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Directions in Rough Analysis

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Abstract. Rough path theory emerged in the 1990s and was developed in the 2000s as an improved approach to understanding the interaction of complex random systems. As a broader alternative to stochastic calculus, it simultaneously settled significant questions and substantially expanded the scope of classical methods in stochastic analysis. Subsequent related developments have had an impact at the highest level, Martin Hairer's work on regularity structures being among the most prominent.

In 2020, rough analysis gained its own AMS classification code, 60L, and this workshop focused on the currently most active areas of the subject among two central strands:

(1) the mathematics of the signature transform, including its applications to data science and finance, and
(2) rough path theory applied to novel areas in stochastic analysis, such as homogenization, SLE and rough PDEs.

Mathematics Subject Classification (2020): 60L10, 60L20, 60L30, 60L40, 60L70, 60L90, 60H07, 60H10, 60H15, 60H20, 62J02, 68T05, 68T45, 76M50, 81Q30, 14A10.

Introduction by the Organizers

The workshop *Directions in Rough Analysis* occurred in November 2024 and was organised by Thomas Cass (Imperial College London), Christa Cuchiero (Vienna University) Peter Friz (TU and WIAS Berlin, Germany). It was attended by participants from a wide distribution of institutions with contributions ranging from long-form presentations, to shorter talks focussed on very recent developments. In keeping with Oberwolfach custom, each day's schedule was finalised the day beforehand allowing participants' to align their talks with the surrounding discussions.

Rough Analysis is a comparatively recent branch of mathematics, having gained its own AMS classification code (60L) in 2020. It encompasses both rough path theory, including its application in stochastic analysis, as well as the mathematics of the signature (Chen-Fliess series). The workshop gathered together leading experts and promising researchers at all career stages for focussed interaction on the current challenges in the area. The discussions helped to establish a set of future research directions in the field.

An important direction, emphasised by several participants', was on applications of the signature to the analysis of streamed data. The workshop began with a presentation by Terry Lyons who gave an overview of recent developments, drawing connections with a new class of signature-based evolution equations that capture transformer architectures in deep learning. Other contributors presented new results on the use of signature-based kernel methods. A further strand of research related to the interaction between signature representations and rough differential equations and methods based on randomised signatures. Explicit series expansions to certain stochastic path-dependent integral equations in terms of the signature of the time-augmented driving Brownian motion were presented by Eduardo Abi-Jaber. Josef Teichmann considered signature transforms from the point of view of invariant theory showing that real analytic functions that are invariant under time reparametrizations admit a convergent signature expansion. In a similar spirit, signature Taylor expansions of sufficiently regular non-anticipative maps of rough paths can be derived, which was presented by Francesca Primavera. Here, the approximation properties of the signature are used as a proof technique to obtain a functional Ito-formula and in turn Taylor expansions for maps of rough paths.

A significant recent offshoot has been to establish suitable analogues of the signature to analyse two-dimensional data such as images or graphs, and to understand the corresponding algebraic, geometric and analytic structures which underpin them. In general, algebraic properties have witnessed a stream of works and are closely related to a number of the participants' interests (Bruned, Ebrahimi-Fard, Tapia). Recent applications of the signature in a real-world setting, in particular in view of statistical regression methods, were presented by Xin Guo.

A second key direction was to make progress on questions at the interface of rough paths and stochastic analysis. Topics in which there is much interest include homogenisation and fast-slow systems prompted by the work of Melbourne and co-authors. Applications from Mathematical Finance, encompassing "rough volatility", robust finance, rough stochastic analysis with jumps and data-driven modeling, were other important themes of the workshop. Furthermore, the interconnection of rough path ideas with SPDEs derived from stochastic calculus, as well as mean-field systems with control and common noise played a central role. Francois Delarue presented for instance a mean field control theoretic approach to analyse the convergence of gradient descent in deep ResNets, a certain type of artificial neural networks. Also, motivated by understanding the gradient descent in deep neural networks, a gradient flow on control space with rough initial condition was analysed by Paul Gassiat, showing how essential questions from deep learning can benefit from techniques of rough path theory.

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Workshop: Directions in Rough Analysis

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Abstracts

A coupling between Sinai's random walk and Brox's diffusion

Samy Tindel
(joint work with Xi Geng, Mihai Gradinaru)

Sinai's random walk is a standard model of 1-dimensional random walk in random environment. Brox diffusion is its continuous counterpart, that is a Brownian diffusion in a Brownian environment. The convergence in law of a properly rescaled version of Sinai's walk to Brox diffusion has been established 20 years ago.

In this talk, I have explained a strategy which yields the convergence of Sinai's walk to Brox diffusion thanks to an explicit coupling. This method, based on rough paths techniques, opens the way to rates of convergence in this demanding context. A large part of the talk has been dedicated to introduce the basic objects we are manipulating. Then I have explained how rough paths can help to quantify the convergence rate. The talk is based on the preprint [1].

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Wong-Zakai approximation of density functions

Yuzuru Inahama

In this talk we discuss the Wong-Zakai approximation of probability density functions of solutions at a fixed time of rough differential equations driven by fractional Brownian rough path with Hurst parameter H $(1/4 < H \leq 1/2)$. Besides rough path theory, we use Hu-Watanabe's approximation theorem in the framework of Watanabe's distributional Malliavin calculus. (See [1].) When $H = 1/2$, the random rough differential equations coincide with the corresponding Stratonovich-type stochastic differential equations. Even in that case, our main result seems new. (This talk is based on the speaker's reecent preprint [2].)

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Functional Ito-formula and Taylor expansions for non-anticipative maps of càdlàg rough paths

Francesca Primavera
(joint work with Christa Cuchiero, Xin Guo)

We rely on the approximation properties of the signature of càdlàg rough paths to derive a functional Ito-formula for non-anticipative maps of rough paths. This leads to a functional extension of the Ito-formula for càdlàg rough paths (by Friz and Zhang (2018)) which coincides with the change of variable formula formulated by Dupire (2009) whenever the functionals representations, the notions of the regularity of the functionals and the integration concepts can be matched. In contrast to these works, by using the concept of vertical Lie derivatives, we can also incorporate path functionals where the second order vertical derivatives do not commute as it is the case for typical signature functionals. As a byproduct, we show that sufficiently regular non-anticipative maps admit a functional Taylor expansion, leading to a far reaching generalization of the recently established results by Dupire and Tissot-Daguette (2022).

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Randomised Path Developments and Signature Kernels as Scaling Limits

William F. Turner
(joint work with Thomas Cass, Samuel Crew, Cristopher Salvi)

Scaling limits of random developments of a path into a matrix Lie group have recently been used to construct signature-based time series kernels. General linear group developments have been shown to be connected to the ordinary signature kernel [2], while unitary developments have been used to construct the path characteristic function distance [3] which has proven a successful discriminator for generative modelling tasks. By leveraging the tools of random matrix theory and free probability theory, we are able to provide a unified treatment of both limits under general assumptions on the randomisation. For unitary developments, we show that the limiting kernel is given by the contraction of a signature against the monomials of freely independent semicircular random variables. Using the Schwinger-Dyson equations, we show that this kernel can be obtained by solving a novel quadratic functional equation. We also present ongoing work with Samuel Crew and Cristopher Salvi where we consider more general driving matrix models which also asymptotically satisfy a Schwinger-Dyson equation. The corresponding limit in this setting solves a path-dependent integro-differential equation.

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Mean Field Approach to Deep ResNets

François Delarue

(joint work with Samuel Daudin)

In this talk, I will present a mean-field approach to deep ResNets in machine learning. These networks have already been the subject of several mathematical works, some of them explicitly based on the theory of rough paths. Conceptually, the mean field approach addresses the overparametrized regime in presence of an infinite number of observations, see [1] an earlier work in this direction. From a mathematical point, only the statistical states of the neurons and the features matter in the analysis. Here, we use the tools from mean field control theory to prove that, for many initial initial conditions, the network exhibit local stability properties, which force the corresponding gradient descent to converge exponentially fast when initialized close to the optimal parameters. This is a joint work with Samuel Daudin, supported by ERC ELISA AdG 101054746 (Programme Horizon Europe).

Talk presented on November 5th 2024, at the MFO conference 'Directions in Rough analysis'.

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Non-explosion and strong completeness

Xue-Mei Li

(joint work with Kexing Ying)

We study solution theory for rough differential equations (RDE) and the strong completeness problem for stochastic differential equations (SDE). An SDE is said to be strongly complete if for each initial value x, there exists a unique global solution $F_t(x)$ (i.e. the SDE is complete), and the map $(t,x) \mapsto F_t(x,\omega)$ is continuous almost surely. Continuity with respect to the initial data implies the existence of a perfect cocycle and is an underlying assumption for numerical solutions.

In case where the Banach fixed point theorem is applicable to the SDE, strong completeness is often solved at the same time (e.g. in the case for SDEs in $\mathbb{R}^d$ with Lipschitz continuous coefficients). It had been a general belief that a 'complete' SDE, with smooth coefficients, is automatically strongly complete, so much so nobody questioned until a first counter example was given by K.D. Elworthy [Elw78]. An open problem immediately arise: provide a criterion for the strong completeness for generic SDEs on non-compact manifolds.

This problem was first solved in [Li94] by coupling the SDE with its linearized equation. In $\mathbb{R}^d$ this is formulated as follows: Let σ, b be locally Lipschitz continuous, (so a local continuous solution flow exists, which is the starting point in [Li94]), we consider the SDE

$$dx_t = b(x_t)\,dt + \sum_{k=1}^m \sigma_k(x_t)\,dW_t^k, \;\; dv_t = Db_{x_t}(v_t)\,dt + \sum_{k=1}^m D\sigma^k_{x_t}(v_t)\,dW_t^k.$$

Fixing the initial condition x_0, the solution to v_t with initial condition v_0 is the derivative of $x \mapsto F_t(x)$ in probability at x_0 in the direction of v_0. It is referred as the derivative flow and denoted by $DF_t(x_0)(v_0)$. Fixing x, $DF_t(x)$ is a $d\times d$-matrix, and we denote $|(DF_t)(x)|$ for its norm .

