Mathematisches Forschungsinstitut Oberwolfach

Report No. 52/2024

DOI: 10.4171/OWR/2024/52

Mini-Workshop: Critical phenomena of the XY model

Organized by
Christophe Garban, Villeurbanne
Trishen S. Gunaratnam, Genève
Eveliina Peltola, Bonn/Aalto
Diederik van Engelenburg, Villeurbanne

17 November – 22 November 2024

Abstract. This mini-workshop focused on recent advances in probability theory concerned with the critical phenomena of the XY model and other related spin systems. There were 18 participants, all working at the forefront of this dynamic field, and from various career stages and a diverse range of institutions as well as backgrounds and gender. The mini-workshop featured talks by almost all participants except organizers as well as an open problem session. The talks consisted of: a) three mini-courses of four lectures each and b) hour long seminars from the remaining participants. The mini-courses covered random currents for the XY model, random walk representations and triviality in the XY and Ising models, and connections between the spherical model and the Gaussian Free Field. The remaining seminars spanned a diverse range of topics, such as novel probabilistic approaches to the BKT transition in 2d, recent progress on high-dimensional spin systems and finite-size effects, and novel geometric representations for correlations in spin systems.

Mathematics Subject Classification (2020): 60K35, 82B20.

Introduction by the Organizers

The XY model is one of the simplest examples of a ferromagnetic spin system with abelian continuous symmetry. It arises as a natural generalisation of the celebrated Ising model to spins valued in the unit circle. In the '70s, Berezinskii [2] and, independently, Kosterlitz and Thouless [1], predicted that the XY model in dimension 2 exhibits an exotic phase transition. The discovery of this so-called BKT transition led to the theory of topological phase transitions, culminating in the Nobel Prize in Physics in 2016. The existence of the BKT transition was rigorously established by Fröhlich and Spencer [3] in the '80s using multiscale analysis.

In the past few years, there has been significant progress towards analysing the BKT transition from a probabilistic point of view. Key breakthroughs, yielding alternative proofs of the existence of the BKT transition, were obtained independently by van Engelenburg and Lis [4], and Aizenman, Harel, Peled, and Shapiro [6]. These new approaches rely on stochastic geometric representations and insights from percolation theory, in this context due to Lammers [7]. Building on these works, a finer understanding of the connection between the BKT phase transition and the localisation-delocalisation transition of a natural height function associated to the XY model has been obtained by Lammers [8], and van Engelenburg and Lis [5].

The purpose of this mini-workshop was to bring together the community involved at the frontier of these new breakthroughs in order to discuss recent progress, attack interesting problems and clarify the major driving problems of the field. There were 18 participants, all working at the forefront of this dynamic field, and from various career stages and a diverse range of institutions (including German) as well as ethnic backgrounds and gender. During the mini-workshop, there were three mini-courses given by Marcin Lis, Piet Lammers, and Juhan Aru. They covered random currents for the XY model, random walk representations and triviality in the XY and Ising models, and connections between the spherical model and the Gaussian Free Field. Almost all participants (except the organizers) gave hour long talks on a diverse range of topics, such as novel probabilistic approaches to the BKT transition in 2d, recent progress on high-dimensional spin systems and finite-size effects, and novel geometric representations for correlations in spin systems. There was additionally a two hour long open problem session hosted by Duminil-Copin.

References

[1] Kosterlitz, J. & Thouless, D. Ordering, metastability and phase transitions in two-dimensional systems. *Journal Of Physics C: Solid State Physics.* **6**, 1181–1203 (1973)

[2] Berezinskii, V. Destruction of long-range order in one-dimensional and two-dimensional systems having a continuous symmetry group I. Classical systems. *Sov. Phys. JETP.* **32**, 493–500 (1971)

[3] Fröhlich, J. & Spencer, T. The Kosterlitz-Thouless transition in two-dimensional abelian spin systems and the Coulomb gas. *Communications In Mathematical Physics.* **81**, 527–602 (1981)

[4] Engelenburg, D. & Lis, M. An elementary proof of phase transition in the planar XY model. *Communications In Mathematical Physics.* **399**, 85–104 (2023)

[5] Engelenburg, D. & Lis, M. On the duality between height functions and continuous spin models. *ArXiv Preprint ArXiv:2303.08596.* (2023)

[6] Aizenman, M., Harel, M., Peled, R. & Shapiro, J. Depinning in integer-restricted Gaussian Fields and BKT phases of two-component spin models. *ArXiv Preprint ArXiv:2110.09498.* (2021)

[7] Lammers, P. Height function delocalisation on cubic planar graphs. *Probability Theory And Related Fields.* **182**, 531–550 (2022)

[8] Lammers, P. Bijecting the BKT transition. *ArXiv Preprint ArXiv:2301.06905.* (2023)

[9] Garban, C. & Sepúlveda, A. Statistical reconstruction of the Gaussian free field and KT transition. *ArXiv Preprint ArXiv:2002.12284.* (2020)

[10] Fröhlich, J. On the triviality of $\lambda\varphi^4$ theories and the approach to the critical point in $d > (=)4$ dimensions. *Nuclear Physics B.* **200**, 281–296 (1982)

[11] Aizenman, M. & Duminil-Copin, H. Marginal triviality of the scaling limits of critical 4D Ising and $\lambda\varphi_4^4$ models. *Annals Of Mathematics.* **194**, 163–235 (2021)

[12] Aizenman, M., Duminil-Copin, H. & Sidoravicius, V. Random currents and continuity of Ising model's spontaneous magnetization. *Communications In Mathematical Physics.* **334** pp. 719–742 (2015)

[13] Duminil-Copin, H. & Tassion, V. A new proof of the sharpness of the phase transition for Bernoulli percolation and the Ising model. *Communications In Mathematical Physics.* **343** pp. 725–745 (2016)

[14] Peled, R. & Spinka, Y. Lectures on the spin and loop O(n) models. *Sojourns In Probability Theory And Statistical Physics-I: Spin Glasses And Statistical Mechanics, A Festschrift For Charles M. Newman.* pp. 246–320 (2019)

[15] Bauerschmidt, R., Park, J. & Rodriguez, P. The Discrete Gaussian model, II. Infinite-volume scaling limit at high temperature. *ArXiv Preprint ArXiv:2202.02287.* (2022)

Mini-Workshop: Critical phenomena of the XY model

Table of Contents

Abstracts

Random currents and the BKT transition in the XY model

Marcin Lis

(joint work with Diederik van Engelenburg)

The **XY model** on a finite graph $G = (V, E)$ is a probability measure on continuous spin configurations $\sigma \in \mathbb{S}^V$, $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$, given by

$$d\mathbb{P}_{\mathrm{XY}}(\sigma) \propto \exp\Big(\sum_{uv\in E} \beta(\sigma_u\bar{\sigma}_v + \bar{\sigma}_u\sigma_v)\Big)\prod_{v\in V} d\sigma_v,$$

where β is the inverse temperature, and $d\sigma_v$ denotes the uniform measure on $\mathbb{S}$. By the Mermin–Wagner theorem [1], in two dimensions the model does not exhibit order at any finite temperature (there is no spontaneous magnetization). Nonetheless it famously undergoes a **Berezinskii–Kosterlitz–Thouless phase transition** from exponential decay of correlations for small β to a power-law decay of for large β [2, 3]. Kosterlitz and Thouless were awarded the 2016 Nobel prize in physics for this discovery, and it was first rigorously confirmed in 1981 in a landmark work of Fröhlich and Spencer [2].

An intrinsically related model is the Fourier–Pontryagin (or Kramers–Wannier) dual – an integer-valued height function on the faces of the graph. To define it, one expands the Boltzmann weights into a power series and integrates out the spin variables over $\mathbb{S}^V$. This leads to a discrete geometric representation through **random currents**. Here a random current is a function $\mathbf{n} : \vec{E} \to \mathbb{N}$, where $\vec{E}$ is the set of directed edges of G, such that the total incoming and outgoing current at each vertex are equal. This **divergence-free** property allows one to define the heigh function on the faces, where the increment between two faces is the net amount of current flowing through the edge separating the faces. The resulting probability measure on currents, and hence also on height functions, is given by

$$\mathbf{P}_{\mathrm{XY}}(\mathbf{n}) \propto \prod_{uv\in\vec{E}} \frac{\beta^{\mathbf{n}_{uv}}}{(\mathbf{n}_{uv})!}. \tag{1}$$

This representation is very closely related to the much better understood **double random current representation** of the **Ising model** that originated in the foundational work of Griffiths, Hurst and Sherman [4]. Its unique combinatorial properties in the form of the celebrated **switching lemma** allow to derive identities for products of spin correlation functions. A canonical application of this technique is the formula for the two point function squared:

$$\langle\sigma_u\sigma_v\rangle^2_{\mathrm{Ising}} = \mathbf{P}_{\mathrm{Ising}}(u \xleftrightarrow{\mathbf{n}} v), \tag{2}$$

where σ_v is the Ising spin at vertex v, $\mathbf{n} : E \to \mathbb{N}$ is the double random current interpreted as a percolation process, and $\{u \xleftrightarrow{\mathbf{n}} v\}$ denotes the event that u is connected to v by a path of open edges in the current (those $e \in E$ with $\mathbf{n}_e > 1$). The current $\mathbf{n}$ appearing in this formula, unlike the one in the XY model, is divergence free only modulo 2 meaning that for every $v \in V$, $\sum_{u\sim v} \mathbf{n}_{uv}$ is even. This identity and its variants have been crucial in the study of the Ising model with applications to questions of sharpness and continuity of phase transition [5, 6], and triviality of scaling limits [7, 8].

From a recent work of van Engelenburg and the author [10], it becomes clear that the right geometric object to study the XY model together with its height function is not the current itself (1) but a finer combinatorial structure built on top of the current. One proceeds independently for each vertex, and samples a uniform random pairing between the incoming and outgoing units of current. This results in a collection of overlapping and self-intersecting directed loops $\mathcal{L}$ whose net flux along each edge is the same as of the underlying current. A **new switching lemma** for such collections of loops implies that

$$\langle \sigma_u^2 \bar{\sigma}_v^2 \rangle_{\mathrm{XY}} \propto \mathbf{P}_{XY}(u \text{ and } v \text{ lie on the same loop in } \mathcal{L}), \quad u, v \in V, \tag{3}$$

where $\bar{\sigma}$ is the complex conjugate, and $\propto$ means equality up to a universal constant independent of G and β. One should note that, compared to (2), the event in (3) is a random-walk-like rather than a percolation-like connectivity.

This offers an alternative point of view on the BKT transition: the fluctuating loops carry statistical information about both the spin model through its correlation functions like in (3), and about the underlying dual height function through the total winding number of all loops around any given face. This together with a recent **delocalisation** result of Lammers [9] was used in [10] to provide and elementary proof of the existence of the BKT phase transition in the XY model. The main idea of this new proof is to argue along the following lines: If the height function delocalises which means that its variance is big in the center of a big box, from the coupling with the loops described above it follows that the total winding of all the loops around the center of the box is large. Furthermore this winding being large implies that the loops must create many connections between points lying on the two sides of the horizontal line going through the center. This means that the probability of lying on one such loop for points that are far away from each other is not that small and hence cannot be exponentially decaying. All in all, a BKT-like transition must occur in the spin model.

