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Mini-Workshop: Geometry of Random Fields and Random Walk Clusters: New Horizons

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Abstract. Critical phenomena represent a central theme in probability. Research has been going on for many decades and remains very active to date. Recently, models involving natural probabilistic objects such as random walks, loop soups, random interlacements and the Gaussian free field have witnessed exciting developments, both in two- and higher-dimensional setups. The purpose of the workshop was to provide an overview of the state of the art in this rapidly evolving research area. The workshop enabled participants to communicate about the most recent advances in the field, and to discuss propitious avenues for future research.

Mathematics Subject Classification (2020): 60-XX, 82-XX.

Introduction by the Organizers

Critical phenomena and their associated random geometric objects constitute a fascinating playground for mathematicians. Next to the continued significant progress on short-range models (such as percolation and Ising models), an emergent line of research has successfully exploited other natural probabilistic objects such as the Gaussian free field (GFF) and related Poissonian ensembles of random walks and loops, in order to exhibit interesting random geometric structures, both in planar and higher-dimensional setups, that lead to a different (long-range) universality class. The mini-workshop brought together 14 researchers interested in this and related topics, more than half of whom were at an "early career" stage.

Following is a very brief overview of the topics covered. We refer to the extended abstracts below for details. *W. Werner* gave a gentle introduction to some of these topics and surveyed facts and conjectures concerning critical loop soup and Gaussian free field clusters, their (possible) scaling limits, and their dependence on dimension. *A. Prévost, Z. Cai and A. Drewitz* all reported on recent progress concerning the critical behavior of metric-graph GFF clusters in dimensions three and higher.

Y. Gao discussed percolation results concerning two-sided level sets of the discrete GFF in dimension two. *A. Sepúlveda* reported on findings regarding the extremal properties of two natural dynamics associated to the GFF.

Q. Vogel focused on loop soup representations of the Bose gas, the onset of condensation in this language, and the validity of a variational principle in the interacting case. *P. Ferrari* derived fluctuations for surface models associated to random hard rod configurations, in terms of a multi-time (Lévy-Chentsov) Brownian field.

J. Aru discussed characterisations of the free field, both in discrete and continuous setups. *A. Jégo* reported on results concerning thick points of branching Brownian motion in four dimensions. *S. Watanabe* presented work on intrinsic volume growth for the three-dimensional uniform spanning tree and on-diagonal heat kernel bounds for the associated random walk.

Finally, *X. Li* reported on sharp asymptotics for favorite sites of the random walk, both in two and in higher dimensions.

The mere role of the organizer(s) was to determine a propitious schedule of talks and to make sure nobody got lost during the traditional Wednesday hike (to St. Roman since the group included various first-timers). Each participant was given an opportunity to present some piece of work. Notwithstanding, the format left ample time for discussions.

The organizers and participants warmly thank the Mathematisches Forschungsinstitut Oberwolfach for enabling this event and for their unwavering support in all aspects of its organization.

Mini-Workshop: Geometry of Random Fields and Random Walk Clusters: New Horizons

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Abstracts

Survey about loop-soup clusters and the Gaussian Free Field – discrete, cable-graph and in the continuum

Wendelin Werner

As requested by the organizers, my two presentations were devoted mostly to a (personal) survey of some aspects of critical loop-soup clusters and their relation to the Gaussian Free Field and its square, with a special emphasis on the way in which the spatial dimension impacts this relation.

So, the first part was more a general introduction to the objects involved [Gaussian Free Field, Occupation times of loop-soups] in the different settings [discrete graphs, cable-graphs, continuum], their definitions and the relation between them, including the by now "classical" results of Le Jan and Lupu [3, 4, 5] and the excursion decomposition of Aru, Lupu and Sepulveda [1]. Among the topics then covered:

- The relation between the resampling property of critical loop-soups from [9] and the Markovian properties of the Gaussian Free Field and its square.
- How Dynkin's key isomorphism theorem can be revisited using the loop-soup construction of the GFF.
- The various predictions from [10] and some of the heuristics behind them. In particular, why the scaling limits of the cable-graph loop-soup clusters are deterministic functions of the continuum loop-soup in dimension $d \leq 4$ and not in dimensions $d \geq 5$, guided by CLE percolation ideas from [6] (i.e. how critical percolation inside fractal-type random sets behaves depending on the frequency of "almost-bottlenecks" in this fractal set).
- How the continuum SLE/CLE ideas and results from [8] are needed and used in the derivation of the actual results about scaling limits of two-dimensional discrete loop-soup clusters (say, in [5]).
- Some ideas of the papers [7, 2] where it is shown that some information is missing in the occupation field in order to determine exactly where the Brownian loops went.
- I also briefly mentioned the case of interacting loops as surveyed in [11].

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Volume critical exponents for the metric graph Gaussian free field

Alexis Prévost
(joint work with Alexander Drewitz and Pierre-François Rodriguez)

Computing the exact values of the critical exponents for a percolation model has proved to be a very challenging problem, especially in transient dimensions below the critical dimension, since it requires a deep understanding of the critical and near-critical phase. I focus on a specific percolation model, for which the rigidity induced by its strong long-range correlation let us derive explicitly some critical exponents.

More precisely, consider the metric (or cable) graph $\widetilde{\mathcal{G}}$ associated to a transient weighted graph $\mathcal{G}$, that is the metric space where each edge of the graph is replaced by a corresponding continuous interval, on which a natural diffusion X can be defined, which can be seen as an equivalent of the Brownian motion on the metric graph. The main object of interest, first introduced in [8], is the Gaussian free field $(\varphi_x)_{x\in\widetilde{\mathcal{G}}}$ on $\widetilde{\mathcal{G}}$, that is the centred Gaussian field with covariance

$$\mathbb{E}[\varphi_x\varphi_y] = g(x,y),\ x,y\in\widetilde{\mathcal{G}}.$$

Here $g(x,y)$ denotes the Green function associated to the diffusion X, that is the average time spent in y when X starts in x. Fixing some $x_0\in\widetilde{\mathcal{G}}$, we consider the connected component of x_0 in the level sets above level h

$$\mathcal{K}^h = \{x\in\widetilde{\mathcal{G}} : x \stackrel{\geq h}{\longleftrightarrow} x_0\},\ h\in\mathbb{R},$$

where $x \stackrel{\geq h}{\longleftrightarrow} x_0$ means that x is connected to x_0 by a continuous path $\pi\subset\widetilde{\mathcal{G}}$ along which $\varphi\geq h$. One can ask the classical percolation question for this random subset of $\widetilde{\mathcal{G}}$: for which values of h is $\mathcal{K}^h$ unbounded with positive probability? The answer is that, if $\mathcal{G}$ is a massless vertex-transitive graph

$$\mathbb{P}(\mathcal{K}^h \text{ is unbounded}) > 0 \text{ if and only if } h < 0.$$

This follows from combining results from [2] and [5], and highlights the interest of this model, since it shows that the associated critical parameter is always equal to 0, and that there is never percolation at criticality.

Under further assumption on the graph $\mathcal{G}$, one can also derive the critical exponents associated to the model. More precisely, let us from now on assume that the weights on $\mathcal{G}$ are bounded, that

$$|B(x,R)| \asymp R^{\alpha} \text{ for some } \alpha > 0,$$

and that

$$g(x,y) \asymp d(x,y)^{-\nu} \text{ for some } \nu \in (0, \alpha-2].$$

Then it was proved that if either $\nu < \alpha/2$, see [7], or $\mathcal{G} = \mathbb{Z}^{\alpha}$, $\alpha < 6$, see [4], then

$$cN^{-\frac{\nu}{2}} \leq \mathbb{P}(\mathrm{rad}(\mathcal{K}^0) \geq N) \leq CN^{-\frac{\nu}{2}}. \tag{1}$$

The lower bound in (1) is actually valid for any possible value of ν and α, see [6]. In other words, (1) proves that the critical exponent ρ associated to the one-arm probability at criticality is equal to $2/\nu$. Given this result, we are interested in obtaining the critical exponents associated to the volume of the critical clusters. More precisely, we focus on the volume of the cluster $\mathcal{K}^0$ of x_0, as well as on M_r^0, the volume of the cluster in $B(x_0, r)$ with the largest volume.

