hence are exact if and only if are exact algebraically. But the algebraic complexes represent Ext-groups in VS-category, where they are known to vanish in degrees at least 2 by Anschütz-Le Bras.

Remark 4. There is an independent, ongoing project of Anschütz-Le Bras-Mann that studies 6-functor formalism for p-adic pro-étale cohomology on analytic varieties. In particular, this should include duality results of the type stated in Theorem 1.

On the converse to Eisenstein's last theorem

Daniel Litt
(joint work with Yeuk Hay Joshua Lam)

This work is motivated by the following question; the earliest place I have been able to find it in print is [1], and see also [2] for a nice overview.

Question. (Fuchs, 1875) When does an algebraic differential equation have solutions which are algebraic functions?

Considering some basic examples shows that this a priori algebraic question is actually of an arithmetic nature; for example, the differential equation

$$\left(\frac{\partial}{\partial z} - \frac{a}{z}\right) f(z) = 0, a \in \mathbb{C}$$

has solutions of the form cz^a, which are algebraic if and only if a is a rational number. For linear differential equations, there is a now-standard conjectural answer:

Conjecture 1. (Grothendieck-Katz p-curvature conjecture [3]) Let A be a matrix with entries in $\mathbb{Q}(z)$. Then all of the solutions to $\left(\frac{\partial}{\partial z} - A\right)\vec{f}(z) = 0$ are algebraic if and only if $\left(\frac{\partial}{\partial z} - A\right)^p \equiv 0 \bmod p$ for almost all primes p.

This conjecture is widely open; the main cases that are known are due to Katz [3] (who proved it for Picard-Fuchs equations) and Chudnovsky-Chudnovsky [4], Bost [5], and André [6], (who proved it in the case of solvable monodromy), though there is other beautiful work handling some sporadic cases.

The main goal of this work is to understand a variant of the conjecture suitable for *non-linear* differential equations, and to try to gather some evidence for it. The conjecture is as follows:

Conjecture 2. Let $f(z) = \sum a_n z^n \in \mathbb{Q}[[z]]$ be a solution to a (possibly non-linear) differential equation

$$f^{(n)}(z) = F(z, f(z), f'(z), \cdots, f^{(n-1)}(z)),$$

with $F \in \mathbb{Q}(z, y_0, \cdots, y_{n-1})$ and $F(0, f(0), \cdots, f^{(n-1)}(0))$ defined. Then the following are equivalent:

(1) $f(z)$ is algebraic over $\mathbb{Q}(z)$,
(2) There exists N such that all $a_i \in \mathbb{Z}[\frac{1}{N}]$, and
(3) There exists $\omega : \text{Primes} \to \mathbb{Z}_{\geq 0}$ with

$$\lim_{p\to\infty} \frac{\omega(p)}{p} = \infty$$

such that $a_0, \cdots, a_{\omega(p)} \in \mathbb{Z}_{(p)}$ for almost all primes p.

That (1) implies (2) is a theorem of Eisenstein, appearing in the final paper he published before his death at the age of 29 [7]. It is trivial to see that (2) implies (3). Of course all of the reverse implications are very difficult. If one could show that (3) implies (1), this would resolve the p-curvature conjecture, but in fact this conjecture is (at least a priori) much stronger than the p-curvature conjecture. For linear ODE, some variant of it has been considered by André and Christol [6].

Before stating our results, we reformulate the conjecture in a more geometric context. Let $\pi : \mathcal{M} \to S$ be a smooth morphism of smooth schemes, and let $\mathcal{F} \subset T_{\mathcal{M}}$ be an integrable foliation on $\mathcal{M}/S$, i.e. a subsheaf closed under the Lie bracket $[-,-]$ such that the natural composition

$$\mathcal{F} \hookrightarrow T_{\mathcal{M}} \to \pi^* T_S$$

is an isomorphism. Let $m \in \mathcal{M}$ be a point.

Conjecture 3. The following are equivalent:

(1) There exists an algebraic leaf of $\mathcal{F}$ through m, and
(2) There exists a descent of $\mathcal{M}, S, \pi, m, \mathcal{F}$ to a finitely-generated $\mathbb{Z}$-algebra R and a formal leaf $\mathrm{Spf} R[[x_1, \cdots, x_{\dim(S)}]] \to \mathcal{M}$ of $\mathcal{F}$ through m.
(3) An appropriate analogue of condition (3) of Conjecture 2 here, which we omit for brevity.

We now explain the situations in which we can verify Conjectures 2 and 3.

Theorem A. (Lam-L–) Let $f : X \to S$ be a smooth proper morphism of smooth $\mathbb{C}$-schemes. Let $(\mathcal{E}, \nabla) = (R^{2k} f_* \Omega^\bullet_{X/S}, \nabla_{GM})$ be the relative de Rham cohomology of f, equipped with its Gauss-Manin connection. Fix $s \in S$ and $m \in \mathcal{E}_s = H^{2k}_{dR}(X_s)$ in the image of the cycle class map

$$Z^k(X_s)_{\mathbb{Q}} \to H^{2k}_{dR}(X_s).$$

Then the formal flat section to the Gauss-Manin connection through m is algebraic if and only if it descends to a formal section over a finitely-generated $\mathbb{Z}$-algebra, i.e. Conjecture 3 is true in this case.

The proof relies on the Mazur-Ogus theorem relating the behavior of the crystalline Frobenius to the Hodge filtration modulo p.

Theorem B. (Lam-L–) Let $f : X \to S$ be a smooth proper morphism of smooth $\mathbb{C}$-schemes, with X/S a relative genus one curve. Let

$$(\mathcal{E}, \nabla) = \mathrm{Sym}^n (R^1 f_* \Omega^\bullet_{X/S}, \nabla_{GM})$$

be a symmetric power of the relative de Rham cohomology of f, equipped with its Gauss-Manin connection. Fix $s \in S, m \in \mathcal{E}_s$. Then the formal flat section ∇ through m is algebraic if and only if it descends to a formal section over a finitely-generated $\mathbb{Z}$-algebra, i.e. Conjecture 3 is true in this case.

For example, this applies to the solutions to Picard-Fuchs equations associated to modular forms (see e.g. [8, Proposition 21]).

There are a number of other cases in which we can verify Conjecture 3 for linear ODE; for example, for Picard-Fuchs equations associated to families of abelian varieties at "CM initial conditions," certain hypergeometric functions, etc. For brevity's sake, we only give one more case, where we verify it for certain *non-linear* differential equations.

Let $f : X \to S$ be a smooth projective morphism (possibly equipped with a relative snc divisor, which we ignore for this abstract). One may associate to f the moduli stack $\mathcal{M}_{dR}(X/S)$, which parametrizes flat bundles on X/S. This stack is a crystal over S; that is, loosely speaking, there is a natural *isomonodromy* foliation on $\mathcal{M}_{dR}(X/S)/S$, whose leaves are families of flat bundles with locally constant mononodromy representation (up to conjugacy). This foliation is the non-abelian analogue of the Gauss-Manin connection, and has been studied classically; for example the Painlevé VI and Schlesinger equations are special cases. Our main theorem in this case is:

Theorem C. (Lam-L–) Let $f : X \to S$ be smooth proper (possibly equipped with a relative snc divisor), and fix $s \in S$. Let $[(\mathcal{E}, \nabla)] \in \mathcal{M}_{dR}(X_s)$ be a Gauss-Manin connection on X_s, i.e. $R^i g_* \Omega_{dR,Y/X_s}$ for some smooth proper $g : Y \to X_s$. Then the leaf of the isomonodromy foliation on $\mathcal{M}_{dR}(X/S)$ through $[(\mathcal{E}, \nabla)]$ is algebraic if and only there exists a formal isomonodromic deformation over a finitely-generated $\mathbb{Z}$-algebra. That is, Conjecture 3 is true in this case.

The proof relies on Ogus-Vologodsky's non-abelian Hodge theory in positive characteristic, and on the theory of the Higgs-de Rham flow (due to Faltings, Lan-Sheng-Zuo, and Esnault-Groechenig).

We end with a small hint relating this case of isomonodromy foliations to the classical p-curvature conjecture:

Proposition D. (Lawrence-L–, Lam-L–) Suppose Conjecture 3 holds for the isomonodromy foliation on the moduli of flat bundles of rank r on the universal curve, $\mathcal{M}_{dR}(\mathcal{C}_g/\mathcal{M}_g, r)$. Then the classical p-curvature conjecture holds for all flat bundles of rank r.

References

[1] Fuchs, L. *Über die linearen Differentialgleichungen zweiter Ordnung welche algebraische Integrale besitzen, und eine neue Anwendung der Invariantentheorie.* Nachrichten von der Königl. Gesellschaft der Wissenschaften und der Georg-Augusts-Universität zu Göttingen 1875 (1875): 568-581.

[2] Gray, Jeremy J. *Fuchs and the theory of differential equations.* Bulletin of the American Mathematical Society 10.1 (1984): 1-26.

[3] Katz, Nicholas M. *Algebraic solutions of differential equations (p-curvature and the Hodge filtration).* Inventiones mathematicae 18.1 (1972): 1-118.

[4] Chudnovsky, David V., and Gregory V. Chudnovsky. *Applications of Padé approximations to the Grothendieck conjecture on linear differential equations.* Number Theory: A Seminar held at the Graduate School and University Center of the City University of New York 1983-84. Berlin, Heidelberg: Springer Berlin Heidelberg, 2006.

[5] Bost, Jean-Benoît. *Algebraic leaves of algebraic foliations over number fields.* Publications Mathématiques de l'IHÉS 93 (2001): 161-221.

[6] André, Yves. *Sur la conjecture des p-courbures de Grothendieck-Katz et un probleme de Dwork.* Geometric aspects of Dwork theory 1 (2004): 55-112.

[7] Eisenstein, Gotthold. *Über eine allgemeine Eigenschaft der Reihen-Entwicklungen aller algebraischen Funktionen.* Bericht der Königl. Preuss. Akademie der Wissenschaften zu Berlin (1852): 441-443.

[8] Bruinier, Jan Hendrik, et al. *Elliptic modular forms and their applications.* The 1-2-3 of modular forms: Lectures at a summer school in Nordfjordeid, Norway (2008): 1-103.

The section conjecture for the toric fundamental group

Giulio Bresciani

The toric fundamental group is the smallest extension of the étale fundamental group which can manage the monodromy of line bundles, in addition to the monodromy of finite étale covers. It is an extension of the étale fundamental group by a projective limit of tori.

We study the analogue of Grothendieck's section conjecture for the toric fundamental group over finite extensions of $\mathbb{Q}_p$. Given a smooth, projective variety over a field k, there exists a space of Galois sections $\mathcal{S}^{\mathrm{tor}}_{X/k}$ of toric Galois sections with natural maps

$$X(k) \to \mathcal{S}^{\mathrm{tor}}_{X/k} \to \mathcal{S}_{X/k}$$

giving a factorization of the usual profinite Kummer map $X(k) \to \mathcal{S}_{X/k}$. Moreover, the natural map $\mathcal{S}^{\mathrm{tor}}_{X/k} \to \mathcal{S}_{X/k}$ is injective if X is a smooth projective curve.

If P is a non-trivial Brauer–Severi variety over k, then $\mathcal{S}^{\mathrm{tor}}_{P/k} = \emptyset$. As a consequence, if X maps to a non-trivial Brauer–Severi variety, we have that $\mathcal{S}^{\mathrm{tor}}_{X/k} = \emptyset$ as well. This makes the toric fundamental group particularly well-suited for studying the Grothendieck's section conjecture using Brauer groups.

We now state the main results. After the talk, Y. Hoshi and A. Tamagawa have found a gap in one of the proofs. Because of this, the main results are conditional on the following statement being true.

Conjecture 1. Let X be a smooth projective curve of genus ≥ 2 embedded in an abelian variety A over a field k finite over $\mathbb{Q}_p$. Assume that $X \subset A$ contains no torsion points of order prime with p. For $n >> 0$ large enough, the inverse image X_n of X along $p^n : A \to A$ has index divisible by p.

We remark that part of the proof of Conjecture 1 remains correct. For instance, we can prove that for every $r > 0$ there exists $n_r >> 0$ such that every closed point of X_{n_r} of ramification index $\leq r$ has degree divisible by p.

Conditional on Conjecture 1, we prove the following results.

Theorem 1. Assume that Conjecture 1 holds. Let X be a smooth projective curve of genus ≥ 2 over a field k finite over $\mathbb{Q}_p$. Then

$$X(k) \to \mathcal{S}^{\mathrm{tor}}_{X/k}$$

is bijective.

Theorem 2. Assume that Conjecture 1 holds. Let X be a smooth projective curve of genus ≥ 2 over a number field k. Then $\mathcal{S}^{\mathrm{tor}}_{X/k} \subset \mathcal{S}^{\mathrm{tor}}_{X/k}$ coincides with the subset of Selmer Galois sections.

Once the theory of toric fundamental groups is established, Theorems 1 and 2 are relatively easy consequences of the following result about standard Galois sections.

Theorem 3. Assume that Conjecture 1 holds. Let X be a smooth projective curve of genus ≥ 2 over a field k finite over $\mathbb{Q}_p$, and $s \in \mathcal{S}_{X/k}$ a Galois section. If s does not come from a point of $X(k)$, there exists a finite étale cover $Y \to X$, a lift $r \in \mathcal{S}_{Y/k}$ of s and a morphism $Y \to P$ where P is a non-trivial Brauer–Severi variety.

References

[1] G. Bresciani, *The section conjecture for the toric fundamental group over p-adic fields*, arxiv:2409.07923 (2024).

Relative mod-n Poincaré duality in nonarchimedean geometry

Emanuel Reinecke

(joint work with Shizhang Li, Bogdan Zavyalov)

This talk is based on [3]. We explain a new, essentially "diagrammatic" proof of finiteness and duality results for étale local systems in nonarchimedean geometry.

1. Introduction

Let us recall (a special case of) the classical statement of finiteness under higher direct images and Poincaré duality for étale local systems in algebraic geometry.

Theorem 1 (SGA 4)**.** *Let $f\colon X \to Y$ be a smooth, proper morphism of schemes. Let $n \in \mathcal{O}_Y^\times$ be an integer that is invertible on Y. Let L be an étale $\mathbf{Z}/n$-local system on X. Then:*

(1) (finiteness) The higher direct images $R^i f_* L$ are étale $\mathbf{Z}/n$-local systems on Y for all $i \geq 0$.