References

[1] Mermin, N. & Wagner, H. Absence of Ferromagnetism or Antiferromagnetism in One- or Two-Dimensional Isotropic Heisenberg Models. *Phys. Rev. Lett.*. **17**, 1133–1136 (1966,11)

[2] Fröhlich, J. & Spencer, T. The Kosterlitz-Thouless transition in two-dimensional Abelian spin systems and the Coulomb gas. *Communications In Mathematical Physics.* **81**, 527–602 (1981)

[3] McBryan, O. & Spencer, T. On the decay of correlations in $SO(n)$-symmetric ferromagnets. *Communications In Mathematical Physics.* **53** pp. 299–302 (1977)

[4] Griffiths, R., Hurst, C. & Sherman, S. Concavity of Magnetization of an Ising Ferromagnet in a Positive External Field. *Journal Of Mathematical Physics.* **11**, 790–795 (1970)

[5] Duminil-Copin, H. & Tassion, V. A New Proof of the Sharpness of the Phase Transition for Bernoulli Percolation and the Ising Model. *Communications In Mathematical Physics.* **343**, 725–745 (2016, 1)

[6] Duminil-Copin, H., Sidoravicius, V. & Tassion, V. Continuity of the Phase Transition for Planar Random-Cluster and Potts Models with $1 \leq q \leq 4$. *Communications In Mathematical Physics.* **349**, 47–107 (2017)

[7] Aizenman, M. Geometric analysis of φ^4 fields and Ising models. I, II. *Comm. Math. Phys..* **86**, 1–48 (1982)

[8] Aizenman, M. & Duminil-Copin, H. Marginal triviality of the scaling limits of critical 4D Ising and $\lambda\varphi_4^4$ models. *Annals Of Mathematics.* **194**, 163–235 (2021)

[9] Lammers, P. Height function delocalisation on cubic planar graphs. *Probability Theory And Related Fields.* (2021)

[10] Engelenburg, D. & Lis, M. An Elementary Proof of Phase Transition in the Planar XY Model. *Communications In Mathematical Physics.* **399**, 85-104 (2023,4,1)

Extract one-arm exponent in FK models from the convergence of height functions to GFF

Jiaming Xia

(joint work with Hugo Duminil-Copin, Hong-bin Chen, Tiancheng He, Francois Jacopin, Dmitry Krachun and Ioan Manolescu)

We consider the 2D FK random cluster models with $q \in [1,4]$ on the square lattice. We assume the convergence of the height functions (of the 6-vertex models associated with the FK models) to GFF and in particular we assume that we know the variance σ^2 of the GFF. Then, we sketch an approach to obtain the one-arm exponent α_1 describing the probability of having a primal crossing of an annulus. The basis for this approach is the BKW coupling relating the height function to the interface loops of FK. We show that by choosing appropriate test functions (viewed as placing charges on the plane), we can get relations between σ^2, α_1, and a factor accounting for the local concentration of small interface loops.

Let F be a nice function such that $\int F = 0$ and let $q \in [1,4]$. Let H be the height function of the six-vertex model. The BKW coupling gives

$$\mathbb{E}_{6V}[e^{i\int FH}] = \mathbb{E}_{FK}\left[\prod_{C\in\mathcal{C}(\omega)} \frac{\cos\left(2\pi\mu + \int_{\mathrm{int}(C)} F\right)}{\cos 2\pi\mu}\right]. \tag{1}$$

In the above display, the product is over all interface loops and

$$\mu = \frac{1}{2\pi}\arccos\left(\frac{\sqrt{q}}{2}\right) \in \left[0, \frac{1}{6}\right]. \tag{2}$$

Sending the mesh size to 0, we expect H to converge to some GFF with some variance σ^2 and to get

$$\mathbb{E}_{6V}\left[e^{i\int FH}\right] \xrightarrow{\delta\to 0} \mathbb{E}_{GFF}\left[e^{i\int Fh}\right]. \tag{3}$$

We want to extract the one-arm exponent from this. Recall the one-arm exponent α_1 defined by

$$\mathbb{P}[\mathrm{Cross}(A(r,R))] = \pi_1(r,R) \sim (r/R)^{\alpha_1 + o(r/R)}, \quad \text{as } r/R \to 0, \tag{4}$$

where $\mathrm{Cross}(A(r,R))$ is the event that there is an open cluster crossing the annulus with radii r and R with $0 < r < R$. Our proof gives the existence of α_1 and its value (in terms of σ^2).

We start from the basic case $q = 4$, in which case we have $\mu = 0$ and the right-hand side of (3) becomes

$$\mathbb{E}_{FK}\left[\prod_{C\in\mathcal{C}(\omega)} \cos\left(\int_{\mathrm{int}(C)} F\right)\right]. \tag{5}$$

We choose F by placing a charge of $\frac{\pi}{2}$ and a charge of $-\frac{\pi}{2}$ respectively on two balls with radii ε at a distance of order 1 on the plane. We can show that the above expectation is approximately

$$\mathbb{P}_{FK}(\text{the charges are connected by an open cluster}) \sim \pi_1(\varepsilon,1)^2. \tag{6}$$

On the other hand, the GFF computation for the left-hand side of (3) gives us

$$\mathbb{E}_{GFF}\left[e^{i\int Fh}\right] \sim \varepsilon^{a(\sigma^2)}, \tag{7}$$

for some simple expression $a(\sigma^2)$ in terms of σ^2. Thus, we obtain $\alpha_1 = a(\sigma^2)/2$.

The other cases of $q \in [1,4)$ require more careful treatments than when $q = 4$. The proof is technical, and we need more than one test function to achieve the goal.

This project is based on the idea of Hugo Duminil-Copin. Many people involved in this project include Hong-Bin Chen, Tiancheng He, Francois Jacopin, Dmitry Krachu, and Ioan Manolescu. Piet Lammers contributed to the discussion at the early stage of the project.

Delocalisation for the discrete long-range Gaussian chain

Paul Dario

1. Overview

The discrete long-range Gaussian chain is a model of discrete interfaces in one dimension with long-range interactions. In order to define the model, three parameters need to be introduced:

- We let $N \in \mathbb{N}$ be a (large) integer which represents the length of the chain;
- We let $\beta \in (0,\infty)$ be the inverse temperature;
- We let $\alpha \in (1,\infty)$ be the range exponent.

We then introduce the set of integer-valued functions

$$\Omega_N := \{\varphi : \mathbb{Z} \to \mathbb{Z} \,:\, \forall\, k \notin \{-N,\dots,N\},\, \varphi(k) = 0\},$$

and define the discrete long-range Gaussian chain of length N at inverse temperature β and with range exponent α to be the probability distribution on the set Ω_N

given by the identity

$$\mu_{N,\beta,\alpha}(\{\varphi\}) = \frac{1}{Z_{N,\beta,\alpha}} \exp\left(-\beta \sum_{\substack{i,j\in\mathbb{Z}\\ i\neq j}} \frac{|\varphi(i)-\varphi(j)|^2}{|i-j|^\alpha}\right), \tag{1}$$

where $Z_{N,\beta,\alpha}$ is the normalizing constant. We will denote by $\mathrm{Var}_{N,\beta,\alpha}$ the variance with respect to $\mu_{N,\beta,\alpha}$.

Various questions can be investigated on the discrete long-range Gaussian chain, and we will be interested in this talk in the localisation/delocalisation of the chain: for fixed inverse temperature $\beta \in (0,\infty)$ and range exponent $\alpha \in (1,\infty)$, how does the variance $\mathrm{Var}_{N,\beta,\alpha}[\varphi(0)]$ behaves as $N \to \infty$?

Before answering this question in more details, we state a few monotonicity properties of this variance which are direct consequences of the Regev-Stephens-Davidowitz monotonicity theory [9]:

(i) The variance $\mathrm{Var}_{N,\beta,\alpha}[\varphi(0)]$ is increasing in N;
(ii) The variance $\mathrm{Var}_{N,\beta,\alpha}[\varphi(0)]$ is decreasing in β;
(iii) The variance $\mathrm{Var}_{N,\beta,\alpha}[\varphi(0)]$ is increasing in α.

In particular, the point (i) implies that, for fixed $\alpha \in (1,\infty)$ and $\beta \in (0,\infty)$, the sequence $N \mapsto \mathrm{Var}_{N,\beta,\alpha}[\varphi(0)]$ is either bounded or diverges to infinity. In the first case, we say that the chain is localised and in the second we say that the chain is delocalised. On a qualitative level, it was proved recently by Garban [7] that the chain is localised for any $\alpha \in (1,2)$ and any $\beta \in (0,\infty)$, and by Coquille-van Enter-Le Ny-Ruszel [3] that the chain is delocalised for any $\alpha \in (2,\infty)$ and for any $\beta \in (0,\infty)$ (the case $\alpha = 2$ is rather singular and the chain exhibits a roughening phase transition between a localised regime for $\beta \gg 1$ and a delocalised regime for $\beta \ll 1$, see [8, 6] and the table below).

On a quantitative level, the behaviour of the variance $\mathrm{Var}_{N,\beta,\alpha}[\varphi(0)]$ is summarised in the following table (upper and lower bounds precise up to multiplicative constants can be obtained)

$\mathrm{Var}_{N,\beta,\alpha}[\varphi(0)]$	$\beta \gg 1$	$\beta \ll 1$
$\alpha \in (1,2)$	1	1
$\alpha = 2$	1	$\ln N$
$\alpha \in (2,3)$	$N^{\alpha-2}$	$N^{\alpha-2}$
$\alpha = 3$	$N/\ln N$	$N/\ln N$
$\alpha > 3$	N	N

Let us make a few bibliographical comments regarding this table:

- The article of Kjaer-Hilhorst [8] proves the result in the case $\alpha = 2$ and $\beta \ll 1$ (together with more precise results);
- The article of Fröhlich-Zergalinski [8] proves the result in the case $\alpha = 2$ and $\beta \gg 1$;

- The article of Garban [7] proves the results in the cases: (i) $\beta \ll 1$ for any $\alpha \in (1, \infty)$ and (ii) for any $\beta \in (0, \infty)$ and $\alpha \in (1,2) \cup (3, \infty)$ (together with various more precise results);
- In the article in preparation [2] with Coquille and Le Ny, the remaining cases are proved: $\beta \gg 1$ and $\alpha \in (2, 3]$.

This talk will be devoted to the argument of [2] and the tools developed there are based on graph surgery techniques which have been recently used by van Engelenburg-Lis [4, 5] and Aizenman-Harel-Peled-Shapiro [1] to study the phase transition of related models: the two-dimensional integer-valued Gaussian free field and the XY and Villain spin systems.

To explain the argument in more details, we first introduce a definition: to each finite connected rooted graph $G := (V, E, x)$ (with vertex set V, edge set E, root $x \in G$) equipped with conductances $\lambda := (\lambda_{ij})_{ij \in E} \in (0, \infty)^E$, we associate a random interface by equipping the set of functions $\Omega_V := \{\varphi : V \to \mathbb{Z} : \varphi(x) = 0\}$ with the probability distribution

$$\mathbb{P}^{\mathrm{IV-GFF}}_{G,\lambda}(\{\varphi\}) := \frac{1}{Z_{G,\lambda}} \sum_{\varphi \in \Omega_V} \exp\left(- \sum_{ij \in E} \lambda_{ij} \left(\varphi(i) - \varphi(j)\right)^2 \right).$$

The discrete long-range Gaussian chain introduced in (1) can be easily expressed in this formalism.

Let us then fix a finite rooted graph $G := (V, E, x)$ equipped with conductances as well as a vertex $v \in V$. The core of the argument of [2] relies on the fact that the following operations on the graph G have a monotonic effect on the variance of the height $\varphi(v)$:

(i) Erasing an edge from the graph G increases the variance of the height $\varphi(v)$;
(ii) Identifying two vertices of the graph G reduces the variance of $\varphi(v)$;
(iii) Adding a new vertex in the middle of an edge while suitably adjusting the conductances reduces the variance of $\varphi(v)$.

