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Combinatorial Optimization

Organized by
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Abstract. Combinatorial optimization deals with optimization problems defined on polyhedral constraints or discrete structures such as graphs and networks. In the past thirty years the topic has developed into a rich mathematical discipline with many connections to other fields of mathematics such as combinatorics, group theory, geometry of numbers, convex analysis or real algebraic geometry. It also has strong ties to theoretical computer science and other more applied sciences (such as game theory and operations research).

Mathematics Subject Classification (2020): 90C27.

Introduction by the Organizers

The workshop *Combinatorial Optimization*, organized by Thomas Rothvoss (Seattle), Laura Sanità (Milano) and Robert Weismantel (Zürich) continued a tradition that the fruitful development of the field is advanced by Oberwolfach Workshops each of which focuses on several specific areas. The focus areas of the workshop in 2024 were mixed integer optimization in high dimensions and algorithmic efficiency in combinatorial and linear optimization.

This workshop aimed at bringing together the most active researchers in combinatorial optimization, and at the same time forging new connections by attracting leading experts from adjacent areas, related to the focus topics of this proposal. It was well attended with over 40 participants with broad geographic representation from several continents. A considerable part of participants was from the younger generation of researchers in the field.

This workshop was a nice blend of researchers with various backgrounds in mathematics and beyond. This breadth of interests of the participants was well reflected by the wealth of topics presented in form of scientific talks or open problems. These topics address important questions in the field of combinatorial optimization that often also have an interface with other disciplines such as graph theory, combinatorics, number theory, geometry of numbers or the design of algorithms that itself is strongly linked with theoretical computer science. The workshop demonstrated that the field is very active and generates surprising links to other areas of mathematics and beyond.

During the workshop, a total of 31 talks were presented. We will provide an overview over a small sample of invited talks, representing the variety of topics presented at this workshop and demonstrating the progress made by the field since the last workshop.

Vera Traub reported on spectacular progress on an important problem in combinatorial optimization called Steiner tree where given a graph $G = (V, E)$ with edge costs and terminals R, the goal is to connect the terminals at minimal cost. The bidirected cut relaxation has long been seen as a promising tool to design approximation algorithms that have a good approximation factor but are yet computationally efficient. For several decades the best upper bound on its integrality gap has been a factor of 2. Vera and co-authors finally managed to break this barrier with a sophisticated algorithm and analysis.

Ola Svensson presented a new approximation algorithm for the k-median problem where a metric space and a parameter k is given and one has to open k facilities that minimize the sum of distances of points to the nearest opened facility. The main technical contribution of their work is a $(2 + \varepsilon)$-approximation which opens just a few facilities – $O(\log n/\varepsilon^2)$ many — more than allowed. Then using a second algorithm for so called stable instances this can be turned into a polynomial time algorithm that truly opens only k facilities. This makes significant progress in understanding this important problem.

Lisa Sauermann reported on tremendous progress in additive combinatorics. A classical question going back to Erdős and Turan (1936) is the following: what is the maximum cardinality $r_3(N)$ of a set $A \subseteq \{1, \ldots, N\}$ that does not contain a three-term arithmetic progression, i.e. there are no distinct $x, y, z \in A$ with $x + z = 2y$. A breakthrough by Kelley and Meka shows that

$$r_n(N) \leq N \cdot \exp(-c(\log N)^{1/12})$$

On the other hand, a result by Behrend from 1946 shows a lower bound of

$$r_3(N) \geq N \cdot 2^{-(2\sqrt{2}+o(1))\sqrt{\log_2 N}}$$

Lisa and co-authors are the first after almost 80 years to improve the leading constant of $2\sqrt{2}$.

Zhuan Khye Koh presented some groundbreaking progress toward resolving the long-standing open problem of finding a strongly polynomial algorithm for linear programming. This work gives a strongly polynomial algorithm for the minimum-cost generalized flow problem, which is equivalent to solving all LPs with at most two variables per inequality (2VPI). Their approach proves that a path-following interior point method terminates in a strongly polynomial number of iterations by bounding the so-called straight-line complexity, that is the minimum number of pieces needed by any piecewise affine curve in order to approximate the central path. This achievement goes beyond previous results, which were limited to feasibility problems, marking a major step toward the aforementioned outstanding open problem.

We conclude with mentioning that the workshop included an open problem session, during which six problems were discussed and compiled in this report to guide future research.

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Workshop: Combinatorial Optimization

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Abstracts

The Bidirected Cut Relaxation for Steiner Tree has Integrality Gap Smaller than 2

Vera Traub

(joint work with Jarosław Byrka, Fabrizio Grandoni)

The Steiner tree problem is one of the most prominent problems in network design, with great practical and theoretical relevance. We are given an undirected graph $G = (V, E)$, with edge costs $c : E \to \mathbb{R}_{\geq 0}$, and a subset of vertices $R \subseteq V$ (the *terminals*). The task is to compute a tree T of minimum cost $c(T) := \sum_{e \in E(T)} c(e)$ which contains R (and possibly other vertices).

The best-known approximation algorithms for Steiner tree [1, 5] involve enumeration of a polynomial but very large number of candidate components and are therefore slow in practice. A promising ingredient for the design of fast and accurate approximation algorithms is the bidirected cut relaxation (BCR), which is one of the oldest and best-studied linear programming relaxations for Steiner tree, see for example [4].

To define BCR, we bidirect all the edges of the given graph G, i.e., we replace each undirected edge $e = \{u, v\} \in E$ with two oppositely directed edges $(u, v), (v, u)$, both with cost $c(e)$. Let $\overrightarrow{E}$ be this set of directed edges and choose an arbitrary terminal $r \in R$ as a root. For $U \subseteq V$, let $\delta^+(U) = \{(u, v) \in \overrightarrow{E} : u \in U, v \notin U\}$. Then BCR is the following linear programming relaxation:

$$
\text{(BCR)} \qquad \begin{array}{llrcl}
\min & \displaystyle\sum_{e \in \overrightarrow{E}} c(e) \cdot x_e & & & \\
\text{s.t.} & \displaystyle\sum_{e \in \delta^+(U)} x_e & \geq & 1 & \text{for all } U \subseteq V \setminus \{r\} \text{ with } R \cap U \neq \emptyset \\
 & x_e & \geq & 0 & \text{for all } e \in \overrightarrow{E}.
\end{array}
$$

BCR is indeed a relaxation of the Steiner tree problem because we can orient every Steiner tree towards the root r and then consider the incidence vector of this oriented tree to obtain a feasible solution of BCR. Moreover, BCR is provably stronger than the natural undirected LP in several interesting special cases. For example, a famous result by Edmonds [3] shows that BCR is integral in the minimum spanning tree case, i.e., when $R = V$. Notice that, in contrast, already on such instances the natural undirected LP has integrality gap 2.

For general Steiner tree instances, however, it was not known whether the integrality gap of BCR is better than the integrality gap of the natural undirected relaxation, which is exactly 2. We resolve this question by proving an upper bound of 1.9988 on the integrality gap of BCR [2].

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Ghost Value Augmentation for k-Edge-Connectivity

Rico Zenklusen

(joint work with D. Ellis Hershkowitz and Nathan Klein)

We give a poly-time algorithm for the k-edge-connected spanning subgraph (k-ECSS) problem that returns a solution of cost no greater than the cheapest $(k+10)$-ECSS on the same graph. Our approach enhances the iterative relaxation framework with a new ingredient, which we call ghost values, that allows for high sparsity in intermediate problems.

Our guarantees improve upon the best-known approximation factor of 2 for k-ECSS whenever the optimal value of $(k+10)$-ECSS is close to that of k-ECSS. This is a property that holds for the closely related problem k-edge-connected spanning multi-subgraph (k-ECSM), which is identical to k-ECSS except edges can be selected multiple times at the same cost. As a consequence, we obtain a $(1+O(\frac{1}{k}))$-approximation algorithm for k-ECSM, which resolves a conjecture of Pritchard [2] and improves upon a recent $(1+O(\frac{1}{\sqrt{k}}))$-approximation algorithm of Karlin, Klein, Oveis Gharan, and Zhang [1]. Moreover, we present a matching lower bound for k-ECSM, showing that our approximation ratio is tight up to the constant factor in $O(\frac{1}{k})$, unless P=NP.

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Almost-linear Time Algorithms for Partially Dynamic Graphs

Rasmus Kyng
(joint work with Brand, Chen, Liu, Meierhans, Probst, Sachdeva)

A partially dynamic graph is a graph that undergoes edge insertions or deletions, but not both. In this talk, I present an unifying framework that yields the first almost-optimal, almost-linear time algorithms for many well-studied problems on partially dynamic graphs. These include cycle detection, strongly connected components, s-t distances, transshipment, bipartite matching, maximum flow, and minimum-cost flow. We achieve this unification by solving the partially dynamic threshold minimum-cost flow problem. In this problem, one is given a fixed demand vector and a threshold F, and tasked with reporting the first time a flow of cost F exists or ceases to exist for insertions and deletions respectively. We give separate algorithms for solving this problem in the edge insertion and deletion cases. Both use extensions of the ℓ_1-interior point method framework introduced as part of the first almost-linear time minimum-cost flow algorithm [1]. For handling edge insertions, we develop new powerful data structures [2] to solve the central min-ratio cycle problem against a general adversary [3]. For handling edge deletions, we take the dual perspective. This leads us to a min-ratio cut problem, which we solve by constructing a dynamic tree that approximately preserves all cuts [4].

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Transitive covers

Stefan Weltge

(joint work with Daniel Dadush, Andrey Kupavskii)

Given a directed graph G, we say that a directed graph H is a *transitive cover* of G if it satisfies

- $V(G) \subseteq V(H)$ and
- for every $s, t \in V(G)$: G has an s-t-path if and only if H has an s-t-path.

Thus, a transitive cover contains the complete reachability information of G. We denote by $tc(G)$ the smallest number of edges of a transitive cover of G. The idea of using transitive covers to compress the reachability information of a graph has been used in different algorithmic contexts [1, 3]. Here, we introduce a formal study of the quantity $\mathrm{tc}(G)$. Using a result by Tuza [4] and a counting argument, one can show

Theorem 1. $\max\{\mathrm{tc}(G) : |V(G)| = n\} = \Theta(\frac{n^2}{\log n})$.

As our main result, we prove the following.

Theorem 2. *If $t \in \mathbb{Z}_{\geq 2}$ and $\mathcal{G}_t$ is the family of directed bipartite graphs*[1] *that are $K_{t,t}$-free, then* $\sup_{G \in \mathcal{G}_t} \frac{|E(G)|}{\mathrm{tc}(G)} = (t-1)^2$.

