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Mini-Workshop: Mixing Times in the Kardar–Parisi–Zhang Universality Class

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Abstract. The workshop covered new developments in the Kardar–Parisi–Zhang (KPZ) class of growth models and markov chain mixing, broadened by talks on random dimers, matrices and parking functions. Mixing is often related to large time fluctuations, which are governed by universal limits such as the KPZ fixed point. The workshop thus focused on the asymptotic behaviour of KPZ models, the characterization of limiting objects, and cutoff.

Mathematics Subject Classification (2020): 60C05, 60B20, 60F05, 60J10, 60K35.

Introduction by the Organizers

The mini-workshop *Mixing Times in the Kardar–Parisi–Zhang Universality Class* assembled 16 researchers primarily from Europe and North America. 11 talks were given in total, some on forthcoming work which was subsequently posted on arxiv. The background of participants ranged from freshly started PhD student to highly established full professor.

To put this diverse group on the same page, we asked two more senior participants, Patrik Ferrari and Nina Gantert, to give a short mini-course consisting of two talks. Patrik Ferrari introduced the KPZ universality class of growth models, a prototypical example being the KPZ equation. Interacting particle systems that also belong to the KPZ class such as the asymmetric simple exclusion process (ASEP) were introduced, ASEP playing a prominent role in several of the workshop's talks. In ASEP on $\mathbb{Z}$, there is at most one particle at each integer. Each particle waits an exponential time and then attempts to jump one step to the right with probability $p > 1/2$, and one step to the right with probability $q = 1-p$. The attempt is successful so long as no more than 1 particle is present at each integer. In the more general colored ASEP, particles have different colors (priorities), such that a particle with higher priority will successfully jump to a position occupied by a particle with lower priority, swapping their positions. Particle configurations can be mapped to height functions, making ASEP a growth model. ASEP is equally considered on finite state spaces, and it is a limit of the stochastic six vertex model from statistical mechanics which appeared in several talks.

Nina Gantert in turn covered some fundamental, general aspects of Markov chain mixing. In particular, various methods to bound mixing times were discussed, as well as the cutoff phenomenon, an abrupt convergence to equilibrium of Markov chains.

The topics of the two mini-courses were brought together in the talk of Jimmy He, who spoke about the ASEP with one open boundary. He reported on the large time fluctuations of the current in this model, and how these can be used to obtain the cutoff profile of ASEP with one open boundary on a finite segment. A related result about ASEP with closed boundaries figured in the talk of Alexey Bufetov, where the stationary Mallows measure of ASEP was investigated in depth.

ASEP on $\mathbb{Z}$ converges to the KPZ fixed point, which is intimately related to the directed landscape. Lingfu Zhang presented an axiomatic approach to the directed landscape akin to the definition of Brownian motion, namely that the directed landscape is the unique directed metric on $\mathbb{R}^2$ with independent increments and KPZ fixed point marginals. Persistence probabilities for a marginal of the KPZ fixed point – specifically, the Airy_1 process – were studied in Min Liu's talk. The speed process for the colored ASEP and the stochastic six vertex model was presented by Hindy Drillick. In this process, particles have random asymptotic speeds. Such colored models were equally presented by Milind Hegde, who established the convergence of colored height functions to the so-called *Airy sheet*, a marginal of the directed landscape.

The scope of the workshop was extended by three talks on related topics: Amanda Priestly presented work on probabilistic parking functions, which are related to the model of activated random walks. Elia Bisi spoke about certain ensembles of $\lambda-$shaped random matrices, where λ is a Young diagram, a decreasing sequence of non-negative integers eventually becoming zero. Bisi characterized the limiting empirical spectral distribution of such matrices via generalizations of Catalan numbers. Finally, Ivailo Hartarsky spoke about local dynamics of dimers and the connectivity properties they induce.

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Mini-Workshop: Mixing Times in the Kardar–Parisi–Zhang Universality Class

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Abstracts

The ASEP with one open boundary

Jimmy He
(joint work with Dominik Schmid)

The asymmetric simple exclusion process (ASEP) is a simple toy model for particle dynamics in one spatial dimension. A possible addition to the model which introduces rich behavior is an open boundary which allows particles to enter/exit the system. More formally, this is a Markov process on configurations of particles on $\mathbb{N}$, where particles move left and right at rates q and 1 respectively ($q < 1$), and where particles enter/exit the system at rates α and β, subject to the exclusion rule that anything that would cause a particle to move on top of another particle is blocked.