Theorem 1. *Assume that* (1) *is satisfied. Then for all* $n \geq 1$,

$$cn^{-\frac{\nu}{2\alpha-\nu}} \leq \mathbb{P}(|\mathcal{K}^0| \geq n) \leq Cn^{-\frac{\nu}{2\alpha-\nu}}, \tag{2}$$

and for all $r, t \geq 1$ *with* $r^{\frac{d+2}{2}} \geq Ct$,

$$\mathbb{P}\big((1/t)r^{\alpha-\frac{\nu}{2}} \leq M_r^0 \leq tr^{\alpha-\frac{\nu}{2}}\big) \geq 1 - Ct^{-c}. \tag{3}$$

To be more concrete, if we focus on the case of $\mathcal{G} = \mathbb{Z}^d$, $d \geq 3$, one can combine Theorem 1 with the results from [12, 3, 4] to obtain the critical exponents associated to both the tail of the volume of the critical cluster of the origin, and the largest cluster in a ball:

$$\delta \stackrel{\text{def.}}{=} \lim_{n\to\infty} \frac{\ln(n)}{\ln(\mathbb{P}(|\mathcal{K}^0| \geq n))} = \begin{cases} \frac{d+2}{d-2} & \text{if } d \leq 6, \\ 2 & \text{if } d \geq 6, \end{cases}$$

and $\mathbb{P}$-a.s.

$$d_f \stackrel{\text{def.}}{=} \lim_{r\to\infty} \frac{\ln(M_r^0)}{\ln(r)} = \begin{cases} \frac{d+2}{2} & \text{if } d \leq 6, \\ 4 & \text{if } d \geq 6. \end{cases}$$

We will now explain the main ideas behind the proof of (3). The upper bound is an easy consequence of the Cauchy-Schwarz inequality and the formula for the two-point function from [8], and we will focus on the lower bound on M_r^0. First using an entropy bound originally used in [1], it is enough to prove that the largest cluster in $B(x_0, r)$ at level $-h$ (instead of 0) has volume at least $cr^{\alpha-\frac{\nu}{2}}$ with constant probability, where $h = r^{-\frac{\nu}{2}}$. The main interest to shift to a negative level is that one can then use the isomorphism [10, 8, 11, 5] with random interlacements [9]. This isomorphism implies in particular that for any $u > 0$

$$\text{the union of the connected component of } \{x \in \widetilde{\mathcal{G}} : |\varphi_x| > 0\} \text{ which intersect } \mathcal{I}^u$$
$$\text{is stochastically dominated by } \{x \in \widetilde{\mathcal{G}} : \varphi_x \geq -\sqrt{2u}\},$$

where $\mathcal{I}^u$ denotes the interlacement set at level u. Hence taking $u = h^2/2 = r^{-\nu}/2$, it is enough to prove that in $B(x_0, r)$, with positive probability the union of the connected component of $\{x \in \widetilde{\mathcal{G}} : |\varphi_x| > 0\}$ which intersect $\mathcal{I}^u$ has a connected component with volume at least $cr^{\alpha - \frac{\nu}{2}} = cur^{\alpha}$. Since the density of $\mathcal{I}^u$ is u, this is not very hard to prove when $\mathcal{I}^u \cap B(x_0, r)$ essentially contains a unique connected component, which is the case [6] when $\nu < \alpha/2$. If $\nu \geq \alpha/2$, one additionally needs to prove that there is a giant cluster of $\{x \in B(x_0, r) : |\varphi_x| > 0\}$ which connects together most of the connected component of $\mathcal{I}^u$. This giant cluster will have capacity cr^{ν}, and the existence of such a cluster is proved by combining (1) with the exact formula for the law of the capacity from [5].

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One-arm probabilities for metric graph Gaussian free fields

Zhenhao Cai
(joint work with Jian Ding)

The Gaussian free field (GFF) on the metric graph was introduced in [13] as a natural extension of the discrete GFF. Referring to [13], such an extension is valid for all transient graphs, where the Green's function is well-defined. In this note, we only focus on the integer lattice $\mathbb{Z}^d$ for $d \geq 3$. We denote the edge set of $\mathbb{Z}^d$ by $\mathbb{L}^d := \{\{x,y\} : x,y \in \mathbb{Z}^d, |x-y| = 1\}$, where $|\cdot|$ is the Euclidean norm. For each $e = \{x,y\} \in \mathbb{L}^d$, consider a compact interval I_e of length d (where the length d is chosen only for convenience and does not cause any difference on the geometry of GFFs) whose two endpoints are identical to x and y respectively. The metric graph of $\mathbb{Z}^d$ (denoted by $\widetilde{\mathbb{Z}}^d$) is defined as the union of all these intervals. The GFF on $\widetilde{\mathbb{Z}}^d$ (denote by $\{\widetilde{\phi}_v\}_{v\in\widetilde{\mathbb{Z}}^d}$) can be constructed in the following two steps:

(1) Restricted to $\mathbb{Z}^d$, $\{\widetilde{\phi}_x\}_{x\in\mathbb{Z}^d}$ is distributed as a discrete GFF on $\mathbb{Z}^d$, i.e., a family of mean-zero Gaussian random variables with covariance

$$\mathbb{E}\big[\widetilde{\phi}_x\widetilde{\phi}_y\big] = G(x,y),\ \forall x,y \in \mathbb{Z}^d. \tag{1}$$

Here the Green's function $G(x,y)$ is the average of visits at y by a simple random walk on $\mathbb{Z}^d$ with starting point x.

(2) For every $e = \{x,y\} \in \mathbb{L}^d$, the values of $\widetilde{\phi}_v$ for $v \in I_e$ are sampled by an independent bridge on I_e of a Brownian motion with variance 2 at time 1, conditioned on the boundary values $\widetilde{\phi}_x$ at x and $\widetilde{\phi}_y$ at y.

Compared to the discrete GFF, the continuity of $\{\widetilde{\phi}_v\}_{v\in\widetilde{\mathbb{Z}}^d}$ provides a crucial advantage in the analysis, particularly for exploration arguments. Two fundamental properties of the percolation of the level-set $\widetilde{E}^{\geq h} := \{v \in \widetilde{\mathbb{Z}}^d : \widetilde{\phi}_v \geq h\}$ ($h \in \mathbb{R}$) were established in [13]:

- (critical level) For any $h < 0$, $\widetilde{E}^{\geq h}$ a.s. percolates (i.e., contains an infinite connected component); for any $h \geq 0$, $\widetilde{E}^{\geq h}$ a.s. does not percolate.
- (two-point function) For any $x \neq y \in \mathbb{Z}^d$,

$$\mathbb{P}\big(x \xleftrightarrow{\geq 0} y\big) = \pi^{-1} \arcsin\left(\frac{G(x,y)}{\sqrt{G(x,x)G(y,y)}}\right) \asymp |x-y|^{2-d}. \tag{2}$$

Here $A_1 \xleftrightarrow{\geq h} A_2$ denotes the event that the level-set $\widetilde{E}^{\geq h}$ contains a path connecting A_1 and A_2, and $f \asymp g$ means that $cg \leq f \leq Cg$ holds for some constants c and C depending only on d.

A powerful coupling between the loop soup and the GFF on the metric graph was also presented in [13], which enriches both models and allows one to derive properties for one model from the other.

The (critical) one-arm probability $\theta_d(N) := \mathbb{P}(\mathbf{0} \xleftrightarrow{\geq 0} \partial B(N))$ has been extensively studied, where $\mathbf{0} := (0,0,...,0)$ is the origin of $\mathbb{Z}^d$, $B(N) := [-N,N]^d \cap \mathbb{Z}^d$, and $\partial A := \{x \in A : \exists y \in \mathbb{Z}^d \setminus A \text{ such that } \{x,y\} \in \mathbb{L}^d\}$. Through the collective efforts from [6, 3, 7, 10, 4] (also see [9] on extensions to more general transient graphs), it was finally established that

(3) when $3 \le d < 6$, $\theta_d(N) \asymp N^{-\frac{d}{2}+1}$;

(4) when $d = 6$, $cN^{-2} \le \theta_6(N) \le CN^{-2+\varsigma(N)}$, where $\varsigma(N) := \frac{\ln\ln(N)}{\ln^{1/2}(N)} \ll 1$;

(5) when $d > 6$, $\theta_d(N) \asymp N^{-2}$.

Notably, it was conjectured in [4, Remark 1.5] that for $d = 6$,

(6) $$\theta_6(N) \asymp N^{-2}\ln^{\delta}(N)$$

for some constant $\delta > 0$.