Theorem 1. [Li94] *Suppose the SDE is complete.*

(1) *It is strongly p-complete, if for each compact set $K \subset \mathbb{R}^d$ and $t > 0$,*

$$\sup_{x\in K} E\left(\sup_{s\le t} |D_x F_s|^p\right) < \infty$$

for some $p > d-1$ ($p = 1$ if $d = 2$).

(2) *Let $g : \mathbb{R}^d \to \mathbb{R}_+$ be such that $\sup_{x\in K} \mathbb{E}\left(e^{6p^2\int_0^t g(F_s(x))ds}\right) < \infty$ for any compact $K \subset \mathbb{R}^d$. If furthermore,*

$$\left\{\begin{array}{l} |D\sigma^k|^2 \le g, \\ 2\langle Db_x(v), v\rangle + (p-2)\sum_{k=1}^m \frac{1}{|v|^2}\langle (D\sigma^k)_x(v), v\rangle^2 \le 6pg(x)|v|^2, \end{array}\right. \tag{1}$$

for all $x, v \in \mathbb{R}^d$, $k = 1, \dots, m$, then for some constant c,

$$\mathbb{E}\left(\sup_{s\le t} |DF_s(x)|^p\right) \le c\mathbb{E}\left(e^{6p^2\int_0^t g(F_s(x))ds}\right) \le \frac{c}{t}\int_0^t \mathbb{E}e^{6p^2 t g(F_s(x))}ds.$$

(3) *If there exists C such that*

$$\frac{1}{2}\sum_1^m |Dg(\sigma^k)|^2 + \frac{1}{2}\sum_1^m D^2g(\sigma^k, \sigma^k) + Dg(b) \le C, \tag{2}$$

then for any $c > 0$, there exists cK such that $\mathbb{E}[e^{cg(x_t)}] \le e^{cg(x_0)+Kt}$.

Consequently (1) *and* (2) *imply strong completeness.*

For example, if σ^k and $\langle b(x), x\rangle$ are bounded above, it suffices to assume that the derivatives are bounded. An example of an SDE is provided where the coefficients are bounded and C^∞ smooth, and yet strong completeness fails (see [LS09]).

Recent investigations into strong completeness have focused on extending the growth conditions on the drift term and reducing the regularity requirements for the coefficients, we mention specially [SS17]. Our results improve current known results on SDE and our results on RDE's builds on that of Lyons, Friz-Victor, Friz-Hairer. Further references will be detailed in our forthcoming article.

Let V, H be Banach spaces. Assume that there exists a non-decreasing function $f \in C(\mathbb{R}_+;\mathbb{R}_+)$ satisfying $\int_r^\infty \frac{1}{f(s)}\,ds = \infty$ for some $r > 0$ and $\beta > 0$ such that, for all $x \in H$ and $t \geq 0$,

$$\left\langle \frac{x}{\|x\|}, b(x) \right\rangle \leq f(\|x\|). \tag{3}$$

Theorem 2. [LY25] *Let $b \in \mathrm{Lip}_{loc}(H;H)$ satisfy (3), $\sigma \in C^3(H;\mathbb{L}(V;H))$ and $\lneqq \in \mathcal{C}^{\alpha+}(\mathbb{R}_+;V)$ for some $\alpha \in (\frac{1}{3},\frac{1}{2})$. Then, if there exist some $\kappa \in [0,\frac{1}{2})$ such that for all $x \in H$,*

- $\|b(x)\| \leq f(\|x\|)^{1+\kappa\alpha}$;
- $\|D^n\sigma(x)\| \lesssim f(\|x\|)^{(1-n\kappa)\alpha-}$ *for $n = 0, 1, 2$,*

the rough differential Equation $dx_t = b(x_t)\,dt + \sigma(x_t)\,d\lneqq_t$ is globally well-posed. Applying to SDE, we obtain strong completeness.

The corresponding result for Young differential Equation is as follows:

Theorem 3. [LY25] *Let $b \in \mathrm{Lip}_{loc}(H;H)$ satisfy (3), $\sigma \in \mathrm{Lip}(H;\mathbb{L}(V;H))$ and $\gamma \in C^\alpha_{loc}(\mathbb{R}_+;V)$ for some $\alpha > \frac{1}{2}$. If there exists some $\kappa \in [0,1)$ such that for all $x \in H$,*

(1) $\|b(x)\| \lesssim f(\|x\|)^{1+\kappa\alpha}$,
(2) $\|D^n\sigma(x)\| \lesssim f(\|x\|)^{(1-n\kappa)\alpha-}$ *for $n = 0, 1$.*

Then, the Young differential Equation $dx_t = b(x_t)\,dt + \sigma(x_t)\,d\gamma_t$ has a global solution for every initial condition. Furthermore, it is unique if $\sigma \in \mathcal{C}^{\left(\frac{\alpha}{1-\alpha}\right)+}$.

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Chain rule symmetry for singular SPDEs

Yvain Bruned
(joint work with Carlo Bellingeri, Vladimir Dotsenko)

We consider the stochastic geometric heat equation given by

$$\partial_t u^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u)\,\partial_x u^\beta \partial_x u^\gamma + h^\alpha(u) + \sigma_i^\alpha(u)\,\xi_i\;, \tag{1}$$

where $i \in \{1, ..., m\}$, $\alpha \in \{1, ..., d\}$ and $u : \mathbb{R}_+ \times \mathbb{T} \mapsto \mathbb{R}^d$. The functions $\Gamma^\alpha_{\beta\gamma} : \mathbb{R}^d \to \mathbb{R}$ with $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$ are smooth Christoffel symbols. The functions $\sigma_i^\alpha : \mathbb{R}^d \to \mathbb{R}$ are the components of a smooth vector field on $\mathbb{R}^d$. The ξ_i are independent space-time noises. The geometric motivation of equation (1) is to provide a natural stochastic process taking values in the space of loops in a compact Riemannian manifold. Its invariant measure is expected to be the Brownian loop measure. The construction for space-time white noise is performed in [5]. The connection with the Brownian loop measure remains an open problem.

For solving this equation, we start by fixing a natural approximation of solutions of (1) via a class of mollifiers that are compactly supported smooth functions $\varrho : \mathbb{R}^2 \to \mathbb{R}$ integrating to 1 with $\varrho(t, -x) = \varrho(t, x)$, $\varrho(t, x) = 0$ for $t \leq 0$. For $\varepsilon > 0$, we replace ξ_i by its regularisation $\xi_i^\varepsilon = \varrho_\varepsilon * \xi_i$, where $*$ is the space-time convolution and

$$\varrho_\varepsilon(t, x) = \varepsilon^{-3}\varrho(\varepsilon^{-2}t, \varepsilon^{-1}x). \tag{2}$$

Here, we have used the parabolic scaling where time counts double in comparison to space. One has the following regularised equation

$$\partial_t u_\varepsilon^\alpha = \partial_x^2 u_\varepsilon^\alpha + \Gamma^\alpha_{\beta\gamma}(u)\,\partial_x u_\varepsilon^\beta \partial_x u_\varepsilon^\gamma + h^\alpha(u_\varepsilon) + \sigma_i^\alpha(u_\varepsilon)\,\xi_i^\varepsilon. \tag{3}$$

The main problem is to find suitable counter-terms such that u_ε converges and such that the geometry of the equation is not broken. This question has been answered only for space-time white noise in [5] by using the complete solution theory provided by Regularity Structures in [11, 7, 9, 2]. In the next theorem, one has a statement for the full subcritical regime:

Theorem 1 ([4]). *There exist renormalisation constants $C_\varepsilon(\tau)$ such that the renormalised equation of (1) is geometric given by:*

$$\begin{aligned}\partial_t u_\varepsilon^\alpha = \partial_x^2 u_\varepsilon^\alpha &+ \Gamma^\alpha_{\beta\gamma}(u_\varepsilon)\,\partial_x u_\varepsilon^\beta \partial_x u_\varepsilon^\gamma \\ &+ h^\alpha(u_\varepsilon) + \sigma_i^\alpha(u_\varepsilon)\,\xi_i^\varepsilon + \sum_{\tau \in \mathfrak{B}_\xi} C_\varepsilon(\tau)\Upsilon^\alpha_{\Gamma,\sigma}[\tau](u_\varepsilon)\,,\end{aligned}$$

where $\mathfrak{B}_\xi$ depends on the ξ_i and the $\Upsilon_{\Gamma,\sigma}[\tau](u_\varepsilon)$ are computed with the σ_i and the covariant derivative $\nabla_X Y$ defined by

$$(\nabla_X Y)^\alpha(u) = X^\beta(u)\,\partial_\beta Y^\alpha(u) + \Gamma^\alpha_{\beta\gamma}(u)\,X^\beta(u)\,Y^\gamma(u)\;.$$

The statement above is valid for any subcritical noises ξ_i which are noises whose space-time Hölder regularity is greater than -2. For convergence theorems concerning the iterated integrals built out of this noise see [9, 14, 12]. These papers show the convergence of the solution of the renormalised equation. Theorem 1 is also valid for quasilinear equations (see [6]) where the chain rule is used for removing non-local counter-terms.

One can make explicit the dimension of these geometric counter-terms via two theorems:

Theorem 2 ([4])**.** *For all subcritical noises ξ_i, sufficiently high dimension d, one can compute the dimension of $(\Upsilon_{\Gamma,\sigma}[\tau])_{\tau\in\mathfrak{B}_\xi}$.*

Theorem 3 ([1])**.** *For all subcritical noise ξ, $d=1$ and $m=1$, the dimension of $(\Upsilon_{\Gamma,\sigma}[\tau])_{\tau\in\mathfrak{B}_\xi}$ is the dimension of the free Novikov algebra.*

The free Novikov algebra is described in [10] see also [3] for an application of this structure to singular SPDEs. When the dimension d is not sufficiently high and not equal to 1, it is very difficult to compute such a dimension. Let us stress that the study of the chain rule symmetry for stochastic geometric heat equation is actually quite general. Indeed, one can look at a general class of subcritical equations:

$$\begin{aligned}\partial_t u^\alpha - Lu^\alpha = &\sum_{(\beta_1,r_1),\dots,(\beta_n,r_n)\in\mathcal{O}_-} \Gamma^\alpha_{(\beta_1,r_1),\dots,(\beta_n,r_n)}(\partial^p u^\gamma\colon(\gamma,p)\in\mathcal{O}_+)\prod_{i=1}^n \partial^{r_i}u^{\beta_i}\\ &+\sigma_i^\alpha(\partial^p u^\gamma\colon(\gamma,p)\in\mathcal{O}_+)\xi_i\,,\end{aligned}$$

where L is a differential operator, $\Gamma^\alpha_{(\beta_1,r_1),\dots,(\beta_n,r_n)}$ and σ_i^α are smooth coefficients. The $\mathcal{O}_+$, $\mathcal{O}_-$ are finite subsets of $\{1,\dots,d\}\times\mathbb{N}^{d+1}$. The $\partial^p u^\gamma$ (resp. $\partial^{r_i}u^{\beta_i}$) are functions (resp. distributions) when $(\gamma,p)\in\mathcal{O}_+$ (resp. $(\beta_i,r_i)\in\mathcal{O}_-$). One can use the same techniques for computing the dimension of geometric counter-terms but there is no natural generating set and it has to be performed case by case.