Using a result by Füredi [2], we obtain the following consequence.

Corollary 3. *There exist directed graphs G with $|E(G)| = \Omega(|V(G)|^{3/2})$ and* $\mathrm{tc}(G) = |E(G)$.

Moreover, we establish the following connection to extended formulations. Here, for a polyhedron P, we denote by $\mathrm{xc}(P)$ the *extension complexity* of P, i.e., the smallest number of facets of a polyhedron that can be linearly projected onto P. For a directed graph G, let $I(G) \in \{0, \pm 1\}^{V(G) \times E(G)}$ denote the node-edge incidence matrix of G.

Proposition 4. *Every directed graph G satisfies* $\mathrm{xc}(\{I(G)x : x \in \mathbb{R}^E_{\geq 0}\}) \leq \mathrm{tc}(G)$.

Given many techniques for establishing lower bounds on extension complexities, we wonder whether this result may provide interesting insights for proving lower bounds on $\mathrm{tc}(G)$.

In our talk, we posed several open questions regarding transitive covers, including the complexity of computing $\mathrm{tc}(G)$. In a subsequent discussion with Rico Zenklusen we found an approximation-preserving reduction from the set cover problem on instances where any two sets intersect in at most one element (containing vertex cover as a special case).

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Extending the Continuum of Six-Colorings

Sebastian Pokutta

(joint work with Konrad Mundinger, Christoph Spiegel, Max Zimmer)

We present two novel six-colorings of the Euclidean plane that avoid monochromatic pairs of points at unit distance in five colors and monochromatic pairs at another specified distance d in the sixth color. Such colorings have previously been known to exist for $0.414 \approx \sqrt{2} - 1 \leq d \leq 1/\sqrt{5} \approx 0.447$. Our results significantly expand that range to $0.354 \leq d \leq 0.657$, the first improvement in 30 years. Notably, the constructions underlying this were derived by formalizing colorings suggested by a custom machine learning approach.

The first coloring is valid for $0.354 \leq d \leq 0.553$ and involves four different polytopal shapes: equidiagonal pentagons, equilateral triangles, octagons, and hexagons. These shapes are colored in a specific pattern to avoid monochromatic pairs at both unit distance and the specified distance d. The coloring is parameterized by d.

The second coloring is valid for $0.418 \leq d \leq 0.657$ and uses four polytopal shapes: axisymmetric pentagons, squares, heptagons, and hexagons. Similar to the first construction, these shapes are arranged and colored to avoid monochromatic pairs at the specified distances. This coloring, independent of d, is valid for the entire range $0.418 \leq d \leq 0.657$.

These constructions were derived using a custom machine learning approach, which we coined "deep annealing." It allows us to explore and formalize new colorings efficiently, resulting in somewhat "unexpected" and irregular patterns compared to previously found colorings.

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Online Algorithms with a Sample

Anupam Gupta

(joint work with C.J. Argue, Alan M. Frieze, and Christopher Seiler)

We consider two central problems in combinatorial optimization: the restricted assignment load-balancing problem, and the Steiner tree network design problem. We consider the online setting, where the input arrives over time, and irrevocable decisions must be made without knowledge of the future. For both these problems, any online algorithm must incur a cost that is approximately $\log |I|$ times the optimal cost in the worst-case, where $|I|$ is the length of the input. But can we go beyond the worst-case? In this talk we give algorithms that perform substantially better when a p-fraction of the input is given as a sample: the algorithm use this sample to *learn* a good strategy to use for the rest of the input. The talk is based on work reported in [1].

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Cost Allocation for Set Covering: The Happy Nucleolus

Jens Vygen

(joint work with Jannis Blauth, Antonia Ellerbrock, Vera Traub)

We consider cost allocation for set covering problems. We allocate as much cost to the elements (players) as possible without violating the group rationality condition, and so that the excess vector is lexicographically maximized. This *happy nucleolus* has several nice properties. In particular, we show that it can be computed considering a small subset of "simple" coalitions only. While computing the nucleolus for set covering is NP-hard, our results imply that the happy nucleolus can be computed in polynomial time [1].

At the end, we briefly discuss applications to real-world logistics, in the context of our vehicle routing heuristic [2].

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Non-distributive lattices, stable matchings, and linear optimization

Yuri Faenza

(joint work with Christopher En)

We show that all finite lattices, including non-distributive lattices, arise as stable matching lattices under standard assumptions on choice functions. Our result proves the converse inclusion of [2] and extends results of [1, 3, 4] on distributive lattices and stable matchings in marriage instances. In the process, we introduce new tools to reason on general lattices for optimization purposes: the *partial representation* of a lattice, which partially extends Birkhoff's representation theorem to non-distributive lattices; the *distributive closure* of a lattice, which gives such a partial representation; and *join constraints*, which can be added to the distributive closure to obtain a representation for the original lattice. Then, we use these techniques to show that the minimum cost stable matching problem under the same standard assumptions on choice functions is NP-hard, by establishing a connection with antimatroid theory.

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Improving Behrend's construction for sets without three-term arithmetic progressions

Lisa Sauermann

(joint work with Christian Elsholtz, Zach Hunter, Laura Proske)

The questions of estimating the maximum possible sizes of subsets of $\{1,\dots,N\}$ and of $\mathbb{F}_p^n$ without three-term arithmetic progressions are among the most central problems in additive combinatorics. Let us denote the maximum possible size of such subsets by $r_3(N)$ and $r_3(\mathbb{F}_p^n)$, respectively. So, formally, $r_3(N)$ is the maximum possible size of a subset $A \subseteq \{1,\dots,N\}$ such that there do not exist distinct $x,y,z \in A$ with $x+z=2y$, and similarly $r_3(\mathbb{F}_p^n)$ is the maximum possible size of a subset $A \subseteq \mathbb{F}_p^n$ such that there do not exist distinct $x,y,z \in A$ with $x+z=2y$. The problem of estimating $r_3(N)$ was raised by Erdős and Turán [8] in 1936, and has been intensively studied since then. It also has connections to problems in communication complexity. The problem for $\mathbb{F}_p^n$ has also been studied for several decades.

In a breakthrough result in 2017, Ellenberg and Gijswijt [6] proved that for any prime $p \geq 3$, there is an upper bound of the form

$$r_3(\mathbb{F}_p^n) \leq (c_p p)^n \tag{1}$$

for some constant $c_p < 1$ only depending on p (and their constant c_p converges to $0.841\dots$ for $p \to \infty$, see [3, Eq. (4.11)]). In the integer setting, in a more recent breakthrough Kelley and Meka [11] proved the upper bound

$$r_3(N) \leq N \cdot \exp(-c(\log N)^{1/12})$$

for all $N \geq 3$, for some absolute constant $c > 0$. This drastically improved upon all the previous bounds, obtained in a long seriesoif works by many authors over several decades. Afterwards, using a modification of the proof of Kelley and Meka, this bound was improved to

$$r_3(N) \leq N \cdot \exp(-c(\log N)^{1/9})$$

by Bloom and Sisask [4].

These upper bounds for $r_3(N)$ match the shape of a classical lower bound for this problem due to Behrend [2] from 1946, which is of the form

$$r_3(N) \geq N \cdot 2^{-(2\sqrt{2}+o(1))\sqrt{\log_2 N}}. \tag{2}$$

Over the past almost eighty years, only the $o(1)$-term in this bound has been improved. In Behrend's original bound, this $o(1)$-term in the exponent encapsulated a factor of $(\log_2 N)^{-1/4}$, so the bound was of the form $r_3(N) \geq \Omega(N \cdot 2^{-2\sqrt{2}\sqrt{\log_2 N}} \cdot (\log_2 N)^{-1/4})$. In 2010, Elkin [5] improved this factor to $(\log_2 N)^{1/4}$ instead, and an alternative proof for the bound with this improved $o(1)$-term was found by Green and Wolf [10].

This talk is based on recent work, giving the first improvement to Behrend's [2] classical lower bound beyond the $o(1)$-term in (2). As stated in the following theorem, we show that the constant factor $2\sqrt{2} \approx 2.828$ in the exponent can be improved to $2\sqrt{\log_2(24/7)} \approx 2.667$, proving that the classical bound (2) is not tight.

Theorem 1. *We have*

$$r_3(N) \geq N \cdot 2^{-(C+o(1))\sqrt{\log_2 N}}$$

with $C = 2\sqrt{\log_2(24/7)} < 2\sqrt{2}$.

Our proof of Theorem 1 is motivated by studying three-term progression free sets in $\mathbb{F}_p^n$, for a fixed relatively large prime p and large n. In this setting, one can adapt Behrend's construction [2] (as noted by Tao and Vu in their book on additive combinatorics [13, Exercise 10.1.3] and also observed by Alon, see [9, Lemma 17]) to show

$$r_3(\mathbb{F}_p^n) \geq \left(\frac{p+1}{2}\right)^{n-o(n)} \tag{3}$$

for any fixed prime p and large n. Alternatively, such a bound can also be shown via an adaptation of an earlier construction in the integer setting due to Salem and Spencer [12] from 1942 (see [1, Theorem 2.13]). The asymptotic notation $o(n)$ in the bound (3) is for $n \to \infty$ with p fixed. The best quantitative bound for the $o(n)$-term in this statement is due to Elsholtz and Pach [7, Theorem 3.10], but beyond the $o(n)$-term, this bound has not been improved (except for specific small primes p).

Comparing the upper and lower bounds for $r_3(\mathbb{F}_p^n)$ for a fixed (reasonably large) prime p and large n in (1) and (3), there is still a large gap. Both of these bounds are roughly of the form $(cp)^n$ with $0 < c < 1$, but with different values of c. For the upper bound, the best known constant due to Ellenberg–Gijswijt [6] is $c \approx 0.85$ (when the fixed prime is large enough). For the lower bound the constant $c = 1/2$ from Behrend's construction [2] or alternatively the Salem–Spencer construction [12] has not been improved in more than eighty years despite a lot of attention, especially after the upper bound of Ellenberg–Gijswijt appeared. This work improves this constant in the lower bound to be strictly larger than $1/2$.

