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# Arbeitsgemeinschaft: Algebraic $K$-Theory and the Telescope Conjecture

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Abstract. The telescope conjecture, first formulated by Ravenel in the late 1970s, is a conjectural classification of smashing localizations of the stable homotopy category. Participants discussed interactions between such localizations and algebraic $K$-theory, culminating in the recent disproof of the telescope conjecture. We ended with some applications and future directions, including to such classical questions as the growth rate of the stable homotopy groups of spheres.

*Mathematics Subject Classification (2020):* 55Nxx, 55Pxx.



## Introduction by the Organizers

Over the course of the week, participants successfully presented the main ingredients disproving the telescope conjecture. The first day began with a talk by Ravenel, introducing the conjecture, its importance to the large scale structure of stable homotopy theory, and some of its history. We then saw a talk by Shai Keidar that reformulated (a part of) the telescope conjecture in terms of Galois descent. Talks by Wickelgren, Krause, and Neuhauser explained several theorems to the effect that chromatically localized algebraic $K$-theory sends Galois extensions to Galois extensions, thereby showing that the telescope conjecture can be disproved by algebraic $K$-theory computations. Later talks described how to compute algebraic $K$-theory using trace methods and the theory of cyclotomic spectra.

The final day concluded with applications of the disproof and proposed further directions for the subject. In particular, Burklund and Levy explained consequences for the growth rate of the $p$-ranks of stable homotopy groups of spheres, as well as the Picard group of the $T(n)$-local category.

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## Arbeitsgemeinschaft: Algebraic $K$-Theory and the Telescope Conjecture

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# Abstracts

## Telescopic localization and the telescope conjecture

Doug Ravenel

### 1. Early 70s

1.1. **Morava K-theory.** In the early 70's Jack Morava discovered the eponynumous spectra $K(n)$. I was lucky enough to spend a lot of time listening to him explain their inner workings.

$K(0)$ is rational cohomology. For each $n > 0$ and each prime $p$, there is a nonconnective complex oriented $p$-local spectrum $K(n)$ with

$$\pi_* K(n) = \mathbb{Z}/p[v_n^{\pm 1}] \qquad \text{where } |v_n| = 2(p^n - 1).$$

It is related to height $n$ formal group laws, and $K(n)_*(K(n))$ is related to the Morava stabilizer group $\mathbb{G}_n$. It is a $p$-adic Lie group and the automorphism group of a height $n$ formal group law over a suitable field of characteristic $p$.

In more detail, a formal group law over a ring $R$ is a power series $F(x, y) = \sum_{i,j} a_{i,j} x^i y^j \in R[\![x, y]\!]$ satisfying

(1) Identity: $F(0, x) = F(x, 0) = x$.
 This means $a_{0,0} = 0$,
 $a_{1,0} = a_{0,1} = 1$ and $a_{i,0} = 0 = a_{0,i}$ for $i > 1$.
(2) Commutativity: $F(y, x) = F(x, y)$. This means $a_{j,i} = a_{i,j}$.
(3) Associativity: $F(F(x, y), z) = F(x, F(y, z))$. This implies complicated relations among the $a_{i,j}$.

Every complex oriented spectrum $E$ has a formal group law over $\pi_* E$ associated with it. Here are two examples.

(1) In 1969 Daniel Quillen showed that its formal group law has a universal property first studied by Michel Lazard in 1955, defined over

$$\pi_* MU = \mathbb{Z}[x_i : i > 0] \text{ with } |x_i| = 2i.$$

(2) $E = K(n)$, the $n$th Morava K-theory. The formal group law is characterized by its $p$-fold formal sum, $[p](x) = v_n x^{p^n}$. This means that its height is $n$. Height is known to be a complete isomorphism invariant for formal group laws over the algebraic closure of $\mathbb{F}_p$.

Formal group laws with $[p](x) = x^{p^n}$ were constructed for all $n$ and $p$ in 1970 by Taira Honda. Hence all heights occur.

1.2. **Morava's vision.** I learned the following from Jack in 1973 and have never forgotten it. You can find a long lost copy of his unpublished AMS Bulletin announcement of it in my archive.

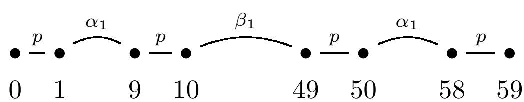
Let $V$ denote the "vector space" of ring homomorphisms $\theta : L \to \overline{\mathbb{F}}_p$, where $L = \pi_* MU$, and let $\mathbb{G}$ be the group of functionally invertible power series in 1 variable over $\overline{\mathbb{F}}_p$.

- Each point $\theta \in V$ induces a formal group law over $\overline{\mathbb{F}}_p$.
- $V$ has an action of $\mathbb{G}$. For $\gamma(x) \in \mathbb{G}$,
$$F(x, y) \mapsto \gamma^{-1} \left( F(\gamma(x), \gamma(y)) \right),$$
which is a formal group law isomorphic to $F$.
- Each $\mathbb{G}$-orbit is an isomorphism class of formal group laws over $\overline{\mathbb{F}}_p$. Hence there is one orbit for each height.
- For each $\theta \in V$, the isotropy or stabilizer group $\mathbb{G}_\theta = \{\gamma \in \mathbb{G} : \gamma(\theta) = \theta\}$ is the automorphism group of the corresponding formal group law. When $\theta$ has height $n$, this group is isomorphic to the Morava stabilizer group $\mathbb{G}_n$.
- There are $\mathbb{G}$-invariant finite codimensional linear subspaces
$$V = V_1 \supset V_2 \supset V_3 \supset \cdots$$
where $V_n = \{\theta \in V : \theta(v_1) = \cdots = \theta(v_{n-1}) = 0\}$. We know now that this filtration of $V$ is related to the chromatic filtation of the stable homotopy category.
- The height $n$ orbit is $V_n - V_{n+1}$. It is the set of $\overline{\mathbb{F}}_p$-valued homomorphisms on $v_n^{-1} L/I_n$.
- The height $\infty$ orbit is the linear subspace
$$\bigcap_{n>0} V_n.$$

1.3. **Smith-Toda complexes.** In 1971 Larry Smith and Hirosi Toda independently constructed the $p$-local finite spectrum $V(n)$, a CW-complex having $2^{n+1}$ cells with

$$MU_* V(n) = MU_*/(v_0 = p, v_1, \ldots v_n) \text{ for } 0 \le n \le 3 \text{ and } p \ge 2n + 1.$$

Cell diagram for $V(2)$ at $p = 5$, where $|v_1| = 8$ and $|v_2| = 48$:




The first 2 cells comprise $V(0)$, the mod $p$ Moore spectrum.
The first 4 cells comprise $V(1)$, and $V(2)/V(1) \simeq \Sigma^{49} V(1)$.
There is a cofiber sequence

$$\Sigma^{|v_n|} V(n-1) \xrightarrow{w_n} V(n-1) \longrightarrow V(n).$$

We know that $K(n)_*V(n-1) \neq 0$ and that $w_n$ is a $K(n)$-equivalence.

These lead to the construction of the $v_n$-periodic families aka Greek letter elements

$$\begin{aligned} &\alpha_t \in \pi_{t|v_1|-1}\mathbb{S} && \text{for } p \geq 3 \\ &\beta_t \in \pi_{t|v_2|-2p}\mathbb{S} && \text{for } p \geq 5 \\ &\gamma_t \in \pi_{t|v_3|-2p^2-2p+1}\mathbb{S} && \text{for } p \geq 7 \end{aligned}$$

$\alpha_t$ is the composite

$$S^{t|v_1|} \xrightarrow{i} \Sigma^{t|v_1|}V(0) \xrightarrow{w_1^t} V(0) \xrightarrow{j} S^1.$$

## 2. Chromatic homotopy theory

**2.1. Algebraic patterns.** The Greek letter elements are nicely displayed in the $E_2$-term the Adams-Novikov spectral sequence. In it there are similar families for all $n$.

In 1977 Haynes Miller, Steve Wilson and I constructed the chromatic spectral sequence converging to this $E_2$-term. It organizes things into layers so that in the $n$th layer everything is $v_n$-periodic. The structure of this $n$th layer is controlled by the cohomology of the $n$th Morava stabilizer group $\mathbb{G}_n$.

**2.2. The chromatic filtration.** Later we learned that the stable homotopy category itself is similarly organized. The key tool here is Bousfield localization, which conveniently appeared in 1978, just in time for us!

Let Sp denote the category of spectra. Given a spectrum $E$, Bousfield constructed an endofunctor $L_E : \mathrm{Sp} \to \mathrm{Sp}$ whose image category $L_E\mathrm{Sp}$ is stable homotopy as seen through the eyes of $E$-theory.

We are interested in the case $E = K(n)$. $L_{K(n)}\mathrm{Sp}$ is much easier to deal with than Sp itself. For example, we can compute $\pi_*L_{K(2)}V(1)$, but we have no hope of computing $\pi_*V(1)$.

**2.3. Enter the telescope conjecture.** Recall the cofiber sequence

$$\Sigma^{|v_n|}V(n-1) \xrightarrow{w_n} V(n-1) \longrightarrow V(n)$$

for $0 \leq n \leq 3$ and $p \geq 2n+1$. Since $K(n)_*w_n$ is an isomorphism, all iterates of $w_n$ are essential. This means that the homotopy colimit of the following is noncontractible.

$$V(n-1) \xrightarrow{w_n} \Sigma^{-|v_n|}V(n-1) \xrightarrow{w_n} \Sigma^{-2|v_n|}V(n-1) \xrightarrow{w_n} \dots$$

We call this the $v_n$-periodic telescope $w_n^{-1}V(n-1)$, often denoted by $T(n)$. The telescope conjecture says it is $L_{K(n)}V(n-1)$. $T(n)$ is more closely related to the homotopy groups of spheres, while $L_{K(n)}V(n-1)$ is more computationally acccessible.

San Francisco earthquake of October 17, 1989

### 2.4. **The Hopkins-Smith periodicity theorem.**

$$\Sigma^{|v_n|}V(n-1) \xrightarrow{w_n} V(n-1) \longrightarrow V(n)$$

Can we generalize this to $n > 3$? Not exactly. To this day, nobody has constructed $V(4)$ at any prime, and in 2010 Lee Nave showed that $V((p+1)/2)$ does not exist.

On the bright side, in 1998 Mike Hopkins and Jeff Smith published the following.

**Periodicity Theorem.** *Let $X$ be a $p$-local type $n$ finite spectrum, meaning that $K(n)_*X \neq 0$ and $K(m)_*X = 0$ for $m < n$. Then for some $d > 0$ (and divisible by $|v_n|$) there is a map*

$$w : \Sigma^d X \to X \qquad \text{where } K(n)_*w \text{ is an isomorphism.}$$

The theorem implies that the cofiber of $w$ has type $n+1$. As before we can form a $v_n$-periodic telescope $w^{-1}X =: T(n)$. It is independent of the choice of $w$ and the corresponding localization functor $L_{T(n)}$ is independent of the choice of $X$.

$V(n-1)$ is an early example of a finite spectrum of type $n$.

Again the telescope conjecture equates the geometrically appealing telescope $w^{-1}X$ with the computationally accessible Bousfield localization $L_{K(n)}X$.

## 3. The telescope conjecture

When I stated the telescope conjecture in 1984, it was known to be true for $n = 0$ and $n = 1$. The height one case was proved around 1980 by Mark Mahowald for $p = 2$ and Haynes Miller for odd primes.

Thus the statement for $n > 1$ seemed to be favored by Occam's razor. However, while I was visiting MSRI (now SLMath) in 1989, something happened that led me to believe it is false for $n \geq 2$.

This failure of the telescope conjecture for $n \geq 2$ is now a theorem of Robert Burklund, Jeremy Hahn, Ishan Levy and Tomer Schlank. Their proof is the subject of this workshop.

Jeremy, Tomer, myself, Ishan and Robert
at Oxford University, June 9, 2023.
Photo by Matteo Barucco.
THANK YOU!

## Ambidexterity and Chromatic Cyclotomic Extensions

Shai Keidar

This talk introduces Rognes' Galois theory [1] and higher semiadditivity as developed by Hopkins and Lurie [2]. Observing the role of semiadditivity in classical cyclotomic extensions, we use higher semiadditivity to construct *higher cyclotomic extensions* following [3]. In the $K(n)$- and $T(n)$-local settings, these higher cyclotomic extensions manifest as faithful Galois extensions, yet the colimit of these extensions is not necessarily faithful. To address this, we introduce the category of *cyclotomically complete* $T(n)$-local spectra, which sits between $\mathrm{Sp}_{K(n)}$ and $\mathrm{Sp}_{T(n)}$, and derive a formula for the cyclotomic completion functor.

### 1. Rognes' Galois theory

Rognes' Galois theory extends the classical notion of Galois extensions to any symmetric monoidal $\infty$-category, defining a Galois extension as an algebra with a group action satisfying simple conditions. Faithful Galois extensions are extensions over which descent theory applies. In classical settings, this framework generalizes regular Galois extensions of rings.

A collection of results from [4], [1], [5], and [6] shows that $\mathbb{S}_{K(n)} \to E_n$ realizes $E_n$ as a faithful Galois closure of $\mathbb{S}_{K(n)}$. This is computationally significant, enabling descent along this extension. The faithfulness of the Galois closure of $\mathbb{S}_{K(n)}$ will turn out to be a key difference between $\mathrm{Sp}_{K(n)}$ and $\mathrm{Sp}_{T(n)}$.

### 2. Higher Semiadditivity

Higher semiadditivity generalizes semiadditivity by allowing the summation (or integration) of maps along $\pi$-finite spaces — truncated spaces finite homotopy groups.

Hopkins and Lurie proved that $\mathrm{Sp}_{K(n)}$ is $\infty$-semiadditive [2], and Carmeli, Schlank, and Yanovski showed the same for $\mathrm{Sp}_{T(n)}$ [7].

## 3. Higher Cyclotomic Extensions

Representation theory gives a construction the $p^r$-th cyclotomic extension of $\mathbb{Q}$ — it can be achieved by inverting an idempotent in the group algebra $\mathbb{Q}[C_{p^r}]$, leveraging the ability to sum over $C_{p^r}$ elements and that $p$ is invertible.

In higher semiadditive contexts, $p$ itself may not be invertible, but higher analogs of $p$ will be. This allows us to mimic the classical construction by replacing the discrete group $C_{p^r}$ with the $\pi$-finite group $B^n C_{p^r}$, defining the higher cyclotomic extensions $\mathbb{S}_{K(n)}[\omega_{p^r}^{(n)}]$ and $\mathbb{S}_{T(n)}[\omega_{p^r}^{(n)}]$. These extensions are faithful $(\mathbb{Z}/p^r)^\times$-Galois extensions in $\mathrm{Sp}_{K(n)}$ and $\mathrm{Sp}_{T(n)}$, respectively.

## 4. Cyclotomic Completion

The $\mathbb{Z}_p^\times$-pro-Galois extension $\mathbb{S}_{T(n)}[\omega_{p^\infty}^{(n)}]$, formed as the colimit of the higher cyclotomic extensions, does not necessarily maintain faithfulness, while $\mathbb{S}_{K(n)}[\omega_{p^\infty}^{(n)}]$ remains faithful. To address this, we define the category $(\mathrm{Sp}_{T(n)})^\wedge_{\mathrm{cyc}} \subseteq \mathrm{Sp}_{T(n)}$ of "cyclotomically complete" $T(n)$-local spectra as the Bousfield localization with respect to $\mathbb{S}_{T(\omega_{p^\infty}^{(n)})}$, i.e. the subcategory in which $\mathbb{S}_{T(\omega_{p^\infty}^{(n)})}$ is faithful. In particular, $\mathrm{Sp}_{K(n)} \subseteq (\mathrm{Sp}_{T(n)})^\wedge_{\mathrm{cyc}}$.

Following [8], we show that the cyclotomic completion functor, which is the localization functor

$$(-)^\wedge_{\mathrm{cyc}} \colon \mathrm{Sp}_{T(n)} \to (\mathrm{Sp}_{T(n)})^\wedge_{\mathrm{cyc}},$$

is a smashing localization with unit $\mathbb{S}_{T(n)}[\omega_{p^\infty}^{(n)}]^{h(T_p \times \mathbb{Z})}$, where $\mathbb{Z}_p^\times = T_p \times \mathbb{Z}_p$ and $T_p$ is the torsion subgroup.

The goal throughout this workshop is to demonstrate that $(\mathrm{Sp}_{T(n)})^\wedge_{\mathrm{cyc}} \subsetneq \mathrm{Sp}_{T(n)}$, thus disproving of the telescope conjecture.

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## Cyclotomic and polygonic spectra

Qingyuan Bai

The disproof depends on understanding of certain K-theory spectrum, and one possible way to approach K-theory is via trace method. Given a ring $R$, there is a trace map

$$\mathrm{K}(R) \longrightarrow \mathrm{THH}(R)$$

which provides a good approximation (there is also trace map to other variants, for example TC). It is thus desirable to get better understanding of THH and related constructions. The category of cyclotomic spectra is a category which hosts interesting structures on $\mathrm{THH}(R)$ and allows us to manipulate them. Following the work of Nikolaus-Scholze and Krause-McCandless-Nikolaus, we define the category of cyclotomic spectra and polygonic spectra. We explain how THH(R) and THH(R;M) provide objects in these categories. We define many functors out of CycSp including TP, TC and TR.

From now on a prime $p$ is fixed and all spectra are assumed to be $p$-complete: in particular Sp would mean the category of $p$-complete spectra. The category of ($p$-typical) cyclotomic spectra is defined as the following lax equalizer:

$$\mathrm{CycSp} := \mathrm{LEq}[(\mathrm{Id}, (-)^{tC_p}) : \mathrm{Sp}^{B\mathbb{T}} \rightrightarrows \mathrm{Sp}^{B\mathbb{T}}]$$

It almost follows directly from the abstract definition that the category CycSp is a presentable stable $\infty$-category with a symmetric monoidal structure. It also follows that one can compute mapping spectra in the category CycSp via an explicit equalizer diagram of mapping spectra in $\mathrm{Sp}^{B\mathbb{T}}$. An example of cyclotomic spectra is $\mathbb{S}^{\mathrm{triv}}$: it has underlying spectrum $\mathbb{S}$ with trivial $\mathbb{T}$-action and the Frobenius map $\mathbb{S} \to \mathbb{S}^{tC_p}$ is the canonical map. We can define the following functors on CycSp, for $X \in$CycSp:

$$\mathrm{TC}^{-}(X) := X^{h\mathbb{T}} \in \mathrm{Sp}$$

$$\mathrm{TP}(X) := X^{t\mathbb{T}} \in \mathrm{Sp}$$

$$\mathrm{TC}(X) := \mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}^{\mathrm{triv}}, X) \in \mathrm{Sp}$$

and $\mathrm{TR}(X) := \lim_k \mathrm{TR}^k(X)$ where

$$\mathrm{TR}^{k+1}(X) := X^{hC_{p^k}} \times_{(X^{tC_p})^{hC_{p^{k-1}}}} X^{hC_{p^{k-1}}} \cdots X^{hC_p} \times_{X^{tC_p}} X \in \mathrm{Sp}.$$

Another source of cyclotomic spectra is THH of a ring spectra $R$: the cyclic bar construction of $R$ comes with amounts of extra information including cyclic structure and Frobenius map, which implies that the spectrum $\mathrm{THH}(R)$ has a $\mathbb{T}$-action and a Frobenius map, so $\mathrm{THH}(R)$ lifts to an object in CycSp. In fact $\mathrm{THH}(-)$ provides a symmetric monoidal functor Alg(Sp)$\to$CycSp.