Let us briefly explain the main steps of the proof of Theorems 1, 2 and 3. One first starts by writing the renormalised equation in the next theorem

Theorem 4 ([2] & [8])**.** *There exist renormalisation constants $C_\varepsilon(\tau)$ and a combinatorial set $\mathfrak{S}_\xi$ associated with some coefficients $(\Upsilon_{\Gamma,\sigma}[\tau](u_\varepsilon))_{\tau\in\mathfrak{S}_\xi}$ such that the renormalised equation of (1) is given by:*

$$\begin{aligned}\partial_t u^\alpha_\varepsilon &= \partial_x^2 u^\alpha_\varepsilon + \Gamma^\alpha_{\beta\gamma}(u_\varepsilon)\,\partial_x u^\beta_\varepsilon\partial_x u^\gamma_\varepsilon\\ &\quad + h^\alpha(u_\varepsilon)+\sigma_i^\alpha(u_\varepsilon)\,\xi_i^\varepsilon+\sum_{\tau\in\mathfrak{S}_\xi}C_\varepsilon(\tau)\Upsilon^\alpha_{\Gamma,\sigma}[\tau](u_\varepsilon)\,.\end{aligned}\tag{4}$$

One wants to find a choice of $C_\varepsilon(\tau)$ such that

$$\sum_{\tau\in\mathfrak{S}_\xi}C_\varepsilon(\tau)\Upsilon_{\Gamma,\sigma}[\tau](u_\varepsilon)=\sum_{\nu\in\mathfrak{B}_\xi}C_\varepsilon(\nu)\Upsilon_{\Gamma,\sigma}[\nu](u_\varepsilon).$$

where the $C_\varepsilon(\nu)$ depend on the $C_\varepsilon(\tau)$. For doing so, one has to first understand the elements $\tau\in\langle\mathfrak{S}_\xi\rangle$ such that $\Upsilon_{\Gamma,\sigma}[\nu](u_\varepsilon)$ is invariant under change of coordinates. In the sequel, we explain this procedure in dimension one but it is the same for higher dimension. Given a diffeomorphism φ, one has

$$(\varphi\cdot\Gamma)(\varphi(u))\,\varphi'(u)^2 = \varphi'(u)\,\Gamma(u)-\varphi''(u)\,,\quad (\varphi\cdot\sigma)(\varphi(u))=\varphi'(u)\,\sigma(u)\,.$$

Then, we define the space $V_{\rm geo} \subset \langle \mathfrak{S}_\xi \rangle$ as consisting of those elements τ such that,

$$\varphi \cdot \Upsilon_{\Gamma,\sigma}[\tau] = \Upsilon_{\varphi\cdot\Gamma,\varphi\cdot\sigma}[\tau].$$

One can replace φ by a family $(\psi_t)_{t\geq 0}$ with $\psi_0 = id, \partial_t \psi|_{t=0} = h$ and get the following equivalence from [5, 1]

$$\begin{aligned} \tau \in V_{\rm geo} &\Leftrightarrow (\partial_t \psi_t \cdot \Upsilon_{\Gamma,\sigma}[\tau])|_{t=0} = (\partial_t \Upsilon_{\psi_t\cdot\Gamma,\psi_t\cdot\sigma}[\tau])|_{t=0} \\ &\Leftrightarrow \Upsilon^h_{\Gamma,\sigma}[\hat{\varphi}_{\rm geo}(\tau)] = 0 \Leftrightarrow \hat{\varphi}_{\rm geo}(\tau) = 0. \end{aligned}$$

where $\Upsilon^h_{\Gamma,\sigma}[\hat{\varphi}_{\rm geo}(\tau)]$ are coeffcients depending on h and the injectivity of the map $\tau \mapsto \Upsilon^h_{\Gamma,\sigma}[\tau]$ is crucially used in the last equivalence. Then, it boils down to find a basis of $\ker \hat{\varphi}_{\rm geo}$ and to compute its dimension. The strategy for the proof depends on the dimension:

- In sufficiently high dimension, the set $\mathfrak{S}_\xi$ is formed of decorated trees. The proof relies on operad theory and homological algebra in [4].
- In dimension one, the set $\mathfrak{S}_\xi$ is formed of multi-indices as introduced in [15, 13]. An elementary proof (linear algebra) is performed in [1].

They are several applications/open problems following these results:

- Itô Isometry in the full subcritical regime for Gaussian noises. One could use operadic and homological tools.
- Global solution in the full subcritical regime.
- Chain rule for small dimensions d ($d \neq 1$).
- Application to conjectures in geometry: In dimension one, geometric elements without Christoffel symbol are generated by Lie Brackets.

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Rough Geometric Integration

Harprit Singh
(joint work with A. Chandra)

Combining ideas from Whitney's geometric integration theory and rough analysis, we introduce spaces of rough differential k–forms on d-manifolds which are formally given by $f = \sum_I f_I \mathrm{d}x^I$ where $(f_I)_I$ belong to a class of genuine distributions of negative regularity. These rough k–forms have several properties desirable of a notion of differential forms:

- they can be integrated over suitably regular k-manifolds,
- they form a module under point-wise multiplication with sufficiently regular functions,
- exterior differentiation as well as the Stokes theorem extend to these spaces,
- they come with natural embeddings into distribution spaces,
- they contain classes of form valued distributional random fields.

Finally, these spaces unify several previous constructions in the literature. In particular, they generalise spaces of α–flat cochains introduced by Whitney [5] and Harrison–Norton [4], they contain the (rough) k-forms $f \cdot dg_1 \wedge ... \wedge dg_k$ introduced by Züst using Young integration [6] and have been revisited for $k = 2$ in [1], and for $d = 2$ and $k = 1$ they are close to the spaces which Chevyrev et al. [3, 2] use to make sense of Yang–Mills connections. Lastly, as a technical tool we introduce a 'simplicial sewing lemma', which provides a coordinate invariant formulation of the (known) multi-dimensional sewing lemma.

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Coming up from $-\infty$ for the KPZ equation

Nicolas Perkowski
(joint work with Carlos Villanueva Mariz)

We consider the KPZ equation

$$\partial_t h = \frac{1}{2}\partial_{xx} h + \frac{1}{2}(\partial_x h)^2 + \xi, \qquad h(0) = h_0,$$

on a torus $\mathbb{T}_L = \mathbb{R}/(L\mathbb{Z})$ for $L > 1$, where ξ is a space-time white noise. Our main results are two estimates that are largely independent of the initial condition h_0, in the spirit of [2]. For the first estimate we assume that $h_0|_{[0,1]} \geq 0$ and obtain a lower bound

$$\inf_{x\in\mathbb{T}_L} h(t,x) \geq -\frac{C(\xi, L, t)}{t},$$

where $C(\xi, L, t) > 0$ is independent of $h_0|_{[0,1]^c}$ and locally bounded in t. Moreover, for $\alpha < \frac{1}{2}$ we control the Hölder semi-norm:

$$\sup_{x\neq y\in\mathbb{T}_L} \frac{|h(t,x) - h(t,y)|}{|x-y|^\alpha} \leq C(\xi, L, \alpha, t),$$

where $C(\xi, L, \alpha, t) > 0$ is independent of h_0. Both of these results are derived via the variational formulation of relative entropy, inspired by the stochastic control formulation in [1]:

$$h(t,x) = \sup\left\{\mathbb{E}_{\mathbb{Q}_x}[h_0(X_t)] - H(\mathbb{Q}_x|\mathbb{P}_x)\right\},$$

where $\mathbb{P}_x$ is the law of $x + W \mod L$ for a Brownian motion W, and where $H(\mathbb{Q}_x|\mathbb{P}_x)$ is the relative entropy.

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Higher order approximation of nonlinear SPDEs with additive space-time white noise

Helena Kremp
(joint work with Ana Djurdjevac, Máté Gerencsér)

In this talk based on [1], we consider strong approximations of 1+1-dimensional stochastic PDEs driven by additive space-time white noise:

$$\partial_t u = \Delta u + f(u) + \xi, \quad u(0, \cdot) = u_0 \in \mathcal{C}^{1/2}(\mathbb{T}).$$

It has been long proposed [2, 3], as well as observed in simulations, that approximation schemes based on samples from the stochastic convolution, rather than from increments of the underlying Wiener processes, should achieve significantly higher convergence rates with respect to the temporal timestep. Utilizing the stochastic sewing lemma [4], we prove that for a large class of nonlinearities f, with possibly superlinear growth, a temporal rate of (almost) 1 can be achieved, a major improvement on the rate 1/4 that is known to be optimal for schemes based on Wiener increments. The spatial rate remains (almost) 1/2 as it is standard in the literature.

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Limit Theorems For Signatures

Yuri Kifer

I will discuss moment and almost sure invariance principles and laws of iterated logarithm for normalized multiple iterated sums and integrals of the form $\mathbb{S}_N^{(v)}(t) = N^{-v/2} \sum_{0 \le k_1 < \cdots < k_v \le N_t} \xi(k_1) \otimes \cdots \otimes \xi(k_v)$, $[t \in [0, T]$ and $\mathbb{S}_N^{(v)}(t) = N^{-v/2} \int_{0 \le s_1 < \cdots < s_v \le N_t} \xi(s_1) \otimes \cdots \otimes \xi(s_v) ds_1 \ldots ds_v$, where $\{\xi(k)\}_{-\infty < k < \infty}$ and $\{\xi(s)\}_{-\infty < s < \infty}$ are centered stationary vector processes with some weak dependence properties which can be generated, in particular, by dynamical systems so that $\xi(k) = g \circ T^k$ or $\xi(s) = g \circ T^s$ for some (vector) function g. The results are applicable, in particular, to hyperbolic and expanding dynamical systems with their Gibbs measures, Gibbs-Markov maps and some systems which can be represented via the Young towers construction. Sequences of iterated sums and integrals were called signatures in recent papers in rough paths, data science and machine learning.