(2) (duality) Assume that f is of equidimension d. Then there exists a trace morphism $\mathrm{tr}_f\colon Rf_*\mathbf{Z}/n(d)[2d] \to \mathbf{Z}/n$ such that the induced pairing

$$Rf_*(L) \otimes^L Rf_*\big(L^\vee(d)[2d]\big) \xrightarrow{\cup} Rf_*\big(L \otimes L^\vee(d)[2d]\big) \to Rf_*\big(\mathbf{Z}/n(d)[2d]\big) \xrightarrow{\mathrm{tr}_f} \mathbf{Z}/n$$

is perfect. That is, the tensor-hom adjoint

$$Rf_*\big(L^\vee(d)[2d]\big) \to R\mathcal{H}om\big(Rf_*(L), \mathbf{Z}/n\big)$$

is an isomorphism.

2. The main statement

We want to discuss a statement in nonarchimedean geometry which is analogous to Theorem 1, replacing schemes by adic spaces.

Example 2. The archetypical example of an adic space is the closed unit disk $\mathbf{D}^1_K$ for some nonarchimedean extension K of $\mathbf{Q}_p$. The functions on it are given by the ring

$$K\langle T\rangle := \big\{ f = \textstyle\sum_{i=0}^\infty a_i T^i \in K[\![T]\!] \;\big|\; \lim_{i\to\infty} |a_i| = 0 \big\}$$

of those power series that converge for all $x \in \mathcal{O}_K$. The points of $\mathbf{D}^1_K$ are equivalence classes of continuous, nonarchimedean valuations $v\colon K\langle T\rangle \to \Gamma \cup \{0\}$ for a totally ordered abelian group Γ such that $v\big(\mathcal{O}_K\langle T\rangle\big) \leq 1$. Here are some examples:

(a) (classical points) for any $x \in \mathcal{O}_K$, the valuation

$$v_x \colon K\langle T\rangle \to \mathbf{R}_{>0} \cup \{0\}, \quad f \mapsto |f(x)|$$

(b) (disks around 0) for any $0 < r \leq 1$, the valuation

$$v_r \colon K\langle T\rangle \to \mathbf{R}_{>0} \cup \{0\}, \quad f = \textstyle\sum_{i=0}^{\infty} a_i T^i \mapsto \max_{|y|\leq r} \big\{|f(y)|\big\} = \max_i \big\{|a_i| r^i\big\}$$

(c) (higher rank points) for any $0 < r < 1$, the valuation

$$v_{r_+} \colon K\langle T\rangle \to \big(\mathbf{R}_{>0} \times (r_+)^{\mathbf{Z}}\big) \cup \{0\}, \quad f = \textstyle\sum_{i=0}^{\infty} a_i T^i \mapsto \max_i \big\{|a_i|(r_+)^i\big\}$$

where $\mathbf{R}_{>0} \times (r_+)^{\mathbf{Z}}$ has the lexicographical ordering (informally: $r < r_+ < r'$ for all $r < r' \in \mathbf{R}_{>0}$).

There exists again a formalism of étale morphisms and the étale topology for adic spaces. In this setting, Theorem 1 admits a direct analog:

Theorem 3 ([1, 2, 5, 6, 4, 3])**.** *Let $f \colon X \to Y$ be a smooth, proper morphism of locally noetherian analytic adic spaces (e.g., rigid-analytic spaces over K). Let $n \in \mathcal{O}_Y^\times$ be an integer that is invertible on Y. Let L be an étale $\mathbf{Z}/n$-local system on X. Then:*

(1) (finiteness) The higher direct images $R^i f_* L$ are étale $\mathbf{Z}/n$-local systems on Y for all $i \geq 0$.

(2) (duality) Assume that f is of equidimension d. Then there exists a trace morphism $\mathrm{tr}_f \colon Rf_*\mathbf{Z}/n(d)[2d] \to \mathbf{Z}/n$ such that the induced pairing

$$Rf_*(L) \otimes^L Rf_*\big(L^\vee(d)[2d]\big) \xrightarrow{\cup} Rf_*\big(L \otimes L^\vee(d)[2d]\big) \to Rf_*\big(\mathbf{Z}/n(d)[2d]\big) \xrightarrow{\mathrm{tr}_f} \mathbf{Z}/n$$

is perfect.

When $n \in \mathcal{O}_Y^{+,\times}$ (e.g., for rigid-analytic spaces over a nonarchimedean extension K of $\mathbf{Q}_p$, this means that $(n, p) = 1$), Theorem 3 was shown by Berkovich [1] and Huber [2]. When $n = p$, the finiteness part was proved by Scholze–Weinstein [5] (based on earlier work of Scholze in the absolute case) and the duality part by Gabber (unpublished), Zavyalov [6], and Mann [4]. All the previous proofs for $n = p$ use the machinery of perfectoid spaces heavily. Our new proof of Theorem 3 works uniformly for all $n \in \mathcal{O}_Y^\times$ and only makes minimal use of perfectoid techniques.

Remark 4. We also prove a version of Theorem 3 for proper (but not necessarily smooth) morphisms of rigid-analytic spaces over K, which involves the formalism of Zariski-constructible sheaves and dualizing complexes. This version confirms a conjecture of Bhatt–Hansen.

3. The trace map

Let us explain a new construction of the trace map featuring in Theorem 3.

Theorem 5 ([3])**.** *There is a unique way to assign to any separated, taut, smooth of equidimension d morphism $f \colon X \to Y$ between locally noetherian analytic adic spaces and any integer $n \in \mathcal{O}_Y^\times$ a trace map $\mathrm{tr}_f \colon Rf_!\mathbf{Z}/n(d)[2d] \to \mathbf{Z}/n$ such that:*

(1) tr_f *is compatible with compositions*
(2) tr_f *is compatible with pullbacks*
(3) if f *is étale, then* tr_f *is the counit of the adjunction between* $f_!$ *and* f^*
(4) if $f\colon \mathbf{P}^{1,\mathrm{an}}_C \to \mathrm{Spa}(C, \mathcal{O}_C)$*, then* tr_f *is the analytification of the algebraic trace.*

Remark 6. When f is in addition partially proper, such a trace map had previously been defined by Berkovich via different methods. While his trace would be sufficient for our proof of Theorem 3, our version for not necessarily smooth morphisms from Remark 4 crucially uses the more general construction in Theorem 5.

The tautness condition in Theorem 5 is a technical one that Huber needs in order to define $Rf_!$; it is for example implied by quasi-paracompactness.

Idea of the proof of Theorem 5. By an SGA 4-type argument, the local structure of smooth morphisms guarantees the uniqueness of tr_f and reduces the construction of tr_f to the case when $f\colon X \to \mathrm{Spa}(C, \mathcal{O}_C)$ is an affinoid curve over an algebraically closed nonarchimedean field C. Let us explain the main idea of our construction when $X = \mathbf{D}^1_C$ is the closed unit disk.

Consider the universal compactification $\mathbf{D}^1_C \subset \overline{\mathbf{D}^1_C}$. The complement $\overline{\mathbf{D}^1_C} \smallsetminus \mathbf{D}^1_C$ consists of one point, which corresponds to the rank-2 valuation

$$v_{1_+}\colon C\langle T\rangle \to \big(\mathbf{R}_{>0} \times (1_+)^{\mathbf{Z}}\big) \cup \{0\}, \quad f = \textstyle\sum_{i=0}^\infty a_i T^i \mapsto \max_i \left\{|a_i|(1_+)^i\right\}$$

where again $1 < 1_+ < r'$ for all $1 < r' \in \mathbf{R}_{>0}$. Now excision and the Artin–Grothendieck vanishing $H^2_{\text{ét}}(\overline{\mathbf{D}^1_C}, \mu_n) = 0$ yield an exact sequence

$$H^1_{\text{ét}}\big(\overline{\mathbf{D}^1_C}, \mu_n\big) \to H^1_{\text{ét}}(\{v_{1_+}\}, \mu_n) \to H^2_{\text{ét},c}(\mathbf{D}^1_C, \mu_n) \to 0.$$

By work of Huber and Kummer theory, $H^1_{\text{ét}}(\{v_{1_+}\}, \mu_n) \simeq \widehat{k(v_{1_+})}^{\mathrm{h},\times}/n$, where $\widehat{k(v_{1_+})}^{\mathrm{h}}$ denotes the henselized completed residue field. The valuation v_{1_+} extends to $\widehat{k(v_{1_+})}^{\mathrm{h}}$ and thus defines a map

$$v_{1_+} \bmod n\colon H^1_{\text{ét}}(\{v_{1_+}\}, \mu_n) \to \big(\mathbf{R}_{>0} \times (1_+)^{\mathbf{Z}}\big)/n \simeq (1_+)^{\mathbf{Z}}/n \simeq \mathbf{Z}/n.$$

Finally, we check that $v_{1_+} \bmod n$ vanishes on $H^1_{\text{ét}}\big(\overline{\mathbf{D}^1_C}, \mu_n\big)$ and therefore descends to the desired map $H^2_{\text{ét},c}(\mathbf{D}^1_C, \mu_n) \to \mathbf{Z}/n$. □

4. Proof of Theorem 3

We present the essential idea of our proof of Theorem 3. Recall that the perfect complexes in $D(X_{\text{ét}}; \mathbf{Z}/n)$ are exactly the dualizable objects. Our proof then boils down to the verification that for any dualizable $\mathcal{E} \in D(X_{\text{ét}}; \mathbf{Z}/n)$, the derived pushforward $Rf_*(\mathcal{E}) \in D(Y_{\text{ét}}; \mathbf{Z}/n)$ is again dualizable, with dual $Rf_*(\mathcal{E}^\vee(d)[2d])$: this will show the finiteness and dualizability in one fell swoop.

For our verification, we need to define an evaluation map $e\colon Rf_*(\mathcal{E}^\vee(d)[2d]) \otimes^L Rf_*(\mathcal{E}) \to \mathbf{Z}/n$ and a coevaluation map $c\colon \mathbf{Z}/n \to Rf_*(\mathcal{E}) \otimes^L Rf_*(\mathcal{E}^\vee(d)[2d])$ such that $(\mathrm{id} \otimes e) \circ (c \otimes \mathrm{id}) = \mathrm{id}_{Rf_*(\mathcal{E})}$ and $(e \otimes \mathrm{id}) \circ (\mathrm{id} \otimes c) = \mathrm{id}_{Rf_*(\mathcal{E}^\vee(d)[2d])}$. The evaluation map is the duality pairing from the statement of Theorem 3. In order to define the coevaluation map, we need a well-behaved theory of cycle classes for adic spaces. It can be set up in a similar way as its analog in algebraic geometry.

Let $\Delta\colon X \to X \times_Y X$ be the diagonal map and $h := (f, f)\colon X \times_Y X \to Y$. Denote by cl_Δ the cycle class of Δ. Then the coevaluation map is

$$c\colon \mathbf{Z}/n \to Rf_*(\mathbf{Z}/n) \to Rf_*\big(\mathcal{E} \otimes^L \mathcal{E}^\vee\big) \simeq Rh_*\big((\mathcal{E} \boxtimes \mathcal{E}^\vee) \otimes^L \Delta_*(\mathbf{Z}/n)\big)$$
$$\xrightarrow{Rh_*(\mathrm{id}\otimes \mathrm{cl}_\Delta)} Rh_*\big(\mathcal{E} \boxtimes \mathcal{E}^\vee(d)[2d]\big) \simeq Rf_*(\mathcal{E}) \otimes^L Rf_*(\mathcal{E}^\vee(d)[2d]\big)$$

where the second map is induced by the coevaluation for the dualizable object $\mathcal{E}$, the first isomorphism follows from the projection formula, and the last isomorphism is the inverse of the Künneth map. The statement that the Künneth map is an isomorphism is the only part of our argument that requires perfectoid methods. Once the evaluation and the coevaluation map are defined, the verification of the identities $(\mathrm{id} \otimes e) \circ (c \otimes \mathrm{id}) = \mathrm{id}$ and $(e \otimes \mathrm{id}) \circ (\mathrm{id} \otimes c) = \mathrm{id}$ amounts to a diagram chase. One crucial input is that $\mathrm{tr}_{\mathrm{pr}} \circ \mathrm{cl}_\Delta = \mathrm{id}$ for a projection $\mathrm{pr}\colon X \times_Y X \to X$.

References

[1] Vladimir G. Berkovich, *Étale cohomology for non-Archimedean analytic spaces*, Inst. Hautes Études Sci. Publ. Math. **78** (1993), 5–161.

[2] Roland Huber, *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, E30, Friedr. Vieweg & Sohn, Braunschweig, 1996.

[3] Shizhang Li, Emanuel Reinecke, and Bogdan Zavyalov, *Relative Poincaré duality in nonarchimedean geometry*, Preprint (2024), available at `https://arxiv.org/abs/2410.08200`.

[4] Lucas Mann, *A p-Adic 6-Functor Formalism in Rigid-Analytic Geometry*, Preprint (2022), available at `https://arxiv.org/abs/2206.02022`.

[5] Peter Scholze and Jared Weinstein. *Berkeley lectures on p-adic geometry*, Annals of Mathematics Studies, vol. 207, Princeton University Press, Princeton, NJ, 2020.

[6] Bogdan Zavyalov. *Mod-p Poincaré duality in p-adic analytic geometry*, Ann. of Math. (2) (to appear) (2024).

Mapping class group and Dehn twists of p-adic fields

Nadav Gropper

1. Motivation

The main motivation for the talk is to understand anabelian questions about a p-adic field K, through the lens of Arithmetic Topology.

In [1], Mochizuki proved the following:

Theorem. *Let K be a finite extension of $\mathbb{Q}_p$, let G_K be its absolute Galois group, and let $Out_{Filt}(G_K)$ be the group of outer automorphism of G_K, which preserve the ramification filtration.*

Then the natural morphism

$$Aut_{\mathbb{Q}_p}(K) \to Out_{Filt}(G_K)$$

is a bijection.

On the other hand, it is known that the absolute Galois group of a p-adic field does not determine the field (see for example [3]). This naturally leads to the following question:

Question 1. *What is the structure of the group $Out(G_K)$?*

In [2] a non trivial element of $Out(G_K)$ was constructed. It was later shown in [8], that if p is odd, and K is an abelian even degree extension of $\mathbb{Q}_p$, then these have infinite order in $Out(G_K)$, and so the outer automorphism constructed in [2] does not preserve the ramification filtration.

Other than that, very little was known about the outer automorphism group.

2. Main Results

In the talk we outline the ideas and tools used [7] to study $Out(G_K)$. These ideas are highly influenced by the philosophy of arithmetic topology.

Arithmetic Topology, first pioneered in [9] by Mazur, draws analogies between number theory and low dimensional topology (see [5] for a more detailed view of arithmetic topology).

Under this analogy, a p-adic field should correspond to a surface, and $Out(G_K)$ should look like the mapping class group of a surface.

Our main result strengthens the above analogy and gives further insight to $Out(G_K)$:

Theorem 1. *Let K be a p-adic field, and let $G_K(p)$ be the maximal pro-p quotient of the absolute Galois group of K.*

There is a family of infinite order elements of $Out(G_K(p))$ which generalize the notion of Dehn twists on a surface.