This model lies in the KPZ universality class, and so in particular has an associated height function $h(t, x)$. Given the interesting results for models without boundary, it is natural to ask what the fluctuations are when one open boundary is present.

Theorem 1 ([1, Theorem 1.1]). *Assume that $\beta < \alpha$. Then depending on ρ, as $t \to \infty$,*

$$(\rho > \tfrac{1}{2}) \qquad \mathbb{P}\left(-\frac{h(t,0) - \mu t}{\sigma t^{1/3}} \leq s\right) \to F_{GSE}(s),$$

$$(\rho = \tfrac{1}{2}) \qquad \mathbb{P}\left(-\frac{h(t,0) - \mu t}{\sigma t^{1/3}} \leq s\right) \to F_{GOE}(s),$$

$$(\rho < \tfrac{1}{2}) \qquad \mathbb{P}\left(-\frac{h(t,0) - \mu_\rho t}{\sigma_\rho t^{1/2}} \leq s\right) \to \Phi(s),$$

where F_{GSE} and F_{GOE} are Tracy–Widom GSE and GOE distributions, Φ is the standard normal distribution, and $\mu, \sigma, \mu_\rho, \sigma_\rho$ are explicit constants.

This result turns out to have applications to have applications to the study of mixing times in a finite version of the model. In particular, if we now consider a finite version of the process on $\{0, 1, \ldots, n\}$ where particles cannot move past n, we obtain an ergodic Markov chain, which thus converges to its stationary measure. A natural question is to ask how long it takes to reach equilibrium.

In joint work with Dominik Schmid [2], sharp asymptotics known as the cutoff profile are obtained for this time to equilibrium, and in particular the shape of the transition to stationarity is given by either the F_{GSE}, F_{GOE}, or Φ distribution functions, depending on the value of ρ.

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Exact decay of the persistence probability in the Airy$_1$ process

Min Liu
(joint work with Patrik Ferrari)

The Airy process $\mathcal{A}_1$ is expected to govern the long-time, large-scale spatial fluctuations in the 1d KPZ universality class for flat non-random initial conditions. Our focus will be on its persistence probability, which quantifies the likelihood that the process stays below a given threshold c over a time span of length L. This probability is anticipated to decay exponentially as $e^{-\kappa(c)L}$. In this talk, we will see how to derive an analytic formula for $\kappa(c)$ when $c \geq 3/2$. Since the formula is analytic only for $c > 0$, we will also discuss how to construct its analytic continuation for $c < 0$. Additionally, we will present numerical results to verify the validity of this continuation.

Mixing measure and ASEP

Alexey Bufetov

We study the statistics of the Mallows measure on permutations in the limit pioneered by Starr (2009). Our main result is the local central limit theorem for its height function. We also re-derive versions of the law of large numbers and the large deviation principle, obtain the standard central limit theorem from the local one, and establish a multi-point version of the local central limit theorem.

We also study the asymptotic behavior of the Asymmetric Simple Exclusion Process (=ASEP) with finitely many particles. It turns out that a certain randomized initial condition is the most amenable to such an analysis. Our main result is the behavior of such an ASEP in the KPZ limit regime. A key technical tool introduced in the paper – the coloring of ASEP particles with the use of random Mallows permutations – may be of independent interest.

These results allow to obtain the mixing times of ASEP with fixed number of particles on a closed segment and fixed parameter q. They also can be important tools for studying the plethora of mixing time for ASEP questions in the limit $q \to 1$.

Random matrices, Young diagrams, and trees

Elia Bisi
(joint work with Fabio Deelan Cunden)

Fix, once and for all, a collection $\{X_{i,j} : i,j \geq 1\}$ of complex i.i.d. random variables with $\mathbb{E}X_{ij} = 0$ and $\mathbb{E}|X_{i,j}|^2 = 1$. Consider the random matrix ensemble given by the sequence $W_N = \frac{1}{N} X_N X_N^*$, where X_N is the $N \times N$ matrix $(X_{i,j})_{i,j=1}^N$ and X_N^* is its adjoint. The random matrix W_N is Hermitian and positive-definite, with eigenvalues $0 \leq x_1 \leq x_2 \leq \cdots \leq x_N$. A classical result [1] states that, with probability 1, the empirical spectral measure $\rho_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ of W_N converges weakly to the measure with density

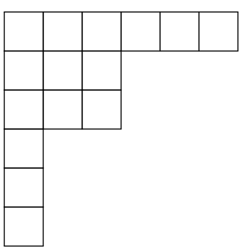
$$\mathrm{MP}(\mathrm{d}x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}\,1_{[0,4]}(x)\,\mathrm{d}x.$$

The latter is called *Marchenko–Pastur distribution* and can be characterised as the unique probability measure on $[0,\infty)$ whose moments are the *Catalan numbers*:

$$\int_0^\infty x^k \mathrm{MP}(\mathrm{d}x) = \frac{1}{k+1}\binom{2k}{k} =: C_k.$$

For later purposes, we recall one of the countless combinatorial objects [2] that Catalan numbers count (pun intended!): C_k is the number of rooted plane trees on $k+1$ vertices.

