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Representations of p-adic Groups

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Abstract. Representation theory of p-adic groups is a topic at a crossroads. It links among others to harmonic analysis, algebraic geometry, number theory, Lie theory, and homological algebra. The atomic objects in the theory are supercuspidal representations. Most of their aspects have a strong arithmetic flavour, related to Galois groups of local fields. All other representations are built from these atoms by parabolic induction, whose study involves Hecke algebras and complex algebraic geometry. In the local Langlands program, connections between various aspects of representations of p-adic groups have been conjectured and avidly studied.

This workshop brought together mathematicians from various backgrounds, who hold the promise to contribute to the solution of open problems in the representation theory of p-adic groups. Topics included explicit local Langlands correspondences, Hecke algebras for Bernstein components, harmonic analysis, covering groups and ℓ-modular representations of reductive p-adic groups.

Mathematics Subject Classification (2020): 20Gxx, 22Exx, 11Fxx.

Introduction by the Organizers

The workshop "Representations of p-adic groups" was organized by Jessica Fintzen, David Schwein, and Maarten Solleveld, and attended by 48 mathematicians spanning all career stages, from early PhD students to established professors. The workshop comprised 22 talks ranging in length from 30 to 90 minutes. Two of those talks were given by a pair of speakers. While most talks discussed recent research, there were also four talks in survey style, designed to introduce all conference participants to a particular topic. In addition, on Monday evening, after dinner, we had an hour of short introductions during which each participant introduced themselves and their mathematics interests in about 50 seconds. Throughout the week there was also much informal discussion among participants.

Within the representation theory of p-adic groups, the workshop had a couple of focus points, which encompassed most talks. It started with a survey talk by Marie-France Vignéras on *ℓ-modular representations*, a topic that she has promoted and contributed to for three decades. Her talk explained similarities and differences between ℓ-modular and complex representation theory. More ℓ-modular representations appeared in the talks of Johannes Girsch and Rose Berry.

Several of the participants were specialists on *supercuspidal representations of reductive p-adic groups*. Extending the current state of the art of the construction of supercuspidal representations in various ways, geometrically or explicitly, was the subject of talks by Charlotte Chan, Alexander Ivanov, and David Schwein. Moreover Geo Kam-Fai Tam and Ekta Tiwari presented results on what happens to such supercuspidal representations under endoscopic lift or restriction.

To understand the whole category of smooth representations of p-adic groups, including the non-supercuspidal representations, Kazuma Ohara gave a presentation on recent results that uses *Hecke algebra* isomorphisms to obtain an equivalence between arbitrary Bernstein blocks and depth-zero Bernstein blocks.

On Tuesday, David Helm and Eugen Hellmann gave a joint survey talk on categorical versions of the *local Langlands correspondence*. They explained the general goals and achievements of the program initiated by Fargues–Scholze, and worked out some examples in low rank. More recent investigations in this direction were discussed by Arnaud Eteve, Chenji Fu, and Yifei Zhao. Cheng-Chiang Tsai reported on current joint work in progress on an explicit local Langlands correspondence for Moy–Prasad types.

To go beyond reductive p-adic groups, Nadya Gurevich and Yifei Zhao exhibited new phenomena and complications that arise for *covering groups* of p-adic groups, and they explained their recent advances to understand these phenomena.

Clifton Cunningham contributed an overview of *ABV-packets and Arthur packets*. It is conjectured that Arthur packets are instances of ABV-packets, and then the difficult, analytically-motivated Arthur packets will be computable by means of the geometrically-defined ABV-packets. Mishty Ray told how far this program has advanced, while Tasho Kaletha and Alberto Mínguez, and separately Wen-Wei Li, reported on Arthur packets from other angles.

The last survey talk was given by Loren Spice, on *harmonic analysis and character formulas* for p-adic groups. He presented a historic overview of the study of characters of representations, ranging from the 19th century to today. Harmonic analysis also featured in the talks of Stephen DeBacker and Emile Okada.

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Workshop: Representations of p-adic Groups

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Abstracts

Survey talk on smooth ℓ-modular representations of connected reductive p-adic groups

Marie-France Vignéras

What is the same as for complex representations and what goes wrong for ℓ-modular representations ? What is known and what is not known?

Let F be a local non-archimedean of finite residue field k_F of characteristic p, $\underline{G}$ be a connected reductive F-group and $G = \underline{G}(F)$, R be a commutative ring.

The category $\mathrm{Mod}_R(G)$ of smooth R-representations of G is abelian Grothendieck of generator $\bigoplus_n C_c(K_n\backslash G, R)$, for any decreasing sequence $(K_n)_n$ of open compact subgroups of trivial intersection.

A representation $V \in \mathrm{Mod}_R(G)$ is called **mod p, ℓ-modular, ℓ-adic**, when $R = \mathbb{F}_p^{\mathrm{ac}}$, $\mathbb{F}_\ell^{\mathrm{ac}}$, $\mathbb{Q}_\ell^{\mathrm{ac}}$ respectively ($\ell \neq p$ and E^{ac} is an algebraic closure of a field E). It is called **admissible** if V^K is a finitely generated R-module for any open compact subgroup K of G.

A prime ℓ is called **banal for** G when it does not divide the pro-order $|K|$ of some K. Let N_G be the product of the non-banal primes. For $\mathrm{SL}_2(F)$ or $\mathrm{GL}_2(F)$, $N_G = p\prod_{\ell|q^2-1}\ell$. For G unramified, $N_G = \prod_{\ell||\underline{G}(k_F)|}\ell$ for a model $\underline{G}$ defined over O_F. When R is an algebraically closed field, an irreducible R-representation has a central character if it is admissible or if R is uncountable. If the characteristic of R is 0 or ℓ banal for G, the ℓ-modular representation theory of G resembles the complex representation theory of G. The p-modular representation theory of G does not resemble the complex representation theory of G because there is a Haar measure on G with values in R if and only if R is a $\mathbb{Z}[1/p]$-algebra.

When R is a field, let $\mathrm{Irr}_R(G)$ denote the set of irreducible representations in $\mathrm{Mod}_R(G)$. If the characteristic of R is $\neq p$, then any $V \in \mathrm{Irr}_R(G)$ is admissible[1], and any finitely generated admissible R-representation of G has finite length.

For an open subgroup K of G, the restriction $\mathrm{Mod}_R(G) \to \mathrm{Mod}_R(K)$ has a left adjoint, the **compact induction** ind_K^G sending $W \in \mathrm{Mod}_R(K)$ to the module of compactly supported functions $G \to W$ right invariant by some K with G acting by right translation. The **Hecke R-algebra of G relative to** K

$$H_R(G,K) = \mathrm{End}_{RG}(\mathrm{ind}_K^G 1_R) \simeq (\mathrm{ind}_K^G 1_R)^K \simeq R[K\backslash G/K]$$

(with the convolution product), is isomorphic to the opposite algebra. If $\mathrm{ind}_K^G(1_R)$ is projective, the **K-invariants functor** $V \mapsto V^K$ gives a bijection

$$\{V \in \mathrm{Irr}_R(G), V^K \neq 0\} \simeq \{M \in \mathrm{Mod}(H_R(G,K)) \text{ simple}\}$$

and a category equivalence from the category $\mathrm{Mod}_R(G,K)$ of $V \in \mathrm{Mod}_R(G)$ generated by V^K is stable by subrepresentations, to $\mathrm{Mod}(H_R(G,K))$. This remains true when $\mathrm{ind}_K^G 1_R$ is only almost projective, as for example for an Iwahori subgroup.

[1] true for mod p representations of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ but false in general

Let $P = MN$ be a parabolic subgroup of G with Levi M and unipotent radical N. The **parabolic induction** $\mathrm{ind}_{P,M}^G : \mathrm{Mod}_R(M) \to \mathrm{Mod}_R(G)$ (inflation from M to P followed by compact induction from P to G) is exact, of left adjoint $\mathrm{res}_{P,M}^G$ (the N-coinvariant functor), has a right adjoint $R_{P,M}^G$, respects admissibility, finite length, and generic irreducibility if R is an algebraically closed field.

When R is a $\mathbb{Z}[1/p]$-algebra, the category $\mathrm{Mod}_R(G)$ is noetherian when R is noetherian. This follows from the second adjointness theorem by Bernstein–Deligne when $R = \mathbb{C}$, and in general by Dat–Helm–Kurinczuk–Moss who proved:

Theorem (Second adjointness)**.** *When R is a $\mathbb{Z}[1/p]$-algebra,*

(i) *(Second adjointness) $R_{P,M}^G = \delta_P^* \,\mathrm{res}_{P^-,M}^G$ where δ_P is the modulus of P and P^- the opposite parabolic of P.*
(ii) *$H_R(G,K)$ is a finitely generated module over its centre, which is a finitely generated R-algebra.*

The proof uses Fargues–Scholze. The theorem implies that $\mathrm{ind}_{P,M}^G$ respects projectivity, finitely generated representations, that $\mathrm{res}_{P,M}^G$ is exact, respects admissibility. We have the geometric lemma.

For mod p representations, $\mathrm{ind}_{P,M}^G$ is fully faithful, does not preserve finitely generated representations, $\mathrm{res}_{P,M}^G$ and $R_{P,M}^G$ fail to be exact, $\mathrm{res}_{P,M}^G$ and $R_{P,M}^G$ send an admissible irreducible to 0 or to an admissible irreducible, $\mathrm{res}_{P,M}^G(V)$ may be not admissible when V is admissible not irreducible, $R_{P,M}^G$ is exact on admissible representations (Hauseux).

In (ii), does one need to invert p ? This is not necessary when K is an Iwahori or a pro-p Iwahori, or G is unramified and K hyperspecial (Xinwen Zhu).

$V \in \mathrm{Irr}_R(G)$ is called **supercuspidal** when it is not a subquotient of $\mathrm{ind}_{P,M}^G W$ for all $W \in \mathrm{Irr}_R(M)$, $M \neq G$, and **cuspidal** if $\mathrm{res}_{P,M}^G = R_{P,M}^G(V) = 0$ for all $P \neq G$. If the characteristic of R is $0, p$ or ℓ banal, cuspidal = supercuspidal. But when $G = \mathrm{GL}_2(\mathbb{Q}_p)$, ℓ divides $p+1$, the ℓ-modular representation $\mathrm{ind}_{B,T}^G(1)$ is indecomposable of length 3, with a cuspidal and non-supercuspidal subquotient.

$W \in \mathrm{Irr}_R(M)$ (super)cuspidal belongs to the **(super)cuspidal support** of $V \in \mathrm{Irr}_R(G)$ if V is a (subquotient) subrepresentation of $\mathrm{ind}_{P,M}^G(W)$ for some P. The cuspidal support is unique up to conjugacy. The supercuspidal support is unique when G is an inner form of GL_n (Mínguez–Sécherre) or G is the unramified unitary group $U(2,1), p \neq 2$ (Kurinczuk), but not for the ℓ-modular representations of $\mathrm{Sp}_8(F)$ for ℓ dividing q^2+1 (Dat).

An irreducible ℓ-adic representation V of G is **integral** (contains a stable admissible $\mathbb{Z}_\ell^{\mathrm{ac}}$-lattice L, necessarily free) if and only if some (any) W in its supercuspidal support is integral if and only if the central character of W is integral. By the Brauer–Nesbitt principle, if V is integral then $L \otimes_{\mathbb{Z}_\ell^{\mathrm{ac}}} \mathbb{F}_\ell^{\mathrm{ac}} \in \mathrm{Mod}_{\mathbb{F}_\ell^{\mathrm{ac}}}(G)$ has finite length, and its semi-simplification $r_\ell(V)$, called the **reduction modulo ℓ of V**, does not depend on the choice of L.

Theorem ([2]). *Each ℓ-modular supercuspidal irreducible representation of G is a subquotient of the reduction modulo ℓ of an integral cuspidal irreducible ℓ-adic representation of G.*

When ℓ is banal, the reduction modulo ℓ gives a surjective map from the irreducible integral cuspidal ℓ-adic representations of G to the irreducible cuspidal ℓ-modular representations of G.

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Positive-depth Deligne–Lusztig induction and positive-depth character sheaves

Charlotte Chan

(joint work with Masao Oi)

I reported on new results studying the representation theory of p-adic groups using geometric methods on parahoric subgroups. One active area of development over the last decade has been the study of positive-depth Deligne–Lusztig induction. In forthcoming work with Masao Oi, we establish—under a largeness assumption on q—that this geometrically realizes Kaletha's regular supercuspidal L-packets associated to unramified tori. Our methodology is based on an elementary analytic argument which allows us to identify representations by their character values on a special locus wherein the formula is remarkably simple. I ended the talk with a brief discussion of the implications of this result in the context of my work with Roman Bezrukavnikov constructing positive-depth character sheaves.

1. Positive-depth Deligne–Lusztig induction

In my work with Alexander Ivanov [1], we defined a class of positive-depth Deligne–Lusztig varieties X_r associated to parahoric subgroups $G_{x,0}$ of G, the F-points of a connected reductive algebraic group $\mathbf{G}$ over a non-archimedean local field F. Assume that x is F-rational and lies in the apartment of an unramified torus $\mathbf{T}$ of $\mathbf{G}$. Then X_r—and hence its cohomology $H_c^i(X_r, \overline{\mathbb{Q}}_\ell)$—enjoys commuting actions by $G_{x,0}$ and the maximal bounded subgroup of $\mathbf{T}(F)$. These actions factor through certain quotients G_r and T_r of these groups. For any character θ of T_r, define the associated positive-depth Deligne–Lusztig induction to be

$$R_{T_r}^{G_r}(\theta) = \sum_{i \geq 0} (-1)^i H_c^i(X_r, \overline{\mathbb{Q}}_\ell)_\theta,$$

where the subscript denotes the θ-isotypic subspace.

The setting $r = 0$ is exactly the classical Deligne–Lusztig setting. Many beautiful facts are known in this setting, but allow me to focus on two facts:

Theorem 1.1 (Deligne–Lusztig)**.**

(1) *If θ has trivial W-stabilizer, then $R_{T_0}^{G_0}(\theta)$ is irreducible.*
(2) *If $\gamma \in G_0$ is regular semisimple, then*

$$\Theta_{R_{T_0}^{G_0}(\theta)}(\gamma) = \sum_{w \in W_{G_0}(Z_\gamma, T_0)} \theta^w(\gamma),$$

where Z_γ denotes the rational points of the connected centralizer of γ.

(1) is a special case of the scalar product formula; (2) is a special case of the Deligne–Lusztig character formula. We remark that for a general element γ, this character formula expresses the nontriviality of $\Theta_{R_{T_0}^{G_0}(\theta)}(\gamma)$ in terms of the nontriviality of Green functions for the connected centralizer of γ. In particular, (2) exactly isolates the γ for which the character formula is the simplest.

Remarkably, it turns out that the simple formula in (2) characterizes $R_{T_0}^{G_0}(\theta)$.

Theorem 1.2 (Lusztig, Henniart, C–Oi)**.** *Let q be sufficiently large and let θ be a character of T_0 with trivial W-stabilizer. If ρ is an irreducible representation of G_0 such that $\Theta_\rho|_{(G_0)_{\mathrm{rss}}} = c \cdot \Theta_{R_{T_0}^{G_0}(\theta)|_{(G_0)_{\mathrm{rss}}}}$ for some $c \in \{\pm 1\}$, then $\rho \cong c \cdot R_{T_0}^{G_0}(\theta)$.*

This theorem is due to Lusztig in a 1977 talk [2] for $q \ggg 0$ (q very large) and to myself and Oi [3] for $q \gg 0$ (q large). In the case that $G_0 = \mathrm{GL}_n$ and T_0 is elliptic, Henniart showed [4] that this theorem holds with essentially no[1 2] assumptions on q.

Our proof in [3] is based on Henniart's proof in [4]. It is so simple that its entirety can be explained in a talk, and a sketch can be presented in an extended abstract. The proof is analytic in nature.

Proof of Theorem 1.2. It is enough to prove $\langle \rho, R_{T_0}^{G_0}(\theta) \rangle \neq 0$. The hypothesis and the Cauchy–Schwarz inequality imply

$$\langle \rho, R_{T_0}^{G_0}(\theta) \rangle_{\mathrm{nrss}} \leq \langle R_{T_0}^{G_0}(\theta), R_{T_0}^{G_0}(\theta) \rangle_{\mathrm{nrss}}.$$

To prove the theorem, it is enough to prove $\langle R_{T_0}^{G_0}(\theta), R_{T_0}^{G_0}(\theta) \rangle_{\mathrm{rss}} > \frac{1}{2}$. Using Theorem 1.1(2), we see that this inequality holds if there are sufficiently many regular semisimple elements in T_0, which holds for $q \gg 0$. □

A key observation is that the proof of Theorem 1.2 relies only on the statements in Theorem 1.1. In particular, if one can prove the analogous Theorem 1.1 for $r > 0$, then the analogous Theorem 1.2 also holds.

[1] There is one case (when $n = 2$ and $q = 3$) where one must additionally specify c in order to get the desired conclusion.

[2] In the talk, I denoted the sentence "for any q" by "$q > 0$" to highlight the parallelism with Lusztig and Chan–Oi. This was slightly controversial.

When $r > 0$, the regular semisimple locus is replaced by the very regular locus (terminology of [4])—the elements in G_r whose image in G_0 is regular semisimple. The $r > 0$ version of Theorem 1.1(2) was established already in [1]. The $r > 0$ Theorem 1.1(1), however, has a much more interesting history. At present, this result has not yet been resolved in full generality, though it is close. Let me assume now that $\mathbf{T}$ is elliptic. A special case of the main theorem in my recent preprint [5] confirms that the $r > 0$ Theorem 1.1(1) holds whenever θ has a Howe factorization. It is a theorem of Kaletha [6] that for all but finitely many p, all θ have a Howe factorization. Moreover, in *op. cit.*, Kaletha constructs L-packets of supercuspidal representations associated to $(\mathbf{T}, \theta)$. Most notably, his construction generalizes the nontrivial correction twist defined by DeBacker–Spice in their construction of "toral" positive-depth L-packets [7]. It is imperative to ask whether the functor $R_{T_r}^{G_r}$ obtains the corrected parametrization. In forthcoming work with Oi, we prove that the answer is "yes", so long as q, the size of the residue field of F, is large enough to guarantee the conclusion of Theorem 1.2:

Theorem 1.3 (C–Oi). *For q sufficiently large, $R_{T_r}^{G_r}$ realizes Kaletha's corrected parametrization of regular supercuspidal representations.*

2. Positive-depth character sheaves

In my recent work with Roman Bezrukavnikov [8], we construct a class of character sheaves $\mathcal{F}$ on parahoric subgroups. Our construction takes as input a geometric version of a Yu-datum: one-dimensional characters being replaced by multiplicative local systems and depth-zero representations being replaced by Lusztig's character sheaves. We expect to establish a complete comparison between the trace-of-Frobenius $\Theta_{\mathcal{F}}$ and the character of the associated (virtual) representation obtained from parahoric Deligne–Lusztig induction. In *op. cit.*, we prove the following compatibility between the character sheaf $\mathcal{F}_\theta$ associated to a "toral" θ and the parahoric Deligne–Lusztig induction of θ:

Theorem 2.1 (Bezrukavnikov–C). *For $q \gg 0$, $\Theta_{\mathcal{F}_\theta} = (-1)^{\dim G_r} \cdot \Theta_{R_{T_r}^{G_r}(\theta)}$.*

The largeness assumption on q here is very mild: we need only guarantee the existence of a single regular element of T_0. The construction of $\mathcal{F}_\theta$ is simply given by geometric parabolic induction. Combining Theorem 2.1 and (a slight strengthening of) Theorem 1.3, we arrive at the following philosophical takeaway:

Geometrically, Yu's construction of supercuspidals
is given by parabolic induction!

The expert will notice that this is not the right statement, due to the correction twist mentioned in the preceding section. The right statement should be that the Fintzen–Kaletha–Spice twist [9] of Yu's construction is geometrically given by parabolic induction. This twist is a genuinely positive-depth phenomenon: in the depth zero setting, the above philosophical takeaway requires no correction, and follows from Lusztig's work on the coincidence of geometric and cohomological Green functions [10].

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Reduction to depth zero for tame p-adic groups via Hecke algebra isomorphisms

Kazuma Ohara
(joint work with Jeffrey D. Adler, Jessica Fintzen, and Manish Mishra)

Let F be a non-archimedean local field and G be a connected reductive group defined over F. The category $\mathrm{Rep}(G(F))$ of smooth, complex representations of $G(F)$ decomposes as

$$\mathrm{Rep}(G(F)) = \prod_{[M,\sigma]_G} \mathrm{Rep}^{[M,\sigma]_G}(G(F)),$$

where $[M,\sigma]_G$ runs through inertial equivalence classes of cuspidal pairs in G, and $\mathrm{Rep}^{[M,\sigma]_G}(G(F))$ denotes the Bernstein block associated to $[M,\sigma]_G$. A pair (K,ρ) of a compact, open subgroup K of $G(F)$ and its irreducible smooth representation ρ is called an $[M,\sigma]_G$-type if $\mathrm{Rep}^{[M,\sigma]_G}(G(F))$ is precisely the full subcategory of $\mathrm{Rep}(G(F))$ consisting of smooth representations that are generated by their ρ-isotypic components. If (K,ρ) is an $[M,\sigma]_G$-type, we have an equivalence of categories

$$\mathrm{Rep}^{[M,\sigma]_G}(G(F)) \simeq \text{Mod-}\mathcal{H}(G(F),(K,\rho)),$$

where $\mathcal{H}(G(F),(K,\rho))$ denotes the Hecke algebra attached to (K,ρ), and Mod-$\mathcal{H}(G(F),(K,\rho))$ denotes the category of right unital modules over $\mathcal{H}(G(F),(K,\rho))$. Therefore, constructing types and knowing the explicit structure of the Hecke algebras attached to the types are essential to understanding the category $\mathrm{Rep}(G(F))$.

In the 1990s, Moy and Prasad ([MP94, MP96]) and, independently, Morris ([Mor99]) provided a construction of a special kind of types called *depth-zero types*. A depth-zero type is an $\mathfrak{s}$-type for an inertial equivalence class $\mathfrak{s}$ of cuspidal pairs in G such that the block $\mathcal{R}^{\mathfrak{s}}(G(F))$ consists of *depth-zero representations*. Depth-zero representations are closely related to the representations of finite reductive groups. Hence, a lot of problems of representations of $G(F)$ become much easier if we restrict them to depth-zero representations.

Building upon the construction of depth-zero types and the construction of supercuspidal representations by Yu ([Yu01]) and using the theory of covers introduced by Bushnell and Kutzko ([BK98]), Kim and Yu ([KY17, Fin21a]) provided a construction of types for a connected reductive group that splits over a tamely ramified extension of F. The construction of Kim and Yu yields types for every Bernstein block if the residue characteristic p of F does not divide the order of the absolute Weyl group W of G by [Fin21b]. Thus, understanding the structure of the corresponding Hecke algebras and their categories of modules leads to an understanding of the whole category of smooth representations of $G(F)$ if G splits over a tamely ramified extension of F and p does not divide the order of W. In their construction, the type (K, ρ) of $G(F)$ is constructed as the tensor product of the inflation of a depth-zero type (K^0, ρ^0) for a twisted Levi subgroup G^0 of G and an irreducible representation κ of K constructed by using the theory of the Heisenberg–Weil representations of the finite symplectic groups. Here, we use the "twisted" construction of κ and ρ by the quadratic character ϵ introduced by [FKS23] instead of their original construction. This twist is necessary for the main result of this talk.

In this talk, I will explain the main result of my joint work [AFMO24a] with Jeffrey D. Adler, Jessica Fintzen, and Manish Mishra that there exists an explicit, support-preserving algebra isomorphism

$$\mathcal{H}(G(F), (K, \rho)) \xrightarrow{\sim} \mathcal{H}(G^0(F), (K^0, \rho^0)).$$

As a direct corollary, we obtain that if G splits over a tamely ramified extension of F and p does not divide $|W|$, then an arbitrary Bernstein block is equivalent to a depth-zero Bernstein block. This allows one to reduce many problems about (the category of) smooth, complex representations of p-adic groups to analogous problems about (the category of) depth-zero representations. I will also explain an explicit description of the structure of the depth-zero Hecke algebra as a semi-direct product of an affine Hecke algebra with a twisted group algebra, which was proved in [AFMO24b] generalizing prior work of Morris ([Mor93]).

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Degenerate representations of GL_n over a p-adic field

Johannes Girsch

(joint work with David Helm)

Let F be a finite extension of $\mathbb{Q}_p$ and let $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$ be the category of smooth complex representations of $\mathrm{GL}_n(F)$ for some integer $n \geq 1$. In this talk I explained how one can define a stratification of $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$ by using the theory of degenerate Whittaker models.

Degenerate Whittaker spaces. Let U_n be the unipotent radical of the standard Borel subgroup of $\mathrm{GL}_n(F)$ and fix a nontrivial additive character $\psi\colon F \to \mathbb{C}^\times$. For any integer partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of n, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$, we can define a character of U_n in the following way. Let S_λ be the subset of $\{1, 2, \dots, n\}$ consisting of integers that cannot be expressed in the form $\lambda_1 + \cdots + \lambda_s$ for some $s \leq r$. We then define the character $\psi_\lambda\colon U_n \to \mathbb{C}^\times$ to be

$$\psi_\lambda(u) = \sum_{i \in S_\lambda} \psi(u_{i,i+1}).$$

The *degenerate Whittaker space* associated to λ is the representation

$$W_\lambda = \text{c-Ind}_{U_n}^{\mathrm{GL}_n(F)}(\psi_\lambda).$$

Note that for the partition (n) of n, we have that $W_{(n)}$ is the Gelfand–Graev representation and an irreducible representation is generic if and only if it admits a nontrivial map from $W_{(n)}$. For an arbitrary irreducible representation $\pi \in$

$\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$ it is then natural to ask if there are partitions λ of n such that there is a nontrivial map from W_λ to π. This can be answered by using the theory of Bernstein–Zelevinsky derivatives.