The Exponential Lie Series and pre-Lie Magnus vector fields

Kurusch Ebrahimi-Fard

(joint work with Frédéric Patras, Anke Wiese)

In 1994, Gaines [7] defined and examined the quasi-shuffle algebra of iterated stochastic integrals, specifically focusing on multiple iterated integrals of Wiener processes. In 2000, Hoffman [8] studied the quasi-shuffle algebra using a Hopf algebraic framework and introduced a rather natural Hopf algebra isomorphism (known as Hoffman's exponential) between the shuffle and quasi-shuffle Hopf algebra. We refer to [6] for details from the viewpoint of deformations of the shuffle Hopf algebra.

In earlier work [4], we consider stochastic differential systems driven by continuous semimartingales and governed by non-commuting vector fields. We show that Hoffman's exponential naturally relates Itô and Fisk–Stratonovich multiple integrals. This permits to express the flowmap in Fisk–Stratonovich form. It is then shown that the logarithm of the flowmap is an exponential Lie series and the corresponding Chen–Strichartz formula is given which provides an explicit formula for the Lie series coefficients. The Chen–Strichartz formula has shown to play a pivotal role in the design of numerical integration schemes that preserve qualitative properties of the solution such as the construction of geometric numerical schemes and in the context of efficient numerical schemes.

In the recent work [5], we extend previous results from [2, 4] by deriving a Chen–Strichartz formula for stochastic differential equations driven by Lévy processes, that is, we derive a series expansion of the logarithm of the flowmap of the stochastic differential equation in terms of commutators of vector fields with stochastic coefficients, and we provide an explicit formula for the components in this series. The stochastic components are generated by the Lévy processes that drive the stochastic differential equation and their quadratic variation and power jumps; the vector fields are given as linear combinations of commutators of elements in the pre-Lie Magnus expansion [1, 3] generated by the original vector fields governing our stochastic differential equation. In particular, we show that the logarithm of the flowmap for Lévy-driven stochastic differential equations is a Lie series.

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Expected Signature on a Riemannian Manifold and Its Geometric Implications

Xi Geng
(joint work with Hao Ni, Chaorui Wang)

On a compact Riemannian manifold M, we show that the Riemannian distance function $d(x,y)$ can be explicitly reconstructed from suitable asymptotics of the expected signature of Brownian bridge from x to y. In addition, by looking into the asymptotic expansion of the fourth level expected signature of the Brownian loop based at $x \in M$, one can explicitly reconstruct both intrinsic (Ricci curvature) and extrinsic (second fundamental form) curvature properties of M at x. As independent interest, we also derive the intrinsic PDE for the expected Brownian signature dynamics on M from the perspective of the Eells-Elworthy-Malliavin horizontal lifting.

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Strong regularization of differential equations with integrable drifts by fractional noise

Khoa Lê
(joint work with Oleg Butkovsky, Toyomu Matsuda)

We consider stochastic differential equations with integrable time-dependent drift driven by additive fractional Brownian noise whose Hurst parameter is less than 1/2. Under some subcriticality conditions, it is shown that such equation have a unique pathwise solution which is also path-by-path unique. Furthermore, stability with respect to all parameters is established and the dynamical description of its gradient flow is investigated. Our strong uniqueness result can be considered as an extension of that from Krylov and Röckner [1] for Brownian motion. It holds under the subcritical regime observed earlier by Galeati and Gerencsér [2], and improves upon previous results of Nualart and Ouknine [3] for dimension one and the second author [4]. Our methods are built around Lyons' rough path theory, Girsanov's theorem, the stochastic sewing lemma and the quantitative

John–Nirenberg inequality for stochastic processes of vanishing mean oscillation [5].

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Degree-7 cubature on Wiener space

Emilio Ferrucci
(joint work with Timothy Herschell, Christian Litterer, Terry Lyons)

We present an explicit degree-7 cubature formula on Wiener space with drift in the sense of Lyons and Victoir [1]. Our formula was derived thanks to the use of the intrinsic Hopf algebra structure on the tensor algebra, in particular the Eulerian idempotent, which replaces the PBW basis.

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Non-Markovian optimal stopping with signatures

Luca Pelizzari
(joint work with Christian Bayer, John Schoenmakers)

We consider the finite horizon optimal stopping problem for α-Hölder continuous state-processes $(X_t : 0 \leq t \leq T)$, that is

$$Y_0 = \sup_{\tau \in \mathcal{S}_0} \mathbb{E}[Z_\tau], \quad Y_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_t} \mathbb{E}[Z_\tau | \mathcal{F}_t^X], \quad 0 < t \leq T, \tag{1}$$

where Z is continuous and $\mathbb{F}^X-$adapted, and $\mathcal{S}_t$ denotes the set of $\mathbb{F}^X-$stopping-times on $[t, T]$. Our particular interest lies in highly non-Markovian processes X, e.g. fractional Brownian motion or asset-price dynamics in rough volatility models, where (at least conventional) Hamilton-Jacobi-Bellman (HJB) equations do not exist.

We present two practical solution methods based on the rough-path signature lift $\mathbf{X}^{<\infty}$ of the time-augmentation (t, X_t), which serves as a universal, Markovian lift, encoding the memory of the state-process. First, replacing $[0,T]$ by some grid $0 = t_0 < t_1 < \cdots < t_N = T$, we define the sequence of stopping times

$$\tau_N^\theta = t_N, \quad \tau_n^\theta = t_n 1_{\{Z_{t_n} \geq f^\theta(\mathbf{X}_{0,t_n}^{<\infty})\}} + \tau_{n+1}^\theta 1_{\{Z_{t_n} < f^\theta(\mathbf{X}_{0,t_n}^{<\infty})\}}, \quad 0 \leq n < N,$$

for $f^\theta(\mathbf{x}) = \langle \theta, \mathbf{x} \rangle$ a linear functional on the tensor algebra. On the other hand, we define the martingales

$$M_t^\theta = \int_0^t f^\theta(\mathbf{X}_{0,s}^{<\infty}) dW_s, \quad 0 \leq t \leq T,$$

where W is a standard Brownian motion. Based on a global $\mathbb{L}^p$-approximation result for signatures [1, Theorem 2.8], we get the following loose formulation of [1, Proposition 3.3 and 3.8].

Theorem 1. *Assuming that* $\sup_{t\leq T} |Z_t| \in \mathbb{L}^2$ *and* $\mathbb{F}^X = \mathbb{F}^W$*, we have*

$$\mathbb{E}[Z_{\tau_0^{\theta_n}}] \nearrow Y_0^N, \quad \mathbb{E}[\max_k (Z_{t_k} - M_{t_k}^{\eta_n})] \searrow Y_0^N, \quad n \to \infty,$$

where Y^N *denotes the discrete-time version of* (1)*, and the sequences* (θ^n) *and* (η^n) *can be obtained from standard minimization problems with respect to the truncated signature* $\mathbf{X}^{\leq n}$.

Similar convergence results can be established by substituting expectations with averages [1, Proposition 3.4 and 3.10], resulting in easy-to-implement algorithms that approximate the optimal stopping value from both above and below. We apply these methodologies for the problem of optimally stopping fractional Brownian motion, and for pricing American options in rough volatility models.

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Duality, signatures, and measure valued processes

Sara Svaluto-Ferro

(joint work with Christa Cuchiero, Josef Teichmann)

In duality theory the moments of a process X are represented through the moments of an auxiliary process U

$$\mathbb{E}[f(U_0, X_T)] = \mathbb{E}[f(U_T, X_0)].$$

We explain how we can apply this principle when X is the signature process of a (jump-)diffusion. We also resource to existence results for sub-probability measures valued processes to construct a dual process U satisfying the needed integrability conditions. More details are available in [1].

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An improved Runge-Kutta method for SDEs with additive noise

James Foster

In this talk, we consider the numerical approximation of additive-noise SDEs,

$$dy_t = f(y_t)\,dt + \sigma\,dW_t\,, \tag{1}$$

where, for simplicity, we assume $y_t \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ is smooth, $\sigma > 0$ and $W_t \in \mathbb{R}$.

Given a sequence of times $0 = t_0 < t_1 < \cdots < t_N = T$, we define $Y_0 := y_0$ and consider the following one-parameter family of stochastic Runge-Kutta methods:

$$\begin{aligned}\widetilde{Y}_n &:= Y_n + c\,\sigma H_n\,,\\ \widetilde{Y}_{n+\alpha} &:= \widetilde{Y}_n + \alpha\big(f(\widetilde{Y}_n)h_n + \sigma W_n + 2(1-c)\sigma H_n\big),\\ Y_{n+1} &:= Y_n + \Big(1 - \frac{1}{2\alpha}\Big)f(\widetilde{Y}_n)h_n + \frac{1}{2\alpha}f(\widetilde{Y}_{n+\alpha})h_n + \sigma W_n\,, \end{aligned} \tag{2}$$

where (W_n, H_n) are the increment and "space-time" Lévy area of Brownian motion over $[t_n, t_{n+1}]$ (see [1]), $h_n := t_{n+1} - t_n$ is a step size, $c \in \mathbb{R}$ and $\alpha := \frac{1}{2} + \frac{1}{3+(1-c)^2}$.