Let us briefly mention a very well-known connection with the KPZ universality class, due to [3]. When the random variables $X_{i,j}$'s are standard complex Gaussian, W_N is a *complex Wishart matrix* (this is also referred to as the *Laguerre Unitary Ensemble*). In this special case, the largest eigenvalue x_N of W_N has the same distribution as the *exponential last passage percolation* on an $N \times N$ grid (a popular KPZ integrable model). Furthermore, after an appropriate rescaling, x_N converges in law to the *Tracy–Widom distribution*, a typical KPZ limiting distribution.



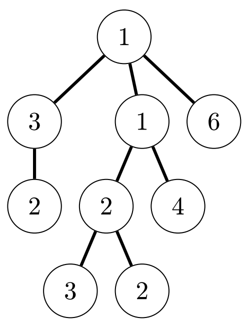


Figure 1. On the left-hand side, the self-conjugate partition $\lambda = (6,3,3,1,1,1)$ of length $\ell = 6$. On the right-hand side, a λ-plane tree.

Here we consider the more general model of Young-diagram-shaped random matrices. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell \geq 1)$ be a self-conjugate integer partition of length ℓ, viewed as the set of integer pairs (i, j) such that $1 \leq i \leq \ell$ and $1 \leq j \leq \lambda_i$; see the Young diagram 'box' representation in Fig. 1. A λ-*shaped random matrix* is an $\ell \times \ell$ matrix whose (i, j)-entry equals $X_{i,j}$ if $(i, j) \in \lambda$, and zero otherwise.

A natural way to construct a sequence of Young-diagram-shaped random matrices whose size grows to infinity and whose asymptotic shape is λ is to consider the *dilation* $N\lambda$ of λ by a factor N (i.e. the partition obtained by replacing every box of λ by an $N \times N$ grid of boxes) and to consider an $(N\lambda)$-shaped random matrix X_N^λ. When $\lambda = (1)$, X_N^λ coincides with the previously defined X_N. The question is now whether the spectral distribution ρ_N^λ of $W_N^\lambda := \frac{1}{N\ell} X_N^\lambda (X_N^\lambda)^*$ has a limit.

Define a λ-*plane tree* to be a rooted plane tree with vertex set V and edge set E, together with a labelling of the vertices $c \colon V \to \{1, \ldots, \ell\}$ such that $(c(u), c(v)) \in \lambda$ for all $\{u, v\} \in E$; see Fig. 1. Then, we have:

Theorem 1 ([4]). *With probability* 1*, the sequence of random measures* ρ_N^λ *converges weakly as* $N \to \infty$ *to the (unique) probability measure* ρ^λ *whose moments are*

$$\int_0^\infty x^k \rho^\lambda(x)\,\mathrm{d}x = \frac{C_k^\lambda}{\ell^{k+1}},$$

where

$$C_k^\lambda := \#\{\lambda\text{-plane trees on } k+1 \text{ vertices}\}.$$

When $\lambda = (1)$, we recover the classical case: $C_k^\lambda = C_k$ for all k and $\rho^\lambda = \mathrm{MP}$.

To extract more information about the combinatorial sequence C_k^λ, two approaches are considered in [4]:

(1) exact formulas: C_k^λ can be expressed as a homogeneous polynomial of degree $k + 1$ in the multirectangular coordinates of λ, using certain enumerations for labelled trees from [5];
(2) generating functions, which solve an algebraic equation (this implies that ρ^λ is *algebraic in the sense of Rao and Edelman* [6]).

Two examples have a more explicit solution. The first one [7] is *staircase partitions*, i.e. those of the type $\lambda = (\ell, \ell - 1, \ldots, 1)$: in this case, C_k^λ has the simple form $\frac{\ell}{k+1}\binom{(\ell+1)k}{k}$, and ρ^λ is the law of a product of $\ell+1$ independent Beta random variables. The second one [4] is *fat hooks*, i.e. Young diagrams made of two rectangular blocks: in this case, ρ^λ can be expressed as a free convolution of measures involving a Marchenko–Pastur and a Bernoulli distribution.

