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Mini-Workshop: Infinite-Dimensional Kac-Moody Lie Algebras in Supergravity and M Theory

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Abstract. This mini-workshop explores the role played by infinite-dimensional Lie algebra of Kac-Moody type in supergravity and M theory. Deep conjectures about the role of affine, hyperbolic and other indefinite Kac-Moody algebras could both provide new methods and models for low-dimensional gravity systems and shed new light on the fundamental structure of gravity and unified theories. The elusive precise mathematical structure of some of the algebras can in turn be informed by physical ideas.

Mathematics Subject Classification (2020): 17B65, 17B67, 20G44, 22E67, 81R10, 83Exx.

Introduction by the Organizers

This mini-workshop, organized by Guillaume Bossard (Paris), Axel Kleinschmidt (Potsdam) and Hermann Nicolai (Potsdam), brought together 16 participants in Oberwolfach as well as two online participants. The audience was mixed between mathematicians and theoretical physicists who together explored topics associated with Kac-Moody symmetries in supergravity. Emphasis was given to explaining the topics in a way accessible to both communities as much as possible. All 18 participants gave presentations and in addition there was one open discussion session as well as a session with the two other mini-workshops being held at the same time.

A prominent topic, reflected in several talks, were hyperbolic Kac-Moody algebras and their realizations using physical ideas. This concerned both the algebra itself (using string theory operators) and their (fermionic) representation theory, covered in the talks of Damour, Malcha, Meissner and Nicolai and touched upon by Ciceri and Feingold. Affine Kac-Moody algebras and their physical significance were discussed by Cesaro, Inverso and König while Lorentzian algebras entered in the talks of Bossard and Kleinschmidt. General results and theorems, covering many algebras and groups at the same time were presented by Köhl, Lautenbacher, Marquis and Palmkvist. Yet other realizations of other large physical symmetries featured in the talks by Alonso-Serrano, Majumdar and Petrini.

The mini-workshop showed the breadth of the field and stimulated many interesting discussions between the participants with different backgrounds. This was particularly visible in the lively discussion session for which the organizers had requested the participants in advance to send open questions and topics. The unique setting of the MFO helped in making this a successful event.

Mini-Workshop: Infinite-Dimensional Kac-Moody Lie Algebras in Supergravity and M Theory

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Abstracts

Quantum field theory in multiply-connected spacetimes

Ana Alonso-Serrano
(joint work with Luis J. Garay, Eduardo Martín-Martínez and Erickson Tjoa)

The analysis of multiply-connected spacetimes in the framework of general relativity becomes relevant when we consider spacetimes which posses closed timelike curves (CTCs) in a region of the (or the whole) spacetime. When the CTCS are restricted to a region on the spacetime, they are protected by Cauchy horizons, which causally disconnect those (problematic) regions from the rest of the spacetime. The existence of these curves challenges our models (because of the nontrivial topological and causal structure), but it also allows us to explore the limits of the theory.

The construction of a quantum field theory in these spacetimes has been largely considered in the literature mainly in relation to the analysis of the stability of the mentioned Cauchy horizons. The main difficulties of the theory are due to the nontrivial topology and that they are not globally hyperbolic. Thus, the general construction of the theory has been studied using the framework of automorphic fields [1, 2]. The strategy is to construct the quantum field theory in the corresponding universal covering space, which has trivial topology. Then, we can translate the resulting quantum field to the multiply-connected spacetime by imposing some automorphic conditions on the field, given by the topological invariants in the construction.

By the sake of simplicity, in this work we focus in 2-dimensional models, where the powerful conformal techniques allow for relevant technical simplifications. We start by constructing the most general $1+1$-dimensional static simply connected spacetime, M, given by the metric $ds^2 = -\alpha(x)^2 dt^2 + dx^2$, with $t, x \in \mathcal{R}$. This spacetime possesses a unique global timelike hypersurface-orthogonal Killing vector field, which allows to foliate M such that $M = \mathcal{R} \times \Sigma$, where Σ is a Cauchy surface orthogonal to the Killing vector field. For the simplicity of the discussion, and without lost of generality (due to a transformation between them encoded in a smooth conformal factor), we focus in a particular case expressed by the metric $ds^2 = -e^{2Wx}dt^2 + dx^2$, where W is a two-parameter constant given by $W = \log A/L \geq 0$, with $A \geq 1$ and $L > 0$.

In order to construct a multiply-connected spacetime, $\tilde{M}$, we identify points in the spacetime M by establishing the equivalence relation $(t, x) \sim (t', x')$, if and only if $t'/t = A$ and $x' - x = L$. So one can understand the multiply-connected spacetime as a strip identified at length L, and at a shifted time given by a warp parameter A. In this way, when $A > 1$ there exists a region of the spacetime, separated by future and past Cauchy horizons, which contain CTCs. When considering the limiting case $A \to 1$ (and so $W \to 0$, provided that L is kept constant), there is no time shift and one recovers the Einstein cylinder (the flat spacetime limit).

The multiply-connected spacetime $\tilde{M}$ is only locally static, i.e., it has a Killing vector field for each simply connected region that cannot be globally extended. It is then convenient to introduce the Killing trajectories, defined from the velocity vector field associated to Killing observers. The velocity vector field, together with the corresponding acceleration vector, can be globally extended throughout $\tilde{M}$ (defining a new foliation in terms of the Killing trajectories [3]).

Once we have defined the classical structure of this spacetime, we need now to define its universal covering spacetime. It fact, it turns that this spacetime results to be the spacetime M already defined. Let me remark at this point also that performing a simple change of coordinates, one can see that this spacetime corresponds to the Poincaré patch of AdS_2 [4]. The spacetime $\tilde{M}$ can be understood as the quotient spacetime $\pi_1(\tilde{M})\backslash M$, where $\pi_1(\tilde{M})$ is the fundamental group of M. The multiply connectedness of the spacetime is captured by the topological invariants such as the fundamental group. In that way, the transformation of the quantum field theory in the universal covering space to a quantum field theory in the multiply-connected spacetime is given by imposing some automorphic conditions on the field, that are defined in terms of the elements of the fundamental group.

We first construct the quantization of a massless scalar field in the Einstein cylinder to control the flat spacetime limit. One of the relevant features of the modes in the Einstein cylinder is the Fourier mode decomposition sum a spatially constant piece zero mode oscillator to the standard oscillator modes. Zero modes can appear naturally in many situations and the issue with them is that they have no Fock representation, so the ground state of this theory is nontrivial. For this reason, they have been sometimes removed by hand in the literature or, more recently, regularized *ad hoc* using squeezed vacuum of a quantum harmonic oscillator [5].

We then construct the quantization of a massless scalar field in the universal covering spacetime M [4]. We impose that the resulting quantum field has to be automorphic under the action of the fundamental group to construct quantized massless scalar field in a fundamental domain of the multiply-connected spacetime $\tilde{M}$. We obtain a Fourier mode decomposition in oscillator modes. Let me stress here that the physical observables defined within this framework are going to be restricted to the region of the spacetime free of CTCs. In this context we can analyze several vacuum two-point functions and study their limit when $A \to 1$, where we consistently recover the Einstein cylinder case, with the corresponding zero mode contribution. More interestingly, when using the two-point functions to compute the renormalized stress-energy tensor, we find a prescription to select the zero mode in a unique way for the limiting case of the Einstein cylinder from the one-parameter, A, family of time-warp identified spacetimes [4].

For a more detailed analysis of the properties of the field, we introduce a localized probe model as an Unruh-de Witt detector [6] to study, mainly, the possible extraction of topological information. That is, to study if a local observer in the region that do not contain CTCs can distinguish the existence of them in a region protected by Cauchy horizons. The fact that the curvature W is regulated by two independent parameters associated to the multiply connectedness, allows to separate topological information from geometrical information and distinguish periodic spacetimes without CTCs, curvature, and spacetimes with topological identifications that enable the appearance of CTCs [7].

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Identities for Kac–Moody algebras

Guillaume Bossard
(joint work with Axel Kleinschmidt and Ergin Sezgin)

It has been proposed by West that it should be possible to write (the bosonic part of) $D = 11$ supergravity in a way that utilises the infinite-dimensional Kac–Moody group E_{11}. The proposed construction involves a non-linear realisation of $E_{11}^{+}/K^*(E_{11})$ where $K^*(E_{11}) \subset E_{11}$ denotes a subgroup that generalises the eleven-dimensional Lorentz group $SO(1,10)$, where E_{11}^{+} denotes the maximally extended Kac–Moody group on the positive Borel. In a different, but not unrelated, strand of research, field theories with extended space-time symmetries and exceptional symmetry groups have been constructed. These so-called exceptional field theories possess fields in a non-linear realisation of E_n (for $n \leq 9$) and these fields depend on extended (internal) coordinates Y^M involving representations of E_n. In an effort to define an exceptional field theory for E_{11} we have proposed a non-linear set of first-order duality equations that can be written as

$$\eta_{IJ} F^J = \Omega_{IJ} F^J \,, \tag{1}$$

where F^I denotes an infinite collection of non-linear field strengths that transform under E_{11} in a representation that is defined by its tensor hierarchy superalgebra. This representation is neither highest nor lowest weight but can be shown to carry a symplectic form that we write as Ω_{IJ} and that generalises at the same time the usual Levi–Civita symbol that appears in duality equations and the symplectic form familiar from electric-magnetic duality relations in $D = 4$. The metric η_{IJ} on the left-hand side is a conjectural $K(E_{11})$-invariant symmetric bilinear form.

The existence of η_{IJ} is a key assumption in our construction. The definition of the field strengths F^I crucially involves an infinite set of constrained fields that go beyond the E_{11} coset fields. These fields are necessary from the construction of the tensor hierarchy algebra and sit in an *indecomposable* representation with the E_{11} coset fields. A similar feature was also observed in the context of E_9 exceptional field theory, for which the theory has been formulated with other authors without involving any conjectural identities [1].

In this talk we emphasise in particular that the E_{11}-covariance of F^I and the mere existence proof of the representation labelled by I depends on the tensor hierarchy $\mathbb{Z}$-graded superalgebra $\mathcal{T}(\mathfrak{e}_{11})$ introduced in [2]. The existence of this superalgebra is based on a local superalgebra. For $\mathfrak{g}_0$ a superalgebra, and $\mathfrak{g}_{\pm 1}$ two distinct representations of $\mathfrak{g}_0$ satisfying that

$$\mathfrak{g}_0 \subset \mathfrak{g}_1 \otimes \mathfrak{g}_{-1} \tag{2}$$

as a representation, one can use the corresponding homomorphism ϕ_0 to define the local superalgebra such that for $\mathbf{x}_i \in \mathfrak{g}_i$,

$$[\mathbf{x}_0, \mathbf{x}_{\pm 1}] = \rho_{\pm 1}(\mathbf{x}_0)\mathbf{x}_{\pm 1}\,, \quad [\mathbf{x}_1, \mathbf{x}_{-1}] = \phi_0(\mathbf{x}_1, \mathbf{x}_{-1}) \in \mathfrak{g}_0\,. \tag{3}$$

From this local superalgebra one defines the freely generated superagebras

$$\mathfrak{n}_\pm = \bigoplus_{n \geq 1} \mathfrak{g}_{\pm n} = \langle \mathfrak{g}_{\pm 1} \rangle \tag{4}$$

and the big superalgebra

$$\hat{\mathfrak{g}} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n\,. \tag{5}$$

The superalgebra $\mathfrak{g}$ is defined by the quotient superalgebra of $\hat{\mathfrak{g}}$ by its maximal ideal not intersecting the local superalgebra $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_0$. Starting from $\mathfrak{g}_0 = W_{11}$ the superlgebra of superdiffeomorphisms in eleven Grassmann variables in V, $\mathfrak{g}_{-1}$ defined as a real scalar superfields and $\mathfrak{g}_1$ by a specific combination of three tensor superfields, in $\wedge^3 V$, $\mathrm{Sym}^2 V$ and V, one gets $\mathfrak{g} = \mathcal{T}(\mathfrak{e}_{11})$. The restriction of $\mathfrak{g}_0 = W_{11}$ to $\mathfrak{gl}_{11}$, $\mathfrak{g}_{-1}$ to $\wedge^3\mathbb{R}^{11}$ and $\wedge^3 V \subset \mathfrak{g}_1$ to $\wedge^3\mathbb{R}^{11}$ produces in the same way the Kac–Moody algebra $\mathfrak{e}_{11}$, proving that $\mathfrak{e}_{11} \subset \mathcal{T}(\mathfrak{e}_{11})$.

We explained in this talk the identities we could prove, the two fact that remain conjectural and provided an idea of the proofs. The conjectural facts are the existence of the bilinear form and the uniqueness of a certain homomorphism from a highest weight module $L(\Lambda_2)$ tensor product with the non-highest weight representation R_θ to the highest weight irreducible module $R(\Lambda_1)$. In our conventions Λ_2 and Λ_1 are the fundamental weight defining the maximal parabolic subalgebras of respective Levi subalgebras $\mathfrak{e}_9 \oplus \mathfrak{sl}_2$ and $\mathfrak{e}_{10}$. The conjecture would follow from the hypothesis that the highest weight module $L(\Lambda_2)$ and the non-highest weight representation R_θ are both irreducible. Assuming the conjecture is true, I explained how to write equations of motions invariant under E_{11} generalised diffeomorphisms, which reproduce the equations of motion of both eleven-dimensional supergravity and type IIB supergravity for appropriate choices of sections [3].

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Integrable auxiliary field deformations of coset models

Mattia Cesàro

(joint work with Axel Kleinschmidt, David Osten)

Non-linear σ-models are ubiquitous in various branches of physics, with appearances in statistical and condensed matter physics, for example as models for random matrix ensembles or disordered metals. They also find important applications in gravity and string theory.