The first step is to look at a group theoretic description of curves on a surface and Dehn twists. We have the following classical fact (see for example [6] theorem 4.12.1):

Theorem. *A simple closed curve on a surface S, gives rise to a splitting of $\pi_1(S)$ as an amalgamated free product over $\mathbb{Z}$, or as an HNN extension over $\mathbb{Z}$. Conversely, all splittings over $\mathbb{Z}$ of $\pi_1(S)$, into HNN extensions and amalgamated free products, arise in such a way.*

Now given a $\mathbb{Z}$ splitting of a group one can define a Dehn automorphism.

Definition 1. *Given a group G and a $\mathbb{Z}$ splitting of it, we define an automorphism of G, called the Dehn twist associated to the splitting, as follows:*

If the $\mathbb{Z}$ splitting is $A *_{<c>} B$, we define the Dehn twist δ to be the automorphism

$$\delta(a) = a, \delta(b) = cbc^{-1}$$

for all $a \in A, b \in B$.

If the the $\mathbb{Z}$ splitting is HNN extension $A*_{<c>}$, we define the Dehn twist δ to be

$$\delta(a) = a, \delta(t) = ct$$

for all $a \in A$ and where t is the stable letter of the HNN extension.

When looking at surface groups, these generalized Dehn twists are the classical Dehn twists. Similar definitions can be made for pro-p groups and $\mathbb{Z}_p$ splittings.

In order to work with surfaces and p-adic fields at the same time, we look at Demushkin groups [4], these are pro-p groups with a Poincaré duality of dimension 2, and they include $G_K(p)$, and the pro-p completion surface groups.

In the talk, we explain the description we have for all $\mathbb{Z}_p$ splittings of a Demushkin group.

By using this description, together with the notion of intersections of splittings, we can show that Dehn twist automorphisms, have infinite order in the outer automorphism group.

We finish with the following:

Question 2. *Does the profinite group, generated by all Dehn twists, have finite index in $Out(G_K(p))$?*

References

[1] Mochizuki, S. A version of the Grothendieck conjecture for p-adic local fields. *International Journal Of Mathematics.* **8**, 499–506 (1997)

[2] Jannsen, U. & Wingberg, K. Die Struktur der absoluten Galoisgruppe p-adischer Zahlkörper. *Inventiones Mathematicae: Invent. Math..* **70**, 71–98 (1982)

[3] Jarden, M. & Ritter, J. On the characterization of local fields by their absolute Galois groups. *Journal Of Number Theory.* **11**, 1–13 (1979)

[4] Labute, J. Classification of Demushkin groups. *Canadian Journal Of Mathematics.* **19** pp. 106–132 (1967)

[5] Morishita, M. Knots and primes. *Universitext.* (2012)

[6] Zieschang, H., Vogt, E. & Coldewey, H. Surfaces and planar discontinuous groups. (Springer,2006)

[7] Gropper, N. Surfaces and p-adic fields I: Dehn twists. *ArXiv Preprint ArXiv:2303.04309.* (2023)

[8] Hoshi, Y. & Nishio, Y. On the outer automorphism groups of the absolute Galois groups of mixed-characteristic local fields. *Research In Number Theory.* **8**, 56 (2022)

[9] Mazur, B. Remarks on the Alexander polynomial. (1963)

Tame fundamental groups of rigid spaces

Piotr Achinger

(joint work with Katharina Hübner, Marcin Lara, and Jakob Stix)

Étale fundamental groups of rigid-analytic spaces can be challenging to understand. For example, $\pi_1^{\text{ét}}(D_{\mathbb{C}_p})$ of the affinoid unit disc over $\mathbb{C}_p$ is not topologically finitely generated, as

$$\mathrm{H}^1_{\text{ét}}(D_{\mathbb{C}_p}, \mathbb{F}_p) = \mathrm{Hom}(\pi_1^{\text{ét}}(D_{\mathbb{C}_p}), \mathbb{F}_p)$$

is infinite. For a proper smooth rigid-analytic space X over $\mathbb{C}_p$, the cohomology groups $\mathrm{H}^*_{\text{ét}}(X, \mathbb{F}_p)$ are finite (as shown by Scholze [15]), and the étale fundamental group is likewise expected to be topologically finitely generated.[1] However, it

[1] We learned of this question from Bogdan Zavyalov.

seems that we currently lack tools to show this, unless X is the analytification of an algebraic variety.

As in the case of schemes [13], one can deal with such issues by considering the tame quotient of the fundamental group. For us, a rigid-analytic space over a non-archimedean field K is an adic space locally of finite type over $\mathrm{Spa}(K, K^+)$. Thus, the residue fields of points of X are equipped with valuations, and the natural definition that presents itself (considered in [7]) is the following: an étale morphism of adic spaces $f\colon Y \to X$ is **tame** if for every $y \in Y$, the finite separable extension of valued fields $k(y)/k(f(y))$ is tamely ramified (meaning that $[k(y)^{\mathrm{sh}} : k(f(y))^{\mathrm{sh}}]$ is prime to the residue characteristic exponent). For X connected, tame finite étale maps $Y \to X$ form a Galois category whose fundamental group $\pi^{\mathrm{t}}(X)$ is a quotient of $\pi_1^{\text{ét}}(X)$.

However, with this definition, the tame fundamental group $\pi_1^{\mathrm{t}}(D_{\mathbb{C}_p})$ is still infinite! Indeed, the coverings defined by

$$y^p - y = \lambda x \qquad (\lambda \in \mathbb{C}_p \text{ with } |\lambda| = 1)$$

are tame (even unramified) and yield an infinite number of maps $\pi_1^{\mathrm{t}}(D_{\mathbb{C}_p}) \to \mathbb{F}_p$. Intuitively, the tameness condition introduced above measures only the ramification along the special fiber of a formal model of X, while in the presented example the wild ramification happens at infinity of the special fiber. We correct this by introducing the following notion: an étale morphism of rigid-analytic spaces $f\colon Y \to X$ is **tame relative to** K if for every maximal point $y \in Y$ and every valuation subring $V \subseteq k(y)^+$ containing K^+, the extension of valued fields $k(y)/k(f(y))$ is tamely ramified with respect to V. Again, for X connected, we obtain a Galois category whose fundamental group $\pi^{\mathrm{t}}(X/K)$ is a quotient of $\pi_1^{\mathrm{t}}(X)$.

For the unit disc $D_{\mathbb{C}_p}$, such test pairs (y, V) consist of points of $D_{\mathbb{C}_p}$ (continuous valuations on $K\langle x\rangle$ which are ≤ 1 on $K^+\langle x\rangle$) and one additional point corresponding to a rank two continuous valuation which is unbounded on $K^+\langle x\rangle$. In fact, if X is quasi-compact and separated, then the test pairs $(y^\circ, k(y)^+)$ form the set of points of an adic space (not a rigid space in general) $\overline{X}$ containing X, the **universal compactification** of X/K defined by Huber [6, §5.1]. Alternatively, $\overline{X}$ can be described as the inverse limit of all compactifications of special fibers of all formal models of X [11]. Thus $\pi_1^{\mathrm{t}}(X/K) = \pi_1^{\mathrm{t}}(\overline{X})$, whenever $\overline{X}$ exists.

Our main result is the following.

Theorem 1. *Let X be a connected qcqs rigid space over a non-archimedean field K. Suppose that the tame Galois group $\pi_1^{\mathrm{t}}(K) = \mathrm{Aut}(K^{\mathrm{t}}/K)$ is topologically finitely generated. Then $\pi_1^{\mathrm{t}}(X/K)$ is topologically finitely generated.*

Similarly, if K is algebraically closed, we can show the Künneth formula

$$\pi_1^{\mathrm{t}}(X \times Y/K) \simeq \pi_1^{\mathrm{t}}(X/K) \times \pi_1^{\mathrm{t}}(Y/K),$$

and that if L is an algebraically closed non-archimedean field containing K, then $\pi_1^{\mathrm{t}}(X_L/L) \simeq \pi_1^{\mathrm{t}}(X/K)$. In light of [3, 14] it is an interesting question whether $\pi_1^{\mathrm{t}}(X/K)$ is topologically finitely presented. Using our methods, we can show that this is the case if X is smooth and admits a semistable model such that the strata of its special fiber admit normal crossings compactifications. In this situation, there is a "van Kampen formula" expressing $\pi_1^{\mathrm{t}}(X/K)$ in terms of the more classically studied tame fundamental groups of the strata.

The proof of Theorem 1 relies on

(1) desingularization techniques [9, 16] which allow us to reduce the finite generation question to the case where $K = \overline{K}$ and X is smooth, with a semistable formal model $\mathfrak{X}$ (treated as a log formal scheme over K^+),
(2) a "semistable Abhyankar's lemma," relating $\pi_1^{\mathrm{t}}(X)$ (not $\pi_1^{\mathrm{t}}(X/K)$!) to the Kummer étale fundamental group $\pi_1^{\text{ét}}(\mathfrak{X}_0)$ of the log special fiber,
(3) an additional argument showing that $\pi_1^{\mathrm{t}}(X/K)$ is isomorphic to the tame Kummer étale fundamental group $\pi_1^{\text{ét},\mathrm{t}}(\mathfrak{X}_0/k)$ (a notion we needed to introduce along the way, and which I will not explain here),
(4) and finally, proving that for a suitable class of log schemes over an algebraically closed field k, the tame Kummer étale fundamental group is topologically finitely generated (see Theorem 2 below).

Logarithmic geometry beyond fs. A major obstacle to this approach is that since $K = \overline{K}$, it is not discretely valued, and hence the log special fiber $\mathfrak{X}_0$ will not be an fs log scheme, precluding the application of most of logarithmic geometry. Recall that a log scheme is fs if it locally admits a chart by an fs (finitely generated and saturated) monoid. Here, the log structure on K^+ admits a chart by the monoid Γ_K^+, the positive part of the value group. This monoid is not finitely generated; however, it is valuative and divisible, which turns out to be quite helpful in this context.

In order to overcome the obstacle, we needed to develop the foundations of logarithmic geometry beyond fs log schemes, which is a project by itself. (Similar, though less comprehensive, approaches appear in some recent papers [1, 2, 12].) The basic notion is that of an sfp morphism. A map of saturated monoids $P \to Q$ is **sfp** (finitely presented up to saturation) if Q is the saturation of a finitely presented monoid over P, or equivalently, if

$$Q = (P \oplus_{P_0} Q_0)^{\mathrm{sat}}$$

for a map of fs monoids $P_0 \to Q_0$ (these are precisely the compact objects of the category of saturated monoids over P). A map of saturated log schemes $Y \to X$ is **locally sfp** if it is étale locally of the form

$$\mathrm{Spec}(Q \to B) \to \mathrm{Spec}(P \to A)$$

where $(P \to A) \to (Q \to B)$ is a map of saturated prelog rings such that $P \to Q$ is sfp and $A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q] \to B$ is finitely presented as a map of rings; it is **sfp** if it is locally sfp and qcqs. Crucially, we show that if $(P \to A) = \varinjlim (P_\alpha \to A_\alpha)$ is a filtered colimit of saturated prelog rings, then the category of sfp log schemes over $\mathrm{Spec}(P \to A)$ is the colimit of the system of categories of sfp log schemes over $\mathrm{Spec}(P_\alpha \to A_\alpha)$. Since for any given $(P \to A)$ we can find such a system with P_α fs and A_α finitely generated over $\mathbb{Z}$, this allows us to extend many known results from fs log schemes to sfp maps (analogously to the elimination of noetherian hypotheses in [5, §8]). Using this, we define smooth, étale, and Kummer étale maps, and develop the theory of the Kummer étale site and the Kummer étale fundamental group (see [8]) for arbitrary saturated log schemes.

Interestingly, sfp or Kummer étale maps might not be locally of finite type as maps of schemes. Indeed, there exist tame extensions of valued fields L/K such that Γ_L^+ is not finitely generated as a monoid over Γ_K^+, and the valuation ring L^+ is not finitely generated over K^+. (For example, let K be a non-archimedean field with $|2| = 1$, with value group $\mathbb{Z} + \mathbb{Z}\sqrt{2} \subseteq \mathbb{R}$, and let $L = K(\sqrt{x}, \sqrt{y})$ where $\nu(x) = 1$ and $\nu(y) = \sqrt{2}$.) However, the map $\Gamma_K^+ \to \Gamma_L^+$ is sfp, and with the natural log structures, the map $\mathrm{Spec}(L^+) \to \mathrm{Spec}(K^+)$ is Kummer étale. Surprisingly, thanks to fundamental results of Kato [10] and Tsuji [17], these difficulties go away for sfp log schemes over a base with a chart given by a divisible valuative monoid, such as $\mathrm{Spec}(K^+)$ for an algebraically closed non-archimedean field K.

Back to tame fundamental groups. With all these preparations, we can finish the proof of Theorem 1 by proving:

Theorem 2. *Let X be a connected log scheme which is sfp over* $\mathrm{Spec}(P \to k)$ *where P is a divisible valuative monoid with finitely many faces and k is an algebraically closed field. Then the tame Kummer étale fundamental group $\pi_1^{\text{ét,t}}(X/k)$ is topologically finitely generated.*

The proof uses "formal gluing" along the log stratification of X to reduce to the case of locally constant log structure, which in turn can be reduced to the case of trivial log structure. In this case, we use alterations to reduce to a result of Esnault and Kindler [4].

Note that, the proofs of all known results of (topological) finite generation of fundamental groups in algebraic geometry eventually rely on the finite generation of the topological fundamental group of smooth curves over $\mathbb{C}$. Ultimately, so does our proof, after a very long pipeline of reductions.

References

[1] Karim Adiprasito, Gaku Liu, Igor Pak, and Michael Temkin. Log smoothness and polystability over valuation rings, 2019.

[2] Hansheng Diao and Zijian Yao. Logarithmic $\mathrm{A}_{\mathrm{inf}}$-cohomology, 2024.

[3] Hélène Esnault, Mark Shusterman, and Vasudevan Srinivas. Finite presentation of the tame fundamental group. *Selecta Mathematica*, 28(2):37, January 2022.

[4] Hélène Esnault and Lars Kindler. Lefschetz theorems for tamely ramified coverings. *Proceedings of the American Mathematical Society*, 144(12):5071–5080, 2016.

[5] Alexander Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III. *Inst. Hautes Études Sci. Publ. Math.*, 28:255, 1966.

[6] Roland Huber. *Étale cohomology of rigid analytic varieties and adic spaces*, volume E30 of *Aspects of Mathematics*. Friedr. Vieweg & Sohn, Braunschweig, 1996.

[7] Katharina Hübner. The adic tame site. *Documenta Mathematica*, 26:873–945, 2021.