This talk aims to present the proof idea of [4, Theorem 1.1], which establishes the upper bounds in (3) and (4). The logic underlying this proof can be described as follows: if $\theta_d(N)$ is large, the symmetry of $\widetilde{\phi}_\cdot$ implies that the negative cluster $\mathcal{C}_N^- := \{v \in \widetilde{\mathbb{Z}}^d : v \xleftrightarrow{\le 0} \partial B(N)\}$ (where $A_1 \xleftrightarrow{\le 0} A_2$ denotes the event that there is a path connecting A_1 and A_2 on which the values of $\widetilde{\phi}_\cdot$ are non-positive), which has the same distribution as the positive one, has a significant probability of being sizable and blocking the positive cluster containing $\mathbf{0}$, thereby in turn preventing $\theta_d(N)$ from being large. This naturally leads us to employ a proof by contradiction. Technically, the contradiction is achieved by the following two ingredients.

(1) Assuming the existence of a finite N_* such that $\theta_d(N_*) > \lambda N^{-\frac{d}{2}+1}$ (where λ is a large constant for $3 \le d \le 5$, and is $N^{\varsigma(N)}$ for $d = 6$), we can find a large integer k_* (which depends on N_* and is implicit) and then employ some coarse-graining methods (where the scale is determined by k_*) to construct an event F (measurable with respect to $\widetilde{\phi}_\cdot$) with $\mathbb{P}(\mathsf{F}) \ge \exp(-k_*^C)$ on which the following two events happen:
   - (i) After sampling $\mathcal{C}_{N_*}^-$, the expected value of $\widetilde{\phi}_{\mathbf{0}}$ is at least $cN_*^{-\frac{d}{2}+1}$.
   - (ii) An independent Brownian motion on $\widetilde{\mathbb{Z}}^d$ reaches $\partial B(\frac{N_*}{10})$ before hitting $\mathcal{C}_{N_*}^-$ with probability at most $\exp(-k_*^{C'})$.

   These two events further imply that after sampling $\mathcal{C}_{N_*}^-$, the expectation of the average of $\widetilde{\phi}_\cdot$ on $\partial B(\frac{N_*}{4})$ will exceed $\exp(k_*^{C'})$ times the standard deviation of this average. Combined with $\mathbb{P}(\mathsf{F}) \ge \exp(-k_*^C)$, it yields that such an excess occurs with probability at least $\exp(-k_*^C)$.

(2) From the tail estimate of the normal distribution, the probability that a normal random variable exceeds $\exp(k_*^{C'})$ times its standard deviation is at most $\exp(-c'e^{2k_*^{C'}})$. Since $\exp(-c'e^{2k_*^{C'}}) \ll \exp(-k_*^C)$, this leads to a contradiction with Item (1).

The heart of this proof is to construct the event F and to show that it implies Event (ii). Notably, during the derivation of the latter part, it surprisingly turns out from a series of estimates that the phenomenon presented in Event (ii) (i.e., the Brownian motion on $\widetilde{\mathbb{Z}}^d$ is typically blocked by the negative cluster) holds if and only if $3 \le d < 6$ (in other words, the $6 - d$ term appears repeatedly and must be positive for these estimates to hold). We believe that much work remains to be done to understand the information hidden in this proof.

In addition to the proof of [4, Theorem 1.1], this talk also introduces recent developments on the incipient infinite cluster (IIC) of $\widetilde{E}^{\geq 0}$, established in [2, 5]. Precisely, [2, Theorem 1.1] shows that for any $d \geq 3$ except $d = 6$, the following four types of IICs (as four limiting probability measures) exist and are equivalent:

$$\mathbb{P}^{(1)}_{d,\mathrm{IIC}}(\cdot) := \lim_{N\to\infty} \mathbb{P}\big(\cdot \mid \mathbf{0} \xleftrightarrow{\geq 0} \partial B(N)\big), \tag{7}$$

$$\mathbb{P}^{(2)}_{d,\mathrm{IIC}}(\cdot) := \lim_{h\uparrow 0} \mathbb{P}\big(\cdot \mid \mathbf{0} \xleftrightarrow{\geq h} \infty\big), \tag{8}$$

$$\mathbb{P}^{(3)}_{d,\mathrm{IIC}}(\cdot) := \lim_{x\to\infty} \mathbb{P}\big(\cdot \mid \mathbf{0} \xleftrightarrow{\geq 0} x\big). \tag{9}$$

$$\mathbb{P}^{(4)}_{d,\mathrm{IIC}}(\cdot) := \lim_{T\to\infty} \mathbb{P}\big(\cdot \mid \mathrm{cap}\big(\mathcal{C}^{\geq 0}(\mathbf{0})\big) \geq T\big). \tag{10}$$

Here $\mathrm{cap}(\cdot)$ is the capacity, and $\mathcal{C}^{\geq 0}(\mathbf{0}) := \{v \in \widetilde{\mathbb{Z}}^d : v \xleftrightarrow{\geq 0} \mathbf{0}\}$. The proof of [2, Theorem 1.1] is based on a robust framework in [1] for constructing IICs and a fundamental property called *quasi-multiplicativity*, which is proved in [5, Theorem 1.1]. More precisely, quasi-multiplicativity implies that for any $d \geq 3$ expect $d = 6$, and any $N \geq 1$, $A_1 \subset B(N^{[(\frac{d-4}{2})\vee 1]^{-1}})$ and $A_2 \subset [B(N^{(\frac{d-4}{2})\vee 1})]^c$,

$$\mathbb{P}\big(A_1 \xleftrightarrow{\geq 0} A_2\big) \asymp N^{(6-d)\wedge 0}\mathbb{P}\big(A_1 \xleftrightarrow{\geq 0} \partial B(N)\big)\mathbb{P}\big(A_2 \xleftrightarrow{\geq 0} \partial B(N)\big). \tag{11}$$

For Bernoulli percolation on $\mathbb{Z}^d$, it was conjectured in [1] that (11) holds for $3 \leq d \leq 5$. In high dimensions (i.e., $d \geq 7$), the similarity between the metric graph GFF and Bernoulli percolation (see [3, 14]) makes it plausible to conjecture that (11) also holds. Readers may refer to [11, 12] for recent progress on the intrinsic geometry of the IIC of $\widetilde{E}^{\geq 0}$ in high dimensions (i.e., $d > 6$).

In [5], during the proof of quasi-multiplicativity, numerous regularity properties of general connecting probabilities, which are interesting on their own right, were also established. E.g., the probability of connecting a point and a general set, as a function of the point, exhibits behaviors similar to harmonic functions, including Harnack's inequality, decay rate, stability on boundary conditions, etc.

It is proved in [2] that conditioned on $\{\mathbf{0} \xleftrightarrow{\geq 0} \partial B(N)\}$, the volume within $B(M)$ of the cluster of $\widetilde{E}^{\geq 0}$ containing $\mathbf{0}$ is typically of order $M^{(\frac{d}{2}+1)\wedge 4}$, regardless of how large N is (provided with $N \gg M$). This phenomenon indicates that the cluster of $\widetilde{E}^{\geq 0}$ exhibits self-similarity, which supports the insightful conjecture in [14] that such cluster has a scaling limit. Moreover, the exponent of the order $M^{(\frac{d}{2}+1)\wedge 4}$ matches the conjectured fractal dimension of the scaling limit presented in [14]. Similar estimates are also derived in a concurrent work [8].

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Percolation of GFF and random walk loop soup in dimension two

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(joint work with Pierre Nolin, Wei Qian)

1. Main results

For $N \geq 1$, let $B_N := [-N, N]^2 \cap \mathbb{Z}^2$ be the discrete box centered on 0 with side length $2N$. Let φ_N be a discrete GFF in B_N with Dirichlet boundary conditions. We are interested in the existence of large-scale paths, for instance crossing B_N from left to right, which are "low" for the field φ_N. That is, nearest-neighbor paths along which $|\varphi_N|$ remains smaller than some given level $\lambda > 0$. For any $0 < n \leq N$, and any subset $A \subseteq B_N$, consider the crossing event

$$(1)\quad \mathcal{C}_n(A) := \{\text{there exists a path in } A \cap B_n \text{ crossing from left to right in } B_n\},$$

where the left side of B_n is defined as the set of leftmost vertices in B_n, and the right side is defined similarly (as well as the top and bottom sides). Our first main result is the following.