Due to its structure, (2) achieves a high order strong convergence rate of $O(h^{\frac{3}{2}})$ and reduces to Rößler's SRA1 scheme [2] when $c = 0$. Taylor expanding (2) gives

$$\begin{aligned} Y_{n+1} = Y_n + f(Y_n)h_n + \sigma W_n + \sigma f'(Y_k)\int_{t_n}^{t_{n+1}}\big(W_t - W_{t_n}\big)\,dt + \frac{1}{2}f'(Y_n)f(Y_n)h_n^2\\ + \sigma^2 f''(Y_n)\underbrace{\Big(\frac{1}{4}\alpha h_n W_n^2 + \frac{1}{2}(c + 2\alpha(1-c))h_n W_n H_n + \frac{1}{2}\big(2c - c^2 + 2\alpha(1-c)^2\big)h_n H_n^2\Big)}_{=:\, I(c)} + R_n\,, \end{aligned}$$

where the remainder term R_n satisfies $\mathbb{E}\big[R_n^2\big] = O\big(h_n^5\big)$. Our main result is then

Theorem 1. *Let $I_{true} := \frac{1}{2}\int_{t_n}^{t_{n+1}}\big(W_t - W_{t_n}\big)^2 dt$. For any c, we have the errors*

$$\mathbb{E}\big[I_{true} - I(c)\big] = 0,$$
$$\mathbb{E}\Big[\big(I_{true} - I(c)\big)^2\Big] = \frac{1}{30}(1-\alpha)h_n^4.$$

Since I_{true} is precisely the integral that appears in the Taylor expansion of (1), we can reduce the asymptotic local mean-squared error of Rößler's scheme by 33% just by setting $c = 1$ (which gives $\alpha = \frac{5}{6}$). The resulting scheme (called ShARK[1])

[1] **Sh**ifted **A**dditive-noise **R**unge-**K**utta

naturally extends to multidimensional additive-noise SDEs and, along with SRA1, has recently been implemented in the Diffrax package for differential equations [3].

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Weak error estimates for rough volatility models

Thomas Wagenhofer

(joint work with Peter K. Friz, William Salkeld)

We consider a class of stochastic processes with rough stochastic volatility, examples of which include the rough Bergomi, see for example [2], and rough Stein-Stein model ([1, 5]), that have gained considerable importance in quantitative finance [3, 8]. We consider the process

$$X_t = X_0 + \int_0^t f\left(W_s^H\right) dB_s.$$

with Riemann-Liouville fractional Brownian motion $W^H = \int_0^t (t-s)^{H-1/2} dW_s$. Here f is a deterministic volatility function and W and B are correlated correlated via $B_t = \rho dW_t + \sqrt{1-\rho^2} W_t^{\perp}$ for some $\rho \in [-1, 1]$.

We consider the standard left-point approximation of X, that is

$$X_{\frac{j}{n}}^{(n)} - X_0 = \sum_{i=0}^{j-1} f\left(W_{\frac{i}{n}}^H\right) B_{\frac{i+1}{n}} - B_{\frac{i}{n}}.$$

A basic question for such (non-Markovian) models concerns efficient numerical schemes. While strong rates are well understood (order H), we tackle the intricate question of weak rates. Over the last years, several authors, for example [4, 5, 7] or [6], have studied this problem. We were able to proof the following

Theorem 1. *Let $f \in \mathcal{C}^N$, $N \in \mathbb{N}$, such that f and its N derivatives have at most exponential growth. Let Φ be a polynomial test function with* $\deg(\Phi) \le N$.

(1) *There is a constant C_N such that, as $n \to \infty$,*

$$\mathbb{E}\Big[\Phi\big(X_T\big)\Big] - \mathbb{E}\Big[\Phi\big(X_T^{(n)}\big)\Big] \le \begin{cases} C_N n^{-3H-1/2} \vee n^{-1} & \text{for } H \neq 1/6, \\ C_N n^{-1} \log(n) & \text{for } H = 1/6. \end{cases}$$

(2) *In the uncorrelated case, with $\rho = 0$, we have*

$$\mathbb{E}\Big[\Phi\big(X_T\big)\Big] - \mathbb{E}\Big[\Phi\big(X_T^{(n)}\big)\Big] \le C_N n^{-1} \qquad \text{any } H > 0.$$

Our results are complemented by a lower bound which show that in the case $\Phi(x) = x^3$, $f(x) = x$ and $H < 1/6$ the obtained weak rate is indeed optimal.

To derive these weak rates we observed, that there is a closed formula for moments of X and $X^{(n)}$. For this closed formula we considered the open simplex Δ_m° in $\mathbb{R}^m$ and functions $F : \mathbb{R}^m \times \Delta_m^\circ \to \mathbb{R}$. For such functions we defined operators $(\mathcal{I}^N F), (\mathcal{J}^N F) : \mathbb{R}^{m+1} \times \Delta_{m+1}^\circ \to \mathbb{R}$ via

$$(\mathcal{I}^N F)(x_1, ..., x_m, y, t_1, ..., t_m, s) = \rho N f(y) \sum_{j=1}^{m} \partial_{x_j} F(x_1, ..., x_m, t_1, ..., t_m) K(t_j, s),$$

as well as

$$(\mathcal{J}^N F)(x_1, ..., x_m, y, t_1, ..., t_m, s) = \frac{N(N-1)}{2} f^2(y) F(x_1, ..., x_m, t_1, ..., t_m).$$

Let $\mathcal{W}$ be an alphabet over words w with letters I, J. Define a length ℓ such that $\ell(I) = 1$, $\ell(J) = 2$ and define an embedding ι such that $\iota(wI) = \iota(w) \circ \mathcal{I}^{\ell(wI)}$ and $\iota(wJ) = \iota(w) \circ \mathcal{J}^{\ell(wJ)}$. Given this construction we end up with the following moment formula:

Theorem 2. *Let $N \geq 1$. Then*

$$\begin{aligned}
&\mathbb{E}\Big[(X_T)^N\Big] \\
&= \sum_{\substack{w \in \mathcal{W} \\ \ell(w) = N}} \int_0^T \int_0^{t_1} \dots \int_0^{t_{|w|-1}} \mathbb{E}\Big[\big(\iota(w)1\big)\big(W_{t_1}^H, ..., W_{t_{|w|}}^H, t_1, ..., t_{|w|}\big)\Big] dt_{|w|} ... dt_1.
\end{aligned}$$

The right hand side now consists of iterated integrals of functions of Gaussian random variables. Together with the following lemma we could show the main theorem.

Lemma 1. *Let $\Sigma : [0, T] \to \mathbb{R}^{d \times d}$ be symmetric and positive semi-definite. Let $g : \mathbb{R}^d \to \mathbb{R}$ smooth and $W(t) \sim \mathcal{N}(0, \Sigma(t))$ and define the function $\varphi(t) = \mathbb{E}\big[g(W(t))\big]$. Then φ is in C^1 and*

$$\partial_t \varphi(t) = \sum_{k,l=1}^{d} \frac{1}{2} \partial_t \Sigma(t)_{k,l} \mathbb{E}\big[\partial_k \partial_l g(W(t))\big].$$

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Branched Itô formula and Itô-Stratonovich isomorphism

Nikolas Tapia
(joint work with Carlo Bellingeri, Emilio Ferrucci)

Branched rough paths, defined as characters over the Connes-Kreimer Hopf algebra $\mathcal{H}_{\text{CK}}$, constitute integration theories that may fail to satisfy the usual integration by parts identity. Using known results on the primitive elements we view it as a commutative cofree Hopf algebra [5] (i.e. a commutative $\mathbf{B}_\infty$-algebra) over the space $\mathcal{P}$ of primitive elements. More precisely, there is a projection $\pi\colon \mathcal{H}_{\text{CK}} \to \mathcal{P}$ which decomposes $\mathcal{H}_{\text{CK}} \cong T(\mathcal{P})$ as a coalgebra, the isomorphism being given by abstract integration maps induced by the so-called natural growth [2, 3]. This allows us to consider Rough Differential Equations (RDEs) with drifts, i.e., of the form

$$\mathrm{d}Y_t = \sum_{\tau\in\mathcal{F}} V_\tau \,\mathrm{d}\mathbf{X}^{\pi(\tau)}.$$

and write an explicit change-of-variable formula for solutions to such RDEs in the form

$$\varphi(Y_t) = \varphi(Y_0) + \sum_{\tau\in\mathcal{F}} \int_0^t \mathbb{V}_\tau\varphi(Y_s)\,\mathrm{d}\mathbf{X}^{\pi(\tau)}$$

for any smooth function φ, and where $\mathbb{V}_\tau$ is a differential operator of order $|\tau|$, defined in terms of the original vector fields V, and in both formulas the integrals are defined as rough integrals of suitable germs.

This formula, which is realised through an explicit morphism from the Grossman–Larson Hopf algebra to the Hopf algebra of differential operators, restricts to the well-known Itô formula in the very special case of semimartingales, and builds on previous work by D. Kelly were a similar idea (the bracket extension) was used [4]. Our approach, however, avoids the need for any additional lifts and is therefore only reliant on the information already contained in $\mathbf{X}$.

In addition, we establish an isomorphism between $\mathcal{H}_{\text{CK}}$ and the shuffle algebra $(T(\mathcal{P}), ⧢, \Delta)$, which extends Hoffman's exponential for the quasi-shuffle algebra, and can therefore be viewed as a far-reaching generalisation of the usual Itô-Stratonovich correction formula for semimartingales. The isormorphism is defined by using the cofreeness of $\mathcal{H}_{\text{CK}}$ to extend the Eulerian idempotent e. In other words, we define a map $\mathrm{Log}\colon \mathcal{H}_{CK} \to T(\mathcal{P})$ induced by the composition $\pi\circ e$. Indeed, this can be stated as a characterisation of the algebra structure of any commutative $\mathbf{B}_\infty$-algebra: *every commutative* $\mathbf{B}_\infty$ *algebra is (isomorphic to) a shuffle algebra (over its primitive space).* Compared to previous approaches, this transformation has the key property of being natural in the underlying vector space and therefore is well suited for extending the theory of RDEs driven by branched rough paths to manifolds with connections.

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Signature transforms from the point of view of invariant theory

Josef Teichmann

(joint work with Valentin Tissot-Daguette, Walter Schachermayer)

Invariant Theory has been an influential subject in mathematics for a long time: it is an important and far reaching question how functions on a geometric object look like which are invariant with respect to a given group action. As guiding example on can consider functions on $\mathbb{R}^n$ invariant under the action of the group of orthogonal transformations. It turns out that invariant polynomials are actually polynomials of the radius square which serves as a generating invariant polynomial among all invariant polynomials, see, e.g., [3].