These operations can then be combined to simplify the graph associated with the model (1) (while having a monotonic effect on the variance of $\varphi(0)$) in order to obtain a simpler graph (essentially a one dimensional nearest-neighbour line) on which the variances of the corresponding model of random interfaces can be explicitly computed.

References

[1] M. Aizenman, M. Harel, R. Peled, and J. Shapiro. Depinning in integer-restricted Gaussian Fields and BKT phases of two-component spin models. *arXiv preprint arXiv:2110.09498*, 2021.

[2] L. Coquille, P. Dario and A. Le Ny. Quantitative delocalisation for the Gaussian and q-SOS long-range chains. *In preparation.*

[3] L. Coquille, A. van Enter, A. Le Ny, and W. M. Ruszel. Absence of Shift-Invariant Gibbs States (Delocalisation) for One-Dimensional $\mathbb{Z}$-Valued Fields With Long-Range Interactions. *Journal of Statistical Physics*, 191(7):80, 2024.

[4] D. van Engelenburg and M. Lis. An elementary proof of phase transition in the planar XY model. *Communications in Mathematical Physics*, 399(1):85–104, 2023.

[5] D. van Engelenburg and M. Lis. On the duality between height functions and continuous spin models. *arXiv preprint arXiv:2303.08596*, 2023.

[6] J. Fröhlich and B. Zegarlinski. The phase transition in the discrete Gaussian chain with $1/r^2$ interaction energy. *Journal of Statistical Physics*, 63:455–485, 1991.

[7] C. Garban. Invisibility of the integers for the discrete Gaussian chain via a Caffarelli-Silvestre extension of the discrete fractional Laplacian. *arXiv preprint arXiv:2312.04536*, 2023.

[8] K. Kjaer and H. Hilhorst. The discrete Gaussian chain with $1/r^n$ interactions: Exact results. *Journal of Statistical Physics*, 28:621–632, 1982.

[9] O. Regev and N. Stephens-Davidowitz. An inequality for Gaussians on lattices. *SIAM Journal on Discrete Mathematics*, 31(2):749–757, 2017.

Polaron and its effective mass

Chiranjib Mukherjee

(joint work with Rodrigo Bazaes, Mark Sellke and S. R. S. Varadhan)

1. The Fröhlich Polaron

The *Polaron problem* in quantum mechanics is inspired by studying the slow movement of a charged particle, e.g. an electron, in a crystal whose lattice sites are polarized by this slow motion. The electron then drags around it a cloud of polarized lattice points which influences and determines the *effective behavior* of the electron.

By Feynman's path integral formulation [6], the Polaron problem can be studied using a probabilistic layout, which we will focus on here. The physically relevant quantities of the Polaron are given by its *ground state energy* $g(\alpha)$ and its *effective mass* $m_{\mathrm{eff}}(\alpha)$. The former is defined as

$$
\begin{aligned}
g(\alpha) &:= \lim_{T\to\infty} \frac{1}{T} \log Z_{\alpha,T}, \\
Z_{\alpha,T} &:= \mathbb{E}\bigg[\exp\bigg\{ \frac{\alpha}{2} \int_{-T}^{T} \int_{-T}^{T} \mathrm{d}s\mathrm{d}t \; \frac{\mathrm{e}^{-|t-s|}}{|\omega(t)-\omega(s)|} \bigg\} \bigg]
\end{aligned}
\tag{1}
$$

where $\mathbb{E}$ denotes expectation w.r.t. the law of increments of a three-dimensional Brownian path. Here $\alpha > 0$ is a constant, known as the *coupling parameter* which measures how strongly the electron is coupled with its neighborhood. The strong coupling limit of $g(\alpha)$ was studied by Pekar in 1949 who also conjectured that the limit

$$
\begin{aligned}
g_0 &:= \lim_{\alpha\to\infty} \frac{g(\alpha)}{\alpha^2} \qquad \text{exists, and} \\
g_0 &= \sup_{\substack{\psi\in H^1(\mathbb{R}^3) \\ \|\psi\|_2=1}} \bigg[\int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\psi^2(x)\psi^2(y)}{|x-y|} \mathrm{d}x\mathrm{d}y - \frac{1}{2}\|\nabla\psi\|_2^2 \bigg].
\end{aligned}
\tag{2}
$$

By a well-known result of E. Lieb [8], the above variational formula g_0 admits a rotationally symmetric, smooth and centered maximizer $\psi_0 \in H^1(\mathbb{R}^3)$ with

$\|\psi_0\|_2 = 1$ which is unique except for spatial translations. Starting with the expression (1) and using large deviation theory from [3], Pekar's conjecture (2) was proved by Donsker and Varadhan [4]. Later a different proof was given by E. Lieb and L. Thomas [10] using a functional analytic approach.

2. Effective mass and Polaron path measure

We turn to the study of the effective mass $m_{\text{eff}}(\alpha)$ (see (9) below for its definition). According to a conjecture of Landau and Pekar [7] from 1948, the effective mass $m_{\text{eff}}(\alpha)$ should diverge like α^4 in the strong coupling limit:

$$m_{\text{eff}}(\alpha) \sim \alpha^4 \qquad \text{as } \alpha \to \infty \tag{3}$$

Studying $m_{\text{eff}}(\alpha)$ and in particular proving its divergence rate turned out to be much more subtle. In 1987, H. Spohn [14] established a link between $m_{\text{eff}}(\alpha)$ and the actual *path behavior* under the *Polaron measure.* Indeed, the exponential weight on the right hand side in (1) defines a tilted measure on the path space of the Brownian motion, or rather, on the space of increments of Brownian paths. More precisely, let $\mathbb{P} = \mathbb{P}_T$ be the law of the Brownian increments $\{\omega(t)-\omega(s)\}_{-T\le s<t\le T}$ for three dimensional Brownian motion. Then the *Polaron measure* is defined as the transformed measure

$$\widehat{\mathbb{P}}_{\alpha,T}(\mathrm{d}\omega) = \frac{1}{Z_{\alpha,T}} \exp\left(\frac{\alpha}{2}\int_{-T}^{T}\int_{-T}^{T} \frac{\mathrm{e}^{-|t-s|}}{|\omega(t)-\omega(s)|}\mathrm{d}t\mathrm{d}s\right)\mathbb{P}(\mathrm{d}\omega) \tag{4}$$

with the partition function $Z_{\alpha,T}$ defined in (1). is the total mass of the exponential weight, or the *partition function.*

It was conjectured by Spohn in [14] that for any fixed coupling $\alpha > 0$ and as $T \to \infty$, the distribution of the diffusively rescaled Brownian path under the Polaron measure should be asymptotically Gaussian with zero mean and variance $\sigma^2(\alpha) > 0$. The following results were shown in [11]: for any $\alpha > 0$, the infinite volume limit

$$\widehat{\mathbb{P}}_\alpha = \lim_{T\to\infty} \widehat{\mathbb{P}}_{\alpha,T} \tag{5}$$

exists and can be identified explicitly. Indeed, both $\widehat{\mathbb{P}}_{\alpha,T}$ and the limit $\widehat{\mathbb{P}}_\alpha$ is an explicit "mixture"

$$\widehat{\mathbb{P}}_{\alpha,T}(\cdot) = \int \mathbf{P}_{\hat{\xi},\hat{u}}(\cdot)\ \widehat{\Theta}_{\alpha,T}(\mathrm{d}\hat{\xi}\,\mathrm{d}\hat{u}), \qquad \widehat{\mathbb{P}}_\alpha(\cdot) = \int \mathbf{P}_{\hat{\xi},\hat{u}}(\cdot)\ \widehat{\Theta}_\alpha(\mathrm{d}\hat{\xi}\,\mathrm{d}\hat{u}) \tag{6}$$

of centered Gaussian measures $\mathbf{P}_{\hat{\xi},\hat{u}}$, indexed by $(\hat{\xi},\hat{u}) = \{[s_i,t_i],u_i\}$ that stand for the realizations of a point process (possibly overlapping intervals $[s_i,t_i]$ in $[-T,T]$, resp. in the entire real line). The *mixing measures* $\widehat{\Theta}_{\alpha,T}$ and $\widehat{\Theta}_\alpha = \lim_{T\to\infty}\widehat{\Theta}_\alpha$ above are the laws of this Poisson point process, with a suitable tilt. We refer to [11] for details.

As a result of the Gaussian representation (5)-(6) and the above point processes structure, the following CLT was also shown in [11] for any $\alpha > 0$:

$$\lim_{T\to\infty} \widehat{\mathbb{P}}_{\alpha,T}\left[\frac{\omega(T)-\omega(-T)}{\sqrt{2T}} \in \cdot\right] = \lim_{T\to\infty} \widehat{\mathbb{P}}_{\alpha}\left[\frac{\omega(T)-\omega(-T)}{\sqrt{2T}} \in \cdot\right] = \mathbf{N}(0,\sigma^2(\alpha)\mathbf{I}_{3\times 3}) \tag{7}$$

where $\mathbf{N}(0,\sigma^2(\alpha)\mathbf{I}_{3\times 3})$ is a three-dimensional Gaussian vector with mean zero and covariance matrix $\sigma^2(\alpha)\mathbf{I}_{3\times 3}$ where the *homogenized variance* admits the variational representation

$$\begin{aligned}\sigma^2(\alpha) &= \lim_{T\to\infty}\frac{1}{2T}\mathbb{E}^{\widehat{\mathbb{P}}_{\alpha,T}}\left[|\omega(T)-\omega(-T)|^2\right] = \lim_{T\to\infty}\frac{1}{2T}\mathbb{E}^{\widehat{\mathbb{P}}_{\alpha}}\left[|\omega(T)-\omega(-T)|^2\right] \\ &= \lim_{T\to\infty}\sup_{f(\cdot)}\left[\frac{f(T)-f(-T)}{\sqrt{2T}} - \frac{1}{2}\int_{-T}^{T}\dot f^2(t)\mathrm{d}t - \frac{1}{2}\sum_{i=1}^{n(\hat\xi,\hat u)} u_i^2|f(t_i)-f(s_i)|^2\right]\end{aligned} \tag{8}$$

Here, the supremum is take over functions $f:[-T,T]\to\mathbb{R}$ with square-integrable derivatives, and the limit in the second line above is taken almost surely and in L^1 w.r.t. the aforementioned tilted law $\widehat{\Theta}_\alpha$ of the point process with realizations $(\hat\xi,\hat u)=\{[s_i,t_i],u_i\}$ (see [11]).

Assuming the validity of the above CLT (7), already in [14] Spohn had proved (see also [5]) a simple relation between the effective mass $m_{\mathrm{eff}}(\alpha)$ and the above CLT variance $\sigma^2(\alpha)$:

$$m_{\mathrm{eff}}(\alpha)^{-1} = \sigma^2(\alpha) \qquad \text{for any } \alpha > 0, \tag{9}$$

In [14] it was also conjectured that the the strong coupling behavior of the infinite-volume limit $\lim_{\alpha\to\infty}\lim_{T\to\infty}\widehat{\mathbb{P}}_{\alpha,T} = \lim_{\alpha\to\infty}\mathbb{P}_\alpha$ (suitably rescaled) should converge to the so-called *Pekar process*, which is a diffusion process with generator

$$\frac{1}{2}\Delta + \frac{\nabla\psi}{\psi}\cdot\nabla, \tag{10}$$

where ψ is any solution of the variational problem (1). It was proved in [12] that (after rescaling) $\widehat{\mathbb{P}}_\alpha$ converges as $\alpha\to\infty$ to a unique limit which is the *increments* of the Pekar process.