Highest derivative partition. Recall for each integer $1 \leq i \leq n$ the i-th Bernstein–Zelevinsky derivative ([1, Section 4]), which is an exact functor

$$(-)^{(i)}\colon \mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F)) \to \mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_{n-i}(F)).$$

The *highest derivative* of a representation $\pi \in \mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$ is the largest integer k such that the k-th Bernstein–Zelevinsky derivative $\pi^{(k)}$ is nonzero. If π is irreducible with highest derivative k, Zelevinsky showed that $\pi^{(k)}$ is an irreducible representation of $\mathrm{GL}_{n-k}(F)$. For any irreducible representation π we can hence define a sequence of numbers $(\lambda_1, \lambda_2, \dots)$, where λ_i is the highest derivative of

$$((\pi^{(\lambda_1)})^{(\lambda_2)\dots})^{(\lambda_{i-1})}.$$

Zelevinsky ([4, Theorem 8.1]) proved that these numbers form an integer partition of n, i.e. $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\lambda_1 + \lambda_2 + \cdots = n$, which we call the *highest derivative partition* of π.

The highest derivative partition is an important invariant of an irreducible representation and equals the partition associated to the maximal member of the wave front set of the representation. Moreover, it can be interpreted under the local Langlands correspondence ([3, Theorem B]).

By using this notion, Mœglin and Waldspurger proved that, if an irreducible representation π has highest derivative partition $\lambda = (\lambda_1, \lambda_2, \dots)$, then

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n(F)}(W_\lambda, \pi) = 1.$$

Moreover, if

$$\mathrm{Hom}_{\mathrm{GL}_n(F)}(W_{\lambda'}, \pi) \neq 0$$

for some partition $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ of n, then λ' precedes λ under the dominance order on partitions, i.e.

$$\sum_{i=1}^{j} \lambda'_i \leq \sum_{i=1}^{j} \lambda_i$$

for all $j \geq 1$.

A Stratification of $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$. Let e be a primitive idempotent in the Bernstein center $\mathfrak{Z}_n$ of $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$ and consider the associated Bernstein block $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$. For any partition λ of n, we consider the following full subcategories of $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$:

- $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))^{\preceq\lambda}$, whose objects are representations in $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$ for which every irreducible subquotient has highest derivative partition λ' satisfying $\lambda' \preceq \lambda$.
- $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))^{\prec\lambda}$, a full subcategory of $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))^{\preceq\lambda}$, with the additional condition that no irreducible subquotient has highest derivative partition λ.

Let $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))^{=\lambda}$ be the Serre quotient

$$e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F)))^{\preceq\lambda}\Big/ e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F)))^{\prec\lambda}.$$

We will now construct a progenerator for $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))^{=\lambda}$. From the projectivity of the Gelfand–Graev representation it follows that the W_λ are projective objects in $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$. Moreover, the functor

$$\begin{aligned} e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F)) &\to e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F)))^{\preceq\lambda} \\ \pi &\mapsto \pi^{\preceq\lambda} = \mathrm{coker}\left(\oplus_{\lambda'\not\preceq\lambda} W'_{\lambda'} \otimes \mathrm{Hom}_{G_n}(W'_{\lambda'}, \pi) \to \pi\right) \end{aligned}$$

is left adjoint to the natural inclusion of $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F)))^{\preceq\lambda}$ into $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$ and hence right exact. This together with the results of Mœglin and Waldspurger mentioned above imply that $(eW_\lambda)^{\preceq\lambda}$ is a progenerator of $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))^{=\lambda}$.

Let eE_λ be the endomorphism ring of $(eW_\lambda)^{\preceq\lambda}$. The upshot of the above is that we have an equivalence of categories between $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))^{=\lambda}$ and the category of right eE_λ-modules. Hence it is natural to study the rings eE_λ and we are able to prove the following result.

Theorem [2]**.** For any primitive idempotent e of $\mathfrak{Z}_n$ and integer partition λ of n the ring eE_λ is commutative and reduced.

Furthermore, we are able to describe the rings eE_λ explicitly, similiar to the description of the Bernstein center due to Bernstein–Deligne.

Example. For the partition (n) we have that $(eW_{(n)})^{\preceq(n)} = eW_{(n)}$ is the component of the Gelfand–Graev representation in $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$ and it is a well known result that its endomorphism ring $eE_{(n)}$ is isomorphic to the commutative and reduced ring $e\mathfrak{Z}_n$.

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Langlands correspondence for Moy–Prasad types

Cheng-Chiang Tsai
(joint work with Tsao-Hsien Chen and Stephen DeBacker)

Let F be any non-archimedean local field with residue field k_F. Let G be a connected reductive group over F. We assume that G splits over a tame extension of F. Moy and Prasad [MP94] attached open compact subgroups $G(F)_{x\geq r}\subset G(F)$ that depend on data (x,r) where $x\in\mathcal{B}(G,F)$ is a point in the Bruhat–Tits building and $r\geq 0$ is a real number. The jumps in r are discrete, so that $G(F)_{x>r}:=G(F)_{x\geq r+\epsilon}$ is defined for $\epsilon>0$ sufficiently small. We have $G(F)_{x>r}\lhd G(F)_{x\geq r}$ and the quotient $G(F)_{x=r}:=G(F)_{x\geq r}/G(F)_{x>r}$ has an explicit structure either as a finite group of Lie type or a vector space over k_F.

When an irreducible smooth $\pi\in\mathrm{Irr}^{\mathrm{sm}}(G(F))$ is such that the fixed subspace $\pi^{G(F)_{x>r}}\neq 0$, we can view $\pi^{G(F)_{x>r}}$ as a representation of $G(F)_{x=r}$. When $r>0$, we have analogous constructions for the Lie algebra and its dual and

$$G(F)_{x=r}\cong\mathfrak{g}(F)_{x=r}=\mathrm{Hom}_{k_F}(\mathfrak{g}^*(F)_{x=-r},k_F).$$

Up to a fixed choice of non-trivial additive character $\psi_F:(k_F,+)\to\mathbb{C}^\times$, representations of $G(F)_{x=r}$ can be understood via elements in $\mathfrak{g}^*(F)_{x=-r}$, a subquotient of the dual Lie algebra $\mathfrak{g}^*(F)$. An $X\in\mathfrak{g}^*(F)_{x=-r}$ is called **non-degenerate** if the coset it represents (as a subset of $\mathfrak{g}^*(F)$) does not contain a nilpotent element. Moy and Prasad proved that starting from $\pi\in\mathrm{Irr}^{\mathrm{sm}}(G(F))$, we have a unique associativity class of non-degenerate $X\in\mathfrak{g}^*(F)_{x=-r}$ [MP94, §5]. In particular $r\in\mathbb{Q}_{\geq 0}$ is unique. Here we remark that a similar but different definition is needed when $r=0$. But since the question to be discussed is actually deeper when $r>0$, for simplicity we will always assume $r>0$.

Meanwhile, the conjectural local Langlands correspondence attaches to π a Langlands parameter $\varphi_\pi:W_F\times\mathrm{SL}_2(\mathbb{C})\to{}^LG$. It is therefore of basic interest to know what does the aforementioned associativity class of X say about φ_π. The work of the given talk aims to address this question.

Recall W_F is the Weil group, which has an upper numbering filtration $W_F^{\geq 0}\supset W_F^{>0}\supset\cdots\supset W_F^{\geq r}\supset W_F^{>r}\supset\cdots$ where $W_F^{\geq 0}$ is the inertia and $W_F^{>0}$ is the wild inertia. Each subgroup is closed and normal in any previous subgroup. Write again $W_F^{=r}:=W_F^{\geq r}/W_F^{>r}$ for the sub-quotient.

Theorem (Chen–DeBacker–Tsai)**.** *There is a well-defined construction $X\mapsto\varphi_X$ where $\varphi_X:W_F^{=r}\to G^\vee$ is a continuous homomorphism. This construction depends only on the associativity class of X and the choice of ψ_F. When φ_π is a supercuspidal Langlands parameter of Kaletha* [Kal19, Kal21] *and the data (X,r) is attached to π like before, we have that $\varphi_\pi|_{W_F^{>r}}\equiv 1$ and $\varphi_\pi|_{W_F^{=r}}=\varphi_X$.*

Naturally, we conjecture that the same is always true.

Conjecture. *Let $\pi \in \mathrm{Irr}^{\mathrm{sm}}(G(F))$. When φ_π is attached to π via the Langlands correspondence and (X, r) is attached to π as before, we have that $\varphi_\pi|_{W_F^{>r}} \equiv 1$ and $\varphi_\pi|_{W_F^{=r}} = \varphi_X$.*

We emphasize that the conjecture is, strictly speaking, a proposed property for *a* Langlands correspondence. For example, one can ask if it is satisfied by the Langlands correspondence of Harris–Taylor for GL_n, or that of Fargues–Scholze. We don't have a strong evidence for the conjecture in these directions yet.

Let us sketch the construction in our theorem. We begin with π and (X, r) where $r \in \mathbb{Q}_{>0}$ and $X \in \mathfrak{g}^*(F)_{x=-r}$. Choose E/F finite Galois such that $r \in \mathrm{val}(E^\times)$. Then it becomes possible to talk about the Jordan decomposition of X in $\mathfrak{g}^*(F)_{x=-r} \subset \mathfrak{g}^*(E)_{x=-r}$ thanks to [KW76] and [Spi21, §3]. The semisimple part $X_s \in \mathfrak{g}^*(E)_{x=-r}$ can be realized to be in $\mathfrak{t}^*(E)_{=-r} \subset \mathfrak{g}^*(E)_{x=-r}$ for some E-torus $T \subset G$; we may assume T is split by a further extension. Using local class field theory we may attach to X_s a continuous homomorphism $\varphi_{X_s,T} : W_F^{=r} \to T^\vee$. We then show that $W_F^{=r} \xrightarrow{\varphi_{X_s,T}} T^\vee \hookrightarrow G^\vee$ is a well-defined map and depends only on π, but is otherwise independent of the choice of E, T and ψ_F, and also independent of X from the associativity class of itself.

(We emphasize that when E/F is not tamely ramified there are significant subtleties, but it can somewhat be circumvented when G is tamely ramified.)

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Types, Hecke algebras and the Wave Front Set

Emile Okada

(joint work with Dan Ciubotaru and Lucas Mason-Brown)

In real analysis the Schwartz–Paley–Wiener theorem gives a powerful characterisation of smooth compactly supported distributions in terms of their Fourier transform.

A compactly supported distribution $D : C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$ is smooth i.e. of the form

$$f \mapsto \int_{\mathbb{R}^n} F_D(x) f(x) dx$$

for some compactly supported smooth function F_D if and only if the Fourier transform $\hat{D}$ of D satisfies the growth condition: $\forall N \in \mathbb{N}, \exists C_N \geq 0 : \forall \xi \in \mathbb{R}^n$

$$|\hat{D}(\xi)| \leq C_N(1 + |\xi|)^{-N}.$$

This enables the measurement of the *directions* in which a distribution D is smooth, and we say D is smooth at $(x, \xi) \in T^*\mathbb{R}^n$ if there is a cutoff function χ centered at x such that the Fourier transform of the distribution $f \mapsto D(\chi f)$ satisfies the growth condition in some open cone around ξ . The *wave front set* of D is then defined to be the subset of the cotangent bundle of $\mathbb{R}^n$ consisting of non-smooth points.

In the representation theory of a connected reductive group G defined over a finite extension F of $\mathbb{Q}_p$, we obtain a distribution from a smooth admissible representation (π, V) by considering

$$D_\pi : C_c^\infty(G) \to \mathbb{C}, \quad f \mapsto \operatorname{tr}\left(\int_G f(g)\pi(g)dg\right).$$

Contrary perhaps to expectation, the distribution character D_π may not be smooth. Harish-Chandra proved that D_π is smooth on the dense open subset of G consisting of regular elements, but there may be singularities elsewhere, in particular at the identity. It turns out these singularities encode significant information about the representation, for example the classical result of Rodier [1] on Whittaker models suggests a deep connection between the wave front set at the identity and the local Langlands correspondence. We denote the wave front set of D_π at the identity by $\mathrm{WF}(\pi)$.

The main focus of my talk will be on my joint work with Dan Ciubotaru and Lucas Mason-Brown where we show that for unipotent representations with real infinitesimal character one can in fact recover the monodromy of the L-parameter of π from $\mathrm{WF}(\pi^*)$ where π^* denotes the Aubert–Zelevinsky dual of π.

The means to recover the monodromy is through a certain map

$$d : \mathcal{N}(\mathfrak{g}^*)/G \to \mathcal{N}(\mathfrak{g}^\vee)/G^\vee$$

from nilpotent co-adjoint orbits of G to nilpotent orbits of the complex Langlands dual group $G^\vee$ of G defined as follows.

For simplicity let us assume G is split. If $\mathcal{O}^*$ is a nilpotent co-adjoint orbit, then by [4] it is the lift of a nilpotent co-adjoint orbit $\underline{\mathcal{O}}^*$ of the reductive quotient of a parahoric subgroup P of G. By [5, Theorem 4.1] one can interpret $\underline{\mathcal{O}}^*$ as an $\mathbb{F}_q$-rational nilpotent orbit of a *pseudo-Levi* subgroup of the mod-p reduction of the hyperspecial parahoric subgroup of G. This puts us squarely in the domain of Sommers' duality map [6], and its application yields the sought-after element $d(\mathcal{O}^*)$ of $\mathcal{N}(\mathfrak{g}^\vee)/G^\vee$.

The monodromy of the L-parameter of π can then be recovered as the minimal orbit of the form $d(\mathcal{O}^*)$ where $\mathcal{O}^* \in \mathrm{WF}(\pi^*)$.

The proof strategy follows the general approach that naturally presents itself when using the test functions of [2] in depth-0 and [3] in positive depth. Roughly speaking this approach subdivides the problem of computing $\mathrm{WF}(\pi)$ into three subproblems of different flavours:

(1) an arithmetic problem,
(2) an algebraic problem,
(3) a combinatorial problem.

The arithmetic problem in this particular case (and in depth-0 in general) was solved by Lusztig using his theory of character sheaves. The algebraic problem takes the form of a branching problem for affine Hecke algebras arising from the theory of types. Finally, the Bernstein presentation of affine Hecke algebras of types that admit representations with real infinitesimal character is closely linked to the generalised Springer correspondence. Through this connection the combinatorial problem reduces to a question about the compatibility between the generalised Springer correspondence and families of Weyl group representations.

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Endoscopic liftings of supercuspidal representations for classical groups: an overview of a calculation

Geo Kam-Fai Tam

1. Abstract

Let G be a classical group (orthogonal, symplectic, or unitary) over a p-adic field F where p is odd, and π be an irreducible, supercuspidal $\mathbb{C}$-representation of $G(F)$. We propose a general strategy to express the endoscopic lift of π, in the sense of Arthur [1] and Mok [14], into the general linear group whose dual expresses the dual group $\hat{G}$ of G as a complex matrix group, with proven examples [23, 22, 5] (partly joint with Corinne Blondel) and work in progress by the speaker built upon and improving the results of Lust–Stevens [10], Blondel–Henniart–Stevens [3], and Oi [16, 17, 18]. These examples include: simple [8], epipelagic [19], certain regular [9], and certain depth-zero non-regular (for example, quadratic unipotent [24]) supercuspidal representations.

2. The strategy

This strategy lies completely on the representation side of the local Langlands correspondence (LLC), i.e., the LHS of the following commutative diagram.

$$\begin{array}{ccc}
\left\{\begin{matrix}\text{`}\sigma\text{-supercuspidal'} \\ \text{irr repres of } \mathrm{GL}_{\hat{N}}(F)\end{matrix}\right\} & \xleftrightarrow{\ \mathrm{LLC}(\mathrm{GL}_{\hat{N}}/F)\ } & \left\{\begin{matrix}\text{sans-trou } \hat{N}\text{-dim} \\ \text{repres of } W_F \times \mathrm{SL}_2(\mathbb{C})\end{matrix}\right\} \\
\text{endoscopic lift} \uparrow & & \uparrow {}^LG \hookrightarrow \mathrm{GL}_{\hat{N}}(\mathbb{C}) \\
\left\{\begin{matrix}\text{irr supercuspidal} \\ \text{repres of } G(F)\end{matrix}\right\} & \xrightarrow{\ \mathrm{LLC}(G/F)\ } & \left\{\begin{matrix}\text{`supercuspidal' parameters} \\ W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG\end{matrix}\right\}
\end{array}$$

In other words, the strategy is in principle equivalent to describing the Langlands parameters of irreducible supercuspidal representations of $G(F)$ as representations of the Weil–Deligne group $W_F \times \mathrm{SL}_2(\mathbb{C})$. We only explain the term '*sans-trou* representations' of $W_F \times \mathrm{SL}_2(\mathbb{C})$ on the parameter side: they are of the form

$$\varphi = \bigoplus_{\tilde{\varphi}} \tilde{\varphi} \otimes (\mathrm{St}(a_{\tilde{\varphi},\varphi}) \oplus \mathrm{St}(a_{\tilde{\varphi},\varphi} - 2) \oplus \cdots),$$

where $\tilde{\varphi}$ ranges over all irreducible representations of W_F of degrees $\leq \hat{N}$, and $\mathrm{St}(a)$ is the a-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$.

We outline the strategy in three parts, each relying on the following tools.

- Mœglin's theory relating Jordan blocks and points of reducibility [12, 13];
- Bushnell–Kutzko's covering types [7];
- Lusztig's descriptions of Hecke algebras and their modules [11].

Fix an irreducible component $\tilde{\varphi}$ as above and let $\tilde{\pi}$ be the supercuspidal representation of $\tilde{G}(F) = \mathrm{GL}_{\dim \tilde{\varphi}}(F)$ corresponding to $\tilde{\varphi}$ via the LLC for $\tilde{G}$. We view $L = \tilde{G} \times G$ as a Levi subgroup of a maximal parabolic subgroup of a classical group $\mathbf{G}$ of the same type as G, and define the parabolically induced representation $I(s, \tilde{\pi}, \pi) := \iota_L^{\mathbf{G}}(\tilde{\pi}\nu^s \times \pi)$ for $s \in \mathbb{C}$, where $\nu : x \mapsto |\det x|$ for $x \in \tilde{G}(F)$. By Mœglin *loc. cit.* and Silberger [20],

$$\begin{aligned} & I(s, \tilde{\pi}, \pi) \text{ is reducible at } s = s_{\tilde{\pi},\pi} \in \mathbb{R}_{\geq 1} \cap \tfrac{1}{2}\mathbb{Z} \\ \Leftrightarrow \quad & \varphi \text{ has a component } \tilde{\varphi} \otimes [[a_{\tilde{\varphi},\varphi}]], \text{ where } a_{\tilde{\varphi},\varphi} = 2s_{\tilde{\pi},\pi} - 1. \end{aligned}$$

Therefore, the calculation of $a_{\tilde{\varphi},\varphi}$ is changed to that of $s_{\tilde{\pi},\pi}$ on the representation side, as a reducibility problem on $I(s, \tilde{\pi}, \pi)$.

We then apply Bushnell–Kutzko's theory to convert the reducibility problem into an analogous problem concerning modules over the Hecke algebra associated with the covering type of the product cuspidal type of $\tilde{\pi} \times \pi$. This conversion is summarized in the following commutative diagram.

$$\begin{array}{ccc}
\mathcal{R}^{\mathfrak{s}}(\mathbf{G}) & \xrightarrow{\ \mathcal{M}_{\mathbf{G}}\ } & \text{Mod-}\mathcal{H}(\mathbf{G}, \lambda_P) \\
\iota_L^{\mathbf{G}} - \uparrow & & \uparrow \mathrm{Hom}_{\mathcal{H}_L}(-, \mathcal{H}_{\mathbf{G}}) \\
\mathcal{R}^{\mathfrak{s}}(L) & \xrightarrow{\ \mathcal{M}_L\ } & \text{Mod-}\mathcal{H}(L, \tilde{\lambda} \times \lambda)
\end{array}$$

Here the parabolic induction functor $\iota_L^{\mathbf{G}}$ takes the Bernstein component $\mathcal{R}^{\mathfrak{s}}(L)$ of $L(F)$, which contains $\tilde{\pi}\nu^s \times \pi$ for all $s \in \mathbb{C}$, into the component $\mathcal{R}^{\mathfrak{s}}(\mathbf{G})$ of $\mathbf{G}(F)$, and is Morita-equivalent to the Hom-functor between the (non-degenerate) module-categories over the respective Hecke algebras $\mathcal{H}_L := \mathcal{H}(L, \tilde{\lambda} \times \lambda)$ and $\mathcal{H}_{\mathbf{G}} := \mathcal{H}(\mathbf{G}, \lambda_P)$. The involved cuspidal types $\tilde{\lambda}$ and λ are described by Bushnell–Kutzko's [6] and Stevens' [21] constructions using semi-simple strata, and λ_P is a covering type of $\tilde{\lambda} \times \lambda$ described also by Stevens *loc. cit.* (see also [3]).

The structures of $\mathcal{H}_L$ and $\mathcal{H}_{\mathbf{G}}$ are known. We have $\mathcal{H}_L \cong \mathbb{C}[X^{\pm 1}]$, where X is an $\mathrm{End}_{\mathbb{C}}(\tilde{\lambda} \times \lambda)$-valued map on $L(F)$ supported at $\varpi_L := \varpi_E \times 1_{G(F)}$, and ϖ_E is a 'uniformizer' in $\tilde{G}(F)$ normalizing $\tilde{\lambda}$. Stevens shows in *loc. cit.* that the structure of $\mathcal{H}_{\mathbf{G}}$ can be reduced to the 'depth-zero' part of λ_P, and eventually obtains $\mathcal{H}_{\mathbf{G}} \cong \mathcal{H}_L \oplus \mathcal{H}_L T_y = \mathcal{H}_L \oplus \mathcal{H}_L T_z$, where the generator $T_w : \mathbf{G}(F) \to \mathrm{End}_{\mathbb{C}}(\lambda_P)$, for each $w \in \{y, z\}$, is supported on a suitable double coset s_w in the normalizer $N_{\mathbf{G}}(\lambda_P)$ over the compact open subgroup underlying λ_P, with the properties that $s_y s_z = \varpi_L$ and the convolution product $T_y * T_z$ is equal to the image of X under a canonical embedding $\mathcal{H}_L \hookrightarrow \mathcal{H}_{\mathbf{G}}$ defined in [7]. By Lusztig [11], each T_w satisfies an equation of the form $(T+1) * (T - q^{r_w}) = 0$ for some half-integer $r_w \geq 0$.

Let D_s, for $s \in \mathbb{C}$, be the $\mathcal{H}_L$-module (which is a character) corresponding to $\tilde{\pi}\nu^s \times \pi$ under $\mathcal{M}_L$, and is co-induced to the $\mathcal{H}_{\mathbf{G}}$-module E_s that corresponds to $I(s, \tilde{\pi}, \pi)$ under $\mathcal{M}_{\mathbf{G}}$. A key observation from [2] tells us that the module E_s is reducible if and only if the value $X|_{D_s}$ is the product of the eigenvalues of $T_y|_{E_s}$ and $T_z|_{E_s}$. Since $\dim_{\mathbb{C}} E_s = \mathrm{rank}_{\mathcal{H}_T}(\mathcal{H}_{\mathbf{G}}) = 2$, there are four possible such products (with multiplicities). With a little more calculation, we obtain the (multi-)set $\mathrm{Red}(\tilde{\pi}, \pi)$ of points of reducibility of $I(s, \tilde{\pi}, \pi)$ inside the domain $s \in \mathbb{C} \bmod \frac{2\pi\sqrt{-1}}{\log q}\mathbb{Z}$.

Main Proposition. *Suppose that $s_y s_z = \varpi_E \times 1_G$, and if $\tilde{\boldsymbol{\lambda}}$ is the inducing type of $\tilde{\pi}$ extending $\tilde{\lambda}$ such that $\tilde{\boldsymbol{\lambda}}(\varpi_E) = \delta T_y(s_y) T_z(s_z)$, then $\delta \in \{\pm 1\}$, and*

$$\mathrm{Red}(\tilde{\pi}, \pi) = \left\{ \pm \frac{r_y + \delta r_z}{2}, \pm \frac{r_y - \delta r_z}{2} + \frac{\pi\sqrt{-1}}{\log q} \right\}.$$

Following [2], [5], we have a formula for $T_w(s_w)$ in terms of an 'orthogonality relation' between the cuspidal types $\tilde{\lambda}$ and λ, and therefore require the knowledge of the character values of representations of reductive groups and Weil–Heisenberg groups, both over a finite field. The value of $T_w(s_w)$ then contains information about the parameters r_w and certain signs for computing $\delta \in \{\pm 1\}$ and also $\mathrm{Red}(\tilde{\pi}, \pi)$.