Several related topics remain to be explored, e.g.: limiting spectral distributions for dilations of non-self-conjugate partitions and, more generally, for sequences of partitions converging to a limit shape; more precise analysis and characterisations of such limiting measures; joint density of the eigenvalues of λ-shaped random matrices, and related asymptotics, in the Gaussian case.

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Scaling limit of the colored ASEP and stochastic six-vertex models

Milind Hegde

(joint work with Amol Aggarwal, Ivan Corwin)

We discuss the scaling limit of the colored asymmetric simple exclusion process (ASEP) and the colored stochastic six-vertex (S6V) model, which is based on joint work with Amol Aggarwal and Ivan Corwin. Both models can be obtained from their standard (uncolored) counterparts by evolving different initial conditions using the same randomness, often called the basic coupling. The observable of interest is called the colored height function, denoted $h(x, 0; y, t)$, and is defined as the number of particles of color at least x to the right of y (in colored ASEP) at time t and the number of arrows exiting horizontally from the vertical ray starting at (t, y) (in colored S6V). Under KPZ rescaling of space and fluctuations at time t, i.e., scaling space by $t^{2/2}$ and fluctuations by scale $t^{1/3}$, the colored height function in both models converges to the Airy sheet $\mathcal{S} : \mathbb{R}^2 \to \mathbb{R}$, a random continuous constructed as the scaling limit of a model known as Brownian last passage percolation in work of Dauvergne-Ortmann-Virág.

The Airy sheet is defined via a variational problem in an object known as the parabolic Airy line ensemble, which is a countable collection of random continuous non-intersecting functions defined on $\mathbb{R}$, and which arises as the edge scaling limit of Dyson Brownian motion. Thus, to prove convergence of the colored ASEP and S6V height functions to $\mathcal{S}$, one must find a representation of them as variational problems in prelimiting versions of the parabolic Airy line ensemble. To do this, we first embed the colored height function, as a function of position with the color argument fixed, as the first curve in a discrete line ensemble (depending on the fixed color). The collection of such discrete line ensembles as we vary the color together forms a structure called a colored line ensemble, defined in work of Aggarwal-Borodin via a certain fused vertex model and the Yang-Baxter equation. Due to the underlying structure of a vertex model, the colored line ensemble structure comes with a certain Gibbs or spatial Markov property, which can be written in terms of the weights of the fused vertex model. Remarkably, by analyzing the weights carefully, we show that the first curve in the constituent line ensembles of the colored line ensemble can be written, up to a random but well-controlled error, as a variational problem in a single line ensemble.

Since this is a prelimiting version of the definition of the Airy sheet in terms of the parabolic Airy line ensemble, the proof of convergence to the Airy sheet reduces to showing that this single line ensemble, appropriately rescaled, converges to the parabolic Airy line ensemble. The prelimiting line ensemble itself has a Gibbs property, which is crucial to showing this convergence. Indeed.,many previous works have shown convergence of prelimiting line ensembles associated with other models by making use of their Gibbs properties. Those proof, however, all made use of monotonicity properties that the Gibbs properties enjoyed; namely, that if two sets of boundary datas are ordered, one larger than the other, then the measures can be coupled to also be ordered in the same way. In contrast, the line ensemble and Gibbs property associated to the ASEP and S6V model does not possess these forms of monotonicity. As a result we are led to develop a new framework to establish convergence of the prelimiting line ensemble, making use of much weaker form of monotonicity that only compares one-point probabilities.

Introduction to the Kardar–Parisi–Zhang universality class

Patrik L. Ferrari

In the introductory lecture we describe the Kardar–Parisi–Zhang (KPZ) universality class of stochastic growth models, some of the large time limit processes, connections with interacting particle systems and directed polymers in a random environment. Finally we describes some useful techniques.

The KPZ universality class contains models describing the evolution of an interface given by a height function $x \mapsto h(x,t)$. Here we focus on space x being one-dimensional. The prototypical effective equation is the KPZ equation

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \xi$$

where ξ is the white noise. As written, the KPZ equation ill-defined since a stationary solution is a two-sided Brownian motion (with diffusion constant 2) and thus the term $(\partial_x h)^2$ is ill-defined. A way to make sense of it is to regularize the noise in space, $\xi \to \xi_\epsilon$, and take $\epsilon \to 0$, but we do not enter in these details in the lecture.

One can also consider discrete (in space and/or time) models, which have the *same physical properties* as the KPZ equations. Then by universality one expects the same large time limit process of the interface.