Theorem 1 ([4]). *There exists $c_0 > 0$, such that for all $\varepsilon > 0$,*

$$\mathbb{P}(\mathcal{C}_N(\{z \in B_N : |\varphi_N(z)| \leq \varepsilon\sqrt{\log N}\})) = 1 - O(N^{-c_0\varepsilon^2}) \quad as \quad N \to \infty. \tag{2}$$

Moreover, for all $\varepsilon > 0$,

$$\liminf_{N\geq 1} \mathbb{P}(\mathcal{C}_{N/2}(\{z \in B_N : |\varphi_N(z)| \leq \varepsilon\sqrt{\log N}\})) > 0. \tag{3}$$

Let X_N^α be the occupation field of a random walk loop soup with intensity α in B_N. The well-known isomorphism theorem [7] states that $\frac{1}{2}|\varphi_N|^2$ has the same distribution as $X_N^{1/2}$. For any $\lambda > 0$, each vertex $z \in B_N$ is called λ-open if $X_N^\alpha(z) \leq \lambda$. Let $q_N^\alpha(\lambda)$ be the probability that there exists a λ-open crossing connecting the left and right sides of B_N. Theorem 1 implies that for any $\epsilon > 0$, we have $q_N^\alpha(\epsilon \log N) = 1 - O(N^{-c_0'\epsilon})$, for some universal $c_0' > 0$.

More generally, we prove in [4] that for any $\alpha \in (0, 1/2)$, there are constants $0 < \lambda_1 < \lambda_2$ and $c > 0$ such that $\lim_{N\to\infty} q_N^\alpha(\lambda) = 0$ if $\lambda \leq \lambda_1$ and $q_N^\alpha(\lambda) = 1 - O(N^{-c})$ if $\lambda \geq \lambda_2$. We also prove an exponential decay property for the crossing probability of a box in the bulk, or of a macroscopic annulus. Finally, we mention that analogous results also hold for loop soups and GFF on metric graphs.

2. Summation argument

Consider $\alpha \in (0, 1/2)$. In order to show the existence of low horizontal crossings of B_N, where $X_N^\alpha \leq \lambda$, we use a Peierls'-type argument. Hence, we show that blocking paths (on the matching graph $(\mathbb{Z}^2)^*$) cannot arise, that is, vertical crossings where X_N^α remains above λ. This is done by considering the connected components (clusters) of random walk loops, and following the "chain" of clusters which are visited by such a potential high path. In particular, we use the *passage edges* connecting any two successive clusters. Around each passage edge, the two clusters originating from it produce a "four-arm" configuration locally. Moreover, such edges have the additional property that the occupation field is high on both their endpoints, which yields an extra cost.

This leads to configurations consisting of sequences of such edges, around each of which four arms can be observed. Summing over all such configurations is challenging, but it can be achieved in a similar way as for another process called self-destructive percolation (s.d.p.). That process was introduced by van den Berg and Brouwer [1] in order to analyze the near-critical behavior of the Drossel-Schwabl forest fire model. Roughly ten years later, it was explained in [6] how to sum the "six-arm" configurations that arise in s.d.p., which has remarkable consequences for that process. Even though the technical details are completely different in our situation, we are able to adapt an induction argument for six-arms in Bernoulli percolation, which was developed in [2], strengthening results from [6]. For both s.d.p. and our own reasonings, the fact that the associated exponent is > 2 plays a fundamental role. In addition, we need to make use of the extra small probabilistic cost produced by each passage edge, coming from the condition $X_N^\alpha > \lambda$ on its endpoints. This gives an extra factor, which can be made arbitrarily small by choosing λ large enough. Note that a completely analogous input in s.d.p. was the cost of recovering vertices, i.e. vertices turning from vacant to occupied during a small time window (that can be made arbitrarily short).

3. Arm events in the random walk loop soup

In order to carry out the summation argument, we need to devise a set of tools to work with arm events in a random walk loop soup (RWLS) with intensity $\alpha \in (0, 1/2]$. For a bounded subset D of $\mathbb{Z}^2$, let $\mathcal{L}_D^\alpha$ be the random walk loop soup in D with intensity α. Suppose $1 \le d_1 \le d_2/2$ and $B_{2d_2} \subset D$. Let $\mathcal{A}_D^\alpha(d_1, d_2)$ denote the (discrete) four-arm event that there are two outermost clusters in $\mathcal{L}_D^\alpha$ crossing $B_{d_2} \setminus B_{d_1}$. We crucially rely on the following upper bound established in [3]: For all $\alpha \in (0, 1/2]$ and all $\epsilon > 0$, there exists $c_1 = c_1(\alpha, \epsilon) > 0$ such that for all $1 \le d_1 < d_2$ and $D \supseteq B_{2d_2}$,

$$\mathbb{P}(\mathcal{A}_D^\alpha(d_1, d_2)) \le c_1 \, (d_2/d_1)^{-\xi(\alpha)+\epsilon}, \tag{4}$$

where $\xi(\alpha) = \eta(\kappa) = (12 - \kappa)(\kappa + 4)/(8\kappa)$ and $\alpha(\kappa) = (3\kappa - 8)(6 - \kappa)/(4\kappa)$. The exponent $\eta(\kappa)$ is the well-known four-arm exponent of SLE_κ. The proof of (4) relies on two inputs.

- The scaling limit of clusters in the random walk loop soup is given by the clusters of the Brownian loop soup [10, 8]. It was shown in [9] that the collection of outer boundaries of the outermost clusters in a Brownian loop soup with intensity $\alpha(\kappa)$ is distributed as a CLE_κ. Since a loop in the CLE_κ locally looks like an SLE_κ curve, it is natural to expect that the four-arm exponent $\xi(\alpha)$ for the CLE_κ is the same as the one for SLE_κ. We establish rigorously this fact in [5], which relies on some non-trivial separation lemma for the Brownian loop soup.
- We establish in [3] a useful *quasi-multiplicativity* upper bound: for any $\alpha > 0$, there is $c_2(\alpha) > 0$ such that for all $1 \le d_1 \le d_2/2 \le d_3/16$ and $D \supseteq B_{2d_3}$,

$$\mathbb{P}(\mathcal{A}_D^\alpha(d_1, d_3)) \le c_2(\alpha) \, \mathbb{P}(\mathcal{A}_{B_{2d_2}}^\alpha(d_1, d_2)) \, \mathbb{P}(\mathcal{A}_D^\alpha(4d_2, d_3)). \tag{5}$$

  This is a key property to connect discrete arm events to the corresponding arm events in the Brownian loop soup.

The upper bound (4), together with the fact that $\xi(\alpha) > 2$ for $\alpha \in (0, 1/2)$, will then allow us to apply the summation argument in Section 2. Finally, we are able to adapt our proof to the critical intensity $\alpha = \frac{1}{2}$, even though the four-arm exponent turns out to be exactly 2 in this case. In order to do so, we let λ grow with N as $\lambda_N = \epsilon \log N$, for any $\epsilon > 0$ (and potentially $C \log\log N$, for a constant C large enough, as we explain in [4]).

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One-arm probability for the percolation problem of the metric graph Gaussian free field at criticality

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(joint work with Alexis Prévost, Pierre-François Rodriguez)

Abstract: Percolation models have been playing a fundamental role in statistical physics for several decades by now. They had initially been investigated in the gelation of polymers during the 1940s by chemistry Nobel laureate Flory and Stockmayer. From a mathematical point of view, the birth of percolation theory was the introduction of Bernoulli percolation by Broadbent and Hammersley in 1957 [2], motivated by research on gas masks for coal miners. One of the key features of this model is the inherent stochastic independence which simplifies its investigation, and which has led to deep mathematical results. On the one hand, this independence greatly simplifies the mathematical computations and as a consequence, the results obtained are impressively profound. On the other hand, this independence also poses a restriction prohibiting the investigation of more realistic models. Thus, one is naturally led to consider percolation models with correlations. While in the case of finite range or fast decaying correlations similar phenomena as in the case of Bernoulli percolation are to be observed, the situation changes drastically when considering models with stronger correlations, so-called *long-range correlations*. Models with long-range dependence exhibit interesting properties which sometimes are in stark contrast to what is observed in Bernoulli percolation. The lack of independence entails further obstacles such as the absence of the finite energy property. Even more dramatically, the BK inequality fails for these models. As a result, many of the techniques which were most essential in the investigation of Bernoulli percolation break down for percolation problems with long-range correlations.

It should be mentioned here that not only are such models oftentimes more realistic but they also lead to beautiful mathematics as well as interesting physics interpretations. Moreover, they sometimes exhibit certain integrability properties, which, somewhat surprisingly, makes them easier to study for some problems than their independent counterpart, and leads to deep results which are unknown for Bernoulli percolation.

As we will elaborate, even though from a probabilistic perspective the strength of correlations seems to make matters a priori only harder, they can also provide certain integrability properties which facilitate their rigorous mathematical study. This opens the door to the study of critical phenomena, notably in non-planar setups, and gives access to questions which so far have remained largely elusive.