Theorem 2. *There is a constant $c > 1/2$ such that for every prime p and every sufficiently large positive integer n (sufficiently large in terms of p), we have $r_3(\mathbb{F}_p^n) \geq (cp)^n$.*

Breaking the barrier of $1/2$ in this result for $r_3(\mathbb{F}_p^n)$ relies on the same key insights as our lower bounds for $r_3(N)$ in Theorem 1 improving Behrend's construction. Our proof of Theorem 2 shows that one can take any $c < \sqrt{7/24}$, for example $c = 0.54$. Even though this may not seem like a large improvement over $1/2$, it is the first qualitative improvement over the constant $1/2$ from the constructions of Salem–Spencer and Behrend from the 1940's. Both of these constructions lead to three-term progression free subsets of $\mathbb{F}_p^n$ only consisting of vectors with all entries in $\{0, 1, \ldots, (p-1)/2\}$, i.e. they only use roughly half of the available elements in $\mathbb{F}_p$ in each coordinate. The restriction of all entries to $\{0, 1, \ldots, (p-1)/2\}$ is crucial in these constructions, as it ensures that there is no "wrap-around" over $\mathbb{F}_p$. However, such an approach cannot be used to obtain three-term progression free subset $A \subseteq \mathbb{F}_p^n$ of size $|A| > ((p+1)/2)^n$, so $c = 1/2$ is a significant barrier for this problem. In light of this, it may actually be considered a surprise that it is possible to obtain a constant $c > 1/2$.

It is plausible that by slight modifications of our method, one can obtain better numerical bounds in our results above. In particular, our construction relies on certain explicit two-dimensional building blocks of area close to 7/24, defined as the union of certain polygons. An improved construction for these building blocks with larger area would automatically carry over to numerical improvements of the constant $C = 2\sqrt{\log_2(24/7)}$ in Theorem 1 and the constant c in Theorem 2. We see the main contribution of this work as introducing this method, and using it to break the lower bounds for $r_3(N)$ and $r_3(\mathbb{F}_p^n)$ in (2) and (3) originating from the 1940's.

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A strongly polynomial algorithm for the minimum cost generalized flow problem

Zhuan Khye Koh

(joint work with Daniel Dadush, Bento Natura, Neil Olver, László A. Végh)

Linear programs (LPs) can be solved in polynomial time using the ellipsoid method [4] or interior point methods [3, 6]. However, it is an outstanding open problem whether there exists a *strongly polynomial* algorithm for linear programming. Given an LP with m constraints, n variables and bit encoding length L, a strongly polynomial algorithm can only use $\mathrm{poly}(m,n)$ elementary arithmetic operations $(+,-,\times,\div,<?)$, and must run in $\mathrm{poly}(m,n,L)$ space. This question was first posed by Megiddo [5], and is often referred to as Smale's 9th problem [7].

In this talk, we focus on the class of LPs with at most 2 variables per inequality (2VPI). By a reduction of Hochbaum [2], this class is strongly polynomially equivalent to the *minimum-cost generalized flow* problem. Given a directed graph $G=(V,E)$ with node demands $b \in \mathbb{R}^V$, arc costs $c \in \mathbb{R}^E$ and gain factors $\gamma \in \mathbb{R}^E_{>0}$, the latter problem can be formulated as the following LP

$$\begin{aligned} &\min \sum_{e\in E} c_e x_e \\ \text{(1)}\qquad &\text{s.t.} \sum_{e\in\delta^{\mathrm{in}}(v)} \gamma_e x_e - \sum_{e\in\delta^{\mathrm{out}}(v)} x_e = b_v \qquad \forall v \in V \\ &\qquad x_e \geq 0 \qquad \forall e \in E. \end{aligned}$$

This is a generalization of the classic minimum cost flow problem, by allowing flow traversing an arc e to be scaled by the corresponding gain factor $\gamma_e > 0$.

We give a strongly polynomial algorithm for solving (1), and consequently all 2VPI LPs. Previously, strongly polynomial algorithms were only known for the primal and dual feasibility problems [8, 5]. Our approach is to show that the path-following interior point method of [1] terminates in a strongly polynomial number of iterations for (1). We achieve this by bounding the *straight line complexity*, which is the minimum number of pieces required by a piecewise affine curve to multiplicatively approximate the central path.

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Exponential Lower Bounds for Many Pivot Rules for the Simplex Method

Alexander E. Black

The existence of a pivot rule for the simplex method that guarantees a polynomial run-time is a longstanding, fundamental open problem in the theory of linear programming. The most popular pivot rule for theoretical analysis is the shadow pivot rule, which solves a linear program by projecting the feasible region onto a polygon. It has been shown to perform in expected strongly polynomial time on uniformly random instances and in smoothed analysis. In practice, the pivot rule of choice is the steepest edge rule, which normalizes the set of improving neighbors and then chooses a maximally improving normalized neighbor. Exponential lower bounds are known for both rules in worst-case analysis [1, 2]. However, for the shadow simplex method, all exponential examples were only proven for one choice of projection, and for the steepest edge rule, the lower bounds were only proven for the Euclidean norm. We construct linear programs for which any choice of projection for shadow rule variants will lead to an exponential run-time and exponential examples for any choice of norm for a steepest edge variant.

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Machine learning augmented MILP

Karen Aardal

(joint work with Lara Scavuzzo, Andrea Lodi, Neil Yorke-Smith)

In the first part of our talk [1] we give a short introduction to how machine learning (ML) is used to solve the so-called variable selection problem within the Branch-and-Bound (B&B) algorithm, i.e., the problem of selecting which of the candidate variables to branch on. Most ML-algorithms are trained to approximate strong branching, but we also mention Reinforcement Learning as a tool to learn how to branch when strong branching is not an appropriate expert.

In the second part [2], we discuss two new ML-based methods for estimating the optimal value of a mixed integer linear program (MILP) once the root node of the B&B-tree has been solved. This estimate is one of the inputs to a classifier that estimates whether the current best known feasible solution is optimal even if no certificate of optimality has yet been found. If we, with high probability, would know that the current best feasible solution is in fact optimal, then the effort of the B&B algorithm can be redirected to more aggressively work on the dual bound. We show computational results from three different problem classes; set cover, combinatorial auctions, and generalized independent set, and observe that our estimates are quite accurate and, compared to other estimates proposed in the literature, they are more well-balanced, in terms of false positives and false negatives.

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Integer and Unsplittable Multiflows in Series-Parallel Digraphs

Martin Skutella

(joint work with Mohammed Majthoub Almoghrabi, Philipp Warode)

For a given digraph with k source-sink pairs, an *unsplittable* multiflow sends the given demand d_i of every commodity $i \in [k] \coloneqq \{1, \dots, k\}$ along a single path P_i from its source s_i to its sink t_i. Unsplittable flow problems represent a compelling extension of disjoint path problems and have received considerable attention in the literature. Even in the apparently straightforward case of a (series-parallel) digraph consisting of one common source and one common sink connected by parallel arcs, determining the existence of a *feasible* unsplittable multiflow (i.e., one that adheres to given arc capacities) is NP-complete. In fact, several classical problems in combinatorial optimization, such as, e.g., Bin Packing, Partition, or parallel machine scheduling with makespan objective occur as special cases; for more details we refer to Kleinberg's PhD thesis [2].

Integer multiflows. When every commodity $i \in [k]$ has unit demand $d_i \coloneqq 1$, unsplittable multiflows precisely correspond to integer multiflows, which have been extensively studied in the literature. In fact, in unit-capacity networks, feasible integer multiflows model arc-disjoint paths, and therefore cannot be computed efficiently, unless P=NP. For example, Vygen [11] shows that the arc-disjoint paths problems is NP-complete even in acyclic and planar digraphs; see [8, Chapter 70.13b+c] for further related complexity results. On the other hand, single-commodity network flows (i.e., $k = 1$) are well-known to feature strong integrality properties. In particular, the network flow polytope is integral, that is, any single-commodity flow $(x_e)_{e \in E}$ on a digraph $G = (V, E)$ with integer excess at every node, is a convex combination of integer flows $(y_e)_{e\in E}$ that maintain the same excesses, such that

$$\lfloor x_e \rfloor \le y_e \le \lceil x_e \rceil \qquad \text{for every arc } e \in E. \tag{1}$$

Arguably, the most important tool for proving the existence of integer multiflows in special graph classes is the Nagamochi-Ibaraki Theorem [6], which relies on the sufficiency of the *cut condition*. This (necessary) condition requires that the capacity of every cut exceeds the total demand of commodities that must cross it. The concept of cut-sufficiency is of interest in its own right and is an active research topic; see, e.g., the recent work of Poremba and Shepherd [7] and references therein. For a fixed digraph, the Nagamochi-Ibaraki Theorem asserts that if, for any integer arc capacities and demands, the cut condition guarantees the existence of a feasible multiflow, it also guarantees the existence of a feasible *integer* multiflow. For most classes of digraphs in which the existence of a feasible multiflow implies the existence of a feasible integer multiflow, the cut condition is indeed sufficient. However, the multiflow instance in Fig. 1 illustrates that the cut condition is generally not sufficient for the class of series-parallel digraphs studied in this paper.

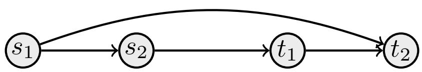

FIGURE 1. Infeasible multiflow instance with unit-demand commodities and unit arc capacities satisfying the cut condition.

Single-source unsplittable flows. Even though our results hold for general multiflow instances in series-parallel digraphs, where each commodity is routed from a distinct source to a distinct sink, our main result is primarily inspired by prior research on single-source unsplittable flows, which we review next. We refer to [3] for a comprehensive overview of results on general unsplittable multiflows.

Single-source unsplittable flows, where all commodities share a common source node s, have first been studied by Kleinberg [2]. Dinitz, Garg, and Goemans [1] prove that a given fractional flow $(x_e)_{e\in E}$ can always be turned into an unsplittable flow $(y_e)_{e\in E}$ (given by s-t_i-paths P_i with $y_e := \sum_{i:e\in P_i} d_i$) such that

$$y_e \le x_e + d_{\max} \qquad \text{for every arc } e \in E, \tag{2}$$

where $d_{\max} := \max_{i\in[k]} d_i$. A famous conjecture of Goemans says that flow $(x_e)_{e\in E}$ can even be expressed as a convex combination of unsplittable flows $(y_e)_{e\in E}$ that satisfy (2). Skutella [9] proves that Goemans' Conjecture is valid when the demands of the commodities are multiples of one another. For acyclic digraphs, Morell and Skutella [5] show that any fractional flow $(x_e)_{e\in E}$ can be turned into an unsplittable flow $(y_e)_{e\in E}$ that satisfies the lower bound

$$y_e \ge x_e - d_{\max} \qquad \text{for every arc } e \in E. \tag{3}$$

Furthermore, they conjecture the existence of an unsplittable flow that satisfies both the upper bounds (2) and the lower bounds (3). Only recently, for the special case of acyclic and planar digraphs, this conjecture has been proved by Traub, Vargas Koch, and Zenklusen [10]. Morell and Skutella also propose the following strengthening to Goemans' conjecture.