We finally turn to the strong coupling behavior of the effective mass $m_{\mathrm{eff}}(\alpha)$. First, using a functional analytic route from [10], it was shown in [9] that $\lim_{\alpha\to\infty} m_{\mathrm{eff}}(\alpha)=\infty$. Using the point process techniques from [11], in [2], this was improved to a bound $m_{\mathrm{eff}}(\alpha)\geq C\alpha^{2/5}$. Using a different probabilistic approach employing Gaussian correlation inequality, [13] showed the almost quartic bound $m_{\mathrm{eff}}(\alpha)\geq C\frac{\alpha^4}{\log(\alpha)^6}$.

Analyzing the point process representation (5)-(6) and the variational formula (8) from [11] in the strong coupling limit and convergence to the Pekar process [12], the *sharp* bound

$$m(\alpha)\geq C\alpha^4$$

was recently shown in [1], verifying the Landau-Pekar conjecture (3) from 1948 upto constants.

References

[1] R. Bazaes, C. Mukherjee, M. Sellke and S. R. S. Varadhan. Effective Mass of the Fröhlich Polaron and the Landau- Pekar- Spohn Conjecture. arXiv: 2307.13058

[2] V. Betz and S. Polzer. Effective mass of the Polaron: a lower bound. arXiv: 2201.06445. *Comm. Math. Phys.* (2023)

[3] M. D. Donsker and S. R S. Varadhan. Asymptotic evaluation of certain Markov process expectations for large time, IV *Comm. Pure Appl. Math.* **3**6 (1983), 183–212.

[4] M. D. Donsker and S. R S. Varadhan. Asymptotics for the Polaron. *C*omm. Pure Appl. Math., 1983, 505-528

[5] W. Dybalski and H. Spohn. Effective Mass of the Polaron- Revisited. *A*nnales Henri Poincaré **21**, 1573–1594 (2020).

[6] R. Feynman. Statistical Mechanics, Benjamin, Reading (1972).

[7] L. D. Landau and S. I. Pekar. Effective mass of a polaron. *Z*h.Eksp.Teor.Fiz. **18**, 419–423 (1948)

[8] E. H. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation. *S*tudies in Appl. Math. **57**, (1976) 93–105

[9] E. H. Lieb and R. Seiringer Divergence of the Effective Mass of a Polaron in the Strong Coupling Limit, *J.* Stat. Phys. **180**, 23–33 (2020)

[10] E. H. Lieb and L. Thomas. Exact ground state energy of the strong-coupling Polaron. *C*omm. Math. Phys., **1**83, (1997), 511–519.

[11] C. Mukherjee and S. R. S. Varadhan. Identification of the Polaron Measure I: Fixed Coupling Regime and the Central Limit Theorem for Large Times *C*omm. Pure Appl. Math., **73** 350–383, (2020), arXiv: 1802.05696

[12] C. Mukherjee and S. R. S. Varadhan. Identification of the Polaron measure in strong coupling and the Pekar variational formula. *A*nn. Probab., **48**(5): 2119–2144 (2020), arXiv: 1812.06927

[13] M. Sellke. Almost Quartic Lower Bound for the Fröhlich Polaron's Effective Mass via Gaussian Domination. arXiv: 2212.14023, *Duke Math. Journal* (2024)

[14] H. Spohn. Effective mass of the polaron: A functional integral approach. *A*nn. Phys. **1**75, (1987), 278–318.

Mini course on mixing and triviality in the XY model

Piet Lammers

The Ising model and the XY model are lattice models in which random spins are assigned to the vertices of the square lattice graph $\mathbb{Z}^d$, with d denoting the dimension. In the Ising model the spins take values in $\{\pm 1\}$, while for the XY model the target space is $\mathbb{S}^1$.

While a variety of tools is available for the analysis of both models, our understanding of the Ising model is now significantly more advanced than that of the XY model. For example, the question of *continuity* of the phase transition is open in all dimensions $d \geq 3$, while it has been proved for the Ising model in any dimension.

The purpose of this mini course is to review some recent developments on the XY model: namely, a new perspective on the Brydges–Fröhlich–Spencer walk.

This new perspective was recently used to prove that the Berezinskii–Kosterlitz–Thouless transition of the two-dimensional XY model coincides exactly with the localisation-delocalisation transition of the corresponding height function. In a collaboration with Paul Dario, Trishen Gunaratnam, and Romain Panis, we aim to adapt the recent proof of Aizenman and Duminil-Copin for *triviality* (Gaussianity) of the scaling limit of the Ising model at criticality in four dimensions to the XY model. We have not yet completed this adaptation, but we have received some interesting partial results, which strongly suggest that it is possible to complete the proof. Most importantly, we obtained *mixing* for the random current representation of the model, meaning that spatially separated regions behave approximately independently.

This course consists of several parts.

- First, we discuss how the BFS random walk can be obtained by developing the exponentials in the partition function, via Poisson point processes.
- Second, we discuss the Ginibre inequality, which implies the FKG lattice condition for the function $J \mapsto Z_J$, where $J = (J_{xy})_{xy \in E(G)}$ is a family of coupling constants and Z_J the corresponding partition function for the XY model. This way of phrasing the Ginibre inequality may be surprising, because the FKG lattice condition is usually considered in the context of the weights of a probabilistic model.
- Third, we discuss some consequences of the Ginibre inequality. For example, it can be used to control the behaviour of the BFS random walk via a series of inequalities. It is important here that the partition function Z_J appears as a weight of this walk. However, the FKG lattice condition is applied to this weight in quite a nontraditional way.
- Then, we discuss Fröhlich's proof of triviality of the Ising and XY models in dimension $d \geq 5$. We believe that this proof is almost equivalent to Aizenman's proof (published earlier in the same year), and in fact our new perspective on the BFS random walk seems to suggest that this random walk is analogous to the backbone exploration used in Aizenman's work.
- Finally, we discuss (very broadly) the proof of Aizenman and Duminil-Copin, and we present the proof of mixing in the context of the XY model, which is the main ingredient of the proof in the context of the Ising model.

Random (tangled) currents for the ϕ^4 model

Franco Severo

(joint work with Trishen S. Gunaratnam, Christoforos Panagiotis, Romain Panis)

The ϕ^4 model is a statistical mechanics model of ferromagnetism with real-valued spins attached to each vertex of a graph whose values are confined according to a quartic single-site potential. The model is described by a probability measure on

$\mathbb{R}^{\mathbb{Z}^d}$ given by

$$\langle F(\phi)\rangle = \frac{1}{Z}\int F(\phi)\,\exp\Big(\beta\sum_{\{x,y\}\in E(\mathbb{Z}^d)}\phi_x\phi_y - \sum_{x\in\mathbb{Z}^d}\big(g\phi_x^4 + a\phi_x^2\big)\Big)\mathrm{d}\phi$$

where $g > 0$ and $a \in \mathbb{R}$ are coupling constants, $\beta > 0$ is the inverse temperature, and Z is a normalisation constant.

This model naturally interpolates between two famous models of statistical physics, namely the Gaussian free field (obtained for $g = 0$ and a large enough) and the Ising model (obtained in the limit $g = a/2 \to \infty$). In fact, he ϕ^4 model is conjectured to be deeply related to the Ising, in the sense that both should belong to the same universality class. A manifestation of this was discovered by Griffiths and Simon in [6], where they show that the ϕ^4 model arises as a certain near-critical scaling limit of mean-field Ising models. Other results towards universality were obtained in dimensions $d \geq 5$ by Aizenman [1] and Fröhlich [4], and more recently in dimension $d = 4$ by Aizenman and Duminil-Copin [2], where triviality of their scaling limits were established. Establishing rigorously universality results in dimensions 2 and 3 (where the scaling limits are expected to be non-trivial) remains a difficult challenge.

In this talk we describe a new geometric representation of the ϕ^4 model, obtained in [7]. This representation, which we call **random tangled currents**, is the analogue of the random current representation of the Ising model, first introduced by Aizenman in [1] and later used in multiple influential works on the Ising model. We obtain such a representation for the ϕ^4 model by properly taking the limit of the Ising random current representation along the aforementioned approximation of Griffiths and Simon [6].

As an application of this representation, we adapt a work of Raoufi [8] on the Ising model, in order to obtain obtain a classification of translation-invariant Gibbs measures. We say that a measure ν on $\mathbb{R}^{\mathbb{Z}^d}$ is a Gibbs measure at inverse temperature $\beta \geq 0$ if it satisfies the corresponding DLR equation

$$\nu(f) = \int_{\eta\in\mathbb{R}^{\mathbb{Z}^d}} \langle f\rangle^{\eta}_{\Lambda,\beta}\mathrm{d}\nu(\eta) \tag{1}$$

for every finite $\Lambda \subset \mathbb{Z}^d$ and $f \in \mathbb{R}^{\Lambda}$ bounded and measurable, where $\langle\cdot\rangle^{\eta}_{\Lambda,\beta}$ is the ϕ^4 measure at $\beta \geq 0$ on Λ with boundary conditions η. For each $\beta \geq 0$, let $\mathcal{G}(\beta)$ be the set of all Gibbs measures at β and $\mathcal{G}_\Gamma \subset \mathcal{G}(\beta)$ be the subset of Gibbs measures which are invariant under all translations γ of $\mathbb{Z}^d$. We prove the following.

Theorem 1. *For every $d \geq 2$, $g > 0$, $a \in \mathbb{R}$ and $\beta \geq 0$, one has $\mathcal{G}_\Gamma(\beta) = \{t\langle\cdot\rangle^+_\beta + (1-t)t\langle\cdot\rangle^-_\beta,\ t \in [0,1]\}$.*

In dimensions $d \geq 3$, we can combine this result with the infrared bound of Fröhlich, Simon, and Spencer [5] to conclude that there exists a unique Gibbs measure at criticality. Such a result is known as continuity of the phase transition, and was established for the Ising model in dimensions $d \geq 3$ in the seminal work of Aizenman, Duminil-Copin, and Sidoravicious [3].

Theorem 2. *For every $d \geq 3$, one has $|\mathcal{G}(\beta_c)| = 1$.*

References

[1] M. Aizenman: *Geometric analysis of ϕ^4 fields and Ising models. Parts I and II.* Communications in Mathematical Physics **86**, 1–48 (1982)

[2] M. Aizenman, H. Duminil-Copin: *Marginal triviality of the scaling limits of critical 4D Ising and ϕ_4^4 models.* Annals of Mathematics **194**, 163–235 (2021)

[3] M. Aizenman, H. Duminil-Copin, V. Sidoravicius: *Random currents and continuity of Ising model's spontaneous magnetization.* Communications in Mathematical Physics **334**, 719–742 (2015)

[4] J. Fröhlich: *On the triviality of $\lambda\phi_d^4$ theories and the approach to the critical point in $d > 4$ dimensions.* Nuclear Physics B **200**, 281–296 (1982)

[5] J. Fröhlich, B. Simon, T. Spencer: *Infrared bounds, phase transitions and continuous symmetry breaking.* Communications in Mathematical Physics **50**, 79–95 (1976)

[6] R. Griffiths, B. Simon: *The $(\phi^4)_2$ field theory as a classical Ising model.* Communications in Mathematical Physics **33**, 145–164 (1973)

[7] T. Gunaratnam, C. Panagiotis, R. Panis, F. Severo: *Random tangled currents for ϕ^4: translation invariant Gibbs measures and continuity of the phase transition*, available at https://arxiv.org/abs/2211.00319, (2022)

[8] A. Raoufi: *Translation-invariant Gibbs states of the Ising model: general setting.* The Annals of Probability **48**, 760–777 (2020)

Thick points of the planar GFF are totally disconnected $\forall \gamma \neq 0$

Ellen Powell

(joint work with Juhan Aru and Léonie Papon)

We study the set of thick points of a GFF h with Dirichlet boundary conditions in an open and simply connected domain $D \subset \mathbb{C}$. This is a centered Gaussian process indexed by smooth functions that are compactly supported in D. Its covariance is given by, for $f, g \in \mathcal{C}_c^\infty(D)$,

$$\mathbb{E}[(h,f)(h,g)] = \int_{D\times D} f(x) G_D(x,y) g(y) dx dy$$

where G_D is the Green function of the Laplacian in D with Dirichlet boundary conditions, normalised so that $\Delta G_D(x,\cdot) = -\Delta_x(\cdot)$. As $G_D(x,x) = \infty$, the process $(h,f)_{f\in\mathcal{C}_c^\infty(D)}$ does not correspond to integration against a pointwise defined function. It does, however, almost surely correspond to an element of the Sobolev space $\mathcal{H}^{-\epsilon}(D)$, $\epsilon > 0$, i.e. a distribution, or generalised function.