3. Recent results

We provide three results from the speaker [23, 22, 5]. In Sec. 3.1, let $\mathbb{F} = \mathfrak{o}_F / \mathfrak{p}_F$ be the residue field of F, and in Sec. 3.2, let μ_F be the subgroup of F of roots of unity such that $\mu_F \cong \mathbb{F}^\times$, and $\left(\frac{\cdot}{\mu_F}\right)$ be the quadratic character of μ_F.

3.1. **Cuspidal quadratic unipotent representations.** Given $a, b \in \mathbb{Z}$ with $a \geq 0$, we denote by $\rho(a, b)$ the cuspidal quadratic unipotent representation of $\mathsf{G}_n = \mathrm{Sp}_{2n}(\mathbb{F})$ defined in [24], where $n = a^2 + a + b^2$. Now, for some integers n_y and n_z, take $G = \mathrm{Sp}_{2(n_y+n_z)}$ over F and $\mathcal{J}$ a maximal parahoric subgroup of $G(F)$ whose reductive quotient is $\mathsf{G}_{n_y} \times \mathsf{G}_{n_z}$. If each $n_w = a_w^2 + a_w + b_w^2$ for some $a_w \in \mathbb{Z}_{\geq 0}$ and $b_w \in \mathbb{Z}$, we define

$$\pi(a_y, b_y; a_z, b_z) = \mathrm{cInd}_{\mathcal{J}}^{G(F)}(\rho(a_y, b_y) \times \rho(a_z, b_z)),$$

which is also called a cuspidal quadratic unipotent representation of $G(F)$.

For $i \in \{1, 2\}$, let ω_i be the two characters of $F^\times$ corresponding to the two ramified quadratic extensions of F. (Implicit here is a choice of a uniformizer ϖ of F, which also determines the position of $\mathcal{J}$.) Fix $m_+, m_- \in \mathbb{Z}_{\geq 0}$ and define two parameters of $G(F)$,

$$\varphi_i = 1 \otimes [[4m_+ + 1]] \oplus \omega_i \otimes [[4m_- - 1]], \quad i = 1 \text{ or } 2.$$

The L-packet Π_{φ_i} of each φ_i contains two supercuspidal representations, which are cuspidal quadratic unipotent. The results in [10] describes the four such representations in $\Pi_{\varphi_1} \sqcup \Pi_{\varphi_2}$. In [23], we use our strategy to further separate them into unique packets.

$$\pi(m_+, m_-; m_+, m_-), \pi(m_+, -m_-; m_+, -m_-) \in \Pi_{\varphi_1};$$
$$\pi(m_+, -m_-; m_+, m_-), \pi(m_+, m_-; m_+, -m_-) \in \Pi_{\varphi_2}.$$

One special feature about the proof is that, with this specific form (called *typically almost symmetric* in [23]) of the parameters, we do not need any deep calculations (e.g., [24]) on the character values.

3.2. **Epipelagic supercuspidal representations.** Let K be the maximal unramified extension of F in $\overline{F}$, and x be the barycenter of a facet in the Bruhat–Tits building of G over K. If r is the minimal positive value of the valuations of x by the K-affine roots, we define the quotients of Moy–Prasad subgroups

$$\mathsf{G}_x := G(K)_{x,0}/G(K)_{x,r}, \quad \mathsf{V}_{x,r} := G(K)_{x,r}/G(K)_{x,r_+}.$$

G_x acts on $\mathsf{V}_{x,r}$ by conjugation. We take a functional $\beta \in \mathsf{V}^*_{x,r}(\mathbb{F})$ which is *stable* for the G_x-action, in the sense of Mumford's GIT [15], and let ψ_β be the character of $G(F)_{x,r}$ pulled-back from β (using a fixed non-trivial additive character ψ of F). Let $G(F)_{x,\beta}$ be the stabilizer of β in $G(F)_{x,0}$, and λ an irreducible component in $\mathrm{Ind}_{G(F)_{x,r}}^{G(F)_{x,\beta}} \psi_\beta$. Then $\pi = \mathrm{cInd}_{G(F)_{x,\beta}}^{G(F)} \lambda$ is an irreducible supercuspidal representation, which is called *epipelagic* in [19] (because of its shallow depth).

In the following result, let $G = \mathrm{Sp}_{2n}$. Given β as above, we view β as a semi-simple element and decompose it into elliptic components $(\beta_i)_{i \in I}$. With a tuple $\delta = (\delta_i)_{i \in I}$, where each $\delta_i \in \{\pm 1\}$, we define a depth-zero character of $F^\times$,

$$\tilde{\lambda}_o|_{\mu_F} = \left(\frac{\cdot}{\mu_F}\right)^{\#I} \quad \text{and} \quad \tilde{\boldsymbol{\lambda}}_o(\varpi) = \prod_{i \in I} \delta_i \left(\frac{\varpi \det \beta_i}{\mu_F}\right).$$

With the help of a Gauss sum $\mathfrak{n}_\psi$, for each pair (β_i, δ_i), we construct an inducing type $\tilde{\boldsymbol{\lambda}}_{(2\beta_i, (\frac{\cdot}{\mu_F}), \delta_i \mathfrak{n}_\psi)}$ of a supercuspidal representation $\tilde{\pi}_i$ of $\mathrm{GL}_{\deg \beta_i}(F)$, and define a(n irreducible) parabolically induced representation of $\mathrm{GL}_{2n+1}(F)$ by $\tilde{\pi}_{(\beta,\delta)} = \left(\prod_{i\in I}\tilde{\pi}_i\right) \times \tilde{\boldsymbol{\lambda}}_o$, then [22] $\tilde{\pi}_{(\beta,\delta)}$ is the endoscopic lifting of the representations, which are all epipelagic supercuspidal, in the packet

$$\Pi_{(\beta,\delta)} := \{\pi_{I_\zeta}\}_{I_\zeta\subseteq I},$$

where each subset I_ζ of I defines an embedding of $G(F)_{x,0}$ into $G(F)$, and π_{I_ζ} is the epipelagic representation constructed as above with the embedded $G(F)_{x,0}$ by I_ζ. This result refines the results for simple supercuspidal representations computed in [18] (and recently [4]).

3.3. **Ramified unitary groups.** Let $F/F_\bullet$ be ramified quadratic, G be the unitary group U_n over $F/F_\bullet$, and $\tilde{G}$ be GL_n over F. We take supercuspidal representations $\tilde{\pi}_{(\tilde{\theta},\tilde{\boldsymbol{\rho}})}$ and $\pi_{(\theta,\rho)}$ of $\tilde{G}(F)$ and $G(F/F_\bullet)$ constructed by the following data (see [6], [21] for details):

- a skew simple stratum $\mathbf{s} = [\Lambda, 0, \beta]$ (and denote $E = F[\beta]$),
- simple characters θ and $\tilde{\theta}$, both associated with $\mathbf{s}$, and
- level-zero components ρ and $\tilde{\boldsymbol{\rho}}$.

Suppose that (θ, ρ) and $(\tilde{\theta}, \tilde{\boldsymbol{\rho}})$ satisfy:

- $\theta = (\tilde{\theta}|_{H^1})^{1/2}$,
- $\tilde{\rho}|_{\mu_E} = \left(\frac{\cdot}{\mu_E}\right)^{f(E/F)-1}$, and
- $\tilde{\boldsymbol{\rho}}(\varpi_E) = \epsilon\rho(-1)$, where $\epsilon = \epsilon_z^P(\varpi_E, \mathbf{s})$ is a sign calculated in [5].

Then the main result of [5] asserts that $\tilde{\pi}_{(\tilde{\theta},\tilde{\boldsymbol{\rho}})}$ is the endoscopic lift of $\pi_{(\theta,\rho)}$.

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An overview of the Fargues–Scholze correspondence

David Helm

The goal of this note is to give a brief and self-contained overview of the Fargues–Scholze categorical local Langlands correspondence. Let F be a p-adic field and G a quasi-split connected reductive group over F. The *Kottwitz set* $B(G)$ is the set of isomorphism classes of G-isocrystals over F. For each $b \in B(G)$ the automorphism group G_b of b is an inner form of a Levi subgroup of G; if this Levi is equal to G we say b is *basic*.

The classical local Langlands correspondence, as formulated by Kaletha ([2], Conjecture F) is a conjectural bijection between triples (b, G_b, π) where $b \in B(G)$ is basic and π an irreducible smooth representation of G, and pairs (ρ, ν), where ρ is an L-homomorphism from W_F to the L-group of G and ν is a representation of a certain quotient of the centralizer of ρ in $\hat{G}$. This bijection is normalized by a choice of Whittaker datum (U, ψ) for G.

The work of Fargues and Scholze [1] geometrizes both sides of this picture. On the left-hand side, they introduce the stack Bun_G of G-bundles on the Fargues–Fontaine curve; it associates to any perfectoid space S over $\mathbb{F}_q$ the groupoid of G-bundles on X_S, where X_S is the relative Fargues–Fontaine curve over S.

The stack Bun_G admits a stratification into locally closed substacks Bun_G^b indexed by $b \in B(G)$; the open strata are precisely those corresponding to basic b. Each Bun_G^b is the quotient of a point by a group $\tilde{G}_b(F)$ which is the extension of $G_b(F)$ by a unipotent group diamond.

For $\Lambda \in \{\mathbb{Z}_\ell, \mathbb{Q}_\ell\}$, Fargues–Scholze describe (derived) categories of sheaves $\mathrm{Shv}(\mathrm{Bun}_G^b, \Lambda)$ and $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ on the strata of Bun_G and Bun_G itself. Each category $\mathrm{Shv}(\mathrm{Bun}_G^b, \Lambda)$ is equivalent to the (derived) category of smooth λ representations of $G_b(F)$, and its inclusion i_b into Bun_G induces a fully faithful embedding:

$$(i_b)_! : \mathrm{Shv}(\mathrm{Bun}_G^b, \Lambda) \to D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda).$$

The category $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ thus admits a semiorthogonal decomposition whose factors are the categories of smooth Λ-representations of the groups $G_b(F)$, for all $b \in B(G)$ (not just the basic ones as in the classical formulation).

On the right hand side of the correspondence, Fargues–Scholze consider the moduli stack $[Z^1(W_F, \hat{G})/\hat{G}]$ of L-homomorphisms from W_F to the L-group of G, up to conjugacy by $\hat{G}$. Their main result constructs a natural "spectral action":

$$\mathcal{F}, \mathcal{G} \mapsto \mathcal{F} * \mathcal{G}$$

of the category $\mathrm{Perf}([Z^1(W_F, \hat{G})_\Lambda/\hat{G}])$ on $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$.

Here Perf denotes the category of perfect complexes; that is the full subcategory of the derived category of coherent sheaves spanned by bounded complexes of vector bundles. The construction of this action follows ideas from the geometric Langlands correspondence: Fargues–Scholze prove a version of geometric Satake that relates representations of $\hat{G}$ (and therefore vector bundles on $[Z^1(W_F, \hat{G})/\hat{G}]$) with sheaves on a certain "Hecke stack". The latter category of sheaves acts naturally by convolution on $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$.

The monoidal unit for this action is the structure sheaf $\mathcal{O}$ on $[Z^1(W_F, \hat{G})/\hat{G}]$. An endomorphism of $\mathcal{O}$ thus yields an endomorphism of $\mathcal{O} * \mathcal{G} = \mathcal{G}$ for any object $\mathcal{G}$ of $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$. This defines a map from the endomorphism ring of $\mathcal{O}$ (that is, the ring of $\hat{G}$ invariant functions on the affine scheme $Z^1(W_F, \hat{G})$) to the center of the category $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$, and hence to the centers of the categories $\mathrm{Rep}_\Lambda(G_b(F))$ of representations of $G_b(F)$ for all $b \in B(G)$.

Thus if E denotes the sections $\Gamma([Z^1(W_F, \hat{G})/\hat{G}], \mathcal{O})$, and Z_b denotes the center of $\mathrm{Rep}_\Lambda(G_b(F))$, we obtain a map: $E \to Z_b$ for each $b \in B(G)$. This map can be thought of as follows: the action of Z_b on an irreducible representation π of $G_b(F)$ over $\overline{\mathbb{Q}}_\ell$ is via a map $f_\pi : Z_b \to \overline{\mathbb{Q}}_\ell$. The composition with the map $E \to Z_b$ then yields a map $E \to \overline{\mathbb{Q}}_\ell$; since E is the ring of $\hat{G}$-invariant functions on the space of Langlands parameters, such a map determines a closed orbit in this space; that is, a semisimple Langlands parameter. This map thus associates a semisimple Langlands parameter to every π, and can thus be thought of as a semisimplified approximation to the local Langlands correspondence. The compatibility of this correspondence with the classical local Langlands correspondence, in cases where such is known, is understood for many groups (particularly general linear groups, unitary groups, and GSp_4), but is an important open problem in general.

This statement, though quite powerful in its own right, is considerably less than is expected. Indeed, Fargues and Scholze conjecture that the spectral action may be upgraded to an equivalence of categories. In particular, for a Whittaker datum (U, ψ), they consider the functor:

$$\mathcal{F} \mapsto \mathcal{F} * (i_e)_! \operatorname{ind}_U^G \psi$$

from $\mathrm{Perf}([Z^1(W_F, \hat{G})/\hat{G}])$ to $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$. They conjecture that this functor extends to a functor on all coherent sheaves, that induces an equivalence of categories:

$$\mathrm{Coh}([Z^1(W_F, \hat{G})/\hat{G}]) \to D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{\mathrm{co}}$$

where the superscript "co" denotes the full subcategory of compact objects. (One can alternatively remove this compactness condition by replacing the left-hand category with that of Ind-coherent sheaves, following Gaitsgory's ideas in the context of classical geometric Langlands.)

The categorical form of these conjectures is currently wide-open, although Hansen and Mann have recently announced a proof for GL_2 (with $\Lambda = \mathbb{Q}_\ell$), and they seem to be close to extending their results to arbitrary general linear groups.

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Examples and Calculations in categorical local Langlands for $\mathbf{GL}_n$

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(joint work with Arthur Bartels)

The categorical Langlands conjecture of Fargues and Scholze [3] predicts an equivalence of categories

$$\mathbf{D}^b_{\mathrm{lisse}}(\mathrm{Bun}_G, \bar{\mathbb{Q}}_\ell)^\omega \xrightarrow{\cong} \mathbf{D}^b_{\mathrm{coh,qc}}(X_{^LG})$$

for a given (quasi-split) reductive group G over a finite extension F of $\mathbb{Q}_p$ (and a prime number $\ell \neq p$). Here Bun_G is the stack (on perfectoid spaces over $\bar{\mathbb{F}}_p$) of G-bundles on the Fargues–Fontaine curve; and $X_{^LG}$ is the stack (on $\bar{\mathbb{Q}}_\ell$-schemes) of L-parameters for G. The $(-)^\omega$ denotes the compact objects in the respective (∞-)category. And the subscript $(-)_{\mathrm{qc}}$ on the right hand side means that we only consider (bounded) complexes of coherent sheaves whose cohomology has quasi-compact support. The equivalence is supposed to be compatible with the *spectral action* and with *parabolic induction* in a way we do not make precise in full generality.

The category $\mathbf{D}^b_{\mathrm{lisse}}(\mathrm{Bun}_G, \bar{\mathbb{Q}}_\ell)^\omega$ has a semi-orthogonal decomposition indexed by the Kottwitz set $B(G)$ into the bounded derived categories with finitely generated cohomology $\mathbf{D}^b_{\mathrm{f.g.}}(\mathrm{Rep}\, G_b)$ of the categories of smooth representations (on $\bar{\mathbb{Q}}_\ell$-vector spaces) of the extended pure inner forms G_b of G defined by $b \in B(G)$. Hence, the Fargues–Scholze conjecture in particular predicts the existence of fully faithful functors

$$R^b : \mathbf{D}^b_{\mathrm{f.g.}}(\mathrm{Rep}\, G_b) \to \mathbf{D}^b_{\mathrm{coh,qc}}(X_{{}^L G}).$$

Let $\mathfrak{Z} = \Gamma(X_{{}^L G}, \mathcal{O}_{X_{{}^L G}})$ denote the ring of invariant functions on $X_{{}^L G}$. Then the spectral action induces for each $b \in B(G)$ a ring homomorphism

$$\mathfrak{Z} \to \mathfrak{Z}_b \tag{1}$$

into the Bernstein center $\mathfrak{Z}_b$ of $\mathrm{Rep}\, G_b$ such that the conjectural functors R^b are linear with respect to the maps (1).

In what follows we restrict to $G = \mathrm{GL}_n$. In this case $X_{{}^L G}$ is the stack of n-dimensional Weil–Deligne representations (as we only deal with characteristic zero coefficients). We denote by rec_{G_b} the local Langlands correspondence for G_b (established thanks to [4, 6, 9]; [2, 8]) which, in this case, is an injection from the set $\Pi(G_b)$ of isomorphism classes of irreducible smooth representations of G_b to the set of isomorphism classes $\mathrm{Par}_{G_b} \subset |X_{{}^L G_b}(\bar{\mathbb{Q}}_\ell)|$ of Frobenius semi-simple L-parameters for G_b. Here $|{-}|$ denotes the set of isomorphism classes of a groupoid.

As G_b is a pure inner form of a Levi subgroup M of G we obtain a map

$$\mathrm{rec}_b : \Pi(G_b) \xrightarrow{\mathrm{rec}_{G_b}} |X_{{}^L G_b}(\bar{\mathbb{Q}}_\ell)| \to |X_{{}^L G}(\bar{\mathbb{Q}}_\ell)|.$$

Then (1) is uniquely characterized by the following property: Let $\Omega \subset \mathrm{Rep}\, G_b$ be a Bernstein block defining a direct factor $\mathfrak{Z}_{b,\Omega}$ of $\mathfrak{Z}_b$. Then $\mathfrak{Z} \to \mathfrak{Z}_{b,\Omega}$ is the unique surjection such that for every $\pi \in \Pi(G_b)$ in Ω the map

$$\mathfrak{Z} \to \mathfrak{Z}_{b,\Omega} \xrightarrow{\chi_\pi} \bar{\mathbb{Q}}_\ell$$

agrees with the map defined by the point $\mathrm{rec}_b(\pi)$. Here χ_π denotes the character via which $\mathfrak{Z}_{b,\Omega}$ acts on π. (In fact the normalization in [3] involves a small twist of the above construction.)

Remark 1. We point out that in general $\mathfrak{Z} \to \mathfrak{Z}_b$ is not surjective. This is the case however, if b is basic. Moreover, if $b = [1]$ is the base point of $B(G)$, then $G_b = \mathrm{GL}_n(F)$ and $\mathfrak{Z} \to \mathfrak{Z}_b$ is an isomorphism. However, this is very special to the choice $G = \mathrm{GL}_n$ and fails for other groups.

Instead of attempting to prove the full conjecture of Fargues and Scholze one can try to understand the functors R^b and study their effect on Grothendieck groups, or more generally: on K-theory.

Theorem 2. *There exists a collection of $\mathfrak{Z}$-linear functors*

$$R^b : \mathbf{D}^b_{\mathrm{f.g.}}(\mathrm{Rep}\, G_b) \to \mathbf{D}^b_{\mathrm{coh,qc}}(X_{{}^L G})$$

that induces an isomorphism

$$K_\bullet(\mathbf{D}^b_{\mathrm{lisse}}(\mathrm{Bun}_G, \bar{\mathbb{Q}}_\ell)^\omega) = \bigoplus_{b\in B(G)} K_\bullet(\mathbf{D}^b_{\mathrm{f.g.}}(\mathrm{Rep}\, G_b)) \xrightarrow{\cong} K_\bullet(\mathbf{D}^b_{\mathrm{coh,qc}}(X_{{}^LG})).$$

Moreover, as G varies through Levi subgroups of GL_n, these isomorphisms are compatible with parabolic induction.

Remark 3.

(i) We do not explicitly spell out here what the compatibility with parabolic induction precisely means. However, once we define the functors R^b on supercuspidal Bernstein blocks of Levi subgroups (which is easy once we assume the local Langlands correspondence), the compatibility with parabolic induction uniquely determines the map on K-theory and implies that this map is an isomorphism.

(ii) The proof of the theorem in fact gives a bit more information than just comparing the K-theory: on both side the K-theory decomposes (using splittings of certain fiber sequences) as a direct sum of the K-theory of rings of the form $\bar{\mathbb{Q}}_\ell[T_1, \dots, T_a, S_1^{\pm 1}, \dots, S_b^{\pm 1}]$. In fact one can match these rings (that can be regarded as the building blocks of the respective categories), not just their K-theory.

Example 4. In the case $G = \mathrm{GL}_2$ the functors R^b can be made rather explicit. For simplicity we restrict to the principal component $Y_{q,2}$ of $X_{{}^LG}$. This is the connected component of 2-dimensional Weil–Deligne representations that are trivial on inertia. Hence $Y_{q,2}$ is the stack quotient of

$$\{(\varphi, N) \in \mathrm{GL}_2 \times \mathrm{Lie}\,\mathrm{GL}_2 \mid N\varphi = q^{-1}\varphi N\}$$

by the diagonal conjugation action of GL_2. Replacing GL_2 by its Borel $\check{B}$ respectively its Levi $\check{T}$ (a maximal torus), one similarly defines stacks $Y_{q,\check{B}}$ and $Y_{q,\check{T}}$ together with morphisms

$$Y_{q,2} \xleftarrow{\beta} Y_{q,\check{B}} \xrightarrow{\alpha} Y_{q,\check{T}}$$

induced by the inclusion $\check{B} \to \mathrm{GL}_2$ respectively the projection $\check{B} \to \check{T}$.
For $G = \mathrm{GL}_2$ we have

$$B(G) = \left\{ (\lambda_1, \lambda_2) \in \mathbb{Q}^2 \;\middle|\; \begin{array}{c} \lambda_1 \leq \lambda_2,\ \lambda_1 + \lambda_2 \in \mathbb{Z} \text{ and} \\ \lambda_1 \neq \lambda_2 \Rightarrow \lambda_1, \lambda_2 \in \mathbb{Z} \end{array} \right\}.$$

and the partial order on $B(G)$ is given as follows: we have $(\lambda_1, \lambda_2) \geq (\mu_1, \mu_2)$ if $\lambda_1 \geq \mu_1$ and $\lambda_1 + \lambda_2 = \mu_1 + \mu_2$. Let $b = (\lambda_1, \lambda_2) \in B(G)$. We then have to distinguish the following cases.

(a) Let $\lambda_1 \neq \lambda_2$. Then $G_b = T = F^\times \times F^\times$ and the Bernstein block Ω_b of $\mathrm{Rep}\, G_b$ corresponding to the chosen connected component $Y_{q,2}$ of $X_{{}^LG}$ is the block of unramified T-representations $(\mathrm{Rep}\, T)_{[T,1]} = \bar{\mathbb{Q}}_\ell[T/T^\circ]$-mod. Here $T^\circ \subset T$ is the (unique) maximal compact subgroup of T and we identify $\mathrm{Spec}\, \bar{\mathbb{Q}}_\ell[T/T^\circ] = \check{T}$ with the dual torus. We note that $Y_{q,\check{T}} = \check{T}/\check{T}$ is the quotient of $\check{T}$ by the trivial action on $\check{T}$ and hence we can identify

$\bar{\mathbb{Q}}_\ell[T/T^\circ]$-mod with the category of quasi-coherent sheaves on $\check{T}$ on which $\check{T} = \mathbb{G}_m^2$ acts via the character (λ_1, λ_2). Finally, we define the functor R^b by composing this identification with $R\beta_* L\alpha^*$.

(b) Let $\lambda_1 = \lambda_2 \in \mathbb{Z}$. Then $G_b = \mathrm{GL}_2(F)$ and the relevant Bernstein block Ω_b of $\mathrm{Rep}\, G_b$ is $(\mathrm{Rep}\, \mathrm{GL}_2(F))_{[T,1]} \cong \mathcal{H}(G, I)$-mod, where $\mathcal{H}(G, I)$ is the Iwahori–Hecke algebra of G (for some choice of an Iwahori subgroup $I \subset G$). Under the identification with $\mathcal{H}(G, I)$-modules (and ignoring a subtlety in passing from left to right modules) the functor R^b is given by

$$\pi \mapsto \pi \otimes^L_{\mathcal{H}(G,I)} \mathcal{M}_b,$$

where $\mathcal{M}_b$ is the $\mathcal{H}(G, I)$-module whose underlying $\mathcal{H}(T, T^\circ) = \Gamma(\check{T}, \mathcal{O}_{\check{T}})$ module is $R\beta_* L\alpha^*(\mathcal{O}_{\check{T}}(\lambda_1 + \lambda_2))$; and this $\mathcal{H}(T, T^\circ)$-module structure is uniquely extended to an $\mathcal{H}(G, I)$-module structure such that generically on $Y_{q,2}$ the fiber of $\mathcal{M}_b$ at a point x equals the Iwahori–Hecke module of the irreducible smooth representation whose L-parameter is given by x, see [5] for details; and see [1] for the more general case of a split group G (and $b = [1]$).