Conjecture 1 (Morell and Skutella [5]). *For the single-source unsplittable flow problem on acyclic digraphs, any fractional flow* $(x_e)_{e\in E}$ *can be expressed as a convex combination of unsplittable flows* $(y_e)_{e\in E}$ *that satisfy both* (2) *and* (3).

Using techniques of Martens, Salazar, and Skutella [4], they show that their Conjecture 1 is valid when the demands of the commodities are multiples of one another. Moreover, for the special case of acyclic and planar digraphs, and under the slightly relaxed lower and upper bounds $x_e - 2d_{\max} \leq y_e \leq x_e + 2d_{\max}$ for $e \in E$, Conjecture 1 is proved by Traub, Vargas Koch, and Zenklusen [10].

Our contributions. We present the following observations on integer multiflows.

Theorem 1. *Consider a multiflow instance on a series-parallel digraph.*
(a) For integer demands, the total arc flows $(x_e)_{e\in E}$ *of any multiflow can be expressed as a convex combination of total arc flows* $(y_e)_{e\in E}$ *of integer multiflows that satisfy* (1).
(b) For (integer) arc capacities, a feasible (integer) multiflow, if one exists, can be efficiently found through a single-commodity flow computation.

In light of the fact that multiflow instances on series-parallel digraphs generally do not satisfy the assumptions of the Nagamochi-Ibaraki Theorem (i.e., the cut condition is not sufficient; see Figure 1), the strong integrality property in Theorem 1(a) might seem surprising. On the other hand, the proof of Theorem 1 relies on the simple observation that, by carefully subdividing commodities, multiflow instances on series-parallel digraphs can be efficiently reduced to a certain subclass of instances which can be solved by single-commodity flow techniques. The integrality property in Theorem 1(a) is then inherited from the integrality of the network flow polytope. To the best of our knowledge, and somewhat surprisingly, these observations have not appeared in the literature before.

Our main result generalizes the strong integrality result in Theorem 1(a) towards unsplittable multiflows.

Theorem 2. *The total arc flows* $(x_e)_{e\in E}$ *of a fractional multiflow in a series-parallel digraph can be expressed as a convex combination of total arc flows* $(y_e)_{e\in E}$ *of unsplittable multiflows that satisfy*

$$x_e - d_{\max} < y_e < x_e + d_{\max} \qquad \textit{for every arc } e \in E.$$

This result implies, in particular, that Conjecture 1 holds for series-parallel digraphs, even for general multiflow instances where commodities have individual source and sink nodes. Even for the weaker conjecture of Goemans, Theorem 2 provides the first proof for a non-trivial class of digraphs.

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Popular Arborescences and Their Matroid Generalization

Yu Yokoi

(joint work with Telikepalli Kavitha, Kazuhisa Makino, Ildikó Schlotter)

Let $G = (V \cup \{r\}, E)$ be a rooted digraph where each vertex in V has a partial order over its incoming edges. An arborescence is *popular* if it does not lose a head-to-head election against any other arborescence, where vertices in V are voters. The popular arborescence problem is to decide whether a given instance admits a popular arborescence or not (and to find one if it exists).

We present a polynomial-time algorithm to solve this problem. In fact, our algorithm solves the more general popular common base problem in the intersection of two matroids, where one matroid is a partition matroid with a partial order on each partition class, and the other is an arbitrary matroid. This problem is a common generalization of the previously studied popular matching problem [1], popular assignment problem [2], and popular branching problem [3]. Our algorithm is combinatorial and can be regarded as a primal-dual algorithm. It searches for a solution along with its dual certificate, a chain of subsets of the ground set E, witnessing its popularity.

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A Better-Than-1.6-Approximation for Prize-Collecting TSP

Nathan Klein
(joint work with Jannis Blauth, Martin Nägele)

Prize-Collecting TSP is a variant of the traveling salesperson problem where one may drop vertices from the tour at the cost of vertex-dependent penalties. The quality of a solution is then measured by adding the length of the tour and the sum of all penalties of vertices that are not visited. We present a polynomial-time approximation algorithm with an approximation guarantee slightly below 1.6, where the guarantee is with respect to the natural linear programming relaxation of the problem. This improves upon the previous best-known approximation ratio of 1.774 [1]. Our approach is based on a known decomposition for solutions of this linear relaxation into rooted trees. Our algorithm takes a tree from this decomposition and then performs a pruning step before doing parity correction on the remainder. Using a simple analysis, we bound the approximation guarantee of the proposed algorithm by $(1+\sqrt{5})/2 \approx 1.618$, the golden ratio. With some additional technical care we further improve it to 1.599.

We also consider the Prize-Collecting Stroll (PCS) problem, in which we want to compute an s-t walk minimizing the cost of the walk plus the cost of the penalties of the unvisited vertices. Here we improve the best known approximation from 1.926 (due to An, Kleinberg, and Shmoys [2]) to 1.6662.

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A modified greedy algorithm for submodular cover

Britta Peis
(joint work with Niklas Rieken, and José Verschae)

Submodular cover is a common generalisation of set cover, integer cover, and the minimum weight spanning tree problem. In the submodular cover problem, we are given a submodular, nonnegative, and monotone non-decreasing function f defined on all subsets of a finite ground set E, and the task is to find a subset S satisfying $f(S) = f(E)$ of minimum cost for some given cost function on E. It is known that a naive greedy heuristic, which starts with the empty set, and iteratively adds elements of optimal ratio between cost and marginal coverage has a performance ratio of of $1 + \ln k$, where k is the maximum f-value of a singleton [1] This performance guarantee is nearly best possible, even in the special case of set cover [2]. We propose and discuss a modification of the greedy algorithm where redundant elements are deleted in each iteration. We show that this modified greedy algorithm is guaranteed to terminate with an optimal solution to the submodular cover problem in case of submodular systems satisfying certain properties which are easily seen to be fulfilled by e.g. laminar set cover and weighted matroid rank functions.

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A $(2+\varepsilon)$-Approximation Algorithm for Metric k-Median

Ola Svensson

(joint work with Vincent Cohen-Addad, Fabrizio Grandoni, Euiwoong Lee, Chris Schwiegelshohn)

In the classical NP-hard (metric) k-median problem, we are given a set of n clients and centers with metric distances between them, along with an integer parameter $k \geq 1$. The objective is to select a subset of k *open* centers that minimizes the total distance from each client to its closest open center.

In their seminal work [1], Jain, Mahdian, Markakis, Saberi, and Vazirani presented the Greedy algorithm for facility location, which implies a 2-approximation algorithm for k-median that opens k centers in *expectation*. Since then, substantial research has aimed at narrowing the gap between their algorithm and the best achievable approximation by an algorithm guaranteed to open *exactly* k centers, as required in the k-median problem. During the last decade, all improvements have been achieved by leveraging their algorithm (or a small improvement thereof), followed by a second step called bi-point rounding, which inherently adds an additional factor to the approximation guarantee.

Our main result closes this gap [2]: for any $\varepsilon > 0$, we present a $(2 + \varepsilon)$-approximation algorithm for the k-median problem, improving the previous best-known approximation factor of 2.613. Our approach builds on a combination of two key algorithms. First, we present a non-trivial modification of the Greedy algorithm that operates with only $O(\log n/\varepsilon^2)$ adaptive phases. Through a novel walk-between-solutions approach, this enables us to construct a $(2 + \varepsilon)$-approximation algorithm for k-median that consistently opens at most $k+O(\log n/\varepsilon^2)$ centers: via known results, this already implies a $(2 + \varepsilon)$-approximation algorithm that runs in quasi-polynomial time. Second, we develop a novel $(2+\varepsilon)$-approximation algorithm tailored for stable instances, where removing any center from an optimal solution increases the cost by at least an $\Omega(\varepsilon^3/\log n)$ fraction. Achieving this involves several ideas, including a sampling approach inspired by the k-means++ algorithm and a reduction to submodular optimization subject to a partition matroid. This allows us to convert the previous result into a polynomial time algorithm that opens exactly k centers while maintaining the $(2+\varepsilon)$-approximation guarantee.

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Sensitivity, Proximity and FPT Algorithms for Exact Matroid Problems

Lars Rohwedder

(joint work with Friedrich Eisenbrand, Karol Węgrzycki)

We consider the problem of finding a basis of a matroid with weight exactly equal to a given target. Here weights can be discrete values from $\{-\Delta, \ldots, \Delta\}$ or more generally m-dimensional vectors of such discrete values. We resolve the parameterized complexity completely, by presenting an FPT algorithm parameterized by Δ and m for arbitrary matroids. Prior to our work, no such algorithms were known even when weights are in $\{0, 1\}$, or arbitrary Δ and $m = 1$. Our main technical contributions are new proximity and sensitivity bounds for matroid problems, independent of the number of elements. These bounds imply FPT algorithms via matroid intersection.

Integer Points of Convex (non-polyhedral!) Cones

Jesús A. De Loera

(joint work with Greg Blekherman, Brittney Marsters, Luze Xu, Shixuan Zhang)

Convex cones play an important role in optimization (e.g., the cone of positive semidefinite matrices is crucial to semidefinite optimization). Given a convex cone $C \subseteq \mathbb{R}^N$, the integer points $S_C := C \cap \mathbb{Z}^N$ form a semigroup which we call the *conical semigroup* of C. Understanding conical semigroups is relevant to the theory of integer conical optimization, specially for extending Hilbert bases techniques used for rational polyhedral cones to a more general setting.

Here I report on two papers (one published in IPCO and another to in progress) that explore the structure of conical semigroups beyond the well-studied case of pointed rational polyhedral cones [2], where we understand there are always unique (finite) Hilbert bases.

In what follows, we denote $\mathrm{GL}(N,\mathbb{Z}) := \{U \in \mathbb{Z}^{N\times N} : \ |\det(U)| = 1\}$. Here is our new key definition of finite generation for conical semigroups:

Definition. Given a conical semigroup $S_C \subset \mathbb{Z}^N$, we call it (R,G)-finitely generated if there is a finite subset $R \subseteq S_C$ and a finitely generated subgroup $G \subseteq \mathrm{GL}(N,\mathbb{Z})$ acting on C linearly such that

(1) both the cone C and the semigroup S_C are invariant under the group action, i.e., $G \cdot C = C$ and $G \cdot S_C = S_C$, and
(2) every element $s \in S_C$ can be represented as
$$s = \sum_{i\in K} \lambda_i T_i \cdot r_i$$
for some $r_i \in R$, $T_i \in G$, and $\lambda_i \in \mathbb{Z}_{\geq 0}$, and where K is a finite index set.