[8] Luc Illusie. An overview of the work of K. Fujiwara, K. Kato, and C. Nakayama on logarithmic étale cohomology. *Astérisque*, 279:271–322, 2002. Cohomologies p-adiques et applications arithmétiques, II.

[9] Luc Illusie. Exposé IX. Uniformisation locale première à ℓ. In *Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents*, number 363-364, pages 161–165. 2014.
[10] Fumiharu Kato. Integral morphisms and log blow-ups. *Israel J. Math.*, 247(2):831–843, 2022.
[11] Mateusz Kobak. Compactifications of rigid analytic spaces through formal models, 2023.
[12] Teruhisa Koshikawa. Logarithmic prismatic cohomology I, 2022.
[13] Moritz Kerz and Alexander Schmidt. On different notions of tameness in arithmetic geometry. *Mathematische Annalen*, 346(MR2578565):641–668, 2010.
[14] Marcin Lara, Vasudevan Srinivas, and Jakob Stix. Fundamental groups of proper varieties are finitely presented. *Comptes Rendus. Mathématique*, 362(G1):51–54, 2024.
[15] Peter Scholze. p-adic Hodge theory for rigid-analytic varieties. *Forum Math. Pi*, 1:e1, 77, 2013.
[16] Michael Temkin. Altered local uniformization of Berkovich spaces. *Israel Journal of Mathematics*, 221(2):585–603, September 2017.
[17] Takeshi Tsuji. Saturated morphisms of logarithmic schemes. *Tunis. J. Math.*, 1(2):185–220, 2019.

Non-abelian Chabauty and the Selmer Section Conjecture

Martin Lüdtke
(joint work with L. Alexander Betts, Theresa Kumpitsch)

Grothendieck's Section Conjecture predicts that for a smooth projective curve $X/\mathbb{Q}$ of genus at least two, every Galois section is induced by a rational point. The Selmer Section Conjecture is a weaker version which assumes that the Galois section locally comes from a $\mathbb{Q}_v$-point for every place v. In this talk I present our results [1] on a new strategy for proving instances of the Selmer Section Conjecture based on non-abelian Chabauty calculations. We show that whenever X satisfies Kim's Conjecture for all choices of auxiliary prime p in a density 1 set of primes, then the Selmer Section Conjecture holds for X. The analogous statement for S-integral points on affine hyperbolic curves is also proved. We demonstrate the viability of our strategy by verifying Kim's Conjecture for $X = \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ over $\mathbb{Z}[1/2]$ and any choice of auxiliary prime $p > 2$.

1. The Selmer Section Conjecture

Let X be a smooth projective curve of genus $g \geq 2$ over a number field K. Grothendieck's Section Conjecture predicts a description of the set of rational points $X(K)$ in terms of the étale fundamental group of X. More precisely, writing $G_K = \mathrm{Gal}(\overline{K}/K)$ for the absolute Galois group of K we have the *fundamental exact sequence*:

$$1 \longrightarrow \pi_1^{\text{ét}}(X_{\overline{K}}) \longrightarrow \pi_1^{\text{ét}}(X) \longrightarrow G_K \longrightarrow 1. \tag{1}$$

Every rational point $x \in X(K)$ induces a section $s_x \colon G_K \to \pi_1^{\text{ét}}(X)$ which is well-defined up to $\pi_1(X_{\overline{K}})$-conjugacy. The assignment $x \mapsto [s_x]$ defines the *profinite Kummer map*

$$\kappa \colon X(K) \longrightarrow \mathscr{S}(X/K) \coloneqq \{\text{conjugacy classes of sections of (1)}\}. \tag{2}$$

It is known that this map is injective. Grothendieck's Section Conjecture states that it is also surjective.

For any place v of K we have a similar *local profinite Kummer map*

$$\kappa_v \colon X(K_v) \longrightarrow \mathscr{S}(X_{K_v}/K_v) \tag{3}$$

which is compatible with the inclusion $X(K) \subseteq X(K_v)$ on the one hand and the restriction of sections to a local decomposition group $G_v \leq G_K$ on the other.

Definition 1. We say that a section $s \in \mathscr{S}(X/K)$ is *Selmer* if $s|_{G_v}$ is in the image of κ_v for every place v of K.

Conjecture 2 (Selmer Section Conjecture)**.** *Every Selmer Section is induced by a rational point.*

Sections which are induced by a rational point are clearly Selmer. Thus, Grothendieck's Section Conjecture is equivalent to the combination of the Selmer Section Conjecture and the statement that every section is Selmer. The latter statement would follow from the p-adic Section Conjecture, as the surjectivity of κ_v is known for real places [2, Cor. 3.13] and trivial for complex places. In this talk I present a result which reduces the Selmer Section Conjecture to a conjecture in non-abelian Chabauty theory.

2. Non-abelian Chabauty

Let X be a smooth projective curve of genus $g \geq 2$ over $\mathbb{Q}$. Assume that $X(\mathbb{Q}) \neq \emptyset$ and fix a rational base point $b \in X(\mathbb{Q})$. Let p be a prime of good reduction for X. The *non-abelian Chabauty* method (also known as the *Chabauty–Kim* method) produces a nested sequence of subsets

$$X(\mathbb{Q}_p) \supseteq X(\mathbb{Q}_p)_1 \supseteq X(\mathbb{Q}_p)_2 \supseteq X(\mathbb{Q}_p)_3 \supseteq \dots$$

which are defined by locally analytic functions on $X(\mathbb{Q}_p)$ and all contain the set of rational points $X(\mathbb{Q})$ [3, 4]. Roughly speaking, the set $X(\mathbb{Q}_p)_1$ corresponds to the classical Chabauty method [5]; it is finite whenever the Mordell–Weil rank $r := \mathrm{rk}_{\mathbb{Z}} \operatorname{Jac}_X(\mathbb{Q})$ satisfies $r < g$. The set $X(\mathbb{Q}_p)_2$ can be computed using *Quadratic Chabauty* [6]; it is finite under the weaker condition $r < g + \rho - 1$, where ρ denotes the Picard number of X.

Conjecture 3. *$X(\mathbb{Q}_p)_n$ is finite for $n \gg 0$.*

The finiteness of $X(\mathbb{Q}_p)_n$ for sufficiently large n is implied by the Bloch–Kato conjecture [4]. It is conjectured that the sets $X(\mathbb{Q}_p)_n$ not only become finite but eventually coincide exactly with the set of rational points [7, Conj. 3.1].

Conjecture 4 (Kim's Conjecture)**.** *$X(\mathbb{Q}_p)_n = X(\mathbb{Q})$ for $n \gg 0$.*

There is a natural variant of the non-abelian Chabauty method for S-integral points on affine hyperbolic curves. For $p \notin S$ of good reduction it produces subsets $X(\mathbb{Z}_p)_{S,n} \subseteq X(\mathbb{Z}_p)$ containing the set of S-integral points $X(\mathbb{Z}_S)$, and Kim's Conjecture in this setting states that $X(\mathbb{Z}_p)_{S,n} = X(\mathbb{Z}_S)$ for $n \gg 0$. We remark that we are working with the *refined* Chabauty–Kim method as introduced by Betts–Dogra [8], which produces potentially smaller sets than the classical Chabauty–Kim method.

3. Main results

Let $X/\mathbb{Q}$ be a smooth projective curve of genus at least two and assume that X has a rational point. Our first result provides a strategy for proving the Selmer Section Conjecture by verifying cases of Kim's Conjecture:

Theorem 5 (Betts–Kumpitsch–L., [1])**.** *Assume that X satisfies Kim's Conjecture for all p in a density 1 set of primes. Then the Selmer Section Conjecture holds for X.*

We actually prove the natural generalisation of this result for S-integral points on any hyperbolic (not necessarily projective) curve. In our second main result, we verify Kim's Conjecture in one case for infinitely many choices of p.

Theorem 6 (Betts–Kumpitsch–L., [1])**.** *Kim's Conjecture holds for $X = \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ over $\mathbb{Z}[1/2]$ for all choices of the auxiliary prime $p > 2$.*

Combining the two theorems we recover a result by Stix [9, Cor. 6] saying that the Selmer Section Conjecture holds for $X = \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ over $\mathbb{Z}[1/2]$. This shows that Theorem 5 provides a viable strategy for proving instances of the Selmer Section Conjecture.

References

[1] L. A. Betts, T. Kumpitsch, and M. Lüdtke, *Chabauty–Kim and the Section Conjecture for locally geometric sections*, arXiv: `2305.09462v1 [math.NT]` (2023).

[2] S. Mochizuki, *Topics surrounding the anabelian geometry of hyperbolic curves*, in: Galois groups and fundamental groups, Math. Sci. Res. Inst. Publ. **41** (1990), 120–140.

[3] M. Kim, *The motivic fundamental group of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ and the theorem of Siegel*, Invent. math. **161** (2005), 629–656.

[4] M. Kim, *The Unipotent Albanese Map and Selmer Varieties for Curves*, Publ. RIMS, Kyoto Univ. **45** (2009), 89–133.

[5] C. Chabauty, *Sur les points rationnels des courbes algébriques de genre supérieur à l'unité*, C. R. Acad. Sci. Paris **212** (1941), 882–885.

[6] J. S. Balakrishnan and N. Dogra, *Quadratic Chabauty and rational points I: p-adic heights*, Duke Math. J. **167**(11) (2018), 1981–2038.

[7] J. S. Balakrishnan, I. Dan-Cohen, M. Kim, and S. Wewers, *A non-abelian conjecture of Tate–Shafarevich type for hyperbolic curves*, Math. Ann. **372**(1-2) (2018), 369–428.

[8] L. A. Betts and N. Dogra, *The local theory of unipotent Kummer maps and refined Selmer schemes*, arXiv: `1909.05734v2 [math.NT]` (2020).

[9] J. Stix, *On the birational section conjecture with local conditions*, Invent. Math. **199** (2015), 239–265.

Log prismatic F-crystals and purity

Koji Shimizu

(joint work with Heng Du, Tong Liu, Yong Suk Moon)

We discuss semistable p-adic local systems on a rigid-analytic variety that admits a semistable formal model based on [5].

Let K be a complete discrete valuation field of mixed characteristic $(0,p)$ with perfect residue field k. Let $\mathcal{O}_K$ denote the ring of integers and fix a uniformizer π.

Let $\mathcal{X}$ be a rigid-analytic variety over K and let $\mathbb{L}$ be a p-adic étale local system on $\mathcal{X}$. We regard $\mathbb{L}$ as a family of Galois representations parametrized by $\mathcal{X}$: for each $x \in \mathcal{X}$ and a geometric point $\overline{x}$ with support x, the absolute Galois group $\mathrm{Gal}_{\kappa(x)}$ acts on the stalk $\mathbb{L}_{\overline{x}}$. We want to understand p-adic Hodge theoretic properties (de Rham, crystalline, semistable,...) of $\mathbb{L}$. Here we focus on semistable local systems. For this, we start with a semistable formal model.

So let us slightly change our setup: let $X \to \mathrm{Spf}\,\mathcal{O}_K$ be a semistable p-adic formal scheme and let $\mathcal{X}$ denote the generic fiber of X. This means that étale locally, X is étale over $\mathrm{Spf}\,R$ where

$$R = \mathcal{O}_K\langle T_1, \ldots, T_m, T_{m+1}^{\pm 1}, \ldots, T_d^{\pm 1}\rangle/(T_1 \cdots T_m - \pi).$$

In this talk, we assume $X = \mathrm{Spf}\,R$ for simplicity. Then the mod π fiber is $\mathrm{Spec}\,k[T_1, \ldots, T_m, T_{m+1}^{\pm 1}, \ldots, T_d^{\pm 1}]/(T_1 \cdots T_m)$, and the generic points of the irreducible components correspond to the prime ideals $(\pi, T_i) \subset R$ $(1 \le i \le m)$. Let $\mathcal{O}_{\mathcal{K}_i}$ denote the p-adic completion of the localization $R_{(\pi, T_i)}$: this is a complete discrete valuation ring with imperfect residue field $k(T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_d)$. Set $\mathcal{K}_i = \mathcal{O}_{\mathcal{K}_i}[p^{-1}]$ and $\xi_i = \mathrm{Spa}(\mathcal{K}_i, \mathcal{O}_{\mathcal{K}_i}) \in \mathcal{X}$. We call the points ξ_i the *X-Shilov points* of $\mathcal{X}$.

Theorem 1 (Semistable purity [5])**.** *An étale $\mathbf{Z}_p$-local system $\mathbb{L}$ on $\mathcal{X}$ is semistable if and only if the $\mathbf{Q}_p$-representation $\mathbb{L}_{\overline{\xi}_i}[p^{-1}]$ of $\mathrm{Gal}_{\mathcal{K}_i}$ is semistable for each i.*

Here the notion of semistable Galois representations for $\mathcal{K}_i$ (which has imperfect residue field) is defined via Fontaine's period ring formalism. However, defining a reasonable notion of semistable $\mathbf{Z}_p$-local systems is not straightforward: it is one of the main goals of this talk and explained in the last paragraph of this note.

We use prismatic theory by Bhatt–Scholze [1] and its log variant by Koshikawa [8]. Equip $\mathcal{X}$ with the log structure $M_X = \mathcal{O}_X \cap (\mathcal{O}_X[p^{-1}])^\times$ and let $(X, M_X)_{\Delta}$ denote the absolute log prismatic site: an object is a log prism $(A, I, M_{\mathrm{Spf}\,A})$ together with morphisms

$$(X, M_X) \leftarrow (\mathrm{Spf}\,A/I, M_{\mathrm{Spf}\,A/I}) \hookrightarrow (\mathrm{Spf}\,A, M_{\mathrm{Spf}\,A}),$$

where the first morphism is strict and the second is an exact closed immersion. We consider flat topology. It comes with the structure sheaf $\mathcal{O}_{\Delta}$ and the ideal sheaf $\mathcal{I}_{\Delta}$ defined by $\mathcal{O}_{\Delta}(A, I, M_{\mathrm{Spf}\,A}) = A$ and $\mathcal{I}_{\Delta}(A, I, M_{\mathrm{Spf}\,A}) = I$. The Frobenius φ_A on A given by the δ-structure defines the Frobenius φ on $\mathcal{O}_{\Delta}$.

Example 2. The following log prisms are important for us.

(1) The Breuil–Kisin log prism: $(\mathfrak{S}_R, (E(u)), M_{\operatorname{Spf}\mathfrak{S}_R})$ where

$$\mathfrak{S}_R = W(k)\langle T_1, \dots, T_m, T_{m+1}^{\pm 1}, \dots, T_d^{\pm 1}\rangle[\![u]\!]/(T_1 \cdots T_m - u),$$

and E is the minimal polynomial of π over $W(k)$; the Frobenius is given by $\varphi_{\mathfrak{S}_R}(T_i) = T_i^p$ and the log structure $M_{\operatorname{Spf}\mathfrak{S}_R}$ is induced by $\mathbf{N}^d \to \mathfrak{S}_R$ sending e_i to T_i.