The connection with the directed polymers goes by considering $Z = \exp(h)$ and first solve the stochastic heat equation with multiplicative noise,

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + Z\xi$$

with initial condition $Z(x, 0) = \exp(h(x, 0))$ and then set $h(x, t) = \ln Z(x, t)$. This is called the Cole-Hopf solution of the KPZ equation. Due to a Feynman-Kac representation, we have an interpretation of Z a the partition function of a Brownian motion (the directed polymer) in a random environment (the white noise). The discrete analogue at zero temperatures is the so-called last passage percolation model (LPP).

The other connection is to interacting particle systems, more specifically with exclusion processes (e.g., the totally asymmetric simple exclusion process (TASEP) or the partially asymmetric version of it (ASEP)). These models can be seen as random interfaces by setting the discrete gradient to be ± 1 depending whether there is a particle or not, and the height function increment during time t is given by twice the current of particles.

Some of the limit processes and properties are discussed, namely for step initial condition $h(x, 0) = |x|$ or flat initial condition $h(x, 0) = 0$. These are called the Airy_2 and Airy_1 processes. For general simple exclusion processes there is a clear KPZ scaling theory which predict how to compute the non-universal scaling coefficients.

Two methods which have been quite useful to study large time asymptotics of observables are also discussed. One is the slow decorrelation, which tells us that along characteristic lines (of the macroscopic PDE) the fluctuations decorrelate only over macroscopic times, unlike the spatial correlations which are of order $t^{2/3}$. A second technique is the comparison with stationarity, first used in LPP models and then for TASEP height function. This allows to control the local increment of the height function for generic initial conditions with the increments of stationary initial conditions (which are explicit). In particular, to show tightness of the limit processes.

Some elementary facts about mixing times and cutoff for Markov chains

Nina Gantert

In this short minicourse, we give an introduction to mixing times and the cutoff effect for Markov chains. All the results that we mention can be found in [3]. The classical theory studies a fixed Markov chain on a finite (but possibly large) state space S. We always assume that the Markov chain is irreducible and aperiodic (for simplicity, we will only consider Markov chains in discrete time). Then, there is a unique invariant distribution π and, for all $x \in S$, the law $P^t(x, \cdot)$ of the chain after t time steps (started from x) converges to π. One can characterize the rate of convergence of $P^t(x, \cdot)$ to π: it is well-known that the exponential rate of decay of the distance of $P^t(x, \cdot)$ to π is given by the *spectral gap* of the Markov chain (which can be defined in the non-reversible case as well).

Fix $\varepsilon \in (0, 1)$. The *mixing time* $t_{\mathrm{mix}}(\varepsilon)$ of a Markov chain is the first time t when the maximal distance to π (the maximum is taken over the starting values) is at most ε. More precisely, we define $d(t) := \max\limits_{x \in S} ||P^t(x, \cdot) - \pi(\cdot)||_{\mathrm{TV}}$ where $||\mu - \nu||_{\mathrm{TV}}$

is the distance of μ and ν in total variation. Then $t \mapsto d(t)$ is non-increasing and we define the mixing time $t_{\text{mix}}(\varepsilon)$ as

$$t_{\text{mix}}(\varepsilon) := \min\{t : d(t) \leq \varepsilon\} .$$

We remark that this definition can be generalized by replacing the total variation by other distances on the space of probability measures on S. The concept of mixing times is particularly useful for *sequences* of Markov chains on growing state spaces. Examples for such sequences are: the simple random walk on the n-cycle, the simple random walk on the n-dimensional hypercube, the symmetric or asymmetric exclusion process with open boundaries on a segment of length n. The mixing time can determine the running time of an algorithm used for simulation. We write $t_{\text{mix}} := t_{\text{mix}}(1/4)$.

We discuss the relation of mixing times with the spectral gap. Then we address the question how to find bounds for mixing times. To this end, we introduce

- coalescent couplings of two Markov chains
- the relation of the mixing time with certain hitting times
- strong stationary times.

These all provide upper bounds for the mixing time. As examples, we consider the simple random walk on the n-cycle, the top-to-random shuffle of a deck of n cards and the simple random walk on the n-dimensional hypercube. In these cases, one can give lower bounds on the mixing times directly.