Note that when C is a (pointed) rational polyhedral cone, then the conical semigroup $S_C = C \cap \mathbb{Z}^N$ is (R,G)-finitely generated by R, its Hilbert basis, and G, the trivial group $\{I_N\}$.

While a non-polyhedral cone cannot be finitely generated in the old usual sense, we showed using a finitely generated group of matrices G allows us to extend our understanding beyond the polyhedral case. Because the possibly infinite generators for S_C can be obtained by group action G on a finite set R and G is finitely generated, this allows for the possibility of algorithmic methods.

In [1] we proved the following two main results pertaining to integer points in the PSD cone $\mathcal{S}^n_+(\mathbb{Z})$, and those in the second order cone $\mathrm{SOC}(n) \cap \mathbb{Z}^n$.

Theorem. The conical semigroup of the cone of $n \times n$ positive semidefinite matrices, $\mathcal{S}^n_+(\mathbb{Z})$, is (R,G)-finitely generated by $G \cong \mathrm{GL}(n,\mathbb{Z})$ where G acts on $X \in \mathcal{S}^n_+(\mathbb{Z})$ by $X \mapsto UXU^T$ for each $U \in \mathrm{GL}(n,\mathbb{Z})$, and by R, the union of any single rank-one matrix and a finite subset of the sporadic points. Moreover,

(1) If $n \leq 5$, then there are no sporadic points. Thus, one single rank one PSD matrix suffices.
(2) If $n = 6$, then R, contains one rank one PSD matrix and the matrix
$$M = \begin{bmatrix} 2 & 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix} \text{ with } \det(M) = 3.$$

Theorem. For dimension $3 \leq n \leq 10$, the conical semigroup $\mathrm{SOC}(n) \cap \mathbb{Z}^n$ is (R,G)-finitely generated (we avoid describing the generators here but they are available in paper).

In a forthcoming second paper, we investigated the natural question which 2-dimensional cones are (R,G)-finitely generated? We have fully answered this and of course the only interesting case is when the rays of the cone are irrational vectors. It turns out that it is a necessary and sufficient condition for this vectors to be eigenvectors of an integer unimodular matrix.

Irrational polyhedral cones in dimensions three and higher are very interesting too. We provided a necessary condition for irrational polyhedral cones to be (R, G)-finitely generated, interesting as a corollary we showed that the even *Fermat cones* defined by $x^{2k} + y^{2k} = z^{2k}$ are not (R, G)-finitely generated.

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Strongly connected orientations and integer lattices

Ahmad Abdi

(joint work with Gérard Cornuéjols, Siyue Liu, Olha Silina)

Let $D = (V, A)$ be a digraph whose underlying graph is 2-edge-connected, and let $\mathrm{SCR}(D)$ be the polytope whose vertices are the incidence vectors of *strengthening sets*, i.e., arc sets whose reversal makes D strongly connected. We study the lattice theoretic properties of the integer points contained in a proper face $\mathrm{SCR}(D)$ not contained in $\{x : x_a = i\}$ for any $a \in A, i \in \{0, 1\}$.

It is known that $\mathrm{SCR}(D)$ is described by capacity constraints $x \geq \mathbf{0}, x \leq \mathbf{1}$, and *cut inequalities*, which are of the form $x(\delta^+(U)) - x(\delta^-(U)) \geq 1 - |\delta^-(U)|$ for U a nonempty proper vertex subset. Let F be a face of $\mathrm{SCR}(D)$ obtained by setting some cut inequalities to equality, where at least one right-hand side value is nonzero; let g be the greatest common divisor of these right-hand side values. Denote by L the lattice generated by the integer linear combinations of $F \cap \mathbb{Z}^A$. In [1], we prove that

(1) L has a lattice basis contained among the generators $F \cap \mathbb{Z}^A$; and
(2) $gx \in L$ for all integral vectors x in the linear hull of F.
(3) Subsequently, if $g = 1$, then $F \cap \mathbb{Z}^A$ contains an *integral basis* B, i.e., B is linearly independent, and any integral vector in the linear hull of F is an integral linear combination of B.

The last result is surprising as the integer points in F do not necessarily contain a (minimal) Hilbert basis. In proving these results, we develop a theory similar to Matching Theory for degree-constrained dijoins in digraphs where every vertex is a source or a sink.

Our results have several consequences, including to a famous conjecture by Woodall [3] that the minimum size of a dicut of D, say τ, is equal to the maximum number of disjoint dijoins. We prove a relaxation of this conjecture, by finding for any prime number $p \geq 2$, a *p-adic* packing y of dijoins of value τ and of support size at most $2|A|$. That is, y_J is a nonnegative rational of the form

$a/p^b, a, b \in \mathbb{Z}$ for every dijoin J, $\sum_{J \ni a} y_J \leq 1$ for every $a \in A$, $\sum_J y_J = \tau$, and $|\{J : y_J \neq 0\}| \leq 2|A|$.

Schrijver [2] conjectures that A can always be partitioned into τ strengthening sets (each of which must intersect every minimum dicut exactly once). As a step towards this, we prove that the all-ones vector belongs to the lattice generated by $F \cap \mathbb{Z}^A$, where F is the face of $\mathrm{SCR}(D)$ satisfying $x(\delta^+(U)) = 1$ for every minimum dicut $\delta^+(U)$.

Finally, let $H = (V, \mathcal{E})$ be a τ-uniform hypergraph such that every nonempty proper vertex subset is crossed by at least τ hyperedges. A *strong orientation* is a mapping $O : \mathcal{E} \to V$ that designates to each hyperedge $E \in \mathcal{E}$ a *head* $O(E) \in E$ inside the hyperedge, where every nonempty proper vertex subset X has a crossing hyperedge whose head is inside X. It has been conjectured by Kristóf Bérczi and Karthik Chandrasekaran (personal communication) that H has τ pairwise head-disjoint strong orientations. That is, there exists an assignment $\lambda_O \in \mathbb{Z}_{\geq 0}$ to every strongly connected orientation $O : \mathcal{E} \to V$ such that

$$\sum_{O(E)=v} \lambda_O = 1 \qquad \forall E \in \mathcal{E}, \forall v \in E.$$

We prove this if the nonnegativity requirement on the λ_O's is dropped.

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On integer feasibility of polyhedra with bounded subdeterminants

Stefan Kuhlmann

(joint work with Marcel Celaya, Martin Henk, Joseph Paat, Robert Weismantel)

Consider a polyhedron P given by the inequality system $Ax \leq b$, where A and b have integer entries, and the absolute values of the full rank subdeterminants of A are bounded by some constant. Our objective is to determine in polynomial time whether P contains an integer vector. When $\Delta = 1$, this problem simplifies to deciding the feasibility of $Ax \leq b$. For $\Delta = 2$, there exists a one-dimensional certificate for integer feasibility [5]. We provide an example demonstrating that a natural extension of this result to $\Delta = 3$ does not hold, despite existing randomized algorithms for integer feasibility when the subdeterminants are in $\pm\{0, \Delta\}$ when $\Delta \leq 4$ [3, 4]. It remains an open problem whether one can determine in polynomial time if P contains an integer vector for general values of $\Delta \geq 3$. To advance the understanding of this problem, we investigate structural properties of P that guarantee the existence of integer vectors. This leads naturally to the notion of lattice width. A first insight is that the lattice width can be approximated in polynomial time within a factor of Δ by considering only the facet-defining rows of A [2]. The minimum width among the facet-defining rows of A is known as the facet width of P. Building on this, we present state-of-the-art bounds in terms of Δ for the facet width [1]. Unlike much of the existing literature, some of these bounds do not depend on the dimension of P. A key technique involves linking the facet width to the proximity between solutions of linear programs and integer solutions of their corresponding integer linear programs.

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Cyclic Transversal Polytopes

Volker Kaibel

(joint work with Jonas Frede, Maximilian Merkert, Jannik Trappe)

We introduce the class of cyclic transversal polytopes (CTP's) and the class of lifted odd set (LOS) inequalities for CTP's [1] that are generalizations of Jeroslow's odd set inequalities for parity polytopes [2]. It turns out that several well-known polytopes that are relevant in Combinatorial Optimization are special cases of CTP's, among them matching polytopes, stable set polytopes, and cut polytopes. We then show that the LOS inequalities are a common generalization of Edmonds' inequalities for matching polytopes, the odd hole inequalities for stable set polytopes, and the cycle inequalities for cut polytopes. We furthermore discuss possibilities for generating relaxation hierarchies via CTP's, where the first level is the relaxation obtained from the LOS inequalities.

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From Incremental Transitive Cover to Strongly Polynomial Maximum Flow

László A. Végh

(joint work with Daniel Dadush, James B. Orlin, Aaron Sidford, László A. Végh)

We provide faster strongly polynomial time algorithms solving maximum flow in structured n-node m-arc networks. Our results imply an an $n^{\omega+o(1)}$-time strongly polynomial time algorithms for computing a maximum bipartite b-matching where ω is the matrix multiplication constant. Additionally, they imply an $m^{1+o(1)}W$-time algorithm for solving the problem on graphs with a given tree decomposition of width W.

We obtain these results by strengthening and efficiently implementing an approach in Orlin's [1] the state-of-the-art $O(mn)$ time maximum flow algorithm. We develop a general framework that reduces solving maximum flow with arbitrary capacities to (1) solving a sequence of maximum flow problems with polynomial bounded capacities and (2) dynamically maintaining a size-bounded supersets of the transitive closure under edge additions; we call this *incremental transitive cover*. Our applications follow by leveraging recent weakly polynomial, almost linear time algorithms for maximum flow due to Chen, Kyng, Liu, Peng, Gutenberg, Sachdeva [2] and Brand, Chen, Kyng, Liu, Peng, Gutenberg, Sachdeva, Sidford [3], and by developing new incremental transitive cover data structures for special cases.