(c) Let $\lambda_1 = \lambda_2 \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. Then $G_b = D^\times$ is the group of units in a quaternion algebra D of dimension 4 (and invariant $\frac{1}{2}$). In this case the relevant Bernstein block Ω_b is the category $(\mathrm{Rep}\, G_b)_{\Omega_b} \cong \bar{\mathbb{Q}}_\ell[F^\times/\mathcal{O}_F^\times]$-mod of $D^\times$-representations that factor through the reduced norm $\mathrm{Nrd} : D^\times \to F^\times$ and such that the corresponding representation of $F^\times$ is unramified. The stack $Y_{q,2}$ has an open substack $Y_{q,2,\min}$ defined by the condition $N \neq 0$ which can easily be identified with $\mathbb{G}_m/\mathbb{G}_m$ for the trivial $\mathbb{G}_m$-action. Then the functor R^b maps the generator $\bar{\mathbb{Q}}_\ell[F^\times/\mathcal{O}_F^\times] = \Gamma(\mathbb{G}_m, \mathcal{O}_{\mathbb{G}_m})$ to the unique extension of $\mathcal{O}_{\mathbb{G}_m}(\lambda_1 + \lambda_2)$ to a coherent sheaf $\mathcal{F}_b$ on $Y_{q,2}$ such that

$$\mathrm{RHom}_{Y_{q,2}}(\mathcal{F}_b, R\beta_* L\alpha^* \mathcal{G}) = 0$$

for all coherent sheaves $\mathcal{G}$ on $Y_{q,\check{T}}$. In fact the sheaf $\mathcal{F}_b$ can also be described rather explicitly using explicit charts for the closure of $Y_{q,2,\min}$.

As the conjectural equivalence of Fargues–Scholze is $\mathfrak{Z}$-linear, we can fix a character $\chi : \mathfrak{Z} \to \bar{\mathbb{Q}}_\ell$ and restrict both sides of the equivalence to the subcategories of objects on which $\mathfrak{Z}$ acts via $\mathfrak{Z} \to \mathfrak{Z}/(\ker \chi)^N$ for some $N \gg 0$. We write $\mathbf{D}^b_{\mathrm{f.g.}}(\mathrm{Rep}\, G_b)_\chi$ respectively $\mathbf{D}^b_{\mathrm{coh}}(X_{^L G})_\chi$ for these categories. The Fargues–Scholze conjecture then predicts that these categories are isomorphic and we can again verify that this is true on the level of K-theory:

Theorem 5. *Let $\chi : \mathfrak{Z} \to \bar{\mathbb{Q}}_\ell$ be a character. Then*

$$K_\bullet(\mathbf{D}^b_{\mathrm{f.g.}}(\mathrm{Rep}\, G_b)_\chi) \cong K_\bullet(\mathbf{D}^b_{\mathrm{coh}}(X_{^L G})_\chi) \tag{2}$$

and both sides are isomorphic to a sum of copies of $K_\bullet(\bar{\mathbb{Q}}_\ell)$. Moreover, one can explicitly match canonical bases on both sides.

Let us give a few more details that explain how the isomorphism (2) is related to the usual local Langlands correspondence.

(i) It is easy to identify

$$K_\bullet(\mathbf{D}^b_{\text{f.g.}}(\operatorname{Rep} G_b)_\chi) = \bigoplus_{b\in B(G)} \bigoplus_{\pi\in\Pi(G_b)_\chi} K_\bullet(\bar{\mathbb{Q}}_\ell), \tag{3}$$

where $\Pi(G_b)_\chi \subset \Pi(G_b)$ is the subset of G_b-representations on which $\mathfrak{Z}$ acts via χ. In particular the irreducible smooth representations provide a canonical basis of the K-theory.

(ii) On the side of stacks of L-parameters the K-theory decomposes as

$$K_\bullet(\mathbf{D}^b_{\text{coh}}(X_{{}^LG})_\chi) = \bigoplus_{(\rho,N)\in\text{Par}_\chi} \bigoplus_{\text{Irr}(C_{(\rho,N)})} K_\bullet(\bar{\mathbb{Q}}_\ell). \tag{4}$$

Here $\text{Par}_\chi \subset |X_{{}^LG}(\bar{\mathbb{Q}}_\ell)|$ is the set of isomorphism classes of Frobenius semi-simple L-parameters (ρ, N) that define a point in $X_{{}^LG}$ on which $\mathfrak{Z}$ acts via χ; and $\text{Irr}(C_{(\rho,N)})$ is the set of isomorphism classes of irreducible algebraic representations of the common centralizer $C_{(\rho,N)}$ of ρ and N in GL_n. The direct sum decomposition (4) is induced by splitting certain fiber sequences in K-theory. These splittings are constructed inductively via the analogue of parabolic induction on the spaces of L-parameters. This construction via parabolic induction gives rise to a canonical basis of (4).

(iii) In order to prove the theorem we need to match the index sets in (3) and (4) and compare the constructed basis in both decompositions. Given the local Langlands correspondences for the various G_b the matching of the index sets is a combinatorial problem. It even holds for general (quasi-split) G (assuming local Langlands for all G_b) by a result of Bertoloni-Meli–Oi [7] that is more involved than in the simple case $G = \text{GL}_n$ that we are using here. The canonical basis in (4) then corresponds to the basis of (3) given by the *standard modules*, i.e. those parabolically induced representation of which a given irreducible representation occurs as the unique irreducible quotient in the Bernstein–Zelevinsky classification.

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The Derived Unipotent Block of $\mathrm{GL}_2(F)$

Rose Berry

Let F be a non-archimedean local field and G the F-points of a reductive group over F. Bernstein showed in [5] that the category of smooth complex representations of G decomposes into blocks, which are parameterised explicitly by inertial supercuspidal support. Furthermore, in almost all cases the structure of these blocks is known via the theory of types ([11, 12]), which assigns to each block a Hecke algebra whose modules are equivalent to the block. Much is then known about the representation theory of these Hecke algebras, which allows the block to be understood.

The unipotent block B_1 is the block containing the trivial representation. For $G = \mathrm{GL}_n(F)$, it was shown, first in [6] and expanded on in [10], that each block of G was equivalent to some $B_1(\prod_j \mathrm{GL}_{m_j}(E_j))$, where E_j are finite extensions of F. Thus, knowing the structure of B_1 for such groups gives us the structure of all other blocks in this case. This is, however, specific to $\mathrm{GL}_n(F)$, and is known to fail for other choices of G. The Hecke Algebra in this case, written $H(n)$, is very well-understood: for example, it is Noetherian and Affine Cellular, and has finite global dimension, and we can describe its irreducible representations.

We can ask about the category of smooth R-representations for R an algebraically closed field of characteristic l. In the case where R is banal, that is, where the order of the reductive quotient of any parahoric subgroup is invertible in R, all the same results hold. However, in the non-banal case much less is known.

Vignéras showed in [7] that, for $G = \mathrm{GL}_n(F)$ and R an algebraically closed field of characteristic not p, there is also the same decomposition into blocks, but this is known to fail for other choices of G. Furthermore, for $\mathrm{GL}_n(F)$, there is the same reduction of blocks to the case of some $B_1(\prod_j \mathrm{GL}_{m_j}(E_j))$ (see [8, 9]). However, the equivalence of B_1 with representations of $H(n)$ fails.

Despite this, some partial results are still known. Vignéras defines in [4] a subcategory B_1' of B_1 given by the modules annihilated by a certain ideal $\mathcal{I}$ of the global Hecke algebra of G. She shows that B_1' is equivalent to modules over a mild extension of $H(n)$, known as the Schur algebra $S(n)$. Furthermore, some power of $\mathcal{I}$ annihilates B_1.

In Vignéras's paper, she shows the equivalence of B_1' with the representations of $S(n)$ by constructing an explicit progenerator Q of B_1', whose endomorphism algebra is $S(n)$. Let $D^b_{\mathrm{fg}}(B_1)$ be the derived category of finitely generated unipotent representations. It is known that representations of G are Noetherian ([3]), and furthermore that $S(n)$ is affine cellular and has finite global dimension ([1]). Using all these existing results, I show that $D^b_{\mathrm{fg}}(B_1)$ is classically generated by Q.

The generator Q has an explicit construction, parahorically induced from Gelfand–Graev representations of the reductive quotients. However, $S(n)$ is also the endomorphism algebra of a much simpler unipotent representation V, which is parahorically induced from trivial representations instead. We may thus hope that V is in fact itself a classical generator. In the case $n = 2$ and l odd dividing $q + 1$, where q is the residual cardinality of F, we were able to show this, that is, that $D^b_{\mathrm{fg}}(B_1)$ is classically generated by V.

The method of proof is to first consider the finite group $G_f = \mathrm{GL}_2(k)$, where k is the residue field of F. Here we may define finite analogues Q_f, V_f, $\mathcal{I}_f$ and $B_{1,f}$ of Q, V, $\mathcal{I}$, and B_1. The structure of irreducible and projective indecomposable representations of $B_{1,f}$ is given explicitly in [13]. Using this I show that V_f and Q_f classically generate the same category, but giving direct constructions of each from triangles and summands of the other.

I then parahorically induce to obtain results about G. While parahoric induction sends V_f to V, a priori it does not relate Q and Q_f. By using the explicit structure of Q_f I show that Q_f induces to a quotient of Q. To show that these in fact agree, it suffices to show that $\mathcal{I}_f$ induces to a subset of $\mathcal{I}$. To prove this, I find an explicit form for the generator of $\mathcal{I}_f$ by combining the explicit structure of $B_{1,f}$ with results of [14], and then calculate directly that this generator lives inside $\mathcal{I}$. This concludes the proof that V classically generates $D^b_{\mathrm{fg}}(B_1)$.

Let $V^\bullet$ be a projective resolution of V. Since we know that $D^b_{\mathrm{fg}}(B_1)$ is classically generated by V, the machinery of [2] gives us a triangulated equivalence $D^b_{\mathrm{fg}}(B_1) \simeq \mathrm{perf}(\text{dg-End}(V^\bullet))$, where dg-End denotes the dg-endomorphism algebra and perf denotes perfect complexes.

The relative simplicity of V allows its projective resolution to be written down. We are thus able to give a complete description of a quasi-isomorphic simplification of dg-End($V^\bullet$), in terms of a restricted tensor product of $S(n)$ with a central contribution arising from dg-endomorphisms of the resolution on the reductive quotient.

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New supercuspidal representations from the Weil representation in characteristic two

David Schwein

Let F be a nonarchimedean local field of residue characteristic p and let G be a reductive F-group, which, for brevity, we identify with the topological group $G(F)$. A (smooth, complex) representation π of G is *supercuspidal* if the matrix coefficients of π are compactly supported modulo the center of G, or equivalently, if π does not arise as a subquotient of a nontrivially parabolically induced representation. This latter characterization, and the close connection between supercuspidal representations and the type theory of Bushnell and Kutzko [BK98], gives these representations a central place in the study of the representation theory of G.

In spite of the mysterious nature of supercuspidal representations, it is expected that they can be constructed explicitly as (compactly) induced representations.

Conjecture 1 ([Kut87, (3.01)]). *Let π be a supercuspidal representation. There exists an open subgroup K of G which is compact modulo the center of G together with an irreducible representation σ of K such that*

$$\pi \simeq \text{c-ind}_K^G(\sigma).$$

Kutzko's conjecture has been the subject of intense research and is known in many cases: for GL_n [BK93a] and SL_n [BK93b]; for U_n, Sp_{2n}, and SO_{2n} when $p \neq 2$ [Ste08]; and whenever G splits over a tamely ramified extension and p is larger than the order of the Weyl group [Yu01, Fin21].

There are many difficulties in extending the construction of supercuspidals to small residue characteristic. One of them is the exceptional behavior of the Heisenberg–Weil representation in characteristic 2. Another is the possible disconnectedness of centralizers of elements of the Lie algebra in low positive characteristic. In joint work with Jessica Fintzen, we overcome these difficulties to construct new supercuspidal representations.

Theorem 2 (Fintzen–S. 2024)**.** *Let Υ be an input to Yu's construction, but where we allow the residue characteristic p to equal 2 and we do not require axiom (GE2). If the residue field of F has cardinality ≥ 3, then the input Υ gives rise to a finite set of supercuspidal representations, each of which is compactly induced from an open subgroup that is compact modulo the center of G.*

Compared to earlier work, there are several new ingredients in Theorem 2.

First, in residue characteristic 2 there is a new obstacle to be overcome in the theory of the Heisenberg–Weil representation: one must replace the symplectic group with Weil's pseudosymplectic group [Wei64, (31)], and even then, the Heisenberg–Weil representation only exists on a double cover. One has to prove that passage to the double cover can be avoided for the relevant representation of a subgroup of G.

Second, in residue characteristic 2 the Heisenberg–Weil representation is self-dual, with Schur indicator determined by the type of the underlying Heisenberg group. We use this additional structure to pin down the Heisenberg–Weil extension up to a character of order at most two, rather than an arbitrary complex character.

Third, even in large residue characteristic, our proof of supercuspidality simplifies earlier proofs by replacing a delicate analysis of the Weil representation, as in [Gér77, Theorem 2.4(b)], with the following simple lemma.

Lemma 3. *Let H be a reductive group over $\mathbb{F}_q$ and let $P = MU$ be a parabolic subgroup. If $q > 3$ then $U(\mathbb{F}_q) \subseteq [P(\mathbb{F}_q), U(\mathbb{F}_q)]$.*

Fourth, axiom (GE2) is satisfied when the residue characteristic p is not a torsion prime for the dual root datum of G, in particular, if $p > 5$. When axiom (GE2) fails, we construct supercuspidal representations using Clifford theory. This aspect of the construction is the reason why it produces several supercuspidal representations, rather than a single one. When $G = \mathrm{GL}_n$, however, there are no choices and our construction produces a single representation.

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The Fargues–Scholze correspondence for tame covers

Yifei Zhao
(joint work with Tony Feng, Ildar Gaisin, Naoki Imai, and Teruhisa Koshikawa)

1. **The LLC for covers.** In this talk, we report on our work-in-progress whose goal is to extend Fargues and Scholze's geometrization of the local Langlands correspondence (*cf.* [4]) to a class of covers of p-adic reductive groups.

Fix a prime p. Let F be a p-adic local field and G be a reductive group F-scheme. Brylinski and Deligne classified central extensions:

$$1 \to \mathsf{K}_2 \to E \to G \to 1 \tag{1}$$

of sheaves on the big Zariski site of $\operatorname{Spec} F$, where K_2 stands for the Zariski sheafified second algebraic K-group (*cf.* [1]).

Fix an integer $n \geq 1$ such that $\mu_n(F)$ has cardinality n. Evaluating (1) at F and pushing out by the nth Hilbert symbol, one obtains a topological cover:

$$1 \to \mu_n(F) \to \widetilde{G} \to G(F) \to 1.$$

If, in addition, an injective character $\zeta : \mu_n(F) \to \overline{\mathbf{Q}}_\ell^\times$ is fixed ($\ell \neq p$), then it makes sense to talk about smooth representations of $\widetilde{G}$ on which $\mu_n(F)$ acts through ζ. Such representations are called ζ-*genuine*.

On the other hand, starting with a Brylinski–Deligne extension (1), Weissman constructs an "L-group", which takes the form of an extension $\widetilde{H}$ of the Weil group W_F by $H(\overline{\mathbf{Q}}_\ell)$, where H is a pinned split reductive group $\overline{\mathbf{Q}}_\ell$-scheme (*cf.* [11]). One may then define L-*parameters* to be $H(\overline{\mathbf{Q}}_\ell)$-conjugacy classes of sections $\sigma : W_F \to \widetilde{H}$ of the natural surjection. Conjecturally, there is a map:

$$\mathrm{LLC} : \left\{ \begin{matrix} \text{irreducible } \zeta\text{-genuine} \\ \text{smooth representations of } \widetilde{G} \end{matrix} \right\} \to \left\{ \begin{matrix} \text{semisimple} \\ \text{L-parameters} \end{matrix} \right\} \tag{2}$$

with similar behavior as the (semisimplified) local Langlands correspondence for linear reductive groups (*cf.* [7, 11]).

2. **Geometrization.** In its most classical manifestation, our result yields a construction of (2) whenever the cover is "tame", *i.e.* the covering degree n is coprime to p. In fact, we construct a much finer geometric structure: a spectral action in the sense of Fargues–Scholze [4].

Let Bun_G denote the moduli stack of G-bundles on the Fargues–Fontaine curve. It contains an open substack of the form $*/G(F)$. Using this open immersion, one embeds smooth representations of $G(F)$ in the ∞-category $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbf{Q}}_\ell)$ of lisse-étale sheaves over Bun_G. Write $\mathrm{LS}_{\check{G}}$ for the moduli stack of L-parameters, defined in terms of the Langlands dual group $\check{G}$. Fargues and Scholze construct an action of $\mathrm{Perf}(\mathrm{LS}_{\check{G}})$ on $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbf{Q}}_\ell)$, called the *spectral action*, which induces the (semisimplified) local Langlands correspondence for G (*cf.* [4]).

In order to incorporate covers into this framework, we need to find a moduli stack $\widetilde{\mathrm{Bun}}_G$ of "$\widetilde{G}$-bundles on the Fargues–Fontaine curve". This object does not exist in the literal sense, because $\widetilde{G}$ is not an algebraic group. On the other hand, whatever $\widetilde{\mathrm{Bun}}_G$ might be, it should contain $*/\widetilde{G}$ as an open substack. This suggests that $\widetilde{\mathrm{Bun}}_G$ is a gerbe over Bun_G banded by the abelian group $\mu_n(F)$.

The initial impetus of our work is the construction of a gerbe on Bun_G associated to each Brylinski–Deligne extension (1) when n is coprime to p. It is given by categorifying the following maps on cohomology:

$$\begin{aligned}(3)\qquad \mathrm{H}^2_e(\mathsf{B}G_{\mathrm{Zar}}, \mathsf{K}_2) &\xleftarrow{\simeq} \mathrm{H}^4_e(\mathsf{B}G_{\mathrm{Zar}}, \mathbf{Z}_{\mathrm{mot}}(2)) \\ &\to \mathrm{H}^4_e(\mathsf{B}G_{\text{ét}}, \mu_n^{\otimes 2}) \xrightarrow{\int_X} \mathrm{H}^2_e(\mathrm{Bun}_G, \mu_n).\end{aligned}$$

Here, H^i_e denotes degree-i *reduced* cohomology, $\mathbf{Z}_{\mathrm{mot}}(2)$ denotes weight-2 integral motivic cohomology, the first isomorphism is a mild generalization of a result of [3], the second map is étale realization, and the last map is "integration along the fiber" of the relative Fargues–Fontaine curve X_S over S. Since there is no structural map from X_S to S, the "integration" map actually invokes the tilting correspondence.

In the context of the geometric Langlands program, there is another (and more classical) construction of the gerbe $\widetilde{\mathrm{Bun}}_G$, which proceeds by constructing a line bundle on Bun_G and taking the gerbe of its nth roots (*cf.* [9, 5]). Unfortunately, this method fails for the Fargues–Fontaine curve.

3. **Étale metaplectic covers.** Because our work requires categorifying the maps in (3), we in particular need to categorify degree-4 (reduced) étale cohomology of the classifying stack $\mathsf{B}G_{\text{ét}}$.

In fact, for any finite abelian group A, there is a 2-groupoid of *A-valued étale metaplectic covers* whose set of isomorphism classes is $\mathrm{H}^4_e(\mathsf{B}G_{\text{ét}}, A(1))$. Formally, they are defined to be morphisms of pointed (higher) étale stacks:

$$(4)\qquad \mu : \mathsf{B}G_{\text{ét}} \to \mathsf{B}^4 A(1)_{\text{ét}}.$$

These objects appeared first in Deligne's work [2] and again as parameters for factorization gerbes in Gaitsgory–Lysenko [6]. An A-valued étale metaplectic cover induces a topological central extension $\widetilde{G}$ of $G(F)$ by A.

For the purpose of constructing the LLC for covers, we find it more advantageous to start with a general A-valued étale metaplectic cover rather than a Brylinski–Deligne extension (which defines a $\mu_n(F)$-valued étale metaplectic cover). Indeed, Weissman's L-group can be defined directly in terms of étale metaplectic covers. Moreover, Kaletha recently defined a class of covers which is largely disjoint from the Brylinski–Deligne ones (*cf.* [8]). The two classes are both captured by étale metaplectic covers and interact in interesting ways.

4. **Spectral action.** Let us state the main result of our work-in-progress.

We fix a prime p, a p-adic local field F, and a reductive group F-scheme. As input for covering data, we take a finite abelian group A and any A-valued étale metaplectic cover (4), subject to the tameness assumption that p does not divide $|A|$. In addition, we fix an injective character $\zeta : A \to \overline{\mathbf{Q}}_\ell^\times$.

Given these data, we construct a gerbe $\widetilde{\mathrm{Bun}_G}$ over Bun_G banded by A as well as a moduli stack $\mathrm{LS}_{\widetilde{H}}$ of Weissman's L-parameters.

We consider the ∞-category $\mathrm{D}_{\mathrm{lis},\zeta}(\widetilde{\mathrm{Bun}_G}, \overline{\mathbf{Q}}_\ell)$ of lisse-étale sheaves over $\widetilde{\mathrm{Bun}_G}$ which are $(*/A)$-equivariant against the character ζ. It contains ζ-genuine smooth representations of $\widetilde{G}$ as a full subcategory.

Theorem. *There is a natural action of* $\mathrm{Perf}(\mathrm{LS}_{\widetilde{H}})$ *on* $\mathrm{D}_{\mathrm{lis},\zeta}(\widetilde{\mathrm{Bun}_G}, \overline{\mathbf{Q}}_\ell)$.

The main ingredient in this theorem is a mixed characteristic version of the twisted geometric Satake equivalence of Finkelberg–Lysenko [5] and Reich [10], which we prove "from scratch".

For split tori, we calculate the spectral action explicitly and discover that the ∞-category of Hecke eigensheaves is nonzero for any L-parameter. This complements Weissman's observation that the map (2) is *not* surjective for certain covers of split tori, indicating that the geometrization of the LLC is better behaved than its classical counterpart.

The tameness assumption is used in our work in two ways: Our construction of $\widetilde{\mathrm{Bun}_G}$ uses étale cohomology of diamonds with prime-to-p coefficients and our proof of the Satake equivalence uses degeneration to characteristic p in some technical steps. Removing this assumption remains a significant problem.

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Spectral action on isocrystals

Arnaud Eteve
(joint work with Dennis Gaitsgory, Alain Genestier, and Vincent Lafforgue)

Let $F = \mathbb{F}_q((t))$ be a local function field of residue characteristic $p > 0$ and let G be a connected reductive group over F. The goal of the talk was to report on work in progress on a construction of a spectral action on the stack of isocrystals. This construction can be understood as a, conjecturally equivalent, analog of the geometrization of local Langlands proposed by Fargues–Scholze [6]. Let $\ell \neq p$ be a prime and $\Lambda \in \{E, \mathcal{O}_E, k_E\}$ be a coefficient ring, where $E/\mathbb{Q}_\ell$ is a finite extension, with ring of integers $\mathcal{O}_E$ and residue field k_E. Let us first introduce the two sides of the local Langlands correspondence.

On the automorphic side, let LG be the loop group of G. We view it as a group ind-scheme over $\overline{\mathbb{F}}_q$ equipped with a Frobenius endomorphism $\sigma : LG \to LG$. We define

$$\mathrm{Isoc}_G = \frac{LG}{\mathrm{Ad}_\sigma \, LG}$$

the stack of G-isocrystals, where Ad_σ is the usual action of σ-conjugation of LG on itself given by $\mathrm{Ad}_\sigma(g)(x) = gx\sigma(g^{-1})$. The underlying space of Isoc_G is Kottwitz' set $\mathrm{B}(G)$, see [13], and the stabilizer of a point $b \in \mathrm{B}(G)$ is $G_b(F)$ the group of rational points of an inner form (of a Levi) of G.

On the Galois side, we let W_F be the Weil group of F, $\hat{G}$ be the Langlands dual group of G, which we view as a reductive group over Λ and ${}^L G$ be the L-group of G. We finally denote by

$$\mathrm{Par}_G = \{\phi : W_F \to {}^L G\}/\hat{G}$$

the stack of local Langlands parameters constructed by [3, 19, 6].

Theorem 1 ([4])**.** *If $\Lambda \in \{\mathcal{O}_E, k_E\}$, assume that ℓ does not divide $|\pi_0(Z(G))|$. There is a canonical action of* $\mathrm{Perf}(\mathrm{Par}_G)$ *on* $\mathrm{Shv}(\mathrm{Isoc}_G, \Lambda)$*.*

Out of such a categorical action, one gets a morphism

$$\mathcal{O}(\mathrm{Par}_G) \to \mathfrak{Z}(G(F)) \tag{1}$$

from the algebra of global sections on Par_G to the Bernstein center of $G(F)$. It is a natural consequence of our construction that this morphism equals the morphism constructed by [9].

Conjecture 2. *Assume that G is quasi-split and let (U, ψ) be a Whittaker datum. There is a canonical* $\mathrm{Perf}(\mathrm{Par}_G)$*-linear equivalence*

$$\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{Par}_G) \simeq \mathrm{Shv}(\mathrm{Isoc}_G, \Lambda)^\omega,$$

where $_{\mathrm{Nilp}}$ *denotes the subcategory of coherent sheaves with nilpotent singular support and* $(-)^{\omega}$ *denotes the category of compact objects. It is normalized so that the structure sheaf is sent to the Whittaker sheaf.*

Relation with Fargues–Scholze. A geometrization of the local Langlands correspondence has also been proposed by Fargues and Scholze [6]. The key difference is that instead of Isoc_G, they construct a spectral action on Bun_G, the stack of G-bundles on the Fargues–Fontaine curve. Both Theorem 1 and Conjecture 2 have their analog in *loc. cit.*.