We are interested in an infinite dimensional analog of this question. Let X denote the Banach space of continuous bounded variation curves on $[0,1]$ with values in $\mathbb{R}^d$ starting at $0 \in \mathbb{R}^d$ at time 0. On bounded variation curves $\omega \in X$ we consider the group action $\omega^\sigma := \omega \circ \sigma$ of continuous reparametrizations σ of $[0,1]$, i.e. monotone increasing homeomorphisms on $[0,1]$, which preserve 0 and 1. We ask for the structure of invariant polynomials and invariant real analytic functions on X (for a convenient definition see [2]) in this case.

The main result is the following: let $f : 0 \in U \subset X \to \mathbb{R}$ be an invariant real analytic function on an open domain containing 0, then its power series expansion at 0 can be solely written in terms of signatures, i.e.

$$f(\omega) = \sum_{k \geq 0} \sum_{|w|=k} a_w \operatorname{Sig}^w(\omega)\,,$$

where signature component $\operatorname{Sig}^w(\omega)$ at length k word $w \in \{1,\dots,d\}^k$ is given by the iterated integral

$$\operatorname{Sig}^w(\omega) = \int_{0 \leq t_1 \leq \ldots \leq t_k \leq 1} d\omega^{w_1}(t_1) \cdots d\omega^{w_k}(t_k)$$

for $\omega \in X$. This fully characterizes the set of all invariant polynomials as well as the set of all invariant real analytic functions on $U \ni 0$. Notice that invariant real analytic functions turn out to be weak-$*$-continuous due to symmetry even though they are only continuous with respect to the strong topology a priori. Notice also that the space of invariant, symmetric k-multilinear, strongly continuous maps on X is therefore actually finite dimensional, namely spanned by signature components of words of length k.

It is furthermore interesting to ask which curves ω have trivial signature, i.e. the same signature as 0. It turns out, following the seminal and beautiful work [1], that those are precisely the tree-like curves. We can now also characterize those curves by the property they can be deformed to the constant curve taking value 0 within their own range with an argument coming from complex analysis.

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A gradient flow on control space with rough initial condition

Paul Gassiat

(joint work with Florin Suciu)

Given smooth vector fields $V_1, \dots, V_d$ on $\mathbb{R}^n$, an initial point $x \in \mathbb{R}^n$, consider, for $u \in L^2 := L^2([0,1], \mathbb{R}^d)$ the controlled ODE

$$X_t = x + \int_0^t \sum_{i=1}^d V_i(X_t) u_t^i dt, \quad t \in [0,1]. \tag{1}$$

We are then given a target point $y \in \mathbb{R}^n$ and consider the following classical deterministic control problem.

$$\text{Find } u \in L^2([0,1], \mathbb{R}^d) \text{ s.t. } X_1 = y. \tag{2}$$

Note that in many situations of interest, the number of vector fields d is smaller than the ambient dimension n, so that, at a given time t, the system is only allowed to move in a subspace of all possible directions. However, assuming the bracket-generating condition

$$\forall z \in \mathbb{R}^n, \quad \mathrm{Lie}(V_1, \dots, V_d)\big|_z = \mathbb{R}^n, \tag{3}$$

the well-known Chow-Rashevskii theorem guarantees the existence of a solution to (2).

Motivated by understanding the gradient descent in deep neural networks, we consider a non-problem specific gradient flow procedure : let $\mathcal{L}$ be defined by

$$\mathcal{L} : u \in L^2([0,1],\mathbb{R}^d) \;\mapsto\; \left|y - X_1^x(u)\right|^2,$$

so that $\mathcal{L} \geq 0$ and any zero of $\mathcal{L}$ is a solution to (2), and given an initial control u_{init}, consider the ODE (valued in $L^2([0,1],\mathbb{R}^d)$)

$$u(0) = u_{init}, \;\; \frac{d}{ds}u(s) = -\nabla_{L^2}\mathcal{L}(u(s)).$$

We are then interested in the following question : can we find conditions guaranteeing that for u as above, it holds that

$$\lim_{s\to\infty} u(s) = u_\infty \text{ with } \mathcal{L}(u_\infty) = 0. \tag{4}$$

A first positive observation is that, thanks to the openness of the endpoint map $u \mapsto X_1^x(u)$ under the bracket-generating condition, $\mathcal{L}$ admits no non-global local minima. However, $\mathcal{L}$ may in general admit **critical points which are not minima** : indeed, since the gradient of $\mathcal{L}$ is seen to be

$$(\nabla\mathcal{L})(u) = (y - X_1^x(u)) \cdot_{\mathbb{R}^n} (\nabla_{L^2} X_1^x)(u), \tag{5}$$

this may happen if u is such that the differential of the endpoint map is not surjective. Such controls are well-known to exist (for instance, $u = 0$ is always one if $d < n$, since then $Im(dX_1^x)$ is spanned by $V_1(x), \ldots, V_d(x)$), and play an important role in sub-Riemannian geometry (they are typically called *singular controls*). Another serious problem when trying to prove convergence is that, since $\mathcal{L}$ does not contain any cost (or penalization) term, its sub-level sets are not bounded (in fact, it is easy to see that it has zeroes of arbitrarily high norm), and there is no a priori guarantee that the trajectory will not **diverge to infinity**.

We are interested in the regime where u_{init} is **rough**, for instance $u_{init} = \dot{B}$ white noise. Note that, in this case u_{init} is not in L^2, but one can still makes sense of the above gradient flow in a robust way via the modern stochastic analysis techniques of **rough path theory**.

Our first result is of a qualitative nature and shows, in a rather general setting, an advantage of initialising from such a rough initial condition.

Theorem 1. *Let $V_1, \ldots, V_d$ be C_b^∞ bracket-generating vector fields on $\mathbb{R}^n$. Let $u_{init} = \dot{B}(\omega)$ where B is a Brownian motion. Then, almost surely :*

(1) There exists $v \in L^2$ such that $\mathcal{L}(u_{init} + v) = 0$.

(2) For any v in L^2, $u_{init} + v$ is not a saddle-point, i.e. $\nabla\mathcal{L}(u_{init} + v) = 0 \Rightarrow \mathcal{L}(u_{init} + v) = 0$.

(3) If the trajectory $(v(s) = u(s) - u_{init})_{s\geq 0}$ is bounded in L^2, then convergence (4) *holds.*

The above result, while a clear hint that rough initial conditions may help, does not guarantee convergence as the gradient flow could still diverge to infinity. Our next theorem, shows (almost sure) convergence in a simple (but non-trivial) case.

Theorem 2. *Let $V_1, \ldots, V_d$ be C_b^∞ bracket-generating vector fields on $\mathbb{R}^n$, with step-2 nilpotent Lie algebra, i.e.*

$$\forall i, j, k \in \{1, \ldots, d\}, \quad [[V_i, V_j], V_k] \equiv 0.$$

Let $u_{init} = \dot{B}(\omega)$ where B is a Brownian motion. Then, almost surely, for any initial and target points $x, y \in \mathbb{R}^n$, convergence (4) holds.

The convergence proof is based on combining ideas from Malliavin calculus with Łojasiewicz inequalities.

A possible motivation for our study comes from the training of deep Residual Neural Nets, in the regime when the number of trainable parameters per layer is smaller than the dimension of the data vector. Our positive results can be seen as a theoretical justification for a choice of initialized weights with SDE-type scaling (see [1, 2, 4] for related recent works)

The talk is based on the article [3].

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Transportation Marketplace Rate Forecast using Signature Transform

Xin Guo

(joint work with Haotian Gu, Tim Jacobs, Phil Kaminsky, Xinyu Li)

Freight transportation marketplace rates are typically challenging to forecast accurately. In this work, we have developed a novel statistical technique based on signature transforms and have built a predictive and adaptive model to forecast these marketplace rates. Our technique is based on two key elements of the signature transform: one being its universal nonlinearity property, which linearizes the feature space and hence translates the forecasting problem into linear regression, and the other being the signature kernel, which allows for comparing computationally efficiently similarities between time series data. Combined, it allows for efficient feature generation and precise identification of seasonality and regime switching in the forecasting process.

An algorithm based on our technique has been deployed by Amazon trucking operations, with far superior forecast accuracy and better interpretability versus commercially available industry models, even during the COVID-19 pandemic and the Ukraine conflict. Furthermore, our technique is able to capture the influence of business cycles and the heterogeneity of the marketplace, improving prediction accuracy by more than fivefold, with an estimated annualized saving of $50 million.

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Some path-dependent processes from signatures

Eduardo Abi Jaber

(joint work with Louis-Amand Gérard, Yuxing Huang)

We provide explicit series expansions to certain stochastic path-dependent integral equations in terms of the path signature of the time augmented driving Brownian motion. Our framework encompasses a large class of stochastic linear Volterra and delay equations and in particular the fractional Brownian motion with a Hurst index $H \in (0, 1)$. Our expressions allow to disentangle an infinite dimensional Markovian structure. In addition they open the door to:

(1) straightforward and simple approximation schemes that we illustrate numerically
(2) representations of certain Fourier-Laplace transforms in terms of a non-standard infinite dimensional Riccati equation with important applications for pricing and hedging in quantitative finance.

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On the signature of an image

Fabian Harang

(joint work with Joscha Diehl, Kurusch Ebrahimi Fard, Samy Tindel)

The 1D signature, introduced by T. Lyons within the framework of rough path theory, has revolutionized the analysis of complex systems. Originally developed to provide a pathwise approach to stochastic differential equations [Lyo98], it has proven to be a powerful tool in extracting fundamental features of paths, see, e.g., [CK16]. Its success lies in its ability to encode the essential structure of paths in an infinite series of iterated integrals, capturing both local and global information. Beyond rough paths, the 1D signature has found remarkable applications in data science, where it serves as an effective feature extraction tool for time series, with successes in tasks ranging from handwriting recognition to psychiatric diagnosis, and more.

This talk will explore how the strengths of the 1D signature are extended to two-dimensional domains through the 2D signature, a novel mathematical framework designed to handle fields such as images. We will discuss the challenges of generalizing the algebraic and analytical principles of the 1D signature, to the 2D signature, such as Chen's relation and shuffle-type products. It also introduces invariances to transformations like translation, stretching, and rotation, ensuring its applicability across diverse datasets.