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Poly-logaritmic Approximations for Directed Steiner Tree and Forest in Planar Digraphs

Chandra Chekuri

(joint work with Rhea Jain, Shubhang Kulkarni, Da Wei Zheng, Weihao Zhu)

In Directed Steiner Tree (DST) the input is a directed graph $G = (V, E)$ with edge costs, a root r and a set of k terminals $T \subseteq V$. The goal is to find a min-cost arborescence rooted at r that contains all the terminals. There is a quasi-polynomial time $O(\log^2 k/\log\log k)$-approximation and essentially matching hardness. Obtaining a poly-time poly-logarithmic approximation for DST is a long-standing open problem. The natural LP has a polynomial factor integrality gap. In 2023, Friggstad and Mousavi [3] obtained a simple and elegant $O(\log k)$-approximation in *planar* digraphs. We show that the ideas in [3] can be used to obtain poly-log approximations for related rooted connectivity problems in planar digraphs [1]. Moreover, we show that the natural LP has an integrality gap of $O(\log^2 k)$.

Subsequently, we address the Directed Steiner Forest (DSF) problem where the goal is to find a min-cost subgraph that connects a given sets of source-destination pairs $(s_1, t_1), \ldots, (s_k, t_k)$. In general digraphs there is an almost poly-factor hardness for DSF. In contrast, we obtain a poly-logarithmic approximation for planar digraphs [2]. This is based on the junction tree technique and the LP result for DSF. The key to the preceding results are insights on planar graph reachability and separators from Thorup's work on shortest path data structure in planar digraphs [4].

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New Bounds for the Integer Carathéodory Rank

Timm Oertel

(joint work with Iskander Aliev, Martin Henk, Mark Hogan and Stefan Kuhlmann)

Let $C \subset \mathbb{R}^n$ be a pointed rational polyhedral cone. A classical result by Carathéodory states that each point of C is a non-negative combination of at most n vectors which lie on extreme rays of C. To derive an integral analogue, consider the inclusion minimal generating set $H \subset C \cap \mathbb{Z}^n$, called the *Hilbert basis* of C, such that any vector in $C \cap \mathbb{Z}^n$ can be expressed as a non-negative integer combination of vectors in H. As the cone C is rational and pointed, this set is finite and unique. Now, analog to Carathéodory's Theorem, one can ask what is the minimum number k of Hilbert basis elements needed to express any given integer vector in the cone C? We will refer to k as the *integer Carathéodory rank* of C, and denote it with $\mathrm{CR}(C)$. In 1986, Cook, Fonlupt, and Schrijver showed that the integer Carathéodory rank can be bounded solely in terms of the dimension n. More precisely they showed $\mathrm{CR}(C) \leq 2n - 1$ [4]. Four years latter Sebő improved this bound by one [5]. It was conjectured that $\mathrm{CR}(C) \leq n$. However, this conjecture was disproven by Bruns et al. in [3], where it was shown that for every n there exists an n-dimensional cone C with $\mathrm{CR}(C) \geq \lfloor \frac{7}{6}n \rfloor$. Since then little improvement has been made. In particular, the 'correct' bound remains unknown.

In this talk, we present new parametric and asymptotic bounds. In particular in an asymptotic setting, where we only consider 'almost all' integer vectors in C, we are able to improve the upper bound significantly [1]. Bruns and Gubeladze introduced first in [2] the *asymptotic integer Carathéodory rank* of C, denoted by $\mathrm{CR}^{\mathrm{a}}(C)$, which is the smallest integer k such that there exists a set $D \subseteq C \cap \mathbb{Z}^n$ such that:

(i) one has

$$\lim_{\delta \to \infty} \frac{\# D \cap [-\delta, \delta]^n}{\# C \cap \mathbb{Z}^n \cap [-\delta, \delta]^n} = 1 \text{ and}$$

(ii) any point in D can be expressed as a non-negative integer combination of at most k Hilbert basis elements.

Bruns and Gubeladze were able to show that $\mathrm{CR}^{\mathrm{a}}(C) \leq 2n - 3$ and that there exist cones C for which $\mathrm{CR}^{\mathrm{a}}(C) > n$ [2]. First we were able to extend the work of [2] and [3] and show that for every n there exists an n-dimensional cone C_n such that

$$\mathrm{CR}^{\mathrm{a}}(C_n) \geq \lfloor \tfrac{7}{6}n \rfloor.$$

Secondly and more importantly, we were able to improve the leading coefficient of the upper-bound of asymptotic integer Carathéodory rank significantly, namely

$$\mathrm{CR}^{\mathrm{a}}(C) \leq \lfloor \tfrac{3}{2}n \rfloor.$$

While closing the gap between the best upper bound and worst case examples for the asymptotic integer Carathéodory rank, the 'correct' bound remains unknown.

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Determinant Maximization and Matroid Intersection Problems

Mohit Singh
(joint work with Adam Brown, Aditi Laddha, Madhusudan Pittu, Prasad Tetali)

Representing data via vectors and matrices and optimizing spectral objectives such as determinants, and traces of naturally associated matrices is a standard paradigm that is utilized in multiple areas including machine learning, statistics, convex geometry, location problems, allocation problems, and network design problems. In this talk, we will look at many of these applications with a focus on the determinant objective. We will then give algorithms for these problems that build on classical matroid intersection algorithms [1].

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Lagrangian dual with zero duality gap that admits decomposition

Santanu S. Dey
(joint work with Diego Cifuentes, Jingye Xu)

Consider a two-block mixed integer program (MIP) with coupling constraints of the following form:

$$\text{OPT} := \min_{(\mathbf{x},\mathbf{y})} \sum_{i\in\{1,2\}} \left\langle \mathbf{c}^{(i)}, \mathbf{x}^{(i)} \right\rangle + \left\langle \mathbf{d}^{(i)}, \mathbf{y}^{(i)} \right\rangle \tag{1a}$$

$$\text{s.t. } (\mathbf{x}^{(i)}, \mathbf{y}^{(i)}) \in \mathcal{X}^{(i)}, \forall i \in \{1,2\}, \tag{1b}$$

$$\mathbf{x}^{(1)} = \mathbf{x}^{(2)} \in \{0,1\}^n. \tag{1c}$$

where $\mathcal{X}^{(i)} := \left\{ (\mathbf{x}^{(i)}, \mathbf{y}^{(i)}) \;\middle|\; \begin{array}{l} A^{(i)}\mathbf{x}^{(i)} + B^{(i)}\mathbf{y}^{(i)} \le \mathbf{b}^{(i)}, \\ \mathbf{y}^{(i)} \text{ is nonnegative and mixed-integer,} \\ \mathbf{x}^{(i)} \in \{0,1\}^n \end{array} \right\}$ with $A^{(i)}$, $B^{(i)}$, $\mathbf{b}^{(i)}$, $\mathbf{c}^{(i)}$, $\mathbf{d}^{(i)}$ being rational data of suitable dimension for each $i \in \{1,2\}$.

If the coupling constraints (1c) are ignored, then the remaining problem can be decomposed into independent optimization tasks over each $\mathcal{X}^{(i)}$. One classic approach that exploits this structure to obtain dual bounds for (1) is that of Lagrangian relaxation. Specifically, by dualizing (1c), we obtain:

$$L(\lambda) := \min_{(\mathbf{x},\mathbf{y})} \left(\sum_{i\in\{1,2\}} \left\langle \mathbf{c}^{(i)}, \mathbf{x}^{(i)} \right\rangle + \left\langle \mathbf{d}^{(i)}, \mathbf{y}^{(i)} \right\rangle \right) + \left\langle \lambda, \mathbf{x}^{(1)} - \mathbf{x}^{(2)} \right\rangle \tag{2}$$
$$\text{s.t. } (\mathbf{x}^{(i)}, \mathbf{y}^{(i)}) \in \mathcal{X}^{(i)}, \forall i \in \{1,2\},$$

and

$$\text{DUAL} := \max_{\lambda} L(\lambda). \tag{3}$$

It is well-known that $L(\lambda)$ is a non-smooth concave function and sub-gradients for $L(\lambda)$ can be obtained by solving (2), which is a collection of independent optimization tasks over $\mathcal{X}^{(i)}$ that could be solved in parallel. Using these sub-gradients, one can solve (3) via non-smooth optimization methods.

Even though weak duality always holds, due to non-convexity, strong duality generally fails, that is, OPT $>$ DUAL. On the other hand, one can solve (1) directly without exploiting decomposability, trivially obtaining zero duality gap. We obtain the following results:

(1) Obtaining zero duality gap and decomposability simultaneously: We design a reformulation of (1) (called M-Lagrangian) whose Lagrangian dual, achieve the twin goal of zero duality gap and decomposability. The M-Lagrangian method is a hierarchy of reformulations of (1) similar to the Reformulation-Linearization-Technique (RLT) but not the same, whose Lagrangian duals achieve zero duality gap in the last step of the hierarchy, while simultaneously each level admits decomposition into sub-problems.
(2) Analysis of bounds: We present multiplicative bounds on the duality gap at different levels of the M-Lagrangian hierarchy for packing and covering problems.
(3) Generalization to arbitrary MIPs: Consider a loosely coupled general MIP where the block structure is revealed using a tree-decomposition of the intersection graphof the constraint matrix. Here, the blocks correspond to smaller problems defined over variables in the bags of the tree-decomposition. We show how to generalize the above results: simultaneously achieving decomposability and strong duality, and multiplicative bounds for packing and covering instances, for the M-Lagrangian dual to this setting.
(4) Preliminary computational results: We illustrate how the proposed Lagrangian duals can outperform classical Lagrangian relaxation and a commercial solver in terms of dual bounds achieved.

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Faster Lattice Basis Computation via a Natural Generalization of the Euclidean Algorithm

Kim-Manuel Klein

(joint work with Janina Reuter)

The Euclidean algorithm is one of the oldest algorithms known to mankind. Given two integral numbers a_1 and a_2, it computes the greatest common divisor (gcd) of a_1 and a_2 in a very elegant way. From a lattice perspective, it computes a basis of the lattice generated by a_1 and a_2 as $\gcd(a_1, a_2)\mathbb{Z} = a_1\mathbb{Z} + a_2\mathbb{Z}$. In this work, we show that the classical Euclidean algorithm can be adapted in a very natural way to compute a basis of a lattice that is generated by vectors $A_1, \ldots, A_n \in \mathbb{Z}^d$ with $n > \operatorname{rank}(A_1, \ldots, A_n)$. Similar to the Euclidean algorithm, our algorithm is easy to describe and implement and can be written within 12 lines of pseudocode.

As our main result, we obtain an algorithm to compute a lattice basis for given vectors $A_1, \ldots, A_n \in \mathbb{Z}^d$ in time (counting bit operations) $LS + \tilde{O}((n-d)d^2 \cdot \log(||A||)$, where LS is the time required to obtain the exact fractional solution of a certain system of linear equalities. The analysis of the running time of our algorithms relies on fundamental statements on the fractionality of solutions of linear systems of equations.