Finally we talk about polygonic spectra: we focus on the case of truncating set $T = \langle p \rangle = \{1, p\}$. In this case we define the category of polygonic spectra to be the oplax limit (with functors $\mathrm{id}_{\mathrm{Sp}}$ and $(-)^{tC_p}$)

$$\mathrm{PgcSp}_{\langle p \rangle} := \mathrm{Sp} \overset{\rightarrow}{\underset{\mathrm{Sp}}{\times}} \mathrm{Sp}^{BC_p}.$$

The category is a presentable stable $\infty$-category with symmetric monoidal structure. The category $\mathrm{PgcSp}_{\langle p\rangle}$ could be identified with the category of genuine $C_p$-spectra. An important collection of examples of polygonic spectra comes from cyclotomic spectra: there is a restriction functor

$$\mathrm{CycSp} \to \mathrm{PgcSp}_{\langle p\rangle}$$

which forgets about the $\mathbb{T}$-action but only remembers the $C_p$-action and the Frobenius map. Another type of examples of polygonic spectra comes from THH with coefficients: for a ring spectra $R$ and an $R$-$R$-bimodule $M$, one can write down a simplicial object of cyclic bar complex for $M$ and define $\mathrm{THH}(R; M)$ to be the colimit. A nontrivial construction with cyclic bar complex implies that $\mathrm{THH}(R; M^{\otimes_R p})$ has a $C_p$ action along with a Frobenius map

$$\mathrm{THH}(R; M) \to \mathrm{THH}(R; M^{\otimes_R p})^{tC_p}.$$

Thus $\mathrm{THH}(R; M)$ lifts to an object in $\mathrm{PgcSp}_{\langle p\rangle}$. In fact $\mathrm{THH}(-;-)$ lifts to a functor BiMod$\to \mathrm{PgcSp}_{\langle p\rangle}$.

## References

[1] Thomas Nikolaus and Peter Scholze, *On Topological Cyclic Homology.*
[2] Achim Krause, Jonas McCandless and Thomas Nikolaus, *Polygonic spectra and TR with coefficients.*

# K-Theory, Land-Tamme, and Levy

Turner McLaurin, Kirsten Wickelgren

## 1. K-Theory and Localizing Invariants

Given a stable $\infty$-category $\mathcal{C}$, one extracts the non-connective and connective $K$-theory spectra, denoted $\mathbb{K}(\mathcal{C})$ and $K(\mathcal{C})$ respectively. Here, $K_0(\mathcal{C})$ admits a tractable description as

$$K_0(\mathcal{C}) = \Big\{\text{free abelian group on symbols } [X] \text{ for } X \in \mathcal{C}\Big\}/\sim$$

where $[X] = [X'] + [X'']$ if there exists a cofiber sequence $X' \to X \to X''$ in $\mathcal{C}$.

We begin by stating the universal property of $K$-theory of Blumberg, Gepner, and Tabuada. Recall that an $\infty$-category $\mathcal{C}$ is *idempotent-complete* if its image under the Yoneda embedding is closed under retracts. Let $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ denote the $\infty$-category of small stable $\infty$-categories, and $\mathrm{Cat}_\infty^{\mathrm{perf}}$ the full subcategory of idempotent-complete small stable $\infty$-categories. The inclusion $\mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Cat}_\infty^{\mathrm{Ex}}$ admits a left adjoint denoted $\mathrm{Idem} : \mathrm{Cat}_\infty^{\mathrm{Ex}} \to \mathrm{Cat}_\infty^{\mathrm{perf}}$. A functor $\mathcal{C} \to \mathcal{D}$ in $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ is said to be a *Morita equivalence* if $\mathrm{Idem}(\mathcal{C}) \to \mathrm{Idem}(\mathcal{D})$ is an equivalence.

**Example 1.** Let $A$ be an $\mathbf{E}_1$-ring, and let perf$(A)$ denote the $\infty$-category of compact objects of $\mathrm{Mod}_A$. Then $\mathrm{perf}(A) \in \mathrm{Cat}_\infty^{\mathrm{perf}}$, and the *connective $K$-theory of $A$* is defined to be

$$K(A) := K(\mathrm{perf}(A))$$

**Definition 2.** A sequence $\mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{p} \mathcal{C}$ in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ is *exact* if $\mathcal{A} \to \mathcal{B}$ is fully faithful, the composite $\mathcal{A} \to \mathcal{C}$ is 0, and $\mathcal{B}/\mathcal{A} \to \mathcal{C}$ is an equivalence. The exact sequence is *split* if both $i$ and $p$ admit left adjoints which compose with $i$ and $p$ to give the respective identities. A sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ in $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ is *exact* (*split exact*) if $\mathrm{Idem}(\mathcal{A}) \to \mathrm{Idem}(\mathcal{B}) \to \mathrm{Idem}(\mathcal{C})$ is exact (*split exact*) in $\mathrm{Cat}_\infty^{\mathrm{perf}}$.

**Definition 3.** A functor $E : \mathrm{Cat}_\infty^{\mathrm{perf}} : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Sp}$ is *localizing* if it sends exact sequences to fiber sequences.

functor $E : \mathrm{Cat}_\infty^{\mathrm{Perf}} \to \mathrm{Sp}$ is an *additive invariant* if it sends split exact sequences to fiber sequences.

**Example 4.** The functors $\mathbb{K}$, THH, and TC are localizing. Every localizing invariant is additive, but the converse is not true. For example, connective $K$-theory is additive, but not localizing.

Here, we follow Land and Tamme's terminology. Blumberg, Gepner and Tabuada also require that localizing invariants preserve filtered colimits, which would exclude TC.

**Theorem 5** (Blumberg-Gepner-Tabuada)**.** [3] *There exist stable presentable $\infty$-categories $\mathcal{M}_{\mathrm{loc}}$ and $\mathcal{M}_{\mathrm{add}}$, and localizing and additive invariants $\mathcal{U}_{\mathrm{loc}} : \mathrm{Cat}_\infty^{\mathrm{Ex}} \to \mathcal{M}_{\mathrm{loc}}$ and $\mathcal{U}_{\mathrm{add}} : \mathrm{Cat}_\infty^{\mathrm{Ex}} \to \mathcal{M}_{\mathrm{add}}$ respectively, which are universal in the following sense: given any stable presentable $\infty$-category $\mathcal{D}$, post-composition induces equivalences*

$$\mathcal{U}_{\mathrm{loc}}^* : \mathrm{Fun}^L(\mathcal{M}_{\mathrm{loc}}, \mathcal{D}) \to \mathrm{Fun}_{\mathrm{loc}}(\mathrm{Cat}_\infty^{\mathrm{Ex}}, \mathcal{D})$$

$$\mathcal{U}_{\mathrm{add}}^* : \mathrm{Fun}^L(\mathcal{M}_{\mathrm{add}}, \mathcal{D}) \to \mathrm{Fun}_{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{Ex}}, \mathcal{D})$$

*where $\mathrm{Fun}^L(\mathcal{M}_{\mathrm{loc}}, \mathcal{D})$ denotes the $\infty$-category of colimit preserving functors, and $\mathrm{Fun}_{\mathrm{loc}}(\mathrm{Cat}_\infty^{\mathrm{Ex}}, \mathcal{D})$ and $\mathrm{Fun}_{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{Ex}}, \mathcal{D})$ denote the $\infty$-categories of localizing and additive invariants, which preserve filtered colimits and invert Motiva equivalences, respectively.*

**Theorem 6** (Blumberg-Gepner-Tabuada)**.** [3] *For any $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$, there is a natural equivalence of spectra*

$$\mathrm{Map}_{\mathcal{M}_{\mathrm{loc}}}(\mathcal{U}_{\mathrm{loc}}(\mathrm{Sp}^\omega), \mathcal{U}_{\mathrm{loc}}(\mathcal{C})) \simeq \mathbb{K}(\mathcal{C})$$

$$\mathrm{Map}_{\mathcal{M}_{\mathrm{add}}}(\mathcal{U}_{add}(\mathrm{Sp}^\omega), \mathcal{U}_{\mathrm{add}}(\mathcal{C})) \simeq K(\mathcal{C})$$

*Moreover, for any additive invariant, which inverts Morita equivalences and preserves filtered colimits, $E : \mathrm{Cat}_\infty^{\mathrm{Ex}} \to \mathrm{Sp}$, there is a natural equivalence*

$$\mathrm{Map}(K, E) \simeq E(\mathrm{Sp}^\omega)$$

In particular, taking $E = \mathrm{THH}$, we obtain $\pi_0\mathrm{Map}(K, \mathrm{THH}) \simeq \pi_0\mathrm{THH}(\mathrm{Sp}^\omega) \simeq \pi_0(\mathbf{S}) \simeq \mathbf{Z}$. The natural transformation $K \to \mathrm{THH}$ given by the image of 1 refines to the Dennis trace $K \to \mathrm{TC}$.

**Theorem 7** (Dundas-Goodwillie-McCarthy)**.** [5] *Let $B \to A$ be a morphism of connective $\mathbf{E}_1$-ring spectra such that $\pi_0(B) \to \pi_0(A)$ is surjective, with kernel a nilpotent ideal. Then the Dennis Trace induces a pullback*

$$\begin{tikzcd} \mathrm{K}(B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{TC}(B) \arrow[d] \\ \mathrm{K}(A) \arrow[r] & \mathrm{TC}(A) \end{tikzcd}$$

Taking $A = \pi_0 B$, we see that computing the spectrum $K(B)$ can be reduced to the more tractable problems of computing $\mathrm{TC}(A)$, $\mathrm{TC}(B)$, and $K(\pi_0 B)$.

When computing with $K$-theory, one is naturally led to the question of when a pullback of rings induces a pullback on $K$-theory spectra. As noted at the beginning of [2], Swann showed that there is no functor $K_2$ for which Milnor squares (which are pullback squares of rings $A' \times'_B B$ with $B \to B'$ surjective) give rise to the long exact excision sequence. Land and Tamme [2] proved that one can obtain pullback diagrams in $K$-theory, or more generally any localizing invariant $E$, by equipping the spectrum $A' \otimes_A B$ with a different ring structure.

**Theorem 8** (Land-Tamme)**.** *Any pullback square*

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d] \\ A' \arrow[r] & B' \end{tikzcd}$$

*of $\mathbf{E}_1$-ring spectra refines naturally to a commutative square*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\ A' \arrow[r] \arrow[drr, bend right] & A' \odot_A^{B'} B \arrow[dr] & \\ & & B' \end{tikzcd}$$

*such that any localizing invariant sends the outer square to a pullback. Furthermore, the underlying spectrum of $A' \odot_A^{B'} B$ is $A' \otimes_A B$.*

**Definition 9.** A localizing invariant $E$ is *truncating* if $E(A) \to E(\pi_0 A)$[1] is an equivalence for any connective $\mathbf{E}_1$-algebra $A$.

**Example 10.** The localizing invariant $K^{\mathrm{inv}} := \mathrm{fib}(K \to \mathrm{TC})$ is truncating by Dundas-Goodwillie-McCarthy.

[1] As with $K$-theory, we denote $E(A) := E(\mathrm{perf}(A))$.

## 2. Topological Cyclic Homology

Ishan Levy extends the Dundas-Goodwillie-McCarthy theorem to the fixed points of connective ring spectra by $\mathbf{Z}$-actions.

The $\infty$-category of spectra Sp has a $t$-structure whose $n$-connective objects can be described as $\mathrm{Sp}_{\geq n} = \{E \in \mathrm{Sp} : \pi_i(E) = 0 \text{ for } i < n\}$. If $R$ is an $\mathbf{E}_1$-ring, then there exists a $t$-structure on $\mathrm{Mod}(R)$ whose connective and coconnective objects admit the following description: $\mathrm{Mod}(R)_{\geq 0}$ is the stable subcategory of $\mathrm{Mod}(R)$ generated by $R$ under colimits and extensions, and $\mathrm{Mod}(R)_{<0}$ consists of those $R$-modules whose underlying spectrum is in $\mathrm{Sp}_{<0}$.

**Lemma 11.** [1, 3.1] *Let $R$ be a $(-1)$-connective $\mathbf{E}_1$-ring. Let $M$ be any $R$-module which is connective as a spectrum. Then*

*(1) $M \in Mod(R)_{\geq 0}$.*
*(2) For any right $R$-module $N$ with $N \in Sp_{\geq 0}$, we have $M \otimes_R N \in Sp_{\geq 0}$.*

*Proof.* (1) The $t$-structure on $\mathrm{Mod}(R)$ supplies a cofiber sequence $\tau_{\geq 0}M \to M \to \tau_{<0}M$. As $\tau_{\geq 0}M \in \mathrm{Mod}(R)$ is built from $R$ by colimits and extensions, and as $R$ is $(-1)$-connective, it follows that the underlying spectrum of $\tau_{\geq 0}M$ is $(-1)$-connective. As $M$ is connective as an underlying spectrum by assumption, it follows that $\tau_{<0}M$ is as well. Since $\tau_{<0}M \in \mathrm{Sp}_{<0}$, it follows that $\tau_{<0}M = 0$, thus $\tau_{\geq 0}M \to M$ is an equivalence; in particular, $M \in \mathrm{Mod}(R)_{\geq 0}$.

(2) By assumption $M$ is generated by $R$ by colimits and extensions, and as $-\otimes_R N$ preserves such constructions, it follows that $M \otimes_R N$ is build out of colimits and extensions by $R \otimes_R N \simeq N$. If $N \in \mathrm{Sp}_{\geq 0}$, it follows that $M \otimes_R N \in \mathrm{Sp}_{\geq 0}$ as well. $\square$

**Lemma 12.** [1, 3.2] *Let $R, S$ be $\mathbf{E}_1$-rings in $\mathrm{Sp}_{\geq -1}$. Suppose that $f : R \to S$ is an $i$-connective map of $\mathbf{E}_1$-rings for $i \geq -1$. Let $M, N$ be right and left $S$-modules respectively, with $M, N \in \mathrm{Mod}(S)_{\geq 0}$. Then $M \otimes_R N \to M \otimes_S N$ is $(i+1)$-connective.*

**Proposition 13.** *(Waldhausen)*[1, 3.3] *Let $f : R \to S$ be an $i$-connective map of connective $\mathbf{E}_1$-spectra for $i \geq 1$. Then $\mathrm{fib}(\mathbb{K}(f))$ is $(i+1)$-connective.*

**Theorem 14.** [1, 3.5] *Let*

$$\begin{array}{ccccc} R_0 & \longrightarrow & R_1 & \longleftarrow & R_2 \\ \downarrow & & \downarrow & & \downarrow \\ S_0 & \longrightarrow & S_1 & \longleftarrow & S_2 \end{array}$$

*be a map of cospans of connective $\mathbf{E}_1$-rings that is levelwise $i$-connective for $i \geq 1$. Then for any truncating localizing invariant $E$, $E(R_0 \times_{R_1} R_2) \to E(S_0 \times_{S_1} S_2)$ is an equivalence, and $\mathrm{TC}(R_0 \times_{R_1} R_2) \to \mathrm{TC}(S_0 \times_{S_1} S_2)$ is $i$-connective.*

*Proof.* Let $R_3 = R_0 \times_{R_1} R_2$ and $S_3 = S_0 \times_{S_1} S_2$, and let $\mathcal{U}'_{\mathrm{loc}}$ denote the version of the universal localizing invariant of [3] that does not necessarily preserve filtered colimits. Note that $R_3$ is $(-1)$-connective. By [2], we have a pullback square

$$\begin{tikzcd}
\mathcal{U}'_{\mathrm{loc}}(R_3) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{U}'_{\mathrm{loc}}(R_0) \arrow[d] \\
\mathcal{U}'_{\mathrm{loc}}(R_2) \arrow[r] & \mathcal{U}'_{\mathrm{loc}}(R_0 \odot_{R_3}^{R_1} R_2)
\end{tikzcd}$$

where the underlying spectrum of $R_0 \odot_{R_3}^{R_1} R_2$ is equivalent to $R_0 \otimes_{R_3} R_2$. Applying Lemma 3.1, we see that $R_0 \odot_{R_3}^{R_1} R_2$ is connective. By assumption, $\mathrm{fib}(R_j \to S_j)$ is $i$-connective for $i \geq 1$, hence $\pi_0(\mathrm{fib}(R_j \to S_j)) \simeq 0$, so that $\pi_0(R_j) \to \pi_0(S_j)$ is an equivalence. Therefore $E(\pi_0(R_j)) \to E(\pi_0(S_j))$ is an equivalence, and as $E$ is truncating, we find that $E(R_j) \to E(S_j)$ is an equivalence. Then $R_3 \to S_3$ is $(i-1)$-connective, and the map $R_0 \otimes_{R_3} R_2 \to S_0 \otimes_{R_3} S_2$ is $i$-connective by Lemma 11. Moreover, by Lemma 11 and Lemma 12, the map $S_0 \otimes_{R_3} S_2 \to S_0 \otimes_{S_3} S_2$ is also $i$-connective. It follows that the composite $R_0 \otimes_{R_3} R_2 \to S_0 \otimes_{S_3} S_2$ is $i$-connective. On underlying spectra, this agrees with the map $R_0 \odot_{R_3}^{R_1} R_2 \to S_0 \odot_{S_3}^{S_1} S_2$, which we conclude is also $i$-connective. Thus $E(R_0 \odot_{R_3}^{R_1} R_2) \to E(S_0 \odot_{S_3}^{S_1} S_2)$ is an equivalence and $\mathrm{TC}(R_0 \odot_{R_3}^{R_1} R_2) \to \mathrm{TC}(S_0 \odot_{S_3}^{S_1} S_2)$ is $(i+1)$-connective by Theorem 7. Finally, by Theorem 8 we deduce that $E(R_3) \to E(S_3)$ is an equivalence, and that $\mathrm{TC}(R_3) \to \mathrm{TC}(S_3)$ is $i$-connective. $\square$

**Remark 15.** Giving a ring $R$ a $\mathbf{Z}$-action is the same as giving an automorphism $\phi : R \to R$. Given the latter, $R^{h\mathbf{Z}}$ fits into the pullback square

$$\begin{tikzcd}
R^{h\mathbf{Z}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "\Delta"] \\
R \arrow[r, "(1,\phi)"'] & R \times R
\end{tikzcd}$$

Applying Theorem 14 to the cospan $R \xrightarrow{\Delta} R \times R \xleftarrow{(1,\phi)} R$ we get the following.

**Theorem 16.** [1, B] *Let $f : R \to S$ be a map of connective $\mathbf{E}_1$-rings with $\mathbf{Z}$-actions, such that $f$ is 1-connective. Then for any truncating invariant $E$, we have that $E(R^{h\mathbf{Z}}) \to E(S^{h\mathbf{Z}})$ is an equivalence. Moreover, if $f$ is $i$-connective, then $\mathrm{TC}(R^{h\mathbf{Z}}) \to \mathrm{TC}(S^{h\mathbf{Z}})$ is also $i$-connective.*

## 3. Purity Theorem

We discuss the Purity results of Land, Mathew, Meier, and Tamme [4]. Fix a prime $p$ and $n \geq 1$, and let $K(i)$ denote the $i$-th Morava K-theory at the prime $p$. Let $V_n$ be a type $n$ complex; that is, a pointed finite CW-complex with $K(i) \otimes V_n = 0$ for $i < n$, and $K(n) \otimes V_n \neq 0$. A self map $\nu_n : \Sigma^d V_n \to V_n$ is called a $\nu_n$-*map* if $K(i)_*(\nu_n)$ is an equivalence for $i = n$, and nilpotent for $i \neq n$. Let $T(n) := \Sigma^\infty V_n[\nu_n^{-1}]$ be the telescope of $\nu_n$. For a spectrum $E$, let $L_E : \mathrm{Sp} \to \mathrm{Sp}_E$ denote the Bousfield localization functor, where $\mathrm{Sp}_E$ denotes the $\infty$-category of $E$-local spectra.