Arguably one of the most important stochastic processes giving rise to percolation models with long-range correlations is the *Gaussian Free Field* (GFF). The GFF, which also goes by the name massless harmonic crystal, has been a fundamental model in statistical mechanics for over half a century, ever since the early days of constructive field theory, for which it serves as a fundamental building block. More recently, its intriguing geometric features have begun to be studied rigorously. One way to think of the GFF is as a generalization of a random walk with Gaussian increments to a process with d-dimensional time. For a long time already, the GFF has had many important applications to other branches of mathematics, with links in two dimensions to objects such as the Schramm-Loewner evolution [10], or to cover times [5] and to theoretical physics (cf. [4, 7]). Literature is abundant, however, and we content ourselves with referring to [12] and the references therein for further details.

The metric graph. It has turned out that when endowing the discrete graph G with a certain continuum structure, the investigation of percolation problems for the GFF becomes in a loose sense more integrable. This continuous structure is the so-called *metric graph*, also sometimes referred to as *cable system*, and denoted by $\widetilde{G}$. Heuristically, $\widetilde{G}$ is obtained from G by adding line segments between neighboring vertices of G so that one obtains a metric graph which is a continuum object. While such a construction goes back to at least [14], it has been re-invigorated in this setting by [9].

The reasons for this model being particularly amenable for a detailed investigation of its percolation are multiple, including among others its Gaussian and continuous character as well as the understanding of the law of the capacity of its level sets and its amenability to advanced isomorphism theorems connecting it to the model of random interlacements (see [6] for the latter two items). The model of Random Interlacements (RI) has been introduced in 2007 by Sznitman, see the article [13]. It has been motivated by investigations in mathematics and theoretical physics on the disconnection [1] and covering [3] of tori and boxes by simple random walk trajectories. In addition, RI serves as a mathematical model for corrosion, and it has found its way into the theoretical physics community also, see [11, 8]. In the context of level set percolation for the GFF it turns out particularly useful as it is supercritical in its entire range of parameters, thereby providing suitable connectivity properties for certain level sets of the GFF also by means of the isomorphism theorems.

We report on the progress in the understanding of the one-arm probability for the level set percolation of the GFF on the metric graph for rather general transient graphs, in low transient dimensions. In particular, we provide the first proof of up-to-constant upper and lower bounds for the previously mentioned one-arm probability on the metric graph pertaining to the Euclidean lattice $\mathbb{Z}^3$. This improves on previous results by Ding and Wirth, as well as by ourselves, where multiplicative logarithmic corrections were present in the respective bounds.

While the lower bound on the one-arm probability follows essentially immediately from the law of the capacity of the cluster of the origin, established previously in joint work together with Prévost and Rodriguez. The main obstacle is the proof of the upper bound. It starts by partitioning the event of connecting the origin and the boundary of a ball into the cluster of the origin having either small or large capacity. The case of large capacity can be conveniently treated by the rather explicit formula for the law of the capacity of that cluster. The probability of the cluster having small capacity (but spanning to the boundary of the ball nevertheless) is then upper bounded by the probability of a random walk not hitting large (in terms of capacity) loops is negligible.

From a technical point of view, this is obtained by invoking, among others, a generalization of Lupu's formula for the two-point function in this model and also the isomorphism theorem between loop soups and the Gaussian free field.

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Dynamics on the thick points of the Gaussian free field

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(joint work with Felipe Espinosa)

This work explores the geometric properties of natural dynamics that have the two-dimensional Gaussian Free Field (GFF) as their stationary distribution. Specifically, we study the behaviour of thick points in two key dynamics: the stationary solution of the additive stochastic heat equation and the Ornstein-Uhlenbeck GFF.

The GFF corresponds to the standard Gaussian of the Hilbert space $H_0^1(D)$, where $D \subseteq \mathbb{C}$ is an open domain. While the GFF is not a function but rather a Schwartz distribution, its geometric properties have been central to the progress in two-dimensional conformally invariant probability theory in recent decades. Among these properties, the behaviour of thick points is particularly fundamental.

Thick points of the GFF. Let Φ is a GFF and $\nu_{z,\epsilon}$ the uniform measure on $\partial B(z,\epsilon)$. The ϵ-average of ϕ is defined as

$$\phi_\epsilon(z) := \langle \phi, \nu_{z,\epsilon} \rangle \text{``}=\text{''} \int_D \phi(y) \nu_{z,\epsilon}(dy).$$

Then, the set of γ-thick points is

$$T_\gamma := \left\{ z \in D : \lim \frac{\phi_\epsilon(z)}{\log(1/\epsilon)} = \gamma \right\}.$$

In a seminal result, Hu, Miller, and Peres (2010) demonstrated that as long as $|\gamma| \leq 2$, T_γ is not empty and its Hausdorff dimension is $2 - \gamma^2/2$. Thick points provide insights into the extreme values of the GFF and play a critical role in Liouville Quantum Gravity.

In this work, we extend the study of thick points to the context of dynamics that have the GFF as its stationary distribution.

The Orstein Uhlenbeck GFF and its thick points. The first of such dynamics is the so-called OU-GFF. That can be constructed as follows. Start by defining the GFF valued Brownian motion X as the unique process such that

- X_t has the law of a GFF times $\sqrt{t}$.
- X has independent and stationary increments.

The Orstein-Uhlenbeck GFF Ψ is then equal to $e^{-t/2} X_{e^t}$ for any $t \in \mathbb{R}$. The thickness of a point z at a time t is

$$T_z^+(t) = \limsup \frac{\Psi_\epsilon(z,t)}{\log(1/\epsilon)}, \qquad T_z(t) = \lim \frac{\Psi_\epsilon(z,t)}{\log(1/\epsilon)}.$$

We show that a.s. for any z the function $t \mapsto T_z^+(t)$ is continuous and its behaviour is characterised via the energy functional

$$\mathcal{E}(f) = \int_{-\infty}^{\infty} \left| f' + \frac{1}{2} f \right|, \qquad \text{for } f \in H_0^1(\mathbb{R}).$$

in the following way:

(1) Almost surely, there is no z such that $\mathcal{E}(T_z(t)) > 4$.
(2) For any $f \in H_0^1(D)$ such that $\mathcal{E}(f) < 4$ a.s. there exists z such that for all t, $f(t) = T_z(t)$. Additionally, the dimension of all such $z \in D$ is $2 - \mathcal{E}(f)/2$.

Stochastic Heat Equation. The second dynamic we study is the Stochastic Heat Equation Field (SHEF) φ, given by the stationary solution of the Stochastic Heat Equation

$$\partial_t \varphi = \frac{1}{2} \Delta \varphi + \xi,$$

where ξ is a white noise and $\varphi(0)$ is a GFF independent of ξ.

In this setting, we show that the thickness function is completely discontinuous. This allow the existence of points that are more than 2-thick, as shown by:

(1) If $\gamma^2 > 8$, a.s. there are no $z \in D$, $t \in \mathbb{R}$ with $T_z^+(t) = \gamma$.
(2) If $\gamma^2 < 8$, a.s. there are points $z \in D$, $t \in \mathbb{R}$ with $T_z(t) = \gamma$. Additionally the space-time dimension of such (z, t) is equal to $\min\{4 - \gamma^2/2, 3 - \gamma^2/4\}$.

Then, we concentrate in the exceptional times. For $N \in \mathbb{N}$, we define the set of N-exceptional time for γ as

$$E_N^\gamma. = \{t \in \mathbb{R} : \text{ there are } z_1, ..., z_N \text{ different points with } T_{z_1}(t) = ... = T_{z_N}(T) = \gamma\}.$$

We show that these sets generate infinitely many phase transitions for the middle

(1) If $\gamma^2 > 4 + 4/N$, a.s. E_N^γ is empty.
(2) If $\gamma^2 < 4 + 4/N$, a.s. E_N^γ is not empty.

Probabilistic representation of the Bose gas

Quirin Vogel

The Bose gas is a system in quantum statistical mechanics that undergoes a phase transition called *Bose–Einstein Condensation* (BEC, for short). More precisely, if one considers a gas of bosons at temperature $\beta > 0$ and density $\rho > 0$, it is predicted to undergo a phase transition in the thermodynamic limit as ρ increases: for sufficiently large ρ, a macroscopic fraction of the particles should occupy the same quantum state.