(2) $(A_{\inf}(\overline{R}), \operatorname{Ker}\theta, M_{\operatorname{Spf}A_{\inf}(\overline{R})})$, where $\overline{R}$ is the integral closure of R in the maximal étale extension of $R[p^{-1}]$, and θ is Fontaine's map $A_{\inf}(\overline{R}) = W(\overline{R}^\flat) \to \overline{R}_p^\wedge$. The Frobenius is given by the Witt-vector Frobenius, and there is a canonical log structure.

We consider certain sheaves with Frobenius structure, following [2, 6]:

Definition 3.

(1) A *Laurent F-crystal* is a pair $(\mathcal{E}, \varphi_{\mathcal{E}})$ where $\mathcal{E}$ is a vector bundle of $\mathcal{O}_{\Delta}[\mathcal{I}_{\Delta}^{-1}]_p^\wedge$-modules and $\varphi_{\mathcal{E}}$ is an isomorphism $\varphi^*\mathcal{E} \xrightarrow{\cong} \mathcal{E}$.

(2) An *analytic prismatic F-crystal* is a compatible system $(\mathcal{E}_{\Delta}, \varphi_{\mathcal{E}_{\Delta}})$ of pairs $(\mathcal{E}_{\Delta,A}, \varphi_{\mathcal{E}_{\Delta},A})$ over $(A, I, M_{\operatorname{Spf}A}) \in (X, M_X)_{\Delta}$ where $\mathcal{E}_{\Delta,A}$ is a vector bundle on $\operatorname{Spec}A \smallsetminus V(p, I)$ and $\varphi_{\mathcal{E}_{\Delta},A}$ is an isomorphism $\varphi_A^*\mathcal{E}_{\Delta,A}[I^{-1}] \xrightarrow{\cong} \mathcal{E}_{\Delta,A}[I^{-1}]$. Let $\operatorname{Vect}^{\varphi,\mathrm{an}}((X, M_X)_{\Delta})$ denote the category of analytic prismatic F-crystals on (X, M_X).

One can associate to an analytic prismatic F-crystal $(\mathcal{E}_{\Delta}, \varphi_{\mathcal{E}_{\Delta}})$ a Laurent F-crystal $(\mathcal{E}, \varphi_{\mathcal{E}})$ by considering $\mathcal{E}_{\Delta,A} \otimes A[I^{-1}]_p^\wedge$ for each $(A, I, M_{\operatorname{Spf}A}) \in (X, M_X)_{\Delta}$. In our semistable case, this functor $(\mathcal{E}_{\Delta}, \varphi_{\mathcal{E}_{\Delta}}) \mapsto (\mathcal{E}, \varphi_{\mathcal{E}})$ is fully faithful.

Theorem 4 (Bhatt–Scholze [2], Koshikawa–Yao [9])**.** *The category of Laurent F-crystals on (X, M_X) is equivalent to the category $\operatorname{Loc}_{\mathbf{Z}_p}(\mathcal{X})$ of étale $\mathbf{Z}_p$-local systems on the generic fiber $\mathcal{X}$. This is given by sending $(\mathcal{E}, \varphi_{\mathcal{E}})$ to the étale $\mathbf{Z}_p$-local system corresponding to the $\operatorname{Gal}(\overline{R}[p^{-1}]/R[p^{-1}])$-module $\mathcal{E}(A_{\inf}(\overline{R}))^{\varphi_{\mathcal{E}}=1}$.*

In particular, we get a fully faithful functor (called the *étale realization*)

$$T\colon \operatorname{Vect}^{\varphi,\mathrm{an}}((X, M_X)_{\Delta}) \to \operatorname{Loc}_{\mathbf{Z}_p}(\mathcal{X}).$$

This functor relates prismatic F-crystals on $\mathcal{O}_K$ to crystalline or semistable Galois representations of K as in (1) and (2) below:

Theorem 5.

(1) (Bhatt–Scholze [2]) *When $X = \operatorname{Spf}\mathcal{O}_K$ with trivial log structure, T induces an equivalence* $\operatorname{Vect}^{\varphi,\mathrm{an}}((X, M_X)_{\Delta}) \xrightarrow{\cong} \operatorname{Rep}_{\mathbf{Z}_p}^{\mathrm{cris}}(\operatorname{Gal}_K)$.

(2) (Du–Liu [3]) *When $X = \operatorname{Spf}\mathcal{O}_K$ with canonical log structure, T induces an equivalence* $\operatorname{Vect}^{\varphi,\mathrm{an}}((X, M_X)_{\Delta}) \xrightarrow{\cong} \operatorname{Rep}_{\mathbf{Z}_p}^{\mathrm{st}}(\operatorname{Gal}_K)$.

(3) (Du–Liu–Moon–S. [4], Guo–Reinecke [6]) *When X is smooth over $\mathcal{O}_K$ with trivial log structure, T induces* $\operatorname{Vect}^{\varphi,\mathrm{an}}((X, M_X)_{\Delta}) \xrightarrow{\cong} \operatorname{Loc}_{\mathbf{Z}_p}^{\mathrm{cris}}(\mathcal{X})$.

Let us go back to the semistable case (X, M_X) with $X = \operatorname{Spf} R$. The above results suggest a prismatic way to define semistable local systems on $\mathcal{X}$. Namely, we say that an étale $\mathbf{Z}_p$-local system on $\mathcal{X}$ is X*-semistable* if it is in the essential image of T. For each $1 \leq i \leq m$, set $\Xi_i = \operatorname{Spf} \mathcal{O}_{\mathcal{K}_i}$ with canonical log structure: it admits a morphism $(\Xi_i, M_{\Xi_i}) \to (X, M_X)$.

Theorem 6 (Prismatic purity [5])**.** *A Laurent F-crystal $(\mathcal{E}, \varphi_{\mathcal{E}})$ on (X, M_X) extends to an analytic prismatic F-crystal if and only if the Laurent F-crystal $(\mathcal{E}, \varphi_{\mathcal{E}})|_{\Xi_i}$ on (Ξ_i, M_{Ξ_i}) extends to an analytic prismatic F-crystal for each i.*

The main strategy of the proof is to describe these sheaves in terms of the values at the Breuil–Kisin log prism and its self-products.

Theorem 6 allows one to compare different notions of semistable $\mathbf{Z}_p$-local systems on $\mathcal{X}$. One can show that an étale $\mathbf{Z}_p$-local system on $\mathcal{X}$ is X-semistable if and only if it is associated to an F-isocrystal on the log crystalline site of the mod p fiber of X (we omit the precise formulation here): the prismatic purity reduces the assertion to the CDVR case, which is relatively easy to verify. Moreover, we can also prove that if $\mathcal{X}$ admits two semistable formal models X and X', then X-semistability is equivalent to X'-semistability. So we refer to X-semistability as *semistability* and obtain a reasonable notion of étale semistable $\mathbf{Z}_p$-local systems on a rigid-analytic variety that admits a semistable formal model. Now Theorem 1 follows from Theorem 6. Finally, we note that a very recent work of Guo and Yang [7] establishes a further equivalent condition using the above purity result.

References

[1] Bhargav Bhatt and Peter Scholze, *Prisms and prismatic cohomology*, Ann. of Math. (2) **196** (2022), 1135–1275.

[2] ———, *Prismatic F-crystals and crystalline Galois representations*, Camb. J. Math. **11** (2023), no. 2, 507–562.

[3] Heng Du and Tong Liu, *A prismatic approach to $(\varphi, \hat{G})$-modules and F-crystals*, 2023, arXiv:2107.12240v3, to appear in J. Eur. Math. Soc.

[4] Heng Du, Tong Liu, Yong Suk Moon, and Koji Shimizu, *Completed prismatic F-crystals and crystalline $\mathbf{Z}_p$-local systems*, Compos. Math. **160** (2024), 1101–1166.

[5] Heng Du, Tong Liu, Yong Suk Moon, and Koji Shimizu, *Log prismatic F-crystals and purity*, 2024, preprint, arXiv:2404.19603.

[6] Haoyang Guo and Emanuel Reinecke, *A prismatic approach to crystalline local systems*, Invent. Math. **236** (2024), 17–164.

[7] Haoyang Guo and Ziquan Yang, *Pointwise criteria of p-adic local systems*, 2024, preprint, arXiv:2409.19742.

[8] Teruhisa Koshikawa, *Logarithmic prismatic cohomology I*, 2022, preprint, arXiv:2007.14037v3.

[9] Teruhisa Koshikawa and Zijian Yao, *Logarithmic prismatic cohomology II*, 2023, preprint, arXiv:2306.00364.

Constructing holomorphic functions on universal covers of complex algebraic varieties

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(joint work with Benjamin Bakker and Jacob Tsimerman)

Starting from dimension 2, the class of complex manifolds that can be realized as the universal cover of a smooth projective complex algebraic variety is mysterious. More modestly, one can try to isolate some properties shared by all complex manifolds in this class. For example, does every non-compact complex manifold in this class admit a non constant holomorphic function? A more precise question was asked by Shafarevich [3, IX.4.3]: is the universal cover $\tilde{X}$ of a smooth projective complex variety X always holomorphically convex, i.e. does $\tilde{X}$ admit a proper holomorphic map to a Stein space?

Shafarevich question has inspired many works. Notably, using techniques from non-abelian Hodge theory and mixed Hodge theory, Eyssidieux, Katzarkov, Pantev and Ramachandran [2] were able to prove that a smooth projective complex variety admitting a faithful representation of its fundamental group has a holomorphically convex universal cover. However, there exist smooth projective complex varieties with a non-linear fundamental group (the first examples were constructed by Toledo [4]).

In my talk, I presented the following quasiprojective version of their result. Since the smooth quasi-projective complex variety obtained by removing a point to the complex projective plane is simply-connected but not holomorphically convex, one needs to be careful when trying to generalize Shafarevich question to possibly non-compact complex varieties.

Theorem 1. *Let X be a connected normal complex algebraic space whose fundamental group admits a faithful finite-dimensional complex linear representation $\rho : \pi_1(X^{\mathrm{an}}) \to \mathrm{GL}_r(\mathbb{C})$. Then there is a partial compactification $X \subset \bar{X}$ by a connected normal Deligne–Mumford stack with isomorphic fundamental group such that the universal cover of $\bar{X}$ is a holomorphically convex complex space. In particular, the universal cover of X is a dense Zariski open subset of a holomorphically convex complex space.*

Allowing X to be quasiprojective makes the theory substantially more difficult due to the presence of the boundary. A key step in our proof is given by the following general existence result for so-called Shafarevich morphisms.

Theorem 2. *Let X be a connected normal complex algebraic space. Let $\rho : \pi_1(X^{\mathrm{an}}) \to \mathrm{GL}_r(\mathbb{C})$ be a nonextendable finite-dimensional complex linear representation with torsion-free image. Then there exists a unique surjective proper algebraic morphism with connected fibres $s\colon X \to Y$ such that for every morphism $g\colon Z \to X$ from a connected complex algebraic variety Z, the representation $g^*\rho$ is trivial if and only if the composition $Z \to X \to Y$ is constant.*

The nonextendability hypothesis says that ρ does not extend to any (strict) partial compactification of X by a connected normal Deligne-Mumford stack. The assumptions that ρ is nonextendable and has torsion-free image can be both removed, at the price of making the statement slightly more technical.

A salient feature of our approach is the use of o-minimal geometry (and in particular of our o-minimal GAGA result [1]) to algebraize holomorphic maps between (possibly non-compact) complex algebraic spaces.

References

[1] B. Bakker, Y. Brunebarbe, J. Tsimerman, *o-minimal GAGA and a conjecture of Griffiths*, Invent. Math. **232** (2023), 163–228.

[2] P. Eyssidieux, L. Katzarkov, T. Pantev, M. Ramachandran, *Linear Shafarevich conjecture*, Ann. of Math. (2) **176** (2012), 1545–1581.

[3] I. Shafarevich, *Basic algebraic geometry. 2*, Springer.

[4] D. Toledo, *Projective varieties with non-residually finite fundamental group*, Publications mathématiques de l'I.H.É.S., tome 77 (1993), 103–119.

Condensed Shape of a Scheme

Catrin Mair

(joint work with Peter Haine, Tim Holzschuh, Marcin Lara, Louis Martini, Sebastian Wolf)

A fundamental notion in classical homotopy theory is the *etale homotopy type* of a (locally noetherian or, more generally, locally connected) scheme. Originally, it was introduced by Artin-Mazur [1] and later refined by Friedlander [2] to the *etale topological type*. These constructions aim to attach to a scheme X a homotopy theoretical invariant recovering the (profinite) *etale fundamental group*

$$\pi_1^{\mathrm{et}}(X, x)$$

introduced by Grothendieck and providing a definition of higher etale homotopy groups. Indeed, the etale topological (resp. homotopy) type is realised as a pro-object in (the homotopy category of) the category of simplicial sets and admits the *extended etale homotopy groups*

$$\pi_i^{\mathrm{et}}(X, x)$$

as homotopy pro-groups. A modern variant of the etale homotopy type arises in the world of ∞-categories via a universal construction in *shape theory*: For every ∞-topos T there exists a (up to a contractible choice) unique geometric morphism

$$T \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathrm{Ani}$$

to the terminal ∞-topos Ani of anima (also referred to as spaces, ∞-groupoids, Kan complexes or homotopy types). Hereby, f^* is the left exact left adjoint, which, in general, does not preserve cofiltered limits. Uniquely extending f^* along cofiltered limits leads to a limit preserving functor

$$\mathrm{Pro}(f^*)\colon \mathrm{Pro}(\mathrm{Ani}) \longrightarrow T,$$

which admits a left adjoint

$$f_!\colon T \longrightarrow \mathrm{Pro}(\mathrm{Ani}).$$

Then, the *shape of* T is defined as the image of the terminal object in T under this left adjoint

$$\Pi_\infty(T) \coloneqq f_!(*_T).$$

Applying this construction to the ∞-topos of hypercomplete sheaves of anima $\mathcal{S}\mathrm{h}_\infty^{\mathrm{hyp}}(X_{\mathrm{et}})$ on the etale site X_{et} of a scheme X defines the *etale shape*

$$\Pi_\infty^{\mathrm{et}}(X) \in \mathrm{Pro}(\mathrm{Ani}).$$

Hoyois [3] has shown that the etale shape is in line with the classical constructions by Artin-Mazur-Friedlander.