A cornerstone of the 2D signature is its universal approximation property, which allows it to approximate continuous functionals on smooth fields with arbitrary precision. This property, combined with its invariances, makes it a compelling tool for feature extraction and classification tasks in image analysis and beyond, as illustrated in the experimental work [ZLT22]. Furthermore, the 2D signature has significant theoretical implications, such as its role in solving hyperbolic partial differential equations with multiplicative noise.

In this talk, we will highlight the mathematical foundation and practical applications of the 2D signature. While addressing the challenges of extending the signature framework to two dimensions–such as handling simplex permutations and ensuring a shuffle product construction, we will outline potential directions for future research and applications. The 2D signature opens exciting new avenues for understanding and processing two-parameter data, blending rigorous mathematics with practical innovation.

The talk is based on the joint work with Joscha Diehl, Kurusch Ebrahimi Fard, and Samy Tindel [DEFHT24]. All authors are grateful to the Center for Advanced Studies (CAS) in Oslo for funding the "Signatures for Images" project over the academic year 2023/2024, during which the writing of this article happened.

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Trees vs. multi-indices in rough paths and regularity structures

Pablo Linares

Given a differential equation of the form

$$\mathrm{d}Y_t = a_{\mathfrak{l}}(Y_t)\mathrm{d}X_t^{\mathfrak{l}}, \quad Y : [0,1] \to \mathbb{R},$$

we consider representations of the solutions in terms of multi-indices, as first introduced in [9] for quasi-linear SPDEs. These take the form

$$Y_t = \sum_{\beta} z^{\beta}[a, Y_s]\mathbf{x}_{st;\,\beta},$$

where the sum runs over multi-indices β of $(\mathfrak{l}, k)$,

$$z^{\beta}[a, Y_s] = \prod_{(\mathfrak{l},k)} \big(\tfrac{1}{k!}\tfrac{\mathrm{d}^k a_{\mathfrak{l}}}{\mathrm{d}y^k}(Y_s)\big)^{\beta(\mathfrak{l},k)},$$

and the components $\mathbf{x}_{st;\,\beta}$ are recursively constructed (assuming smoothness) solving an infinite family of linear ODEs. The goal is to compare such expansions with those based on trees in branched rough paths [6].

We make the comparison at the level of the underlying pre-Lie algebra structure. While tree-based expansions rely on the free pre-Lie algebra [4], (populated) multi-indices together with the pre-Lie product defined by the derivation $D = \sum_{(\mathfrak{l},k)}(k+1)z_{(\mathfrak{l},k+1)}\partial_{z_{(\mathfrak{l},k)}}$ give the structure of the free Novikov algebra [5]. The pre-Lie algebra structure allows us to rewrite the hierarchy of defining equations for $\mathbf{x}$ in terms of a differential equation of the form

$$\mathrm{d}\mathbf{x}_{st} = \rho_D(\exp_D(\mathbf{x}_{st}))z_{(\mathfrak{l},0)}\mathrm{d}X_t^{\mathfrak{l}}, \quad \mathbf{x}_{ss} = 0,$$

cf. [8], which is the pre-Lie version of Cartan's development for Hopf-algebraic smooth rough paths [1], and which is based on the Guin–Oudom procedure [7]. We also construct an analogue of the insertion pre-Lie product and the extraction-contraction Hopf algebra in trees [3] for multi-indices, based on the pre-Lie product given by derivations of the form

$$\sum_{k} \big(\tfrac{1}{k!}D^k z^{\gamma}\big)\partial_{z_{(\mathfrak{l},k)}},$$

cf. [8]. This allows us to define groups of algebraic renormalization of rough paths.

These ideas can be extended to the context of regularity structures, [2].

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Regularization by multiplicative noise

Chengcheng Ling
(joint work with Konstantinos Dareiotis, Máté Gerenscér, Gerald Lampl, Khoa Lê)

Understanding the effect of random perturbations in deterministic dynamics is a central topic of research in stochastic analysis. *Regularization by noise* (RBN) refers to the beneficial effects produced by random perturbations for various types of systems. The simplest instance of this phenomenon are stochastic differential equations (SDEs) driven by an additive Brownian motion W:

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}W_t, \quad X_0 = x \in \mathbb{R}^d$$

where strong well-posedness holds even when the vector field b fails to be Lipschitz continuous. A concrete example clearly given is in one dimensional Euclidean space $\mathbb{R}$ let us consider the ordinary differential equation (ODE) and SDE with the singular drift $b(x) = 2\mathrm{sign}(x)\sqrt{|x|^2}$ which has singular point at $x = 0$:

$$dX_t = 2\mathrm{sign}(X_t)\sqrt{|X_t|^2}dt, \quad dX_t = 2\mathrm{sign}(X_t)\sqrt{|X_t|^2}dt + dW_t, \quad X_0 = 0.$$

Without noise the ODE has many solutions, e.g. $X_t = 0$ and another two extremal solutions $X_t = \pm t^2$. Surprisingly the corresponding SDE has a unique strong solution, therefore it is well-posed.

In general, for an ODE with singular non-Lipschitz coefficient, we know from the classical Peano theory that there may exist non-unique solutions. Get along with the development of Itô's martingale theory, it has been shown that adding further Brownian type noise could save the ill-posed equation via smoothing out the singularity and make it to be well-posed. Recently this field has been extensively explored among different concepts from stochastic calculus. In this talk we will introduce the ideas from classical Itô calculus involving the theory from PDEs and modern tools from rough path theory and Malliavin calculus for tackling the problems on well-posedness theory and numerics of a class of singular SDEs driven by multiplicative Brownian [1, 2] and fractional Brownian noise [3].

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Loop invariants and conjugation

Rosa Preiss
(joint work with Joscha Diehl, Jeremy Reizenstein)

Observing a closed trajectory, it is natural to ask which properties, like the enclosed area, remain the same under changing the simultanous start- and endpoint across the loop. Let $S(X) \in T((\mathbb{R}^d)) = T(\mathbb{R}^d)^*$ denote the iterated-integrals signature of the path X.

Definition 1. *We call $u \in T(\mathbb{R}^d)$ a* loop invariant *if*

$$\langle S(X), u\rangle = S(X'), u\rangle$$

holds for all loops X and X' such that X and X' only differ by moving the start- and endpoint.

We relate these to conjugation invariants, which are a canonical object of study when treating (tree reduced) paths as a group with multiplication given by the concatenation. Let $\overleftarrow{X}$ denote the path X traversed backwards and $X \sqcup Y$ the concatenation of the paths X and Y.

Definition 2. *We call $v \in T(\mathbb{R}^d)$ a* conjugation invariant *if*

$$\langle S(Z \sqcup Y \sqcup \overleftarrow{Z}), v\rangle = \langle S(Y), v\rangle$$

holds for all paths Y and Z.

If we restrict the condition to loops Y, then we call the larger set of v satisfying the relation *conjugation invariants for loops.*

A key observation is the following.

Proposition 1. *$u \in T(\mathbb{R}^d)$ is a conjugation invariant for loops if and only if it is a loop invariant.*

During the talk, the proof was sketched pictorially.

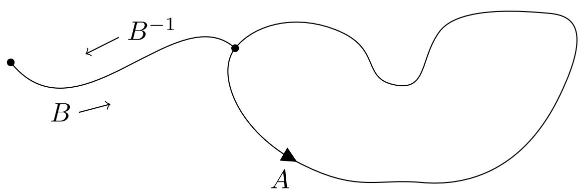

Figure 1: A closed path A conjugated by a path B [1, Figure 3.2].

We conjecture that all loop invariants can be written as shuffles of trivial loop invariants (shuffles of anything with letters), signed areas (`ij`−`ji`) and conjugation invariants.

Conjugation invariants, however, are fully charaterized in terms of combinatorial necklaces. For an introduction to combinatorial necklaces see [3, Chapter 7].

One of our main theorems is then the following.

Theorem 1. *There are infinitely many algebraically independent loop invariants.*

Another open problem that remains is to characterize the equivalence relation $X \sim X'$ defined by

$$\langle S(X), u\rangle = \langle S(X'), u\rangle \text{ for all loop-invariants } u.$$

The talk was based on the preprint [1] and featured an excursion on some contents of [2].

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Controlled RSDEs, pathwise stochastic control, dynamic programming principles

Huilin Zhang

(joint work with Peter Friz and Khoa Lê)

We study stochastic optimal control of rough stochastic differential equations (RSDEs). This is in the spirit of the pathwise control problem (Lions–Souganidis 1998, Buckdahn–Ma 2007; also Davis–Burstein 1992). With renewed interest from a variety of fields, including filtering, reinforcement learning and SPDE theory, we establish regularity of *rough* value functions, validity of a *rough* dynamic programming principles and new *rough* stability results for HJB equations, removing excessive regularity demands previously imposed by flow transformation methods. Measurable selection is used to see that RSDEs have jointly measurable version, allowing us to relate RSDEs to "doubly stochastic" SDEs under conditioning. In contrast to previous works, Brownian statistics for the to-be-conditioned-on noise are not required, aligned with the "pathwise" intuition that these should not matter upon conditioning. Depending on the chosen class of admissible controls, the involved processes may also be anticipating. The resulting stochastic value functions coincide in great generality for different classes of controls. RSDEs theory offers a powerful and unified perspective on this problem class.