A sequence of Markov chains satisfies *cutoff* if the distance of $P^t(x, \cdot)$ to π drops from being close to 1 to being close to 0 over a time interval which is (asymptotically) smaller than the the mixing time. More precisely, there is cutoff if

$$\lim_{n\to\infty} \frac{t^{(n)}_{\text{mix}}(\varepsilon)}{t^{(n)}_{\text{mix}}(1-\varepsilon)} = 1,$$

where we write in $t^{(n)}_{\text{mix}}(\varepsilon)$ for the mixing time of the n-th Markov chain. In other words, there is cutoff if the first order of the mixing time does not depend on ε. This can be shown easily to be equivalent to

$$\lim_{n\to\infty} d^{(n)}(ct^{(n)}_{\text{mix}}) = \begin{cases} 1 & \text{if } c < 1 \\ 0 & \text{if } c > 1, \end{cases}$$

where we write $d^{(n)}(\cdot)$ for the distance belonging to the n-th Markov chain and $t^{(n)}_{\text{mix}}$ for the mixing time of the n-th Markov chain. Going back to our examples, the simple random walk on the n-cycle does not satisfy cutoff while the top-to-random shuffle and the random walk on the n-dimensional hypercube do. The question whether a given sequence of Markov chains satisfies cutoff is in general not easy. We refer to several recent and ongoing works for the symmetric or asymmetric exclusion process with open boundaries on a segment of length n.

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Characterization of the directed landscape from the KPZ fixed point

Lingfu Zhang

(joint work with Duncan Dauvergne)

Two central objects in the KPZ universality class are the KPZ fixed point and the directed landscape, which are the scaling limits of the growth process and directed metric, respectively. It is known that the KPZ fixed point gives marginals of the directed landscape, and the directed landscape is a natural coupling of multiple instances of the KPZ fixed point.

I will discuss a work in preparation with Duncan Dauvergne, where we show that the directed landscape is the unique process with the KPZ fixed point as its marginals, satisfying four natural properties: independent increments, a semigroup structure, monotonicity, and shift commutativity. This gives a characterization of the directed landscape, and provides a framework for proving convergence to the directed landscape given convergence to the KPZ fixed point. We apply this framework to prove convergence to the directed landscape for 1D exclusion processes with potentially non-nearest neighbour interactions, exotic couplings of ASEP, the Brownian web and random walk web distance, and directed polymers. In particular, we give new proofs of directed landscape convergence for colored ASEP and the KPZ equation.

Local dynamics for 3d dimers

Ivailo Hartarsky

(joint work with Lyuben Lichev, Fabio Toninelli)

Consider the state space of dimer configurations (perfect matchings) on a hypercubic box $[n]^d$. A local move consists in successively switching the dimers along an alternating cycle of small length. We study the connectivity properties of the state space equipped with these transitions. We establish that each configuration admits at least order n^{d-2} alternating cycles of length at most $4d-2$. Furthermore, cycles of length at most $4d-4$ are sufficient to connect the configuration space on the unit hypercube $[2]^d$. The conjecture that cycles of length 4 and 6 are sufficient for three dimensional boxes remains open, obstructing the study of the associated Markov chain.

It is important to note that, while our proofs can also be viewed as new proofs of known facts in two dimensions, in the higher dimensional setting, standard tools such as height functions become unavailable. Instead, our approach is purely combinatorial.

The stochastic six-vertex model speed process

Hindy Drillick
(joint work with Levi Haunschmid-Sibitz)

For the stochastic six-vertex model on the quadrant $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ with step initial conditions and a single second-class particle at the origin, we show almost sure convergence of the speed of the second-class particle to a random limit taking values in the rarefaction fan. This allows us to define the stochastic six-vertex model speed process, whose law is ergodic and stationary for the dynamics of the multi-class stochastic six-vertex process. We also prove that the fluctuations of the second-class particle around its asymptotic limit are bounded on the order of $t^{7/9}$.

Speed processes were first constructed for TASEP [Amir-Angel-Valko '08], a zero temperature model. For positive temperature models, new tools were required. The ASEP speed process was recently constructed in [Aggarwal-Corwin-Ghosal '23] using moderate deviation tail bounds and Rezakhanlou's coupling for ASEP [Rezakhanlou '95]. To construct the stochastic six-vertex model speed process, we develop analogous results for the stochastic six-vertex model.

In particular, we need the following two ingredients, which are the key novelties of this paper:

- A geometric stochastic domination result that states that a second-class particle to the right of any number of third-class particles will at any fixed time be overtaken by at most a geometric number of third-class particles.
- Upper and lower moderate deviation tail bounds that quantify how close the height function of the stochastic six-vertex model started from step initial conditions will be to its limit shape.