Remarkably, despite being non-linear, interacting quantum field theories, in two dimensions they can be classically integrable, with prime examples being the principal chiral model or the symmetric space σ-model. In the latter case, the theory is given by maps from the two-dimensional 'world-sheet' to a target space G/K where G is a Lie group and K a subgroup fixed by an involution of G, for example K can be the maximal compact subgroup of G. It is therefore interesting to consider whether there are further, related integrable models. These are often referred to as integrable deformations. So far, the focus in the literature lay on deformations inside the class of σ-models, most prominently so-called Yang–Baxter [3, 4] and λ-deformations [5].

Recently, another type of deformations has been introduced for principal chiral models [6, 7]. These deformations are obtained by introducing an auxiliary field and adding an arbitrary function of an invariant built from it to the Lagrangian. In comparison with the above mentioned examples, one can identify two main characteristics: First, these deformations leave the realm of σ-models since they introduce additional fields. Second, they can still be considered to be quite mild deformations of σ-models, as they still share some features. On the level of classical integrability, the latter can be understood by the fact that the original and deformed level both have the same twist function.

In this presentation, I show how to extend these deformations from principal chiral models to coset σ-models, where both the symmetric space cosets (with an underlying $\mathbb{Z}_2$-automorphism) and their $\mathbb{Z}_N$-generalisation are considered, leading to $\mathbb{Z}_N$-coset spaces. This is achieved first of all by finding a Lax connection representation whose flatness is equivalent to the Euler-Lagrange equations of the system: this ensures the existence of an infinite tower of conserved charges. Subsequently, the charges are proven to be in involution by displaying that the

Poisson brackets of the spatial components of the Lax connection assume the non-ultralocal form of Maillet [8], [9]. This leads to the construction of an infinite family of integrable models.

In the second part, I adopt a "reversed logic" approach that generalises the method of [6]: I avoid to start from a Lagrangian deformation, but rather consider a deformation of the Lax formulation of the dynamics. The latter is a less constraining option of which the Lagrangian deformation in [6] represents a subcase, and its straightforward generalisations for coset spaces considered in the first part of the talk are also a special case.

I conclude by speculating on possible extensions of this type of integrable deformations to the Kaluza-Klein reduction of $D = 4$ general relativity down to $D = 2$ along two Killing isometries [2], and on the intriguing chance to construct their uplift back to $D = 4$.

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Sphere reductions of gravity to low dimensions

Franz Ciceri
(joint work with Henning Samtleben)

Consistent reductions of higher-dimensional gravity theories have a long history nourished by the emergence of extra compactified dimensions in supergravity and string theory. In some cases, the infinite tower of massive Kaluza-Klein modes associated to a compactification on a given internal space can be consistently truncated to a finite set of modes. The term 'consistent' refers here to the requirement that the retained modes do not source those that have been discarded. In other words, such a consistent reduction means that the compactified gravitational theory can be truncated to a finite set of fields whose dynamics is described by a lower-dimensional action, such that any solution of the lower-dimensional theory can be uplifted to a solution of the original, higher-dimensional theory. In this talk, we will importantly focus on consistent reductions that retain all the massless Yang-Mills fields that gauge the isometry group of the internal space. These are sometimes referred to as Pauli reductions.

Unlike for toroidal reductions, where the retained modes are all singlets under the $U(1)^d$ isometry of the torus T^d and whose consistency is guaranteed by a simple group-theoretical argument, the question becomes much more involved for non-trivial spaces such as spheres. In fact, the reduction of pure gravity on a d-sphere S^d is known to be inconsistent for any d. The obstruction stems from certain couplings in the Einstein-Hilbert action in which a quadratic product of $SO(d+1)$ gauge fields act as a source for massive spin-2 fields. This triggers a chain reaction which ultimately implies that an infinite number of massive fields have to be retained.

Consistent sphere reductions become possible when the higher-dimensional gravity theory is coupled to matter. In an important paper [1], Cvetič, Lü and Pope have classified and explicitly constructed such reductions on S^d that retain all the $SO(d+1)$ Yang-Mills fields. These classification includes well-known examples that have played an important role for the study of holography, such as the reduction of type IIB supergravity on S^5. A strong necessary condition for the existence of such sphere reductions can be found from the toroidal reduction of the relevant higher-dimensional theory. Its global symmetry group must accommodate an $SO(d+1)$ subgroup such that gauging of the latter describes the theory obtained from reduction on the sphere. In general, this requires some global symmetry enhancement that only occurs for specific matter content and couplings of the higher-dimensional theory.

In this talk I will discuss sphere reductions to two spacetime dimensions [2], and emphasize the peculiarities of this case which had been left out from the classification of [1]. More explicitly, I will present the consistent reduction on S^d of Einstein-Maxwell-dilaton gravity in $2+d$ dimensions. The resulting two-dimensional theory is an $SO(d+1)$ gauged non-linear sigma model coupled to dilaton gravity that possesses a non-trivial potential for the various scalars fields.

Along the way, I will explain the role played by the unique symmetry enhancement that occurs when reducing the relevant higher-dimensional theory on T^d. In this case the global symmetry group becomes the affine extension of $SL(d+1)$, and allows for a non-trivial embedding of the sphere isometry group $SO(d+1)$. I will conclude by presenting preliminary results on the sphere truncation of matter-coupled gravity theories to one time dimension.

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Hidden Hyperbolic Kac-Moody Structures in Supergravity

Thibault Damour

The various indications of the presence of hyperbolic Kac-Moody structures in supergravity were reviewed.

After reminding the field actions that define Einsteinian gravity (General Relativity; GR), and its Supergravity (sugra) generalizations (notably in spacetime dimensions $D \equiv d+1 = 4, 5$ and 11), the talk started by recalling the Belinsky-Khalatnikov-Lifshitz (BKL) approximate description of the structure of the spacetime metric near generic inhomogeneous spacelike singularities in $D = 3+1$ GR [1]. The main point is that, near a spacelike singularity, different spatial points become causally disconnected, giving rise to a "Carollian structure". This allows one to set up an expansion in which spatial gradients are considered as being parametrically smaller than time derivatives ("gradient expansion"). At leading order in the latter BKL gradient expansion, the dynamical evolution of the spacetime metric near a spacelike singularity is captured by a "billiard description", namely the dynamics of a "massless particle" moving within an auxiliary d-dimensional Lorentzian space and reflecting upon some hyperplanar "walls". [See, e.g., Ref. [2] for a review of cosmological billiards, and references to the literature.]

The first indication of the presence of hidden hyperbolic Kac-Moody structures in (super)gravity came from Ref. [3], which found that the billiard walls describing a generic inhomogeneous spacelike singularities in $D = 11$ sugra (or its $D = 10$ superstring-theory avatars) could be identified with the Weyl chamber of the last hyperbolic Kac-Moody algebra, namely E_{10}. Further work [4] proved that the dynamics of the bosonic sector of $D = 11$-supergravity (considered in a BKL gradient expansion) was in precise mathematical correspondence (up to the 30th order in root height) with the dynamics of a massless particle moving on the infinite-dimensional coset space $E_{10}(\mathbb{R})/K(E_{10}(\mathbb{R}))$. [The latter infinite-dimensional coset space, where $K(E_{10}(\mathbb{R}))$ denotes the "maximal compact subgroup" of $E_{10}(\mathbb{R})$, is endowed with a pseudo-Riemannian metric, of signature $-, +, +, +, +, +, +, +, \cdots$.] This led Ref. [4] to conjecture the existence of a putative, exact gravity/Kac-Moody-coset correspondence. Further evidence for this conjecture came from the study of the fermionic sector of supergravity. Namely, Refs. [5–7] found that the sugra-predicted dynamics of the gravitino (when neglecting cubically nonlinear terms, and spatial gradients of the gravitino) was described by a simple parallel transport (with a $K(E_{10})$-valued connection) of a coset analog of the gravitino, along the null geodesic of the coset space $E_{10}/K(E_{10})$ describing the bosonic sector. This led to conjecture that $D = 11$ supergravity (or, more ambitiously, "M-theory") was in correspondence with the motion of a massless *spinning* particle on the infinite-dimensional coset space $E_{10}(\mathbb{R})/K(E_{10}(\mathbb{R}))$.

Since then, the latter grand gravity/coset conjecture has received some further partial confirmations (e.g. from the study of nonlinear-in-fermions effects [8–10]), and some completions [11,12], but, as of today, the precise role, extent, and meaning of the hyperbolic Kac-Moody structures in supergravity remain unclear: are they the echo of a hidden, or broken, underlying Kac-Moody-related symmetry, or a red herring linked to the well-established presence of Cremmer-Julia-type E_7, E_8 and E_9 symmetries in toroidally compactified supergravity [13–17]. Recently developed new approaches to the role of Kac-Moody structures in supergravity, such as Exceptional Field Theories [18–21], might shed a new light on the meaning of the findings obtained by zooming (à la BKL) on the dynamical behavior near cosmological singularities. See the abstracts of G. Bossard and A. Kleinschmidt.

While, from the physics point of view, the jury is still out, the work on the gravity/coset conjecture has unearthed potentially interesting mathematical structures, notably the existence of finite-dimensional spinorial representations of $K(E_{10})$ (and other involutory subalgebras of Kac-Moody algebras) [5–7, 22–24], and a corresponding "spinorial" generalization of the Weyl group, generated by spinorial reflection operators of the form $\mathcal{R}_i = \exp \frac{\pi}{2}(e_i - f_i)$ [25]. [The latter (finite-dimensional) spinorial reflection operators satisfy $\mathcal{R}_i^8 = 1$ and Kac-Peterson-like braid relations [10].] See the abstracts of R. Köhl, and of R. Lautenbacher for more information. Let us finally mention that it would be interesting to study the (mathematical) conjecture formulated in [11] that there exists an hyperbolic Kac-Moody generalization of the Sugawara construction, and a corresponding hyperbolic generalization of the Virasoro algebra: see Eqs (2.13), (2.14) and (2.16) in Ref. [11]. The talk had started by citing the nice article "Missed Opportunities" [26], in which Freeman Dyson mentions several cases where "mathematicians and physicists lost chances of making discoveries by neglecting to talk to each other." One can hope that the format, and framework, of this workshop, as provided by the Mathematisches Forschungsinstitut Oberwolfach, has gone a long way towards avoiding losing chances to make new discoveries related to hyperbolic Kac-Moody algebras.

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Piecewise Isometry Groups of Tessellations from Weyl groups

Alex Feingold
(joint work with Robert Bieri, Daniel Studenmund)

Abstract
Weyl groups are Coxeter groups generated by reflections determined by a Cartan matrix of a Kac-Moody (KM) Lie algebra. We study infinite Weyl groups acting on a Euclidean or a hyperbolic space so that a fundamental domain tessellates the space. In joint work with Robert Bieri and Daniel Studenmund, we are are investigating the geometry of such tessellations in order to define groups of piecewise isometries of the tessellations. A known example is the tessellation of the Poincaré disk by ideal triangles where the Weyl group is the hyperbolic triangle group $T(\infty,\infty,\infty)$ and the piecewise isometry group is a Thompson group which can be viewed as $PPSL(2,\mathbb{Z})$. For an application in physics see [8].

Introduction
Combinatorial group theory is a very active subject involving ideas from geometry, topology and algebra. Thompson groups of **piecewise** linear maps on the unit interval or the circle give examples with interesting properties. For any geometric object, A, the group $Isom(A)$ can have a piecewise extension, $PIsom(A)$. For example, the Poincaré disk tessellated by ideal triangles has isometry group, $PSL(2,\mathbb{Z})$, whose piecewise extension was studied by several people (Thurston, Penner, Schneps, Lochak, Kontsevich) and seen to be isomorphic to a Thompson group. An application of that example in physics was found by Osborne and Stiegemann [8]. But $PSL(2,\mathbb{Z})$ is the even subgroup of the hyperbolic Weyl group,

$$PGL(2,\mathbb{Z}) \cong T(2,3,\infty)$$

of a rank 3 hyperbolic KM algebra, $\mathcal{F} = AE3 = A_1^{++}$, studied by Feingold and Frenkel [3].

Weyl groups of two rank 4 hyperbolic KM algebras contain finite index subgroups $PSL(2,E)$ and $PSL(2,\mathbb{Z}[\mathbf{i}])$, where $E = \mathbb{Z}[e^{\pi\mathbf{i}/3}]$ is the ring of Eisenstein integers and $\mathbb{Z}[\mathbf{i}]$ is the ring of Gaussian integers. These Weyl groups were studied by Feingold-Kleinschmidt-Nicolai [4–6] in relation to normed division algebras, and by Feingold-Vallières [7] in relation to Clifford algebras. They are arithmetic subgroups of $PSL(2,\mathbb{C})$ which act properly discontinuously on hyperbolic 3-space, H^3, and have fundamental domains studied long ago. We wish to find piecewise isometry groups acting on the tessellations of H^3 by these hyperbolic Weyl groups, giving us mathematical definitions of new groups which could be called $PPSL(2,E)$ and $PPSL(2,\mathbb{Z}[\mathbf{i}])$.

On a grand scale, we envision the study of a wide class of piecewise isometry groups defined for all infinite Coxeter groups. For now we focus on the Weyl groups of affine and hyperbolic KM algebras where the geometry of the tessellation allows us to define finite decompositions into convex pieces which can be permuted by piecewise local isometries. This talk is a glimpse into a work in progress.

The boundary of H^3 is the 2-sphere in the Poincaré ball model, and is $\mathbb{C} \cup \{\infty\}$ in the upper half-space model. The rank 4 hyperbolic Weyl groups mentioned above have parabolic subgroups fixing the "north pole" (∞), and are isomorphic to affine Weyl groups of type A_2^+ or B_2^+ whose tessellations are easy to visualize in $\mathbb{R}^2$.