**Theorem 17** (Land, Mathew, Meier, Tamme). *Let $A$ be a ring spectrum. For $n \geq 1$, the canonical map $A \to L_{T(n-1)\oplus T(n)}A$ induces an equivalence in $T(n)$-local $K$-theory.*

One can use such purity results to reprove the following theorem of Mitchell.

**Theorem 18** (Mitchell). *Let $E$ be a module over $K(\mathbf{Z})$. Then $K(n)_*E \simeq 0$ for all $n \geq 0$.*

**Lemma 19.** [4, Lemma 2.3] *Let $R$ be a ring spectrum and $n \geq 1$. Then $R$ is $K(n)$-acyclic if and only if $R$ is $T(n)$-acyclic.*

**Corollary 20.** *For any ordinary ring $R$, $L_{T(n)}K(R) \simeq 0$ and $L_{T(n)}\mathrm{TC}(R) \simeq 0$.*

*Proof.* Note that both $K(R)$ and $\mathrm{TC}(R)$ are modules over $K(\mathbf{Z})$. By Theorem 18, $K(n)_*K(R) \simeq 0$ and $K(n)_*\mathrm{TC}(R) \simeq 0$, and by Lemma 19, we find that $T(n)_*K(R) \simeq 0$ and $T(n)_*\mathrm{TC}(R) \simeq 0$. □

**Proposition 21.** [4, Cor 4.30] *Let $n \geq 2$ and let $A$ be a commutative ring spectrum. Then $L_{T(n)}K(A) \to L_{T(n)}\mathrm{TC}(A)$ is an equivalence.*

*Proof.* By the Dundas-Goodwillie-McCarthy theorem, we have a bicartesian square

$$\begin{tikzcd} K(A) \arrow[r] \arrow[d] & \mathrm{TC}(A) \arrow[d] \\ K(\pi_0(A)) \arrow[r] & \mathrm{TC}(\pi_0(A)) \end{tikzcd}$$

As the localization functor $L_{T(n)}$ is a left adjoint, applying it to the diagram above yields a bicartesian square

$$\begin{tikzcd} L_{T(n)}K(A) \arrow[r] \arrow[d] & L_{T(n)}\mathrm{TC}(A) \arrow[d] \\ L_{T(n)}K(\pi_0(A)) \arrow[r] & L_{T(n)}\mathrm{TC}(\pi_0(A)) \end{tikzcd}$$

Finally, by Theorem 18, $L_{T(n)}K(\pi_0(A)) \simeq 0$ and $L_{T(n)}\mathrm{TC}(\pi_0(A)) \simeq 0$, so that $\mathrm{cofib}(L_{T(n)}K(A) \to L_{T(n)}\mathrm{TC}(A)) \simeq 0$, and thus $L_{T(n)}K(A) \to L_{T(n)}\mathrm{TC}(A)$ is an equivalence. □

**Proposition 22.** *Let $n \geq 2$ and let $A$ be a connective $\mathbf{E}_1$-ring spectrum with $\mathbf{Z}$-action. Then $L_{T(n)}K(A^{h\mathbf{Z}}) \to L_{T(n)}TC(A^{h\mathbf{Z}})$ is an equivalence.*

*Proof.* Let $f : A \to \pi_0(A)$ denote the canonical map; as $\mathrm{fib}(f) \in \mathrm{Sp}_{\geq 1}$, $f$ is 1-connective. Then by Theorem 16, $K^{\mathrm{inv}}(A^{h\mathbf{Z}}) \to K^{\mathrm{inv}}(\pi_0(A)^{h\mathbf{Z}})$ is an equivalence. By definition, $K^{\mathrm{inv}}(A^{h\mathbf{Z}})$ fits in a fiber sequence $K^{\mathrm{inv}}(A^{h\mathbf{Z}}) \to K(A^{h\mathbf{Z}}) \to \mathrm{TC}(A^{h\mathbf{Z}})$; applying $L_{T(n)}$ yields a fiber sequence

$$L_{T(n)}K^{\mathrm{inv}}(A^{h\mathbf{Z}}) \to L_{T(n)}K(A^{h\mathbf{Z}}) \to L_{T(n)}\mathrm{TC}(A^{h\mathbf{Z}})$$

To prove that $L_{T(n)}K(A^{h\mathbf{Z}}) \to L_{T(n)}\mathrm{TC}(A^{h\mathbf{Z}})$ is an equivalence, it suffices to show that $L_{T(n)}K^{\mathrm{inv}}(A^{h\mathbf{Z}}) \simeq 0$, or equivalently $L_{T(n)}K^{\mathrm{inv}}(\pi_0(A)^{h\mathbf{Z}}) \simeq 0$. Observe that $\pi_0(A)^{h\mathbf{Z}}$ is a $\mathbf{Z}$-module, hence both $K(\pi_0(A)^{h\mathbf{Z}})$ and $\mathrm{TC}(\pi_0(A)^{h\mathbf{Z}})$ are $K(\mathbf{Z})$-modules. By Mitchell's theorem, $K(n)_*K(\pi_0(A)^{h\mathbf{Z}})$ and $K(n)_*\mathrm{TC}(\pi_0(A)^{h\mathbf{Z}})$ both vanish, which by Theorem 19 implies $L_{T(n)}K(\pi_0(A)^{h\mathbf{Z}})$ and $L_{T(n)}\mathrm{TC}(\pi_0(A)^{h\mathbf{Z}})$ vanish, thus $L_{T(n)}K^{\mathrm{inv}}(\pi_0(A)^{h\mathbf{Z}}) \simeq 0$. □

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# The cyclotomic t-structure

Piotr Pstrągowski

We discuss the Antieau-Nikolaus t-structure on cyclotomic spectra introduced in [1]. Following the work of Hahn-Wilson and Burklund-Hahn-Levy-Schlank [2, 3], we relate the condition of being cyclotomically bounded to various other properties a cyclotomic spectrum might possess, such as satisfying the Segal conjecture, canonical vanishing, or the existence of Bökstedt elements.

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# Disassembling the disproof

Jeremy Hahn

We give an overview of the disproof of the telescope conjecture, explaining how Lichtenbaum-Quillen properties & cyclotomic redshift are both applied. This is a preview for the rest of the week's talks.

## Purity and Galois descent

Achim Krause

This talk was about recent results regarding telescopically localized algebraic $K$-theory $L_{T(n)}K(-)$, specifically about the Purity theorem of [1], which says that $L_{T(n)}K(R)$ only depends on $L_{T(n)\oplus T(n-1)}(R)$, and the descent results of [2], which say that $L_{T(n+1)}K(-)$ of $L_{n,f}$-local rings (or categories) has Galois descent.

Early indicators that $K$-theory plays well with chromatic localisation (in addition to the role topological $K$-theory played in the origins of chromatic homotopy theory) were a result of Mitchell on the vanishing of $L_{T(n)}K(R)$ for $n \geq 2$ and ordinary rings $R$, as well as a result of Thomason that $L_{K(1)}K(A) = L_{K(1)}K(B)^{hG}$ for a $G$-Galois extension $B/A$, i.e. that $K(1)$-local algebraic $K$-theory of ordinary rings satisfies Galois descent. So the main theorems of [1] and [2] are generalisations of those results to higher height:

**Theorem 1** (Purity theorem, [1])**.**

$$L_{T(n)}K(R) \simeq L_{T(n)}K(L_{T(n)\oplus T(n-1)}R).$$

If one defines the chromatic height of an $E_\infty$ ring as the largest $n$ such that $L_{T(n)}R \neq 0$, this says in particular that $K(-)$ increases chromatic height by (at most) 1. This is Rognes' "redshift" philosophy, and one can view purity as a strengthening of redshift in the sense that the above says that the height $n$ information contained in $K(R)$ *only* depends on the height $n$ and $n-1$ information in $R$.

For an $L_{n,f}$-local ring $R$, purity in particular says that $L_{T(m)}(K(R)) = 0$ for $m \geq n+2$. The descent result by [2] essentially provides a more refined analysis of the top nonvanishing chromatic layer of $K(R)$, proving that $L_{T(n+1)}K(-)$ has Galois descent. More precisely, the statement is the following:

**Theorem 2** ($T(n+1)$-local Galois descent, [2])**.** *Let $G$ be a finite $p$-group, and let $\mathcal{C}$ be a stable idempotent complete $L_{n,f}$-local $\infty$-category with $G$-action. Then*

$$L_{T(n+1)}K(\mathcal{C}^{hG}) \simeq L_{T(n+1)}K(\mathcal{C})^{hG}.$$

In particular, for a $L_{n,f}$-local $G$-Galois extension $R'$ of $R$, we have $\mathrm{Perf}(R) \simeq \mathrm{Perf}(R')^{hG}$, and so the above implies Galois descent for $p$-groups. For general finite groups, the statement is wrong in higher height (but true in Thomason's case $n = 1$).

Not only are these two results somewhat related, the proofs in the two papers [1] and [2] are also subtly entangled. We will proceed by first proving a weak form of purity, namely that $L_{T(n)}K(R)$ only depends on $L_{n,f}(R)$. This will then be used to establish a key inductive ingredient in the proof of Galois descent. As part of the inductive proof of Galois descent, we will then also establish a vanishing result which can be used to deduce the full purity result.

## 1. Proof of a weak version of the purity theorem

We want to check that $L_{T(n)}K(R)$ only depends on the $L_{n,f}$-localisation of $R$. We first prove a "highly connective" preliminary version:

**Lemma 3.** *Fix $n$. There exists $m$ such that if $A \to B$ is any $m$-connective $L_{n,f}$-equivalence between connective spectra,*

$$K(A) \to K(B)$$

*is an $L_{n,f}$-equivalence.*

*Proof.* Due to an argument with the Bousfield-Kun functor, it suffices to check that $\Sigma^\infty\Omega^\infty K(A) \to \Sigma^\infty\Omega^\infty K(B)$ is a $L_{T(n)}$-equivalence. For connective rings, $\Omega^\infty\tau_{\geq 1}K(A) = B\mathrm{GL}(A)^+$, and the plus construction goes away under suspension. So we need to prove that

$$\Sigma^\infty B\mathrm{GL}(A) \to \Sigma^\infty B\mathrm{GL}(B)$$

is an $L_{T(n)}$-equivalence. Since $A \to B$ is an $L_{T(i)}$-equivalence for $i \leq n$, we get that $\Omega^\infty A \to \Omega^\infty B$ induces an isomorphism on $v_i$-periodic homotopy groups (i.e. $[V,-][v^{-1}]$ for a type $i$ complex $V$ and $v_i$ self-map $v$). Since $\mathrm{GL}(A) \subset \mathrm{M}(A)$ is a union of connected components and $\mathrm{M}(A)$ is a filtered colimit of $\Omega^\infty A^{r\times r}$, we also get that $\mathrm{GL}(A) \to \mathrm{GL}(B)$ and hence $B\mathrm{GL}(A) \to B\mathrm{GL}(B)$ is an iso on $v_i$-periodic homotopy groups. Now a priori this is not strong enough to give a $L_{T(n)}$-equivalence on the suspension spectra, but Bousfield proves that it is if the map is highly connective (with an explicit bound $m$ that only depends on $n$). $\square$

To remove the connectivity hypotheses on the map $A \to B$ one can then use Land-Tamme excision. Specifically, we show the following special case:

**Lemma 4.** *If $n \geq 1$ and $R$ is connective and $L_{n,f}$-acyclic, then $L_{T(n)}K(R) = 0$.*

*Proof.* For any $k \geq 0$, consider the following statement: For *all* $L_{n,f}$-acyclic connective $R$, $L_{T(n)}K(R) \to L_{T(n)}K(\tau_{\leq k}R)$ is an equivalence.

Above we have proven this for $k = m$ where $m$ only depended on $n$. Now we proceed by downwards induction, using that $\tau_{\leq k}R$ is a square-zero extension over $\tau_{\leq k-1}R$ and is classified by a pullback square

$$\begin{tikzcd}
\tau_{\leq k}R \arrow[r] \arrow[d] & \tau_{\leq k-1}R \arrow[d] \\
\tau_{\leq k-1}R \arrow[r] & \tau_{\leq k-1}R \oplus \pi_k R[k+1].
\end{tikzcd}$$

To a pullback square of rings, Land-Tamme associate a new square where the bottom right term is replaced by the circle-dot product $\tau_{\leq k-1}R \odot_{\tau_{\leq k}R} \tau_{\leq k-1}R$. The map from $\tau_{\leq k-1}R$ to the circle-dot ring is an equivalence under $\tau_{\leq k}$ and hence by the inductive assumption gives an equivalence on $L_{T(n)}K(-)$. So from the Land-Tamme pullback square we get that $L_{T(n)}K(\tau_{\leq k}R) \to L_{T(n)}K(\tau_{\leq k-1}R)$ is an equivalence, and we can proceed inductively.

For $k = 0$ we learn that $L_{T(n)}K(R)$ is truncating, i.e. depends only on $\pi_0(R)$. If $L_{n,f}R = 0$, in particular $L_{T(0)}R = 0$, so $R$ is a $\mathbb{Z}/p^m$-algebra for some $m$ and it suffices to check $L_{T(n)}K(\mathbb{Z}/p^m) = 0$. By a similar argument as above starting with the square-zero extension $\mathbb{Z}/p^m \to \mathbb{Z}/p^{m-1}$, we can reduce to $m = 1$, and $L_{T(n)}K(\mathbb{F}_p) = 0$ follows from Quillen's computation since $K(\mathbb{F}_p)$ is $p$-adically discrete. □

For the general (possibly nonconnective) result, one directly shows that for a stable (or just additive) category $\mathcal{C}$ where $\mathrm{map}_{\mathcal{C}}(X, Y)$ is $L_{n,f}$-acyclic for all objects, one has $L_{T(n)}K^{\mathrm{add}}(\mathcal{C}) = 0$, using Schwede-Shipley-Morita theory and that additive $K$-theory only depends on the connective endomorphism spectrum of a generator. For a general stable category one may then use that $K^{\mathrm{add}}(\mathcal{C})$ acts on $K(\mathcal{C})$.

**Lemma 5.** *If $\mathcal{C}$ is an $L_{n,f}$-acyclic stable category, $L_{T(n)}K(\mathcal{C}) = 0$.*

**Theorem 6** (Weak purity theorem)**.** *$L_{T(n)}K(R)$ only depends on $L_{n,f}R$.*

*Proof.* Tensoring the Verdier sequence

$$\ker L_{n,f} \to \mathrm{Perf}(\mathbb{S}) \to \mathrm{Perf}(L_{n,f}\mathbb{S})$$

with $\mathrm{Perf}(R)$, and using that $L_{n,f}$ is smashing, we get

$$\ker L_{n,f} \otimes \mathrm{Perf}(R) \to \mathrm{Perf}(R) \to \mathrm{Perf}(L_{n,f}R),$$

and since the left hand category if $L_{n,f}$-acyclic, the result follows from the previous lemma. □

## 2. Galois descent

To study Galois descent of $K$-theory, we may organize $K$-theory into an equivariant spectrum. Through the formalism of spectral Mackey functors, this is very straightforward: For $\mathcal{C}$ a small stable idempotent complete $\infty$-category with $G$-action, we have a Mackey functor with values in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ which takes

$$G/H \mapsto \mathcal{C}^{hH}$$

This is a category-valued Mackey functor, and if we apply any additive invariant like $K$, we get a spectral Mackey functor $K_G(\mathcal{C})$ with $K_G(\mathcal{C})^H = K(\mathcal{C}^{hH})$.

To show Galois descent we want to compare $L_{T(i)}K_G(\mathcal{C})^H$ and $L_{T(i)}K(\mathcal{C})^{hH}$, i.e. whether this spectrum is Borel. In fact, we will inductively reduce these questions to $G = C_p$ using solvability of $p$-groups. More generally, for a genuine $C_p$-spectrum $M$, we may compare $M^{C_p}$ to $M_{hC_p}$ and $M^{hC_p}$ using assembly and coassembly maps. In the diagram

$$L_{T(i)}M_{hC_p} \to L_{T(i)}M^{C_p} \to L_{T(i)}M^{hC_p}$$

(1) The first morphism is an equivalence if and only if $L_{T(i)}\Phi_{C_p}M = 0$.
(2) The second morphism is an equivalence if and only if $L_{T(i)}M$ is Borel.
(3) The composite is an equivalence if and only if $L_{T(i)}M^{tC_p} = 0$.

If our input is suitably localized, the third condition will be implied by Kuhn's blueshift theorem:

**Theorem 7** (Kuhn). *If $R$ is $L_{n,f}$-local, then $L_{T(n)}R^{tC_p} = 0$, and $R^{tC_p}$ is $L_{n-1,f}$-local.*

(See [3] for an elegant proof using little more than the Bousfield-Kuhn functor and a version of the Kahn-Priddy theorem.)

So we will deduce Galois descent from vanishing of geometric fixed points. The key ingredient is the following lemma, which facilitates the inductive step:

**Lemma 8.** *For an $E_\infty$ ring $R$ and $i \geq 1$, we have the implications* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *in:*

*(1)* $L_{T(i)}R = 0$ *and* $L_{T(i)}K(R^{tC_p}) = 0$.
*(2)* $L_{T(i)}\Phi^{C_p}(K_{C_p}(R)) = 0$.
*(3)* $L_{T(j)}K(R) = 0$ *for all* $j \geq i+1$.

*Proof.* For (1) $\Rightarrow$ (2) one may assume $R$ to be connective (this uses the weak form of purity above!). We have the maps

$$K(R)_{hC_p} \to K(R[C_p]) \to K(\mathrm{Perf}(R)^{hC_p}).$$

The second map has cofiber given by $K(\mathrm{stmod}_R(C_p))$, by the associated Verdier sequence. This category is $R^{tC_p}$-linear, and so $L_{T(i)}(\mathrm{stmod}_R(C_p)) = 0$ if $L_{T(i)}(R^{tC_p}) = 0$. The first map is the assembly, and here one may use Dundas-Goodwillie-McCarthy to reduce to the following two statements:

- The assembly gives an iso $L_{T(i)}K(\pi_0 R)_{hC_p} \to L_{T(i)}K(\pi_0 R[C_p])$. By Mitchell's special case of purity for ordinary rings, this is a nontrivial statement only for $i = 1$, in which it is an explicit computation.
- The assembly gives an iso $L_{T(i)}\,\mathrm{TC}(R)_{hC_p} \to L_{T(i)}\,\mathrm{TC}(R[C_p])$, and same for $\pi_0 R$ instead of $R$. This relies on a description by Hesselholt-Nikolaus on the cofiber of the assembly on TC as $\mathrm{THH}(R;\mathbb{Z}_p)_{hS^1}[1] \otimes C_p$, which is an $R$-module and thus vanishes $T(i)$-locally by assumption.

For (2) $\Rightarrow$ (3), $K(R)^{tC_p}$ is a module over geometric fixed points. If we had $L_{T(j)}K(R) \neq 0$, by the chromatic Nullstellensatz we would get a ring map $K(R) \to E_j$ and hence $\Phi^{C_p}(K_{C_p}(R)) \to K(R)^{tC_p} \to E_j^{tC_p}$, where $E_j$ is height $j$ Morava $E$-theory associated to some algebraically closed field $k$ of characteristic $p$. But it is not hard to check that $L_{T(i)}E_j^{tC_p} \neq 0$ whenever $i \leq j-1$, proving the claim. (The paper uses a more direct, pre-Nullstellensatz argument based on [4].) □

**Theorem 9** (Galois descent and vanishing in high heights). *Let $\mathcal{C}$ be an $L_{n,f}$-local stable $\infty$-category, then $L_{T(m)}K(\mathcal{C}) = 0$ for $m \geq n+2$, and*

$$L_{T(n+1)}(\mathcal{C}^{hG}) \simeq L_{T(n+1)}(\mathcal{C})^{hG}$$

*for any finite $p$-group acting on $\mathcal{C}$.*

*Proof.* For the vanishing part, $K(L_{n,f}\mathbb{S})$ acts on $K(\mathcal{C})$, and so it suffices to know $L_{T(m)}K(L_{n,f}\mathbb{S}) = 0$. To deduce this from part (3) of the previous result, we need $L_{T(n+1)}L_{n,f}\mathbb{S} = 0$ (clear), and $L_{T(n+1)}K(L_{n,f}\mathbb{S}^{tC_p}) = 0$. Since $L_{n,f}\mathbb{S}^{tC_p}$ is $L_{n-1,f}$-local by Kuhn, and so an $L_{n-1,f}\mathbb{S}$-module, this follows inductively.