These results will be used in the following way. We want to control the behavior of a single second-class particle. Hydrodynamic theory allows us to control the bulk behavior of many particles, so we augment our system by filling up all empty positions to the left of X_t with third-class particles. We then use our effective hydrodynamic estimates to control the union of the second- and third-class particles. Finally, we can revert this back to an estimate of the position of the second-class particle since we know that our second-class particle is to the left of at most a geometric number of the third-class particles. A similar argument can be made to bound the position of the second-class particle from the left.

A Probabilistic Parking Processes

Amanda Priestley
(joint work with Pamela E. Harris, Thiago Holleben, J. Carlos Martínez Mori, Keith Sullivan, and Per Wagenius)

1. Introduction

In 1966 Konheim and Weiss [1] introduced a now classical parking problem which is defined as follows. Fix $n \in \mathbb{N}$ and consider a sequence of n cars, each given labels in $[n] := \{1, \dots, n\}$, attempting to park on the sites of the line segment $[n]$ according to the following protocol. Initially, each car records its preferred parking spot in the preference vector $\alpha := (\alpha_1, \dots, \alpha_n) \in [n]^n$ with α_i being the preference of car i. The cars then enter the street from left to right, and try to park in increasing order of their labels. If their preferred parking spot is unoccupied, they are able to immediately park in their desired spot. Otherwise, the car continues to drive to the next available spot to the right and parks there. If there is no such available parking spot, then the car exits the street and is unable to park. The set of preference vectors α such that all cars are able to park under the aforementioned protocol are called *parking functions*, denoted by PF_n. To give a few simple but illustrative examples, the vector $(1, 1, 1 \dots, 1)$ is a parking function, as is any permutation in S_n. However the vector $(n, n, n \dots, n)$ is not a parking function. Konheim and Weiss initiated the study of these combinatorial objects by proving that $|\mathrm{PF}_n| = (n+1)^{n-1}$, and since, classical parking functions, and several new variants [2, 3, 4] have been of great interest to the enumerative combinatorics community.

1.1. **A Model of Stochastic Parking.** In the forthcoming work of Harris, Holleben, Martinez Mori, Priestley, Sullivan, and Wagenius [5] the authors introduce a model of stochastic parking based on the classical protocol of Konheim and Weiss. Again letting $n \in \mathbb{N}$ represent both the number of cars attempting to park and the size of the parking lot, and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in [n]^n$ denote a preference vector, the case of successful parking is the same. However, in the case that a car, say car i's, preferred spot is occupied, with probability $0 \leq p \leq 1$, it takes one step to the right and attempts to park in spot $\alpha_i + 1 \in [n]$. Similarly, with probability $1 - p$, it takes one step to the left and attempts to park in spot $\alpha_i - 1$ as long as $\alpha_i - 1 \in [n]$. In this way, car i performs a random walk on the segment starting from its preferred spot α_i, and parks in the first unoccupied spot in $[n]$ it encounters, if any. Note that the case in which $p = 1$ reduces to the original protocol of Konheim and Weiss [1]. As in the classical model, cars arrive one at a time, and a car may only enter the street only after its predecessor has completed the search process. Now, we distinguish two different variants of this model depending on the dynamics at the *boundaries*, namely spots 1 and n: Recall that in the classical parking protocol, a preference vector was deterministically either a member of the set of parking functions, or it was not. In this probabilistic model, every function in $[n]^n$ now has some probability attributed to it of allowing all of the cars to park, and determining this probability and the expected running time of the process, are now the central questions of interest.

1.2. **Contributions.** Before describing the contributions in [5], we first must give one more preliminary definition. A *weakly increasing parking function* is exactly as it sounds, a parking function which whoes entries are weakly increasing order. Equivalently, a weakly increasing parking function is a vector $\alpha \in [n]^n$ such that $\alpha_i \leq i$, for all cars $i \in [n]$, and we denote the set of all such functions by $\mathsf{PF}_n^{\uparrow}$. One can see using this second definition, that if α is a weakly increasing parking function, then the *outcome permutation*, or the final order of the parked cars in the lot at the end of the classical parking process, must be the identity permutation. In other words, at the end of the protocol, the first car will have parked in spot one, the second in spot two, and so on.