So far, the fastest algorithm for lattice basis computation was due to Storjohann and Labahn (ISSAC 1996) having a running time of $\tilde{O}(nd^\omega \log ||A||)$, where ω denotes the matrix multiplication exponent. We can improve upon their running time as our algorithm requires at most $\tilde{O}(\max\{n-d, d^2\} d^{\omega(2)-1} \log ||A||)$ bit operations, where $\omega(2)$ denotes the exponent for multiplying a $n \times n$ matrix with an $n \times n^2$ matrix. For current values of ω and $\omega(2)$, our algorithm improves the running time therefore by a factor of at least $d^{0.12}$ (since $n > d$) providing the first general runtime improvement for lattice basis computation in nearly 30 years. In the cases of either few additional vectors, e.g. $n - d \in d^{o(1)}$, or a very large number of additional vectors, e.g. $n - d \in \Omega(d^k)$ and $k > 1$, the run time improves even further in comparison.

At last, we present a postprocessing procedure which yields an improved size bound of $\sqrt{d}||A||$ for vectors of the resulting basis matrix. The procedure only requires $\tilde{O}(d^3 \log ||A||)$ bit operations. By this we improve upon the running time of previous results by a factor of at least $d^{0.74}$.

Recent Progress for Correlation Clustering

Alantha Newman

(joint work with Nairen Cao, Vincent Cohen-Addad, Euiwoong Lee, Shi Li, Lukas Vogl)

In the correlation clustering problem, we are given a complete graph with each edge labeled as + (similar) or − (dissimilar). The goal is to find a clustering (a partition) of the vertices that minimizes the number of *disagreements*, which are dissimilar intracluster edges and similar intercluster edges. The approximability of this problem has been studied [1, 2, 3], culminating in a 2.06-approximation algorithm, based on a modification of the natural LP-based pivot algorithm [4]. Until recently, this was the best-known approximation factor; since the integrality gap for the natural LP formulation is at least 2, it had remained a tantalizing open question to determine if the approximation factor of 2 can be reached or even breached.

We now know how to go below 1.5 using various new relaxations and rounding tools [5, 6, 7]. Our new contributions include: the use of the *correlated rounding* procedure for rounding a solution for relaxations based on the *Sherali-Adams* hierarchy, which bypasses the limitations of the previous independent rounding of the natural LP formulation; a *preclustering* subroutine to absorb the rounding error from the correlated rounding; and the formulation of the *cluster LP*, which is a new relaxation, solvable via an intermediate Sherali-Adams relaxation, that provides a clean framework in which to study rounding algorithms for the correlation clustering problem. In order to analyze our algorithms for rounding the cluster LP, we introduce additional tools such as *budget functions for edges*, allowing for an edge-by-edge analysis (as opposed to previous triangle-by-triangle analysis). To go below the approximation ratio of 1.5, we use a *global triangle charging* argument. In this talk, we give an overview of these contributions and discussed some of the open questions.

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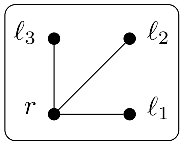
A characterization of unimodular hypergraphs with disjoint hyperedges

Joseph Paat

(joint work with Marco Caoduro, Meike Neuwohner)

Grossman et al. [1] show that the subdeterminants of the incidence matrix of a graph can be characterized using the graph's odd cycle packing number. In particular, a graph's incidence matrix is totally unimodular if and only if the graph is bipartite. We extend the characterization of total unimodularity to disjoint hypergraphs, i.e., hypergraphs whose hyperedges of size at least four are pairwise disjoint. Disjoint hypergraphs interpolate between graphs and hypergraphs, which correspond to arbitrary $\{0, 1\}$-matrices. We prove that total unimodularity for disjoint hypergraphs is equivalent to forbidding both odd cycles and a structure we call an *odd tree house*. The following figure, which comes from S83.3 in [2], provides an example of an odd tree house.



Our result extends to disjoint directed hypergraphs, i.e., those whose incidence matrices allow for $\{0, \pm 1\}$-entries. As a corollary, we resolve a conjecture on almost totally unimodular matrices, formulated by Padberg [3] and Cornuéjols & Zuluaga [4], in the special case where columns with at least four non-zero elements have pairwise disjoint supports.

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Thin trees for laminar families

Neil Olver

(joint work with Nathan Klein)

In the laminar-constrained spanning tree problem, the goal is to find a minimum-cost spanning tree which respects upper bounds on the number of times each cut in a given laminar family is crossed. This generalizes the well-studied degree-bounded spanning tree problem [1, 4], as well as a previously studied setting where a chain of cuts is given [3]. We give the first constant-factor approximation algorithm; in particular we show how to obtain a multiplicative violation of the crossing bounds of less than 22 while losing less than a factor of 5 in terms of cost.

Our result compares to the natural LP relaxation. That is, given a point x in the spanning tree polytope of G, and a laminar family $\mathcal{L}$, we produce a spanning tree T such that $|\delta(S) \cap T| = O(x(\delta(S))$ and $c(T) = O(c(x))$. As a consequence (by exploiting that in a k-edge-connected graph, the point $x \in \mathbb{R}^E$ defined by $x_e = 2/k$ for each e is in the dominant of the spanning tree polytope), our results show that given a k-edge-connected graph and a laminar family $\mathcal{L} \subseteq 2^V$ of cuts, there exists a spanning tree which contains only an $O(1/k)$ fraction of the edges across every cut in $\mathcal{L}$. This can be viewed as progress towards the *Thin Tree Conjecture* [2], which (in a strong form) states that this guarantee can be obtained for all cuts simultaneously.

Our approach is based on linear programming relaxation. In this approach, a version of the LP relaxation is repeatedly solved, and a basic optimal solution is found. Using sparsity properties of this solution, we need to argue that a cut constraint can be removed in a "safe" way, where we can guarantee that the violation of this cut by the end of the algorithm is within the claimed multiplicative factor. As long as a constraint can be safely dropped, progress is made; a new LP is obtained, with a new basic optimal solution, and the process continues.

In order to succesfully apply this method, a crucial first step was required. We show that the problem can be reduced to a special case that we call $\mathcal{L}$-*aligned*. Here, the initial fractional solution x has the property that its restriction to every set $S \in \mathcal{L}$ is itself in the spanning tree polytope of $G[S]$.

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Open Problem: Is Zonotope Containment fixed-parameter tractable with respect to the dimension d?

Martin Skutella

A *zonotope* is a Minkowski sum of line segments, i.e., given a matrix $\mathbf{G} \in \mathbb{R}^{d \times n}$, the corresponding zonotope is given by

$$Z(\mathbf{G}) := \sum_{i=1}^{n} \operatorname{conv}\{\mathbf{0}, \mathbf{g}_i\},$$

where $\mathbf{g}_1, \ldots, \mathbf{g}_n \in \mathbb{R}^d$ are the columns of $\mathbf{G}$. We are interested in the following Zonotope Containment Problem: Given two matrices $\mathbf{G} \in \mathbb{R}^{d \times n}$ and $\mathbf{H} \in \mathbb{R}^{d \times m}$, decide whether or not $Z(\mathbf{G}) \subseteq Z(\mathbf{H})$. It is known that the Zonotope Containment Problem is coNP-complete [1]. We ask if the Zonotope Containment Problem is *fixed-parameter tractable* with respect to the dimension d. In other words:

> Is there an algorithm that decides Zonotope Containment with running time $\mathrm{poly}(n+m) \cdot f(d)$ for an arbitrary function f?

We are interested in this problem as it comes up in the mathematical study of neural networks with rectified linear units (ReLUs), a widely used model in deep learning. More details can be found in our recent manuscript [2], where the problem is stated.

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Open Problem: The δ-distance Game

Sophie Huiberts

We say that a full-dimensional simplex $\mathrm{conv}(a_1, \ldots, a_{d+1}) \subset \mathbb{R}^d$ has the δ-*distance property* if for each $i = 1, \ldots, d+1$ the Euclidean distance between vertex a_i and the supporting hyperplane of the opposite face $\mathrm{affhull}(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{d+1})$ is at least δ.

The δ-*distance game* is played between two players, Peter Pivot and Ronald Ratio. We start with a set of points in the unit ball $A^0 \subset \mathbb{B}^d, |A| = d+1$, whose convex hull satisfies the δ-distance property. In round $k \in \mathbb{N}$ Peter will first pick some leaving point $l \in A^0$, after which Ronald may pick some entering point $e \in \mathbb{B}^d$ and set $A^{k+1} = A^k \cup \{e\} \setminus \{l\}$. In his choice, Roland must satisfy two rules:

(1) the convex hull of A^{k+1} must once again satisfy the δ-distance property, and

(2) e and l must be on opposite sides of the remaining face, i.e.,

$$\mathrm{conv}(A^k) \cap \mathrm{conv}(A^{k+1}) = \mathrm{conv}(A^k \setminus \{l\}).$$

The game ends as soon as the simplex contains the origin $0 \in \mathrm{conv}(A^{k+1})$. Peter wants to finish as soon as possible, while Roland aims to delay this. Does Peter have a finite strategy? And if so, does Peter have a strategy that finishes in $\mathrm{poly}(d, \delta^{-1})$ rounds?

The δ-distance game is a model for the simplex method, where Peter is a model for the pivot rule and Roland for the ratio test. In the case where the feasible region of a linear program is *non-degenerate* then the ratio test always has a single eligible choice, in which case there is a pivot rule that makes the simplex method run in expected time $O(d\delta^{-1}\log(d/\delta))$. This can be found in the first reference. The idea behind the δ-distance is to identify if a similar type of result may exist when the ratio test is a more unpredictable algorithm, such as proposed by Paula Harris in the second reference.

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Open Problem: Maximum balanced independent sets in the boolean cube

Monique Laurent

(joint work with Sven Polak, Luis Felipe Vargas)

Given a bipartite graph $G = (V = U \cup W, E)$, an independent (a.k.a. stable) set $S \subseteq V$ is called *balanced* if $|S \cap U| = |S \cap W|$ holds. Then, $\alpha(G)$ denotes the maximum cardinality of an independent set in G and we let $\alpha_{\text{bal}}(G)$ denote the maximum cardinality of a *balanced* independent set. As is well-known, one can compute the parameter $\alpha(G)$ in polynomial time for bipartite graphs. On the other hand, computing the balanced parameter is hard: given G bipartite and an integer $k \in \mathbb{N}$, deciding whether $\alpha_{\text{bal}}(G) \geq k$ is an NP-complete problem [5]. Approximating the balanced parameter is also known to be hard (under well-known complexity assumptions); see, e.g., [1, 3, 7].