Conjecture 3. *There is a canonical* $\mathrm{Perf}(\mathrm{Par}_G)$*-linear equivalence of categories*

$$\mathrm{Shv}(\mathrm{Bun}_G, \Lambda) = \mathrm{Shv}(\mathrm{Isoc}_G, \Lambda). \tag{2}$$

While the full equivalence is conjectural, it is a theorem of [17] that the morphism (1) constructed out of the spectral action of Fargues–Scholze agrees with the one of Genestier–Lafforgue.

Higher nearby cycles. It follows from the spectral decomposition theorems [6, 1], that the data of the spectral action is equivalent to the data of the Hecke functors. That is, functorially in finite sets I, we have a $\otimes$-functor

$$T_I : \mathrm{Rep}_{\Lambda}({}^L G)^{\otimes I} \to \mathrm{End}(\mathrm{Shv}(\mathrm{Isoc}_G)) \otimes \mathrm{Lisse}(\mathbb{D}^{\times})^{\otimes I},$$

where $\mathrm{Lisse}(\mathbb{D}^{\times})$ denotes the category of Weil local systems on $\mathbb{D}^{\times} = \mathrm{Spec}(\overline{\mathbb{F}}(\!(t)\!))$. To construct our spectral action, we define these functors using a nearby cycles procedure.

When $I = \{*\}$ is a point, we introduce a deformation $f : \mathrm{Isoc}_{G,\mathbb{A}^1} \to \mathbb{A}^1$ of Isoc_G whose fiber over 0 is Isoc_G and whose fiber over $\mathbb{A}^1 - 0$ is $\mathrm{Isoc}_G \times \mathrm{Hk}^{\mathrm{loc}}$, where the second factor is the local Hecke stack. The functor T_I is then defined to be, for $V \in \mathrm{Rep}_{\Lambda}({}^L G)$ and $A \in \mathrm{Shv}(\mathrm{Isoc}_G)$,

$$T_I(V)(A) = \Psi_f(A \otimes \mathrm{Sat}_V)$$

where Ψ_f denotes the nearby cycle functor (defined with respect to f) and $\mathrm{Sat}_V \in \mathrm{Shv}(\mathrm{Hk}^{\mathrm{loc}})$ is the Satake sheaf.

To define the functors T_I when $|I| > 1$, we use the theory of nearby cycles over general bases due to Deligne, Laumon [16], Gabber, Orgogozo [18], Illusie [12] and its reinterpretation by Hansen–Scholze [10]. When dealing with nearby cycles over general bases, one usually gets control theorems for the nearby cycles only after modifying the base. In our situation, to get a canonical construction, we prove that the 'naïve' nearby cycles (i.e. without introducing a modification) are well behaved. The key part for this control involves showing that the space that controls the difference between naïve and non-naïve nearby cycles is contractible (in a suitable sense).

There is a second difficulty stemming from the fact that nearby cycles, a priori, are only equipped with an action of the inertia $I_F \subset W_F$. The difference between the actions of these two groups is encoded by the action of the partial Frobeniuses. The problem here is essentially combinatorial as it requires organizing the partial Frobenius correspondences in a coherent way with respect to the nearby cycles.

Relation with Hemo–Zhu. Hemo and Zhu [11] have announced a construction of a piece of the categorical equivalence of conjecture 2. Their equivalence is obtained by taking the categorical trace of Frobenius on Bezrukavnikov's equivalences [2].

Theorem 4 ([11]). *Assume that G is unramified. There is an equivalence of categories*

$$\mathrm{Shv}(\mathrm{Isoc}_G, E)^{\mathrm{unip}} = \mathrm{IndCoh}(\mathrm{Par}_G^{\mathrm{unip}}), \tag{3}$$

where the LHS is the full subcategory of $\mathrm{Shv}(\mathrm{Bun}_G, E)$ composed of all sheaves whose restrictions to all strata are unipotent representations and in the RHS is the $\mathrm{Par}_G^{\mathrm{unip}}$ is the substack of Par_G of unipotent parameters.

It follows from this equivalence of categories that the LHS of (3) is equipped with an action of $\mathrm{Perf}(\mathrm{Par}_G)$. This action can essentially be tracked down to the construction of Gaitsgory's central functor [8]. It is then a consequence of the compatibility of the nearby cycles with proper pushforward and smooth pullback that the action coming from Theorem 4 is the same as the one provided by Theorem 1.

Local–Global compatibility. Finally, let us mention some the compatibility with the global situation of [14]. Let X be a smooth projective curve over $\mathbb{F}_q$ and let $v \in X$ be a place together with an identification $F = \mathbb{F}_q(X)_v$ between F and the completion of the function field of X at v. Let us also assume that G is the restriction to F of a reductive group over X.

Let $\mathrm{Par}_G^{\mathrm{glob}}(X - v)$ be the stack of global Langlands parameters that are unramified outside of v, see [19]. As proven (1-categorically) in [15], the cohomology sheaves of stacks of global shtukas with level structures at v can be organized as an object $\mathrm{Drinf}^v \in \mathrm{Coh}(\mathrm{Par}_G^{\mathrm{glob}}(X - v)) \otimes \mathrm{Shv}(\mathrm{pt}/(G(F)))$. Note however that the derived part of this statement is still open.

As proposed by [7], after replacing the stacks of shtukas with level at v by stacks of shtukas up to isogeny at v, Drinf^v should upgrade to an object

$$\mathrm{Drinf}^{\mathrm{enh}} \in \mathrm{IndCoh}(\mathrm{Par}_G^{\mathrm{glob}}(X - v)) \otimes \mathrm{Shv}(\mathrm{Isoc}_G).$$

Conjecture 5 (Local–global compatibility). *The (conjectural) object $\mathrm{Drinf}^{\mathrm{enh}}$ lifts to an object*

$$\mathrm{Drinf}^{\mathrm{ult}} \in \mathrm{IndCoh}(\mathrm{Par}_G^{\mathrm{glob}}(X - v)) \otimes_{\mathrm{Qcoh}(\mathrm{Par}_G)} \mathrm{Shv}(\mathrm{Isoc}_G),$$

where $\mathrm{Qcoh}(\mathrm{Par}_G)$ acts on $\mathrm{IndCoh}(\mathrm{Par}_G^{\mathrm{glob}}(X-v))$ by pullback along the restriction $\mathrm{Par}_G^{\mathrm{glob}}(X - v)) \to \mathrm{Par}_G$.

The strategy for the proof of this conjecture can be broken down in two steps : a combinatorial one to construct $\mathrm{Drinf}^{\mathrm{enh}}$ and a geometric one to control nearby cycles on stacks of global shutkas. The latter is established in [5].

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Failure of uniqueness of Whittaker model for covering groups: Iwahori component

Nadya Gurevich

(joint work with Fan Gao and Edmund Karasiewicz)

The main object of this talk is the Iwahori component of the Gelfand–Graev representation of a central covering group. We begin with introducing some notation.

(1) Let F be a p-adic field, containing μ_n, the group of n-roots of unity. Let G be a connected reductive algebraic group, split over F. We fix a maximal split torus T of G. Let X and Y be the character and cocharacter lattices of T, $(X, Y, R, R^\vee)$ be the root datum of G and W be the Weyl group.

(2) Let $\bar{G}$ be a Brylinski-Deligne central covering of the group G of degree n with $(n,p)=1$. Such a covering is partially defined by a W-invariant quadratic form $Q:Y\to\mathbb{Z}$ with associated bilinear form B_Q. Let $Y_{Q,n}=nY^*\cap Y\subset Y$ be the modified lattice. We denote by σ the permutation representation of W on the quotient $Y/Y_{Q,n}$.

(3) We fix a Borel subgroup $B=TU$ of G. The groups U,U^{op} split canonically in the covering. We fix a splitting in $\bar{G}$ of the maximal compact subgroup $K=G(\mathcal{O}_F)$ and all its subgroups. The inverse image $\bar{T}$ of the torus T is not commutative and its center $Z(\bar{T})$ is governed by the lattice $Y_{Q,n}$.

(4) We fix an embedding $\epsilon:\mu_n\hookrightarrow\mathbb{C}$ and consider the category $\mathrm{Rep}_\epsilon(\bar{G})$ of smooth ϵ-genuine representations of $\bar{G}$.

The Gelfand–Graev representation $\mathcal{V}=\mathrm{ind}_{\mu_n U^{\mathrm{op}}}^{\bar{G}}\epsilon\otimes\psi$, where ψ is a non-degenerate character of U^{op} of conductor $\mathcal{P}$ is representing the Whittaker functor $\mathcal{W}$ defined by $\mathcal{W}(\pi)=\mathrm{Hom}_{U^{\mathrm{op}}}(\pi,\mathbb{C}_\psi)$.

(5) We fix an Iwahori subgroup $I\subset K$, that is the inverse image of the Borel subgroup over a finite field under the reduction map. Let $\mathcal{H}$ denote the genuine Iwahori–Hecke algebra on $\bar{G}$. Its support is governed by $W\ltimes Y_{Q,n}$. The finite Hecke subalgebra is denoted by $\mathcal{H}_f$.

The failure of uniqueness of Whittaker model for irreducible representations of $\bar{G}$ is tightly related to the fact that $\bar{T}$ is not commutative. The latter phenomenon can be measured by the permutation representation $(\sigma,W,\mathbb{C}[Y/Y_{Q,n}])$ defined above.

To compute Whittaker dimension we study the Gelfand–Graev representation $\mathcal{V}$. In the talk we restrict our attention to its Iwahori component as a representation of $\mathcal{H}$.

The structure of this model for algebraic groups, i.e. in the case $n=1$ and $Y_{Q,n}=Y$, was proven by Chan and Savin.

Theorem ([1])**.** *There is an isomorphism of $\mathcal{H}$ modules*

$$\mathcal{V}^I=\mathrm{sgn}\otimes_{\mathcal{H}_f}\mathcal{H}.$$

We extend this result for arbitrary Brylinski Deligne tame coverings.

Theorem ([2])**.**

(1) *There is a decomposition of $\mathcal{V}^I$ as a direct sum of $\mathcal{H}$ modules*

$$\mathcal{V}^I=\bigoplus_{\mathcal{O}}\mathcal{V}^I_{\mathcal{O}},$$

where $\mathcal{O}$ runs over the set of right $W\ltimes Y_{Q,n}$ orbits on $Y=W\backslash W\ltimes Y$.

(2) *One has $\mathcal{V}_{\mathcal{O}}=\chi_y\otimes_{\mathcal{H}_y}\mathcal{H}$, where y is a minimal representative in the orbit $\mathcal{O}$ and the subalgebra $\mathcal{H}_y$ is generated by the elements supported on the stabilizer of y in $W\ltimes Y_{Q,n}$. In addition χ_y is a character of $\mathcal{H}_y$.*

In particular, if $\mathcal{O}$ is a free orbit then one has $\mathcal{V}^I_{\mathcal{O}} \simeq \mathcal{H}$ in which case $\mathcal{V}^I$ becomes a progenerator of the category of $\mathcal{H}$-modules.

The set of orbits $\mathcal{O}$ can be identified with the set of W orbits on $Y/Y_{Q,n}$. If an orbit is splitting, i.e. there exists a W equivariant splitting map $\mathcal{O} \to Y$, the algebra $\mathcal{H}_y$ is contained in $\mathcal{H}_f$ and χ_y is the sign character of it. We give a table of covering groups for which all the orbits $\mathcal{O}$ are splitting.

Next we apply the above result to determine the Whittaker dimension of constituents of unramified regular principal series in terms of representation σ. From now on we assume that any orbit in $Y/Y_{Q,n}$ is splitting.

For any unramified character χ of $Z(\bar{T})$ extended trivially to $Z(\bar{T})T(\mathcal{O})$ let $i(\chi)$ be the induced representation of $\bar{T}$ of dimension $|Y/Y_{Q,n}|$. Let $I(\chi) = \mathrm{Ind}_{\bar{B}}^{\bar{G}}\, i(\chi)$. The space $\mathcal{W}(I(\chi))$ of Whittaker functionals on $I(\chi)$ is isomorphic to the vector space $\mathrm{Hom}_{\mathcal{H}_I}(\mathcal{V}^I, I(\chi^{-1}))$ of dimension $|Y/Y_{Q,n}|$.

The decomposition of $I(\chi)$ for a regular character χ is governed by the set

$$\Phi(\chi) = \{\alpha \in R \mid \langle \chi, n_\alpha \alpha^\vee\rangle = 1\} \subset \Delta, \quad n_\alpha = n/\mathrm{gcd}(n, Q(\alpha)).$$

Precisely, by Rodier's result, the semisimplification of $I(\chi)$ is multiplicity free and there is a natural bijection $\mathcal{P}(\Phi(\chi)) \to \mathrm{JH}(I(\chi)), \quad S \to \pi_S$ determined by the Jacquet module of π_S. Here $\mathcal{P}(\Phi(\chi))$ is the power set of $\Phi(\chi)$ and $\mathrm{JH}(I(\chi))$ is the Jordan–Hölder set of $I(\chi)$.

We show that $\mathcal{W}(\pi_S) = \langle \sigma_S, \sigma\rangle_W$ where σ_S is a sum of certain Kazhdan–Lusztig representations of W naturally associated with S.

If $\Phi(\chi) = \Delta$, the set of simple roots in R then $I(\chi)$ has a unique irreducible quotient that is denoted by $\Theta(\chi)$. It is the minimal representation of $\bar{G}$.

Corollary.

(1) *The Whittaker dimension of $\Theta(\chi)$ equals to the number of free W-orbits in $Y/Y_{Q,n}$.*
(2) *The Whittaker dimension of any irreducible genuine representation of $\bar{G}$ having non-zero Iwahori-fixed vectors is bigger or equal than the Whittaker dimension of $\Theta(\chi)$.*

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An introduction to ABV-packets for p-adic groups

Clifton Cunningham

This talk provides a gentle introduction to ABV-packets for p-adic groups, as presented in [2], building on [8], including updates on work in progress.

1. Basic properties of ABV-packets

Let $F/\mathbb{Q}_p$ be a p-adic field and let G be a conntected reductive algebraic group over F. In this talk I will assume G is quasi-split only to simplify exposition. We assume the Langlands correspondence is known for $G(F)$.

Let $\Pi(G)$ denote the set of equivalence classes of irreducible representations in the category $\mathrm{Rep}(G)$ of smooth representations of $G(F)$. Let $\Phi(G)$ denote the set of $\widehat{G}(\mathbb{C})$-conjugacy classes of Langlands parameters for $G(F)$; these equivalence classes are called L-parameters. ABV-packets $\Pi_\phi^{\mathrm{ABV}}(G)$ relate to L-packets $\Pi_\phi(G)$ as follows:

Theorem 1.1 ([2]). *For every $\phi \in \Phi(G)$,*

$$\Pi_\phi(G) \subseteq \Pi_\phi^{ABV}(G) \subseteq \Pi(G).$$

We refer to the set difference $\Pi_\phi^{\mathrm{ABV}}(G) \setminus \Pi_\phi(G)$ as the *corona* to the L-packet $\Pi_\phi(G)$.

It is useful to tighten the inclusion $\Pi_\phi^{\mathrm{ABV}}(G) \subseteq \Pi(G)$ as follows. For every Langlands parameter $\phi : W_F' \to \widehat{G}(\mathbb{C}) \rtimes W_F$, define $\lambda_\phi : W_F \to \widehat{G}(\mathbb{C}) \rtimes W_F$ by $\lambda_\phi(w) := \phi(w, \mathrm{diag}(|w|^{1/2}, |w|^{-1/2})$, where $W_F' = W_F \times \mathrm{SL}_2(\mathbb{C})$; then λ_ϕ is called the *infinitesimal parameter* of ϕ. The function $\phi \mapsto \lambda_\phi$ is finite-to-one. If we write λ_π for the infinitesimal parameter of the Langlands parameter of π, then we may set $\Pi_{\lambda_\phi}(G) := \{\pi \in \Pi(G) \mid \lambda_\pi = \lambda_\phi\}$.

Theorem 1.2 ([2]). *For every $\phi \in \Phi(G)$,*

$$\Pi_\phi^{ABV}(G) \subseteq \Pi_{\lambda_\phi}(G).$$

In fact, this result follows directly from the definition of $\Pi_\phi^{\mathrm{ABV}}(G)$, which we will review below.

For the next basic result, we need the following notion: a Langlands parameter ϕ is *open* if $L(s, \phi, \mathrm{Ad})$ is holomorphic at $s = 1$. All tempered Langlands parameters are open, but the converse is not true. See [4] for more on open parameters.

Theorem 1.3 ([4]). *If ϕ is an open Langlands parameter for G then*

$$\Pi_\phi^{ABV}(G) = \Pi_\phi(G).$$

In order to state the last basic property discussed in this talk, let use write $\mathrm{AZ} : \Pi(G) \to \Pi(G)$ for the Aubert–Zelevinsky involution. The following result will appear in a paper joint with José Cruz, Alex Hazeltine and Chi-Heng Lo:

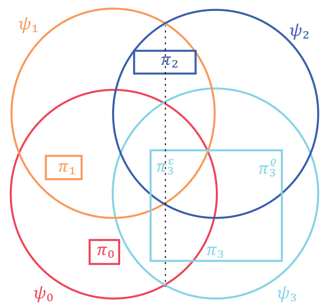
Theorem 1.4. *For any quasi-split classical group G and for any $\phi \in \Phi(G)$,*

$$\mathrm{AZ}\left(\Pi_\phi^{ABV}(G)\right) = \Pi_{\hat{\phi}}^{ABV}(G),$$

where $\hat{\phi} \in \Phi(G)$ is the Pyasetskii dual L-parameter for ϕ.

2. Example: exceptional group G_2

For $G = G_2$ (exceptional group), let π_3^ε be the unique supercuspidal, irreducible representation of $G(F)$ with Langlands parameter ϕ_3 for which the component group $A_{\phi_3} := \pi_0(Z_{\widehat{G}(\mathbb{C})}(\phi_3))$ is the symmetric group S_3; see [5] for more on this representation. Then $\Pi_{\lambda_{\phi_3}}(G) = \{\pi_0, \pi_1, \pi_2, \pi_3, \pi_3^\varrho, \pi_3^\varepsilon\}$. The diagram here, taken from [5], shows all ABV-packets (circles) and L-packets (squares) containing these representations. Reflection about the dotted line describes the Aubert-Zelevinsky involution on these representations.



3. Example: GL_{16}

Consider $\sigma \in \Pi(\mathrm{GL}_8)$ with multisegment $\{[-2,-1],[0],[1,2],[-1,0,1]\}$. The representation of $\mathrm{GL}_{16}(F)$ induced parabolically from $\sigma \otimes \sigma$ on $(\mathrm{GL}_8 \times \mathrm{GL}_8)(F)$ is reducible; in fact,

$$\mathrm{Ind}_{P(F)}^{\mathrm{GL}_{16}(F)}(\sigma \otimes \sigma) = \pi_{\mathrm{KS}} \oplus \pi_\psi,$$

where π_{KS} and π_ϕ are irreducible representations of $\mathrm{GL}_{16}(F)$ described in [3]. While π_ψ is of Arthur-type, π_{KS} is not. Let ϕ_{KS} be the L-parameter for π_{KS}. Then

$$\Pi_{\phi_{\mathrm{KS}}}^{\mathrm{ABV}}(\mathrm{GL}_{16}) = \{\pi_{\mathrm{KS}}, \pi_\psi\}.$$

This example shows that ABV-packets for general linear groups are not all singletons. Indeed, it can be shown that for $N \geq 16$, there is always a non-singleton ABV-packet for $\mathrm{GL}_N(F)$.

4. Definition of ABV-packet

The definition of the ABV-packet $\Pi_\phi^{\mathrm{ABV}}(G)$ requires Vogan's version [8] of the local Langlands correspondence, as we now briefly recall. Consider the variety of Langlands parameters with infinitesimal parameter λ_ϕ

$$V_{\lambda_\phi} := \{x \in \mathrm{Lie}\, \widehat{G}(\mathbb{C}) \mid \mathrm{Ad}(\lambda_\phi(w))x = |w|\, x,\ \forall w \in W_F\},$$

which comes equipped with a natural action by the group

$$H_{\lambda_\phi} := \{g \in \widehat{G}(\mathbb{C}) \mid \mathrm{Inn}(\lambda_\phi(w))g = g,\ \forall w \in W_F\}.$$

The local Langlands correspondence determines a function

$$\mathcal{P} : \Pi_{\lambda_\phi}(G) \to \mathrm{Per}_{H_{\lambda_\phi}}(V_{\lambda_\phi})_{/\mathrm{iso}}^{\mathrm{simple}}$$

to the set of isomorphism classes of simple H_{λ_ϕ}-equivariant perverse sheaves on V_{λ_ϕ}. The Langlands parameter ϕ determines an H_{λ_ϕ}-orbit $C_\phi \subset V_{\lambda_\phi}$ and the conormal bundle $\Lambda_{C_\phi} \subset V_{\lambda_\phi} \times V_{\lambda_\phi}^*$ to C_ϕ plays an important role in the theory. In

[2] we explain how to find a *generic* $(x_\phi, y_\phi) \in \Lambda_{C_\phi}$ with which we may define the ABV-packet for ϕ

$$\Pi_\phi^{\mathrm{ABV}}(G) := \{\pi \in \Pi_{\lambda_\phi}(G) \mid \left(R\Phi_{y_\phi}\mathcal{P}(\pi)\right)_{x_\phi} \neq 0\},$$

where $R\Phi_{y_\phi}$ is the vanishing cycles functor determined by $y_\phi : V_{\lambda_\phi} \to \mathbb{C}$.

5. Vogan's conjecture on Arthur packets

In [2] we consider the H_{λ_ϕ}-equivariant fundamental group A_ϕ^{ABV} of the conormal bundle Λ_{C_ϕ} at the base point (x_ϕ, y_ϕ) and define a functor $\mathrm{NEvs}_\phi : \mathrm{Per}_{H_{\lambda_\phi}}(V_{\lambda_\phi}) \to \mathrm{Rep}(A_\phi^{\mathrm{ABV}})$ which, in turn, defines a function $\mathrm{NEvs}_\phi \circ \mathcal{P} : \Pi_\phi^{\mathrm{ABV}}(G) \to \mathrm{Rep}(A_\phi^{\mathrm{ABV}})$. The theorem below will appear in a paper joint with Alexander Hazeltine, Chi-Heng Lo, Baiying Liu, Mishty Ray and Bin Xu. For general linear groups, the analogous theorem was established in [6] and [7].

Theorem 5.1. *If G is a split odd orthogonal or symplectic group and $\phi \in \Phi(G)$ is of Arthur type ψ then*

$$\Pi_\phi^{ABV}(G) = \Pi_\psi(G).$$

Moreover, $A_\phi^{ABV} = Z_{\widehat{G}(\mathbb{C})}(\psi)/Z_{\widehat{G}(\mathbb{C})}(\psi)^\circ =: A_\psi$ and the function

$$\mathrm{NEvs}_\phi \circ \mathcal{P} : \Pi_\phi^{ABV}(G) \to \mathrm{Rep}(A_\psi)$$

coincides with the function $\Pi_\psi(G) \to \mathrm{Rep}(A_\psi)$ defined by Arthur's theory [1].

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Geometric analogues to local Arthur packets: p-adic classical groups

Mishty Ray

ABV-packets, named after Jeff Adams, Dan Barbasch, and David Vogan, are proposed generalizations to local Arthur packets. They arise out of Vogan's geometric perspective on the local Langlands correspondence for p-adic groups. This proposal says that an ABV-packet for a Langlands parameter of Arthur-type coincides with the corresponding local Arthur packet. This is often stated as Vogan's conjecture; it was articulated in [1] as Conjecture 1 where it was attributed to Vogan. This talk aims to report on the progress of Vogan's conjecture. In joint work with Cunningham [2, 3], we prove this conjecture for p-adic GL_n. The proof of the conjecture is in progress for p-adic classical groups in a joint project with Cunningham, Hazeltine, Liu, Lo, and Xu. In this talk, we discuss an important piece of the proof for GL_n and its adaptation to the more general setting of classical groups.

For simplicity, assume G is GL_n, SO_{2n+1}, or Sp_{2n}. The field F is a nonarchimedean local field of characteristic 0; we dub groups considered over such F as p-adic groups. The enhanced local Langlands correspondence is a bijection between the set $\Pi^{\mathrm{pure}}(G)$ of equivalence classes of smooth irreducible representations of the F-points of G along with pure inner forms G_δ of G, as δ varies over elements of $H^1(F,G)$, and the set $\Phi^e(G)$ of equivalence classes of (ϕ,τ), where $\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$ is a Langlands parameter and $\tau \in \mathrm{Irrep}(A_\phi)$. Here $A_\phi = \pi_0(Z_{\widehat{G}}(\phi))$.