We have details about piecewise isometries of the A_2^+ tessellation whose fundamental domain is an equilateral triangle, as well as that of A_3^+ whose fundamental domain is a tetrahedron, T, with a pair of opposite edges of length 1 and four other edges of length $\sqrt{3}/2$, so that 24 copies of T fill up a **rhombic dodecahedron**.

The example of A_1^+ is just the infinite dihedral group, D_∞, which acts on the tessellation of $\mathbb{R}$ into unit intervals. The piecewise isometries of that tessellation are an example of a Houghton group.

The most important source of ideas in our project has come from the work of Bieri and Sach [1, 2]. They set up the ground work for general study of piecewise defined groups. In particular, they did the Euclidean cases where $\mathbb{R}^n$ is tessellated by a generalized cube, $[0, 1]^n$, so the associated affine Weyl group would be the direct product of n copies of the infinite dihedral group, D_∞.

A first goal in the affine case is to study the group of piecewise isometries of the tessellation of $\mathbb{R}^n$ coming from the Weyl group of type A_n^+ for $n \geq 1$.

In H^3 we are studying its tessellation by an **ideal tetrahedron** whose vertices are all in the boundary. That tessellation comes from the hyperbolic Weyl group whose Dynkin diagram is a tetrahedron, the complete graph on four vertices. We have learned a lot about the geometry of that tessellation, and can show some beautiful pictures and a 3D printed model, but we are not yet prepared to present theorems about the hyperbolic case.

Coxeter Groups and piecewise isometries of their tessellations
The **Coxeter** group $G = G(M)$ associated with an $n \times n$ Coxeter matrix, $M = [m_{ij}]$, is the group generated by **reflections**,

$$\langle r_1, \cdots, r_n \mid (r_i r_j)^{m_{ij}} = 1, \quad 1 \leq i \leq j \leq n \rangle$$

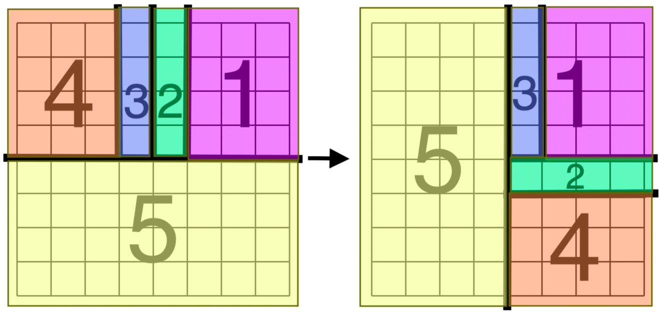
When G acts as isometries on a Euclidean or hyperbolic space, X, each reflection r_i has a **hyperplane** h_i of fixed points which divides X into two **half-spaces**, $h_i^\pm$. A **fundamental domain** for G acting on X is the intersection

$$D = \bigcap_{i=1}^{n} h_i^+ \quad \text{such that} \quad X = \bigcup_{g \in G} g(D)$$

is the **tessellation**, $Tess(D)$, of X into **tiles**, $g(D)$. Let $\bar{X}$ be the union of the interiors of all tiles in $Tess(D)$. For any $g \in G$ the conjugates $g r_i g^{-1}$, $1 \leq i \leq n$, are reflections with associated fixed hyperplanes. A **piece** of $Tess(D)$ is a tiled intersection of a finite number of open half-spaces bounded by such hyperplanes. A **finite decomposition** of $Tess(D)$ is a disjoint union of finitely many pieces (finite or infinite) which equals $\bar{X}$.

A **piecewise isometry** of $Tess(D)$ is a bijective map $f : \bar{X} \to \bar{X}$ such that: (1) there is a finite decomposition and on each of its pieces f is the restriction of a global isometry of $Tess(D)$, (2) the images of the pieces are a finite decomposition of $Tess(D)$.

Example: Let $\mathbb{R}^2$ be tessellated by unit squares whose vertices have integral coordinates, $\mathbb{Z} \times \mathbb{Z}$. Then $D = [0,1] \times [0,1]$ and the reflection hyperplanes of $Tess(D)$ are the vertical and horizontal lines, $x = m$ and $y = n$ for all $m, n \in \mathbb{Z}$. Below is a finite decomposition of $\mathbb{R}^2$ into five pieces, one half-space, two strips and two quadrants. An example of a piecewise isometry is shown.



Outline of this talk

(1) Affine Weyl Group $W(A_1^+)$
(2) Piecewise isometries of $Tess(I)$
(3) Affine Weyl Group $W(B_2^+)$
(4) Hyperbolic Weyl Group $W(A_1^{++})$ and $PPSL(2,\mathbb{Z})$
(5) Affine Weyl Group $W(A_2^+)$ and piecewise isometries
(6) Affine Weyl Group $W(A_3^+)$
(7) Rank 4 Ideal Hyperbolic Weyl Group in $W(A_2^{++})$

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E_9 and two-dimensional supergravity

Gianluca Inverso
(joint work with Guillaume Bossard, Franz Ciceri, Axel Kleinschmidt)

Gauged supergravities are strictly related to brane solutions and their near horizon geometries. The two-dimensional case is peculiar in many ways. Two-dimensional gauged supergravities are the realm of (near) AdS_2 solutions and hence one can expect that many such models arise from near-horizon geometries of extremal black-hole configurations. A concrete example for this talk will be the physics of D0 branes in IIA supergravity. The near horizon geometry of concident D0 branes is conformal to $\mathrm{AdS}_2 \times S^8$ (with a running dilaton) and the system is described by a matrix model [1–3] holographically dual to the above geometry [4]. A consistent truncation of IIA supergravity on S^8, leading to a two-dimensional maximal supergravity with SO(9) gauge group [5] was long expected to exist, and one of the aims of this talk is to describe the tools and structures that lead to the explicit proof of such truncation.

The properties of two-dimensional supergravities, and of the maximally supersymmetric one in particular, are quite interesting *per-se*. The theory is classically integrable, which is reflected in its global symmetries which form the affine Kac–Moody group E_9. This is the maximally supersymmetric version of the Geroch group of axisymmetric solutions of General Relativity and is conveniently captured by the Bretenlohner–Maison linear system or its extension to maximal supersymmetry [6, 7]. Such formulation relies on a fixed $K(E_9)$ gauge to parametrise the infinity of selfdual scalar fields. For applications to gauged supergravity and dimensional reductions a covariant formulation is necessary, and we present it in the talk based on [8–10]. An Hermitian current $\mathcal{P}$ is defined based on the coset space $\frac{\widehat{E}_8 \rtimes \mathrm{Vir}^-}{K(E_9)}$ which involves half a Virasoro algebra. Then a twisted self-duality constraint can be imposed of the form $\star\mathcal{P} = S_1(\mathcal{P}) + \chi_1$ with an operator shifting the loop number of generators and an auxiliary one-form along the central element. This is covariant and equivalent to the original linear system. An invariant pseudoaction is constructed by combining the integrability of the current $\mathcal{P}$ with a shift operator. Physical actions are extracted by making a choice of $K(E_9)$ gauge (akin to a choice of duality frame) and rewriting the pseudoaction as a physical one plus squares of the selfduality constraint.

In contrast with higher-dimensional models, a systematic construction of gauged supergravity lagrangians based on the embedding tensor formalism [11] has long been beyond reach, because of the complicated representation theory of $K(E_9)$. This makes it impossible to construct supergravity actions by completing the supersymmetry transformations of fermions, which would require to decompose the embedding tensor into $K(E_9)$ modules. The general structure of the bosonic sector of $D = 2$ gauged supergravities was first described in [12] relying on the BM linear system, but without control over supersymmetry the scalar potential of such theories could not be identified. The only complete and supersymmetric construction has been achieved (bypassing a duality covariant formulation) for SO(9) gauged supergravity (and its analytic continuations) [5, 13, 14]. This model arises from consistent truncation of IIA supergravity on S^8 and includes a 1/2 BPS solution lifting to the near-horizon limit of the D0 brane (or its 11d uplift) [15, 16].

In this talk it is first presented how the structure of twisted selfduality and pseudoaction lends itself immediately to gauging, by covaraintsation of the same structures. It is then explained how the very same system can be applied to formulate the dynamics of E_9 exceptional field theory [10, 17, 18] – a rewriting of 11d and IIB supergravity in a formally E_9 covariant way, apt at carrying out consistent Kaluza–Klein truncations. Fields depend on ten or eleven coordinates but are arranged as the fields of 2d supergravity. Gauge symmetries along the internal space are captured in this formalism by a generalised Lie derivative, and consistent truncations are based on factorising the internal dependence in terms of an E_9 valued *twist matrix*, subject to a differential constraint based on the generalised Lie derivative. These so-called *generalised Scherk–Schwarz reductions* capture truncations to gauged maximal supergravities in higher dimensions as well [19, 20] and have only recently been formulated for reductions to $D = 2$ [21, 22]. With these tools, one can bypass the supersymmetry analysis and compute the scalar potential of any $D = 2$ gauged maximal supergravity admitting a geometric uplift. The embedding tensor is an element θ_M in the basic representation of E_9 [12] and one arrives at the deceivingly simple expression

$$V = \frac{1}{2\rho^3}\,\theta_M\theta_N\,\mathcal{M}^{MN} + \frac{1}{2\rho}\,[\eta_{-2}]^M{}_P{}^N{}_Q\,\theta_M\theta_N\,\mathcal{M}^{PQ} \tag{1}$$

with $[\eta_{-2}]^M{}_P{}^N{}_Q$ (proportional to) the level 2 coset Virasoro generator L^{coset}_{-2}, M^{MN} a 'generalised metric' parametrising the scalar manifold and ρ the 2d dilaton.

With these tools at hand, the consistent truncation of IIA supergravity in S^8 (or 11d supergravity on $S^8 \times S^1$) to SO(9) gauged supergravity is proved explicitly and complete uplift formulæ presented [21, 23]. Furthermore a plethora of new gauged supergravities and their associated uplift geometries can now be analysed. In fact, a full analysis is still missing of the conditions for a given 2d gauging to admit an uplift. However it can be reasonably expected to work along the same lines as their higher dimensional counterparts [24, 25] and this is also briefly sketched in the talk, showing that the uplift requirements reduce to algebraic conditions on the embedding tensor itself.

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Spinors, bilinears and dualities

Axel Kleinschmidt

(joint work with Guillaume Bossard and Ergin Sezgin)

Maximal supergravity or M-theory has been conjectured to exhibit infinite-dimensional Kac–Moody symmetries. This has been shown for the toroidal reduction of $D = 11$ supergravity to $D = 2$ dimensions where an affine E_9 (affine Kac–Moody extension of the split real Lie group E_8) appears as a global symmetry [1,2]. The much more far-reaching conjectures of [3] and [4] concern the potential role that the hyperbolic E_{10} or the Lorentzian E_{11} may play, respectively, in the full, uncompactified theory. In a related development, called Exceptional Field Theory

(ExFT) [5, 6] such symmetries have been used to derive families of theories that are written with an action of the E_n symmetries but without the claim that a given theory on its own (like $D = 11$ supergravity) is invariant under E_n. Such theories have been constructed in particular for E_9 and E_{11} [7, 8].

As the more fundamental conjectures and ExFT concern *supersymmetric* theories, it is necessary to include *spinor* fields into the construction. The way the symmetries are thought to be realised, the group acting on spinors is a cover of the subgroup $K(E_n)$ of the exceptional group E_n, where $K(E_n)$ is defined using an anti-involution on E_n that derives from a Cartan–Chevalley type involution at the level of the Lie algebra. The analogy to keep in mind is that for $SL(n, \mathbb{R})$ has the maximal subgroup $K(SL(n, \mathbb{R})) = SO(n)$ defined using transposition and spinors are known to transform in representations of the spin cover $Spin(n)$ of $SO(n)$.

In the Kac–Moody case, the Lie algebra $\mathfrak{k} = \mathrm{Lie}\, K(E_n)$ for $n \geq 9$ is an infinite-dimensional Lie algebra, however, it falls outside the class of Kac–Moody algebras, thus making many of the standard tools for studying its representation theory unavailable. It has been shown that *finite-dimensional* spinorial (a.k.a. fermionic) representations of $\mathfrak{k}$ exist [9–11], including the lifting to the covering group [12, 13]. The existence of such unfaithful representations means that the Lie algebra $\mathfrak{k}$ is not simple but has Lie algebra ideals of finite co-dimension.

In this talk, based loosely on [14], representation-theoretic constraints implied by the way the spinorial representations have to appear in a supersymmetric theory with E_n symmetry are discussed. There are three principal constraints on tensor products of spinorial representations, where two different types of spinor representations play a role. The first one, called spin-1/2 and denoted ϵ, is associated with the supersymmetry parameter of the theory, while the second one, called spin-3/2 and denoted Ψ, is associated with the propagating fermionic fields of the theory.

Since supersymmetry relates bosons and fermions, we also have to consider the representations carried by the bosonic fields. There are bosonic fields associated with the orthogonal complement $\mathfrak{p}$ of $\mathfrak{k}$ in $\mathrm{Lie}\, E_n$ and the first constraint, arising from their supersymmetry transformation, is that

(RC I) $\quad \epsilon \otimes \Psi$ must be a quotient of $\mathfrak{p}$ or representation extension of it

The extension is relevant and associated with Virasoro-type generators [15, 16]. It has only been proved for the affine case E_9.

The second constraint, arising from closure of supersymmetry into (generalised) diffeomorphisms, is

(RC II) $\quad \epsilon \otimes \epsilon$ must be a quotient of $R(\Lambda_1)$,

where $R(\Lambda_1)$ is a highest-weight representation of $\mathrm{Lie}\, E_n$ associated with a certain fundamental weight. This has been shown in the affine case [17] with evidence in the Kac–Moody case [14, 18].

The third constraint, arising from the structure of the twisted self-duality equations, is

(RC III) $\Psi \otimes \Psi$ must be a quotient of FS,

where FS denotes the field-strength representation of the bosonic fields constructed for E_{11} in [19]. It is neither highest nor lowest weight and evidence for the fulfilment of this constraint was given in [14].