For the descent part, we reduce to $G = C_p$ by solvability, and then use that in the diagram

$$L_{T(n+1)}K(\mathcal{C})_{hC_p} \to L_{T(n+1)}K(\mathcal{C}^{hC_p}) \to L_{T(n+1)}K(\mathcal{C})^{hC_p}$$

the cofiber of the left map is given by the geometric fixed points of $L_{T(n+1)}\Phi_{C_p}K_{C_p}(\mathcal{C})$, and of the long composite by $L_{T(n+1)}K(\mathcal{C})^{tC_p}$. On these, we have $L_{T(n+1)}\Phi_{C_p}K_{C_p}(L_{n,f}\mathbb{S})$ acting. But this vanishes by part (2) of the previous lemma. So the left map and the composite are equivalences, and Galois descent follows. □

Combining the weak purity result above with this descent result, one may replace the $L_{n,f}$-locality assumption on $\mathcal{C}$ by a weaker version where one replaces $\mathcal{C}$ by an $R$-linear category for $R$ an $E_\infty$ ring with $L_{T(n)}R^{tC_p} = 0$.

We also can prove the strong version of purity:

**Theorem 10** (Purity theorem)**.** *$L_{T(n)}K(R) \to L_{T(n)}K(L_{T(n-1)\oplus T(n)}R)$ is an equivalence.*

*Proof.* By the weak form of purity, we may assume $R$ to be $L_{n,f}$-local already.

Tensoring the sequence

$$\ker L_{n-2,f} \to \mathrm{Perf}(\mathbb{S}) \to \mathrm{Perf}(L_{n-2,f}\mathbb{S})$$

with $\mathrm{Perf}(R)$, we obtain a sequence

$$\ker L_{n-2,f} \otimes \mathrm{Perf}(R) \to \mathrm{Perf}(R) \to \mathrm{Perf}(L_{n-2,f}R).$$

Tensoring instead with $\mathrm{Perf}(L_{T(n-1)\oplus T(n)}R)$, we obtain the same sequence with $R$ replaced by $L_{T(n-1)\oplus T(n)}R$, but crucially the kernel does not change: This is because $\ker L_{n-2,f}$ is generated by a type $n-1$ complex, and tensoring with a type $n-1$ complex turns $R \to L_{T(n-1)\oplus T(n)}R$ into an equivalence (since $R$ is $L_{n,f}$-local by assumption). So on $K$-theory, one gets a pullback square

$$\begin{tikzcd}
K(R) \arrow[r] \arrow[d] & K(L_{T(n-1)\oplus T(n)}R) \arrow[d] \\
K(L_{n-2,f}R) \arrow[r] & K(L_{n-2,f}L_{T(n-1)\oplus T(n)}R).
\end{tikzcd}$$

After applying $L_{T(n)}$, the bottom row vanishes by the vanishing result, and so the claim follows. □

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## Cyclotomic Redshift

Leor Neuhauser

This talk is entirely based on [BMCSY23]. The disproof of the telescope conjecture consists of two claims:

(1) every $K(n+1)$-local spectra is cyclotomically complete, and
(2) there exists some $T(n)$-local ring spectrum $R$ such that $K_{T(n+1)}(R) := L_{T(n+1)}K(R)$ is not cyclotomically complete.

To approach (2), we need to understand how $T(n+1)$-local $K$-theory interacts with higher height cyclotomic extensions. Our goal in this talk is to prove that $T(n+1)$-local $K$-theory satisfies redshift:

**Theorem 1.** *Let $R$ be a $T(n)$-local ring spectrum, then there is an isomorphism*

$$K_{T(n+1)}(R[\omega_{p^\infty}^{(n)}]) \simeq K_{T(n+1)}(R)[\omega_{p^\infty}^{(n+1)}]$$

The main tool in proving this theorem will be descent for $T(n+1)$-local $K$-theory. In [CMNN22], the authors proved descent along finite $p$-groups:

**Theorem 2.** *Let $\mathcal{C}$ be an $L_n^f$-local category (i.e. a stable category with $L_n^f$-local mapping spectra) with $G$-action for a finite $p$-group $G$, then*

$$K_{T(n+1)}(\mathcal{C}^{hG}) \xrightarrow{\sim} K_{T(n+1)}(\mathcal{C})^{hG}.$$

To consider arbitrary higher cyclotomic extensions, we need to generalize this result for groups in spaces. In fact, we will prove a vast generalization, including all limits and colimits along $\pi$-finite $p$-spaces.

**Theorem 3.** *The functor $K_{T(n+1)}\colon \mathrm{Cat}_{L_n^f} \to \mathrm{Sp}_{T(n+1)}$ commutes with $\pi$-finite $p$-space indexed (co)limits.*

The proof of Theorem 1 then proceeds by applying Theorem 3 to $\mathrm{Perf}(R)$ with a constant $B^{n+1}C_{p^{r+1}}$ action to deduce

$$K_{T(n+1)}(R[B^nC_{p^r}]) \xrightarrow{\sim} K_{T(n+1)}(R)[B^{n+1}C_{p^{r+1}}]$$

Recall that the cyclotomic extension $R[\omega_{p^r}^{(n)}]$ was defined by splitting an idempotent on $R[B^nC_{p^r}]$. By showing that the idempotents are compatible on both sides of the above isomorphism, we conclude cyclotomic redshift

$$K_{T(n+1)}(R[\omega_{p^r}^{(n)}]) \xrightarrow{\sim} K_{T(n+1)}(R)[\omega_{p^r}^{(n+1)}]$$

with the case $r=\infty$ achieved as a filtered colimit.

To prove Theorem 3, we first use the purity result of [LMMT24] to reduce to the claim to $n$-monochromatic categories. A category $\mathcal{C}$ is $n$-monochromatic if it is $L_n^f$-local and $L_{n-1}^f\mathcal{C}=0$. The inclusion of $n$-monochromatic categories in $L_n^f$-local categories $\mathrm{Cat}_{M_n^f} \hookrightarrow \mathrm{Cat}_{L_n^f}$ has a right adjoint denoted $M_n^f : \mathrm{Cat}_{L_n^f} \to \mathrm{Cat}_{M_n^f}$, and purity tells us that for every $\mathcal{C} \in \mathrm{Cat}_{L_n^f}$ $K_{T(n+1)}(\mathcal{C}) \simeq K_{T(n+1)}(M_n^f(\mathcal{C}))$. The functor $M_n^f : \mathrm{Cat}_{L_n^f} \to \mathrm{Cat}_{M_n^f}$ commutes with all limits and colimits, so we reduce Theorem 3 to proving that $K_{T(n+1)} \colon \mathrm{Cat}_{M_n^f} \to \mathrm{Sp}_{T(n+1)}$ commutes with $\pi$-finite $p$-space indexed (co)limits.

We now use the fact that both $\mathrm{Cat}_{M_n^f}$ and $\mathrm{Sp}_{T(n+1)}$ are $\infty$-semiadditive categories. Immediately this implies that the $\pi$-finite $p$-space indexed limits and colimits agree, but moreover, using the $\infty$-semiadditive structure, we can reduce the problem to only proving that $K_{T(n+1)} \colon \mathrm{Cat}_{M_n^f} \to \mathrm{Sp}_{T(n+1)}$ commutes with constant colimits indexed on a $\pi$-finite $p$-space concentrated in a single homotopy degree. This is then done by induction on the homotopy degree, bootstrapping Theorem 2.

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## Basic Examples of Lichtenbaum–Quillen

Gabriel Angelini-Knoll

Lichtenbaum [5] and Quillen [8] conjectured that there should be a relationship between algebraic K-theory and étale cohomolology providing K-theoretic descriptions of special values of Dedekind zeta functions. This can be phrased as the question of whether the fiber of the localization map

$$\mathrm{K}(\mathcal{O}_F[1/p]) \longrightarrow L_1^f\mathrm{K}(\mathcal{O}_F[1/p])$$

is bounded above where K denotes $p$-complete algebraic K-theory. More generally, we say that a $p$-complete spectrum $X$ is *asymptotically $L_n^f$-local* if the localization map $X \to L_n^f X$ has bounded above fiber. If $X$ is $p$-cyclotomic, then $\mathrm{TR}(X)$ is asymptotically $L_{n+1}^f X$-local if and only if $V \otimes X$ is cyclotomically bounded for some $p$-local, type $n+2$ finite spectrum. For example, if $X = \mathrm{THH}(R)$ for a ring spectrum $R$ then this implies that $\mathrm{TC}(R)$ is asymptotically $L_{n+1}^f$-local and one can often show that $\mathrm{K}(R)$ is asymptotically $L_{n+1}^f$-local.

In this talk, I showed that $\mathrm{TC}(\mathbb{F}_p)$ is asymptocially $L_0^f$-local, $\mathrm{TC}(\mathbb{Z}_p)$ is asymptotically $L_1^f$-local, and $\mathrm{TC}(\ell)$ is asymptotically $L_2^f$-local. We also discussed redshift in each case. Along the way, we observed that $\mathrm{THH}(\mathbb{F}_p)/p$, $\mathrm{THH}(\mathbb{Z}_p)/(p,v_1)$ and $\mathrm{THH}(\ell)/(p,v_1,v_2)$ are cyclotomically bounded and gave explicit computations of $\mathrm{TC}(\mathbb{F}_p)$, $\pi_*\mathrm{TC}(\mathbb{Z}_p)/p$ of $\pi_*\mathrm{TC}(\ell)/(p,v_1)$. These results are originally due to Hesselholt–Madsen [4], Bökstedt–Madsen [3], and Ausoni–Rognes [1] respectively. For simplicity, fix $p \geq 7$ and $p$-complete all invariants.

First, we recall a result of Hopkins and Mahowald [6] that there is an equivalence

$$\mathbb{F}_p \simeq \mathrm{Th}(\Omega^2 S^3 \xrightarrow{1+p} BGL_1\mathbb{S}_p)$$

of $\mathbb{E}_2$-rings. By Blumberg–Cohen–Schlichtkrull [2], this implies that

$$\mathrm{THH}(\mathbb{F}_p) \simeq \mathbb{F}_p \otimes \Omega S^3_+$$

and from this we can deduce that there are preferred isomorphisms of $\mathbb{F}_p$-algebras

$$\mathrm{THH}_*(\mathbb{F}_p) \cong \mathbb{F}_p[\mu]\,, \mathrm{THH}_*(\mathbb{Z}_p) \cong \mathbb{F}_p[\mu^p]\langle\lambda_1\rangle\,, \mathrm{THH}_*(\ell) \cong \mathbb{F}_p[\mu^{p^2}]\langle\lambda_1,\lambda_2\rangle$$

where $\mathbb{F}_p[\mu^{p^k}]$ denotes polynomial algebra on a class $\mu^{p^k}$ in degree $2p^k$ and $\mathbb{F}_p\langle\lambda_j\rangle$ denotes an exterior algebra on a class in degree $2p^j-1$.

Fix $R \in \{\mathbb{F}_p, \mathbb{Z}, \ell\}$ and let $F$ be a suitable Smith–Toda complex. We will consider the unit map $\eta : \pi_*F \longrightarrow \pi_*F \otimes \mathrm{TC}^-(R)$ and say that $y \in \pi_*F \otimes \mathrm{TC}^-(R)$ is in the Hurewicz image of $x \in \pi_*F$ if the unit map sends $x$ to $y$. In my talk, I proved the following results:

(1) There is a non-trivial class in $\pi_0\mathrm{TC}^-(\mathbb{F}_p)$ in the Hurewicz image of $p \in \pi_0\mathbb{S}_p$, which is detected by $t\mu \in H^2_{\mathrm{gp}}(S^1, \mathrm{THH}_2(\mathbb{F}_p))$.
(2) There is a non-trivial class in $\pi_{2p-3}\mathrm{TC}^-(\mathbb{Z}_p)$ in the Hurewicz image of $\alpha_1 \in \pi_{2p-3}\mathbb{S}_p$, which is detected by $t\lambda_1 \in H^2_{\mathrm{gp}}(S^1, \mathrm{THH}_{2p-1}(\mathbb{Z}_p))$.
(3) There is a non-trivial class in $\pi_{2p-2}\mathrm{TC}^-(\mathbb{Z}_p)/p$ in the Hurewicz image of $v_1 \in \pi_{2p-2}\mathbb{S}/p$, which is detected by $t\mu^p \in H^2_{\mathrm{gp}}(S^1, \mathrm{THH}_{2p}(\mathbb{Z}_p)/p)$.
(4) There is a non-trivial class in $\pi_{2p-3}\mathrm{TC}^-(\ell)$ in the Hurewicz image of $\alpha_1 \in \pi_{2p-3}\mathbb{S}_p$, which is detected by $t\lambda_1 \in H^2_{\mathrm{gp}}(S^1, \mathrm{THH}_{2p-1}(\ell))$.
(5) There is a non-trivial class in $\pi_{2p^2-2p-1}\mathrm{TC}^-(\ell)/p$ in the Hurewicz image of $\beta_1' \in \pi_{2p^2-2p-1}\mathbb{S}/p$, which is detected by $t^p\lambda_2 \in H^{2p}_{\mathrm{gp}}(S^1, \mathrm{THH}_{2p^2-1}(\ell)/p)$.
(6) There is a non-trivial class in $\pi_{2p^2-2}\mathrm{TC}^-(\ell)/(p,v_1)$ in the Hurewicz image of $v_2 \in \pi_{2p^2-2p-1}\mathbb{S}/p$, which is detected by $t^p\lambda_2 \in H^{2p}_{\mathrm{gp}}(S^1, \mathrm{THH}_{2p^2-1}(\ell)/p)$.

In each case, except for the proof of Item (5), the proof follows from a careful analysis of the map of Adams spectral sequences

$$\begin{tikzcd}
\mathrm{Cotor}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_p, \pi_*\mathbb{F}_p \otimes F) \arrow[r] \arrow[d, Rightarrow] & \mathrm{Cotor}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_p, \pi_*\mathbb{F}_p \otimes F \otimes (\lim_{\mathbb{C}P^1} \mathrm{THH}(\mathbb{F}_p))) \arrow[d, Rightarrow] \\
\pi_*F \arrow[r, "\eta"] & \pi_*(\lim_{\mathbb{C}P^1} \mathrm{THH}(\mathbb{F}_p))
\end{tikzcd}$$

where $\eta : F \to F \otimes \mathrm{TC}^-(R) \to F \otimes \lim_{\mathbb{C}P^1} \mathrm{THH}(R)$ is the unit map. The case Item (5) relies on a power operation

$$\mathcal{P}^k : \pi_{2k-1}(\mathbb{S}/p \otimes -) \longrightarrow \pi_{2pk-1}(\mathbb{S}/(p, v_1) \otimes -).$$

Consider the diagram of spectral sequences

$$\begin{tikzcd}
F_*\mathrm{THH}(R)^{tC_p}[t] \arrow[d, Rightarrow] & F_*\mathrm{THH}(R)[t] \arrow[l, "(\varphi_p)[t]"'] \arrow[r, "(-)[t^{-1}]"] \arrow[d, Rightarrow] & F_*\mathrm{THH}(R)[t, t^{-1}] \arrow[d, Rightarrow] \\
F_*(\mathrm{THH}(R)^{tC_p})^{hS^1} & F_*\mathrm{TC}^-(R) \arrow[l, "\varphi_p^{hS^1}"'] \arrow[r, "\mathrm{can}"] & F_*\mathrm{TP}(R)
\end{tikzcd}$$

and let $E_1 := F_*\mathrm{THH}(R)[t]$, $\widehat{E}_1 := F_*\mathrm{THH}(R)[t]$ and $\mu^{-1}E_1 := F_*\mathrm{THH}(R)^{tC_p}[t]$. Let $G : (\mathrm{THH}(R)^{tC_p})^{hS^1} \simeq \mathrm{TP}(R)$ be the equivalence from [7, Lemma II.4.2].

For $\mathbb{F}_p$, we observe that for bidegree reasons the middle and right spectral sequence collapse at the first page. Using the relation $t\mu = p$ we can resolve multiplicative extensions to show show that the canonical map

$$\mathrm{can} : \mathrm{TC}^-_*(\mathbb{F}_p) \longrightarrow \mathrm{TP}_*(\mathbb{F}_p)$$

is given in by the map $\mathbb{Z}_p[t, \mu]/(t\mu - p) \to \mathbb{Z}_p[t, t^{-1}]$ which inverts $t$. Consequently, $\mu \in \pi_2\mathrm{THH}(\mathbb{F}_p)/p$ is a Bökstedt class. I showed that the map $\varphi_p^{hS^1}$ is the map $\mathbb{Z}_p[t, \mu]/(t\mu - p) \to \mathbb{Z}_p[\mu, \mu^{-1}]$ given by inverting $\mu$ and $G(\mu) \doteq t^{-1}$ where $\doteq$ means equals up to a unit. We concluded that $\mathrm{TC}_*(\mathbb{F}_p) = \mathbb{Z}_p\langle\partial\rangle$ with $|\partial| = -1$. This implies that $L_{T(0)}\mathrm{K}(\mathbb{F}_p) \neq 0$ and $L_{T(1)}\mathrm{K}(\mathbb{F}_p) = 0$ proving redshift.

As a consequence, there is a map $\mathbb{Z}_p^{\mathrm{triv}} \longrightarrow \mathrm{THH}(\mathbb{F}_p)$ of $\mathbb{E}_\infty$-algebras in $p$-cyclotomic spectra. Using this, one can show that the cyclotomic Frobenius map

$$\varphi_p : \mathrm{THH}(R) \longrightarrow \mathrm{THH}(R)^{tC_p}$$

is given by inverting $\mu$ on $\pi_*$ when $R = \mathbb{F}_p$, inverting $\mu^p$ on mod $p$-homotopy when $R = \mathbb{Z}_p$, and inverting $\mu^{p^2}$ on mod $(p, v_1)$-homotopy when $R = \ell$. This implies the Segal conjecture, so $\mathrm{THH}(\mathbb{F}_p)/p$ is bounded in the cyclotomic t-structure. To show that $\mathrm{THH}(\mathbb{Z}_p)/(p, v_1)$ and $\mathrm{THH}(\ell)/(p, v_1, v_2)$ are bounded in the cyclotomic $t$-structure, we will observe that $\mu^p \in \pi_{2p}\mathrm{THH}(\mathbb{Z}_p)/(p, v_1)$ and $\mu^{p^2} \in \pi_{2p^2}\mathrm{THH}(\ell)/(p, v_1, v_2)$ are Bökstedt classes.