More recently, Bhatt-Scholze [4] introduced a refined version of the etale fundamental group via the pro-etale topology: The pro-etale fundamental group

$$\pi_1^{\mathrm{proet}}(X, x) \in \mathrm{TopGrps}$$

of a (locally topologically noetherian) scheme X is a topological group whose topology is (in general) finer than just profinite. Particularly, it is a Noohi-group. These are topological groups G that are already determined by the category $G-$ Set of discrete sets with a continuous action by G. For geometrically unibranch schemes, it exactly recovers the profinite etale fundamental group. In contrary to the etale version, the pro-etale fundamental group of any scheme cannot be regained via the classical notion of shape as its refined topological structure is not seen by pro-objects. This fact motivates the involvement of *condensed mathematics* into homotopy theory of schemes as it constitutes a suitable framework for dealing with algebraic structures that carry a topology.

The world of condensed mathematics allows to think of a homotopy type of a scheme, which we refer to as *condensed shape*, carrying additional information in a topological sense. This is reflected by the fact that from such a homotopy type we cannot only recover the pro-etale fundamental group, but we can even define higher homotopy groups inside the condensed world. The condensed shape is realised as an object in the ∞-category Cond(Ani) of *condensed anima*, in which the topological direction of *condensed sets* and the homotopy theoretical direction of anima are combined.

$$\begin{tikzcd}
\mathrm{Cond}(\mathrm{Set}) \arrow[dr, hook, "\text{topological direction}"'] & & \mathrm{Ani} \arrow[dl, hook', "\text{homotopy direction}"] \\
& \mathrm{Cond}(\mathrm{Ani}) &
\end{tikzcd}$$

There are different approaches to define the condensed shape of a (qcqs) scheme:

1. **Relative Shape Theory**
   This approach imitates the construction of shape relative to the new base ∞-topos Cond(Ani) using that the hypercomplete pro-etale ∞-topos $\mathcal{S}\mathrm{h}^{\mathrm{hyp}}_{\infty}(X_{\mathrm{proet}})$ is equipped with a canonical geometric morphism mapping from Cond(Ani) and admitting an additional left adjoint.
2. **Condensed Classifying Space**
   This approach defines the condensed shape as condensed classifying space of the *Galois category*

   $$\mathrm{Gal}(X) \in \mathrm{Cond}(\mathrm{Cat}),$$

   a condensed version of the category of points the etale ∞-topos introduced by Barwick-Haine-Glasman [5], by means of levelwise inversion of morphisms.

For any qcqs scheme X, both approaches result in the same condensed shape

$$\Pi^{\mathrm{cond}}_{\infty}(X) \in \mathrm{Cond}(\mathrm{Ani}).$$

One can extract different homotopy theoretical objects from the condensed shape.

$$\begin{tikzcd} & \mathrm{Cond}(\mathrm{Ani}) \arrow[dl, "\text{condensed homotopy groups}"'] \arrow[dr, "\text{pro-homotopy types}"] & \\ \mathrm{Cond}(\mathrm{Grps}) & & \mathrm{Pro}(\mathrm{Ani}) \end{tikzcd}$$

Proposition. For any scheme X, such that the subsequent notions are defined, the condensed shape $\Pi^{\mathrm{cond}}_{\infty}(X)$ recovers

i) the etale homotopy type $\Pi^{\mathrm{et}}_{\infty}(X) \in \mathrm{Pro}(\mathrm{Ani})$ via the pro-homotopy type, up to protruncation.
ii) the pro-etale fundamental group $\pi_1^{\mathrm{proet}}(X, x) \in \mathrm{TopGrps}$ via the condensed fundamental group, up to Noohi completion.

In the joint project we examine further properties of the condensed shape and give concrete computations.

Example. Let $X = \mathrm{Spec}(A)$ be an affine scheme.

a) If A is a w-contractible ring, then

$$\Pi^{\mathrm{cond}}_{\infty}(X) = \pi_0(X) \in \mathrm{Pro}(\mathrm{FinSets}) \subset \mathrm{Cond}(\mathrm{Ani})$$

is the extremally disconnected profinite set of connected components.

b) If $A = k$ is a field, then

$$\Pi^{\mathrm{cond}}_{\infty}(X) = BG_k \in \mathrm{Ani} \subset \mathrm{Cond}(\mathrm{Ani})$$

is the classifying space of the absolute Galois group G_k of k.

Proposition. If X is a scheme in one of the following classes, the condensed shape of X is trivial

$$\Pi^{\mathrm{cond}}_{\infty}(X) = *.$$

i) X is the spectrum of a strictly henselian local ring $\iff$ Gal(X) has an initial object.

ii) X is everywhere strictly local and irreducible $\iff$ Gal(X) has a terminal object.

References

[1] M. Artin, B. Mazur *Etale homotopy*, Springer Lecture Notes in Mathematics **100** (1969), Berlin

[2] E. Friedlander *Etale homotopy of simplicial schemes*, Annals of Mathematics Studies **104** (1982), Princeton University Press, Princeton, N.J.

[3] M. Hoyois *Higher Galois theory*, Journal of Pure and Applied Algebra **222, Issue 7** 1859-1877 (2018)

[4] B. Bhatt, P. Scholze *The pro-etale topology of schemes*, Astérisque **369** 99–201 (2015)

[5] C. Barwick, P. Haine, S. Glasman *Exodromy*, arxiv **1807.03281** (2020), https://arxiv.org/abs/1807.03281

Point objects on abelian varieties

Martin Olsson

(joint work with A.J. de Jong)

For a noetherian scheme X let $D(X)$ denote the bounded derived category of coherent sheaves on X.

Classical work of Mukai, Bondal, and Orlov shows that one can have two smooth projective nonisomorphic varieties X and Y over an algebraically closed field k with equivalent derived categories $D(X) \simeq D(Y)$ (as k-linear triangulated categories). That said, there are many instances where one knows that $D(X)$ determines the variety X. Most notably, Bondal and Orlov [1] showed that if the canonical bundle ω_X is either ample or anti-ample then $D(X)$ determines X. Their argument can be viewed as a type of "reconstruction" result. They characterize the skyscraper sheaves of points of X as certain *point objects* of $D(X)$, thereby showing how to rebuild X from $D(X)$.

It is natural to try to understand the point objects of $D(X)$ for other varieties X. Let k be a field, let A/k be an abelian variety of dimension d, and let X be an A-torsor. We say that an object $K \in D(X)$ is a *point object* if the following conditions hold:

(1) $\mathrm{Ext}^i(K,K) = 0$ for $i < 0$.

(2) The natural map $k \to \mathrm{Ext}^0(K,K)$ is an isomorphism and the k-dimension of $\mathrm{Ext}^1(K,K)$ is $\leq d$.

Theorem 1. *If K is a point object on X then $K \simeq i_*\mathscr{E}[r]$, where $i : Y \hookrightarrow X$ is a torsor under a subabelian variety $B \subset A$, $\mathscr{E}$ is a simple semihomogenous vector bundle on Y, and r is an integer. Conversely, such a sheaf on X is a point object.*

Recall from [3] that when k is algebraically closed a vector bundle $\mathscr{E}$ on a torsor X for an abelian variety A is called *semi-homogeneous* if for every point $a \in A(k)$ there exists a line bundle $\mathscr{L}$ on X such that $t_a^*\mathscr{E} \simeq \mathscr{E} \otimes \mathscr{L}$. If k is not algebraically closed we say that a vector bundle $\mathscr{E}$ is semi-homogeneous if it becomes so after base change to an algebraic closure.

This theorem also gives a complete description of all the Fourier-Mukai partners of X; that is, a classification of smooth projective varieties X'/k with $D(X) \simeq D(X')$. Namely, they are all given as moduli spaces for simple semi-homogeneous vector bundles on subtorsors $Y \subset X$ for a subabelian variety $B \subset A$.

References

[1] A. Bondal and D. Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalences*, Compositio Math. **125** (2001), no. 3, 327–344. MR 1818984

[2] M. G. Gulbrandsen, *Fourier-Mukai transforms of line bundles on derived equivalent abelian varieties*, Matematiche (Catania) **63** (2008), no. 1, 123–137. MR 2512513

[3] S. Mukai, *Semi-homogeneous vector bundles on an Abelian variety*, J. Math. Kyoto Univ. **18** (1978), no. 2, 239–272. MR 498572

[4] Shigeru Mukai, *Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves*, Nagoya Math. J. **81** (1981), 153–175. MR 607081

[5] ———, *Lagrangian-invariant sheaves and functors for abelian varieties*, Derived categories in algebraic geometry, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012, pp. 197–250. MR 3050705

[6] ———, *Phases of Lagrangian-invariant objects in the derived category of an abelian variety*, Kyoto J. Math. **54** (2014), no. 2, 427–482. MR 3215574

Characteristic classes of étale local systems

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(joint work with Lue Pan)

To a local system $\mathbb{L}$ of $\mathbb{C}$-vector spaces on a smooth manifold M one can attach Cheeger-Chern-Simons characteristic classes $\widehat{c}_i(\mathbb{L}) \in H^{2i-1}(M, \mathbb{C}/\mathbb{Z})$ (cf. [1, Théorème 1]). They refine Chern classes of the complex vector bundle on M associated to $\mathbb{L}$: the image of $\widehat{c}_i(\mathbb{L})$ under the connecting homomorphism $H^{2i-1}(M, \mathbb{C}/\mathbb{Z}) \to H^{2i}(M, \mathbb{Z})$ is equal to the class $c_i(\mathbb{L} \otimes_{\underline{\mathbb{R}}} \mathcal{O}_M)$.

The data of a rank n local system $\mathbb{L}$ is equivalent to the data of a representation $\rho_{\mathbb{L}} \colon \pi_1(M) \to GL_n(\mathbb{C})$ of the fundamental group (if M is connected), and class $\widehat{c}_i(\mathbb{L})$ arises as the image of the universal class $\widehat{c}_i \in H^{2i-1}_{\mathrm{grp}}(GL_n(\mathbb{C}), \mathbb{C}/\mathbb{Z})$ in group cohomology under the map $H^{2i-1}_{\mathrm{grp}}(GL_n(\mathbb{C}), \mathbb{C}/\mathbb{Z}) \xrightarrow{\rho_{\mathbb{L}}^*} H^{2i-1}(M, \mathbb{C}/\mathbb{Z})$.

We investigate a p-adic analog of this theory. A crucial difference in the scope of it is that local systems with pro-finite coefficients can be considered not only on manifolds or topological spaces, but also on arithmetic objects such as varieties over non-algebraically closed fields.

For a connected scheme X consider an étale $\mathbb{Z}_p$-local system $\mathbb{L}$ of rank n on X. The data of $\mathbb{L}$ is equivalent to the data of a continuous representation $\rho_{\mathbb{L}}$: $\pi_1^{\text{ét}}(X) \to GL_n(\mathbb{Z}_p)$ of the étale fundamental group of X. This representation defines a map from the continuous cohomology of the group $GL_n(\mathbb{Z}_p)$ to the étale cohomology of X:

$$\rho_{\mathbb{L}}^* : H^{\bullet}_{\mathrm{cont}}(GL_n(\mathbb{Z}_p), \mathbb{Q}_p) \to H^{\bullet}_{\text{ét}}(X, \mathbb{Q}_p)$$

By a theorem of Lazard [2, Théorème V.2.4.9] continuous cohomology of $GL_n(\mathbb{Z}_p)$ is the free exterior algebra $\Lambda^{\bullet}_{\mathbb{Q}_p}(\ell_1, \ldots, \ell_n)$ on n generators in degrees $\deg \ell_i = 2i-1$.

Definition. *Characteristic classes $\ell_i(\mathbb{L}) \in H^{2i-1}_{\text{ét}}(X, \mathbb{Q}_p)$ of a local system $\mathbb{L}$ on X are defined as the images of the classes ℓ_i under the map $\rho^*_{\mathbb{L}}$.*

This definition was introduced by Pappas [3, 4.4.2], and closely related constructions of characteristic classes of Galois representations have been considered previously by Kim [4].

The degree 1 class $\ell_1(\mathbb{L}) \in H^1_{\text{ét}}(X, \mathbb{Q}_p)$ is simply the result of applying the p-adic logarithm map $\mathbb{Z}_p^\times \to \mathbb{Q}_p$ to the determinant $\det \rho_{\mathbb{L}} \in H^1_{\text{ét}}(X, \mathbb{Z}_p^\times)$ of the representation $\rho_{\mathbb{L}}$. Our first main result is a partial calculation of characteristic classes for $\mathbb{Z}_p$-local systems on varieties over $\mathbb{Q}_p$:

Theorem 1. *Let X be a smooth proper geometrically connected variety over $\mathbb{Q}_p$ of dimension d. For a Hodge-Tate $\mathbb{Z}_p$-local system $\mathbb{L}$ on X its top degree characteristic class $\ell_{d+1}(\mathbb{L}) \in H^{2d+1}_{\text{ét}}(X, \mathbb{Q}_p) \simeq H^1(G_{\mathbb{Q}_p}, H^{2d}_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)) \simeq H^1(G_{\mathbb{Q}_p}, \mathbb{Q}_p(-d)) \simeq \mathbb{Q}_p$ is equal to the following integer:*

$$d! \sum_{m \in \mathbb{Z}} m \cdot \mathrm{ch}_d(\mathrm{gr}^m D_{\mathrm{HT}}(\mathbb{L})) \in \mathbb{Z} \subset \mathbb{Q}_p$$

where $D_{\mathrm{HT}}(\mathbb{L}) \simeq \bigoplus_m \mathrm{gr}^m D_{\mathrm{HT}}(\mathbb{L})$ is the graded Higgs bundle associated to $\mathbb{L}$, and $\mathrm{ch}_d(E) \in \frac{1}{d!}\mathbb{Z}$ for a vector bundle E denote its top degree Chern character.

One source of Hodge-Tate local systems is cohomology of families of varieties: for any smooth proper morphism $f: Y \to X$ the local system $\mathbb{L} = R^i f_* \mathbb{Z}_p$ of relative étale cohomology is Hodge-Tate with $D_{\mathrm{HT}}(\mathbb{L}) \simeq \bigoplus_{m \geq 0} R^{i-m} f_* \Omega^m_{Y/X}$. On the contrary, for local systems on varieties over an algebraically closed field the characteristic classes are zero in degrees > 1:

Theorem 2. *Let X be a smooth variety over an algebraically closed field $k = \overline{k}$ of characteristic zero. For a fixed rank n there exists a constant $c(n)$ such that for all primes $p > c(n)$ the class $\ell_i(\mathbb{L}) \in H^{2i-1}_{\text{ét}}(X, \mathbb{Q}_p)$ vanishes for $i > 1$ for all $\mathbb{Z}_p$-local systems $\mathbb{L}$ of rank n.*

This is a p-adic analog of a result of Reznikov [5] asserting that characteristic classes $\widehat{c}_i$ of all complex local systems on a smooth proper algebraic variety X over $\mathbb{C}$ vanish in $H^{2i-1}(X(\mathbb{C}), \mathbb{C}/\mathbb{Q})$ for $i > 1$.