General proofs that these constraints can be satisfied are open problems and their investigation can also lead to new insights into representations of $\mathfrak{k}$.

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The spin and string groups in the Kac–Moody context

Ralf Köhl

The spin group in the Kac–Moody context has been postulated by Damour–Hillmann [3], constructed by Ghatei–Horn–K.–Weiß [5], and for simply-laced diagrams (and, more generally, diagrams in which edges only admit odd entries in the generalized Cartan matrix) observed to be topologically simply connected by Harring–K. [6].

This is the group to which the various spin representations constructed in [1], [2], [8], [9] integrate; see also the report by Robin Lautenbacher.

Techniques involving homotopy exact sequences based on Palais' slice theorem [11] as used in [6] allow to prove the following theorem:

Theorem 1. *Let G be a simply-laced split real Kac–Moody group and let $G = KAN$ be the Iwasawa decomposition. Then G and K admit a Whitehead tower which in the case of type E_n for K can be chosen as*

$$\cdots \to \mathrm{String} \to \mathrm{Spin} \to K.$$

Proof. By [6, Proposition 4.9] the two-fold cover

$$\mathrm{Spin} \to K$$

is universal with respect to τ. Hence by [12, Proposition 1.14] and [12, Theorem 2.13] it suffices to observe that K has countable homotopy groups. In case of type E_n, additionally Theorem 4 below applies. □

A recurring theme in this note will be fibrations of the form

$$K_J \to K \to K/K_J$$

where $K_J := K \cap P_J$ for a spherical J-parabolic subgroup P_J of G.

Proposition 2. *K and* Spin *both admit a CW decomposition.*

Proof. By [6, Proposition 3.7] the building admits a CW decomposition

$$K/(T \cap K) \cong G/P_\emptyset = G/B = \bigsqcup_{w \in W} BwB/B,$$

where the homeomorphism $K/(T \cap K) \cong G/B$ follows from [6, Lemma 4.1]. Since $T \cap K$ is finite ([4, Lemma 3.26]), the surjections $\mathrm{Spin} \to K \to K/(T \cap K)$ in fact are coverings onto a CW complex, yielding CW decompositions of K and Spin. □

Theorem 3. *Let $n \geq 8$ and let $K(E_n)$ be the maximal compact subgroup of type E_n. Then for $1 \leq k \leq 6$ one has*

$$\pi_k(K(E_n)) = \pi_k(\mathrm{SO}(16)).$$

Proof. One has $\mathrm{SO}(9)/\mathrm{SO}(8) \cong \mathbb{S}^8$. Considering $\mathrm{SO}(n)$ as the maximal compact subgroup of $\mathrm{SL}(n,\mathbb{R})$, one obtains $K(A_8)/K(A_7) \cong \mathbb{S}^8$. Comparing cells as in [10, p. 173] yields identical cell decompositions in $K(E_9)/K(E_8)$ in low dimension (in the sense of [6, Proposition 3.7]); hence in low dimensions one has

$$\pi_k(K(E_9)/K(E_8)) = \pi_k(\mathbb{S}^8) = \{1\}.$$

Palais' slice theorem [11] yields the homotopy exact sequence

$$\{1\} = \pi_{k+1}(K(E_9)/K(E_8)) \to \pi_k(K(E_8)) \to \pi_k(K(E_9)) \to \pi_k(K(E_9)/K(E_8)) = \{1\}$$

from which the assertion of the theorem follows for $n = 9$. A straightforward induction on n finishes the proof. □

Theorem 4. *For $n \geq 8$ there exists a topological group $\mathrm{String}(E_n)$ with vanishing homotopy up to and including dimension 6 and homotopy identical to $\mathrm{Spin}(E_n)$ beyond that admitting a quotient map $\mathrm{String}(E_n) \to \mathrm{Spin}(E_n)$.*

The proof follows the strategy of [13, Proof of Theorem 5.1], reproduced here for the reader's convenience as a series of the following results:

Proposition 5. *Let H be an infinite-dimensional separable complex Hilbert space and define $\mathrm{PU}(H) := \mathrm{U}(H)/\{\lambda \cdot \mathrm{id} \mid \lambda \in \mathrm{U}_1(\mathbb{C})\}$ with respect to the norm topology on the unitary group $\mathrm{U}(H)$. Then $\mathrm{U}_1(\mathbb{C}) \to \mathrm{U}(H) \to \mathrm{PU}(H)$ is locally trivial, and the homotopy of $\mathrm{PU}(H)$ is trivial with the exception of $\pi_2(\mathrm{PU}(H)) = \mathbb{Z}$.*

Proof. The norm topology turns $\mathrm{U}(H)$ into a contractible group (since H is infinite-dimensional). The embedded circle group $\mathrm{U}_1(\mathbb{C})$ carries the Lie topology, and hence has trivial homotopy with the exception of $\pi_1(\mathrm{U}_1(\mathbb{C})) = \mathbb{Z}$. By Palais' slice theorem [11] the fiber bundle $\mathrm{U}_1(\mathbb{C}) \to \mathrm{U}(H) \to \mathrm{PU}(H)$ is locally trivial. The resulting homotopy exact sequence

$$\{1\} = \pi_2(\mathrm{U}(H)) \to \pi_2(\mathrm{PU}(H)) \to \pi_1(\mathrm{U}_1(\mathbb{C})) \to \pi_1(\mathrm{U}(H)) = \{1\}$$

completes the proof. □

Corollary 6. *Let $\mathrm{EPU}(H) \to \mathrm{BPU}(H)$ be a universal $\mathrm{PU}(H)$-bundle (in the sense of, e.g., [7, Theorem 4.11.2]). Then $\mathrm{BPU}(H)$ is an Eilenberg–MacLane space $\mathrm{K}(\mathbb{Z},3)$.*

Proof. The locally trivial fiber bundle $\mathrm{PU}(H) \to \mathrm{EPU}(H) \to \mathrm{BPU}(H)$ induces the homotopy exact sequence

$$\{1\} = \pi_3(\mathrm{EPU}(H)) \to \pi_3(\mathrm{BPU}(H)) \to \pi_2(\mathrm{PU}(H)) \to \pi_2(\mathrm{EPU}(H)) = \{1\},$$

and the claim follows from Proposition 5. □

Corollary 7. *The isomorphism classes of principal $\mathrm{PU}(H)$-bundles over $\mathrm{Spin}(E_n)$ are in one-to-one correspondence to the elements of $H^3(\mathrm{Spin}(E_n),\mathbb{Z})$ (the so-called characteristic classes).*

Proof. This follows from the classification theorem of principal bundles (see [14, Theorem 14.4.1], also [7, Theorem 4.13.1]) combined with the interplay between Eilenberg–MacLane spaces and cohomology (see [14, Theorem 17.5.1]). Note here that $\mathrm{Spin}(E_n)$ is paracompact (finite covers of k_ω-spaces are k_ω) and admits a CW decomposition by Proposition 2; moreover, $\mathrm{BPU}(H)$ is an Eilenberg–Maclane space $\mathrm{K}(\mathbb{Z}, 3)$ by the preceding corollary. □

Proof of Theorem 4. We follow [13, Proof of Theorem 5.1]. Consider the principal $\mathrm{PU}(H)$-bundle $P \to \mathrm{Spin}(E_n)$ corresponding to a generator of $H^3(\mathrm{Spin}(E_n), \mathbb{Z}) \cong \mathbb{Z}$. Moreover, let $\mathrm{Aut}(P)$ be the group of $\mathrm{PU}(H)$-equivariant homeomorphisms $P \to P$. Each equivariant homeomorphism f of P induces a homeomorphism $\tilde{f}$ of the base space $\mathrm{Spin}(E_n)$, yielding a continuous group homomorphism $\pi : \mathrm{Aut}(P) \to \mathrm{Homeo}(\mathrm{Spin}(E_n))$. Define

$$\mathrm{String}(E_n) := \{f \in \mathrm{Aut}(P) : \pi(f) \in \mathrm{Spin}(E_n) \subset \mathrm{Homeo}(\mathrm{Spin}(E_n))\},$$

considering the embedding $\mathrm{Spin}(E_n) \subset \mathrm{Homeo}(\mathrm{Spin}(E_n))$ given by right multiplication $g \mapsto \{x \mapsto xg\}$. This yields the gauge bundle

$$\{1\} \to \mathrm{Gauge}(P) \to \mathrm{String}(E_n) \to \mathrm{Spin}(E_n) \to \{1\},$$

killing π_3 by [13, Lemma 5.6]. □

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$\mathfrak{k}$-structure of basic representation of affine algebras

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Kac-Moody algebras are by definition complex, but allow for different real slices from real forms. One of these real forms is the split real form in which the Chevalley-Serre generators are real. The split real Kac-Moody algebra is equipped with a Cartan-Chevalley involution that defines the maximal compact subalgebra of the Kac-Moody algebra as its fixed point subalgebra. For finite split real Kac-Moody algebras the maximal compact subalgebra is also a Kac-Moody algebra and its representation theory is very well understood. However, for infinite dimensional Kac-Moody algebras, the maximal compact subalgebra is not of Kac-Moody type and the construction of its representations requires novel methods.

The (split real) Kac-Moody algebras posses standard highest weight representations and it is a natural question how these representations 'decompose' under the maximal compact subalgebra. While this is very well known for finite Kac-Moody algebras, this was a longstanding problem for infinite dimensional algebras, due to the much more complicated structure of the maximal Kac-Moody algebra.

In this presentation we achieve the first result in this direction: we develop a novel relation between the basic representation of split real simply-laced affine Kac-Moody algebras and finite dimensional representations of its maximal compact subalgebra $\mathfrak{k}$. We provide infinitely many $\mathfrak{k}$-subrepresentations of the basic representation and we prove that these are all the finite dimensional $\mathfrak{k}$-subrepresentations of the basic representation, such that the quotient of the basic representation by the subrepresentation is a finite dimensional representation of a certain parabolic algebra and of the maximal compact subalgebra. By this result we provide an infinite composition series with a cosocle filtration of the basic representation. Finally, we present examples of the results and applications to supergravity.

These results find fascinating applications in mathematics and physics. In physics, the results solve the representation theoretical difficulties underlying gauged two dimensional supergravity. In mathematics, this work is a starting point for further considerations. First, we expect the construction and results to extend to the basic representation of non-simply-laced affine Kac-Moody algebras. Second, in a more sophisticated development, the construction may be generalized to all highest-weight representations of affine Kac-Moody algebras. Third, understanding the $\mathfrak{k}$-subrepresentations of highest weight representations of an affine Kac-Moody algebra paves the way to understand hyperbolic Kac-Moody algebras under the action of its maximal compact subalgebra. This is of particular interest since hyperbolic Kac-Moody algebras are expected to be the underlying mathematical structure of spacetime.

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A Weyl group-based perspective on the higher spin representations of $\mathfrak{k}$

Robin Lautenbacher
(joint work with Ralf Köhl)

The involutory subalgebra $\mathfrak{k}(A)$ w.r.t. the Chevalley involution of a split-real Kac-Moody algebra $\mathfrak{g}(A)$ (cp. [7]) is typically referred to as its *maximal compact subalgebra*. If A is a generalized Cartan matrix of finite type, $\mathfrak{g}(A)$ is a semi-simple Lie algebra and $\mathfrak{k}(A)$ is indeed its maximal compact subalgebra. If A is not of finite type, then both $\mathfrak{g}(A)$ and $\mathfrak{k}(A)$ are infinite-dimensional and $\mathfrak{k}(A)$ admits an invariant, negative definite bilinear form, but it is not compact in a topological sense, i.e., it is not the Lie algebra of a compact Lie group. There are at least three reasons to study the representations of $\mathfrak{k}(A)$: First, its representations and among these in particular the finite-dimensional ones reveal parts of the structure theory of $\mathfrak{k}(A)$. Second, its finite-dimensional representations occur as a symmetry in theories of quantum gravity and therefore a well-developed representation theory of $\mathfrak{k}(A)$ is required there. Third, the representation theory of $\mathfrak{k}(A)$ is expected to be important to the theory of Kac-Moody symmetric spaces, similar to the finite-dimensional case.

An early result concerning the structure theory of $\mathfrak{k}(A)$ is a presentation by generators and relations given in [1]. The major challenge in the study of $\mathfrak{k}(A)$ is that it is in general not graded by a root system as $\mathfrak{g}(A)$ is, but only admits a filtered structure w.r.t. the roots of $\mathfrak{g}(A)$. In particular, $\mathfrak{k}(A)$ is not a simple Kac-Moody algebra or sum thereof if A is not of finite type, and as a consequence the standard tools of representation theory such as highest weight representations and character formulas are not applicable. It was observed in [10] that the $\mathfrak{k}(E_n)$-series can be characterized as the quotient of a generalized intersection matrix algebra (cp. [14]) but the representation theory of these is also rather poorly understood. It is not obvious that $\mathfrak{k}(A)$ even possesses finite-dimensional representations if A is not of finite type, but of course, these provide interesting ideals of $\mathfrak{k}(A)$. At some point it may be possible to characterize $\mathfrak{k}(A)$ as the co-limit of ideals of finite-dimensional representations. For the affine case, this has been shown in [10] and the ideal structure of affine $\mathfrak{k}(A)$ has been used in [11] to construct a cosocle filtration for the basic representation of $\mathfrak{g}(A)$ restricted to $\mathfrak{k}(A)$.

Concerning the case that A is an indefinite generalized Cartan matrix, there are currently four *elementary* representations known which are the talk's subject. The basic one has been first described in the physics literature ([3] and [2]) under the name of the $K(E_{10})$-Dirac spinor. It has been studied in a mathematical setting and generalized to arbitrary symmetrizable types A in [6], where they were referred to as generalized spin representations. Both names, Dirac spinor as well as generalized spin representation, stem from the fact that the first and most important example is the representation of $\mathfrak{k}(E_{10})$ which extends the standard spinor representation of its naturally contained $\mathfrak{so}(10)$-subalgebra.