We now highlight some of the results in [5] for the models introduced in Section 1.1. More specifically, we give expressions for the probability of parking and expected time to park (i.e., a random variable analogue of the displacement statistic), conditioned on all cars ultimately parking. In the following, we let X_α^i be an indicator random variable which is 1 if car i parks under the stated protocol and preference tuple α, and 0 otherwise. Similarly, we let X_α be an indicator random variable which is 1 if all of the cars park under the stated protocol and preference tuple α, and 0 otherwise. Finally, let Z_t^i be the random variable keeping track of the position of car i at time t. For each car, we define the stopping times

$$\tau_\alpha^i := \min\{t \ : Z_t^i = i \text{ or } Z_t^i = 0\}$$

and let τ_a denote the time it takes the entire protocol to complete. In particular, we have that

$$\mathbb{E}\left[\tau_\alpha \mid X_\alpha = 1\right] = \sum_{i=1}^{n} \mathbb{E}\left[\tau_\alpha^i \mid Z_{\tau_\alpha^i}^i = i\right] . \tag{1.1}$$

Theorem 1 (Open Boundaries)**.** *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathsf{PF}_n^{\uparrow}$. Under the probabilisitc parking protocol with open boundaries, we have*

(1) $\Pr[X_\alpha^1 = 1] = 1$ *and, for any* $1 < i \leq n$,

$$\Pr\left[X_\alpha^i = 1 \mid \prod_{j=1}^{i-1} X_\alpha^j = 1\right] = \begin{cases} \prod\limits_{k=1}^{i} \frac{\alpha_k}{k} & \textit{if} \quad p = \frac{1}{2} \\ \prod\limits_{k=1}^{i} p^{k-\alpha_k} \frac{p^{\alpha_k} - (1-p)^{\alpha_k}}{p^k - (1-p)^k} & \textit{otherwise.} \end{cases}$$

(2) $\mathbb{E}[\tau_\alpha^1] = 0$ *and, for any* $1 < i \leq n$, *we have*

$$\mathbb{E}[\tau_\alpha | \prod_{j=1}^{i-1} X_\alpha^j = 1] = \begin{cases} \frac{1}{3} \sum\limits_{k=1}^{i} (k^2 - \alpha_k^2) = \frac{2n^3+3n^2+n}{18} - \frac{1}{3} \sum\limits_{i}^{n} a_k^2 & \textit{if} \quad p = \frac{1}{2} \\ \frac{1}{p-(1-p)} \sum\limits_{k=1}^{i} \frac{k\left(1+\left(\frac{(1-p)}{p}\right)^k\right)}{\left(1-\left(\frac{(1-p)}{p}\right)^k\right)} - \frac{\alpha_k\left(1+\left(\frac{(1-p)}{p}\right)^{\alpha_k}\right)}{\left(1-\left(\frac{(1-p)}{p}\right)^{\alpha_k}\right)} & \textit{otherwise.} \end{cases}$$

A similar result holds when starting with vectors of the form $(n, i \in [n-1, n], i \in [n-2, n], \ldots)$, and the authors give analogous results in the case of the Unbounded Model. Moreover, the authors also show that the random variables X^i_α are negatively correlated, and give concentration results for the probability of any subset of cars parking.

1.3. **Related Work.** As one familiar with the subject might intuit, the probabilistic parking protocol described can easily be interpreted as an *interacting particle system* (for a thorough treatment of the subject, refer to [6, 7]). In particular, preference vectors translate to initial configurations of particles (i.e., cars) on the line segment such that multiple particles are allowed on the same site. Furthermore, the dynamics of the process are closely reminiscent of Activated Random Walks [8] with sleeping rate $\lambda = \infty$ (the case in which each particle immediately falls asleep upon landing on an unoccupied site) otherwise known as IDLA [10, 9]. However, for the dynamics described in Section 1.1, the cars, or particles, are distinct. That is, when several cars are assigned to the same initial spot, the car with the smallest number in the order is allowed to park immediately, while other cars must perform random walks in increasing order of their labels. This seems to have the effect of making the model more sensitive to the initial configuration α.

Recently, and independently, Varin introduced the Golf Model [11], an interacting particle system that can be seen as a generalization of IDLA. In their work, Varin also introduces a probabilistic parking protocol on the cycle $\mathbb{Z}/n\mathbb{Z}$ as a special case of the Golf Model in which each car starts at a uniformly random spot on the circular parking lot. The cars move to the right with probability p and to the left with probability $(1-p)$ to find the nearest open parking spot, as in the protocol described in Section 1.1. However, the work of Varin differs in several ways. Perhaps most significantly, they focus on the setting in which there are strictly more parking spots than cars. Thus, they are interested in studying the distribution of empty parking spots remaining at the end of the process. Furthermore, the parking process of Varin is essentially equivalent to the IDLA model, and as such, lets the cars be indistinguishable.

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