In the case of $\mathbb{Z}_p$, we further qutient by $t\mu^p = v_1$ and consider

$$\widehat{\mathrm{E}}_1 = \mathbb{F}_p[t, t^{-1}]\langle\lambda_1\rangle \implies \mathrm{TP}_*(\mathbb{Z}_p)/(p, v_1)\,.$$

The fact that $\alpha_1$ is detected by $t\lambda_1$ implies $d_p(t^{1-p}) = t\lambda_1$ and the spectral sequence collapses after this page. We conclude that

$$\mathrm{can} : \mathbb{F}_p[t^p, \mu^p]/(t^p\mu^p)\langle\lambda_1\rangle \longrightarrow \mathbb{F}_p[t^p, t^{-p}]\langle\lambda_1\rangle$$

is given by inverting $t^p$ and $\varphi_p^{hS^1}$ is given by inverting $\mu^p$. This shows that $\mu^p \in \mathrm{THH}_{2p}(\mathbb{Z}_p)/(p, v_1)$ is a Bökstedt class.

By a result of Bökstedt, we know that the map $\mathrm{TC}_{2p-1}(\mathbb{Z}_p) \to \mathrm{THH}_{2p-1}(\mathbb{Z}_p)$ is surjective so $G(\lambda_1) = \lambda_1$ and more generally $G(\lambda_1^a t^{-jp}) \doteq \lambda_1^a \mu^{jp}$ for $a \in \{0, 1\}$,

$j \in \mathbb{Z}$. We conclude

$$\mathrm{TC}_*(\mathbb{Z}_p)/(p, v_1) \cong \mathbb{F}_p\langle\lambda_1, \partial\rangle \oplus \mathbb{F}_p\{t^d\lambda_1 : 0 < d < p\}$$

and the $v_1$-Bockstein spectral sequence collapses (e.g. use the motivic filtration). This implies $L_{T(1)}\mathrm{K}(\mathbb{Z}_p) \neq 0$ and $L_{T(2)}\mathrm{K}(\mathbb{Z}_p) = 0$ proving redshift.

In the case of $\ell$, we also quotient by $t\mu^{p^2} = v_2$ and consider

$$\widehat{\mathrm{E}}_1 = \mathbb{F}_p[t, t^{-1}]\langle\lambda_1, \lambda_2\rangle \implies \mathrm{TP}_*(\ell)/(p, v_1, v_2), .$$

The fact that $\alpha_1$ is detected by $t\lambda_1$ and $\beta_1'$ is detected by $t^p\lambda_2$ implies differentials $d_p(t^{1-p}) = t\lambda_1$ and $d_{p^2}(t^{p-p^2}) = t^p\lambda_2$. We conclude that

$$\mathrm{can} : \mathbb{F}_p[t^{p^2}, \mu^{p^2}]/(t^{p^2}\mu^{p^2})\langle\lambda_1, \lambda_2\rangle \longrightarrow \mathbb{F}_p[t^{p^2}, t^{-p^2}]\langle\lambda_1, \lambda_2\rangle$$

is given by inverting $t^{p^2}$ and $\varphi_p^{hS^1}$ is given by inverting $\mu^{p^2}$. This shows that $\mu^{p^2} \in \mathrm{THH}_{2p^2}(\ell)/(p, v_1, v_2)$ is a Bökstedt class. Using the same power operation as before and Bökstedt's result, we can determine that $G(\lambda_s) = \lambda_s$ for $s = 1, 2$ and more generally $G(\lambda_1^a\lambda_2^b t^{-jp^2}) \doteq \lambda_1^a\lambda_2^b\mu^{jp^2}$ for $a, b \in \{0, 1\}$ and $j \in \mathbb{Z}$. From this, we can determine that $\mathrm{TC}_*(\ell)/(p, v_1, v_2)$ is

$$\mathbb{F}_p\langle\lambda_1, \lambda_2, \partial\rangle \oplus \mathbb{F}_p\langle\lambda_2\rangle\{t^d\lambda_1 : 0 < d < p\} \oplus \mathbb{F}_p\langle\lambda_1\rangle\{t^{dp}\lambda_2 : 0 < d < p\}$$

and the $v_2$-Bockstein spectral sequence collapses (e.g. use the motivic filtration). This implies that $L_{T(2)}\mathrm{K}(\ell) \neq 0$ and $L_{T(3)}\mathrm{K}(\ell) = 0$ proving redshift.

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## Adams operations on $BP\langle n\rangle$

Lennart Meier

Our goal is to construct Adams operations on $BP\langle n\rangle$. We will first recall the definition of $BP\langle n\rangle$.

### 1. Forms of $BP\langle n\rangle$

After localizing at $p$, the complex cobordism spectrum $MU$ splits into (shifted) copies of $BP$, where $\pi_*BP \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots]$. From now on, we implicitly localize everywhere at $p$. A complex-oriented ring spectrum $R$ is called a *form of $BP\langle n\rangle$* if the map $BP \to R$ induces an isomorphism $\mathbb{Z}[v_1, \dots, v_n] \to R$.

Classic examples of forms of $BP\langle n\rangle$ are the Adams summand of $ku$ (at height 1) and the spectrum $tmf_1(3)$ (at height 2, for $p = 2$). These are actually $E_\infty$-ring spectra. By work of Lawson and Senger, we know that for general $p$ and $n$, there are no $E_\infty$-forms of $BP\langle n\rangle$; but work of Hahn and Wilson [4, Theorem A] establishes forms of $BP\langle n\rangle$ with an $\mathcal{E}_3$-$MU$-algebra structure. In the following, we will denote by $BP\langle n\rangle$ always a form of $BP\langle n\rangle$ with this structure.

### 2. Adams operations

We give several examples of Adams operations. For this, fix a prime $l \neq p$.

(1) The most classical Adams operations are the $E_\infty$-operations $\psi^l\colon ku \to ku$. These also induce operations $\psi^l$ on the Adams summand (and thus on some $BP\langle 1\rangle$).

(2) Let $F$ be a formal group law of height $n$ over $\overline{\mathbb{F}}_p$ and $E_n$ the associated Lubin–Tate spectrum. The $l$-series $[l]$ induces an automorphism of $F$ and hence (by the Goerss–Hopkins–Miller theorem) an $E_\infty$-automorphism of $E_n$, which we call $\psi^l$.

(3) Jack Davies constructed $E_\infty$-Adams operations on many variants of topological modular forms; applying this to $tmf_1(3)$ yields $E_\infty$-Adams operations on some $BP\langle 2\rangle$.

(4) The stable Adams conjecture provides a homotopy between $J$ and $J\psi^l$ as $E_\infty$-maps $BU \to BSL_1(\mathbb{S})$.[1] Thus, $\psi^l$ becomes a map over $BSL_1(\mathbb{S})_{(p)}$ and Thomifying yields an $E_\infty$-Adams operation $\psi^l\colon MU \to MU$. We denote by $MU^\psi$ the $E_\infty$-ring spectrum $MU$ together with its $\mathbb{Z}$-action by $\psi^l$.

Our goal is to prove the following two theorems.

**Theorem 1** ([2], Corollary 5.16)**.** *Together with its Adams operation $\psi^l$, the spectrum $E_n$ lifts to an $E_2$-$MU^\psi$-algebra.*

[1] The stable Adams conjecture has a complicated history. Friedlander claimed a proof in 1980, which however is incorrect. Bhattachariya and Kitchloo [1, Theorem 1.8] provides a corrected proof with additional conditions on the prime $l$. Friedlander [3] sheds doubts on part of the argument of [1] and provides a version for general $l$; however, he seems never to state that his spectrum $b\mathbf{S}$ agrees with $bsl_1(\mathbb{S})$. We treat the stable Adams conjecture as a hypothesis here.

**Theorem 2** (Main theorem; [2], Theorem 5.4)**.** *There exists an Adams operation $\psi^l$ on $BP\langle n\rangle$ making it into an $\mathcal{E}_1 \otimes A_2$-$MU^{\psi}$-algebra, and the map $BP\langle n\rangle \to E_n$ lifts to an $\mathcal{E}_1$-$MU^{\psi}$-algebra map.*

Here, the map $BP\langle n\rangle \to E_n$ comes from an identification of $L_{K(n)}BP\langle n\rangle$ with $E_n^{h\mu_{p^n} \rtimes \mathrm{Gal}(\mathbb{F}_p)}$.

Concretely, the $\mathcal{E}_1 \otimes A_2$-algebra structure on $BP\langle n\rangle$ means that the multiplication map $BP\langle n\rangle \otimes_{MU} BP\langle n\rangle \to BP\langle n\rangle$ is $\mathcal{E}_1$ and thus $THH(BP\langle n\rangle)$ still retains an $A_2$-structure (i.e. a homotopy unital multiplication).

Lastly, we remark that the surjection $\pi_* MU \to \pi_* BP\langle n\rangle$ implies that $\psi^l$ has to act on $\pi_{2k} BP\langle n\rangle$ as it does on $\pi_{2k} MU$, i.e. by multiplication by $l^k$.

## 3. Proof of the main theorem

Assuming Theorem 1, we will sketch the proof of the $\mathcal{E}_1$-part of Theorem 2. We will construct the Adams operations cell by cell.

**Proposition 3** (Odd cells; stated in the proof of Proposition 5.29 of [2])**.** *There is a filtration*

$$MU = R_0 \to R_1 \to \cdots \to R_\infty = BP\langle n\rangle$$

*of $\mathcal{E}_1$-$MU$-algebras together with pushouts*

$$\begin{tikzcd} MU\{\bigoplus \Sigma^{2i-2} MU\} \arrow[r] \arrow[d] & MU \arrow[d] \\ R_{i-1} \arrow[r] & R_i \end{tikzcd}$$

*of $\mathcal{E}_1$-$MU$-algebras, where $MU\{M\}$ denotes the free $\mathcal{E}_1$-$MU$-algebra on an $MU$-module $M$.*

The proof idea is similar to how one shows that some space has only cells in odd dimensions, i.e. using homology. Here our homology theory sends connective $MU$-algebras $A$ (with $\pi_0 A = \mathbb{Z}$) to $\mathrm{fib}(\mathbb{Z} \to \mathbb{Z} \otimes_{A \otimes_{MU} \mathbb{Z}} \mathbb{Z})$. For $A = BP\langle n\rangle$, one computes by applying a Tor-spectral sequence twice that its homology is concentrated in odd degrees, where it consists of free $\mathbb{Z}$-modules. A suitable Hurewicz theorem allows to inductively construct the cell-structure of Proposition 3.

Showing the $\mathcal{E}_1$-part of Theorem 2 amounts to completing the square

$$\begin{tikzcd} BP\langle n\rangle \arrow[r, dashed] \arrow[d] & \psi^l_* BP\langle n\rangle \arrow[d] \\ E_n \arrow[r] & \psi^l_* E_n \end{tikzcd}$$

in $\mathcal{E}_1$-$MU$-algebras. Applying Proposition 3 to do this cell-by-cell, we see obstructions in $\pi_{2i-2} \psi^l_* BP\langle n\rangle = \pi_{2i-2} BP\langle n\rangle$. As this injects into $\pi_{2i-2} E_n$, these obstructions have to vanish by Theorem 1. Together with an argument about the connectedness of the space of nullhomotopies, this shows Theorem 2.

## 4. Adams operations on Lubin–Tate spectra

We will sketch here a proof of Theorem 1. By the map

$$L_{K(n)}BP\langle n\rangle \simeq E_n^{h\mu_{p^n}\rtimes \mathrm{Gal}(\mathbb{F}_p)} \to E_n,$$

the Lubin–Tate spectrum obtains the structure of an $\mathcal{E}_3$-$MU$-algebra. This is equivalent structure to an $\mathcal{E}_4$-map to the $\mathcal{E}_3$-center of $E_n$, which happens to agree with $E_n$ (essentially since $L_{K(n)}\mathbb{S} \to E_n$ is pro-Galois); thus we obtain an $\mathcal{E}_4$-map $MU \to E_n$. By the universal property of Thom spectra, this corresponds to an $\mathcal{E}_4$-nullhomotopy of $BU \to BSL_1\mathbb{S} \to BSL_1E_n$ and thus to a pointed nullhomotopy of $BU\langle 6\rangle \simeq B^4BU \to B^4BSL_1\mathbb{S} \to B^4BBSL_1E_n$.

To show Theorem 1, we need to produce a $\mathbb{Z}$-equivariant refinement of this nullhomotopy. Here, we act on $E_n$ via $\psi^l$ (to obtain a $\mathbb{Z}$-equivariant spectrum $E_n^\psi$). Denoting by $S^{1,[l^2]}$ the circle with a $\mathbb{Z}$-action of degree $l^2$, we further use a suitable $\mathbb{Z}$-equivariant version of $B^4$ so that $\mathrm{Map}_*(S^3 \wedge S^{1,[l^2]}, -)$ reproduces $BU \to BSL_1\mathbb{S} \to BSL_1E_n^\psi$. Thus, if the nullhomotopy refines $\mathbb{Z}$-equivariantly, we obtain an $\mathcal{E}_3$-map $MU^\psi \to E_n^\psi$, yielding Theorem 1.

To produce the $\mathbb{Z}$-equivariant refinement, we have to compute

$$\pi_*\mathrm{Map}(B^4BU, B^5SL_1E_n)^{h\mathbb{Z}}.$$

As $B^4BU \simeq BU\langle 6\rangle$ has even cells, this can be done via an Atiyah–Hirzebruch spectral sequence, and the groups inject into their rationalization. As $B^5SL_1\mathbb{S}$ is rationally trivial, a $\mathbb{Z}$-equivariant nullhomotopy must exist and one can show it to be compatible with the original non-equivariant nullhomotopy.

**Remark 4.** It would have been good enough for our purposes to have an $\mathcal{E}_1$-$MU^\psi$-algebra structure in Theorem 1. But the proof of this would have been no easier since $B^3BU$ does not have the nice even cell structure of $B^4BU \simeq BU\langle 6\rangle$.

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# Lichtenbaum–Quillen for truncated Brown–Peterson spectra

John Rognes

Let $p$ be any prime and $n \geq 0$ an integer. Recall from Basterra–Mandell [2] that $BP$ is a retract of $MU_{(p)}$ in $\mathbb{E}_4$-rings. Following Hahn–Wilson [3, Thm. A], let

$$R := BP\langle n\rangle$$

be an $\mathbb{E}_3$-$BP$-algebra such that the composite ring homomorphism

$$\mathbb{Z}_{(p)}[v_1, \dots, v_n] \subset BP_* \to R_*$$

is an isomorphism. Let $C_{p^k}$ denote the subgroup of $\mathbb{T}$ of order $p^k$ when $0 \le k < \infty$, and $\mathbb{T}$ itself when $k = \infty$.

The topological Hochschild homology spectrum $THH(R)$ is a cyclotomic $\mathbb{E}_2$-$THH(BP)$-algebra, with ($p$-)cyclotomic structure map

$$\varphi\colon THH(R) \longrightarrow THH(R)^{tC_p}\,,$$

and canonical maps

$$\mathrm{can}\colon THH(R)^{hC_{p^k}} \longrightarrow THH(R)^{tC_{p^k}}$$

for $0 \le k \le \infty$, all compatible with the (residual) $\mathbb{T}$-actions.

**Theorem 1** (Segal conjecture, [3, Thm. C, Thm. 4.0.1])**.** *Let $U$ be any type $\ge n+1$ finite $p$-local spectrum. The cyclotomic structure map $U \otimes \varphi$ is truncated, i.e., induces an isomorphism*

$$U_*\varphi\colon U_*THH(R) \xrightarrow{\cong} U_*THH(R)^{tC_p}$$

*in all sufficiently large degrees $* \gg 0$.*

**Proposition 2** ([3, Prop. 6.2.1])**.** *There is a finite $p$-local $\mathbb{E}_1$-ring $U$ with a non-nilpotent central $v_{n+1}$-element $v \in U_*$ of degree $|v| = (2p^{n+1} - 2)e$, such that*

*(1) $v$ has Adams filtration $e$;*
*(2) $U \otimes R$ splits as an $R$-module as a finite sum of suspensions of $\mathbb{F}_p$;*
*(3) the homomorphism $U_*BP \to U_*R$ is surjective.*

The cofiber $U/v$ is a type $n+2$ finite $p$-local spectrum.

**Theorem 3** (Canonical vanishing, [3, Thm. D, Thm. 6.3.1])**.** *There are $U$ and $v$ as above, and an integer $d$, such that for each $0 \le k \le \infty$ the canonical homomorphism*

$$(U/v)_*\mathrm{can}\colon (U/v)_*THH(R)^{hC_{p^k}} \xrightarrow{0} (U/v)_*THH(R)^{tC_{p^k}}$$

*is zero whenever $* \ge d$.*

The Segal conjecture and canonical vanishing together imply cyclotomic boundedness.

**Corollary 4** (Bounded $TR$, [3, Thm. G, Thm. 3.3.2(f)])**.** *For each type $n+2$ finite $p$-local spectrum $V$, the graded abelian group $V_*TR(R)$ is bounded.*

This conclusion is equivalent to saying that $V \otimes THH(R)$ is bounded in the cyclotomic $t$-structure, by Antieau–Nikolaus [1, Thm. 9].

The relative topological Hochschild homology

$$THH(R/BP) = R \otimes_{R \otimes_{BP} R^{op}} R$$

is an $\mathbb{E}_2$-$BP$-algebra with $\mathbb{T}$-action, with homotopy fixed points $TC^-(R/BP) = THH(R/BP)^{h\mathbb{T}}$. Letting $v_{n+1}$ be the lowest-degree generator of

$$(v_{n+1}, v_{n+2}, \dots) = \ker(BP_* \to R_*)\,,$$

its suspension $\sigma v_{n+1} \in [v_{n+1}]$ is the lowest-degree generator of

$$\ker(\pi_*(R \otimes_{BP} R^{op}) \to R_*)\,,$$

and its double suspension $\sigma^2 v_{n+1} \in [\sigma v_{n+1}]$ is the lowest-degree generator of

$$\ker(\pi_* THH(R/BP) \to R_*)\,.$$

**Theorem 5** (Polynomial THH, [3, Thm. E, Thm. 2.5.4])**.** *There is an isomorphism of even $R_*$-algebras*

$$\pi_* THH(R/BP) \cong R_*[\gamma_{p^i}\sigma^2 v_{n+1} \mid i \geq 0]\,,$$

*with lowest-degree generator $\sigma^2 v_{n+1}$ in degree $2p^{n+1}$.*

**Theorem 6** (Detection, [3, Thm. F, Thm. 5.0.1])**.** *There is an isomorphism of even $R_*$-algebras*

$$\pi_* TC^-(R/BP) \cong \pi_* THH(R/BP)\,[[t]]$$

*with $|t| = -2$. The unit map $\iota\colon BP \to TC^-(R/BP)$ takes $v_{n+1}$ to $t \cdot \sigma^2 v_{n+1}$.*

The $\mathbb{E}_2$-ring maps

$$TC(R) \xrightarrow{\pi} TC^-(R) \longrightarrow TC^-(R/BP)$$

lead to the following variant of [3, Thm. B], where we may assume $T(n+1) = v^{-1}U$.

**Corollary 7.** *Multiplication by $v$ acts non-nilpotently on $U_*TC^-(R/BP)$, so $T(n+1)_*TC^-(R/BP) \neq 0$ and $T(n+1)_*TC(R) \neq 0$.*

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## THH of cochains on the circle

Neil P. Strickland

Given a space $X$ we write $D_+(X)$ for the mapping spectrum from $X$ to the sphere spectrum $S$. This might also be denoted by $F(X_+, S)$ or $S^X$, and called the spherical cochain ring for $X$. The spectrum $A = D_+(S^1) = D_+(B\mathbb{Z}) = S^{B\mathbb{Z}}$ can also be regarded as the homotopy fixed point spectrum $S^{h\mathbb{Z}}$ for the trivial action of $\mathbb{Z}$ on $S$. Let $R = \mathrm{THH}(A)$ be the topological Hochschild homology spectrum of $A$. If $B$ is any other $\mathbb{Z}$-equivariant $A_\infty$ ring spectrum then there is a natural map $R \to \mathrm{THH}(B^{h\mathbb{Z}})$; because of this, $R$ (or its $p$-adic completion) plays a role in the disproof of the Telescope Conjecture. In this note we discuss some relevant structure of $R$.