The so-called higher spin representations $\mathcal{S}_{\frac{3}{2}}$, $\mathcal{S}_{\frac{5}{2}}$, and $\mathcal{S}_{\frac{7}{2}}$ of $\mathfrak{k}(A)$ with the exception of $\mathcal{S}_{\frac{3}{2}}$ were introduced first in [8], again in a physics setting. In [12], we had derived a coordinate-free formulation of $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{5}{2}}$ in the setting of simply-laced A but a similar formulation for $\mathcal{S}_{\frac{7}{2}}$ remained elusive back then. However, it became apparent that the Weyl group plays a central role in this construction, which builds these representations on top of the generalized spin representation $\mathcal{S}_{\frac{1}{2}}$ from [3] and [2]. We provide a detailed construction and description of all these representations including $\mathcal{S}_{\frac{7}{2}}$ and spell out the importance of the Weyl group in this construction as clearly as possible using a mathematical and coordinate-free formulation.

Afterwards, the Weyl group-based perspective is used to analyze these representations. It is shown that $\mathcal{S}_{\frac{3}{2}}$ is irreducible if A is indecomposable, regular and simply-laced and that the image of $\mathfrak{k}(A)$ under this representation is a semi-simple Lie-algebra. Concerning $\mathcal{S}_{\frac{5}{2}}$ it is shown to be completely reducible, to always contain an invariant submodule isomorphic to $\mathcal{S}_{\frac{1}{2}}$ and that its other invariant factors are controlled by the representation theory of $W(A)$, namely how the symmetric product $\mathrm{Sym}^2(\mathfrak{h}^*)$ of the dual Cartan subalgebra decomposes as a $W(A)$-module. As for $\mathcal{S}_{\frac{3}{2}}$, we show that the image of $\mathfrak{k}(A)$ under this representation is semi-simple. We also show that the kernels of some of these representations are not contained in each other and that their tensor product has a smaller kernel than the individual representations.

Eventually, we study the spin representations' lift to the group level. We confirm the common belief (cp. [2,3,8,9,12]) that these representations are spinorial in the sense that they do not lift to the involutory subgroup $K(A) = G(A)^{\theta}$, where $G(A)$ is the minimal split-real Kac-Moody group of type A and θ is its Chevalley involution, but instead lift only to its spin cover $Spin(A)$ introduced in [5]. This belief is plausible if one compares the one-parameter subgroups induced by $\exp(\phi\sigma(X_i))$ and $\exp(\phi\mathrm{ad}(X_i))$, where σ is a spin representation and X_i is a so-called Berman generator of $\mathfrak{k}(A)$. We show that it indeed suffices to look at these one-parameter subgroups. Afterwards, we demonstrate that the spin representations' lift realizes an action of the spin-extended Weyl group from [5] on the modules $\mathcal{S}_{\frac{n}{2}}$. Using that the action of $Spin(A)$ on $\mathfrak{k}(A)$ factors through the adjoint action of $K(A)$ on $\mathfrak{k}(A)$, one is able to derive the representation matrices up to sign of all elements in the $W^{ext}(A)$-orbit of the Berman generators (introduced in [1]), where $W^{ext}(A)$ is the extended Weyl group. This amounts to providing the representation matrices up to sign of all $x \in \mathfrak{k}_\alpha = \mathfrak{k}(A) \cap \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ for α a positive real root.

A point for further research is uniqueness of these representations which has not been studied systematically yet. Another open question is, if there exists a substitute for the Casimir element within the universal enveloping algebra of $\mathfrak{k}(A)$ or a completion thereof. The occurrence of mass terms that are quartic in fermion fields (cp. [4]) could be an indicator that such an element needs to be fourth order in a basis of $\mathfrak{k}(A)$.

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A string-like realization of hyperbolic Kac-Moody algebras

Hannes Malcha and Hermann Nicolai

(joint work with Saverio Capolongo, Axel Kleinschmidt)

The two talks are based on the paper [1]. This is their joint summary.

We propose a new approach to study hyperbolic Kac-Moody algebras, focussing on the rank-3 algebra $\mathfrak{F}$ first studied by Feingold and Frenkel [2]. $\mathfrak{F} \equiv \mathfrak{g}(A)$ is associated with the indefinite Cartan matrix of rank three

$$(A_{ij}) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}. \tag{1}$$

The Kac-Moody algebra based on the Cartan matrix (1) is the simplest hyperbolic Kac-Moody algebra with a null root, and thus admits a distinguished affine subalgebra $A_1^{(1)} \equiv A_1^+$. Although not much is known about $\mathfrak{F}$, the following facts have been established [2]. The 'germ' of the algebra $\mathfrak{F}$ resides in the beginnings of its graded decomposition w.r.t. its distinguished affine subalgebra for levels $|\ell| \leq 1$

$$\overline{\mathrm{V}} \,\oplus\, \mathfrak{F}^{(0)} \,\oplus\, \mathrm{V}\,, \tag{2}$$

where at the center we have the affine subalgebra $\mathfrak{F}^{(0)} \equiv A_1^{(1)} \subset \mathfrak{F}$. At level one, $\mathrm{V} \equiv \mathfrak{F}^{(1)} = L(\boldsymbol{\Lambda}_0 + 2\boldsymbol{\delta})$ is the basic representation, while $\overline{\mathrm{V}} \equiv \mathfrak{F}^{(-1)}$ is the conjugate representation. The algebra $\mathfrak{F}$ can then be generated by multiply commuting V and $\overline{\mathrm{V}}$. This task is, however, complicated enormously by the need to divide out ideals generated by the Serre relations. At level 2, this is still relatively simple, and we have [2]

$$\mathfrak{F}^{(2)} \,\cong\, \mathrm{V} \wedge \mathrm{V} \big/ \mathcal{J}_2 \,\equiv\, \mathfrak{F}^{(1)} \wedge \mathfrak{F}^{(1)} \big/ \mathcal{J}_2\,, \tag{3}$$

where $\mathcal{J}_2$ is the ideal generated by the Serre relation involving the over-extended root with index -1 and $\mathcal{J}_2$ carries an action of the affine algebra $\mathfrak{F}^{(0)}$. The above formula (2) is only the beginning of an infinite string of vector subspaces $\mathfrak{F}^{(\ell)}$ extending in both directions with $\ell \in \mathbb{Z}$, where each subspace $\mathfrak{F}^{(\ell)}$ consists of an infinite sum of affine representation spaces for $|\ell| > 1$. Consequently, the main obstacle towards a more 'global' understanding of $\mathfrak{F}$ is that the procedure of dividing out Serre relations gets more and more cumbersome with higher levels already for levels $\ell = 3$ and $\ell = 4$ [3,4].

As for products of affine representations, it has long been known that [5]

$$L(\boldsymbol{\Lambda}_0 + 2\boldsymbol{\delta}) \wedge L(\boldsymbol{\Lambda}_0 + 2\boldsymbol{\delta}) \,=\, \mathrm{Vir}(\tfrac{1}{2}, \tfrac{1}{2}) \otimes L(2\boldsymbol{\Lambda}_1 + 3\boldsymbol{\delta})\,, \tag{4}$$

where $\mathrm{Vir}(\frac{1}{2}, \frac{1}{2})$ is the minimal representation of the coset Virasoro algebra with central charge $c = \frac{1}{2}$ and $h = \frac{1}{2}$; we recall that such a coset Virasoro algebra always accompanies the product of affine representations [6]. However, in the algebra $\mathfrak{F}$ the nice product structure of the r.h.s. is lost because one has to remove the top state associated with the Serre relation, and thus one whole affine representation $L(2\boldsymbol{\Lambda}_1 + 3\boldsymbol{\delta})$, so that by (3) the level-2 sector of $\mathfrak{F}$ has the vector space structure [2]

$$\mathfrak{F}^{(2)} \,=\, \Big(\mathrm{Vir}(\tfrac{1}{2}, \tfrac{1}{2}) \ominus \mathbb{R}v_0\Big) \otimes L(2\boldsymbol{\Lambda}_1 + 3\boldsymbol{\delta})\,, \tag{5}$$

where v_0 is the vacuum state of the $\mathrm{Vir}(\frac{1}{2}, \frac{1}{2})$ representation. Taking out the subspace $\mathbb{R}v_0$ leaves a 'hole' in the coset Virasoro representation space $\mathrm{Vir}(\frac{1}{2}, \frac{1}{2})$, as a result of which the level-2 sector of the KMA is *not* a representation of the coset Virasoro algebra anymore. Indeed the Virasoro algebra is no longer obeyed on the truncated representation space, a statement which extends to all levels of the Lie algebra $\mathfrak{F}$.

As a consequence, there is no 'easy' way to construct the algebra by simply multiplying affine representations as in (4), and to obtain the Lie algebra elements of a given level-ℓ sector by application of the affine and coset Virasoro raising generators to a given set of ground states that belong to $\mathfrak{F}$. In order to circumvent this difficulty, one main new tool we employ in [1] is to fill the 'holes' by introducing 'virtual states' which belong to the relevant tensor products (corresponding to the l.h.s. of (4)), but vanish as elements of the Lie algebra, in this way restoring the full coset Virasoro representation.

At least for levels $|\ell| \leq 2$ this trick enables us to generate the whole level-ℓ sector by acting with the coset Virasoro algebra and the affine algebra on a finite set of states that we will refer to as 'maximal ground states'. For levels $\ell > 2$ we encounter a vector space structure similar to (5) but with a 'pile-up' of coset Virasoro representations stemming from calculations similar to (4). This pile-up generates infinitely many copies of the finitely many maximal ground states. We call these copies 'Virasoro ground states'. The application of only affine and coset Virasoro raising generators does not allow us to generate these additional Virasoro ground states from the maximal ground states. Hence for level $\ell = 3$ we propose yet another set of operators that does exactly this. We conjecture that there exists a generalization of this operator for all $\ell > 3$. Together with the affine and coset Virasoro raising operators these operators would allow us to generate any level-ℓ sector $\mathfrak{F}^{(\ell)}$ of $\mathfrak{F}$ from the finite set of maximal ground states. An interactive visualization of the associated root systems is presented in [7].

A second new tool we rely on is the vertex operator formalism in the specific version developed in [8, 9], which builds on the seminal work of [10–12]. In this formalism, the Lie algebra is realized as a subspace of a certain Hilbert space of physical string states, such that the elements of the Lie algebra are explicitly given in terms of DDF states built on certain tachyonic ground states, rather than in terms of multi-commutators (the DDF formalism [13] is a well known and convenient tool to generate physical states in string theory). A key feature first pointed out in [8] is that for all levels $|\ell| > 1$, there also appear *longitudinal* DDF states in the algebra, in addition to the transversal DDF states familiar from the critical string. One main advantage of the vertex operator algebra formalism is that we do not have to worry about Jacobi identities and the Serre relations as these are automatically taken care of. That is, unlike in [2, 14, 15] there is no need to take out affine representations 'by hand', subtracting sub-representations and compensating for over-subtractions. Here, we will give explicit expressions for the maximal ground states for $\ell \leq 4$ in terms of the DDF basis. In this way, we seek to develop a perspective on hyperbolic KMAs different from the one usually taken in the mathematics literature, with the aim of gaining a more 'global' understanding of its structure, as well as a more concrete realization of the algebra itself (as opposed to merely counting root multiplicities).

The Virasoro ground states are here determined by imposing the suitable vacuum conditions on a given ansatz in terms of DDF states. With increasing level this method becomes more and more unwieldy. Therefore it would be desirable to determine the maximal ground states by independent and more efficient means. If this can be done, we would have an efficient tool to explore higher levels. On top of unbounded pile-up of coset Virasoro representations described above, there is the added difficulty that in the final product for general level ℓ certain subspaces of affine representations must be taken out, in analogy with (5). The real complication is therefore not so much with products of affine representations but with the ‘holes’ in the coset Virasoro representations, which become more and more difficult to deal with as the level is increased. This proliferation of complications is reminiscent of the fractal structure of a Mandelbrot set, although we know of no Lie algebra analog of the self-similarity features.

Our approach is based on the concrete realization of this Lie algebra in terms of a Hilbert space of transverse and longitudinal physical string states, which are expressed in a basis using DDF operators. When decomposed under its affine subalgebra $A_1^{(1)}$, the algebra $\mathfrak{F}$ decomposes into an infinite sum of affine representation spaces of $A_1^{(1)}$ for all levels $\ell \in \mathbb{Z}$. For $|\ell| > 1$ there appear in addition coset Virasoro representations for all minimal models of central charge $c < 1$, but the different level-ℓ sectors of $\mathfrak{F}$ do not form proper representations of these because they are incompletely realized in $\mathfrak{F}$. To get around this problem, we propose to nevertheless exploit the coset Virasoro algebra for each level by identifying for each level a finite set of ‘maximal ground states’ that are not necessarily elements of $\mathfrak{F}$ (in which case we refer to them as ‘virtual’), but from which the level-ℓ sectors of $\mathfrak{F}$ can be fully generated by the joint action of affine and coset Virasoro raising operators.

Our results hint at an intriguing but so far elusive secret behind Einstein’s theory of gravity, with possibly important implications for quantum cosmology. More specifically, from the Cartan matrix (1) we see that $\mathfrak{F}$ possesses two distinguished rank-two subalgebras, both of which appear in the dimensional reduction of Einstein’s theory to lower dimensions. Namely, the upper 2-by-2 submatrix corresponding to a $\mathfrak{sl}(3)$ subalgebra is associated with the Matzner-Misner SL(3) (actually GL(3)) group obtained by reducing Einstein gravity from four to one dimension.