Probabilistic representations of the Bose gas were introduced in [Fe53] and are based on the relationship between differential operators and stochastic processes. See [G71] for a rigorous mathematical reference, where the Bosonic loop process $P_{\Lambda,\beta,\rho}$ is constructed for bounded subsets Λ of $\mathbb{R}^d$. The measure $P_{\Lambda,\beta,\rho}$ is a conditioned Poisson point process of Brownian bridges (each of duration in $\beta\mathbb{N}$) in the domain Λ with total duration $\beta\rho|\Lambda|$. To model the forces between atoms, these bridges interact via a spatial potential $V\colon \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$.

It was long conjectured that in the absence of interaction, for $d \geq 3$ and $\rho > \rho_c$ (with $\rho_c = (4\pi\beta)^{-d/2}\zeta(d/2)$), "infinite loops" appear in the limit $\Lambda \uparrow \mathbb{R}^d$. Owing to the seminal work of Sznitman, infinite loops can nowadays be interpreted as random interlacements. Indeed, in [V23], it is shown that for $\rho > \rho_c$, the canonical representation $P_{\Lambda,\beta,\rho}$ of the Bose gas converges to the independent superposition of the free Bose gas with zero chemical potential and the (beaded) random interlacements at density $\rho - \rho_c$ (see [AFY21] for a definition of this latter process). This result was then extended to a (non-spatially) interacting model; see [DV24]. However, while it is generally expected that infinite loops appear at sufficiently high densities, it is not expected that the law of these paths follows the beaded interlacement process. It remains an open question for which class of interactions infinite paths appear in the limit.

Feynman already conjectured that the appearance of infinite paths is an order parameter for BEC. However, to this day, no proof exists for a spatially interacting gas of bosons. Previous works in probability were limited to the subcritical regime, where no infinite or open paths are present (see, for example, [ACK11]).

In [BDM24], the authors developed a new tool that simplifies the study of infinite interacting paths. The idea is to consider a reference Poisson point process of Brownian paths (not loops) and then enforce consistency (i.e., that the paths either form loops or escape to infinity) via an additional energy term. The change of measure between the original process and this new process can be explicitly calculated. The locality of this new reference process allows the authors to use many tools from classical point process theory. They show the existence of Gibbs measures for any density, although they cannot prove or disprove the existence of long loops.

Together with G. Bellot, I have extended this technique to prove the existence of the free energy of an interacting gas of bosons at any density. Moreover, we can write it as the following variational problem:

$$-\inf\left\{\mathrm{SFE}(P)\,|\,P \in \mathcal{M}\right\},$$

where $\mathcal{M}$ denotes the translation-invariant probability measures on collections of loops and infinite paths, and SFE is the *specific free energy* of P, defined as the limit of the average specific free energy (entropy plus energy) in finite boxes. Our proof uses the decomposition of infinite paths into finite paths, which can then be glued together at sufficiently small cost for the specific free energy. This hinges delicately on the interplay between energy and the surface scale occupied by the infinite paths.

The above result is significant because it covers the case where BEC is expected to occur. A proof of the existence of infinite paths in the limit can then be carried out by showing that minimizers of the above variational problem assign positive mass to infinite paths—a project we aim to pursue in future investigations.

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Multi-time random walks and Brownian fields, and hard rod hydrodynamics

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(joint work with Stefano Olla)

Inspired by Buffon needle problem, Crofton proposed in 1868 a method to compute the length of a planar curve, involving randomly drawing straight lines within a plane [3], [5]. Let $\ell(\theta, p)$ be the line that has a distance of p from the origin, and whose perpendicular, drawn from the origin of $\mathbb{R}^2$, makes an angle θ with the x-axis. This map and the Lebesgue measure ν on the strip $\mathbb{R}_+ \times [0, 2\pi)$ induce a measure on the set of lines contained in $\mathbb{R}^2$ that is invariant by isometries.

In 1945 Lévy proposes a random surface $\eta : \mathbb{R}^2 \to \mathbb{R}$ called Brownian motion with several (time) parameters, a centered Gaussian process with covariances $\mathrm{Cov}(\eta(a), \eta(b)) = \frac{1}{2}(|a| + |b| - |a - b|)$, where $|\cdot|$ is the Euclidean norm. The one-dimensional trajectory obtained by cutting the surface with a vertical plane is one dimensional Brownian motion, explaining the name [11, 12, 13].

White noise in $\mathbb{R}^d$ with control measure μ is a centered Gaussian process ω indexed by Borel sets, with covariances $\mathrm{Cov}(\omega(A), \omega(B)) = \mu(A \cap B)$. Chentsov describes the Lévy's multi-time Brownian field as the surface $\eta(b) := \omega(\underline{ob})$, where $\underline{ob}$ is the set of points (θ, p) mapped to lines crossing the segment $\overline{ob}$; o is the origin of $\mathbb{R}^2$, and ω is white noise with control measure ν [4].

In 1975 Mandelbrot defined $M(b) := N_1(\underline{ob}) := \sum_{(\theta,p,r)\in X} r\, \mathbb{1}\{(\theta, p) \in \underline{ob}\}$, where X is a marked Poisson process with intensity ν and iid marks r, and proved that the rescaled version of M converges to the multi-time Brownian field [14, 15]. Related results were obtained by Ossiander and Pyke [16] and Lantuejoul [9, 10].

Franceschini, Grevino, Spohn and the first author introduced a surface H, as a function of a non homogeneous marked Poisson process, and applied it to show hydrodynamics of hard rods [7]. We review the approach and extend the diffusive limits of Olla and the first author to the non-homogeneous case [8].

To better match the relation with hard rods, we map points in the space-velocity $\mathbb{R}^2$ to lines contained in the space-time $\mathbb{R}^2$, by defining $\ell(x,v) = \{(x+vt,t) : t \in \mathbb{R}\}$, the line intersecting the x-axis at the point x and having inclination $\alpha(v) := \arctan(1/v)$. To incorporate marks, we add a dimension to our space. A point (x,v,r) in the space-velocity-mark $\mathbb{R}^3$ represents the line $\ell(x,v)$ with mark r. For a,b in the space-time plane $\mathbb{R}^2$, denote by ab the set of points (x,v,r) such that the line $\ell(x,v)$ intersects the segment $\overline{ab}$. Orient the lines in the positive direction of time and denote the half-planes to the left and right of the line $\ell(x,v)$ by $\mathrm{left}(x,v)$ and $\mathrm{right}(x,v)$, respectively. Each line in ab belongs to one of the sets

$$ab_+ := \big\{(x,v,r) \in \mathbb{R}^3 : a \in \mathrm{left}(x,v),\ b \in \mathrm{right}(x,v)\big\}, \tag{1}$$

$$ab_- := \big\{(x,v,r) \in \mathbb{R}^3 : a \in \mathrm{right}(x,v),\ b \in \mathrm{left}(x,v)\big\}. \tag{2}$$

The marked line associated to the point (x,v,r) induces a surface $h_{x,v,r} : \mathbb{R}^2 \to \mathbb{R}$ which is constant in each half-plane determined by $\ell(x,r)$. The height difference between the right and left half-planes is r, and the height of the half-plane containing the origin o is zero,

$$h_{x,v,r}(b) := r\big(\mathbb{1}\{(x,v) \in ob_+\} - \mathbb{1}\{(x,v) \in ob_-\}\big). \tag{3}$$

Let N be the empirical measure of a point configuration X and define

$$H_N(b) := \sum_{(x,v,r)\in X} h_{x,v,r}(b) = N_1(ob_+) - N_1(ob_-), \tag{4}$$

where $N_k := \sum_{(x,v,r)\in X} r^k\, \delta_{(x,v,r)}$, $N_0 = N$. H_N is well defined for $N \in \mathcal{M}$,

$$\mathcal{M} := \big\{\mu \text{ on } \mathcal{B}(\mathbb{R}^3) :\ \mu(ab) + \mu_2(ab) < \infty,\ \text{ for all } a,b \in \mathbb{R}^2\big\}. \tag{5}$$

where $d\mu_k(x,v,r) := r^k\, d\mu(x,v,r)$, $\mu_0 = \mu$. For $\mu \in \mathcal{M}$ define

$$H_\mu(b) := \iiint h_{x,v,r}(b)\, d\mu(x,v,r) = \mu_1(ob_+) - \mu_1(ob_-). \tag{6}$$

The discrete field H_N in (4) is a particular case of (6).