The proof of Theorem 1 relies on the notion of Chern classes for pro-étale vector bundles on X that we introduce. Given a $\mathbb{Z}_p$-local system on a rigid-analytic variety X over a p-adic field K we can form the associated pro-étale vector bundle $\mathbb{L} \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_X$ on the pro-étale site of X. As a consequence of the work of Huber-Kings [6], we prove that characteristic classes of $\mathbb{L}$ are related to Chern classes $c_i(\mathbb{L} \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_X) \in H^{2i}_{\text{ét}}(X, \mathbb{Q}_p(i))$ of the corresponding pro-étale vector bundle via the formula:

$$c_i(\mathbb{L} \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_X) = \ell_i(\mathbb{L}) \cdot \kappa_i$$

where $\kappa_i \in H^1(G_{\mathbb{Q}_p}, \mathbb{Q}_p(i))$, for each $i \geq 0$, is a certain class in Galois cohomology (independent of X and $\mathbb{L}$). It can be described explicitly as the image of $(-1)^i \in \mathbb{Q}_p = (B^+_{\mathrm{dR}}/t^i B^+_{\mathrm{dR}})^{G_{\mathbb{Q}_p}}$ under the Bloch-Kato exponential map, which is the connecting homomorphism arising from the exact sequence of Galois modules $0 \to \mathbb{Q}_p(i) \to B_{\mathrm{cris}}^{+,\varphi=p^i} \to B_{\mathrm{dR}}^+/t^i B_{\mathrm{dR}}^+ \to 0$.

For a Hodge-Tate local system $\mathbb{L}$, classes $c_i(\mathbb{L} \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_X)$ can be calculated using the Hodge-Tate filtration on this pro-étale vector bundle. In the setting of Theorem 1 the class $\ell_{d+1}(\mathbb{L})$ can be recovered from the product $\ell_{d+1}(\mathbb{L}) \cdot \kappa_{d+1}$. However, in general more information than just the Chern classes of $\mathbb{L} \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_X$ is needed to recover the classes $\ell_i(\mathbb{L})$. One could ask if an analog of Theorem 1 nonetheless holds in the following sense.

For a smooth algebraic variety X over a finite extension K of $\mathbb{Q}_p$ we have a natural map

$$\alpha_X : H^n_{\text{ét}}(X, \mathbb{Q}_p) \to H^1(G_K, H^{n-1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}) \simeq H^{n-1}_{\mathrm{dR}}(X/K)$$

If X has good reduction over $\mathcal{O}_K$, this map is an isomorphism for $n > 1$.

Question. For a Hodge-Tate local system $\mathbb{L}$ on X, is it true that the image of the characteristic class $\ell_i(\mathbb{L}) \in H^{2i-1}_{\text{ét}}(X, \mathbb{Q}_p)$ under the map α_X equals

$$(i-1)! \sum_{m \in \mathbb{Z}} m \cdot \mathrm{ch}_{i-1}(\mathrm{gr}^m D_{\mathrm{HT}}(\mathbb{L}))?$$

Here ch_{i-1} denotes the degree $2(i-1)$ component of the Chern character in de Rham cohomology.

References

[1] C. Soulé, Classes caractéristiques secondaires des fibrés plats, Astérisque No. 241 (1997), Exp. No. 819, 5, 411–424
[2] M. Lazard, Groupes analytiques p-adiques, Inst. Hautes Études Sci. Publ. Math. No. 26 (1965), 389–603
[3] G. Pappas, Volume and symplectic structure for ℓ-adic local systems, Adv. Math. **387** (2021), Paper No. 107836, 70 pp.
[4] M. Kim, Arithmetic Chern-Simons theory I, in *Galois covers, Grothendieck-Teichmüller Theory and Dessins d'Enfants*, 155–180, Springer Proc. Math. Stat., 330, Springer, Cham,
[5] A. G. Reznikov, All regulators of flat bundles are torsion, Ann. of Math. (2) **141** (1995), no. 2, 373–386
[6] A. Huber-Klawitter and G. Kings, A p-adic analogue of the Borel regulator and the Bloch-Kato exponential map, J. Inst. Math. Jussieu **10** (2011), no. 1, 149–190

Convergent Decomposition Groups and an $\mathfrak{S}$-adic Shafarevich Conjecture

Andrew Kwon

Ax [1] initiated the study of the theory of pseudo-algebraically closed (PAC) fields: we say k is PAC if every geometrically irreducible variety X over k has a k-rational point. One may think of this as a "trivial" local-global principle, where the local condition is vacuous. This was generalized to pseudo-real closed (P$\mathbb{R}$C) fields by Prestel [11], Basarab [2], Haran–Jarden [5], Ershov [4], where the local condition that guarantees a k-point on X is the presence of points over all real closures of k. The case of considering p-adic closures, hence studying the theory of pseudo-p closed (PpC) fields, was done by Haran–Jarden [6], Efrat–Jarden [3]. Heinemann-Prestel [7] also introduced pseudo-$\mathcal{L}$ closed fields, where $\mathcal{L}$ is a finite set of localities, i.e., real or p-adic closures, of k. All these results were unified by Pop in [9] by generalizing the theory of pseudo-$\mathcal{L}$ closed fields to the case where $\mathcal{L}$ is a quasicompact set (in the so-called *étale topology*) of localities of k. Specializing to the case $\mathcal{L} = \emptyset$ recovers the theory of PAC fields. If $\mathcal{L}$ is instead taken to be the quasicompact set of all real (resp. p-adic) closures then one gets the P$\mathbb{R}$C (resp. PpC) fields; and evidently any finite set is quasicompact.

To briefly summarize the consequences for the Galois theory of these fields: if k is PAC, then the absolute Galois group G_k of k is projective [1], i.e., G_k has solutions for every embedding problem. Appropriate notions of real, resp. p-adically projective groups were formulated and it was shown that if k is pseudo-real closed, then G_k is real projective, and similarly for PpC fields [5, 6]. More generally, if k is pseudo-$\mathcal{L}$ closed ($\mathcal{L}$ quasicompact), then G_k is $\mathcal{G}_{\mathcal{L}}$-projective, where $\mathcal{G}_{\mathcal{L}}$ is the set of decomposition groups corresponding to the localities in $\mathcal{L}$ [9]. These results essentially say that local-global principles for points induce local-global principles for embedding problems.

This general abstract theory was made concrete by work of Rumely [12], Moret-Bailly [8] and Pop [10], who finally gave many explicit examples of pseudo-$\mathcal{L}$ closed fields as follows. Let k denote a global field and $\mathfrak{S}$ a finite set of places of k. The field of totally $\mathfrak{S}$-adic numbers is the maximal (Galois) extension of k where all primes of $\mathfrak{S}$ are totally split, denoted by $k^{\mathfrak{S}}$. If $\mathcal{L}^{\mathfrak{p}}$ denotes the set of all $\mathfrak{p}$-adic closures of k and $\mathcal{L}^{\mathfrak{S}} = \cup_{\mathfrak{p}\in\mathfrak{S}}\mathcal{L}^{\mathfrak{p}}$, then Pop showed that $k^{\mathfrak{S}}$ is pseudo-$\mathcal{L}^{\mathfrak{S}}$ closed. Concretely, this is equivalent to saying that for every smooth, geometrically integral k-variety X, one has

$$X(k^{\mathfrak{S}}) \neq \emptyset \Leftrightarrow \text{for all } \mathfrak{p} \in \mathfrak{S}, X(k_{\mathfrak{p}}) \neq \emptyset,$$

where $k_{\mathfrak{p}}$ is the completion at $\mathfrak{p}$.

In this talk, I extend the previous results by considering infinite sets of places $\mathfrak{S}$ whose *decomposition groups converge to 1*. That is, for every finite Galois extension $K|k$, all but finitely many $\mathfrak{p} \in \mathfrak{S}$ are totally split in $K|k$, or equivalently, the decomposition groups $D_{\mathfrak{p}}$ tend to 1 (in the space of closed subgroups of G_k) as $|\mathfrak{p}| \to \infty$. Precisely, let $\mathfrak{S}_0$ be any nonempty finite set of primes and $\mathfrak{S}' \supset \mathfrak{S}_0$ be any infinite set of primes whose decomposition groups converge to 1. Then there are "many" infinite subsets $\mathfrak{S} \subset \mathfrak{S}'$ such that $k^{\mathfrak{S}}$ satisfies a local-global principle for rational points. We note that in this setting, $\mathcal{L}^{\mathfrak{S}}$ is a union of infinitely many $\mathcal{L}^{\mathfrak{p}}$ that is still quasicompact because of the convergence property of $\mathfrak{S}$.

One of the main motivations for this direction of inquiry is that the fields $k^{\mathfrak{S},\mathrm{cyc}}$ "approximate" k^{cyc} as $\mathfrak{S}$ ranges over larger sets of primes, hence one gets an "approximation" of absolute Galois groups as well. The following is new evidence for the freeness of $G_{k^{\mathrm{cyc}}}$ that was previously known only for finite sets of primes.

Theorem 1. *With $\mathfrak{S}$ as above, the absolute Galois group $G_{k^{\mathfrak{S},\mathrm{cyc}}}$ of the maximal cyclotomic extension $k^{\mathfrak{S},\mathrm{cyc}}$ of $k^{\mathfrak{S}}$ is profinite free of countably infinite rank.*

References

[1] Ax, J., *The elementary theory of finite fields,* Ann. of Math. **88** (1968), 239–271.
[2] Basarab, Ş., *Axioms for pseudo-real-closed fields,* Rev. Roumaine Math. Pures Appl. **29** (1984), 449–456.
[3] Efrat, I. and Jarden, M., *Free pseudo p-adically closed fields of finite corank,* J. Algebra **133** (1990), 132–150.
[4] Ershov, Y., *PRC-fields with small absolute Galois groups,* Ann. Pure Appl. Logic **43** (1989), 197–208.
[5] Haran, D. and Jarden, M., *The absolute Galois group of a pseudo real closed field,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. **12** (1985), 449–489.
[6] Haran, D. and Jarden, M., *The absolute Galois group of a pseudo p-adically closed field,* J. Reine Angew. Math. **383** (1988), 147–206.
[7] Heinemann, B. and Prestel, A., *Fields regularly closed with respect to finitely many valuations and orderings,* CMS Conf. Proc., **4** (1984), 297–336.
[8] Moret-Bailly, L., *Groupes de Picard et problèmes de Skolem,* Ann. Sci. École Norm. Sup. **22** (1989), 161–179, 181–194.
[9] Pop, F., *Classically projective groups and pseudo classically closed fields,* preprint, Heidelberg, 1990.
[10] Pop, F., *Embedding problems over large fields,* Ann. of Math. **144** (1996), 1–34.
[11] Prestel, A., *Set theory and model theory,* Lect. Notes in Math., **872** (1981), 127–156.
[12] Rumely, R., *Arithmetic over the ring of all algebraic integers,* J. Reine Angew. Math. **368** (1986), 127–133.

Arithmetic Finiteness and Big Monodromy on Abelian Varieties

Thomas Krämer
(joint work with Ariyan Javanpeykar, Christian Lehn and Marco Maculan)

At the ICM in 1962, Shafarevich conjectured that over any number field there are only finitely many isomorphism classes of smooth projective curves of fixed genus > 1 with good reduction outside a given finite set of primes. This was proven by Faltings in 1983 together with the corresponding result for abelian varieties and the Mordell conjecture. In fact such finiteness results are expected in much greater generality: The Lang-Vojta conjecture, one of the major open problems in Diophantine geometry, predicts the finiteness of integral points on any hyperbolic variety over a number field. By the work of Campana-Păun, moduli spaces of canonically polarized varieties are hyperbolic, so the Shafarevich conjecture for a given class of canonically polarized varieties is equivalent to the Lang-Vojta conjecture for their moduli space. Even 40 years later, the Shafarevich conjecture is known only in very few cases beyond curves and abelian varieties, such as for K3 and del Pezzo surfaces, certain hyperkähler varieties or flag varieties. In the talk I discussed the following new class of varieties of general type:

Definition. *A smooth projective variety over a field K is* very irregular *if the Albanese morphism $X_{\overline{K}} \to \mathrm{Alb}(X_{\overline{K}})$ for any chosen base point over the algebraic closure $\overline{K}$ is a closed immersion with non-zero ample normal bundle.*

Very irregular varieties form a very large class of canonically polarized varieties that includes smooth projective curves of genus > 1, smooth complete intersections of ample divisors on abelian varieties, and plenty of other examples. In joint work with Maculan [5], we prove the Lang-Vojta conjecture for their moduli spaces:

Theorem 1. *Over any number field K, there are up to isomorphism only finitely many very irregular varieties X with*

- *good reduction outside a given finite set of places of K,*
- *given Hilbert polynomial with respect to the canonical bundle ω_X,*
- $h^0(X, \Omega^1_X) \geq 2\dim X + 2$ *(plus some mild numerical conditions that hold in all known cases, for the precise formulation we refer to loc. cit.).*

This is the first instance of a finiteness result in higher dimension that holds for such a large class of varieties. An analogous result for smooth ample hypersurfaces on abelian varieties was proven by Lawrence and Sawin [8], and as in their paper we use the Lawrence-Venkatesh method [9]: We establish a general criterion for the nondensity of integral points on varieties with a local system of geometric origin that has big monodromy. In our case the relevant local systems arise from a family of subvarieties of a fixed abelian variety as follows:

Let S be a smooth complex variety. Let A be a g-dimensional complex abelian variety and $\mathcal{X} \subset A_S$ a subvariety such that the projection $f\colon \mathcal{X} \to S$ is a smooth morphism with connected fibers of dimension d. For a character $\chi\colon \pi_1(A, 0) \to \mathbb{C}^\times$ let L_χ be the associated local system on A. For any tuple $\underline{\chi} = (\chi_1, \ldots, \chi_n)$ of such characters, we get a local system

$$V_{\underline{\chi}} \;:=\; R^d f_* \mathrm{pr}_A^*(L_{\chi_1} \oplus \cdots \oplus L_{\chi_n}) \;=\; \bigoplus_{i=1}^n R^d f_* \mathrm{pr}_A^*(L_{\chi_i})$$

on S. Its monodromy preserves the direct summands on the right hand side, and for symmetric subvarieties the monodromy of each summand lies in a symplectic or orthogonal group by Poincaré duality. In joint work with Javanpeykar, Lehn and Maculan we show that for generic $\underline{\chi}$ the local system $V_{\underline{\chi}}$ has *big monodromy* in the sense that its algebraic monodromy group is as big as possible given the above restrictions — in particular it is a product of classical groups acting on each summand via the standard representation [1]:

Theorem 2. *Suppose the geometric generic fiber $X := \mathcal{X}_{\bar\eta} \subset A_{S,\bar\eta}$ has ample normal bundle and dimension $d < (g-1)/4$. Then the following are equivalent:*

(a) X is nondivisible, not constant up to translation, not a symmetric power of a curve and not a product;

(b) $V_{\underline{\chi}}$ has big monodromy for most torsion n-tuples of characters $\underline{\chi}$.