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Deep Reinforcement Learning for Games with Controlled Jump-diffusion Dynamics

Ruimeng Hu

(joint work with Liwei Lu, Xu Yang, Yi Zhu)

Many real-world multi-party decision-making problems are subject to sudden exogenous events–such as wars, central bank decisions, or global crises like COVID-19–that can cause major shifts in the system, impacting all players simultaneously. These scenarios can be modeled mathematically as games with controlled jump-diffusion dynamics. In [1], we introduce a computational framework using the actor-critic method in deep reinforcement learning to solve stochastic control problems with jumps. More precisely, for a generic stochastic control problem under the Lévy model, where the state process $X_t \in \mathbb{R}^d$ follows a controlled Itô-Lévy process:

$$\mathrm{d}X_t = b(X_{t-}, u_t)\,\mathrm{d}t + \sigma(X_{t-}, u_t)\,\mathrm{d}W_t + \int_{\mathbb{R}^d} G(X_{t-}, z, u_t)\widetilde{N}(\mathrm{d}t, \mathrm{d}z), \tag{1}$$

where $\{W_t\}_{t=0}^T$ is a d-dimensional standard Brownian motion, N is a Poisson random measure with the Lévy measure ν and $\widetilde{N}(\mathrm{d}t, \mathrm{d}z) := N(\mathrm{d}t, \mathrm{d}z) - \nu(\mathrm{d}z)\,\mathrm{d}t$ is the compensated Poisson random measure, we aim to find the optimal u^* that maximize the expected utility of running and terminal rewards:

$$J^u(t, x) = \mathbb{E}^{t,x}\left[\int_t^T f(s, X_s, u_s)\,\mathrm{d}s + g(X_T)\right]. \tag{2}$$

We developed an actor-critic framework that employs neural networks to parameterize both the value function (critic) and the control (actor). Policy evaluation and policy improvement are applied iteratively to update these networks. We further extend this algorithm to handle multi-agent games with jumps, utilizing parallel computing to improve computational efficiency. To illustrate the accuracy, efficiency, and robustness of our approach, we provide numerical examples including the Merton problem with jumps, linear quadratic regulators, and the optimal investment game under various conditions.

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An adaptive algorithm for rough differential equations

Christian Bayer
(joint work with Simon Breneis, Terry Lyons)

We consider a system y controlled by a rough path $\mathbf{x}$ in the form of a *rough differential equation*, say

$$dy_t = f(y_t)d\mathbf{x}_t.$$

Except for trivial cases, rough differential equations do not allow for closed form expressions, and, hence, numerical approximations are required. In this work, we discretize rough differential equations using the *log-ODE method*, see [2]. In a nutshell, the log-ODE method replaces the above rough differential equation by a controlled ordinary equation, where the rough path $\mathbf{x}$ is locally, i.e., on a short time interval $[t_i, t_{i+1}]$, is replaced by a smooth path which generates the same *signature* up to a fixed level N. The resulting ODE is then solved by a classical ODE solver.

The framework of rough differential equations is flexible enough to accommodate highly varying situations including

- driving paths changing their roughness over time;
- singularities in the vector field f in time or space.

For this reason, *adaptive* versions of the log-ODE method have large potential. In this specific case, we want to adaptively choose both the local time steps $\Delta t(s) = t_{i+1} - t_i$, $s \in [t_i, t_{i+1}]$, as well as the local degree N on $[t_i, t_{i+1}]$.

We use the approach of A. Szepessy and his co-authors, see, for instance, [3]. In their approach, they consider the problem of optimally (adaptively) choosing the numerical parameters – here, Δt and N – as an *optimal control problem* for minimizing the computational cost for a given error tolerance. In order to obtain the necessary error estimates – more specifically, the error propagation–, the dual backward equation is solved.

In [1], we

- derive well-posedness results for the corresponding dual backward equation, generalizing the classical well-posedness theory for rough differential equations;
- derive conditions under which the error in our a-posteriori error estimate based on a log-ODE discretization of the *dual equation* is asymptotically smaller than the error if the discretization of y, allowing us to *correct* the error using the a-posteriori error estimate;
- provide several numerical examples involving singularities in either the driving path or the vector fields, the driving path changing roughness, or stiffness of the rough differential equation.

The examples show that using the adaptive algorithm can drastically reduce the computational cost for approximating rough differential equations, provided that there is some inhomogeneity along the solution over time.

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Expected signature kernels of Lévy processes

Paul P. Hager
(joint work with Peter K. Friz)

The expected signature kernel arises in statistical learning tasks as a similarity measure of probability measures on path space. Computing this kernel for known classes of stochastic processes is an important problem that, in particular, can help in reducing computational costs. Building on the representation of the expected signature of inhomogeneous Lévy processes as the development of a smooth path in the extended tensor algebra [1], we extend the arguments developed for smooth rough paths in [2] to derive a PDE system for the expected signature of inhomogeneous Lévy processes. As a specific example, we demonstrate that the expected signature kernel of Gaussian martingales satisfy a Goursat PDE.

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A semigroup approach for stochastic quasilinear equations driven by rough noise

Alexandra Neamţu
(joint work with Antoine Hocquet)

We consider stochastic parabolic quasilinear equations perturbed by nonlinear multiplicative noise. Exploring semigroup methods and combining techniques from functional analysis with tools from rough path theory, we establish the pathwise well-posedness of such equations. One key technical tool is to introduce an appropriate space of controlled rough paths tailored to the parabolic quasilinear problem. We apply our results to the stochastic Landau-Lifshitz-Gilbert equation which models the magnetization of a ferromagnetic material. Moreover, we emphasize the advantage of rough path theory in the study of the long-time behavior of such systems.

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Rough Stochastic Analysis with Jumps

Andrew Allan
(joint work with Jost Pieper)

Rough path theory provides a framework for the study of nonlinear systems driven by highly oscillatory (deterministic) signals. The corresponding analysis is inherently distinct from that of classical stochastic calculus, and neither theory alone is able to satisfactorily handle hybrid systems driven by both rough and stochastic noise. The introduction of the stochastic sewing lemma by Lê [3] has paved the way for a theory which can efficiently handle such hybrid systems. In particular, intrinsic solutions to rough stochastic differential equations (RSDEs) under optimal regularity conditions on the vector fields were established by Friz, Hocquet and Lê [2].

In this talk, we discuss how this can be done in a general setting which allows for jump discontinuities in both sources of noise. Specifically, in [1] we establish existence, uniqueness and stability of solutions to RSDEs of the form

$$\mathrm{d}Y_t = b(Y_t)\,\mathrm{d}t + \sigma(Y_t)\,\mathrm{d}M_t + f(Y_t)\,\mathrm{d}\mathbf{X}_t,$$

where M is a sufficiently integrable càdlàg martingale, and $\mathbf{X} = (X, \mathbb{X})$ is a càdlàg rough path. This is based on a new version of the stochastic sewing lemma, which can handle multiple discontinuous control functions, and provides a significant generalization of [2] in terms of the permissible classes of driving noise.

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High Rank Path Development: an approach of learning the filtration of stochastic processes

Hao Ni

(joint work with Jiajie Tao, Chong Liu)

Since the weak convergence for stochastic processes does not account for the growth of information over time, which is represented by the underlying filtration, a slightly erroneous stochastic model in weak topology may cause huge loss in multi-periods decision making problems. To address such discontinuities Aldous introduced the extended weak convergence, which can fully characterise all essential properties, including the filtration, of stochastic processes; however was considered to be hard to find efficient numerical implementations. In this talk, we introduce a novel metric called High Rank PCF Distance (HRPCFD) for extended weak convergence based on the high rank path development method from rough path theory, which also defines the characteristic function for measure-valued processes. We then show that such HRPCFD admits many favourable analytic properties which allows us to design efficient algorithms to ensure the stability and feasibility in training. Finally, by using such metric as the discriminator in hypothesis testing and generative modelling, our numerical experiments validate the out-performance of the approach based on HRPCFD compared with several state-of-the-art methods designed from the perspective of weak convergence and therefore demonstrate the potential applications of this approach in many classical financial and economic circumstances such as optimal stopping or utility maximisation problems, where the weak convergence fails and the extended weak convergence is needed. This talk is based on [1].

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Dean-Kawasaki models for particle systems

Ana Djurdjevac

(joint work with Xiaohao Ji, Helena Kremp, Nicolas Perkowski)

Interacting particle systems provide flexible and powerful models that are useful in many application areas such as sociology (agents [3]), molecular dynamics (proteins) etc. We consider systems of N particles presented by distribution dependent SDEs

$$dX_t^i = b(X_t^i, \mu^N)dt + \sqrt{2}\Sigma(X_t^i, \mu_t^N) \cdot dW_t^i, \quad i = 1, \dots, N \tag{1}$$

with empirical distribution μ^N.

However, particle systems with large numbers of particles are very complex and difficult to handle, both analytically and computationally. Therefore, a common strategy is to derive effective equations that describe the time evolution of the empirical particle density μ^N.

A prototypical example that we will consider is the formal identification of a finite system of particles with the singular Dean-Kawasaki equation [1, 4]

$$du = \frac{1}{2}\nabla^2 : (a(\cdot,u)u)dt - \nabla\cdot(b(\cdot,u)u)dt + \frac{1}{\sqrt{N}}\nabla\cdot(\sqrt{u}\Sigma(\cdot,u)dW). \tag{2}$$

Our aim is to introduce a well-behaved nonlinear SPDE that approximates (2) for a particle system with mean-field interaction both in the drift and the noise term (1). We want to study the well-posedness of these nonlinear SPDE models and to control the weak error of the SPDE approximation with respect to the particle system using the technique of transport equations on the space of probability measures.

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The Surface Signature

Darrick Lee

(joint work with Harald Oberhauser)

Path development is a representation of paths in terms of matrix groups, which is defined by the solution of a linear controlled differential equation. The path signature is the universal path development: given any path development map, it uniquely factors through the path signature. These representations preserve the underlying concatenation structure of paths. In addition, they satisfy universality and characteristicness properties, allowing us to approximate functions on path space, and characterize the law of stochastic processes. In this talk, we consider the higher dimensional generalization of these constructions to surfaces, preserving both horizontal and vertical concatenations. We discuss the notion of *surface development* [1], which satisfies similar universality and characteristicness properties as path development, and the corresponding universal object: the surface signature [2].

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A representation for the Expected Signature of Brownian motion up to the first exit time of the planar unit disc

Horatio Boedihardjo

(joint work with He Lin, Lisa Wang)

The signature of a sample path is a formal series of iterated integrals along the path. The expected signature of a stochastic process gives a summary of the process that is especially useful for studying stochastic differential equations driven by the process. Lyons-Ni derived a partial differential equation for the expected signature of Brownian motion, starting at a point z in a bounded domain, until it hits to boundary of the domain. We focus on the domain of planar unit disc centred at 0. Motivated by recently found explicit formulae for some terms in the expected signature of this process in terms of Bessel functions, we derive a tensor series representation for this expected signature, coming from from studying Lyons-Ni's PDE. Although the representation is rather involved, it simplifies significantly to give a formula for the polynomial leading order term in each tensor component of the expected signature.

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