As a general feature of THH, the spectrum $R$ has an action of the circle. We call this the *outer circle* and denote it by $\mathbb{T}$; it will be convenient to distinguish it notationally from the circle $S^1$ in the definition of $A$, which we call the *inner circle*. Both circles have group structures which we write additively.

Because $A$ is commutative, the spectrum $R$ also has a commutative ring structure with the following universal property. For each $a \in \mathbb{T}$ we have a ring map $i_a : A \to R$, depending continuously on $a$. Moreover, if we have a continuous family of ring maps $f_a$ from $A$ to another commutative ring $B$, there is a unique $f : R \to B$ with $fi_a = f_a$ for all $a$. In other words, $R$ is the $\mathbb{T}$-fold coproduct of $A$.

Using this we can construct many ring endomorphisms of $R$. Put

$$\Gamma = \mathrm{End}(\mathbb{T}) \times \mathrm{End}(S^1) \times \mathrm{Hom}(\mathbb{T}, S^1) \times \mathbb{T} \times S^1 \simeq \mathbb{Z}^3 \times (S^1)^2.$$

For $p \in \mathrm{End}(S^1)$ and $b \in S^1$ we define $\alpha_{p,b} : S^1 \to S^1$ by $\alpha_{p,b}(t) = pt + b$. For $(n, p, q, a, b) \in \Gamma$ we define $\gamma(n, p, q, a, b) : R \to R$ to be the unique ring map such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "\alpha^*_{p,qs+b}"] \arrow[d, "i_s"'] & A \arrow[d, "i_{ns+a}"] \\ R \arrow[r, "\gamma(n,p,q,a,b)^*"'] & R \end{tikzcd}$$

One can make $\Gamma$ into a monoid in such a way that this gives a monoid action. The Dehn twist, which plays an important part the disproof of the Telescope Conjecture, is closely related to the action of $\mathrm{Hom}(\mathbb{T}, S^1) \subset \Gamma$. The cyclotomic structure of $R$ is related to the action of $p.1_{\mathbb{T}} \in \mathrm{End}(\mathbb{T})$ and the translation action of $C_p = \ker(p.1_{\mathbb{T}})$, both of which are part of the $\Gamma$-action. Most other constructions mentioned here are equivariant for appropriate actions of $\Gamma$. In particular, $\Gamma$ acts on the space $\mathrm{Hom}(\mathbb{T}, S^1) \times S^1$. The group $\mathbb{T}$ is a submonoid of $\Gamma$, and $\mathrm{Hom}(\mathbb{T}, S^1) \times S^1$ decomposes $\mathbb{T}$-equivariantly as $S^1 \amalg \coprod_{q \neq 0} \mathbb{T}/\ker(q)$ (where $q$ runs over $\mathrm{Hom}(\mathbb{T}, S^1)$). There is a canonical $\Gamma$-equivariant ring map $\phi$ from $R$ to the ring

$$\widehat{R} = D_+(\mathrm{Hom}(\mathbb{T}, S^1) \times S^1) = A \times \prod_{q \neq 0} D_+(\mathbb{T}/\ker(q)).$$

It turns out that $\widehat{R}$ is a kind of completion: the relationship between $R$ and $\widehat{R}$ is like the relationship between an infinite direct sum and the corresponding infinite product. There are various cyclotomic constructions that can be done easily for $\widehat{R}$, and our problem is to decomplete the answer to get corresponding results for $R$.

There is a Hochschild filtration $A = F_0R \leq F_1R \leq \cdots \leq R$ of $R$, in which $F_iR.F_jR \leq F_{i+j}R$ and $F_iR/F_{i-1}R \simeq A \otimes \Sigma^i(A/S)^{\otimes i}$ but $A/S = S^{-1}$ so $F_iR/F_{i-1}R \simeq A$. Using this we see that $F_iR \simeq \bigoplus_{j \leq i} A$ as an $A$-module, and so $R \simeq \bigoplus_{j \in \mathbb{N}} A$. We can understand the ring structure of the homotopy by comparison with the ring

$$\pi_*(\widehat{R}) = \mathrm{Map}(\mathrm{Hom}(\mathbb{T}, S^1), \pi_*(S)\langle\epsilon\rangle) \simeq \mathrm{Map}(\mathbb{Z}, \mathbb{Z}) \otimes \pi_*(S)\langle\epsilon\rangle.$$

(Here $\epsilon$ is an exterior generator in degree $-1$ corresponding to the generator of $H_{-1}(A)$.) Let $NP < \mathrm{Map}(\mathbb{Z}, \mathbb{Z})$ be the ring $\mathbb{Q}[x] \cap \mathrm{Map}(\mathbb{Z}, \mathbb{Z})$ of numerical polynomials, which is spanned by the functions $b_k(x) = \binom{x}{k}$. It can be shown that the map $\phi\colon R \to \widehat{R}$ identifies $\pi_*(R)$ with $NP \otimes \pi_*(S)\langle\epsilon\rangle$. (The ring $NP$ also appears in algebraic topology as $KU_0(BS^1)$, and one can show that there is an isomorphism $KU \otimes R \simeq KU \otimes (S^1 \times BS^1)$ of $KU$-algebras that is compatible with this.)

We now $p$-complete everywhere. It is a standard fact that the $p$-completion of $NP$ can be identified with the ring of $p$-adically continuous functions from $\mathbb{Z}$ to $\mathbb{Z}_p$. When we perform cyclotomic constructions on the factors $F((\mathbb{T}/\ker(q))_+, S_p)$ in $\widehat{R}_p$, many features depend only on the $p$-adic valuation of the element $q \in \mathrm{Hom}(\mathbb{T}, S^1) \simeq \mathbb{Z}$. Because the set $p^v\mathbb{Z}_p^\times$ is open and closed in $\mathbb{Z}_p$, its characteristic function is continuous, and so comes from an idempotent element on $\pi_0(R_p)$, which we can use to split $R_p$ as a product. We can also do the same (more obviously) for $\widehat{R}_p$. The fact that some features are constant on $p^v\mathbb{Z}_p^\times$ makes it meaningful to ask whether other features vary continuously on that set. The main conclusion is that the cyclotomic invariants of $R$ are the $p$-adically continuous parts of the corresponding invariants for $\widehat{R}$. Using this we can compute enough about the cyclotomic invariants of $R$ to see that the sphere $S$ lies in the thick subcategory of cyclotomic spectra generated by the fibre of the canonical map $R \to A$. This is the key fact about $R$ needed elsewhere in the disproof of the telescope conjecture.

In more detail, the results can be described as follows. We write $C_m$ for the the cyclic subgroup of order $p$ in $\mathbb{T}$, and $\zeta$ for any map induced by the map $p.1_{\mathbb{T}}\colon \mathbb{T} \to \mathbb{T}/C \to \mathbb{T}$. We then have

$$\begin{aligned}
\widehat{R} &= D_+(S^1) \times \prod_{q\neq 0} D_+(\mathbb{T}/\ker(q)) \\
\widehat{R}^{h\mathbb{T}} &= D_+(B\mathbb{T} \times S^1) \times \prod_{q\neq 0} D_+(B(\ker(q))) \\
\widehat{R}^{hC_p} &= \prod_{p|q} D_+(BC \times S^1) \times \prod_{p\nmid q} D_+(\mathbb{T}/\ker(pq)) \\
\widehat{R}^{tC_p} &= D_+(S^1)^\wedge_p \times \prod_{p|q\neq 0} D_+(\mathbb{T}/\ker(q))^\wedge_p \simeq \widehat{R}^\wedge_p \text{ via } \zeta \\
(\widehat{R}^{tC_p})^{h\mathbb{T}/C} &= D_+(B(\mathbb{T}/C_p) \times S^1)^\wedge_p \times \prod_{p|q\neq 0} D_+(B(\ker(q)/C_p))^\wedge_p \simeq (\widehat{R}^\wedge_p)^{h\mathbb{T}} \text{ via } \zeta
\end{aligned}$$

Let $U_v$ be the set of elements of $p$-adic valuation $v$ in $\mathrm{Hom}(\mathbb{T}, S^1)$. The first of the above isomorphisms gives

$$\pi_0(\widehat{R}^\wedge_p) = \pi_0(D_+(S^1)^\wedge_p) \times \prod_{v\geq 0} \mathrm{Map}(U_v, \pi_0(D_+(\mathbb{T}/C_{p^v}))).$$

We will write

$$\pi_0(R^\wedge_p) \sim \pi_0(D_+(S^1)^\wedge_p) \times \prod_{v\geq 0} C(U_v, \pi_0(D_+(\mathbb{T}/C_{p^v}))).$$

Here $C(U_v, -)$ refers to functions that are continuous with respect to the $p$-adic topology on $U_v$, and we write $\sim$ rather than $=$ because $\pi_0(R^\wedge_p)$ is in fact the subring of the product defined by a certain asymptotic condition as $v \to \infty$. The rings $\pi_*(R^{h\mathbb{T}}$, $\pi_*(R^{hC_p})$, $\pi_*(R^{tC_p})$ and $\pi_*((R^{tC_p})^{h\mathbb{T}/C_p})$ can be described in similar terms.

## Locally unipotent $\mathbb{Z}$-actions

Liam Keenan

### 1. Why locally unipotent $\mathbb{Z}$-actions?

One of the basic technical notions used in Burklund–Hahn–Levy–Schlank's disproof of the telescope conjecture is that of a *locally unipotent $\mathbb{Z}$-action.* The telescope conjecture, if true, predicts that $T(n+1)$-local topological cyclic homology satisfies descent for certain *covers* of the form $R^{h\mathbb{Z}} \to R$, where satisfying descent is equivalent to the coassembly map

$$T(n+1) \otimes \mathrm{TC}(R^{h\mathbb{Z}}) \to T(n+1) \otimes \mathrm{TC}(R)^{h\mathbb{Z}} \tag{1}$$

being an equivalence. The overall strategy of the disproof is to produce one of these coverings for which the map (1) is not an equivalence. By a relatively direct analysis ([2, Section 3.4]) combined with deep computations of Hahn–Wilson [4], the map (1) fails to be an equivalence for $R = \mathrm{BP}\langle n\rangle$ with the trivial $\mathbb{Z}$-action **but** the map $\mathrm{BP}\langle n\rangle^{B\mathbb{Z}} \to \mathrm{BP}\langle n\rangle$ is not a covering, so we have not yet produced a counterexample. Therefore, if it was possible to "trivialize" the $\mathbb{Z}$-action for a suitable covering of $\mathrm{BP}\langle n\rangle$, the disproof would be well under way. Enter locally unipotent $\mathbb{Z}$-actions.

**Slogan:** Locally unipotent $\mathbb{Z}$-actions on finite objects can be trivialized up to rescaling the $\mathbb{Z}$-action.

### 2. Definition and basic properties

Throughout, we work with presentable stable categories, $C$, and view objects with $\mathbb{Z}$-action as diagrams of the form $X : B\mathbb{Z} \to C$. We will often speak of such a diagram in terms of an object $X \in C$ and an automorphism $\psi : X \xrightarrow{\sim} X$, obtained from $1 \in \mathbb{Z} \to \mathrm{Map}_C(X, X)$.

**Definition 1.** Let $C$ be a presentable stable category. We let $C^{B\mathbb{Z},u}$ denote the localizing subcategory of $C^{B\mathbb{Z}}$ generated by the image of $(-)^{\mathrm{triv}} : C \to C^{B\mathbb{Z}}$. The we denote the inclusion $C^{B\mathbb{Z},u} \to C^{B\mathbb{Z}}$ by $\iota$.

**Example 2.** Certainly, $\mathbb{S}$ with the trivial $\mathbb{Z}$-action is locally unipotent. Given any $\mathbb{Z}$-action $\psi : X \xrightarrow{\sim} X$ where $\psi - 1$ is an equivalence, this action will fail to be locally unipotent; for instance, if $X$ is a $\mathbb{Q}$-module then multiplication by a rational number on $X$ is not locally unipotent.

**Remark 3.** By the adjoint functor theorem, $\iota$ admits a right adjoint $(-)^u$ : $C^{B\mathbb{Z}} \to C^{B\mathbb{Z},u}$. When $C$ is additionally presentably symmetric monoidal, then $\iota$ is symmetric monoidal and $(-)^u$ is lax symmetric monoidal.

In nice cases, it is possible to give a characterization of locally unipotent $\mathbb{Z}$-actions in line with the familiar notion of local unipotence from algebra.

**Example 4.** ([2, A.25]) For $M$ an abelian group with $\mathbb{Z}$-action, $M$ is locally unipotent if and only if for all $m \in M$, we have that $(\psi - 1)^{\circ N}(m) = 0$ for some $N \gg 0$.

**Example 5.** ([2, A.27]) If $C$ has a compact generator, $V$, then $X \in C^{B\mathbb{Z},u}$ if and only if the $\mathbb{Z}$-action on the homotopy groups $\pi_0 \mathrm{Map}_C(\Sigma^k V, X)$ is locally unipotent for all $k$. Consequently, the $\mathbb{Z}$-action on a spectrum, $X$, is locally unipotent if and only if the $\mathbb{Z}$-action on $\pi_* X$ is locally unipotent.

**Example 6.** ([2, Theorem 5.4]) Burklund–Hahn–Levy–Schlank construct a $\mathbb{Z}$-action on $\mathrm{BP}\langle n\rangle_p^\wedge$ by a certain Adams operation $\Psi^{\ell_p}$; let $\ell_p = 3$ if $p = 2$ and $\ell_p = 2$ if $p > 2$. For any $k \geq 0$, it follows from the construction of these Adams operations ([2, Lemma 5.30, Corollary 5.31]) that

$$\Psi^{\ell_p} : \pi_{2k}\mathrm{BP}\langle n\rangle_p^\wedge \to \pi_{2k}\mathrm{BP}\langle n\rangle_p^\wedge$$

is multiplication by $\ell_p^k$. However, as $k$ is always divisible by $p-1$, Examples 4 and 5 show that this action is locally unipotent.

It is profitable to study locally unipotent actions in terms of categories of modules. Let $\mathbb{S}[t^{\pm 1}] = \mathbb{S}[\mathbb{Z}]$, with $t \in \pi_0(\mathbb{S}[t^{\pm 1}]) = \mathbb{Z}[t^{\pm 1}]$. Clearly, $\mathbb{S}[t^{\pm 1}]$ determines a spectrum with $\mathbb{Z}$-action, and by a standard Morita theory argument there is a symmetric monoidal identification

$$\mathrm{Sp}^{B\mathbb{Z}} \simeq \mathrm{Mod}(\mathbb{S}[t^{\pm 1}]),$$

given by $X \mapsto \mathrm{map}_{\mathbb{Z}}(\mathbb{S}[t^{\pm 1}], X)$; importantly, we use the symmetric monoidal structure coming from the fact that $\mathbb{S}[t^{\pm 1}]$ is a bicommutative bialgebra in spectra. Under this identification, the full subcategory $\mathrm{Sp}^{B\mathbb{Z},u}$ can be identified with the full subcategory of $(t-1)$-nilpotent $\mathbb{S}[t^{\pm 1}]$-modules, which has orthogonal complement given by modules over $\mathbb{S}[t^{\pm 1}][(1-t)^{-1}]$. These subcategories form a recollement of $\mathrm{Mod}(\mathbb{S}[t^{\pm 1}])$, and consequently for any $\psi : X \xrightarrow{\sim} X$, we have a natural "gluing" cofiber sequence

$$\iota(X^u) \to X \to X[(\psi - 1)^{-1}].$$

More generally, by tensoring with a presentable stable category $C$, we obtain a similar recollement for $C^{B\mathbb{Z}}$ as well as a gluing cofiber sequence, which shows that $X^u \simeq 0$ if and only if $\psi - 1$ is an equivalence [2, A.22, A.24].

**Lemma 7.** [2, Lemma A.21] *The functor $(-)^{h\mathbb{Z}} : C^{B\mathbb{Z},u} \to C$ induces a symmetric monoidal equivalence*

$$C^{B\mathbb{Z},u} \xrightarrow{\sim} \mathrm{Mod}(C; \mathbb{S}^{B\mathbb{Z}}),$$

*compatible with restriction to any $n\mathbb{Z} \subseteq \mathbb{Z}$. Consequently, the adjunction $(\iota \dashv (-)^u)$ can be expressed as*

$$\mathbb{S} \otimes_{\mathbb{S}^{B\mathbb{Z}}} (-) : \mathrm{Mod}(C; \mathbb{S}^{B\mathbb{Z}}) \rightleftharpoons \mathrm{Mod}(C; \mathbb{S}[\mathbb{Z}]) : \mathrm{map}_{\mathbb{S}[\mathbb{Z}]}(\mathbb{1}_C, -),$$

*where $\mathbb{S}$ is viewed as a $(\mathbb{S}[\mathbb{Z}], \mathbb{S}^{B\mathbb{Z}})$-bimodule.*

*Proof.* Using the Lurie tensor product as above, we can reduce to the universal case of $C = \mathrm{Sp}$, where the claim follows from [1, A.4] combined with the observation that $(-)^{h\mathbb{Z}} : \mathrm{Sp}^{B\mathbb{Z},u} \to \mathrm{Sp}$ is conservative. $\square$

## 3. Trivializing locally unipotent actions

We now specialize to the case where $C$ is a stable, $p$-complete, presentable category, e.g. $C = \mathrm{Sp}_p^\wedge$, the category of $p$-complete spectra obtained by Bousfield localization at $\mathbb{S}/p$. Throughout, we will often leave the prime $p$ implicit and we hope this does not cause confusion for the reader.

In the $p$-complete setting, the local unipotence of a $\mathbb{Z}$-action $\psi : X \xrightarrow{\sim} X$ is equivalent to the local unipotence of the induced $p\mathbb{Z}$-action on $X$; this follows from the observation that in the $p$-complete setting inverting $(\psi - 1)^p$ is the same as inverting $\psi^p - 1$. This observation will allow us to compute the underlying object of $\iota X^u$.