On the other hand, the lower 2-by-2 submatrix is associated with an $A_1^{(1)}$ affine symmetry, which is just the Lie algebra underlying the Geroch group of general relativity, obtained by reducing Einstein’s theory to two dimensions [16, 17]. The lower-most diagonal entry corresponds to the Ehlers $\mathfrak{sl}(2)$ symmetry obtained by dualizing the Kaluza-Klein vector in three dimensions. The Geroch algebra and the Matzner-Misner $\mathfrak{sl}(3)$ intersect in the middle entry corresponding to the Matzner-Misner SL(2), which likewise has been known for a long time in general relativity.

All this suggests that one might try to find a concrete physical realization of $\mathfrak{F}$ by simply combining the Matzner-Misner SL(3) and the Geroch symmetry [18]. However, it turns out that a simple dimensional reduction to one dimension cannot accomplish this because to realize the Geroch group, we need *two* coordinates for the duality transformations ([18] tried to circumvent this problem by means of a null reduction, but again finds that the bulk of $\mathfrak{F}$ is realized only trivially; see also [19] for a recent related investigation). The conclusion is that we cannot find a non-trivial realization of $\mathfrak{F}$ by sticking with Einstein’s theory and standard notions of space-time based field theory, but need an extension from which standard general relativity ‘emerges’ only in a specific limit. Hints of such a theory have emerged from the study of cosmological billiards [20]. In particular, the celebrated

BKL analysis [21] of cosmological singularities can be rephrased in terms of a cosmological billiard that takes place in the Weyl chamber of the Weyl group of $\mathfrak{F}$ (it is a main result of [2] that the even part of the Weyl group for $\mathfrak{F}$ is the modular group $\mathrm{PSL}_2(\mathbb{Z})$).

In view of the compelling links with Einstein gravity on the one hand [20] and the horrendous complexity of $\mathfrak{F}$ on the other, one may also ask about possible implications for Big Bang cosmology. From a physics perspective, the pile-up of truncated Virasoro modules with increasing level may indicate that more and more degrees of freedom 'open up' in the approach towards the cosmological singularity. It is for this reason that [22] conjectured the emergence of a mathematically well-defined notion of non-computability towards the singularity which may thwart attempts at mathematically understanding the beginning of time, unless a more 'global' description of $\mathfrak{F}$ can be found. At the very least this shows that the restriction to finitely many degrees of freedom that underlies most investigations in quantum cosmology (*e.g.* by means of a mini-superspace approximation, where keeping only diagonal metric degrees of freedom would correspond to restricting $\mathfrak{F}$ to its Cartan subalgebra) may be far too naïve to understand the quantum origin of our universe.

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Kinematical Lie algebras on the light front

Sucheta Majumdar

Kinematical Lie algebras describe the algebraic structures of symmetry transformations that govern spacetime kinematics and evolution of physical systems. Starting with basic assumptions of isotropy of space, rotational and translational invariance, and invariance under inertial transformations, one can classify all possible kinematical algebras relevant to various physical systems, such as the Poincaré algebra for spacetimes with zero curvature [1]. Of particular interest are the kinematical Lie algebras arising from an Inönü-Wigner contraction of the Poincaré algebra – the Galilei algebra obtained from a 'non-relativistic' speed of light, $c \to \infty$ limit of Poincaré algebra and the Carroll algebra which follows from the 'ultra-relativistic' $c \to 0$ limit . More generally, a $(n+1)$-dimensional kinematical Lie algebra is characterized by the generators $\mathcal{K} = \{L_{ab}, B_a, P_a, H\}$, where L_{ab} spans an $\mathfrak{so}(n)$ algebra, P_a and B_a are vectors under $\mathfrak{so}(n)$, and H is a scalar. The specific kinematical Lie algebras are defined through their commutation relations [2]. For example, the Galilei algebra, which governs spacetime symmetries in non-relativistic systems, is defined by $[B_a, B_b] = 0$ and $[B_a, H] = -P_a$. The Carroll algebra, on the other hand, defined by the relations $[B_a, B_b] = 0$ and $[B_a, P_b] = \delta_{ab} H$, has been linked to symmetries of null hypersurfaces, the BMS symmetry of gravity, and spacelike singularities in cosmology, among other interesting physical systems

The Galilei algebra admits a central extension, called the Bargmann algebra, featuring an additional commutation relation $[B_a, P_b] = \delta_{ab} Z$, with Z being the central element. Although not derived through the group contraction method, the Bargmann algebra plays a crucial role in unifying the Carroll and Galilei algebras, as this algebra may also be obtained through an extension-by-derivation of the Carroll algebra [3]. These Lie algebras may be promoted to Lie groups and associated with homogeneous spaces, leading to interesting kinematical spacetimes, often referred to as 'non-Lorentzian spacetimes'. Geometrically, Bargmann spacetimes encompass both Galilean and Carrollian spacetimes: the former emerges via null reduction à la Kaluza-Klein, while the latter may be viewed as an embedded null hyperplane [4].

Motivated by recent advances in non-Lorentzian physics, we revisit the light-cone or light-front formulation of quantum field theories and its connections to non-Lorentzian symmetries of the Bargmann, Galilei and Carroll types. Dirac, in his pioneering work on Hamiltonian dynamics [5], proposed treating one of the null directions, $x^{\pm} \sim (t \pm z)$ along the light-cone as the time coordinate for describing the evolution of relativistic theories. The double-null nature of the light-cone coordinates corresponds to a $\mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}$ split of the $(d+1)$ Minkowski spacetime. In this framework, the Minkowski spacetime exhibits a flat Bargmann structure, which combined with the choice of light-cone time, gives rise to a rich array of interesting kinematical Lie algebras within its Poincaré algebra. Depending on which light-cone coordinate is taken to be the 'Newtonian' time, one can identify two distinct sets of Galilei, Carroll, and Bargmann subalgebras within the Poincaré algebra [6]. Thus, in a $(d+1)$-dimensional spacetime, the light-cone Poincaré algebra $\mathfrak{p}$, with $\dim(\mathfrak{p}) = \frac{(d+1)(d+2)}{2}$ generators, contains d-dimensional subalgebras $(\mathfrak{b}_{\pm}, \mathfrak{c}_{\pm}, \mathfrak{g}_{\pm})$, which are mapped into one another upon exchanging x^+ with x^-. The group dimensions are $\dim(\mathfrak{b}_{\pm}) = \frac{d(d+1)}{2} + 1$ and $\dim(\mathfrak{g}_{\pm}) = \dim(\mathfrak{c}_{\pm}) = \frac{d(d+1)}{2}$. We further explore certain aspects of field theories defined on a light front, emphasizing their connections to the hypersurface deformation algebra associated with Carrollian geometries [7].

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From Kac–Moody algebras to Kac–Moody groups and back again

Timothée Marquis

The purpose of this talk is to explain how one can construct a Kac–Moody group attached to a Kac–Moody algebra and, once this is done, how one can "parametrise" the elements of the group.

The setting is as follows. Let $A = (a_{ij})_{i,j \in I}$ be a generalised Cartan matrix (say for simplicity of the presentation that det $A \mathfrak{n} eq 0$, although this does not play any fundamental role in what follows), and fix a base field $\mathbb{K}$ of characteristic zero, say $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathfrak{g}_{A,\mathbb{K}}$ be the Kac-Moody algebra of type A over $\mathbb{K}$, with

Chevalley generators e_i, f_i $(i \in I)$ and simple coroots $h_i := [e_i, f_i]$. Then $\mathfrak{g}_{A,\mathbb{K}}$ has a root space decomposition

$$\mathfrak{g}_{A,\mathbb{K}} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$$

with respect to the adjoint action of the Cartan subalgebra $\mathfrak{h} = \sum_{i\in I} \mathbb{K}h_i$, with set of roots $\Delta \subseteq \mathfrak{h}^*$. We write $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) for the subalgebra of $\mathfrak{g}_{A,\mathbb{K}}$ generated by the e_i (resp. f_i), corresponding to the set Δ^+ of positive roots (resp. Δ^- of negative roots). We let $\Pi = \{\alpha_i \mid i \in I\}$ be the set of simple roots, defined by $\alpha_j(h_i) = a_{ij}$. Let also

$$W = \langle s_i : \mathfrak{h}^* \to \mathfrak{h}^* : \lambda \mapsto \lambda - \lambda(h_i)\alpha_i \mid i \in I\rangle \leq \mathrm{GL}(\mathfrak{h}^*)$$

be the Weyl group of A. Finally, let $\Delta^{\mathrm{re}} := W\Pi$ be the set of real roots and set $\Delta^{\mathrm{re}+} := \Delta^{\mathrm{re}} \cap \Delta^+$.

Consider a representation $\pi : \mathfrak{g}_{A,\mathbb{K}} \to \mathrm{End}(V^\pi)$ of $\mathfrak{g}_{A,\mathbb{K}}$ on a $\mathbb{K}$-vector space V^π such that π is *integrable*, in the sense that for every $v \in V^\pi$, there exists $N \in \mathbb{N}$ such that $\pi(e_i)^N v = 0 = \pi(f_i)^N v$ for all $i \in I$. This is for instance the case for the adjoint representation $(\pi, V^\pi) = (\mathrm{ad}, \mathfrak{g}_{A,\mathbb{K}})$, or for irreducible highest-weight representations $(\pi, V^\pi) = (\pi_\lambda, V^\lambda)$ with highest weight λ satisfying $\lambda(h_i) \in \mathbb{N}$ for all $i \in I$, or for direct sums of such representations.

In case A is of type A_{n-1}, the algebra $\mathfrak{g}_{A,\mathbb{K}}$ is isomorphic to $\mathfrak{sl}_n(\mathbb{K})$; letting $\pi : \mathfrak{g}_{A,\mathbb{K}} \to \mathrm{End}(\mathbb{K}^n)$ denote the corresponding matrix representation, we then obtain a corresponding group $G^\pi_A(\mathbb{K}) = \mathrm{SL}_n(\mathbb{K})$ as the group

$$G^\pi_A(\mathbb{K}) = \left\langle \exp(\pi(x)) = \sum_{r\geq 0} \pi(x)^n/n! \;\middle|\; x \in \mathfrak{g}_{A,\mathbb{K}} \right\rangle \leq \mathrm{Aut}(\mathbb{K}^n).$$

If we try to apply the same procedure with an integrable representation (π, V^π) of a general $\mathfrak{g}_{A,\mathbb{K}}$, say $(\pi, V^\pi) = (\mathrm{ad}, \mathfrak{g}_{A,\mathbb{K}})$, we run into the following problem: given a nonzero element $x \in \mathfrak{g}_\alpha$ for some $\alpha \in \Delta$, the sum $\exp(\mathrm{ad}(x))y = \sum_{n\geq 0} \frac{\mathrm{ad}(x)^n}{n!}y$ is finite for every $y \in \mathfrak{g}_{A,\mathbb{K}}$ if and only if α is a real root. In other words, $\exp(\mathrm{ad}(x))$ is a well-defined automorphism of $\mathfrak{g}_{A,\mathbb{K}}$ if and only if $\alpha \in \Delta^{\mathrm{re}}$. Note that the Cartan subalgebra can also be integrated to a **torus** $T := \mathrm{Hom}_{\mathrm{gr}}(\mathfrak{h}^*, \mathbb{K}^\times)$ acting diagonally on V^π: for instance, in the adjoint representation, $t \cdot x_\alpha = t(\alpha)x_\alpha$ for all $t \in T$ and $x_\alpha \in \mathfrak{g}_\alpha$ $(\alpha \in \Delta \cup \{0\})$, and for $h \in \mathfrak{h}$ the exponential $\exp(\mathrm{ad}h)$ corresponds to the torus element t mapping α to $e^{\alpha(h)}$. For an integrable representation (π, V^π) of $\mathfrak{g}_{A,\mathbb{K}}$, we then define the associated (minimal) **Kac–Moody group**

$$G^\pi_A(\mathbb{K}) := \langle T, \exp(\pi(x)) \mid x \in \mathfrak{g}_\alpha,\ \alpha \in \Delta^{\mathrm{re}}\rangle \leq \mathrm{Aut}(V^\pi)$$

over $\mathbb{K}$. Note that one can show that the group $G^\pi_A(\mathbb{K})$ is in fact independent of the choice of representation π (up to a difference in the torus, and with obvious nontriviality assumptions on π).

Now that we constructed a Kac–Moody group $G = G^\pi_A(\mathbb{K})$, the next question we will address is whether we can write any element g of G in a unique, standard way. For this, we need additional notations. For each $\alpha \in \Delta^{re}$, define the **root**

group $U_\alpha := \exp(\pi(\mathfrak{g}_\alpha)) \cong (\mathbb{K}, +)$, and set $U^+ := \langle U_\alpha \mid \alpha \in \Delta^{\mathrm{re}+}\rangle \leq G$ and $B^+ := TU^+$. For each $i \in I$, set $\widetilde{s}_i := \exp(\pi(f_i))\exp(-\pi(e_i))\exp(\pi(f_i)) \in G$ and $N := \langle T,\ \widetilde{s}_i \mid i \in I\rangle \leq G$. [For instance, if $G = \mathrm{SL}_n(\mathbb{K})$, then T is the subgroup of diagonal matrices, U^+ is the subgroup of upper triangular matrices with $1's$ on the diagonal, B^+ is the subgroup of upper triangular matrices, N the subgroup of monomial matrices and $W \cong \mathrm{Sym}(n)$ can be identified with the permutation matrices.]