For h a Dirichlet GFF in D, the set of thick points of h is a special set of points at which, loosely speaking, h takes atypically high or low values. As h is not defined pointwise, this set must be defined by regularisation. Let $z \in D$ and $r > 0$ and denote by ρ_r^z the uniform measure on $\partial B(z,r)$ where $B(z,r)$ is the ball centered at z of radius r. We consider the random variable $h_r(z) := (h, \rho_r^z)$ which is well-defined, e.g. by taking limits, since the integral $\int G_D(x,y)\rho_r^z(dx)\rho_r^z(dy)$ is finite.

In fact, by [1, Proposition 2.1], if h is a GFF with Dirichlet boundary conditions in the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, then $(h_r(z))_{r,z}$ has a version that is a.s. jointly Hölder continuous in r and z. We will only work with this version of the circle average process.

For fixed $z \in D$, a direct calculation shows that the process $h_{e^{-t}}(z)$ actually evolves as a linear Brownian motion in t. In particular, $\lim_{r\to 0} h_r(z)/\log(1/r) = 0$ almost surely. However, this does not rule out the existence of exceptional points at which this limit is non-zero: these points are called the thick points of h. It is natural to define, for $\gamma \in \mathbb{R}$, the set of γ-thick points of h by

$$\mathcal{T}_\gamma(h) := \{z \in D : \lim_{r\to 0} \frac{h_r(z)}{\log 1/r} = \frac{\gamma}{\sqrt{2\pi}}\} \tag{1}$$

where the factor $1/\sqrt{2\pi}$ comes from our choice of normalisation for the Green function. Note that since we work with a Hölder continuous version of the circle average process, there is an event of probability one on which we can determine the existence (or not) of the limit in (1) for all z in D simultaneously. That is, the set $\mathcal{T}_\gamma(h)$ is well defined with probability one.

It was shown in [1] that if $|\gamma| > 2$, this set is almost surely empty, and if $\gamma \in [-2, 2]$, it almost surely has Hausdorff dimension $2 - \gamma^2/2$. In particular, if $\gamma = 0$, then $\mathcal{T}_0(h)$ almost surely has Hausdorff dimension 2: 0-thick points are typical, as discussed above.

We consider another geometric property of $\mathcal{T}_\gamma(h)$. Recall that a set U is said to be totally disconnected if for each point $x \in U$, the connected component of x in U consists of just that point x. A sufficient condition for a set to be totally disconnected is that this set has Hausdorff dimension strictly less than 1. In particular, observe that if $|\gamma| > \sqrt{2}$, then $\mathcal{T}_\gamma(h)$ has almost sure Hausdorff dimension strictly less than 1, which therefore implies that $\mathcal{T}_\gamma(h)$ is almost surely totally disconnected. One may wonder whether this property extends to the full range $\gamma \in [-2, 2] \setminus \{0\}$. The answer to this question is positive and this is the main result discussed in the talk. By conformal invariance, we may restrict ourselves to the case where h is a Dirichlet GFF in $D = \mathbb{D}$.

Theorem 1. Let h be a GFF with Dirichlet boundary conditions in $\mathbb{D}$. Then almost surely, $\mathcal{T}_\gamma(h)$ is totally disconnected for all $\gamma \in [-2, 2] \setminus \{0\}$.

Theorem 1 is stated for a GFF with Dirichlet boundary conditions in $\mathbb{D}$. However, one can deduce from this result that a similar statement holds for a GFF with other boundary conditions.

The proof of Theorem 1 is based on a coupling of the Dirichlet GFF with a nested version of CLE_4. This coupling, and the construction of nested CLE_κ gives rise to a *different*, but natural, definition of the set of γ-thick points for the GFF, $\Phi_\gamma(h)$, defined via its so-called weighted CLE_4 nesting field, as studied in [2]. The key geometric property of CLE that we need is the following.

Theorem 2. Let $\kappa \in (8/3, 4]$ and let Γ be a nested CLE_κ in $\mathbb{D}$. Then the complement of Γ is almost surely totally disconnected.

Given Theorem 2, Theorem 1 is then a consequence of the following result, that is of independent interest, and can be thought of as a universality statement for different notions of GFF-thick points.

Theorem 3. Let h be a GFF in $\mathbb{D}$ with Dirichlet boundary conditions. Then, with probability one, $\mathcal{T}_\gamma(h) \equiv \Phi_\gamma(h)$ for every $\gamma \in [-2,2] \setminus \{0\}$.

One consequence of these results is in the probabilistic approach to *Liouville quantum gravity.* One of the most important objects in this context is the so-called Liouville quantum gravity (LQG) measure. It depends on a parameter $\gamma \in (0,2)$ and can informally be defined as $\mu_\gamma(dz) = e^{\gamma h(z)}dz$ where h is a Dirichlet GFF. As h is not defined pointwise, the rigorous construction of μ_γ involves a regularisation procedure. This measure is intimately connected to thick points of the underlying field. Indeed, if one samples a point according to the normalised LQG measure μ_γ with parameter $\gamma \in (0,2)$, then this point is almost surely a γ-thick point of the field used to construct μ_γ.

Another object of interest in the context of Liouville quantum gravity is the so-called LQG metric which can be thought of as a conformal perturbation of the Euclidean metric by the exponential of the GFF. For a parameter $\xi > 0$, the LQG metric in a disk $\mathbb{D}$ is formally defined by

$$D_h^\xi(z,w) = \inf_{P:z\to w} \int_0^1 e^{\xi h(P(t))} |P'(t)| dt, \tag{2}$$

where the infimum is over all continuous paths from z to w inside $\mathbb{D}$ and h is a GFF. The definition (2) is purely formal as h is not defined pointwise. A proper construction via regularisation is more involved than the construction of μ_h, but has now been succesfully carried out in a series of works by Ding, Dubédat, Dunlap, Falconet, Gwynne and Miller.

The properties of the LQG metric crucially depend on the parameter ξ in (2). In particular, by [3], there exists a unique $\xi_{crit} > 0$ such that if $\xi > \xi_{crit}$, then the metric with parameter ξ almost surely does *not* induce the Euclidean topology on $\mathbb{D}$. Instead, such a metric, called supercritical, admits a set of singular points: these points are at infinite distance from every other point. We denote by $S_h^\xi(\mathbb{D})$ this set of singular points of D_h^ξ, that is $S_h^\xi(\mathbb{D}) := \{z \in \mathbb{D} : D_h^\xi(z,w) = \infty \quad \forall\, w \in \mathbb{D} \setminus \{z\}\}$. This set is intimately related to thick points of h. Indeed, [3, Proposition 1.11] shows that there exists $Q(\xi) \in (0,2)$ such that

$$S_h^\xi(\mathbb{D}) = \{z \in \mathbb{D} : \limsup_{r\to 0} \frac{h_r(z)}{\log 1/r} > Q(\xi)\} \quad \text{almost surely.}$$

A consequence of the proof of Theorem 1 is the almost sure total disconnectedness of $S_h^\xi(\mathbb{D})$.

Proposition. Let $\xi > \xi_{crit}$. Then $S_h^\xi(\mathbb{D})$ is almost surely totally disconnected.

References

[1] X. Hu, J. Miller and Y. Peres *Thick points of the Gaussian free field*, The Annals of Probability **38** (2010).

[2] J. Miller, S.S. Watson and D.B. Wilson *The conformal loop ensemble nesting field*, Probability Theory and Related Fields **163** (2015).

[3] J. Pfeffer *Weak Liouville quantum gravity metrics with matter central charge $c \in (-\infty, 25)$*, Probability and Mathematical Physics **5** (2024).

Infinite volume limit of the Spherical Model: Appearance of the Gaussian Free Field

Juhan Aru, Aleksandra Korzhenkova

Let $\mathbb{T}_n^d$ be a d-dimensional discrete torus with side-length n. To each vertex $x \in \Lambda$, associate a continuous "spin" $\theta_x \in \mathbb{R}$ in such a way that the whole configuration $\boldsymbol{\theta} = (\theta_x)_{x\in\mathbb{T}_n^d}$ lives on the (n^d-1)-dimensional sphere of radius $\sqrt{n^d}$, i.e., $\|\boldsymbol{\theta}\|_2^2 = \sum_x \theta_x^2 = n^d$. We call $\boldsymbol{\theta}$ a configuration of the *spherical model* on $\mathbb{T}_n^d$ at the inverse temperature $\beta \geq 0$ if

$$\boldsymbol{\theta} \sim \nu_{\mathbb{T}_n^d,\beta}(\mathrm{d}\boldsymbol{\theta}) = \frac{1}{Z_{\mathbb{T}_n^d,\beta}} \exp\left(\frac{\beta}{2}\sum_{x\sim y}\theta_x\theta_y\right)\mathrm{Unif}_{\sqrt{n^d}\mathbb{S}^{n^d-1}}(\mathrm{d}\boldsymbol{\theta}),$$

where $x \sim y$ stands for the neighbouring vertices. This model was first introduced by Berlin and Kac [BK52] in 1952 as a simplification of the Ising model that still exhibits a phase transition in $d \geq 3$, but whose free energy can be analytically studied in the infinite volume limit. The two main methods used in the literature to analyze the model in the thermodynamic limit are variations of the steepest-descent method [BK52, SM75] and the so-called mean-spherical approach [LW52, YW65, BD87], where instead of conditioning on the spin configuration, one adds a mass term $(s\sum_x \theta_x^2)$ to the Hamiltonian in such a way that on average $\|\boldsymbol{\theta}\|_2^2$ equals n^d. Both methods, under some unverified technical assumptions (in the low-temperature regime in the absence of a magnetic field), may also lead to our main result, which we prove using probabilistic tools.

Theorem 1 (Infinite volume limit of the spherical model)**.** *Let $d \geq 2$. The spherical model on $\mathbb{T}_n^d$ at inverse temperature $\beta > 0$, $[\boldsymbol{\theta}]_n = (\theta_x)_{x\in\Lambda}$, converges in law uniformly over compact subsets of $\mathbb{Z}^d$ as $n \to \infty$ to:*

(1) *$\beta < \beta_c$: a massive Gaussian free field (GFF) on $\mathbb{Z}^d$ scaled by $1/\sqrt{\beta}$ with the mass $m^2 > 0$ depending on β and d in a specific way;*
(2) *$\beta = \beta_c$: a GFF on $\mathbb{Z}^d$ scaled by $1/\sqrt{\beta_c}$;*
(3) *$\beta > \beta_c$: a GFF on $\mathbb{Z}^d$ scaled by $1/\sqrt{\beta}$ plus an independent constant random drift $\sqrt{\frac{\beta-\beta_c}{\beta}}X$ with X being a Rademacher random variable.*

Furthermore, all local correlations of spins converge.