To talk about the geometric perspective on the local Langlands correspondence, we fix an infinitesimal parameter $\lambda : W_F \to {}^LG$. Given a Langlands parameter ϕ, its infinitesimal parameter λ_ϕ is given by the formula $\lambda_\phi(w) = \phi\left(w, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2}\end{pmatrix}\right)$. Given λ, one defines the Vogan variety V_λ with the group action of H_λ, as in Chapter 4 of [1]. There is a bijection $\phi \mapsto C_\phi$ between equivalence classes of Langlands parameters (L-parameters) ϕ and H_λ-orbits C_ϕ in V_λ. To every enhanced Langlands parameter (ϕ,τ), one may attach a simple equivariant perverse sheaf $\mathcal{P} = \mathcal{IC}(\mathcal{L}^\tau) \in \mathrm{Per}_{H_\lambda}(V_\lambda)^{\mathrm{simple}}_{/\mathrm{iso}}$, where $\mathcal{L}^\tau$ is the local system on C_ϕ corresponding to the irreducible representation τ of A_ϕ, also the equivariant fundamental group attached to C_ϕ. Let $\Pi^{\mathrm{pure}}_\lambda(G)$ denote all the elements of $\Pi^{\mathrm{pure}}(G)$ which correspond to a Langlands parameter with infinitesimal parameter λ. One may summarize the geometric perspective on the local Langlands correspondence as follows.

$$\Pi^{\mathrm{pure}}_\lambda(G) \longrightarrow \Phi^e_\lambda(G) \longrightarrow \mathrm{Per}_{H_\lambda}(V_\lambda)^{\mathrm{simple}}_{/\mathrm{iso}}$$
$$\pi \mapsto (\phi,\tau) \mapsto \mathcal{P}(\pi) = \mathcal{IC}(\mathcal{L}^\tau_{C_\phi}).$$

This identification determines a pairing between Grothendieck groups

$$\langle\cdot,\cdot\rangle : K\Pi^{\mathrm{pure}}_\lambda(G) \times K\mathrm{Per}_{H_\lambda}(V_\lambda) \to \mathbb{C},$$

defined in [1], Equation (8.6).

Now let us consider a local Arthur parameter $\psi : W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$. Given ψ, we obtain a Langlands parameter ϕ_ψ according to $\phi_\psi(w,x) = \psi\left(w, x, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}\right)$. Our current focus is on the quasisplit form G. The local Arthur packet $\Pi_\psi(G)$ is a set of smooth irreps of $G(F)$ defined in terms of Arthur's endoscopic classification for classical groups; for $G = \mathrm{GL}_n$, the packet $\Pi_\psi(G) = \Pi_{\phi_\psi}(G) = \{\pi_\psi\}$ where $\Pi_{\phi_\psi}(G)$ is the singleton L-packet for ϕ_ψ. Given a Langlands parameter ϕ_ψ, one defines the ABV-packet

$$\Pi^{\mathrm{ABV}}_{\phi_\psi}(G) = \{\pi \in \Pi_\lambda(G) : \mathrm{Evs}_\psi(\mathcal{P}(\pi)) \neq 0\}$$

using the functor

$$\mathrm{Evs}_\psi : \mathrm{Per}_{H_\lambda}(V_\lambda) \to \mathrm{Rep}(A_\psi)$$

for $A_\psi = \pi_0(Z_{\widehat{G}}(\psi))$ defined using Deligne's vanishing cycles; see [3], Equation (3). ABV-packets are defined for any Langlands parameter, not just those of Arthur-type, but we specialize here to parameters of Arthur-type as it pertains to Vogan's conjecture. We formulate the quasisplit version (for trivial δ) of Vogan's conjecture below.

Conjecture 1 (See Conjecture 1 in [1]). *Let G be GL_n, SO_{2n+1}, or Sp_{2n}.*

(a) *The local Arthur packet for ψ coincides with the ABV-packet for ϕ_ψ: $\Pi_\psi(G) = \Pi^{\mathrm{ABV}}_{\phi_\psi}(G)$.*
(b) *Arthur's stable distributions are calculated by* Evs*: $\eta_\psi = \eta^{\mathrm{Evs}}_{\phi_\psi}$.*
(c) *The endoscopic transfer of Arthur's stable distributions are calculated by* NEvs*: $\eta_{\psi,s} = \eta^{\mathrm{NEvs}}_{\phi_\psi,s}$, for every semisimple $s \in Z_{\widehat{G}}(\psi)$.*

Here, $\eta_\psi, \eta_{\psi,s} \in K\Pi_\lambda(G)$ are virtual representations which serve as analogues to Arthur's stable distributions as defined in Section 3.9 of [1]. The virtual representations $\eta^{\mathrm{Evs}}_{\phi_\psi}$, $\eta^{\mathrm{NEvs}}_{\phi_\psi,s}$ have coefficients given by Evs or NEvs, where the latter is a normalization of the Evs functor; see [1], Section 8.2.

For $G = \mathrm{GL}_n$, Conjecture 1 is a theorem due to [3]. As we do not have to worry about stability of virtual distributions or pure inner forms for general linear groups, the theorem can be stated more simply.

Theorem 2. *Let G denote p-adic GL_n. Let ψ be a local Arthur parameter of G with corresponding Langlands parameter ϕ_ψ. Then, $\Pi_\psi(G) = \Pi_{\phi_\psi}(G)$.*

We aim to show that $\Pi^{\mathrm{ABV}}_{\phi_\psi}(G) = \Pi_\psi(G) = \{\pi_\psi\}$. Instead of working with sets, we work with virtual representations. The virtual representation $\eta^{\mathrm{Evs}}_{\phi_\psi} \in K\Pi_\lambda(G)$ is

$$\eta^{\mathrm{Evs}}_{\phi_\psi} = \sum_{\pi \in \Pi^{\mathrm{ABV}}_{\phi_\psi}(G)} \pm \dim(\mathrm{Evs}_\psi(\mathcal{P}(\pi)) \cdot [\pi].$$

Since A-packets are singletons for GL_n, we set $\eta_\psi := [\pi_\psi]$. It is enough to show $\eta^{\mathrm{Evs}}_{\phi_\psi} = \eta_\psi = \pi_\psi$. As the pairing $\langle \cdot, \cdot \rangle_\lambda$ is non-degenerate, this is equivalent to showing that $\langle \eta^{\mathrm{Evs}}_{\phi_\psi}, \mathcal{F} \rangle = \langle \eta_\psi, \mathcal{F} \rangle$, for all $\mathcal{F} \in K\mathrm{Per}_{H_\lambda}(V_\lambda)$. An arbitrary Arthur parameter of GL_n factors through an irreducible parameter ψ' of a Levi subgroup

G' of G, that is, $\psi' = \varepsilon \circ \psi$ where $\varepsilon : \widehat{G'} \hookrightarrow \widehat{G}$. A key step in the proof of Theorem 2 is to reduce the proof of Vogan's conjecture to an irreducible parameter of a Levi subgroup G'. Write ϕ_ψ (resp. $\phi_{\psi'}$) and λ (resp λ') for the Langlands and infinitesimal parameters of G (resp. G'). The inclusion $\varepsilon : V_{\lambda'} \hookrightarrow V_\lambda$ induces the restriction $\varepsilon^* : K\mathrm{Per}_{H_\lambda}(V_\lambda) \to K\mathrm{Per}_{H_{\lambda'}}(V_{\lambda'})$. We reduce the proof via the three equalities below.

$$\langle \eta^{\mathrm{Evs}}_{\phi_\psi}, \mathcal{F} \rangle \stackrel{\mathrm{A}}{=} \langle \eta^{\mathrm{Evs}}_{\phi_{\psi'}}, \varepsilon^* \mathcal{F} \rangle \stackrel{\mathrm{B}}{=} \langle \eta_{\psi'}, \varepsilon^* \mathcal{F} \rangle \stackrel{\mathrm{C}}{=} \langle \eta_\psi, \mathcal{F} \rangle$$

The equality A follows from a fixed point formula, as explained in [3], Section 3. The equality B is Vogan's conjecture for irreducible parameters of G', which follows from the main result of [2]. The equality C is the most interesting. We get $\langle \eta_{\psi'}, \varepsilon^* \mathcal{F} \rangle = \langle {}^t\varepsilon^* \eta_{\psi'}, \mathcal{F} \rangle$ and we set $\mathrm{Lift}^G_{G'} := {}^t\varepsilon^*$. We show in [3], Proposition 4.5 that

$$\mathrm{Lift}^G_{G'} = \mathrm{Ind}^G_P, \tag{1}$$

where P is the standard parabolic subgroup of G with Levi component G', and Ind^G_P is normalized parabolic induction, interpreted as a linear map between Grothendieck groups. To prove this, we use the dual bases consisting of standard representations and constructible sheaves. To go between the bases of standard and simple objects, we use the p-adic analogue of the Kazhdan–Lusztig hypothesis for GL_n, which is a theorem. Now we can finish equality C as

$$\mathrm{Lift}^G_{G'}(\eta_{\psi'}) = \mathrm{Ind}^G_P(\pi_{\psi'}) = \pi_\psi = \eta_\psi, \quad \text{in } K\Pi_\lambda(G). \tag{2}$$

To progress with $G = \mathrm{SO}_{2n+1}$, Sp_{2n}, we must generalize (1) and (2). With apologies to the reader for the vague treatment, we explain the next steps, with details to be made precise in upcoming work. Given a semisimple $s \in Z_{\widehat{G}}(\psi)$, we obtain the endoscopic datum $(G', {}^LG', s, \varepsilon)$ of G, with $\psi' = \varepsilon \circ \psi$ and $\phi_{\psi'}$, λ', ε^* as before. Endoscopic lifting is now

$$\mathrm{Lift}^G_{G',s} : K\Pi_{\lambda'}(G')^{\mathrm{st}} \xrightarrow{\text{upgrade}} K\Pi^{\mathrm{pure}}_{\lambda'}(G') \xrightarrow{{}^t\varepsilon^*} K\Pi^{\mathrm{pure}}_{\lambda}(G) \xrightarrow{\text{project}} K\Pi_\lambda(G).$$

Here, $K\Pi_{\lambda'}(G')^{\mathrm{st}}$ is the subspace of $K\Pi_{\lambda'}(G')$ spanned by stable virtual representations. Let $\mathrm{Trans}^G_{G',s}$ denote the Langlands–Shelstad transfer map as used in Arthur's theory. We are able to show that $\mathrm{Lift}^G_{G',s} = \mathrm{Trans}^G_{G',s}$. As for GL_n, we show this equality on the dual basis of standard objects, and use the p-adic analogue of the Kazhdan–Lusztig hypothesis to apply it to the basis of simple objects. We must replicate the same argument with Kottwitz–Shelstad transfer, where we consider G to correspond to a twisted endoscopic datum for GL_{2n}. This will give us the analogue to (1). To finish the argument, we must prove the analogue to (2), in other words, Conjecture 1 parts (b) and (c). This will require further reductions all the way to GL_{2n}, and utilize Theorem 2. Parts (b) and (c) imply (a), which will allow us to conclude the argument.

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Supercuspidal Characters: What and How(e)

Loren Spice

In this extended abstract, we consider only complex representations. This is meant to be, not a comprehensive history, but a story of the development of one strand of thought in the modern character theory of p-adic groups. As such, important work had to be omitted during the talk. Because of space limitations, even more has to be omitted here. I mean no slight to anyone whose work is not directly mentioned. All claims about the history of the ideas discussed here should be viewed as implicitly preceded by "to the best of my knowledge", even when I do not explicitly say so.

What do characters look like?

Weyl type. A Weyl-type character formula involves parameterising representations of a group G (in Weyl's case, a compact Lie group) by certain pairs (T, ϕ) of a maximal torus T in G and a complex-valued, linear character ϕ of G, and then computing the character of $\pi_{(T,\phi)}$ at an element γ in terms of a weighted sum of values of ϕ at conjugates of γ. Weyl's original work in 1920, and the work of Harish-Chandra on discrete-series characters of real semisimple Lie groups in 1966 and of Kazhdan on twisted characters of p-adic general linear groups in 1984, in fact computed characters *before* constructing representations!

For p-adic groups, other Weyl-type formulæ were proven by Jacquet–Langlands for multiplicative groups of quaternionic division algebras in 1970; by Gérardin for what we would now call "unramified, toral very supercuspidals" (I think—the meanings of some of these modifiers have shifted over time) in 1975; by Corwin–Moy–Sally for multiplicative groups of prime-degree division algebras in 1994; and by DeBacker for supercuspidals of general linear groups of prime size in 1997. In addition to other restrictions, most of which are just artifacts of the technology available at the time, all of these latter results hold only far from the identity. This is an essential restriction that reflects the qualitatively different behaviour of a character in different ranges.

Kirillov type. A Kirillov-type character formula involves parameterising representations of a group G by coadjoint orbits, and then computing the character of π_{X^*} in terms of the Fourier transform of the orbital integral of X^* (pushed forward to the group *via* an exponential map). This phenomenon was completely realised by Kirillov for real nilpotent Lie groups in 1962, inaugurating his "orbit method", which crops up in a wide variety of contexts. In the context of p-adic reductive groups, this theme was most thoroughly explored by the exhaustive work of Murnaghan in 1995–1996 on Kirillov-type formulæ for classical groups. These formulæ hold only near the identity, again reflecting the qualitatively different behaviour of a character in different ranges.

Deligne–Luztig type. In its simplest form, Deligne–Lusztig theory once again parameterises certain representations by pairs (T, ϕ), as in a Weyl-type formula. However, in the Deligne–Lusztig setting of finite groups of Lie type, one does not seem to know everything about a character of G if one knows its values only on regular semisimple elements in G. (See, however, the work of Chan and Oi [2], including Chan's abstract in these proceedings.) Now one needs to compute the character of $\pi_{(T,\phi)}$ at an arbitrary element γ of G, and it is given as a weighted sum, over conjugates of γ with appropriate Jordan-type decomposition su, of products of the form $\phi(s)Q_T^{C_G(s)^\circ}(u)$, where Q is a certain special function attached to the maximal torus T in the connected centraliser group $C_G(s)^\circ$.

The first work in this direction was done by Green in 1955, who, as far as I know just on the basis of Steinberg's work on finite GL_3 and GL_4, figured out the characters of *all* finite general linear groups. It is in his honour that the special functions Q are known as Green's functions. Building on his work, Deligne and Lusztig handled arbitrary finite groups of Lie type in 1976.

In the talk, I focussed on Deligne–Lusztig-type formulæ for supercuspidal representations of p-adic reductive groups.

How to compute characters

An integral formula. In this section, we always work with p-adic reductive groups. The only generally available tool for computing characters in this setting is an integral formula, first proven by Harish-Chandra for supercuspidal representations in 1970, and later generalised by Rader and Silberger in 1993 to handle discrete-series representations. (The form of the Harish-Chandra integral formula is most pleasant when applied to a compactly induced supercuspidal representation. Schwein's abstract in these proceedings discusses the conjecture that all supercuspidals arise in this way.) As far as I know, the first time this was used to compute a supercuspidal character was in DeBacker's thesis in 1997. Note that this is just after the Moy–Prasad theory had become available, in 1994 and 1996. The work of Adler and me in 2008–2009 evaluated the integral formula for supercuspidal representations satisfying a strong "compactness" condition. This already took a lot of work, and it seemed likely that explicitly evaluating the integral formula was not likely to be successful in general.

Asymptotic expansions. Using, as far as I know, just the evidence of Silberger's 1970 computations for PGL_2, Howe made an audacious conjecture in 1977 about the blow-up of characters of general linear groups near the singular set. This conjecture not only was correct for general linear groups, but admitted a natural generalisation to *arbitrary* reductive groups, as what is known as the Harish-Chandra–Howe local character expansion (proven in this generality by Harish-Chandra in 1977). It qualitatively describes the character values near the identity as a linear combination of Fourier transforms of orbital integrals.

A quantitative strategy introduced for classical groups by Waldspurger (unpublished), then carried out for general depth-0 representations by Barbasch and Moy in 1997 and for arbitrary-depth representations by DeBacker in 2002, uses cleverly chosen test functions to achieve more quantitative results, with total control over the range of validity and some control over the coefficients in the linear combination. (See also the work of Ciubotaru and Okada [3], including Okada's abstract in these proceedings, on a positive-depth analogue of the Barbasch–Moy test functions.)

In 2007, Adler and Korman generalized DeBacker's techniques to provide a lower bound on the range of validity of the local character expansion near an arbitrary semisimple element. In 2003 and 2006, Kim and Murnaghan generalized the techniques in another direction to prove the existence of, and compute the range of validity for, a different kind of expansion that they called "Γ-asymptotic", which is a middle ground between the Harish-Chandra–Howe local character expansion and a classical Murnaghan–Kirillov-type formula. They did not compute the coefficients in full generality, but their work showed how appropriate Hecke-algebra isomorphisms could be used to reduce the coefficients in the asymptotic expansion of an arbitrary-depth representation to the depth-0 case. In particular, the Hecke-algebra isomorphisms of Howe and Moy allow the completely explicit computation of the Γ-asymptotic expansion of a positive-depth representation of a general linear group in terms of the local character expansion of a depth-0 unipotent representation of a product of smaller general linear groups.

In [5], the same techniques are pushed one step farther, proving analogous results for Γ-asymptotic expansions centred at arbitrary semisimple elements, not just at the identity. These character computations unexpectedly revealed the need for a modification of the Yu construction, carried out jointly with Fintzen and Kaletha in [4].

Again, appropriate Hecke-algebra isomorphisms would allow the computation of the coefficients in such an expansion in terms of asymptotic expansions, including local character expansions, of depth-0 representations. Such expansions are not yet available in general (although see the work of Adler–Fintzen–Mishra–Ohara [1], including Ohara's abstract in these proceedings), but I was able to prove a satisfactory substitute for supercuspidal representations, and so provide an inductive recipe for their characters. With the aid of an analogous recipe for orbital integrals, this inductive recipe is unwound in [6].

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Arthur packets for Mp(2n)

Wen-Wei Li

The talk is based on the eponymous article [4].

For a local field F of characteristic zero and a symplectic F-vector space W of $2n$, the metaplectic group is a central extension of locally compact groups

$$1 \to \mu_2 \to \mathrm{Mp}(W) \to \mathrm{Sp}(W) \to 1,$$

which splits if and only if $F = \mathbb{C}$. For a number field $\dot{F}$ with adèle ring $\mathbb{A}$ and a symplectic $\dot{F}$-vector space $\dot{W}$, the adelic metaplectic group is a central extension

$$1 \to \mu_2 \to \mathrm{Mp}(\dot{W}, \mathbb{A}) \to \mathrm{Sp}(\dot{W}, \mathbb{A}) \to 1,$$

which splits uniquely over $\mathrm{Sp}(\dot{W})$ and can be identified with the restricted product of the local ones modulo $\{(z_v)_v \in \bigoplus_v \mu_2 : \prod_v z_v = 1\}$. The notation $\mathrm{Mp}(2n)$ is frequently used.

These groups are not F or $\mathbb{A}$-points of a linear algebraic group, yet they play a prominent role in the study of automorphic forms, for example in the recent formulation of relative Langlands duality. My aim is to understand *Arthur's conjectures* for $\mathrm{Mp}(W)$ in both the local and global aspects.

Let us fix the symplectic form on W and an additive character $\underline{\psi}$ of F or $\mathbb{A}/F$. Put $G := \mathrm{Sp}(W)$ and define the Langlands dual group of $\tilde{G}$ as

$$\tilde{G}^\vee := \mathrm{Sp}(2n, \mathbb{C}).$$

A representation or automorphic form of $\tilde{G}$ is said to be genuine if every $z \in \mu_2$ acts by multiplication by z; a basic example is the Weil representation $\omega_{\underline{\psi}} = \omega_{\underline{\psi}}^+ \oplus \omega_{\underline{\psi}}^-$.

For tempered genuine representations, a local Langlands correspondence for metaplectic groups has been obtained by Adams–Barbasch (Archimedean) and by Gan–Savin (non-Archimedean). Furthermore, for discrete global Arthur parameters $\dot{\psi}$ that are trivial on $\mathrm{SL}(2, \mathbb{C})$, Gan–Ichino [2] obtained a multiplicity formula for $L^2_{\dot{\psi}}$.

The aforementioned works are all based on Θ-correspondence. If we adhere to Arthur's vision, these statements should be deduced from the stable trace formula. We shall follow a somewhat different route, by combining the stable trace formula in [3] with Θ-correspondence [2] to yield the desired results. The works [5, 6] also serves as the crucial local inputs.

Main local results. Let F be local of characteristic zero. Denote the set of Arthur parameters of $\tilde{G}$ by $\Psi(\tilde{G})$, taken up to $\tilde{G}^\vee$-conjugacy. Let $n = \frac{1}{2}\dim W$. Elliptic endoscopic data of $\tilde{G}$ are in bijection with conjugacy classes of elements $s \in \tilde{G}^\vee$ satisfying $s^2 = 1$, which are in turn classified by pairs $(n', n'') \in \mathbb{Z}^2_{\geq 0}$ such that $n' + n'' = n$. The endoscopic group underlying an elliptic endoscopic datum $\mathbf{G}^!$ corresponding to (n', n'') is $G^! = \mathrm{SO}(2n'+1) \times \mathrm{SO}(2n''+1)$ (split). We have $(G^!)^\vee \simeq Z_{\tilde{G}^\vee}(s) \subset \tilde{G}^\vee$.

The transfer $\mathcal{T}_{\mathbf{G}^!, \tilde{G}}$ of orbital integrals dualizes to $\mathcal{T}^\vee_{\mathbf{G}^!, \tilde{G}}$, sending stable virtual characters on $G^!(F)$ to genuine virtual characters on $\tilde{G}$.

Given a pair $(\mathbf{G}^!, \psi^!)$ where $\psi^! \in \Psi(G^!)$, we obtain (ψ, s) where $\psi \in \Psi^+(\tilde{G})$ is the image of $\psi^!$. Every pair (ψ, s) arises uniquely in this way. Arthur's theory for $G^!$ affords a stable virtual character $S\Theta^{G^!}_{\psi^!}$. Consider the (-1)-eigenspace of $\mathrm{std} \circ \psi$ under s and set

$$\epsilon(\psi^{s=-1}) := \epsilon\left(\frac{1}{2}, \mathrm{std} \circ \psi^{s=-1}|_{\mathcal{L}_F}, \underline{\psi}\right),$$
$$T_{\psi,s} := \epsilon\left(\psi^{s=-1}\right) \cdot \mathcal{T}^\vee_{\mathbf{G}^!, \tilde{G}}\left(S\Theta^{G^!}_{\psi^!}\right).$$

One shows that $T_{\psi,s}$ depends only on ψ and the image x of s in $\mathcal{S}_\psi$. Every $x \in \mathcal{S}_\psi$ arises from some $s \in S_\psi$ with $s^2 = 1$, hence we may consider the Fourier expansion of $x \mapsto T_{\psi,x}$ or its translates.

Theorem. *Given $\psi \in \Psi^+(\tilde{G})$, set $s_\psi := \psi\left(1, \left(\begin{smallmatrix} -1 & \\ & -1 \end{smallmatrix}\right)\right) \in S_\psi$ and let $x_\psi \in \mathcal{S}_\psi$ be its image. Then*

$$\pi_{\psi,\chi} := |\mathcal{S}_\psi|^{-1} \sum_{x \in \mathcal{S}_\psi} \chi(x_\psi x) T_{\psi,x}$$

is a non-negative integral combination of genuine irreducible characters of $\tilde{G}$, for all χ. If $\psi \in \Psi(\tilde{G})$, then these irreducible characters are unitary.

One can then define the Arthur packet Π_ψ as a finite multi-set of genuine irreducible representations, equipped with two maps

$$\Pi_{\mathrm{genuine}}(\tilde{G}) \leftarrow \Pi_\psi \rightarrow \mathcal{S}^\vee_\psi.$$

If ψ is a bounded L-parameter of $\tilde{G}$, then the local Langlands correspondence of Adams–Barbasch and Gan–Savin is recovered.

The following information on $\pi_{\psi,\chi}$ is also available.

- Reduction to good parity by parabolic induction (cf. Mœglin's works).
- Specified infinitesimal character when F is Archimedean.
- Uniqueness of spherical constituents in the unramified situation.
- Description of central characters in terms of certain local root numbers.
- The L-packet Π_{ϕ_ψ} within Π_ψ.
- Effect of variation of additive characters $\underline{\psi}$, cf. [1, Theorem 12.1].

Main global results. Consider the adelic metaplectic covering

$$1 \to \mu_8 \to \left(\tilde{G} := \mathrm{Mp}(\dot{W}, \mathbb{A})\right) \to G(\mathbb{A}) \to 1.$$

Let $\dot{\Psi}(\tilde{G})$ be the set of global Arthur parameters for $\tilde{G}$. For each $\dot{\psi} \in \dot{\Psi}(\tilde{G})$, one can define its centralizer and component groups $S_{\dot{\psi}}$ and $\mathcal{S}_{\dot{\psi}}$, related to their local avatars by localization maps, as well as the element $s_{\dot{\psi}} \in S_{\dot{\psi}}$.

For almost all places v, one attaches a Satake parameter to $\dot{\psi}_v$ using the metaplectic Satake isomorphism. This extracts from the genuine discrete L^2-automorphic spectrum the summands $L^2_{\dot{\psi}}$.

Theorem. *Let $\dot{\Psi}_2(\tilde{G})$ denote the set of discrete global Arthur parameters of $\tilde{G}$, i.e. the parameters which cannot factor through any proper Levi. Then*

$$L^2_{\mathrm{disc,genuine}}(G(\dot{F})\backslash\tilde{G}) = \widehat{\bigoplus}_{\dot{\psi}\in\dot{\Psi}_2(\tilde{G})} L^2_{\dot{\psi}},$$

The result above is actually due to Gan–Ichino [2, Theorem 1.1]. The hardest part is to show that only discrete parameters contribute. Arthur called it the "no embedded Hecke eigenvalues" property.