Recall that we have a surjective group morphism $p : N \to W : \widetilde{s}_i \mapsto s_i$ with kernel T. In particular, $W \cong N/T$, and for each $w \in W$ it makes sense to define the subset $wT := nT$ of G for some choice of $n \in p^{-1}(w)$. Then G has a **Bruhat decomposition**

$$G = \coprod_{w\in W} B^+ wB^+ = \coprod_{w\in W} U_w wTU^+ \tag{1}$$

where each element u_w of the group $U_w := \langle U_\alpha \mid \alpha \in \Phi_w\rangle$ can be uniquely written as a product

$$u_w = \prod_{\alpha\in\Phi_w} u_\alpha \quad (u_\alpha \in U_\alpha) \tag{2}$$

parametrised by the finite set of real roots $\Phi_w = \{\alpha \in \Delta^+ \mid w^{-1}\alpha \in \Delta^-\}$ (on which some total order is fixed). Moreover, there is uniqueness of writing on the right-hand side of (1): each element $g \in G$ can be written in a unique way as a product

$$g = u_w wtu_+ \quad \text{for some } u_w \in U_w,\ w \in W,\ t \in T \text{ and } u_+ \in U^+. \tag{3}$$

In view of (3) and (2), our initial question then boils down to finding "standard" decompositions for the elements of U^+. To this latter question, we propose two different answers.

First, we can go to a completion of U^+: allowing infinite sums on the positive side of $\mathfrak{g}_{A,\mathbb{K}}$ by considering its positive completion $\widehat{\mathfrak{g}}_{A,\mathbb{K}} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \prod_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$, we can also exponentiate the positive imaginary root spaces of $\mathfrak{g}_{A,\mathbb{K}}$. More precisely, we fix a basis $\mathcal{B}$ of $\mathfrak{n}^+$ consisting of homogeneous elements $x_\alpha \in \mathfrak{g}_\alpha$ ($\alpha \in \Delta^+$) and we set $n_{x_\alpha} := \mathrm{ht}(\alpha)$. Fix a total order on $\mathcal{B}$ such that $n_x < n_y \implies x < y$, and set

$$U_A^{ma+}(\mathbb{K}) := \Big\{ \prod_{x\in\mathcal{B}} \exp(\mathrm{ad}(\lambda_x x)) \ \Big|\ \lambda_x \in \mathbb{K}\Big\} \subseteq \mathrm{Aut}(\widehat{\mathfrak{g}}_{A,\mathbb{K}}),$$

where the products are given in the prescribed order on $\mathcal{B}$. Then one can show that $U_A^{ma+}(\mathbb{K})$ is a subgroup (thus containing U^+: it is actually the completion of U^+ with respect to the topology with basis of identity neighbourhoods of 1_G the subgroups $U_n^{ma} := \{\prod_{n_x\geq n} \exp(\mathrm{ad}(\lambda_x x)) \mid \lambda_x \in \mathbb{K}\}$ for $n \in \mathbb{N}$). Moreover, each $u \in U_A^{ma+}(\mathbb{K})$ can be written in a unique way as a product

$$u = \prod_{x\in\mathcal{B}} \exp(\mathrm{ad}(\lambda_x x)) \quad \text{for some } \lambda_x \in \mathbb{K}. \tag{4}$$

We refer to [2] for more details.

For the second answer we propose, we have to assume that $\mathbb{K} = \mathbb{Q}$ and that $a_{ij}a_{ji} \leq 3$ for all $i \neq j$. In this case, let $(\Gamma_n)_{n\geq 1}$ denote the lower central series of U^+, that is, Γ_n is the subgroup of U^+ generated by n-fold commutators $[u_1, [u_2, \dots, u_k]..]$ of elements $u_i \in U_+$ (where $[g, h] := g^{-1}h^{-1}gh$). Then each quotient Γ_n/Γ_{n+1} is an abelian group, hence a $\mathbb{Z}$-module. One can then show (see [1]) that the $\mathbb{Z}$-module $\mathrm{L}(U^+) := \bigoplus_{n\geq 1} \Gamma_n/\Gamma_{n+1}$ admits a natural $\mathbb{Q}$-Lie algebra structure, with Lie bracket $[g\Gamma_{m+1}, h\Gamma_{n+1}] := [g, h]\Gamma_{m+n+1}$ for all $g \in \Gamma_m$ and $h \in \Gamma_n$, and with scalar multiplication

$$\lambda \cdot [x_{i_1}(r_1), [x_{i_2}(r_2), \dots, x_{i_n}(r_n)]..]\Gamma_{n+1} = [x_{i_1}(\lambda r_1), [x_{i_2}(r_2), \dots, x_{i_n}(r_n)]..]\Gamma_{n+1}$$

for all $\lambda, r_i \in \mathbb{Q}$, $n \in \mathbb{N}$ and $i_j \in I$, where we set $x_i(r) := \exp(r\pi(e_i)) \in U_{\alpha_i}$. Moreover, the map $\mathfrak{n}^+ \to \mathrm{L}(U^+) : [e_{i_1}, \dots, e_{i_n}] \mapsto [x_{i_1}(1), \dots, x_{i_n}(1)]\Gamma_{n+1}$ defines a $\mathbb{Q}$-Lie algebra isomorphism. We now rephrase this in terms of standard forms for the elements of U^+: choose a basis $\mathcal{B}$ of $\mathfrak{n}^+$ consisting of elements of the form $x = [e_{i_1}, [e_{i_2}, \dots, e_{i_n}]..]$ (we then set $n_x := n$ and $u_x(\lambda) := [x_{i_1}(\lambda), [x_{i_2}(1), \dots, x_{i_n}(1)]..] \in U^+$ for all $\lambda \in \mathbb{Q}$). Order as before $\mathcal{B}$ so that $n_x < n_y \implies x < y$. Then each element $u \in U^+$ can be written in a unique way as a product

$$u = \prod_{x \in \mathcal{B}} u_x(\lambda_x) \quad \text{for some } \lambda_x \in \mathbb{Q}. \tag{5}$$

The advantage of the writing (5) compared to (4) is that it is "intrinsic" to U^+: for each $n \geq 1$, the partial product $\prod_{n_x \geq n} u_x(\lambda_x)$ belongs to $\Gamma_n \subseteq U^+$, and there is a clear algorithm on how to transform the expression of u as a (finite) product of elements $u_i(r)$ ($i \in I$, $r \in \mathbb{Q}$) into the standard form (5), namely using the isomorphism $\mathfrak{n}^+ \xrightarrow{\sim} \mathrm{L}(U^+)$.

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Dark Matter and $E(10)$

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(joint work with Hermann Nicolai)

Dark Matter remains a very mysterious component of the Universe. The arguments for its existence are very compelling, starting from the observations of Fritz Zwicky in 1933 (too fast rotations of edge galaxies in clusters of galaxies) and Vera Rubin in 1970 (too fast rotations of edge stars in Andromeda Galaxy) up to modern observations: flat rotation curves of hydrogen outside visible parts of almost all galaxies, Bullet Cluster (collision of galaxies) and indirect argument from LIGO/Virgo result combined with the other observations that the Universe is composed of 33% of matter, 5% luminous, 28% dark, with the rest most probably cosmological constant.

In 1982 Murray Gell-Mann conjectured [1] that $N = 8$ supergravity can be relevant for the connection with the Standard Model (the conjecture was extended by Nicolai and Warner [2]). It has shown that the breaking of $SO(8)$ into $SU(3) \times U(1)$ gives proper group assignments for both quarks and leptons, and there are 6 and only 6 quarks and leptons in this scheme, but the electric charges are shifted by $\pm 1/6$ with respect to the SM. The required correction was found in [3] and turned out to be very natural. However, it is outside of $SU(8)$ (as R-symmetry of $N = 8$ SUGRA) but it was proven in [4] that it is inside $K(E_{10})$ what points out to the crucial role of this group in this scheme.

In this scheme there are also 8 gravitini – 6 strongly interacting with electric charge $\pm\frac{1}{3}$ and 2 interacting only electromagnetically with charge $\pm\frac{2}{3}$. All of these gravitini should have masses of the order of the Planck mass and they cannot decay since there are no Standard Model particles they could decay to. Therefore we proposed that charge $\pm\frac{2}{3}$ gravitini can be Dark matter candidates even though they are charged. Electric charge of particles as candidates for Dark Matter up to TeV mass range is very strongly constrained by data. and the DM candidates usually discussed (axion-like or WIMP-like) assumed to have masses $< O(1)$ TeV the allowed charges are extremely small. However, if we extrapolate this formula to the Planck scale then the bound does not forbid charges of order 1. Planck mass gravitinos of charge $\pm\frac{2}{3}$ can be DM particles bound to the Solar System ($v \sim 30$ km/s) or our Galaxy ($v \sim 400$ km/s).

Estimated mass density of DM in the proximity of the Solar System is $\sim 0.3 \cdot 10^6$ GeV/m $^{-3}$ (it is million times larger than the average DM density in the Universe). For velocity ~ 400 km/s we arrive at a flux estimate $\Phi \sim 10^{-9}$ m^{-2}s^{-1}sr^{-1} $\sim$ 0.03 m^{-2}yr^{-1}sr^{-1} Being charged and extremely massive they can go through any medium along straight lines exciting and ionizing but with Planck mass they can cross the Earth without significant change in kinetic energy.

Since these gravitini are charged (in distinction to other proposals for Dark Matter candidates) they can eventually be discovered in deep underground neutrino experiments. There are two such experiments presently built: JUNO experiment in China, starting next year with 20 kt with oil as liquid scintillator and DUNE experiment in USA starting presumably in 2026 with 70 kt of liquid argon. In both the signal for the charged gravitino traversing the detector should be clearly visible by thousands of emitted photons with the emissions lasting many microseconds. If this proposal is true $K(E_{10})$ would play a crucial role in the description of the Standard Model and moreover Dark Matter should change name to Rare Matter...

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Non-associative structures in extended geometry

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(joint work with Martin Cederwall)

Let A be the Weyl–Clifford superalgebra with $2n$ even generators $\tilde{E}_a, \tilde{F}^a$ and $2n$ odd generators E_a, F^a $(a = 1, \ldots, n)$ satisfying the commutation relations

$$[\tilde{E}_a, \tilde{F}^b] = [E_a, F^b] = \delta_a{}^b\,, \tag{1}$$

and otherwise commuting. A polynomial vector field on $\mathbb{R}^n$ can be considered as an element in A of the form $U = U^a E_a$ where $U^a \in \mathbb{R}[\tilde{F}^b] \subset A$. Define an odd differential d (derivation that squares to zero) on A by $dE_a = \tilde{E}_a$ and $d\tilde{F}^a = F^a$. The Lie derivative of a vector field V parameterised by another U can then be written as a *derived bracket* [1]

$$\begin{aligned}\mathscr{L}_U V = [dU, V] &= [dU^a E_a, V^b E_b] + [U^a dE_a, V^b E_b]\\ &= [\partial_c U^a F^c E_a, V^b E_b] + [U^a dE_a, V^b E_b]\\ &= \partial_c U^a V^b [F^c E_a, E_b] + U^a [dE_a, V^b] E_b\\ &= \partial_c U^a V^b (-\delta_b{}^c E_a) + U^a \partial_a V^b E_b\\ &= (-V^a \partial_a U^b + U^a \partial_a V^b) E_b\,. \end{aligned}\tag{2}$$

It follows that

$$\begin{aligned}\mathscr{L}_U(\mathscr{L}_V W) &= [dU, [dV, W]] = [[dU, dV], W] + [dV, [dU, W]]\\ &= [d[dU, V]] + [dV, [dU, W]] = \mathscr{L}_{\mathscr{L}_U V} W + \mathscr{L}_V(\mathscr{L}_U W)\end{aligned}\tag{3}$$

and

$$\mathscr{L}_U V = [dU, V] = d[U, V] + [U, dV] = -[dV, U] = -\mathscr{L}_V U\,. \tag{4}$$

Thus the set of all vector fields form a Lie algebra with the bracket $[\![U, V]\!] = \mathscr{L}_U V$.

The elements $F^a E_b$ can be considered as basis elements $K^a{}_b$ of $\mathfrak{gl}(n) = \mathfrak{sl}(n) \oplus \mathbb{R}$, where $\mathfrak{sl}(n)$ is the split real form of the finite Kac–Moody algebra $\mathfrak{a}_r$, with $r = n-1$. The generators are basis elements of n-dimensional modules over $\mathfrak{a}_r$ with highest weight Λ_1 or lowest weight $-\Lambda_1$.

In *extended geometry*, the idea is to generalise geometrical concepts such as vector fields and Lie derivatives from $\mathfrak{a}_r$ and Λ_1 to any Kac–Moody algebra $\mathfrak{g}$ of rank r and any dominant integral weight λ of $\mathfrak{g}$, with corresponding irreducible highest- and lowest-weight modules $R(\lambda)$ and $\overline{R(\lambda)}$ [2]. We require that in the case of *exceptional geometry* (where $\mathfrak{g} = \mathfrak{e}_n$ and λ is the fundamental weight associated to the vertex to which we attach another one in the extension to $\mathfrak{e}_{n+1}$), the transformation of a generalised vielbein one-form under a generalised Lie derivative unifies n of the eleven diffeomorphisms in eleven-dimensional supergravity with gauge transformations associated to the three-form gauge field [3–7]. It turns out that the generalised vector fields in general do not form a Lie algebra with respect to a bracket $[\![U,V]\!] = \mathscr{L}_U V$. The question that we aim to answer is whether the generalised Lie derivative nevertheless can be written as a derived bracket $[dU,V]$. If so, d can no longer be an odd differential on an associative algebra.