Let us explain the reasoning leading to the proof. We start by observing that for any n and $m^2 > 0$, since $\sum_x \theta_x^2 = n^d$,

$$\nu_{\mathbb{T}_n^d,\beta}(\mathrm{d}\boldsymbol{\theta}) \propto \exp\left(-\frac{\beta}{2}\langle\boldsymbol{\theta},(-\Delta+m^2)\boldsymbol{\theta}\rangle\right)\mathrm{Unif}_{\sqrt{n^d}\mathbb{S}^{n^d-1}}(\mathrm{d}\boldsymbol{\theta}).$$

Hence, the law of $\sqrt{\beta}[\boldsymbol{\theta}]_n$ is related to a massive GFF $\boldsymbol{\phi}$ (for an arbitrary mass $m^2 > 0$) through conditioning on its norm to be equal to $\sqrt{\beta n^d}$, i.e., $\|\boldsymbol{\phi}\|_2^2 = \beta n^d$. Now the idea is somewhat similar to the mean-spherical approach, but from a probabilistic point of view: namely, we choose mass m_n^2 in such a way that $\mathbb{E}[\|\boldsymbol{\phi}\|_2^2] = \beta n^d$ (or equivalently, $G_{\mathbb{T}_n^d,m_n^2}(0,0) = \beta$ for the massive Green's function $G_{\mathbb{T}_n^d,m_n^2}$ on $\mathbb{T}_n^d$), which is natural due to concentration of Gaussian measure.

For large n and fixed $m > 0$, $G_{\mathbb{T}_n^d,m^2}$ is roughly comparable to $G_{\mathbb{Z}^d,m^2}$; and thus, the question of the existence of a critical point boils down to understanding whether there is an $m^2 \geq 0$ such that $G_{\mathbb{Z}^d,m^2} = \beta$ or not. Notice that this explains the difference between the 2D and higher-dimensional cases. Indeed, since in $d = 2$, $G_{\mathbb{Z}^d,0}(0,0)$ is infinite, by choosing the mass small enough, one could get an arbitrarily large value of the variance $G_{\mathbb{Z}^d,m^2}(0,0)$; whereas in $d \geq 3$, there is a maximal possible value $G_{\mathbb{Z}^d,0}(0,0) < \infty$.

More precisely, we observe that in the high-temperature regime $\beta < \beta_c := G_{\mathbb{Z}^d,0}(0,0)$, the aforementioned sequence of masses $(m_n^2)_n$ converges to a positive number $m^2 > 0$ in such a way that

$$G_{\mathbb{T}_n^d,m_n^2}(0,0) \to G_{\mathbb{Z}^d,m^2}(0,0) = \beta.$$

In this case, the concentration of measure for $\|\boldsymbol{\phi}\|_2^2$ is sufficient to make the conditioning disappear in the limit, and we obtain a purely Gaussian limit. Somewhat refined versions of classical limiting theorems help us handle this phase rather directly.

The low-temperature phase $\beta > \beta_c$ is already trickier. One can check that

$$G_{\mathbb{T}_n^d,m_n^2}(x,y) \sim G^{\text{0-avg.}}_{\mathbb{T}_n^d}(x,y) + \beta - \beta_c \overset{n\to\infty}{\sim} G_{\mathbb{Z}^d,0}(x,y) + \beta - \beta_c,$$

where $G^{\text{0-avg.}}_{\mathbb{T}_n^d}$ is the correlation function of the zero-average GFF on torus. This heuristically allows us to restate the problem in terms of the zero-average GFF $\boldsymbol{\gamma}$ plus an independent zero mode, a constant (in space) Gaussian drift $Z\mathbf{1}_{T_n^d}$, conditioned on the norm of the sum being $\sqrt{\beta n^d}$. Since the drift is constant, almost surely $\frac{1}{n^d}\|\boldsymbol{\gamma} + Z\mathbf{1}\|_2^2 = \frac{1}{n^d}\|\boldsymbol{\gamma}\|_2^2 + |Z|^2$. Due to high concentration of $\frac{1}{n^d}\|\boldsymbol{\gamma}\|_2^2$ around β_c, conditioning on the norm of the sum forces $|Z|$ to become constant in the limit; and by symmetry, we obtain a Rademacher random variable for the zero mode. Making this precise is slightly more subtle, but after finding the right angle, the proof follows from combining basic concentration of measure results with relatively direct density bounds.

Finally, in the critical case, we observe that $G^{\text{0-avg.}}_{\mathbb{T}_n^d}(x,y) \sim G_{\mathbb{Z}^d,0}(x,y)$; however, further refinement of this relation is necessary. In this direction we prove the following error estimates on the zero-average Green's function on the torus in $d \geq 3$:

(1) $G^{\text{0-avg.}}_{\mathbb{T}_n^d}(x,y) = G_{\mathbb{Z}^d,0}(x,y) + O(n^{2-d})$;

(2) $d = 3$: $G^{\text{0-avg.}}_{\mathbb{T}_n^d}(x,x) < G_{\mathbb{Z}^d,0}(x,x)$ uniformly for large n.

Obtaining the sign was surprisingly tricky and is related to what is called Madelung constant for electrostatic potential in certain salts – a quantity of interest in chemical physics introduced in the beginning of the 20th century [Mad19].

References

[BD87] J.G. Brankov and D.M. Danchev. On the limit Gibbs states of the spherical model. *Journal of Physics A: Mathematical and General*, 20(14):4901, oct 1987.

[BK52] T. H. Berlin and M. Kac. The Spherical Model of a Ferromagnet. *Phys. Rev.*, 86:821–835, Jun 1952.

[LW52] H. W. Lewis and G. H. Wannier. Spherical Model of a Ferromagnet. *Phys. Rev.*, 88:682–683, Nov 1952.
[Mad19] Erwin Madelung. Das elektrische Feld in Systemen von regelmäßig angeordneten Punktladungen. *Physikalische Zeitschrift*, 19:524–533, 1919.
[SM75] Yu.N. Sudarev S.A. Molchanov. Gibbs states in the spherical model. *Dokl. Akad. Nauk SSSR*, 224:536–539, 1975.
[YW65] C. C. Yan and G. H. Wannier. Observations on the Spherical Model of a Ferromagnet. *Journal of Mathematical Physics*, 6(11):1833–1838, 11 1965.

High-dimensional spin models may not be so trivial after all....

Romain Panis

The Ising model has been studied for a century as a fundamental example of a phase transition in statistical mechanics. We are interested here in the finite-size scaling of the Ising model on a finite box in the Euclidean lattice $\mathbb{Z}^d$ in dimensions $d > 4$, particularly in the case of periodic boundary conditions which make the box into a torus.

For $d \geq 1$ and an integer $r \geq 3$, we write Λ_r for the discrete box $[-r/2, r/2)^d \cap \mathbb{Z}^d$ of volume r^d, and write $\mathbb{T}_r = (\mathbb{Z}/r\mathbb{Z})^d$ for the d-dimensional discrete torus of period r. Let $\mathcal{G} = (V, E)$ be a finite graph with vertex set V and edge set E. The Ising model on $\mathcal{G}$ is a family of probability measures on spin configurations $\sigma : V \to \{-1, +1\}$, defined using the *Hamiltonian*

$$H^{\mathcal{G}}(\sigma) := -\sum_{xy \in E} \sigma_x \sigma_y. \tag{1}$$

The expectation of a function F of spin configurations, at inverse temperature $\beta > 0$, is defined by

$$\langle F \rangle_\beta^{\mathcal{G}} := \frac{1}{Z_\beta^{\mathcal{G}}} \sum_\sigma F(\sigma) e^{-\beta H^{\mathcal{G}}(\sigma)}, \tag{2}$$

where $Z_\beta^{\mathcal{G}} := \sum_\sigma e^{-\beta H^{\mathcal{G}}(\sigma)}$ is the *partition function.* Free boundary conditions (FBC) on the box correspond to $\mathcal{G} = \Lambda_r$ with edges connecting nearest-neighbours in Λ_r. Periodic boundary conditions (PBC) correspond to $\mathcal{G} = \mathbb{T}_r$ with edges connecting nearest-neighbours in $\mathbb{T}_r$, which include all edges of Λ_r and additional edges that join opposite sides of Λ_r. The *two-point functions* are defined by

$$\tau_\beta^{\Lambda_r}(x, y) := \langle \sigma_x \sigma_y \rangle_\beta^{\Lambda_r}, \qquad \tau_\beta^{\mathbb{T}_r}(x, y) := \langle \sigma_x \sigma_y \rangle_\beta^{\mathbb{T}_r}, \tag{3}$$

and the finite-volume *susceptibilities* are defined by

$$\chi^{\Lambda_r}(\beta) := \sum_{x \in \Lambda_r} \tau_\beta^{\Lambda_r}(0, x), \qquad \chi^{\mathbb{T}_r}(\beta) := \sum_{x \in \mathbb{T}_r} \tau_\beta^{\mathbb{T}_r}(0, x). \tag{4}$$

It follows from the Griffiths inequalities that $\tau_\beta^{\Lambda_r}$ is nonnegative and increasing in r, so that the infinite-volume two-point function $\tau_\beta(x, y) = \lim_{r \to \infty} \tau_\beta^{\Lambda_r}(x, y)$ and infinite-volume susceptibility $\chi(\beta) := \lim_{r \to \infty} \chi^{\Lambda_r}(\beta)$ exist in $[0, \infty]$. Moreover, it is known that there is a critical inverse temperature β_c, below which $\tau_\beta(x,y)$ decays exponentially as $|x-y| \to \infty$ and $\chi(\beta)$ is finite.

The setting of dimensions $d > 4$ corresponds to the *mean-field* regime of the model. It is characterised by a simplification of the critical behaviour of the model. This simplification can be observed either qualitatively or quantitatively. In two independent works, Aizenman [1] and Fröhlich [2] proved that the scaling limits (in an appropriate sense) of the critical Ising model (with FBC) in dimensions $d > 4$ are Gaussian, or *trivial*. This simplified behaviour can be quantified through the computation of *critical exponents*. From [1], we know that for $d > 4$ and $\beta < \beta_c$,

$$\chi(\beta) \asymp \frac{1}{\beta_c - \beta}. \tag{5}$$

Very recently, it was proved in [3] that, for $d > 4$, there are positive constants c_0, C_0 such that for all $\beta \leq \beta_c$,

$$\frac{c_0}{(1 \vee |x|)^{d-2}} \leq \tau_{\beta_c}(0,x) \leq \frac{C_0}{(1 \vee |x|)^{d-2}} \qquad (x \in \mathbb{Z}^d). \tag{6}$$

These results concern the infinite-volume Ising model. It is natural to ask how they are modified when working in a finite setting. In this setting, different boundary conditions may generate very different (near-)critical pictures. The influence of boundary conditions on finite-size critical behaviour in dimensions $d \geq 4$ has been studied extensively in the physics literature for Ising and related models. In particular, it has been observed numerically and via physics scaling arguments that, for $d > 4$,

$$\begin{aligned} &\tau_{\beta_c}^{\Lambda_r}(0,x) \asymp \frac{1}{(1 \vee |x|)^{d-2}}, \qquad && \tau_{\beta_c}^{\mathbb{T}_r}(0,x) \asymp \frac{1}{(1 \vee |x|)^{d-2}} + \frac{1}{r^{d/2}}, \\ &\chi^{\Lambda_r}(\beta_c) \asymp r^2, && \chi^{\mathbb{T}_r}(\beta_c) \asymp r^{d/2}. \end{aligned} \tag{7}$$

(The claim for $\tau_{\beta_c}^{\Lambda_r}(0,x)$ is for x not too close to the boundary of the box). The constant term $r^{-d/2}$ in $\tau_{\beta_c}^{\mathbb{T}_r}(0,x)$ is referred to as the *plateau*, and is responsible for the larger susceptibility for PBC compared to FBC. For the Ising model in dimensions $d > 4$, the absence of a plateau for FBC was proven in [4]. In recent years, the plateau for PBC has been proven to exist for simple random walk in dimensions $d > 2$ [5, 6], self-avoiding walk for $d > 4$ [5, 7], and percolation for $d > 6$ [8, 9]. A general theory of the effect of FBC vs PBC in the setting of the weakly-coupled hierarchical n-component $|\varphi|^4$ model in dimensions $d \geq 4$ was developed in [10, 11]. Together with Yucheng Liu and Gordon Slade [12] we investigated this observation and obtained the following results.