To decompose $L^2_{\dot{\psi}}$ further, take a representative $s \in S_{\dot{\psi}}$ of $x \in \mathcal{S}_{\dot{\psi}}$ satisfying $s^2 = 1$. The global root number $\nu_{\dot{\psi}}(x) := \epsilon\left(\dot{\psi}^{s=-1}\right) = \prod_v \epsilon\left(\dot{\psi}_v^{s=-1}\right)$ is independent of the choice of s, and gives a character $\nu_{\dot{\psi}}$ of $\mathcal{S}_{\dot{\psi}}$.

Denote by $\epsilon^{\mathrm{Art}}_{\dot{\psi}} \in \mathcal{S}^\vee_{\dot{\psi}}$ Arthur's sign character for $\mathrm{SO}(2n+1)$. Put $\epsilon_{\dot{\psi}} := \epsilon^{\mathrm{Art}}_{\dot{\psi}}\nu_{\dot{\psi}} \in \mathcal{S}^\vee_{\dot{\psi}}$. Denote by diag the diagonal map $\mathcal{S}_{\dot{\psi}} \to \prod_v \mathcal{S}_{\dot{\psi}_v}$. It makes sense to define $\Pi_{\dot{\psi}}$ as the set of tuples $(\dot{\pi}_v)_v$ with $\dot{\pi}_v \in \Pi_{\dot{\psi}_v}$, spherical at almost all v, and put

$$\Pi_{\dot{\psi}}(\epsilon_{\dot{\psi}}) := \left\{ \dot{\pi} \in \Pi_{\dot{\psi}} \;\middle|\; \mathrm{diag}^* \prod_v \langle \cdot, \dot{\pi}_v \rangle = \epsilon_{\dot{\psi}} \right\}.$$

Theorem. *Let $\dot{\psi} \in \dot{\Psi}_2(\tilde{G})$. With the notations above, one has*

$$L^2_{\dot{\psi}} \simeq \bigoplus_{\dot{\pi}\in\Pi_{\dot{\psi}}(\epsilon_{\dot{\psi}})} \dot{\pi}.$$

This generalizes the multiplicity formula in [2, Theorem 1.4]; it also confirms Gan's Arthur conjecture for $\tilde{G}$ in his ICM 2014 lecture.

Finally, when $n = 1, 2$, the local and global theorems above are shown to be compatible with known results by Waldspurger and Gan–Ichino.

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Branching rules for supercuspidal representations of unramified $U(1,1)$

Ekta Tiwari

1. Introduction

The study of branching rules involves examining how an irreducible representation breaks down when restricted to a specific subgroup. The aim is to uncover hidden internal symmetries within the original representation and to highlight shared features across related families of representations.

As part of my PhD thesis, I prove that for the rank-one quasi-split unramified unitary group G, the restriction of a supercuspidal representation of $G(F)$ to a maximal compact subgroup $\mathcal{K}$ is multiplicity-free. I provide an explicit decomposition into irreducible representations of $\mathcal{K}$ characterized by unique depth and degree, something known only for two other groups (both of rank one). Moreover, upon restriction to a smaller subgroup of $\mathcal{K}$ the decomposition can be described entirely using representations constructed from the nilpotent orbits of the Lie algebra of $G(F)$, thereby proving a new case of a novel recent conjecture in the literature. In this talk, I present an explicit description of this decomposition for positive depth supercuspidal representations of $G(F)$.

1.1. **Notation.** Let F be a non archimedean local field, and let $E = F[\sqrt{\epsilon}]$ be the quadratic unramified extension of F, where ϵ is a non-square element of $F^{\times}$. We fix ϖ to be a uniformizer of F and we normalize the valuation ν on F such that $\nu(\varpi) = 1$. We denote the ring of integers of F by $\mathcal{O}_F$, the ring of integers of E by $\mathcal{O}_E$, and their respective maximal ideals are denoted by $\mathfrak{p}_F$, and $\mathfrak{p}_E$. We denote the norm one elements of E over F by $U(1)_{E/F}$.

Let G denote the quasi-split unitary group $U(1,1)$ of rank 1, and let $G(F)$ be the group of F-points of G. Let $\mathcal{B}(G,F)$ denote the Bruhat–Tits building of $G(F)$. For $r \geq 0$, and $x \in \mathcal{B}(G,F)$, we denote the Moy–Prasad filtration subgroup of depth r at x by $G(F)_{x,r}$.

Up to conjugacy, $G(F)$ has four classes of anisotropic tori, and each anisotropic torus $\mathcal{T}$ corresponds to a point y in $\mathcal{B}(G,F)$ as summarized in the table below.

Isomorphism Classes of tori $\mathcal{T}$	Conjugacy Classes $\mathcal{T}_{\gamma_1,\gamma_2} = \left(\begin{smallmatrix} a & b\gamma_1 \\ b\gamma_2 & a \end{smallmatrix}\right)$	$\mathcal{A}(G,\mathcal{T},F) = \{y\}$
$U(1)_{E/F} \times U(1)_{E/F}$ unramified torus	$\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$ $\left(\begin{smallmatrix} a & b\varpi^{-1} \\ b\varpi & a \end{smallmatrix}\right)$	$y = 0$ $y = 1$
$U(1)_{E[\sqrt{\varpi}]/F[\sqrt{\varpi}]}$ ramified torus	$\left(\begin{smallmatrix} a & b \\ b\varpi & a \end{smallmatrix}\right)$	$y = \frac{1}{2}$
$U(1)_{E[\sqrt{\epsilon\varpi}]/F[\sqrt{\epsilon\varpi}]}$ ramified torus	$\left(\begin{smallmatrix} a & b\epsilon^{-1} \\ b\varpi & a \end{smallmatrix}\right)$	$y = \frac{1}{2}$

2. Branching rules for positive depth supercuspidal representations of G

For $r > 0$, and ϕ a G-generic character of $\mathcal{T}$ of depth r, let $\rho = \rho(\mathcal{T}, y, r, \phi)$ denote the irreducible representation of $\mathcal{T}G(F)_{y,\frac{r}{2}}$ constructed using the Adler–Fintzen–Yu method.

Theorem 2.1. *All positive depth supercuspidal representations of $G(F)$ have the form*

$$\theta \otimes \text{c-Ind}_{\mathcal{T}G_{y,\frac{r}{2}}}^{G} \rho$$

for some quasi-character θ of $G(F)$, and ρ as above, or $\theta \times \pi_0$, where π_0 is a depth zero supercuspidal representation of $G(F)$.

Let $\mathcal{K} = G(\mathcal{O}_E)$. Then $\mathcal{K}$ is a maximal compact subgroup of $G(F)$, and it can also be realized as the stabilizer of the point $x = 0$ in $\mathcal{B}(G,F)$. Set $s := r/2$. We now describe the branching of $\text{c-Ind}_{\mathcal{T}G_{y,s}}^{G}$ when restricted to $\mathcal{K}$.

Theorem 2.2. *Let $\rho = \rho(\mathcal{T}, y, r, \phi)$. Upon restriction to $\mathcal{K}$, $\text{c-Ind}_{\mathcal{T}G_{y,s}}^{G} \rho$ decomposes as a direct sum of inequivalent irreducible representations. More precisely,*

$$\text{Res}_{\mathcal{K}}^{G} \text{c-Ind}_{\mathcal{T}G_{y,s}}^{G} \rho \cong \bigoplus_{g \in \mathcal{K}\backslash G/\mathcal{T}G_{y,s}} \text{c-Ind}_{\mathcal{K}\cap(\mathcal{T}G_{y,s})^g}^{\mathcal{K}} \rho^g,$$

and each of the components $\text{c-Ind}_{\mathcal{K}\cap(\mathcal{T}G_{y,s})^g}^{\mathcal{K}} \rho^g$ occurring in the decomposition is irreducible of depth $d = d(\mathcal{T}, g, r)$ (distinct) and degree $q^{d-1}(q^2-1)$.

We prove this by providing an explicit description of $\text{c-Ind}_{\mathcal{K}\cap(\mathcal{T}G_{y,s}^{\mathcal{K}})^g} \rho^g$ up to equivalence. More explicitly, for a double coset representative g of $\mathcal{K}\backslash G/\mathcal{T}G_{y,s}$, we construct an irreducible representation $S_d(\phi^g)$ of $\mathcal{K}$ of the same depth and degree as $\text{c-Ind}_{\mathcal{K}\cap(\mathcal{T}G_{y,s})^g}^{\mathcal{K}} \rho^g$, and we show that the two representations intertwine.

A set of representatives for the double coset space $\mathcal{K}\backslash G/\mathcal{T}G_{y,s}$ is given as follows

$$M(\mathcal{T}) := \left\{ \begin{array}{ll} \{\alpha^t \mid t \geq 0\} & \text{if } \mathcal{T} \text{ is unramified} \\ \{I, \alpha^t, \alpha^t \mathrm{w} \mid t > 0\} & \text{if } \mathcal{T} \text{ is ramified} \end{array} \right\},$$

$$\text{where } \alpha^t = \begin{pmatrix} \varpi^{-t} & 0 \\ 0 & \varpi^t \end{pmatrix}, \text{ and } \mathrm{w} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The following is a more explicit form of Theorem 2.2.

Theorem 2.3. *Let $\rho = \rho(\mathcal{T}, y, r, \phi)$. Then the decomposition of the restriction of the associated supercuspidal representation of G into irreducible $\mathcal{K}$ representations is given as follows:*

$$\mathrm{Res}^G_{\mathcal{K}}\, \text{c-Ind}^G_{\mathcal{T}\mathcal{K}_s}\, \rho \cong \mathrm{Ind}^{\mathcal{K}}_{\mathcal{T}\mathcal{K}_s}\, \rho \oplus \bigoplus_{t>0} \mathcal{S}_{r+2t}(\phi^{\alpha^t}),$$

$$\mathrm{Res}^G_{\mathcal{K}}\, \text{c-Ind}^G_{\mathcal{T}G_{1,s}}\, \rho \cong \bigoplus_{t\geq 0} \mathcal{S}_{r+2t+1}(\phi^{\alpha^t}),$$

$$\mathrm{Res}^G_{\mathcal{K}}\, \text{c-Ind}^G_{\mathcal{T}G_{\frac{1}{2},s}}\, \rho \cong \mathcal{S}_{r+\frac{1}{2}} \oplus \bigoplus_{t>0} \left(\mathcal{S}_{r+2t+\frac{1}{2}}(\phi^{\alpha^t}) \oplus \mathcal{S}_{r+2t-\frac{1}{2}}(\phi^{\alpha^t \mathrm{w}}) \right).$$

The nilpotent orbits of G on $\mathfrak{g}$ are parameterized by the elements

$$\left\{ X(\delta) := \begin{pmatrix} 0 & \delta\sqrt{\epsilon} \\ 0 & 0 \end{pmatrix} \;\middle|\; \delta \in \{0, 1, \varpi\} \right\}.$$

Let us denote the G orbit of the element $X(\delta)$ by $\mathcal{N}_\delta$. Then each $\mathcal{N}_\delta$ further decomposes into the following disjoint $\mathcal{K}$ orbits:

$$\mathcal{N}_0 = \mathcal{K} \cdot \mathcal{N}_0, \qquad \mathcal{N}_1 = \bigsqcup_{m \in 2\mathbb{Z}} \mathcal{K} \cdot X(\varpi^m), \qquad \mathcal{N}_\varpi = \bigsqcup_{m \in 2\mathbb{Z}+1} \mathcal{K} \cdot X(\varpi^m).$$

Given a nilpotent orbit $\mathcal{N}$ of the Lie algebra of $G(F)$ and a character η of the center of $G(F)$, we construct an infinite family of finite-dimensional irreducible representations of $\mathcal{K}$, whose sum $\tau(\eta, \mathcal{N})$ depends only on η and $\mathcal{N}$. Then upon restricting a supercuspidal representation π of $G(F)$ of depth $r \geq 0$ to the smaller subgroup $\mathcal{K}_{r+} := G(F)_{x,r+}$ of $\mathcal{K}$ we obtain the following decomposition.

Theorem 2.4. *Let (π, V) be a supercuspidal representation of $G(F)$ of depth r, and let η be its central character. Then*

$$\mathrm{Res}^G_{\mathcal{K}_{r+}}\, \pi \cong \begin{cases} \pi^{\mathcal{K}_{r+}} \oplus \mathrm{Res}_{\mathcal{K}_{r+}}\, \tau(\eta, \mathcal{N}_1)_{m \geq r+1} & \text{if } r \text{ even, } \mathcal{T} = \mathcal{T}_{1,1}, \\ & \text{or } r \text{ odd, } \mathcal{T} = \mathcal{T}_{\varpi^{-1},\varpi} \\ \pi^{\mathcal{K}_{r+}} \oplus \mathrm{Res}_{\mathcal{K}_{r+}}\, \tau(\eta, \mathcal{N}_\varpi)_{m \geq r+1} & \text{if } r \text{ odd, } \mathcal{T} = \mathcal{T}_{1,1}, \\ & \text{or } r \text{ even, } \mathcal{T} = \mathcal{T}_{\varpi^{-1},\varpi} \\ \pi^{\mathcal{K}_{r+}} \oplus \mathrm{Res}_{\mathcal{K}_{r+}}\, \tau(\eta, \mathcal{N}_\varpi)_{m \geq r+1} & \\ \quad \oplus \mathrm{Res}_{\mathcal{K}_{r+}}\, \tau(\eta, \mathcal{N}_1)_{m \geq r+1} & \text{if } \mathcal{T} \text{ is ramified.} \end{cases}$$

3. Acknowledgements

Theorems 2.3 and 2.4 were previously known for SL(2) by [4], [5]. I would like to thank Prof. Monica Nevins, my PhD supervisor, for suggesting this topic and for her invaluable guidance throughout the development of this work.

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Kac coordinates for elliptic Weyl group elements

Stephen DeBacker

(joint work with Jacob Haley)

In [2] Jacob Haley and I show that the map from the set of elliptic conjugacy classes in a Weyl group to the set of Kac coordinates is injective. In this note I discuss what motivated us to think about this and then present a nice application.

Notation

Suppose k is a finite extension of a p-adic field and $\bar{k}$ is an algebraic closure of k. Let $K \leq \bar{k}$ denote the maximal unramified extension of k and $k^t \leq \bar{k}$ the maximal tame extension. Fix a uniformizer $\varpi \in k$. Let Fr be a topological generator for $\mathrm{Gal}(K/k)$, and let σ be a topological generator for $\mathrm{Gal}(k^t/K)$.

Let $\mathbf{G}$ denote a connected reductive k-group. We set $G = \mathbf{G}(K)$. Note that $\mathbf{G}(k) = G^{\mathrm{Fr}}$. Fix a maximal K-split k-torus $\mathbf{A}$ of $\mathbf{G}$ that contains a maximal k-split torus of $\mathbf{G}$. Such a torus is unique up to G^{Fr}-conjugacy. Since $\mathbf{G}$ is K-quasisplit, $\mathbf{A}^\sharp := C_{\mathbf{G}}(\mathbf{A})$ is a maximal K-torus in $\mathbf{G}$. Fix a Fr-stable alcove C in the apartment of $\mathbf{A}$ in $\mathcal{B}(G)$, the reduced building of G. Let x_0 be an absolutely special vertex in the closure, $\bar{C}$, of C. We let $G_{x_0,0}$ denote the parahoric subgroup of G attached to x_0, and we denote by $G_{x_0,0:0^+}$ its associated reductive quotient.

Totally ramified tori

A maximal K-torus is called K-minisotropic provided that $\mathbf{X}^*(\mathbf{T}/\mathbf{Z})^{\mathrm{Gal}(\bar{k}/K)}$ is trivial. Here $\mathbf{Z}$ is the center of $\mathbf{G}$. Fix a maximal K-minisotropic K-torus $\mathbf{T}$ of $\mathbf{G}$.

If $\mathbf{T}$ splits over a Galois extension E of K, then there is a unique fixed point for the action of $\mathrm{Gal}(E/K)$ on the apartment of $\mathbf{T}(E)$. This point in $\mathcal{B}(\mathbf{G}(E))$ may not be an element of $\mathcal{B}(G)$, but, thanks to the non-positive curvature of Bruhat–Tits buildings, there will be a unique $x_T \in \mathcal{B}(G)$ that is closest to this fixed point.

Since the point x_T is independent of the isogeny type of $\mathbf{G}$, the G-orbit of x_T intersects the closure of C exactly once [1, Lemma 2.2.2]. Going forward, we assume $x_T \in \bar{C}$.

A natural question

Given the above, it is natural to ask:

Where is x_T located in $\bar{C}$?

At present, in order to answer this question we must assume Sally's hypothesis $\mathcal{W}$: p does not divide the order of the Weyl group $W = N_{\mathbf{G}}(\mathbf{A})/\mathbf{A}^\sharp$. We also assume that $\tilde{K}$, the spitting field of $\mathbf{A}^\sharp$, is a tame extension of K.

There is a bijective map φ_σ from $W^{\text{ell}}_{\sim\sigma}$, the set of σ-conjugacy classes of σ-elliptic Weyl group elements, to $\mathcal{C}^{\text{ell}}$, the set of G-conjugacy classes of maximal K-minisotropic K-tori. Mark Reeder showed me that if you realize this bijective correspondence carefully, then you can read off the point x_T.

Here is how Reeder's argument works: Suppose w is a σ-elliptic element of W. We can choose $n \in N_{\mathbf{G}(\tilde{K})_{x_0,0}}(\mathbf{A})$ that belongs to a σ-stable Tits section of W. Suppose ℓ denotes the order of n, and let E denote the tame extension of K of degree ℓ. Fix a uniformizer $\Pi \in E$ such that $\Pi^\ell = \varpi$ and set $\xi := \sigma(\Pi)/\Pi$. Then there exists $h \in \mathbf{G}(E)_{x_0,0}$ such that $h^{-1} n \sigma(h) = \lambda(\xi)$ for some $\lambda \in \mathbf{X}_*(\mathbf{A})$. Put $g = \lambda(\Pi) h^{-1} \lambda(\Pi)^{-1}$. Then $\mathbf{T}_w := \text{Int}(g)\mathbf{A}^\sharp$ is a maximal K-minisotropic K-torus in $\mathbf{G}$, and the G-conjugacy class of $\mathbf{T}_w$ is independent of all choices; that is, the orbit $\text{Int}(G)\mathbf{T}_w$ depends only on the σ-conjugacy class of w. Moreover, the one-parameter subgroup λ may be chosen so that

$$x_T = x_w := x_0 + \lambda/\ell.$$

This is part of a much larger, beautiful story. The point x_T is encoding what are known as the Kac coordinates of $\bar{n}\sigma$, where $\bar{n}$ is the image of n in the reductive quotient $\mathbf{G}(\tilde{K})_{x_0,0:0^+}$. In our language, Kac showed that to every finite order automorphism of $\mathbf{G}(\tilde{K})_{x_0,0:0^+} \rtimes \sigma$ we can associate a point in $\bar{C}$ with rational barycentric coordinates [5] (see also [4, Chapter X, Section 5], [6, Chapter 8], [8, 7° of Section 4 of Chapter 4], or [9, Theorem 3.7]). Moreover, when $\mathbf{G}$ is adjoint, this association is, up to natural equivalencies, bijective. In some gorgeous work, Reeder and Yu [10, §4] connect the theory of Moy–Prasad filtrations to the theory of Kac coordinates; I highly recommend reading this.

Examples. For K-split groups of type A, the Coxeter conjugacy class is the only elliptic conjugacy class in W, and the associated point in $\bar{C}$ is the barycenter of $\bar{C}$.

In Figure 1 we identify the points x_w in the closure of C for the group SU_4 ramified. Since the σ-elliptic σ-conjugacy classes in the Weyl group of SU_4 ramified correspond to the partitions $(3,1)$ and $(1,1,1,1)$ of 4, we use these partitions to label the points x_w.

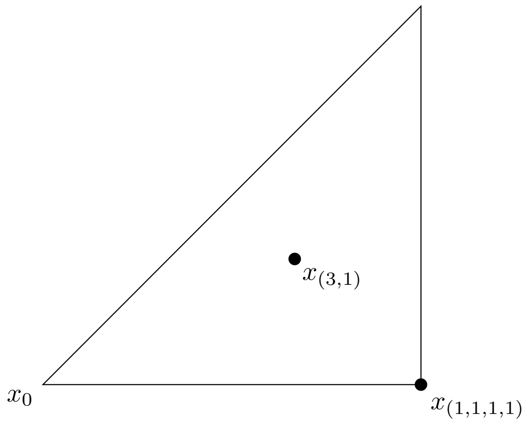

Figure 1. The location of the points x_w for groups of type ${}^2A_3$

A commutative diagram. The elements Fr and σ interact as expected. For example, if $\mathbf{G}$ is k-split, q denotes the cardinality of the residue field of k, and $W^{\mathrm{ell}}_{\sim}$ denotes the set of W-conjugacy classes in W^{ell}, then the following diagram commutes.

$$\begin{CD} W^{\mathrm{ell}}_{\sim} @>{(\,)^q}>> W^{\mathrm{ell}}_{\sim} \\ @V{\varphi_\sigma}VV @VV{\varphi_\sigma}V \\ \mathcal{C}^{\mathrm{ell}} @>{\mathrm{Fr}}>> \mathcal{C}^{\mathrm{ell}} \end{CD} \tag{1}$$

Note that we are implicitly using here that if $w \in W$ is elliptic, then w^q is elliptic as well. This can be shown without appealing to the fact that W is rational (see the last section of this note for the definition of the term rational).

A result

Many natural questions like, "Does the location of x_w in $\bar{C}$ tell us something about the smallest Weyl subgroup of W to which w belongs?" do not seem to have good answers. However, after generating enough examples one notices that

Theorem (DB–Haley)**.** *The map $\psi\colon W^{\mathrm{ell}}_{\sim\sigma} \to \bar{C}$ induced by sending $w \in W$ to x_w is injective.*

The proof of this result boils down to answering the question: if w and w' are two elements of W with lifts n and n' in $N_{\mathbf{G}(\bar{k})}(\mathbf{A})$, is it true that w and w' are σ-conjugate in W if and only if n and n' are σ-conjugate in $\mathbf{G}(\bar{k})$? Warning: the answer to this question can be no if we do not assume that **both** w and w' are σ-elliptic.

An application

A finite group L is said to be *rational* provided that for all $\ell \in L$ and for all $m \in \mathbb{N}$ we have: if the order of ℓ is relatively prime to m, then ℓ^m is L-conjugate to ℓ.

Theorem (Springer [11], Kletzing [7], DB–Haley)**.** *Every Weyl group is rational.*

We will prove this by induction. Suppose W is not trivial and the theorem holds for every proper parabolic subgroup of W.

Fix $w \in W$ and $m \in \mathbb{N}$ such that m is relatively prime to the order of w. Thanks to Dirichlet's theorem on primes in arithmetic progressions [3], we may assume that $p = m$ is prime and p is larger than the cardinality of W. If w is not elliptic, then it belongs to a proper parabolic subgroup of W and the result follows. So, we now assume that w is elliptic.

Let $k = \mathbb{Q}_p$ and assume that $\mathbf{G}$ is a k-split semisimple group with Weyl group W. Since σ acts trivially on W, we have $W^{\mathrm{ell}}_{\sim\sigma} = W^{\mathrm{ell}}_{\sim}$, the set of conjugacy classes in W. The commutativity of the diagram in (1) implies that $\varphi_\sigma(w^p) = \mathrm{Fr}(\varphi_\sigma(w))$. Since for all maximal K-minisotropic K-tori $\mathbf{T}$ the G-orbit of x_T intersects $\bar{C}$ exactly once, we conclude that $x_{w^p} = \mathrm{Fr}(x_w)$. Since $\mathbf{G}$ is k-split and $x_w \in \bar{C}$, we have $\mathrm{Fr}(x_w) = x_w$. The injectivity of $\psi\colon W^{\mathrm{ell}}_{\sim} \to \bar{C}$ then implies that w must be W-conjugate to w^p.

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Stability of elliptic Fargues–Scholze *L*-packets

Chenji Fu

The local Langlands correspondence conjecturally partitions the irreducible representations of a p-adic group into the so-called L-packets. Such a partition is conjecturally characterized by the stability condition, which has been proven in many cases using the theory of endoscopy. The aim of this talk is to approach the stability from a new point of view using [2], which geometrizes the representations as sheaves on Bun_G. Given an irreducible representation of $G(F)$, we define a sheaf on Bun_G by averaging over the automorphisms of the corresponding L-parameter, which we observe is a Hecke eigensheaf. Combining this with a formula of [1], we reduce the problem of stability to showing equi-distribution properties of the weight multiplicities of highest weight representations of an algebraic group. Our proof of equi-distribution properties might be of independent interest.

To explain our results in more detail, let F be a non-archimedean local field and G be a connected reductive F-group with Langlands dual group $\widehat{G}$ over $\overline{\mathbb{Q}}_\ell$. $\widehat{G}$ comes canonically with a torus and a Borel $\widehat{T} \subseteq \widehat{B} \subseteq \widehat{G}$. Let $\mathbb{T}_{\mathrm{univ}}$ be the universal Cartan of G (see [2, Section VI. 11]). The cocharacter lattice of $\mathbb{T}_{\mathrm{univ}}$ is canonically isomorphic to the character lattice of $\widehat{T}$:

$$X_*(\mathbb{T}_{\mathrm{univ}}) \cong X^*(\widehat{T}).$$

To state our theorem for general G, we need to take care of all the extended pure inner forms of G. For simplicity, let us make the following assumption.