**Lemma 8.** [2, Lemma A.32] *Let $C$ be a stable $p$-complete, presentable category, and let $X \in C^{B\mathbb{Z}}$. There is a natural equivalence*

$$\mathrm{colim}_k X^{hp^k\mathbb{Z}} \xrightarrow{\sim} \iota X^u$$

*of objects in $C$. Consequently, $X$ is locally unipotent if and only if the map*

$$\mathrm{colim}_k X^{hp^k\mathbb{Z}} \to X^{he} = X,$$

*is an equivalence.*

*Proof.* By colimit interchange combined with the fact that $\mathrm{colim}_k \mathbb{S}^{Bp^k\mathbb{Z}} \simeq \mathbb{S}$, we have natural equivalences

$$\mathrm{colim}_k X^{hp^k\mathbb{Z}} \simeq \mathrm{colim_k} \mathbb{S} \otimes_{\mathrm{colim}_k \mathbb{S}^{Bp^k\mathbb{Z}}} X^{hp^k\mathbb{Z}} \simeq \mathrm{colim}_k \left( \mathbb{S} \otimes_{\mathbb{S}^{Bp^k\mathbb{Z}}} X^{hp^k\mathbb{Z}} \right) \simeq \iota X^u,$$

which follow from Lemma 7 and $\mathrm{res}^{p^k\mathbb{Z}}_{p^{k+1}\mathbb{Z}} \iota X^u \simeq \iota(\mathrm{res}^{p^k\mathbb{Z}}_{p^{k+1}\mathbb{Z}} X)^u$. The second claim follows easily from the first. $\square$

**Lemma 9.** *Let $C$ be a stable $p$-complete, presentable category.*

*(1)* [2, Lemma A.33] *If $X \in C$ is compact and $X$ has a locally unipotent $\mathbb{Z}$-action, then for $k \gg 0$ the action of $p^k\mathbb{Z}$ on $X$ is trivializable.*

*(2)* [2, Lemma A.34] *Assume $C$ is equipped with a t-structure. If $X \in C$ is bounded for the t-structure, compact in $C$, and has a locally unipotent $\mathbb{Z}$-action, then for $k \gg 0$ the action of $p^k\mathbb{Z}$ on $X$ is trivializable.*

*Proof.* We prove (1) and (2) follows mutatis mutandis after unraveling the definitions. By Lemma 8, we can choose $k \gg 0$ so that $X$ is a retract of $X^{hp^k\mathbb{Z}}$. This provides a nullhomotopy $\psi^{p^k} - \mathrm{id} \simeq 0$, whence the claim. $\square$

3.1. **A special example.** Consider the ring of continuous functions $C^0(\mathbb{Z}_p, \mathbb{F}_p)$, where $\mathbb{Z}_p$ carries the $p$-adic topology and $\mathbb{F}_p$ has the discrete topology, so that continuity is tantamount to local constancy. For brevity, we write $C^0(\mathbb{Z}_p)$ for this ring, and note that $\mathbb{Z}_p$, and thus $C^0(\mathbb{Z}_p)$, carries a natural $\mathbb{Z}_p$-action from $+1 : \mathbb{Z}_p \to \mathbb{Z}_p$. We denote these objects with their $\mathbb{Z}$-actions by $\overrightarrow{\mathbb{Z}_p}$, and $C^0(\overrightarrow{\mathbb{Z}_p})$, respectively. This ring is relevant in virtue of the identification of commutative algebras

$$\mathbb{W}C^0(\mathbb{Z}_p) \otimes \mathbb{S}^{B\mathbb{Z}} \xrightarrow{\sim} \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$$

from [2, Lemma 3.6]. We conclude this section by studying some of the basic structural features of $\mathbb{W}C^0(\overrightarrow{\mathbb{Z}_p})$ which see use in the proof of the cyclotomic asymptotic constancy theorem. Given $a \in \mathbb{Z}_p$, evaluation at $a$ induces a ring homomorphism $C^0(\mathbb{Z}_p) \to \mathbb{F}_p$, and thus a map $\mathrm{ev}_a : \mathbb{W}C^0(\mathbb{Z}_p) \to \mathbb{W}(\mathbb{F}_p) \simeq \mathbb{S}_p^\wedge$; we denote base-change along this map by $(-)|_a$.

**Proposition 10.** [2, Proposition A.35] *Let $C$ be a stable, $p$-complete, and presentably symmetric monoidal category.*

*(1) The $\mathbb{Z}$-action on $\mathbb{W}C^0(\overrightarrow{\mathbb{Z}_p})$ is locally unipotent.*
*(2) The unit map $\mathbb{S} \to \mathbb{W}C^0(\overrightarrow{\mathbb{Z}_p})^{h\mathbb{Z}}$ is an equivalence.*
*(3) There is a symmetric monoidal equivalence of categories*

$$C \xrightarrow{\sim} \mathrm{Mod}(C^{B\mathbb{Z},u}; \mathbb{W}C^0(\overrightarrow{\mathbb{Z}_p})),$$

*given by $X \mapsto X \otimes \mathbb{W}C^0(\overrightarrow{\mathbb{Z}_p})$ with inverse $(-)^{h\mathbb{Z}}$.*[1]
*(4) For each $a \in \mathbb{Z}_p$, there is a natural equivalence of functors of the form $\mathrm{Mod}(\mathrm{Sp}^{B\mathbb{Z},u}; \mathbb{W}C^0(\mathbb{Z}_p)) \to \mathrm{Sp}$,*

$$(-)^{h\mathbb{Z}} \simeq (-)|_a \circ \mathrm{fgt}.$$

*Proof.* For (1), as $\mathbb{W}$ perserves colimits, we can write

$$\mathbb{W}C^0(\overrightarrow{\mathbb{Z}_p}) \simeq \mathrm{colim}_k \mathbb{W}C^0(\overrightarrow{\mathbb{Z}/p^k})$$

by the definition of locally constant functions on $\mathbb{Z}_p$. The $+1$-action on $\overrightarrow{\mathbb{Z}/p^k}$ is trivial after restriction to $p^k\mathbb{Z}$, so the colimit above is a colimit of locally unipotent objects. (2) follows by smashing the unit map $\mathbb{S} \to \mathbb{W}C^0(\overrightarrow{\mathbb{Z}_p})^{h\mathbb{Z}}$ with $\mathbb{F}_p$, and observing that

$$0 \to \mathbb{F}_p \to C^0(\mathbb{Z}_p) \xrightarrow{\psi-1} C^0(\mathbb{Z}_p) \to 0$$

is exact.[2] For (3) and (4) it suffices to consider the case $C = \mathrm{Sp}$. To see (3), first observe that $\mathbb{W}C^0(\overrightarrow{\mathbb{Z}_p}) \otimes (-)$ is valued in spectra with locally unipotent $\mathbb{Z}$-action. From (2) combined with the finiteness of $B\mathbb{Z}$, we see that the unit map

$$X \to (\mathbb{W}C^0(\overrightarrow{\mathbb{Z}_p}) \otimes X)^{h\mathbb{Z}}$$

[1] In particular, every $\mathbb{W}C^0(\overrightarrow{\mathbb{Z}_p})$-module in $C^{B\mathbb{Z},u}$ is free.

[2] If $f$ a locally constant and $f(x+1) = f(x)$ for all $x \in \mathbb{Z}_p$, we find $f$ must have been constant by translating neighborhoods.

is an equivalence, so we may conclude by the conservativity of $(-)^{h\mathbb{Z}}$ on locally unipotent objects. Finally, for (4), we have a chain of natural equivalences

$$\begin{aligned}(-)|_a = \mathbb{S} \otimes_{\mathbb{W}C^0(\mathbb{Z}_p),\mathrm{ev}_a} (-) &\simeq \mathbb{S} \otimes_{\mathbb{W}C^0(\mathbb{Z}_p),\mathrm{ev}_a} \mathbb{W}C^0(\mathbb{Z}_p) \otimes (-)^{h\mathbb{Z}} \\ &\simeq \mathbb{W}C^0(\mathbb{Z}_p)|_a \otimes (-)^{h\mathbb{Z}} \simeq (-)^{h\mathbb{Z}}\end{aligned}$$

where we have used (3). □

## 4. Interactions with THH

We conclude by discussing the interaction of THH and TC with locally unipotent $\mathbb{Z}$-actions, with everything implicitly $p$-complete. Additionally we will use the notation $W_k = \mathrm{THH}(\mathbb{S}^{Bp^k\mathbb{Z}})$, setting $W = W_0$, and for $X$ a $W_k$-module, we define $X|_0 = \mathbb{S}^{Bp^k\mathbb{Z}} \otimes_{W_k} X$.

**Lemma 11.** [2, Lemma 4.7] *Let $R \in \mathrm{Alg}(\mathrm{Sp}^{B\mathbb{Z},u})$. Then, the induced $\mathbb{Z}$-action on* $\mathrm{THH}(R)$ *is locally unipotent, and if $R$ is connective, the induced action on* $\mathrm{TC}(R)$ *is locally unipotent as well.*

*Proof.* As spectra with locally unipotent $\mathbb{Z}$-action are closed under tensor product and colimits in $\mathrm{Sp}^{B\mathbb{Z}}$, the claim follows for the underlying spectrum of $\mathrm{THH}(R)$, but the forgetful functor $\mathrm{CycSp} \to \mathrm{Sp}$ is conservative so it detects local unipotence. By $p$-completeness, we can check the claim for TC modulo $p$. By [3, Theorem 2.7], $\mathrm{TC}/p$ preserves small colimits of connective cyclotomic spectra, and colimit preserving functors send locally unipotent $\mathbb{Z}$-actions to locally unipotent $\mathbb{Z}$-actions. □

**Lemma 12.** [2, Lemma 4.8] *Let $R \in \mathrm{Alg}(\mathrm{Sp}^{B\mathbb{Z},u})$. The natural map*

$$W_k \otimes_W \mathrm{THH}(R^{h\mathbb{Z}}) \to \mathrm{THH}(R^{hp^k\mathbb{Z}}),$$

*is an equivalence of $W_k$-modules.*

*Proof.* The map above is obtained by applying THH to the map

$$\mathbb{S}^{Bp^k\mathbb{Z}} \otimes_{\mathbb{S}^{B\mathbb{Z}}} R^{h\mathbb{Z}} \to R^{hp^k\mathbb{Z}},$$

which is an equivalence by the compatibility specified in Lemma 7. □

**Lemma 13.** [2, Lemma 4.9] *Let $R \in \mathrm{Alg}(\mathrm{Sp}^{B\mathbb{Z},u})$. Then, there is a natural factorization of the coassembly map*

$$\mathrm{THH}(R^{h\mathbb{Z}}) \to \mathrm{THH}(R^{h\mathbb{Z}})|_0 \xrightarrow{\sim} \mathrm{THH}(R)^{h\mathbb{Z}}.$$

*In particular, if $R$ has the trivial $\mathbb{Z}$-action, then the coassembly map has the form*

$$\mathbb{W}C^0(\mathbb{Z}_p) \otimes \mathbb{S}^{B\mathbb{Z}} \otimes \mathrm{THH}(R) \to \mathbb{S}^{B\mathbb{Z}} \otimes \mathrm{THH}(R).$$

*Proof.* Using the $\mathbb{S}^{B\mathbb{Z}}$-module structure on $\mathrm{THH}(R)^{h\mathbb{Z}}$, we have a natural factorization of the coassembly map

$$\mathrm{THH}(R^{h\mathbb{Z}}) \to \mathbb{S}^{B\mathbb{Z}} \otimes_W \mathrm{THH}(R^{h\mathbb{Z}}) \to \mathrm{THH}(R)^{h\mathbb{Z}},$$

and it remains to prove that the second map is an equivalence. As the $\mathbb{Z}$-action on $\mathrm{THH}(R)$ is locally unipotent, we can do so after applying $\mathbb{S} \otimes_{\mathbb{S}^{B\mathbb{Z}}} (-)$, and the same hypothesis implies that

$$\mathrm{THH}(R) \simeq \mathbb{S} \otimes_W \mathrm{THH}(R^{h\mathbb{Z}}) \to \mathbb{S} \otimes_{\mathbb{S}^{B\mathbb{Z}}} \mathrm{THH}(R)^{h\mathbb{Z}} \simeq \mathrm{THH}(R),$$

is an equivalence, whence the claim. □

These ideas and properties are leveraged in the proof of [2, Theorem 4.4], which asserts that when $R$ is a suitable ring spectrum with locally unipotent $\mathbb{Z}$-action satisfying very strong chromatic finiteness conditions, then we can choose $k \gg 0$ so that for $V$ a finite spectrum of type $n+2$, there is an equivalence of cyclotomic spectra

$$V \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}}) \simeq V \otimes \mathrm{THH}(R^{Bp^k\mathbb{Z}}),$$

i.e., after rescaling, the functor $V \otimes \mathrm{THH}(-)$ "trivializes" the $\mathbb{Z}$-action on $R$.

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## Almost compactness of cyclotomic spectra

Christoph Winges

As indicated in the preceding talk, almost compactness is a sufficient finiteness condition which guarantees that a locally unipotent $\mathbb{Z}$-action will become trivial upon restriction to $p^k\mathbb{Z}$ for sufficiently large $k$. In this talk, we will review the notion of almost compactness and discuss characterisations of almost objects in the category of spectra and cyclotomic spectra, following [3, Section 2.4 & Appendix A.2].

Almost compactness is essentally a weaking of compactness in which we do not require that a corepresented functor commutes with all filtered colimits, but only with filtered colimits of uniformly bounded objects. More precisely, let $\mathcal{C}$ be a cocomplete, stable $\infty$-category with a t-structure. The main examples to keep in mind are the categories of spectra Sp (with the Postnikov t-structure), of spectra with a $\mathbb{T}$-action $\mathrm{Sp}^{B\mathbb{T}}$ (with the pointwise Postnikov t-structure), and of cyclotomic spectra CycSp with the cyclotomic t-structure of [1]. Recall that this always refers to the $p$-complete version of these gadgets, and that cyclotomic spectra are $p$-typical cyclotomic spectra.

**Definition.** [3, Definition A.6] An object $X \in \mathcal{C}$ is *almost compact* if $X$ is bounded below and for all $c \leq b$ and all filtered diagrams $F\colon I \to \mathcal{C}_{[c,b]}$ the comparison map

$$\operatorname*{colim}_I \operatorname{Map}(X, F) \to \operatorname{Map}(X, \operatorname*{colim}_I F) \tag{1}$$

is an equivalence.

The following observations are rather straightforward:

(1) If $X \in \mathcal{C}$ is bounded below and compact, then it is almost compact.
(2) In the definition of almost compactness, we may equivalently require that the analogue of map (1) for mapping spectra is an equivalence (in the category of not-necessarily-$p$-complete spectra).
(3) If the subcategory $\mathcal{C}_{\leq 0}$ of coconnective objects is closed under filtered colimits, an object $X \in \mathcal{C}$ is almost compact if and only if it is bounded below and $\tau_{\leq n}X$ is compact in $\mathcal{C}_{\leq n}$ for all $n$.
(4) The full subcategory of almost compact objects is closed under finite limits, finite colimits, and sequential colimits $X_0 \to X_1 \to \dots$ such that $\operatorname{cofib}(X_k \to X_{k+1}) \in \mathcal{C}_{\geq k+1}$. In particular, geometric realisations of almost compact, uniformly bounded below objects are almost compact.

This leads us to the folllowing characterisation of almost compactness in Sp.

**Proposition.** [3, Example A.7 & Lemma A.10]

(1) An object $X \in \mathrm{Sp}$ is almost compact if and only if it is bounded below and all its homotopy groups are finite $p$-groups.
(2) If $R$ is a connective ring spectrum such that $\pi_*(R/p)$ is degreewise finitely generated, and $V$ is any almost compact spectrum, then $V \otimes \mathrm{THH}(R)$ is almost compact.

*Proof.* If $X$ is bounded below and its homotopy groups are finite $p$-groups, one can choose a cell decomposition of $X$ whose filtration quotients are sums of shifts of Moore spectra $\mathbb{S}/p^k$. Applying the observation about sequential colimits, it follows that $X$ is almost compact.

Conversely, the bottom homotopy group of an almost compact spectrum must be finitely generated (and is therefore necessarily a finite $p$-group). By the previous step, Eilenberg–Mac Lane spectra on finite $p$-groups are almost compact, so the fibre of the first non-trivial Postnikov section of $X$ is almost compact. Now proceed by induction.

For the second assertion, one chooses a cell decomposition of $V$ as before. Since $R/p$ has finitely generated homotopy groups, the criterion for almost compactness implies that $R/p^k$ is almost compact for all $k$. The chosen cell decomposition of $V$ induces a cell decomposition for $V \otimes R$ whose filtration quotients are finite sums of shifts of $R/p^k$, so $V \otimes R$ is almost compact by the closure properties of almost compact objects. Using these closure properties again, it follows that $V \otimes \mathrm{THH}(R) \simeq |V \otimes R^{\otimes n}|$ is almost compact. □

**Theorem.** [3, Lemma 2.43 & Proposition 2.42] Let $(X, \phi)$ be a cyclotomic spectrum which is bounded below.

(1) If $(X, \phi)$ is almost compact, then $X \in \mathrm{Sp}$ is almost compact.
(2) If $X \in \mathrm{Sp}$ is almost compact and $p^m \simeq 0\colon X \to X$ for some $m$, then $(X, \phi)$ is almost compact.

*Proof.* Observe that an exact functor $F\colon \mathcal{C} \to \mathcal{D}$ preserves almost compact objects if it admits a right adjoint $G$ which preserves colimits and has *bounded t-amplitude*, meaning there exist $r$ and $s$ such that $G(\mathcal{D}_{\geq 0}) \subseteq \mathcal{C}_{\geq r}$ and $G(\mathcal{D}_{\leq 0}) \subseteq \mathcal{C}_{\leq s}$. We apply this criterion to the forgetful functor $\mathrm{CycSp} \to \mathrm{Sp}$.

Recall the cofibre sequence $\Sigma^\infty_+ \mathbb{T} \to \mathbb{S} \to \mathbb{S}^{(1)}$ in $\mathrm{Sp}^{B\mathbb{T}}$. Dualising this sequence, we obtain a cofibre sequence $\mathbb{S}^{-(1)} \to \mathbb{S} \to \mathbb{S}^{\mathbb{T}}$, where $\mathbb{S}^{\mathbb{T}}$ is the image of $\mathbb{S}$ under the right adjoint to the forgetful functor $\mathrm{Sp}^{B\mathbb{T}} \to \mathrm{Sp}$. This right adjoint is easily seen to preserve colimits, so it is given by $Y \mapsto \mathbb{S}^{\mathbb{T}} \otimes Y$. Observing that $(\mathbb{S}^{\mathbb{T}} \otimes Y)^{\mathrm{t}C_p} \simeq 0$ (since $\mathbb{T}$ admits the structure of a free $C_p$-CW-complex), we obtain a cyclotomic spectrum $(\mathbb{S}^{\mathbb{T}} \otimes Y, \mathbb{S}^{\mathbb{T}} \otimes Y \to 0)$ for each $Y \in \mathrm{Sp}$, and we have

$$\mathrm{map}_{\mathrm{CycSp}}((X, \phi), (\mathbb{S}^{\mathbb{T}} \otimes Y, 0)) \simeq \mathrm{map}_{\mathbb{T}}(X, \mathbb{S}^{\mathbb{T}} \otimes Y) \simeq \mathrm{map}(X, Y).$$

The resulting right adjoint $G$ preserves colimits. As $\mathbb{S}^{\mathbb{T}}$ is (-1)-connective, it follows that this right adjoint has t-amplitude $[-1, 0]$; for the upper bound, observe that

$$\mathrm{map}_{\mathrm{CycSp}}((X, \phi), G(Y)) \simeq \mathrm{map}_{\mathrm{Sp}}(X, Y)$$

is coconnective for all $(X, \phi) \in \mathrm{CycSp}_{\geq 0}$ because connectivity of cyclotomic spectra is detected on the underlying spectrum. This proves the first assertion.

For the second assertion, one first shows that $X$ is almost compact as an object of $\mathrm{Sp}^{B\mathbb{T}}$: for an appropriate filtered diagram $F$, the relevant comparison map can be identified with

$$\operatorname*{colim}_I \mathrm{map}_{\mathrm{Sp}}(X, F)^{\mathrm{h}\mathbb{T}} \to \mathrm{map}_{\mathrm{Sp}}(X, \operatorname*{colim}_I F)^{\mathrm{h}\mathbb{T}},$$

which is an equivalence because $(-)^{\mathrm{h}\mathbb{T}}$ preserves arbitrary sums of uniformly bounded above spectra.

Since $p^m \simeq 0$, the object $(X, \phi)$ is a retract of $(X, \phi) \otimes \mathrm{D}(\mathbb{S}/p^m)$, so we may consider the latter. The comparison map for $(X, \phi)$ can then be identified with

$$\operatorname*{colim}_I \mathrm{map}_{\mathrm{CycSp}}((X, \phi), \mathbb{S}/p^m \otimes F) \to \mathrm{map}_{\mathrm{CycSp}}((X, \phi), \operatorname*{colim}_I \mathbb{S}/p^m \otimes F).$$

By a theorem of Burklund [2], $\mathbb{S}/p^m$ admits an associative ring structure for $m \geq 3$, so we may assume without loss of generality that $F$ takes values in $\mathrm{Mod}_{\mathbb{S}/p^m}(\mathrm{CycSp}_{[c,b]})$.