Theorem 1. *Let $d > 4$. There exists $c = c(d) > 0$ such that, letting*

$$\beta^* := \beta_c - c r^{-d/2}, \tag{8}$$

for every r large, every $x \in \mathbb{T}_r$,

$$\tau_{\beta^*}^{\mathbb{T}_r}(0,x) \asymp (1 \vee |x|)^{-(d-2)} + r^{-d/2}. \tag{9}$$

In particular, $\chi^{\mathbb{T}_r}(\beta^*) \asymp r^{d/2}$.

Let $S_r := r^{-d} \sum_{x \in \mathbb{T}_r} \sigma_x$ denote the average spin on the torus. Define the *renormalised coupling constant* by

$$g^{\mathbb{T}_r}(\beta) = -\frac{\langle S_r^4 \rangle_\beta^{\mathbb{T}_r} - 3(\langle S_r^2 \rangle_\beta^{\mathbb{T}_r})^2}{(\langle S_r^2 \rangle_\beta^{\mathbb{T}_r})^2}. \tag{10}$$

By the Lebowitz inequality [13], the above numerator is non-positive, so $g^{\mathbb{T}_r}(\beta) \geq 0$. By the second Griffiths inequality, $\langle S_r^4 \rangle_\beta^{\mathbb{T}_r} \geq (\langle S_r^2 \rangle_\beta^{\mathbb{T}_r})^2$, so $g^{\mathbb{T}_r}(\beta) \leq 2$. The following theorem shows that $g^{\mathbb{T}_r}(\beta_*)$ is bounded away from zero. This indicates a non-Gaussian limit for the average field at $\beta = \beta_*$: for a Gaussian random variable the numerator in (10) is zero. This is in contrast to the situation at (and below) β_c with free boundary conditions [1, 2, 14, 15], where the limit is Gaussian.

Theorem 2. *Let $d > 4$. There is a constant $c_g > 0$ such that for all r large,*

$$0 < c_g \leq g^{\mathbb{T}_r}(\beta_*) \leq 2. \tag{11}$$

References

[1] Aizenman, M. Geometric analysis of φ^4 fields and Ising models. Parts I and II. *Communications In Mathematical Physics.* **86**, 1–48 (1982)

[2] Fröhlich, J. On the triviality of $\lambda\varphi^4$ theories and the approach to the critical point in d¿4 dimensions. *Nuclear Physics B.* **200**, 281–296 (1982)

[3] Duminil-Copin, H. & Panis, R. New lower bounds for the (near) critical Ising and φ^4 models' two-point functions. *ArXiv Preprint ArXiv:2404.05700.* (2024)

[4] Camia, F., Jiang, J. & Newman, C. The effect of free boundary conditions on the Ising model in high dimensions. *Probability Theory And Related Fields.* **181** pp. 311–328

[5] Slade, G. The near-critical two-point function and the torus plateau for weakly self-avoiding walk in high dimensions. *Mathematical Physics, Analysis And Geometry.* **26**, 6 (2023)

[6] Deng, Y., Garoni, T., Grimm, J. & Zhou, Z. Two-point functions of random-length random walk on high-dimensional boxes. *Journal Of Statistical Mechanics: Theory And Experiment.* **23203**

[7] Liu, Y. A general approach to massive upper bound for two-point function with application to self-avoiding walk torus plateau. *ArXiv Preprint ArXiv:2310.17321.* (2023)

[8] Hofstad, R. & Sapozhnikov, A. Cycle structure of percolation on high-dimensional tori. *Annales De L'Institut Henri Poincaré: Probabilités Et Statistiques.* **50** pp. 999–1027

[9] Hutchcroft, T., Michta, E. & Slade, G. High-dimensional near-critical percolation and the torus plateau. *The Annals Of Probability.* **51**, 580–625 (2023)

[10] Michta, E., Park, J. & Slade, G. Boundary conditions and universal finite-size scaling for the hierarchical $|\varphi|^4$ model in dimensions 4 and higher. *ArXiv Preprint ArXiv:2306.00896.* (2023)

[11] Park, J. & Slade, G. Boundary conditions and the two-point function plateau for the hierarchical $|\varphi|^4$ model in dimensions 4 and higher. *ArXiv Preprint ArXiv:2405.17344.* (2024)

[12] Liu, Y., Panis, R. & Slade, G. The torus plateau for the high-dimensional Ising model. *ArXiv Preprint ArXiv 2405.17353.* (2024)

[13] Lebowitz, J. GHS and other inequalities. *Communications In Mathematical Physics.* **35**, 87–92 (1974)

[14] Aizenman, M. & Duminil-Copin, H. Marginal triviality of the scaling limits of critical 4D Ising and φ_4^4 models. *Annals Of Mathematics.* **194**, 163–235 (2021)

[15] Panis, R. Triviality of the scaling limits of critical Ising and φ^4 models with effective dimension at least four. *ArXiv Preprint ArXiv:2309.05797.* (2023)

Vortex fluctuations in 2d Coulomb gas and maximum of the IV-GFF

Avelio Sepúlveda
(joint work with Christophe Garban)

Vortices play a fundamental role in the large scale fluctuations of statistical physics models in 2d such as the XY (plane rotator) model or the Villain model. Their statistics, especially in the case of the Villain model, are described by a celebrated statistical physics model called the (lattice)-2d Coulomb gas. Upper bounds on the fluctuations of these systems in the low temperature regime have been analyzed in the seminal work by Fröhlich and Spencer [1] and lead to the first rigorous proof of the existence of a Berezinskii-Kosterlitz-Thouless phase transition. (See also the recent proofs [2, 3] which rely on the delocalization result from [4]). There is no direct way to tune the proof from [1] to provide lower bounds on fluctuations. In the case of the Villain model, lower bounds on fluctuations are equivalent to upper bounds on the two-point function $\langle \sigma_x \sigma_y \rangle$ and the best upper bounds known so far on the latter are given by the celebrated McBryan-Spencer estimate [5]. These bounds capture the fluctuations produced by the *Gaussian spin-wave* but do not quantify the amount of fluctuations coming from the vortices (i.e. the topological defects).

We study the influence of the vortices on the fluctuations of 2d systems such as the Coulomb gas, the Villain model or the integer-valued Gaussian free field. In the case of the 2d Villain model, we prove that the fluctuations induced by the vortices are at least of the same order of magnitude as the ones produced by the spin-wave. We obtain the following quantitative upper-bound on the two-point correlation in $\mathbb{Z}^2$ when $\beta > 1$

$$\langle \sigma_x \sigma_y \rangle_\beta^{\text{Villain}} \leq C \left(\frac{1}{\|x-y\|_2} \right)^{\frac{1}{2\pi\beta}\left(1+\beta e^{-\frac{(2\pi)^2}{2}\beta}\right)}$$

The proof is entirely non-perturbative. Furthermore it provides a new and algorithmically efficient way of sampling the 2d Coulomb gas. For the 2d Coulomb gas, we obtain the following lower bound on its fluctuations at high inverse temperature

$$\mathbb{E}_\beta^{\text{Coul}}[\langle \Delta^{-1} q, g \rangle] \geq \exp(-\pi^2 \beta + o(\beta)) \langle g, (-\Delta)^{-1} g \rangle.$$

This estimate coincides with the predictions based on a RG analysis from [6] and suggests that the Coulomb potential $\Delta^{-1} q$ at inverse temperature β should scale like a Gaussian free field of inverse temperature of order $\exp(\pi^2 \beta)$.

Finally, we transfer the above vortex fluctuations via a duality identity to the integer-valued GFF by showing that its maximum deviates in a quantitative way from the maximum of a usual GFF. More precisely, we show that with high probability when $\beta > 1$

$$\max_{x \in [-n,n]^2} \Psi_n(x) \leq \sqrt{\frac{2\beta}{\pi}\left(1 - \beta e^{-\frac{(2\pi)^2 \beta}{2}}\right)} \log n \,.$$

where Ψ_n is an integer-valued GFF in the box $[-n,n]^2$ at inverse temperature β^{-1}. Applications to the free-energies of the Coulomb gas, the Villain model and the integer-valued GFF are also considered.

References

[1] Fröhlich, J. & Spencer, T. The Kosterlitz-Thouless transition in two-dimensional Abelian spin systems and the Coulomb gas. *Communications In Mathematical Physics.* **81**, 527-602 (1981)

[2] Aizenman, M., Harel, M., Peled, R. & Shapiro, J. Depinning in the integer-valued Gaussian field and the BKT phase of the 2D Villain model. *ArXiv Preprint ArXiv:2110.09498.* (2021)

[3] Engelenburg, D. & Lis, M. An Elementary Proof of Phase Transition in the Planar XY Model. *Communications In Mathematical Physics.* **399**, 85-104 (2023,4,1)

[4] Lammers, P. Height function delocalisation on cubic planar graphs. *Probability Theory And Related Fields.* (2021)

[5] McBryan, O. & Spencer, T. On the decay of correlations in $SO(n)$-symmetric ferromagnets. *Communications In Mathematical Physics.* **53** pp. 299-302 (1977)

[6] José, J., Kadanoff, L., Kirkpatrick, S. & Nelson, D. Renormalization, vortices, and symmetry-breaking perturbations in the two-dimensional planar model. *Physical Review B.* **16**, 1217 (1977)

Two loop Loewner potentials

Sid Maibach

(joint work with Yan Luo)

In recent work under the same title [9] with Yan Luo, we introduce two-loop Loewner potentials, which is are functionals of pairs of non-intersecting Jordan curves in the Riemann sphere. Heuristically, they are the probability densities of a pair of curves appearing as interfaces in CFTs / lattice models. In this sense, the definition is

$$H^Z_{\hat{\mathbb{C}},2}(\gamma_1,\gamma_2) = \frac{2}{\mathbf{c}}\log\frac{Z(D_1)Z(A)Z(D_2)}{Z(\hat{\mathbb{C}})}\ , \tag{1}$$

where D_1 is a simply connected domain, $D_2 \sqcup A = \mathbb{C}\setminus D_1$ and A is an annulus bounded by the closed simple curves γ_1 and γ_2, and $Z(\cdot)$ are the partition functions of the (boundary) CFT with central charge $\mathbf{c}$ in the respective domains and boundary conditions such that indeed γ_1 and γ_2 become interfaces.