Let $\mu \in \mathcal{M}$ and consider a Poisson process X with intensity measure μ, and empirical measure $N = N[X]$. We have $\mathbb{E}H_N = H_\mu$ and $\mathrm{Cov}(H_N(a), H_N(b)) = \mu_2(oa \cap ob) = \frac{1}{2}(\mu_2(oa) + \mu_2(ob) - \mu_2(ab))$. The field H_N is called multi time random walk field, because the one dimensional marginals consist of non homogeneous continuous time random walks.

Fix a rescaling parameter $\varepsilon > 0$, later tending to 0, and consider a Poisson process X^ε with intensity measure $\varepsilon^{-1}\mu$, and rescaled empirical measure

$$N^\varepsilon \varphi := \varepsilon \sum_{(x,v,r)\in X^\varepsilon} \varphi(x,v,r), \qquad N^\varepsilon(A) := N^\varepsilon \mathbb{1}_A. \tag{7}$$

Notice that $\mathbb{E}N_1^\varepsilon(ob_\pm) = \mu_1(ob_\pm)$ for all ε, which implies $\mathbb{E}H_{N^\varepsilon}(b) = H_\mu(b)$. Define the empirical fluctuation fields

$$\eta^\varepsilon(b) := \varepsilon^{-1}\big(H_{N^{\varepsilon^2}}(b) - H_\mu(b)\big), \tag{8}$$

$$\widehat{\eta}^\varepsilon(b) := \varepsilon^{-1/2}\big(H_{N^{\varepsilon^2}}(\varepsilon b) - H_\mu(\varepsilon b)\big), \tag{9}$$

The processes η^ε and $\widehat{\eta}^\varepsilon$ are functions of the same X^{ε^2}. Denote by $d\mu(v,r|z)$ the conditioned law of μ given the space coordinate z.

Let $\mu \in \mathcal{M}$ and ω be white noise with control measure μ_2. The Gaussian process $\eta(b) := \omega(ob)$ is called multi-time nonhomogeneous Brownian field associated to the distance $\mathrm{d}(a,b) := \mu_2(ab)$. The covariances are given by $\mathrm{Cov}(\eta(a),\eta(b)) = \mu_2(oa \cap ob) = \frac{1}{2}(\mu_2(oa) + \mu_2(ob) - \mu_2(ab))$.

Theorem 1 (Scaling limits for multi-time fields)**.** *Assume $\mu \in \mathcal{M}$, then, as $\varepsilon \to 0$,*

$$H_{N^\varepsilon}(b) \xrightarrow{\text{a.s.}} H_\mu(b), \tag{10}$$

$$(\eta^\varepsilon(a), \widehat{\eta}^\varepsilon(b)) \xrightarrow{\text{law}} (\eta(a), \widehat{\eta}(b)), \tag{11}$$

where η and $\widehat{\eta}$ are the multi-time Brownian fields associated to the distance $\mathrm{d}(a,b) = \mu_2(ab)$ and $\widehat{\mathrm{d}}(a,b) = \widehat{\mu}_2(ab)$, respectively. Furthermore, the processes η and $\widehat{\eta}$ are independent.

Ideal gas and hard rod dynamics. We describe the hard rod dynamics in function of the multi-time random walk. A point (x,v,r) codifies a length zero particle sitting at x at time zero travelling at speed v, and carring a mark r. The ideal gas dynamics of a configuration X is defined by

$$T_tX := \{(x+vt, v, r) : (x,v,t) \in X\}, \quad t \in \mathbb{R}. \tag{12}$$

There is no interaction between particles. The trajectory $(T_tX)_{t\in\mathbb{R}}$ coincides with the set of lines $\ell(x,v)$ carrying the mark r. The ideal gas conserves Poisson processes. If μ is space translation invariant, the Poisson process with intensity measure μ is invariant for the ideal gas.

To a given a point (y,v,r) associate a rod, the interval $(y, y+r)$, carrying a velocity v. A hard rod configuration is a set $Y \subset \mathbb{R}^3$ satisfying that distinct rods do not intersect. The set of hard rod configurations is denoted

$$\mathcal{Y} := \{Y \subset \mathbb{R}^3 : (y, y+r) \cap (\tilde{y}, \tilde{y}+\tilde{r}) = \emptyset \text{ if } (y,v,r) \neq (\tilde{y},\tilde{v},\tilde{r}) \in Y\}. \tag{13}$$

Starting with an $Y \in \mathcal{Y}$, each rod travels deterministically with its velocity, until collision with another, faster or slower, rod. Just before collision time, the right extreme of the fast rod coincides with the left extreme of the slow one. At collision time the rods swap positions, the left extreme of the updated slow rod goes to the left extreme of the fast rod, and the right extreme of the updated fast rod goes to right extreme of the slow rod. After collision, each rod continues moving at its original velocity. Given a hard rod configuration $Y \in \mathcal{Y}$, the hard rod configuration at time t is denoted U_tY. The set $\mathcal{Y}$, is conserved by the dynamics, $Y \in \mathcal{Y}$ if and only if $U_tY \in \mathcal{Y}$.

Theorem 2 (Surface representation of the hard rod evolution [7])**.** *Let X be a point configuration with empirical measure $N \in \mathcal{M}$. Define*

$$D_0X := \big\{(x + H_N(x, 0), v, r) : (x, v, r) \in X\big\}. \tag{14}$$

Then, $Y = D_0X$ is a hard rod configuration, $Y \in \mathcal{Y}$, and U_tY is given by

$$U_tY = \big\{(x + vt + H_N(x + vt, t), v, r) : (x, v, r) \in X\big\}. \tag{15}$$

The configuration D_0X is the dilation of X with respect to the origin. We prove known and new theorems for the length and fluctuation fields of hard rods, by combining Theorems 1 and 2.

The results extend previous results by Boldrighini, Dobrushin and Suhov [2] and Presutti and Wick [17]. See the hard rod section of the classical book of Spohn [18] and the monograph on generalized hydrodynamics by Doyon [6].

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Where is the soul of the GFF, or how to characterise the Gaussian free field?

Juhan Aru
(joint work with W. Werner, G. Woessner)

The Gaussian free field on a graph or a subdomain of $\mathbb{R}^d$ is the Gaussian process with the covariance given by the Green's function of the Laplacian / the simple random walk. In the case of subdomains of $\mathbb{R}^d$ with $d \geq 2$, the covariance blows up on the diagonal and thus this Gaussian process is defined only as a generalized function. In this talk we discussed which simple properties one might use to characterise the free field both in the continuous and the discrete set-up.

As a first example or analogy one can think of the characterisation theorems of Brownian motion, which is actually just the Gaussian free field on the half-line. In this case there are many characterisations:

- Brownian motion (with drift) is the only continuous process with stationary independent increments
- Brownian motion is the only continuous local martingale with quadratic variation equal to t.

The crux is that in both cases the Gaussianity of the process is a byproduct of other properties. It is this type of characterisation we are looking for also for the GFF.

The discrete case

In the discrete case we presented the following slightly surprising result (joint with W. Werner). To state this consider $G = (V, E)$ be a finite graph and suppose in addition that V_i contains no isolated vertices, i.e. that each connected component of V_i contains at least two vertices. Then the following resampling property identifies the random process $\Gamma(x)$ (up to scaling and a deterministic drift) as the GFF:

- For every $x \in V_i$, the random variable $\Gamma(x) - \overline{\Gamma}(x)$ is independent of $(\Gamma(y), y \neq x)$.

This condition can be restated as follows: it is possible to find a law $\mathcal{L}_x$ on $\mathbb{R}$, such that if a random variable Z_x has this law and is independent of Γ, then the process $(\tilde{\Gamma}(y), y \in V)$ defined by $\tilde{\Gamma}(y) = \Gamma(y)$ when $y \neq x$ and $\tilde{\Gamma}(x) = \overline{\Gamma}(x) + Z_x$ has the same law as Γ.

This might sound a bit surprising to begin with, as compared to the usual DLR condition for the GFF, it does not presume any Gaussianity. Notice that importantly each connected component of V_i has to contain at least two vertices.

On the other hand, as explained in the talk, a posteriori it is a very simple result and such a characterisation works much more generally for a large family of discrete Gaussian processes and as such maybe does not touch upon the soul of the GFF.

The continuous case

In the continuous case there have been several characterisation theorems:

- In [3, 4] the authors characterised the 2D continuum GFF on simply-connected domains using conformal invariance and a certain domain Markov property.
- In [1] the authors characterised the continuum GFF on any n-dimensional ball using just a domain Markov property with the right scaling. This can be seen as a generalization of the stationary independent increments type of characterisation of the Brownian motion.