Let the Cartan matrix of $\mathfrak{g}$ be of co-rank at most 1 and symmetrisable, so that, for some non-zero numbers d_i, multiplication from the left with $D = \mathrm{diag}(d_i)$ yields a symmetric matrix. Let $\hat{\lambda}$ be the weight with Dynkin labels $\hat{\lambda}_i = d_i\lambda_i$. Extend the derived algebra $\mathfrak{g}'$ of $\mathfrak{g}$ to $\mathscr{B}_0 = \mathfrak{g}' \oplus \mathbb{R}$ by adding a basis element h_0 such that $[h_0, e_i] = -\hat{\lambda}_i e_i$ and $[h_0, e_i] = \hat{\lambda}_i f_i$. Let $\mathscr{B}_{\pm 1}$ be $\mathfrak{g}'$-modules, $\mathscr{B}_{-1} \simeq R(\lambda)$ with highest-weight vector q, and $\mathscr{B}_1 \simeq \overline{R(\lambda)}$ with lowest-weight vector p. Extend $\mathscr{B}_{\pm 1}$ to modules over $\mathscr{B}_0$ by $h_0 \cdot p = h_0 \cdot q = 0$. Let k be a linear combination of $h_0, h_1, \ldots, h_r$ commuting with $\mathfrak{g}'$ and such that $k \cdot p = p$ and $k \cdot q = -q$.

Let $\mathscr{C}$ be the $\mathbb{Z}$-graded superalgebra with $\mathscr{B}_{\pm 1}$ as odd subspaces of $\mathscr{C}_{\pm 1}$ and $\mathscr{B}_0$ as an even subspace of $\mathscr{C}_0$, generated by $\mathscr{B}_{\pm 1}$ and $\mathscr{B}_0$ modulo the relations

(i) $[x, y] = [x, y]_{\mathfrak{g}}$ and $[x, z] = x \cdot z$ for all $x, y \in \mathscr{B}_0$ and $z \in \mathscr{B}_{\pm 1}$,
(ii) $(xy)z = x(yz)$ for all $x, y, z \in \mathscr{C}$ such that $x, y \in \mathscr{C}_0 \oplus \mathscr{C}_{\pm}$ or $y, z \in \mathscr{C}_0 \oplus \mathscr{C}_{\pm}$,
(iii) $[p, q] = 1$ and $pq = 1 + k - h_0$,

where $[-,-]$ denotes the (graded) commutator and $[-,-]_{\mathfrak{g}}$ the bracket in $\mathfrak{g}$.

The existence of the algebra $\mathscr{C}$ was proven in refs. [8, 9]. It is in general not associative, and not even Lie admissible, so we do not get a Lie superalgebra directly from it as its commutator algebra. However, since the relations (ii) hold, the subspace $\mathscr{C}_{-1} \oplus \mathscr{C}_0 \oplus \mathscr{C}_1$ forms what is called a *local* Lie superalgebra, and from it, we can get a Lie superalgebra [10].

It follows from the relations that the universal enveloping algebra $\mathscr{U}(\mathscr{B}_0)$ can be identified with $\mathscr{C}_0$, and that the tensor algebra $\mathscr{T}(\mathscr{B}_{\pm 1})$ can be considered as a subalgebra of $\bigoplus_{k=0}^{\infty} \mathscr{C}_{\pm k}$. For each $k \geq 0$, the subspace $\mathscr{C}_{\pm k}$ is then equal to both $\mathscr{U}(\mathscr{B}_0)\mathscr{T}(\mathscr{B}_{\pm 1})_{|k|}$ and $\mathscr{T}(\mathscr{B}_{\pm 1})_{|k|}\mathscr{U}(\mathscr{B}_0)$ with the natural $\mathbb{Z}$-grading of $\mathscr{T}(\mathscr{B}_{\pm 1})$.

Let $\tilde{\mathscr{B}}_{\pm 1}$ be $\mathfrak{g}$-modules isomorphic to $\mathscr{B}_{\pm 1}$ with an isomorphism $x \mapsto \tilde{x}$ for any $x \in \mathscr{B}_{\pm 1}$. Let $\mathscr{W}$ be the associative even superalgebra generated by the subspaces $\tilde{\mathscr{B}}_{\pm 1}$ modulo the relations $[\tilde{\mathscr{B}}_{\pm 1}, \tilde{\mathscr{B}}_{\pm 1}] = 0$ and $[x, y] = \langle x|y\rangle$ for $x \in \tilde{\mathscr{B}}_1$ and $y \in \tilde{\mathscr{B}}_{-1}$, where $\langle -|-\rangle$ is the natural pairing between the conjugate $\mathfrak{g}'$-modules given by $\langle \tilde{p}|\tilde{q}\rangle = 1$ and invariance under $\mathfrak{g}'$. Thus $\mathscr{W}$ is a Weyl algebra with a number of generators equal to twice the dimension of $R(\lambda)$.

Set $\mathscr{A} = \mathscr{W} \otimes \mathscr{C}$. We consider $\mathscr{W}$ and $\mathscr{C}$ as subalgebras of $\mathscr{A}$ and write an element $x \otimes y$ in $\mathscr{W} \otimes \mathscr{C}$ simply as $xy = yx$. We define a linear map d : $\mathscr{W}\mathscr{T}(\mathscr{B}_1) + \mathscr{W}\mathscr{T}(\mathscr{B}_{-1}) \to \mathscr{W} \otimes \mathscr{C}$ in three steps. First, on $\mathscr{B}_{\pm 1}$ and $\tilde{\mathscr{B}}_{\pm 1}$ by

$$(5) \qquad d\tilde{x} = \begin{cases} x & \text{if } \tilde{x} \in \tilde{\mathscr{B}}_{-1}\,, \\ 0 & \text{if } \tilde{x} \in \tilde{\mathscr{B}}_1\,, \end{cases} \qquad\qquad dx = \begin{cases} 0 & \text{if } x \in \mathscr{B}_{-1}\,, \\ \tilde{x} & \text{if } x \in \mathscr{B}_1\,. \end{cases}$$

Second, as an odd derivation on the subspaces $\mathscr{W}$ and $\mathscr{T}(\mathscr{B}_1) + \mathscr{T}(\mathscr{B}_{-1})$. Third, by $d(xy) = (dx)y + x(dy)$ for $x \in \mathscr{W}$ and $y \in \mathscr{T}(\mathscr{B}_1) + \mathscr{T}(\mathscr{B}_{-1})$.

We can now define a (generalised) vector field as an element in $\mathscr{S}(\tilde{\mathscr{B}}_{-1})\mathscr{B}_1 \subset \mathscr{A}$ and the (generalised) Lie derivative of a vector field V with respect to another one U as $\mathscr{L}_U V = [dU, V]$. This reproduces the known expressions in exceptional geometry, also in infinite-dimensional cases.

In the cases where $\mathfrak{g}$ is finite-dimensional with highest root θ and $(\lambda, \theta) = 1$, where $(-,-)$ is the inner product induced by the invariant symmetric bilinear form $\kappa(e_i, f_i) = \delta_{ij}/d_i$, we then have $\mathscr{L}_U \mathscr{L}_V W = \mathscr{L}_{\mathscr{L}_U V} W + \mathscr{L}_V(\mathscr{L}_U W)$ up to the *section condition*, which in a $\mathfrak{g}$-covariant way reduces the coordinate dependence to an n-dimensional subspace of $R(\lambda)$. In all other cases, the commutator of two Lie derivatives yields not only another Lie derivative, but also an additional *ancillary* $\mathfrak{g}$-transformation.

Let $\mathscr{V}$ be the commutator algebra generated by $p, q\mathscr{B}_0, \mathscr{B}_0 \in \mathscr{C}$. Its local part is a local Lie superalgebra. Then there is a unique $\mathbb{Z}$-graded Lie superalgebra W which is bitransitive (meaning that no $\mathbb{Z}$-graded ideal of W is included in $W_{\geq 0}$ or $W_{\leq 0}$) with a surjective local Lie superalgebra homomorphism from the local part of $\mathscr{V}$ to the local part of W. In the non-ancillary cases $(\lambda, \theta) = 1$, the Lie superalgebra W agrees with the *tensor hierarchy algebra* associated to the pair $(\mathfrak{g}, \lambda)$, previously defined in different ways (along with the related tensor hierarchy algebra S), also for infinite-dimensional $\mathfrak{g}$ [11–13]. Apart from intriguing mathematical features, the tensor hierarchy algebras seem to play a fundamental role in extended geometry by encoding the gauge structure of the generalised Lie derivatives [14]. The construction of the latter as derived brackets will hopefully lead to a better understanding of this role.

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Gauged Supergravities, Consistent Truncations and Generalised Geometry

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(joint work with D. Cassani, G. Josse, E. Malek, D. Waldram)

Supergravity theories are field theories of matter and gauge fields coupled to gravity in a supersymmetric way. Even if they are non-renormalisable, they are of interest as effective theories. Of particular interest are 10 and 11-dimensional supergravities, as they correspond to the low energy limit of string or M-theory.

In many applications of string theory, such as the construction of four dimensional theories to be compared with the Standard Model or in the context of the AdS/CFT correspondence, we are interested in configurations of 10 and 11-dimensional supergravities where the space time is the product

$$X_{10/11} = X_D \times M_d \tag{1}$$

where X_D is the external non-compact space-time and M is a compact space spanning the extra space dimensions.

Since M is compact, the full 10/11-dimensional theory can be reinterpreted as D-dimensional theory with an infinite number of fields organised into representations of $GL(d)$, the structure group of M. They are the so-called Kaluza-Klein modes. A finite set of such modes is massless while the massive ones have masses proportional to the inverse size of the internal manifold M.

If the size of the internal manifold can be made arbitrarily small with respect to the size of the physical space X_D, the massive states become very heavy and can be truncated away to obtain an effective field theory for the massless ones. This is the case for Calabi-Yau and flux compactifications to Minkowski space-time, which are used to construct effective models of phenomenological interest.

However for compactifications to Anti de Sitter space, which are relevant for the AdS/CFT correspondence, there is no such separation of scales and the truncated theory is obtained via a consistent truncation.

A consistent truncation is a procedure to truncate the theory to a finite set of modes in such a way that all truncated modes decouple form the lower-dimensional equations of motion and that no dependence on the internal manifold is left. If the truncation is consistent any solution of the truncated theory can be uplifted to a solution of the hight-dimensional theory.

The main difficulty in constructing consistent truncations is to find a principle to select the set of modes to be kept in the truncated theory. In our work we show that the general principle underlying consistent truncations of supergravity theories is that of generalised G_S-structures in Exceptional Generalised Geometry (EGG).

EGG is a reformulation of supergravity that treats in a geometric way the gauge transformations of the supergravity potentials on the internal manifold M_d [1, 2]. The idea is to replace the tangent bundle of M_d in (1) with a generalised tangent bundle E, which is an extension of TM by appropriate exterior powers of the cotangent bundle and has as structure group the exceptional group $E_{d(d)}$. The full set of diffeomorphisms on M_d and p-form gauge transformations is generated by an extension of the usual Lie derivative, called generalised Lie derivative or Dorfman derivative.

All ordinary notions of tensors, connections, G-structure and intrinsic torsion are naturally extended to E. In particular, generalised tensors are constructed as appropriate products of E and its dual E^*, while a generalised G_S-structure on M_d is defined as a reduction of the structure group $E_{d(d)}$ to a subgroup G_S.

More precisely, a generalised G_S-structure defines a G_S-principal sub-bundle P of the $E_{d(d)}$ frame bundle. In most cases, this is equivalent to the existence of a set of G_S-invariant, nowhere vanishing tensors $\{\Xi_i\}$. For instance, an H_d structure, with H_d the maximally compact subgroup of $E_{d(d)}$, is associated to existence of a globally defined generalised metric G.

A given G_S-structure is characterised by its intrinsic torsion W_{int}, which is defined as the part of the torsion of a G_S-compatible connection D ($D\{\Xi_i\} = 0$) that cannot be changed by varying the choice of connection. In other words, it is the obstruction to finding a generalised connection which is torsion-free and compatible with the G_S-structure. When W_{int} vanishes, the G_S-structure is integrable or torsion-free.

The intrinsic torsion can be decomposed into representations of the structure group G_S. For consistent truncations we are interested in manifolds that admit G_S structures whose intrinsic torsion only contains the singlet representation and is constant.

The main result is the following theorem [3]:

Let M be a d-dimensional (respectively (d - 1)-dimensional) manifold with a generalised $G_S \subseteq H_d$-structure defining a set of invariant tensors $\{\Xi_i\}$ and only constant, singlet intrinsic torsion. Then there is a consistent truncation of eleven- dimensional (respectively type II) supergravity on M defined by expanding all bosonic fields in terms of the invariant tensors. If $\tilde{H}_d$ is the double cover of H_d, acting on fermions the structure group lifts to $\tilde{G}_S \subseteq \tilde{H}_d$ and the truncation extends to the fermionic sector, provided again one expands the spinor fermion fields in terms of $\tilde{G}_S$ singlets.

The data of the generalised G_S-structure fully determine the truncated lower dimensional theory, namely field content, gauge group and supersymmetry

Different amounts of supersymmetry correspond to different choices of of structure group G_S.

All maximally supersymmetric truncations are associated to generalised identity structure and therefore can be seen as generalised Scherk-Schwarz reductions [4, 5]. All maximally supersymmetric truncations of sphere are unified in this class: truncations of 11-dimensional supergravity on S^7 and S^4, and IIB supergravity on S^5 [4, 6] and massive IIA on spheres [7, 8].

Similar classifications can be given for lower supersymmetry by considering larger generalised structure groups [3, 9–12]. It turns out the number of supergravity theories that can be obtained as consistent truncations of 11 and 10-dimensional supergravity is very limited.

For instance, half-maximal truncations to 5-dimensional supergravity with no matter fields correspond to $SO(5)$ structures, while only theories with up to 4 vector multiplets can be obtained and they correspond to $SO(5-n)$-structures with $n = 1, \ldots, 4$ [3]. Similarly, truncations to pure $\mathcal{N} = 2$ supergravities in 5-dimensions correspond to $Usp(6)$ structures, while allowing for $G_S \subset Usp(6)$ gives supergravities with hyper and/or vector multiplets. The number of allowed truncations is larger than for the half-maximal ones, but still covers a very limited subset of the 5-dimensional $\mathcal{N} = 2$ supergravities that one can construct from a purely 5-dimensional point of view.

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