We now claim that there exists for sufficiently large $e$ (which depends on $m$, $b$ and $c$) a map of fibre sequences

$$\begin{array}{ccccc}
\operatorname*{colim}_I \mathrm{map}_{\mathrm{CycSp}}((X, \phi), F) & \to & \operatorname*{colim}_I \mathrm{map}_{\mathbb{T}}(X, \tau_{\leq e} F) & \longrightarrow & \operatorname*{colim}_I \mathrm{map}_{\mathbb{T}}(X, \tau_{\leq e}(F^{\mathrm{t}C_p})) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{map}_{\mathrm{CycSp}}((X, \phi), \operatorname*{colim}_I F) & \to & \mathrm{map}_{\mathbb{T}}(X, \tau_{\leq e}(\operatorname*{colim}_I F)) & \to & \mathrm{map}_{\mathbb{T}}(X, \tau_{\leq e}((\operatorname*{colim}_I F)^{\mathrm{t}C_p}))
\end{array}$$

The middle vertical map is an equivalence since we have already shown $X$ to be almost compact in $\mathrm{Sp}^{B\mathbb{T}}$. We additionally claim that $(-)^{\mathrm{t}C_p}\colon \mathrm{CycSp} \to \mathrm{Sp}^{B\mathbb{T}}$ preserves filtered colimits of uniformly bounded objects, which implies that the right vertical map is also an equivalence.

This claim follows relatively easily after showing that the natural transformation $(-)^{\mathrm{t}C_p} \Rightarrow (\tau_{\leq e}-)^{\mathrm{t}C_p}$ induced by truncation admits a retraction, which is a consequence of a sufficiently natural version of strong canonical vanishing. This retraction is also used in the definition of the right-hand horizontal maps (see [3, Construction 2.40] for details). The proof of the existence of the horizontal fibre sequences additionally relies on the validity of the Segal conjecture for bounded cyclotomic spectra. □

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## Asymptotic constancy I: The Dehn twist

Maxime Ramzi

### Introduction

In order to study the failure of descent for $T(n+1)$-local TC, our goal was to relate $\mathrm{THH}(R^{h\mathbb{Z}})$ and $\mathrm{THH}(R^{B\mathbb{Z}})$ as cyclotomic spectra, for a suitably chosen ring $R$ with $\mathbb{Z}$-action, at least upon tensoring with a suitably chosen finite spectrum. We already observed that in our situation, $\mathrm{THH}(R)$ has a locally unipotent $\mathbb{Z}$-action and therefore its fixed points are well-approximated by $\mathrm{THH}(R)^{B\mathbb{Z}}$, at least upon restricting to some small enough $p^k\mathbb{Z} \subset \mathbb{Z}$.

The strategy at this point is to express $\mathrm{THH}(R^{h\mathbb{Z}})$ *in terms* of $\mathrm{THH}(R)^{h\mathbb{Z}}$, so that we can later compare it to $\mathrm{THH}(R)^{B\mathbb{Z}}$ and hence to $\mathrm{THH}(R^{B\mathbb{Z}})$. The goal of this talk was to go over Theorem 4.11 of the paper, which implements precisely this in larger generality, but with restricted structure, namely taking into account the cyclotomic Frobenius, but not the $S^1$-action.

The $S^1$-action (and the remaining details of this strategy) is then taken care of by the second "Asymptotic constancy" talk by Sil. In the statement of the theorem which we recall below, UAlg is the category informally described as tuples consisting of an algebra with locally unipotent $\mathbb{Z}$-action $R \in \mathrm{Alg}(\mathrm{Sp}^{B\mathbb{Z},u})$, an algebra which is dualizable as a ($p$-complete) spectrum $V \in \mathrm{Alg}(\mathrm{Sp}^{\diamond})$ and a trivialization of the $\mathbb{Z}$-action on $R \otimes V \in \mathrm{Alg}(\mathrm{Sp}^{B\mathbb{Z}})$.

**Theorem 1** (Theorem 4.11)**.** *There is a natural square of lax symmetric monoidal functors on* UAlg *of the form :*

$$\begin{tikzcd}
\mathrm{THH}(R)^{h\mathbb{Z}} \otimes V \otimes \mathbb{W}(C^0(\mathbb{Z}_p)) \arrow[r, "\eta_0"] \arrow[d, "\varphi_p"'] & \mathrm{THH}(R^{h\mathbb{Z}}) \otimes V \arrow[d, "\varphi_p"] \\
(\mathrm{THH}(R)^{h\mathbb{Z}})^{tC_p} \otimes V \otimes \mathbb{W}(C^0(\mathbb{Z}_p)) \arrow[r, "\eta_1"'] & \mathrm{THH}(R^{h\mathbb{Z}})^{tC_p} \otimes V
\end{tikzcd}$$

*where the vertical arrows have their canonical lax symmetric monoidal structure. In this square, $\eta_0$ is a natural equivalence, and $\eta_1$ is an equivalence on a pair $(R, V)$ as soon as the assembly map $\bigoplus_{\mathbb{N}}(V \otimes \mathrm{THH}(R)^{tC_p}) \to (\bigoplus_{\mathbb{N}} V \otimes \mathrm{THH}(R))^{tC_p}$ is an equivalence.*

Thus the failure of descent for the $\mathbb{Z}$-action on $R$ is essentially the same as the failure of descent for the trivial $\mathbb{Z}$-action on $\mathbb{S}$ (at least up to tensoring with $V$).

A reformulation of the theorem is that, as a module over $\mathbb{W}(C^0(\mathbb{Z}_p))$, $\mathrm{THH}(R^{h\mathbb{Z}})$ is free. From Liam's talk about local unipotence, we know that this amounts to finding a suitably compatible (locally unipotent) $\mathbb{Z}$-action on $\mathrm{THH}(R^{h\mathbb{Z}})$, as this action is exactly the datum of a preimage under $\mathbb{W}(C^0(\mathbb{Z}_p))$ (see Proposition A.35 in the paper). This is precisely what we construct, and this action is what is called "the Dehn twist" in the paper.

## 1. Recollections on $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$

There is an equivalence of commutative algebras $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}}) \simeq \mathbb{S}^{B\mathbb{Z}} \otimes \mathbb{W}(C^0(\mathbb{Z}_p))$, which makes the relevant cyclotomic spectra $\mathrm{THH}(R^{h\mathbb{Z}})$ (potentially tensored with a finite spectrum) modules over $\mathbb{W}(C^0(\mathbb{Z}_p))$. Furthermore, basechanging along evaluation at 0: $C^0(\mathbb{Z}_p) \to \mathbb{F}_p$ induces an equivalence $\mathrm{THH}(R^{h\mathbb{Z}}) \otimes_{\mathbb{W}C^0(\mathbb{Z}_p)} \mathbb{S} \simeq \mathrm{THH}(R)^{h\mathbb{Z}}$ - it follows that if $\mathrm{THH}(R^{h\mathbb{Z}})$ is free as a $\mathbb{W}(C^0(\mathbb{Z}_p))$-module, it is indeed free on $\mathrm{THH}(R)^{h\mathbb{Z}}$.

The idea is now to try and interpret the $+1$-action on $C^0(\mathbb{Z}_p)$ *in terms* of $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$, to ultimately define a compatible action on $\mathrm{THH}(R^{h\mathbb{Z}})$.

Part of the talk was spent on recollections concerning the structure of $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$ as a $\langle p\rangle$-polygonic spectrum.

## 2. Constructing the Dehn twist

To make it compatible with all the relevant structure, the Dehn twist is constructed at an "earlier" level. Recall that if $R \rightrightarrows S$ are two algebra maps, one can make $S$ into an $(R, R)$-bimodule by using the top map to act on the left, and the bottom map to act on the right. Making this precise provides a functor $\mathrm{Alg}(\mathrm{Sp})^{\mathrm{EQ}} \to \mathrm{BiMod}$, where EQ is the walking equalizer, that is, the 1-category pictorially described as $\bullet \rightrightarrows \bullet$.

Now we are interested in

$$\mathrm{THH}(R^{h\mathbb{Z}}) \otimes V \simeq \mathrm{THH}(R^{h\mathbb{Z}}, V \otimes R^{h\mathbb{Z}}) \simeq \mathrm{THH}(R^{h\mathbb{Z}}, (R \otimes V)^{h\mathbb{Z}})$$

where the action on $R \otimes V$ is given a trivialization, so the functor of interest factors as a composite

$$\mathrm{UAlg} \to \mathrm{Alg}(\mathrm{Sp})^{(B\mathbb{Z})^{\triangleright}} \to \mathrm{Alg}(\mathrm{Sp})^{\mathrm{EQ}} \to \mathrm{BiMod} \to \mathrm{PgSp}_{\langle p\rangle}$$

Where the first functor is $(R, V) \mapsto (R \to (R \otimes V))$, the second functor is $(R \to S) \mapsto (R^{h\mathbb{Z}} \rightrightarrows S^{B\mathbb{Z}})$, the third functor is the one described above, that is $(A \rightrightarrows B) \mapsto (A,_f B_g)$ and finally the last functor is $(R, M) \mapsto (\mathrm{THH}(R, M))$. The Dehn twist is a lift of the functor $\mathrm{Alg}(\mathrm{Sp})^{(B\mathbb{Z})^{\triangleright}} \to \mathrm{Alg}(\mathrm{Sp})^{\mathrm{EQ}}$ to $\mathrm{Alg}(\mathrm{Sp})^{\mathrm{EQ} \times B\mathbb{Z}}$, that is, it produces an automorphism of $R^{h\mathbb{Z}} \rightrightarrows S^{B\mathbb{Z}}$ - in fact, this is constructed in greater generality, that is, with Alg(Sp) replaced by any category with finite limits. The construction relies on an automorphism $\sigma_0$ of $\mathrm{EQ} \times B\mathbb{Z}$ which is compatible with the projections to EQ and $(B\mathbb{Z})^{\triangleright}$.

The second part of the talk was dedicated to describing this lift and its properties (namely local unipotence, and its value on the sphere with trivial $\mathbb{Z}$-action). Combining these with the description of $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$ recalled earlier, the talk concluded with a proof of Theorem 4.11.

## Asymptotic Constancy II

Sil Linskens

In this talk, which follows [1, Sections 4.3,4.4], we completed the proof of “Asymptotic cyclotomic constancy” and deduced the consequences relevant for the disproof of the telescope conjecture given in [1]. Let us first recall the relevant assumptions for asymptotic cyclotomic constancy, followed by the statement of the theorem.

**Convention 1.** *We fix $R \in \mathrm{Alg}_{\mathbb{E}_1 \otimes \mathbb{A}_2}(\mathrm{Sp})$ with locally unipotent $\mathbb{Z}$-action, which we further assume*

1. *is connective;*
2. *is of fp-type $n \geq -1$, i.e. $R \otimes U$ is $\pi$-finite, where $U$ is of type $n+1$;*
3. *satisfies the height $n$ Lichtenbaum–Quillen property, i.e.* $\mathrm{THH}(R) \otimes V$ *is bounded in the cyclotomic t-structure, where $V$ is of type $n+2$.*

*We also fix $U$ and $V$ finite spectra of type $n+1$ and $n+2$ respectively.*

The key example of $R$ which satisfies these assumptions is $\mathrm{BP}\langle n\rangle$, equipped with the Adams operations constructed in [1]. Given these conventions we may state the main theorem.

**Theorem 2** (Cyclotomic asymptotic constancy)**.** *There exists a $K \in \mathbb{N}$ such that for all $k > K$, there is an isomorphism*

$$V \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}}) \simeq V \otimes W_k \otimes \mathrm{THH}(R)$$

*of $W_k = \mathrm{THH}(\mathbb{S}^{hp^k\mathbb{Z}})$-modules in cyclotomic spectra.*

Crucially, this result tells us that for the purposes of understanding the height $n+1$ telescopically localized topological cyclic homology of $R^{hp^k\mathbb{Z}}$, it suffices to consider the case where $R$ has the trivial action. In this case one has much more control. For example, it is shown in [1] that the thick subcategory of the fiber of the coassembly map

$$L_{T(n+1)}\mathrm{TC}(R^{\mathrm{B}\mathbb{Z}}) \to L_{T(n+1)}\mathrm{TC}(R)^{\mathrm{B}\mathbb{Z}}$$

always contains $L_{T(n+1)}\mathrm{TC}(R)$. By [2], $L_{T(n+1)}\mathrm{TC}(\mathrm{BP}\langle n\rangle)$ is not zero, and so we conclude that the assembly map above cannot be an equivalence in this case. By Cyclotomic asymptotic constancy, this implies that the assembly map

$$L_{T(n+1)}\mathrm{TC}(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}}) \to L_{T(n+1)}\mathrm{TC}(R)^{hp^k\mathbb{Z}}$$

for $\mathrm{BP}\langle n\rangle$ with its action by Adams operations is also not an equivalence. This is the crucial fact which is used to disprove the telescope conjecture. Namely the paper proceeds by showing that the coassembly map above exhibits the target as the so called "cyclotomic completion" of the source. This implies that $L_{T(n+1)}\mathrm{TC}(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}})$ is not cyclotomically complete, and therefore also not $K(n+1)$-local.

Moving on, let us now discuss the proof of Theorem 2. Applying the Dehn twist, together with the fact that locally unipotent $\mathbb{Z}$-actions on almost compact objects are trivial after restricting to some $p^k\mathbb{Z}$, we conclude that there is an equivalence

$$V \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}}) \simeq V \otimes W_k \otimes \mathrm{THH}(R)$$

of $\mathbb{A}_2$-algebras in $\mathrm{Mod}_{W_k}(\mathrm{Sp})$. Therefore the crucial question is how to lift this equivalence to an equivalence of cyclotomic spectra. The crucial tool is the following result:

**Proposition 3.** *Let $X$ be a $p$-nilpotent $W$-module in* CycSp*, and let $X_j = W_j \otimes_W X$. Suppose that*

*(1) The $X_j$ are uniformally bounded;*
*(2) There is an isomorphism $X \simeq W \otimes X_\infty$ of $W$-modules in spectra;*
*(3) $X_\infty$ is almost compact as a cyclotomic spectrum.*

*Then there is an isomorphism*

$$X_j \simeq W_j \otimes X_\infty$$

*in* $\mathrm{Mod}_{W_j}(\mathrm{CycSp})$ *for $k$ sufficiently large.*

This proposition is not so difficult to prove. Simply put, the assumptions of uniformally bounded and almost compact allow one to obtain comparison maps $\phi_j\colon X_j \to W_j \otimes X_\infty$ for $j \gg 0$. One then computes on homotopy groups that for $j$ potentially even larger, the map $\phi_j$ is an equivalence. Here one uses that one knows the result "at infinity" by assumption (2), and the fact that the maps $\phi$ are locally constant in $C^0(\mathbb{Z}_l)$ to conclude that it must be true in some neighborhood of infinity.

We apply this result to $V \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}})$ for $k \gg 0$. Assumption (2) is precisely the equivalence $V \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}}) \simeq V \otimes W_k \otimes \mathrm{THH}(R)$ obtained before, while (3) is easily verified. Therefore the crucial fact which remains to be shown is that the cyclotomic spectra

$$X_j = W_j \otimes_{W_k} V \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}}) \simeq V \otimes \mathrm{THH}(R^{hp^{j+k}\mathbb{Z}})$$

are uniformally bounded. Here we apply the machinery developed in [1, Section 2], which allows us to prove the following proposition.

**Proposition 4.** *Let $m \geq m_p^{\mathbb{A}_2}$ and let $S$ be a $h\mathbb{A}_2$-ring in $W$-modules in cyclotomic spectra with $p^m = 0$. Suppose that*

*(1) $S_{|0} = S \otimes_W \mathbb{S}^{B\mathbb{Z}}$ is cyclotomically bounded in a range;*
*(2) There is an isomorphism*

$$\pi_* S \simeq \pi_0(\mathbb{W}C^0(\mathbb{Z}_p)) \otimes \pi_* S_{|0}$$

*of graded $\mathbb{A}_2$-rings such that the map $S \to S_{|0}$ is given by restriction to zero;*
*(3) $S$ satisfies* Segal($\leq b'$). *Then $S$ is $e$-truncated for some $e$ depending on $m$, $b'$ and the range of $S_{|0}$.*

Once again we apply this to $R = \mathrm{THH}(R^{hp^k\mathbb{Z}}) \otimes V$ for $k \gg 0$:

(1) follows immediately from the height $n$ LQ property;
(2) follows from the equivalence

$$V \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}}) \simeq V \otimes W_k \otimes \mathrm{THH}(R)$$

of spectra obtained before.
(3) follows from the fact that the equivalence above also identifies the frobenius on $V \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}})$ with that of $V \otimes W_k \otimes \mathrm{THH}(R)$. Because $\mathrm{THH}(R) \otimes V$ is bounded, it satisfies Segal($\leq b'$) for some $b'$. Therefore so does $V \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}})$.

As explained, this suffices to conclude the proof of cyclotomic asymptotic constancy.

## References

[1] *R. Burklund, J. Hahn, I. Levy* and *T. M. Schlank*, $K$-theoretic counterexamples to Ravenel's telescope conjecture, arXiv:2310.17459 (2023).
[2] *J. Hahn* and *D. Wilson*, Redshift and multiplication for truncated Brown-Peterson spectra, Ann. Math. (2) 196, No. 3, 1277–1351 (2022)

## Assembling the proof

Niko Naumann

We will finish the construction of a counter-example to the telescope conjecture at a prime $p$ at height $n + 1 \geq 2$ by showing that the coassembly map (known at this point not to be an isomorphism) is isomorphic to the cyclotomic completion map.

## Future Directions I

Ishan Levy

(joint work with Robert Burklund, Shachar Carmeli, Jeremy Hahn, Tomer Schlank, Lior Yanovski)

In this talk I explained some follow up work extracting some consequences of the failure of the telescope conjecture. The mechanism of the failure of the telescope conjecture that we showed was that at heights $\geq 2$, the $T(n)$-local category doesn't have Galois hyperdescent with respect to the cyclotomic extensions.

In this talk I explained how one can construct certain obstructions to hyperdescent with respect to $\mathbb{Z}_p$-Galois extensions. There is a universal example of a $\mathbb{Z}_p$-Galois extension in a presentably symmetric monoidal stable category, which is a version of $\mathbb{Z}_p$-equivariant spectra, constructed as additive sheaves of spectra on finite $\mathbb{Z}_p$-sets in the surjection topology. In this category, the fiber of the hypercompletion map can be explicitly analysed and can be seen to be an $\mathbb{N}$-indexed colimit of Picard elements $P_i$. In the case that the finite extensions making up the $\mathbb{Z}_p$-Galois extension are descendable, the maps between the $P_i$ are sent to nilpotent maps.

So if a $\mathbb{Z}_p$-extension isn't hypercomplete, this analysis shows that either most of the Picard elements are distinct from each other, or the nilradical of the endomorphism ring of the unit isn't nilpotent. In the case of the $T(n+1)$-local algebraic $K$-theory of $\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}}$, which form a non-hypercomplete $\mathbb{Z}_p$-extension in the $T(n+1)$-local category, the nilradical is nilpotent, so most of the $P_i$ are distinct. There are obstructions $\tau_i$ in $\pi_1$ of the unit which are obstructions to trivializing the $P_i$ so these must also be nonzero. A slightly refined argument shows that in the homotopy groups of a telescope that is a ring, these $\tau_i$ generated an infinitely generated subgroup of $\pi_1$.

As a consequence, the homotopy groups of telescopes of type $n$ finite complexes are never finitely generated for $n \geq 1$. This in turn implies that the average ranks of the homotopy groups of finite complexes diverges.

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