In this talk we discussed what can be seen as an analogue of the Lévy type of characterisation, based on joint work with G. Woessner [2]. This result characterises the GFF on any multiply-connected domains in any dimensions using a martingale-type of resampling property.

I leave the interested reader to check-out the precise formulation of this martingale-type of resampling property, but roughly it says that inside any ball we can resample the field by taking the harmonic extension from the outside field and adding some field with right variance and zero conditional mean. Importantly we do not assume any independence in the resampling property nor anything very detailed on the law. In comparison to previous results, this allows us to treat the case of multiple-connected domains. Moreover, the proof via dynamics is quite robust, working also for example for the fractional Gaussian free fields.

Importantly, no Gaussianity is presumed in none of these works, it is again just a consequence of the other properties. Still, all the proofs separate the two steps of Gaussianity and the covariance kernel, making one doubt if the soul has escaped our grasp again...but maybe GFF really has these two sides - being Gaussian and having the covariance given by the Green's function of the random walk - as explained in the very beginning?

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Thick points of 4D critical branching Brownian motion

Antoine Jego
(joint work with Nathanaël Berestycki, Tom Hutchcroft)

I will describe a recent work in which we prove that branching Brownian motion in dimension four is governed by a nontrivial multifractal geometry and compute the associated exponents. As a part of this, we establish very precise estimates on the probability that a ball is hit by an unusually large number of particles, sharpening earlier works by Angel, Hutchcroft, and Járai [1] and Asselah and Schapira [2] and allowing us to compute the Hausdorff dimension of the set of "a-thick" points for each $a > 0$. Surprisingly, we find that the exponent for the probability of a unit ball to be "a-thick" has a phase transition where it is differentiable but not twice differentiable at $a = 2$, while the dimension of the set of thick points is positive until $a = 4$. If time permits, we will also discuss a new strong coupling theorem for branching random walk that allows us to prove analogues of some of our results in the discrete case.

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Volume and heat kernel fluctuations for the three-dimensional uniform spanning tree

Satomi Watanabe
(joint work with Daisuke Shiraishi)

The uniform spanning tree (UST) of a connected graph is defined as a uniformly chosen random spanning tree of the graph. A spanning tree is a subtree containing all the vertices of the original graph, while a forest satisfying the same condition is called a spanning forest. UST was first defined on finite graphs and was discovered to be related to harmonic analysis on graphs. Pemantle [6] extended the notion of spanning tree to $\mathbb{Z}^d$ as the weak limit of the uniform spanning tree on sequences of finite subgraphs that exhaust $\mathbb{Z}^d$, with the known result that the limit measure concentrates on the set of spanning trees for $d \leq 4$ and spanning forests for $d \geq 5$. In this talk, we discuss the asymptotic behavior of the volume of the three-dimensional UST and the heat kernel of the simple random walk on it.

Several algorithms have been developed to investigate the structure of UST. Wilson's algorithm was introduced by Wilson [10] and later extended to transient graphs by Benjamini, Lyons, Peres and Schramm [4]. Using Wilson's algorithm, UST can be constructed with loop-erased random walk (LERW) paths, allowing for applying LERW estimates in the analysis of UST. Consequently, UST is one of the few probabilistic models of critical phenomena originating in statistical physics that have been analyzed precisely and rigorously even in three dimensions, which is considered the most challenging setting.

The aim of this talk is to examine the volume and heat kernel fluctuations of the three-dimensional UST. Such fluctuation results are known for the simple random walk on the critical Galton-Watson tree [3, 5] and the two-dimensional UST [2]. On the other hand, tail estimates of the volume of the three-dimensional uniform spanning tree and the spectral dimension for the simple random walk on the graph were obtained in [1].

We now introduce the notation that we need to state our main result. Let $\mathcal{U}$ be the uniform spanning tree on $\mathbb{Z}^3$. We write $B_{\mathcal{U}}(0,r)$ be the intrinsic ball (with respect to the graph distance) in $\mathcal{U}$ of radius r centered at the origin and write $p_n^{\mathcal{U}}(x,y)$ for the heat kernel of the simple random walk on $\mathcal{U}$. We also let $\beta \in (1, 5/3]$ be the growth exponent that governs the time-space scaling of the three-dimensional LERW, see [8, 9] for details.

Our main result consists of two theorems: volume and heat kernel fluctuations.

Theorem 1. *There exist deterministic constants $a_1, a_2 > 0$ such that one has*

$$\liminf_{r\to\infty} (\log\log r)^{a_1}\, r^{-\frac{3}{\beta}}\, |B_{\mathcal{U}}(0,r)| = 0, \tag{1}$$

and also

$$\limsup_{r\to\infty} (\log\log r)^{-a_2}\, r^{-\frac{3}{\beta}}\, |B_{\mathcal{U}}(0,r)| = \infty, \tag{2}$$

almost surely, where $|A|$ stands for the cardinality of the set A.

Theorem 2. *There exist deterministic constants $a_3, a_4 > 0$ such that one has*

$$\liminf_{n\to\infty} (\log\log n)^{a_3} n^{\frac{3}{3+\beta}} p_{2n}^{\mathcal{U}}(0,0) = 0, \tag{3}$$

and also

$$\limsup_{n\to\infty} (\log\log n)^{-a_4} n^{\frac{3}{3+\beta}} p_{2n}^{\mathcal{U}}(0,0) = \infty, \tag{4}$$

almost surely.

The idea of the proof is inspired by [2], which constructs via Wilson's algorithm the atypical events where either a large or small volume of the UST is observed to prove volume fluctuations. Heat kernel fluctuations follow from the volume fluctuations and some basic estimates of the heat kernel. In contrast to the two-dimensional case, it is essential to control the hitting probability of the partially constructed subtree at each step of Wilson's algorithm. We address this issue by applying Beurling-type estimates for the three-dimensional LERW given in [7].

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Favorite sites for simple random walk in two and more dimensions

Xinyi Li
(joint work with Chenxu Hao, Izumi Okada, Yushu Zheng)

Let $(S_n)_{n\geq 0}$ be a discrete-time simple random walk on the integer lattice $\mathbb{Z}^d$ for $d \geq 1$. We write

$$\xi(x,n) = \#\{0 \leq j \leq n : S_j = x\}; \qquad \xi^*(n) := \sup_{y\in\mathbb{Z}^d} \xi(y,n)$$

for the local time of site x at time n and maximal local time at time n, respectively.

There is a considerable amount of interest and a huge literature on the asymptotics of $\xi^*(n)$ and the distribution of thick points. Erdős and Taylor [5] gave sharp bounds on $\xi^*(n)$ for $d \geq 3$ and derived up-to-constants bounds for $d = 2$, with upper bound conjectured to be appropriate. Nearly forty years later, the seminal work [1] by Dembo, Peres, Rosen and Zeitouni confirmed this conjecture and also established the growth exponent of thick points.

For any $n \geq 0$, we denote by

$$\mathcal{K}^{(d)}(n) := \left\{x \in \mathbb{Z}^d : \xi(x,n) = \xi^*(n)\right\}$$

the set of favorite sites at time n. A major direction of research is the cardinality of $\mathcal{K}^{(d)}(n)$. A classical question of Erdős and Révész [3] reads as follows:

$$\text{(1)} \qquad \text{Can } \#\mathcal{K}^{(d)}(n) = r \text{ occur infinitely often for } r \geq 3 \text{ and } d \geq 1?$$

For $d \geq 3$, Erdős and Révész themselves gave a positive answer in [4]. For $d = 1$, it was proved [7, 6] that a.s. $\#\mathcal{K}^{(1)}(n) \leq 3$ eventually and it remained open for a long time whether $\#\mathcal{K}^{(1)}(n) = 3$ i.o. Much later, Ding and Shen showed in [2] that this is indeed the case.

In this talk, I will discuss our recent preprint that gives a complete answer to the open question (1) of Erdős and Révész.

For $d = 2$, the number of favorite sites has the same behavior as the $d = 1$ case: almost surely,

$$\limsup_{n\to\infty} \#\mathcal{K}^{(2)}(n) = 3.$$

For $d \geq 3$, the following sharp asymptotics of $\#\mathcal{K}^{(d)}(n)$ are obtained: almost surely,

$$\limsup_{n\to\infty} \frac{\#\mathcal{K}^{(d)}(n)}{\log\log n} = -\frac{1}{\log \gamma_d},$$

where $\gamma_d := P[S_n \neq S_0,\ \forall n \geq 1]$, which is the probability that a simple random walk never returns to the starting point.

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