To motivate this definition, we first consider the probabilistic two-loop Loewner potential involving renormalized Brownian loop measure Λ^*, see [1], and the one-loop Loewner potential $H_{\hat{\mathbb{C}},1}$ defined in [2, 3], which emerges from the generalization of SLE loop measure to two loops under the assumption of a "cascade relation":

$$H_{\hat{\mathbb{C}},2}(\gamma_1,\gamma_2) = H_{\hat{\mathbb{C}},1}(\gamma_1) + H_{\hat{\mathbb{C}},1}(\gamma_2) + \Lambda^*(\gamma_1,\gamma_2). \tag{2}$$

The main question in our work is what is the infimum of the two-loop Loewner potential? The answer is relevant for large deviations theories of two-loop SLE measures, where a Loewner energy / rate function would be defined by

$$I_{\hat{\mathbb{C}},2}(\gamma_1,\gamma_2) = H_{\hat{\mathbb{C}},2}(\gamma_1,\gamma_2) - \inf_{\eta_1,\eta_2} H_{\hat{\mathbb{C}},2}(\eta_1,\eta_2) \tag{3}$$

and also pertains to the more geometric objective of finding conformally natural embeddings of the loops into the Riemann sphere [8].

We introduce four equivalent formulas for $H_{\hat{\mathbb{C}},2}(\gamma_1,\gamma_2)$, one of which is in terms of zeta-regularized determinants of the Dirichlet b.c. Laplacian in the respective domain. We single this one out, because it leads to a variational formula much like in [5] for the one-loop case,

(4)
$$\partial_\varepsilon H_{\hat{\mathbb{C}},2}(\omega^{\varepsilon\nu}(\gamma_1),\omega^{\varepsilon\nu}(\gamma_2))\big|_{\varepsilon=0} = -\frac{1}{3\pi}\Re\bigg(\iint_{D_1} \nu\,\mathcal{S}[f_1^{-1}]\,|\mathrm{d}z|^2 + \iint_{D_2} \nu\,\mathcal{S}[f_2^{-1}]\,|\mathrm{d}z|^2\bigg)$$

where ν is a Beltrami differential with support in $D_1 \cup D_2$ and f_1, f_2 are Riemann mappings of the domains D_1 and D_2 respectively. Note that at any critical point of $H_{\hat{\mathbb{C}},2}$, all the variations above and hence the Schwarzian derivatives $\mathcal{S}[f_1^{-1}]$ and $\mathcal{S}[f_2^{-1}]$ must vanish. This implies that f_1 and f_2 are Möbius transformations and consequently $\gamma_1 = f_1(S^1)$ and $\gamma_2 = f_2(S^1)$ are circles. By Möbius invariance, we reduce our question of finding the minimizer to the concentric circles S^1 and $e^{-2\pi\tau}S^1$, where τ is the modulus of the annulus enclosed by the circles. Using explicit expressions for the zeta-regularized determinants of Laplacians in this setup, we find that

$$\begin{aligned} &H_{\hat{\mathbb{C}},2}(e^{-2\pi\tau}S^1, S^1) \to -\infty, \quad \text{as } \tau \to \infty, \\ &H_{\hat{\mathbb{C}},2}(e^{-2\pi\tau}S^1, S^1) \to \infty, \qquad \text{as } \tau \to 0. \end{aligned} \tag{5}$$

Therefore, the definition of $H_{\hat{\mathbb{C}},2}$ does not lead to a notion of Loewner energy.

Our proposed solution is to turn to the two-loop Loewner potentials $H^Z_{\hat{\mathbb{C}},2}$ based on CFT partition functions. By geometric considerations, they differ from $H_{\hat{\mathbb{C}},2}$ only by a function of the modulus τ of the annulus between the loops (not just for circles). Mathematically, this definition also emerges from the real determinant line bundle [6, 4]. We find that a minimizer for $H^Z_{\hat{\mathbb{C}},2}$ must also be a circle and that it exists if and only if

$$e^{-\frac{\pi}{3}\mathbf{c}\tau} Z_{\mathrm{d}z\mathrm{d}\bar{z}}(\{e^{-2\pi\tau} \le |z| \le 1\}) \tag{6}$$

has a global minimum for $\tau \in (0,\infty)$. Since this condition in somewhat open-ended, let us pose the following questions: Do we find anything interesting by asking whether the condition above holds by known examples of CFT partition functions? How does the two-loop SLE measure relate to specific models such as the $O(n)$ loop model or to the counting measure on CLE as in [7]?

References

[1] Field, L. & Lawler, G. Reversed radial SLE and the Brownian loop measure. *J. Stat. Phys.*. **150**, 1030-1062 (2013)

[2] Wang, Y. Equivalent descriptions of the loewner energy. *Inventiones Mathematicae*. **218**, 573–621 (2019,11)

[3] Peltola, E. & Wang, Y. Large deviations of multichordal SLE, real rational functions, and zeta-regularized determinants of Laplacians. *Journal Of The European Mathematical Society.* **26**, 469–535 (2023,4)
[4] Maibach, S. & Peltola, E. From the Conformal Anomaly to the Virasoro Algebra. (arXiv,2024)
[5] Takhtajan, L. & Teo, L. Weil-Petersson metric on the universal Teichmüller space. *Memoirs Of The American Mathematical Society.* **183** (2006)
[6] Kontsevich, M. & Suhov, Y. On Malliavin measures, SLE, and CFT. *Proceedings Of The Steklov Institute Of Mathematics.* **258**, 100–146 (2007)
[7] Ang, M., Cai, G., Sun, X. & Wu, B. SLE Loop Measure and Liouville Quantum Gravity. (arXiv,2024,9)
[8] Bishop, C. Weil-Petersson Curves, Conformal Energies, β-Numbers, and Minimal Surfaces. (2020)
[9] Luo, Y. & Maibach, S. Two-loop Loewner potentials. (arXiv,2024)

Relation between the geometry of sign clusters of the 2D GFF and its Wick powers

Titus Lupu

In 1990 Le Gall showed an asymptotic expansion of the epsilon-neighborhood of a planar Brownian trajectory (Wiener sausage) of finite duration into integer powers of $1/|\log \varepsilon|$ [LG90]. The coefficient of the leading term is the occupation measure of the Brownian path. Higher order terms involve more complicated renormalized quantities, the renormalized self-intersection local times, which can be thought of as measures on multiple points of the Brownian motion, renormalized through polynomial compensation so as to remove the divergences.

In my talk I will present an analogue of this result in the case of the two-dimension continuum Gaussian free field (GFF). Consider $D \subset \mathbb{C}$ a bounded open simply-connected domain, and Φ a GFF on D with 0 boundary conditions. As shown in [ALS23], Φ admits a decomposition into sign clusters: there is a countable collection of K_i two-by-two disjoint compact subsets of D, such that

$$\Phi = \sum_i \sigma_i \, \nu_i,$$

where the σ_i are i.i.d. uniform signs in $\{-1, 1\}$, and the ν_i are positive measure supported on K_i, actually Minkowski contents in the gauge $|\log \varepsilon|^{1/2}\varepsilon^2$.

These sign clusters K_i can be further expanded into half-integer powers of $1/|\log \varepsilon|$. We define an epsilon-neighborhood of K_i via the conformal radius:

$$K_{i,\varepsilon} = \{z \in D |\, \mathrm{CR}(z, D \setminus K_i) < \varepsilon \, \mathrm{CR}(z, D)\}.$$

Then, for every $N \geq 1$ and every fixed test function f,

$$\int_{K_{i,\varepsilon}} f(z)\, d^2 z = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{N} (-1)^n \frac{(2\pi)^{n+1/2}}{2^n n!(n+1/2)} \frac{\langle \psi_{i,2n+1}, v \rangle}{|\log \varepsilon|^{n+1/2}} + o(|\log \varepsilon|^{-(N+1/2)}),$$

where the $o(|\log \varepsilon|^{-(N+1/2)})$ is in the L^2 sense. In the expansion above, for $n = 0$, $\psi_{i,1} = \nu_i$, and for $n \geq 1$, the $\psi_{i,2n+1}$ are generalized functions supported on K_i.

Actually, these are restriction of odd Wick powers of Φ to K_i:

$$\psi_{i,2n+1} = \lim_{\varepsilon \to 0} \mathbf{1}_{K_{i,\varepsilon}} : \Phi^{2n+1} : ,$$

with

$$: \Phi^{2n+1} := \sum_i \sigma_i \, \psi_{i,2n+1}.$$

References

[ALS23] Juhan Aru, Titus Lupu, and Avelio Sepúlveda. Excursion decomposition of the 2D continuum GFF. arXiv:2304.03150, 2023.

[LG90] Jean-François Le Gall. Wiener sausage and self-intersection local times. *Journal of Functional Analysis*, 88(2):299–341, 1990.

Reporter: Trishen S. Gunaratnam

Participants

Juhan Aru
Institute of Mathematics
EPFL
1015 Lausanne
SWITZERLAND

Dr. Paul Dario
Université Paris Est Créteil
61 avenue du Général de Gaulle
94010 Créteil Cedex
FRANCE

Prof. Dr. Hugo Duminil-Copin
Institut des Hautes Etudes Scientifiques
(IHES), Le Bois-Marie
35, route de Chartres
91440 Bures-sur-Yvette
FRANCE

Prof. Dr. Christophe Garban
Institut Camille Jordan
Université de Lyon I
43 Blvd. du 11 Novembre 1918
69622 Villeurbanne Cedex
FRANCE

Dr. Trishen Gunaratnam
Université de Genève
Section de Mathématiques
UNI DUFOUR
24, Rue du Général Dufour
Case Postale 64
1211 Genève 4
SWITZERLAND

Aleksandra Korzhenkova
EPFL
SB MATH RGM
MA A2 407 (Bâtiment MA)
Station 8
1015 Lausanne
SWITZERLAND

Dr. Piet Lammers
LPSM, Sorbonne Université
Campus Pierre et Marie Curie
4, place Jussieu
75005 Paris
FRANCE

Dr. Marcin Lis
Technische Universität Wien
Wiedner Hauptstraße 8 - 10
1040 Wien
AUSTRIA

Dr. Titus Lupu
CNRS
LPSM, Sorbonne Université
75005 Paris
FRANCE

Sid Maibach
Institut für Angewandte Mathematik
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Chiranjib Mukherjee
Institute of Mathematical Stochastics,
Department of Mathematics and
Computer Science,
Universität Münster
Einsteinstraße 62
48149 Münster
GERMANY

Romain Panis
University of Geneva
Section de Mathématiques
P.O. Box 64
1211 Genève 4
SWITZERLAND

Prof. Dr. Eveliina Peltola
Institut für Angewandte Mathematik
Rheinische
Friedrich-Wilhelms-Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Ellen G. Powell
Dept. of Mathematical Sciences
Durham University
Science Laboratories
Stockton Road
Durham DH1 3LE
UNITED KINGDOM

Dr. Avelio Sepulveda
Departamento de Ingeniería
Matemáticas
Universidad de Chile
Beaucheff 851
Estación Central Santiago 5555
CHILE

Franco Severo
Institut Camille Jordan
43 bd du 11 novembre 1918
69622 Villeurbanne
FRANCE

Dr. Diederik van Engelenburg
Institut Camille Jordan
UMR 5208 du CNRS
Université Claude Bernard Lyon 1
21, bd. Claude Bernard
69622 Villeurbanne Cedex
FRANCE

Dr. Jiaming Xia
IHÉS
Institut des Hautes Ètudes Scientifiques
Le Bois-Marie 35, route de Chartres
91440 Bures-sur-Yvette
FRANCE