Here *most* means that the claim holds outside a finite union of translates of linear subvarieties of the character variety $\mathrm{Char}(A^n)$, as usual in generic vanishing theory [6, 7]. The condition on the dimension can be relaxed to $d < (g-1)/2$ under very weak numerical conditions, and via the Lawrence-Venkatesh method we obtain theorem 1. The starting point for the proof of theorem 2 is the same as in [8]: We reduce the statement about monodromy groups to a statement for

Tannaka groups of perverse sheaves by an analogue of the theorem of the fixed part in Hodge theory. The relevant Tannaka groups arise as follows: For any complex abelian variety A the sum morphism endows the category of perverse sheaves with a convolution product, which makes it a neutral Tannakian category by [6]. In particular, for any subvariety $X \subset A$ the associated perverse intersection complex generates a semisimple neutral Tannakian subcategory. We fix a fiber functor ω on this category and denote by $G_{X,\omega}$ the corresponding Tannaka group. Theorem 2 is then obtained from the following result [1, 4]:

Theorem 3. *Let $X \subset A$ be a smooth subvariety with ample normal bundle and dimension $d < (g-1)/4$. Then the following are equivalent:*

- *(a) X is nondivisible, not a symmetric power of a curve and not a product;*
- *(b) The Tannaka group $G_{X,\omega}$ is big.*

The key ingredient of the proof is the general correspondence in [2, 3] between characteristic cycles of perverse sheaves and Weyl group orbits of weights for a maximal torus in the Tannaka group. This was already crucial for the case of divisors in [8], but for the high codimension subvarieties in theorem 3 the conormal geometry becomes much more involved. Using characteristic cycles, we show that the Tannaka group is simple and acts via a minuscule representation; a careful study of the conormal geometry also allows us to rule out wedge powers of the standard representation in type A_n. The only remaining non-big cases are spin representations and the minimal representations of E_6 and E_7, which we exclude by our numerical assumptions. It is known that the exceptional group E_6 does occur for the Fano surface on the intermediate Jacobian of a smooth cubic threefold, but here $d = 2$ and $g = 5$ which is beyond the dimension range of the previous theorem. In a recent work with Lehn and Maculan, we show that for $d < g/2$ this is the only source of examples [4]:

Theorem 4. *Let $X \subset A$ be a smooth irreducible subvariety with ample normal bundle and dimension $< g/2$. Then the following are equivalent:*

- *(a) $X \subset A$ is nondivisible with Tannaka group $G^*_{X,\omega} \simeq E_6$.*
- *(b) X is isomorphic to the Fano surface of lines on a smooth cubic threefold, and the canonical morphism $\mathrm{Alb}(X) \to A$ is an isogeny.*

For the proof we show that the Hodge decomposition on cohomology is defined by a cocharacter of the Tannaka group of complex Hodge modules in the sense of Sabbah-Schnell, and play off cocharacters of E_6 against Hodge number estimates for irregular varieties à la Lazarsfeld-Popa and Lombardi. These techniques are of a general nature and will be useful also in other situations: For instance, do there exist smooth subvarieties $X \subset A$ whose Tannaka group $G_{X,\omega}$ is of type E_7 or a spin group? How about subvarieties of smaller codimension in abelian varieties, or more generally varieties with a finite morphism to an abelian variety?

References

[1] A. Javanpeykar, T. Krämer, C. Lehn and M. Maculan, *The monodromy of families of subvarieties on abelian varieties*, Duke Math. J. (to appear), `arxiv.org/abs/2210.05166`.

[2] T. Krämer, *Characteristic cycles and the microlocal geometry of the Gauss map, I.* Ann. Sci. ENS **55** (2022) 1475-1527.

[3] T. Krämer, *Characteristic cycles and the microlocal geometry of the Gauss map, II.* J. reine angew. Math. **774** (2021) 53-92.

[4] T. Krämer, C. Lehn and M. Maculan, *The Tannaka group E_6 only arises from cubic threefolds*, `arxiv.org/abs/2406.07401`.

[5] T. Krämer and M. Maculan, *Arithmetic finiteness of very irregular varieties*, `arxiv.org/abs/2310.08485`.

[6] T. Krämer and R. Weissauer, *Vanishing theorems for constructible sheaves on abelian varieties.* J. Alg. Geom. **24** (2015) 531-568.

[7] C. Schnell, *Holonomic $\mathcal{D}$-modules on abelian varieties.* Publ. Math. IHÉS **121** (2015), 1-55.

[8] B. Lawrence and W. Sawin, *The Shafarevich conjecture for hypersurfaces in abelian varieties*, `arxiv.org/abs/2004.09046`.

[9] B. Lawrence and A. Venkatesh, *Diophantine problems and p-adic period mappings*, Invent. Math. **221** (2020), 893–999.

Singular supports in positive and mixed characteristics

Takeshi Saito

1. Positive characteristic

Let X be a smooth scheme over a field k of characteristic $p > 0$. Let Λ be a finite field of characteristic $\ell \neq p$ and by abuse of terminology we call a bounded constructible complex $\mathcal{F}$ of Λ-modules on $X_{\text{ét}}$ a sheaf on X.

A closed subset C of a vector bundle E on X is said to be conical if it is stable under the $\mathbb{G}_m$-action. A closed conical subset C is uniquely determined by the intersection $C \cap X$ with the 0-section, called the base of C, and the projectivization $\mathbb{P}(C) \subset \mathbb{P}(E)$.

Let $h\colon W \to X$ be a morphism of smooth schemes over k. For a closed conical subset $C \subset T^*X$, its pull-back $h^*C \subset T^*X \times_X W$ is defined to be the inverse image by $T^*X \times_X W \to T^*X$. The morphism h is called C-transversal if the intersection $h^*C \cap \operatorname{Ker}(T^*X \times_X W \to T^*W)$ is a subset of the 0-section of $T^*X \times_X W$.

For example, if C is the conormal bundle $T^*_Z X = \operatorname{Ker}(T^*X|_Z \to T^*Z)$ of a closed subscheme $Z \subset X$ smooth over k, then h is C-transversal if and only if h is transversal to $Z \to X$; namely, $V = Z \times_X W$ is smooth over k and $\operatorname{codim}_W V = \operatorname{codim}_X Z$.

We say that a separated morphism h is $\mathcal{F}$-transversal, if the canonical morphism $h^*\mathcal{F} \otimes Rh^!\Lambda \to Rh^!\mathcal{F}$ is an isomorphism. We say that $\mathcal{F}$ is micro supported on C if the following conditions is satisfied:

For every pair of separated morphisms $h\colon W \to X$ and $f\colon W \to Y$ of smooth schemes over k, if $(h, f)\colon W \to X \times Y$ is $C \times T^*Y$-transversal, then (h, f) is $\mathrm{pr}_1^*\mathcal{F} \otimes \mathrm{pr}_2^*\mathcal{G}$-transversal for every sheaf $\mathcal{G}$ on Y.

If the smallest closed conical subset $C \subset T^*X$ on which $\mathcal{F}$ is micro supported exists, we call $C = SS\mathcal{F}$ the singular support of $\mathcal{F}$. The existence is non-trivial because $\mathcal{F}$ being micro supported on C_1 and C_2 does not imply a priori $\mathcal{F}$ being micro supported on the intersection $C_1 \cap C_2$.

Theorem 1. (Beilinson [1])
1. *$SS\mathcal{F}$ always exists.*
2. *Every irreducible component of $SS\mathcal{F}$ has the same dimension as X.*

If X is a curve, an irreducible component of dimension 1 of a closed conical subset of a line bundle T^*X over X is either the 0-section or the fiber of a closed point. The 0-section appears in $SS\mathcal{F}$ if and only if the sheaf $\mathcal{F}$ is generically non-zero. The fiber of a closed point appears if and only if the sheaf ramifies there. In higher dimension, $SS\mathcal{F}$ can be more complicated.

A key tool in the proof by Beilinson of Theorem 1 is the Radon transform. First, we reduce the proof to the case where X is a projective space $\mathbb{P}^n$. The dual projective space $\mathbb{P}^{n\vee}$ is the moduli of hyperplanes in $\mathbb{P}^n$. The universal family Q of hyperplanes is canonically identified with the projectivizations $\mathbb{P}(T^*\mathbb{P}^n) = \mathbb{P}(T^*\mathbb{P}^{n\vee})$. Since the base of $SS\mathcal{F}$ equals the support of $\mathcal{F}$, the singular support is essentially determined by its projectivization $\mathbb{P}(SS\mathcal{F}) \subset Q$. Using this fact and analyzing the projections $Q \to \mathbb{P}^n, \mathbb{P}^{n\vee}$ as h and f in the definition of micro support, one can prove Theorem 1.

2. Mixed characteristic

Let X be a regular noetherian scheme over $\mathbb{Z}_{(p)}$. To consider singular supports in mixed characteristic case, we first need to solve the problem: Where $SS\mathcal{F}$ should live? In the geometric case, the vector bundle T^*X is defined by $\Omega^1_{X/k}$. In mixed characteristic, Ω^1_X may not be locally free. Even if it is, it will be too small, e.g. for $X = \operatorname{Spec} \mathbb{Z}_{(p)}$.

A solution is given by the Frobenius–Witt differentials. The sheaf Ω^1_X of Kähler differentials is defined by the universality for the usual derivations satisfying $d(x+y) = dx + dy$ and $d(xy) = xdy + ydx$. The sheaf $F\Omega^1_X$ of FW differentials is defined by replacing these relations by $d(x+y) = dx + dy + ((x+y)^p - x^p - y^p)/p \cdot dp$ and $d(xy) = x^p dy + y^p dx$. The fraction in the first equality means the substitution to the quotient as a polynomial.

We assume the following finiteness condition:

(F) The reduced part $X_{\mathbb{F}_p,\mathrm{red}}$ of the characteristic p fiber is of finite type over a field k of finite p-basis $[k : k^p] < \infty$.

Then, the $\mathcal{O}_X$-module $F\Omega^1_X$ is a locally free $\mathcal{O}_{X_{\mathbb{F}_p}}$-module of finite type. For $x \in X_{\mathbb{F}_p}$, we have a short exact sequence $0 \to F^*\mathfrak{m}_x/\mathfrak{m}_x^2 \to F\Omega^1_{X,x} \otimes k(x) \to F^*\Omega^1_{k(x)} \to 0$ where F^* denotes the Frobenius pull-back. For example, if X is of finite type over a complete discrete valuation ring of mixed characteristic with perfect residue field, the rank of the locally free $\mathcal{O}_{X_{\mathbb{F}_p}}$-module $F\Omega^1_X$ is $\dim X$.

In the following, we assume the finiteness condition (F) above and define the FW-cotangent bundle FT^*X on $X_{\mathbb{F}_p}$ to be the vector bundle corresponding to

$F\Omega^1_X$. Although it is restricted to the characteristic p fiber, the vector bundle has the correct rank.

Let $h\colon W \to X$ be a separated morphism of finite type of regular noetherian schemes. For a closed conical subset $C \subset FT^*X$, we say that h is C-transversal if the intersection $h^*C \cap \mathrm{Ker}(FT^*X \times_X W \to FT^*W)$ is a subset of the 0-section of $FT^*X \times_X W$. We say that a sheaf $\mathcal{F}$ on X is micro supported on C if the following conditions are satisfied:

(i) The intersection of the support of $\mathcal{F}$ with $X_{\mathbb{F}_p}$ is a subset of the base of C.

(ii) For every separated morphism $h\colon W \to X$ of finite type of regular noetherian scheme if h is C-transversal, then h is $\mathcal{F}$-transversal on a neighborhood of $W_{\mathbb{F}_p}$.

To use the Radon transform, we fix a regular noetherian scheme S over $\mathbb{Z}_{(p)}$ satisfying (F) and introduce a relative version. For a closed conical subset $C \subset FT^*X$, we say that a pair (h, f) of morphisms $h\colon W \to X$ and $f\colon W \to Y$ of regular schemes of finite type over S such that Y is smooth over S is C-acyclic if we have an inclusion

$$\begin{aligned}&(h^*C \times_W (FT^*Y \times_Y W)) \cap \mathrm{Ker}((FT^*X \times_X W) \times_W (FT^*Y \times_Y W) \to FT^*W)\\ &\subset \mathrm{Ker}((FT^*X \times_X W) \times_W (FT^*Y \times_Y W) \to FT^*(X \times_S Y) \times_{X\times_S Y} W).\end{aligned}$$

We say that a sheaf $\mathcal{F}$ on X is S-micro supported on C if the following condition is satisfied:

For every C-acyclic pair (h, f) as above and for every sheaf $\mathcal{G}$ on Y micro supported on some closed conical subset $C' \subset FT^*Y$ such that $C' \cap \mathrm{Im}(FT^*S \times_S Y \to FT^*Y)$ is a subset of the 0-section of FT^*Y, the morphism $(h, f)\colon W \to X \times_S Y$ is $\mathrm{pr}_1^*\mathcal{F} \otimes \mathrm{pr}_2^*\mathcal{G}$-transversal on a neighborhood of $W_{\mathbb{F}_p}$.

We define $SS\mathcal{F}$ and $SS_S\mathcal{F}$ to be the smallest closed conical subsets of FT^*X on which $\mathcal{F}$ is micro supported and is S-micro supported respectively. We say that a closed conical subset $C \subset FT^*X$ is S-stable if C is stable under the action of $FT^*S \times_S X$. We also define $SS^{\mathrm{sat}}_S\mathcal{F}$ to be the smallest S-stable closed conical subset of FT^*X on which $\mathcal{F}$ is S-micro supported. Although we don't know the existence, we expect to have inclusions $SS_S\mathcal{F} \subset SS\mathcal{F} \subset SS^{\mathrm{sat}}_S\mathcal{F}$.

By adopting Beilinson's argument using Radon transform, we obtain the following.

Theorem 2. *If X is smooth over S, $SS^{\mathrm{sat}}_S\mathcal{F}$ exists.*

References

[1] A. Beilinson, *Constructible sheaves are holonomic*, Selecta Math. **22** (4), (2016), 1797–1819.

[2] T. Saito, *Cotangent bundles and micro-supports in mixed characteristic case*, Algebra & Number Theory **16**-2 (2022), 335–368.

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