In [6] we investigate the parameter $\alpha_{\text{bal}}(G)$ and some close relatives, relevant to bi-independent pairs and to bicliques in general graphs. In particular, we show the following sharper hardness result: given a bipartite graph G, it is NP-complete to decide whether $\alpha_{\text{bal}}(G) = \alpha(G)$. We also show that this implies hardness of the following parameters:

$$g(G) = \max\{|A| \cdot |B| : A \subseteq U, B \subseteq W, A \cup B \text{ is independent}\},$$
$$h(G) = \max\left\{\frac{|A| \cdot |B|}{|A \cup B|} : A \subseteq U, B \subseteq W, A \cup B \text{ is independent}\right\}.$$

Hardness of $g(\cdot)$ was shown earlier in [9] in terms of the maximum edge biclique problem. The parameter $h(\cdot)$ is relevant, in particular, to work by Gowers dealing with maximum product-free sets in groups (see [6] for details).

Semidefinite bounds and eigenvalue bounds are given in [6] for the parameters $\alpha_{\text{bal}}(G)$ and its relatives. In particular, if G is a bipartite r-regular graph, then

$$\alpha_{\text{bal}}(G) \leq |V| \frac{\lambda_2}{r + \lambda_2}, \tag{1}$$

where λ_2 is the second largest eigenvalue of the adjacency matrix of G.

Maximum balanced independent sets in the boolean cube Q_r. Let Q_r denote the boolean cube, whose vertex set is $V = \{0,1\}^r$ and whose edges are the pairs $\{x, y\} \subseteq V$ where x and y are at Hamming distance 1. So, the bipartition is given by

$$U = \mathrm{Ev}(r) = \{x \in \{0,1\}^r : |x| \text{ is even}\},\ W = \mathrm{Odd}(r) = \{y \in \{0,1\}^r : |y| \text{ is odd}\}.$$

Here, $|x|$ denotes the Hamming weight of $x \in V$. The boolean cube Q_r is r-regular and its second largest eigenvalue is $\lambda_2 = r - 2$. Hence, the eigenvalue bound (1) reads:

$$\alpha_{\mathrm{bal}}(Q_r) \leq 2^{r-1}\frac{r-2}{r-1}. \tag{2}$$

On the other hand, the following lower bound is shown in [6]:

$$\alpha_{\mathrm{bal}}(Q_r) \geq a(r-1), \tag{3}$$

where the sequence $a(r)_{r\geq 1}$ is defined iteratively by

$$a(1) = 0, \quad a(2r) = 2^{2r} - \binom{2r}{r}, \quad a(2r+1) = 2 \cdot a(2r) \quad \text{for } r \geq 1. \tag{4}$$

To see this, consider the subsets of $\{0,1\}^{2r}$:

$$S = \{x \in \{0,1\}^{2r} : |x| \leq r-1\}, \quad T = \{y \in \{0,1\}^{2r} : |y| \geq r+1\}.$$

Then, $|S| = |T|$ and $|S \cup T| = 2^{2r} - \binom{2r}{r} = a(2r)$. Define the subsets of $\{0,1\}^{2r+1}$:

$$S_{\mathrm{even}} = \{(x, |x| \bmod 2) : x \in S\} \subseteq \mathrm{Ev}(2r+1),$$
$$T_{\mathrm{odd}} = \{(y, |y| + 1 \bmod 2) : y \in T\} \subseteq \mathrm{Odd}(2r+1).$$

Then, the set $S_{\mathrm{even}} \cup T_{\mathrm{odd}}$ is balanced and independent in Q_{2r+1}, which shows $\alpha_{\mathrm{bal}}(Q_{2r+1}) \geq a(2r)$. Next, set $S' = S \times \{0,1\}, T' = T \times \{0,1\} \subseteq \{0,1\}^{2r+1}$, and define analogously the subsets of $\{0,1\}^{2r+2}$:

$$S'_{\mathrm{even}} = \{\{(x, |x| \bmod 2) : x \in S'\} \subseteq \mathrm{Ev}(2r+2),$$
$$T'_{\mathrm{odd}} = \{(y, |y| + 1 \bmod 2) : y \in T'\} \subseteq \mathrm{Odd}(2r+2).$$

Then, $|S'_{\mathrm{even}}| = 2|S|$, $|T'_{\mathrm{odd}}| = 2|T|$, the set $S'_{\mathrm{even}} \cup T'_{\mathrm{odd}}$ is balanced and independent in Q_{2r+2}, which shows $\alpha_{\mathrm{bal}}(Q_{2r+2}) \geq 2 \cdot a(2r) = a(2r+1)$. Hence, the lower bound (3) holds. Using the upper bound (2) one can show that

$$\lim_{r\to\infty} \frac{\alpha_{\mathrm{bal}}(Q_r)}{a(r-1)} = 1.$$

This indicates that the lower bound (3) might be tight and motivates the following conjecture of [6], which has been verified (numerically) for $r \leq 13$.

Conjecture. For any $r \geq 1$, we have $\alpha_{\mathrm{bal}}(Q_r) = a(r-1)$.

It is interesting to note that the sequence $a(r)$ in (4) corresponds to the sequence A307768 in OEIS Foundation Inc. [8]. This sequence counts the number of walks of length r along the line, starting at the origin and returning to it at least once, where at each step one may walk left or right. The sequence $a(r)$ is also related to other combinatorial counting problems (see, e.g., [2], [4]). Settling the above conjecture about the balanced independence number of the boolean cube would also establish a new interesting combinatorial interpretation for the sequence $a(r)$.

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Open Problem: Fare Zone Assignment

Britta Peis

(joint work with Britta Peis, Philipp Pabst, Lennart Kauther, and Sam Fiorini)

We are given a tree $G = (V, E)$ and a set of commodities (P_i, w_i, u_i), where each $P_i \subseteq E$ is a path in G, w_i denotes the weight of P_i and u_i denotes the "maximum number of allowed cuts on P_i". We want to solve the following problem:

$$\begin{aligned} \max \ & \sum_{e \in E} y_e \cdot w_e \\ \text{s.t.} \ & \sum_{e \in P_i} y_e \leq u_i \quad \forall i \\ & y_e \in \{0,1\} \quad \forall e \end{aligned}$$

where $w_e = \sum_{\{i \mid e \in P_i\}} w_i$.

This problem is polytime solvable when G is a path and strongly NP-hard when G is a star. There is a 2-approximation when all u_i are even (implying a 3-approximation when all $u_i \geq 2$). What else can we say about approximation results on trees? In particulare, we would like to understand the approximability of the problem when the "congestion", i.e., the maximum number of paths that share the same edge, is bounded by some parameter k.

Open Problem: Complexity of 2-coloring light tournaments

Alantha Newman
(joint work with Felix Klingehoefer)

A *tournament* $T = (V, A)$ is an orientation of a complete graph: for each pair of vertices $u, v \in V$, either arc (u, v) or arc (v, u) belongs to the arc set A. The *neighborhood* of an arc (u, v), denoted by $N(u, v)$, contains all vertices $w \in V$ such that u, v and w form a directed triangle. If $N(u, v)$ contains no directed triangles (i.e., it is acyclic), then we say that the arc (u, v) is *light*. If all arcs of a tournament T are light, then we say that T is a *light tournament*.

A k-coloring of a tournament is a partition of its vertex set into k acyclic sets (i.e., sets whose induced subgraph contains no directed cycle). The problem of deciding if a given tournament is 2-colorable is NP-hard [1], and it is even hard to color a 2-colorable tournament with three colors [2, 3]. There are polynomial-time algorithms to color 2-colorable tournaments with ten colors and to color 2-colorable light tournaments with five colors [2, 3]. The open question is to determine the complexity of 2-coloring light tournaments: Is there a polynomial-time algorithm to color a 2-colorable light tournament with two colors? Or is it NP-hard to decide if a light tournament is 2-colorable?

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Open Problem: Rounding Algorithms for Feedback Vertex Set and Subset Feedback Vertex Set

Chandra Chekuri

The Feedback Vertex Set problem (FVS) in undirected graphs is the following: given a graph $G = (V, E)$ with vertex weights $w : V \to \mathbb{Z}_+$, find a min-weight vertex subset $S \subseteq V$ such that $G - S$ is acyclic. In Oberwolfach 2021 workshop on Combinatorial Optimization, Sam Fiorini raised an open question about extreme point solution to an LP formulation for FVS suggested by Chudak, Goemans, Hochbaum and Williamson [10] who established an integrality gap of 2 for that formulation via the primal-dual method (inspired by the local-ratio based algorithm and analysis from [6, 7]). The LP from [10] is not known to be solvable in polynomial-time. Fiorini asked if every basic feasible solution to the LP formulation in [10] has a variable with value at least $1/2$ (for non-trivial instances). That open problem led to a collaboration and to a paper on polyhedral formulations for FVS and related problems — see [9]. In particular, [9] obtained polynomial-time solvable LP relaxations for FVS with an upper bound of 2 on their integrality gap.

The paper also showed that a previous LP relaxation suggested by Chekuri and Madan [4] also has integrality gap of 2 and this resolved an open problem. However, the integrality gap bounds for the new LP formulations are shown indirectly by proving that the new LP formulations are at least as strong as the one in [10]. We do not know primal rounding proofs that yield a 2-approximation for the LP formulations. In particular, the original question of Fiorini is not yet resolved. Some machinery towards answering this question has been developed in [9]. Not only is the question technically interesting on its own right but it is also relevant to the Subset Feedback Vertex Set problem (SFVS) which generalizes FVS. In SFVS the input is a vertex-weighted graph $G = (V, E)$ and a set of terminal vertices $T \subseteq V$, and the goal is to remove a min-weight subset of the vertices $S \subseteq V$ such that $G - S$ does not have any cycles containing a terminal. For SFVS the best known approximation is 8 [5] while the known lower bound is only 2. Chekuri and Madan [4] proposed the first LP relaxation for SFVS and proved that its integrality gap is at most 13 via a primal-rounding algorithm. No integrality gap example with a ratio worse than 2 is known for this LP relaxation. Although we now know that the LP in [4] has gap at most 2 for the special case of FVS (via [9]), as we mentioned earlier, we do not know a direct primal rounding algorithm to achieve the tight bound. We hope that progress on these questions can lead to improved approximation for SFVS via the LP in [4] (or via other techniques) and also other useful insights on FVS and related problems.

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