Assumption 1. *We assume that G is F-split, semisimple, and simply connected.*

Under this assumption, G has no non-trivial extended pure inner forms.

Let W_F be the Weil group of F. The local Langlands conjecture predicts that there exists a surjective finite-to-one map from the set of equivalence classes of irreducible smooth representations of $G(F)$ to the set of *L-parameters*, i.e., $\widehat{G}$-conjugacy classes of ℓ-adically continuous 1-cocycles $\varphi : W_F \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ ([6, Section 1.1]). The fiber over φ is called an *L-packet*, denoted by $\Pi_\varphi(G)$. This conjectural association is expected to satisfy a list of properties, one of which is the stability of L-packets.

Conjecture 2 ([3, Conjecture 2.2])**.** *Any discrete series L-packet is* ***stable****, i.e., there exists a linear combination of the Harish-Chandra characters of its members that is a stable distribution.*

Explicit constructions of the local Langlands correspondence are known in many special cases and Conjecture 2 is proven in some of those cases (for example, [4, 5, 7, 8, 9, 10, 11, 12, 13]). All these proofs rely on some deep results from the theory of endoscopy.

Using a geometric approach, Fargues–Scholze [2] attached a semisimplified L-parameter $\varphi_{\pi'}^{\mathrm{FS}}$ to any smooth irreducible representation π' of $G(F)$, giving a general candidate for the local Langlands correspondence.

Moreover, they constructed the so-called *spectral action* ([2, Chapter X]), denoted by $*$. Let φ be an elliptic L-parameter. The spectral action in particular gives an action of the derived category of perfect complexes of S_φ-representations on the derived category of $G(F)$-representations, where S_φ is the centralizer of φ. For every $\pi \in \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell} G(F)$ such that $\varphi_\pi^{\mathrm{FS}} = \varphi$ (note that it is not known whether such π exists or not), let us denote

$$\pi_0 := \mathcal{O}(S_\varphi) * \pi,$$

where $\mathcal{O}(S_\varphi)$ is the regular representation of S_φ. π_0 is an object in the derived category of $G(F)$-representations, which turns out to be a finite direct sum of irreducible representations with Fargues–Scholze L-parameter φ up to degree shifts. We define its *Harish-Chandra character* Θ_{π_0} as the alternating sum of the Harish-Chandra characters of the cohomologies. Θ_{π_0} is thus a $\mathbb{Z}$-linear combination of Harish-Chandra characters of members of the (Fargues–Scholze) L-packet of φ, as required in Conjecture 2.

The main result we prove is:

Theorem 3. *Let φ be an elliptic L-parameter. For every $\pi \in \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell} G(F)$ such that $\varphi_\pi^{\mathrm{FS}} = \varphi$, the Harish-Chandra character Θ_{π_0} of $\pi_0 := \mathcal{O}(S_\varphi) * \pi$ is stable, i.e. Θ_{π_0} is invariant under $G(\overline{F})$-conjugacy as a function on the elliptic regular semisimple elements $G(F)_{\mathrm{ell}}$ of $G(F)$.*

Our proof consists of two ideas. First, we observe that π_0 is a Hecke eigensheaf, which is formal by the definition of π_0. Second, we invoke the formula [1, Theorem 6.5.2] to reduce to the following Theorem.

Theorem 4. *For $m \in \mathbb{Z}_{\geq 1}$, let $\mu_m = 4m\rho_G$, where ρ_G is the half sum of positive roots of $\widehat{G}$. Let $g \in G(F)_{\mathrm{ell}}$ and $T := \mathrm{Cent}(g, G)$. Then for any $h \in X_*(T)_\Gamma$, the limit*

$$c(h) := \lim_{m\to\infty} \frac{\sum_{\lambda \in X_*(T),\, \overline{\lambda}=h} \dim(V_{\mu_m}[\lambda])}{\dim V_{\mu_m}}$$

exists and is independent of $h \in X_*(T)_\Gamma$.

Example 5. For $G = \mathrm{SL}_2$, $\widehat{G} = \mathrm{PGL}_2$. $X_*(T)_\Gamma = \{\overline{0}, \overline{1}\}$. One can see that $c(\overline{0}) = c(\overline{1}) = 1/2$.

Now the question is purely combinatoric. We resolve this by showing the following proposition using Fourier analysis on finite abelian groups and the Weyl character formula.

Proposition 6. *Let k be the number of positive roots.*

(1) $\dim(V_{\mu_m})$ *is a polynomial in m of degree k.*
(2) *For any nontrivial character χ of $X_*(T)_\Gamma$, $\chi(\mathrm{Char}\, V_{\mu_m})$ is bounded by a polynomial in m of degree less than or equal to $k-1$.*

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On p-adic Deligne–Lusztig representations

Alexander B. Ivanov

This talk is based on the work of the author [7, 8], as well as on several joint works with Charlotte Chan [3], Olivier Dudas [5], Sian Nie [9] and Sian Nie and Panjun Tan [10].

Classical Deligne–Lusztig theory [6] gives a uniform construction of all irreducible representations of all finite groups of Lie type ($\mathrm{GL}_n(\mathbb{F}_q)$, $\mathrm{Sp}_{2n}(\mathbb{F}_q)$, ...) in the ℓ-adic cohomology of certain varieties over $\overline{\mathbb{F}}_q$. Similarly, p-adic Deligne–Lusztig theory makes the same idea work for a relatively big class of smooth representations of p-adic reductive groups. Moreover, p-adic Deligne–Lusztig spaces organize into a stacky family X_w naturally living over the stack of G-isocrystals Isoc_G. First, we sketch the construction of this stack. Then we explain recent results about the cohomology of p-adic Deligne–Lusztig spaces, which give rise to some new supercuspidal representations of p-adic reductive groups for small p.

Let k be a local non-archimedean field with ring of integers $\mathcal{O}_k$ and residue field $\mathbb{F}_q$ of characteristic p with q elements. Denote by $\breve{k}$ the completion of the maximal unramified extension of k, by $\mathcal{O}_{\breve{k}}$ its integers and by $F\colon \breve{k} \to \breve{k}$ the Frobenius automorphism over k. For any k-scheme X, we have its loop space LX. Formally, LX is an arc-sheaf on perfect $\mathbb{F}_q$-schemes; less formally one can think of it as an object over $\mathbb{F}_q$ whose geometric points are $LX(\overline{\mathbb{F}}_q) = X(\breve{k})$. Let G_0 be a unramified (connected) reductive group over k. Let $T_0 \subseteq B_0 \subseteq G_0$ be a k-rational quasi-split maximal torus of G_0 and a k-rational Borel subgroup with unipotent radical U_0. Let W be the Weyl group of T_0 in G_0. For $w \in W$ let T_w be the $\breve{k}/k$-form of T_0 equipped with the Frobenius $\mathrm{Ad}(w) \circ F$, and let $\mathcal{O}(w) \subseteq (G_0/B_0)^2$ be the G-orbit corresponding to w.

For any $b \in G_0(\breve{k})$, the *p-adic Deligne–Lusztig space* $X_w(b)$ is the intersection in $L(G_0/B_0)^2$ of the image of $x \mapsto (x, bF(g))\colon L(G_0/B_0) \to L(G_0/B_0)^2$ with $L\mathcal{O}(w) \subseteq L(G_0/B_0)^2$. These spaces were introduced and studied in [7]. Let us mention that the formation of $X_w(b)$ is (in a reasonable sense) well-behaved under parabolic induction. Letting the parameter b vary, we obtain a family $\mathfrak{X}_w$ parametrized over LG_0. Now, $\mathfrak{X}_w$ admits a natural LG_0-action and $\mathfrak{X}_w \to LG_0$ is LG_0-equivariant, when the target is equipped with the F-twisted conjugation action (this simply means that $X_w(b)$ essentially only depends on the F-conjugacy class of b rather than on b). Passing to quotients modulo LG_0 we obtain the *p-adic Deligne–Lusztig stack* $X_w \to \mathrm{Isoc}_{G_0} \cong LG_0/_{\mathrm{Ad},F}LG_0$, which lives over the stack of G_0-isocrystals. The stack X_w was introduced in [3, §9].

The stack X_w comes equipped with two maps $\mathrm{Isoc}_{T_w} \leftarrow X_w \to \mathrm{Isoc}_{G_0}$. Once a sufficiently nice sheaf-theoretic formalism for these huge spaces is constructed (this is work in progress with L. Mann), pulling and pushing forward a sheaf from Isoc_{T_w} to Isoc_{G_0} realizes an automorphic induction from $T_w(k)$ to (simultaneously) *all* inner forms of all Levi subgroups of $G_0(k)$. This procedure for inner forms of $G_0(k)$ (that is, restricted to the closed points of Isoc_{G_0}) and sufficiently regular characters of elliptic tori $T_w(k)$ was the subject of intensive investigations by various authors in the last decades, including Boyarchenko, Chan, Lusztig, Nie and the author, see for example [1, 2, 3, 4, 5, 12, 14].

To explain some new results in this direction, we slightly change our point of view. Let G be an (extended pure) inner form of G_0 and assume that there is a k-embedding $T_w \hookrightarrow G$. Let T denote its image (so, $T \subseteq G$ is a k-rational *unramified* maximal torus in G of type w), and let U be the unipotent radical of a Borel subgroup B such that $B, F(B)$ are in relative position w. Suppose that w is a Coxeter element, so in particular T_w is elliptic. Write $* = \mathrm{Spec}\,\overline{\mathbb{F}}_q$. Pulling X_w first along $* \to [*/G(k)] \subseteq \mathrm{Isoc}_{G_0}$ and then along $* \to [*/T_w(k)] \subseteq \mathrm{Isoc}_{T_w}$ gives us (up to cohomologically inessential affine spaces) an ind-scheme, which is a disjoint union (indexed over the rational conjugacy classes of embeddings $T_w \hookrightarrow G$) of copies of

$$X_{T,U} = \{g \in LG\colon g^{-1}F(g) \in L(\overline{U} \cap FU)\},$$

where $\overline{U}$ is the unipotent radical of the opposed Borel subgroup. This has a $G(k) \times T(k)$-action given by $(h,t)\colon g \mapsto hgt$. The apartment of T in the reduced Bruhat–Tits building of G over k consists of one point; let $\mathcal{G}$ be the (connected) parahoric model of G at this point and let $\mathcal{T}$, $\mathcal{U}$ denote the closures of T, U in it.

The following geometric result expresses the ind-scheme $X_{T,U}$ in terms of smaller and more accessible objects.

Theorem 1 (I. [8], Nie [13])**.** *If w is Coxeter, we have*

$$X_{T,U} = \coprod_{\gamma \in G(k)/\mathcal{G}(\mathcal{O}_k)} \gamma X^{\mathcal{G}}_{T,U},$$

where $X^{\mathcal{G}}_{T,U} = \{g \in \mathcal{G}(\mathcal{O}_{\breve{k}}) \colon g^{-1}F(g) \in \overline{\mathcal{U}}(\mathcal{O}_{\breve{k}}) \cap F\mathcal{U}(\mathcal{O}_{\breve{k}})\}$ is an affine $\overline{\mathbb{F}}_q$-scheme.

Note that $X^{\mathcal{G}}_{T,U}$ is an inverse limit of pfp, perfectly smooth $\overline{\mathbb{F}}_q$-schemes and has a $\mathcal{G}(\mathcal{O}_k) \times \mathcal{T}(\mathcal{O}_k)$-action. Let $\theta \colon T(k) \to \overline{\mathbb{Q}}_\ell^\times$ be a smooth character. We can consider the θ-part $R_T^G(\theta)$ of the (co)homology of $X_{T,U}$. Let Z denote the centre of G. Due to Theorem 1, $R_T^G(\theta) = \mathrm{cInd}_{Z(k)\mathcal{G}(\mathcal{O}_k)}^{G(k)} H^*(X^{\mathcal{G}}_{T,U,r}, \overline{\mathbb{Q}}_\ell)[\theta]$ is compactly induced from the usual compactly supported ℓ-adic cohomology of the r-truncation $X^{\mathcal{G}}_{T,U,r}$ of $X^{\mathcal{G}}_{T,U}$, where r is any integer such that $\mathrm{depth}(\theta) \leq r$.

Theorem 2 (Dudas–I. [5], I.–Nie–Tan [10])**.** *Assume that G is unramified, T is Coxeter and $\mathrm{Stab}_{W(T)^F}(\theta) = \{1\}$. Suppose that $q > 2$ (resp. $q > 3$; resp. $q > 5$) if the absolute Dynkin diagram of G has a component of type B_n, C_n or D_n (resp. of type E_6, E_7, F_4 or G_2; resp. of type E_8). Then the following hold.*

(1) $\pm H^*(X^{\mathcal{G}}_{T,U,r}, \overline{\mathbb{Q}}_\ell)[\theta]$ *is an irreducible $\mathcal{G}(\mathcal{O}_k)$-representation.*
(2) $\pm R_T^G(\theta)$ *is a direct sum of finitely many irreducible $G(k)$-representations.*

We emphasize the absence of any assumptions on p, resp. on the existence of a Howe factorization (cf. [11, §3.6]) of θ in Theorem 2. If θ admits a Howe factorization, then —in many cases— it is known that $R_T^G(\theta)$ coincides with the representation attached to (T, θ) by the construction(s) of J.-K. Yu and Kaletha (and hence is irreducible). But when the Howe factorization does not exist (e.g. for small p), the construction from Theorem 2 seems to be new. Note also that the proof of this theorem involves neither any sequence of (twisted Levi) subgroups of G attached to θ, nor any computation of traces of various elements in $R_T^G(\theta)$. With minimally modified assumptions on q, Theorem 2(1) also holds when G is not unramified.

Finally, let $\mathcal{G}^+$ denote the pro-unipotent radical of $\mathcal{G}$ and $\mathcal{T}^+, \mathcal{U}^+$ the closures of T, U in it. We can consider the closed subscheme $X^+_{T,U} = \{g \in \mathcal{G}^+(\mathcal{O}_{\breve{k}}) \colon g^{-1}F(g) \in \overline{\mathcal{U}}^+(\mathcal{O}_{\breve{k}}) \cap F\mathcal{U}^+(\mathcal{O}_{\breve{k}})\}$ of $X^{\mathcal{G}}_{T,U}$. It admits the action of $\mathcal{G}^+(\mathcal{O}_k) \times \mathcal{T}^+(\mathcal{O}_k)$ by $(g,t) \mapsto gxt$. It seems most elegant to express the following result in terms of the homology $f_\natural \overline{\mathbb{Q}}_\ell$ of the structure morphism $f \colon X^+_{T,U} \to \mathrm{Spec}\, \overline{\mathbb{F}}_q$ (which will be introduced in the work in progress with L. Mann). Let $N \geq 1$ be such that F^N stabilizes $X^+_{T,U}$.

Theorem 3 (I.–Nie [9]). *For a smooth character χ of $\mathcal{T}^+(\mathcal{O}_k)$ the following hold.*

(1) *Suppose p is not a torsion prime for G.*[1] *The homology $f_\natural\overline{\mathbb{Q}}_\ell[\chi]$ is concentrated in precisely one degree $s_\chi \geq 0$.*
(2) *Suppose p is as in (1). The Frobenius F^N acts on $f_\natural\overline{\mathbb{Q}}_\ell[\chi]$ by multiplication with $(-1)^{s_\chi}q^{s_\chi N/2}$. All Moy–Prasad quotients on $X^+_{T,U}$ are maximal $\mathbb{F}_{q^N}$-varieties.*
(3) *For varying χ, $\pm f_\natural\overline{\mathbb{Q}}_\ell[\chi]$ varies through pairwise non-isomorphic irreducible smooth $\mathcal{G}(\mathcal{O}_k)$-representations.*

Note the absence of any conditions on χ. Also note that Theorem 3 implies that $\dim_{\overline{\mathbb{Q}}_\ell} f_\natural\overline{\mathbb{Q}}_\ell[\chi] = q^{s_\chi N/2}$. This allows a purely geometric proof of the formal degree formulas, obtained recently by Schwein [15] via an algebraic method, for many supercuspidal representations.

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[1] In [9] the assumption on p is stronger, but it will be relaxed in a future work.

Local intertwining relations and co-tempered L-packets for classical groups

Tasho Kaletha, Alberto Mínguez
(joint work with Hiraku Atobe, Wee Teck Gan, Atsushi Ichino, and Sug Woo Shin)

This talk is a report on joint work with Hiraku Atobe, Wee Teck Gan, Atsushi Ichino, and Sug Woo Shin, which renders the endoscopic classification of representations of classical groups, a result of Arthur from 2013, conditional only on the weighted fundamental lemma.

Arthur's results

The Arthur–Langlands conjectures: rough statements. Let $F/\mathbb{Q}$ be a finite extension and let G be a connected reductive F-group. A central goal in the theory of automorphic forms is to understand the Hilbert space $L^2(G) := L^2(G(F)\backslash G(\mathbb{A}))$ as a representation of the group $G(\mathbb{A})$. Langlands' theory of Eisenstein series reduces the problem to the discrete part of $L^2(G)$. The rough conjectural description is then as follows.

Conjecture (Reciprocity, rough)**.** *There are correspondences*

$$\begin{array}{ccc} \mathrm{IrrConst}(L^2_{disc}(G)) & \longleftrightarrow & \mathrm{Irr}(\mathrm{Gal}_F, \hat{G}) \\ \downarrow{\scriptstyle \pi\mapsto\pi_v} & & \downarrow{\scriptstyle \psi\mapsto\psi|_{\Gamma_{F_v}}} \\ \mathrm{Irr}(G(F_v)) & \longleftrightarrow & \mathrm{Rep}(\mathrm{Gal}_{F_v}, \hat{G}) \end{array}$$

satisfying $L(s,\pi,\tau) = L(s,\tau\circ\psi)$ for each $\tau : \hat{G} \to \mathrm{GL}_N(\mathbb{C})$.

Conjecture (Functoriality, rough)**.** *A homomorphism $\xi : \hat{H} \to \hat{G}$ produces a transfer of automorphic representations $\xi_* : \mathrm{AutRep}(H) \to \mathrm{AutRep}(G)$ such that*

$$L(s, \xi_*(\pi), \tau) = L(s, \pi, \tau\circ\xi).$$

Slightly more precisely, one has to replace $\mathrm{Irr}(\mathrm{Gal}, \hat{G}) \subset \mathrm{Rep}(\mathrm{Gal}, \hat{G})$ by sets

$$\Psi_2(G) \subset \Psi(G) = \{\psi : L_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG\}/\hat{G}$$

of *Arthur-parameters*, where F is the local or global field in question, and L_F is the *Langlands group* for F. When F is local this group is unconditionally defined as

$$L_F = \begin{cases} W_F, & F/\mathbb{Q} \\ W_F \times \mathrm{SL}_2(\mathbb{C}), & F/\mathbb{Q}_p. \end{cases}$$

When $F/\mathbb{Q}$ the group L_F, hence the set $\Psi(G)$, is conjectural.

The appearance of the group SL_2 in addition to L_F was introduced by Arthur in order to measure the non-temperedness of local components of discrete automorphic representations, and in particular the failure of the naïve Ramanujan conjecture beyond GL_N, i.e. the existence of non-tempered *cuspidal* automorphic representations. For general groups it is expected that the cuspidal automorphic representation that are *generic*, i.e. have a Whittaker model, have tempered local components. Such representations should correspond to the parameters

$$\Phi(G) = \{\phi \in \Psi(G) \mid \phi|_{\mathrm{SL}_2} = 1\} \subset \Psi(G),$$

which are called *generic*, or *tempered*. Furthermore, if one wants to describe all *discrete* automorphic representations, rather than just the cuspidal ones, then the group SL_2 is required even when dealing with GL_N, as initially conjectured by Arthur and later proved by Mœglin–Waldspurger.

Precise statements for classical groups. Arthur proves the following precise version of the Reciprocity Conjecture. Let G be a quasi-split symplectic or orthogonal group (or a quasi-split unitary group, handled by Mok). One can define a substitute of $\Psi(G)$, called the "set of formal global parameters for G", using cuspidal automorphic representations of GL_N. There is a subset $\Phi(G) \subset \Psi(G)$ of *generic*, equivalently *tempered* parameters.

Theorem. *Under certain assumptions, the following is true.*

(1) $L^2_{disc}(G) = \bigoplus_{\psi \in \Psi_2(G)} L^2_\psi(G)$.
(2) $\mathrm{Irr}_{temp}(G(\mathbb{Q}_v)) = \coprod_{\varphi \in \Phi(G_v)} \Pi_\varphi(G_v)$ *and* $\mathrm{Irr}(G(\mathbb{Q}_v)) = \bigcup_{\psi \in \Psi(G_v)} \Pi_\psi(G_v)$.
(3) *There is a map* $\Pi_\psi(G_v) \to \mathrm{Irr}(S_\psi)$, *where* $S_\psi = \mathrm{Cent}(\psi, \hat{G})$. *We denote it by* $\pi_v \mapsto \rho_{\pi_v}$.
(4) *There is a localization map* $\mathrm{loc}_v : \Psi(G) \to \Psi(G_v)$ *and*

$$L^2_\psi(G) = \bigoplus \pi^{m(\psi,\pi)},$$

where π *runs over all irreducible admissible representations* $\pi = \bigotimes' \pi_v$ *of* $G(\mathbb{A})$ *for which* $\mathrm{loc}_v(\psi) = \psi_{\pi_v}$ *and*

$$m(\psi, \pi) = \mathrm{mult}(\epsilon_\psi, \bigotimes_v (\rho_{\pi_v}|_{S_\psi})),$$

and ϵ_ψ *is a certain explicit but complicated sign character of* S_ψ, *trivial when* $\psi = \varphi \in \Phi(G)$.

The proof of this theorem is a long and complicated induction on the dimension of the standard representation of $\hat{G}$, based on comparisons of trace formulas. These comparisons require certain local information that we now review.

Geometric comparison: transfer of (weighted) orbital integrals. To compare the geometric sides, one needs identities between the orbital integrals and their weighted analogs.

The identity of orbital integrals we need is

$$\text{(GT)} \qquad SO_{\gamma'}(f') = \sum_\gamma \Delta(\gamma', \gamma) O_\gamma(f).$$

Since the pairs (γ, f) are adelic, and the orbital integrals decompose along the places, such identities reduce to local identities, i.e. identities over local fields. For almost all places the test function is the characteristic function of $G(\mathbb{Z}_p)$. Then this identity is known as the Fundamental Lemma, a celebrated result of many people, most notably Ngô.

A similar result is needed to compare the weighted orbital integrals, and is called the weighted fundamental lemma: Here one has a Levi $M \subset G$ and an endoscopic group M' for the Levi and one wants

$$\text{(WFL)} \qquad \sum_{G' \in \mathcal{E}(M')} \iota(G', G) SWO_{M'}^{G'}(\gamma', 1_{K'}) = \sum_{\gamma} \Delta(\gamma', \gamma) WO_M^G(\gamma, 1_K).$$

Spectral comparison: transfer of (weighed) characters. In the comparison of spectral sides, the identity for usual characters is (ECR):

$$\text{(ECR)} \qquad \sum_{\pi \in \Pi_\psi(G)} \langle \pi, s \cdot s_\psi \rangle_\psi \Theta_\pi(f) = S\Theta_{\psi'}(f').$$

Here $\langle \pi, - \rangle$ is the trace of ρ_π and $s_\psi = \psi(1, -1) \in S_\psi$.

A similar result is required to handle the weighted characters, which reflect the contributions of the Eisenstein series. It is called the *local intertwining relation*, and is the following identity

$$\text{(LIR)} \qquad \sum_{\pi_M \in \Pi_\psi(M)} \operatorname{tr}(R_P(w) I_P(\pi_M)(f)) = S\Theta_\psi(f'),$$

where $R_P(w)$ is a normalized intertwining operator. It is an analog of (ECR) and is again essential in the comparison of the spectral sides of the trace formulas.

Assumptions. The assumptions under which Arthur's theorem holds are (mostly) listed in the bibliography of his book as the unwritten references [A24–A27]. Reference [A24] is the stabilization of the twisted trace formula, which has subsequently been achieved by work of Mœglin–Waldspurger conditional on the validity of the weighted fundamental lemma, which is currently only available for split groups. References [A25–A27] assert various versions of (ECR) and (LIR).

Our results

In our work, we supply the missing results that constitute [A25, A26, A27], as well as further results that are assumed in Arthur's book without being listed as such references. Most of [A25, A26, A27] consists of claiming various cases of (LIR). We will focus on [A25], which is by far the hardest part. It claims both (ECR) and (LIR) for certain non-tempered Arthur parameters.

For $\psi \in \Psi(G)$ define its Aubert dual as $\hat{\psi}(w, x, y) = \psi(w, y, x)$. For $\pi \in \operatorname{Irr}(G)$ let $\hat{\pi} \in \operatorname{Irr}(G)$ denote its (covariant) Aubert dual.

Theorem ([AGIKMS])**.** *Let ψ be a co-tempered parameter, i.e. one such that $\varphi := \hat{\psi} \in \Phi(G)$. Define the set*

$$\Pi_\psi(G) := \{\hat{\pi} \mid \pi \in \Pi_\varphi(G)\}.$$

(a) *There exists an explicit map $\Pi_\psi(G) \to \mathrm{Irr}(S_\psi)$ for which* (ECR) *holds.*
(b) *If ψ factors through ${}^L M \to {}^L G$, then* (LIR) *holds.*

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