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Convex Geometry and its Applications

Organized by
Shiri Artstein-Avidan, Tel Aviv
Daniel Hug, Karlsruhe
Elisabeth Werner, Cleveland

15 December – 20 December 2024

Abstract. The geometry of convex domains in Euclidean space plays a central role in several branches of mathematics: functional and harmonic analysis, the theory of PDEs, linear programming and, increasingly, in the study of algorithms in computer science. Convex Geometry has experienced a series of striking developments in the past few years: for example, the new tools from stochastic localization, the huge progress around the slicing problem, the measure transportation perspective on old problems, progress on conjectured geometric and functional inequalities and new applications of methods and results to a wide range of fields, including random matrices and statistical learning. The purpose of this meeting is to bring together researchers from the analytic, geometric and probabilistic groups who have contributed to the latest exciting results, to exchange ideas and pave the path for future developments. The meeting will continue a tradition of more than 50 years of Oberwolfach meetings with Convex Geometry in the title, at the same time emphasizing the new directions and developments, and new connections to other mathematical fields.

Mathematics Subject Classification (2020): 52Axx (60D05, 68Q25).

Introduction by the Organizers

The meeting Convex Geometry and its Applications, organized by Shiri Artstein-Avidan, Daniel Hug, and Elisabeth Werner, was held from December 15 to December 20, 2024, just before the winter break. It was attended by 46 participants working in all areas of convex geometry and related fields. Of these 28% were female and a large proportion were younger participants. There were 5 plenary lectures of 50 minutes duration (including discussion) and 23 shorter lectures, each lasting 35 minutes (including discussion).

Convex geometry has always contributed to and benefited from connections to other fields, both within mathematics and in applied disciplines. The workshop reflected the diversity of the topics and tools employed by researchers in the field. Remarkably, major progress has been reported on several long-standing difficult problems in convex geometry and in particular the final step in the (positive) resolution of one of the major conjectures, Bourgain's slicing problem, has been announced during the workshop via an arXiv publication by Boaz Klartag and Joseph Lehec, building on recent work by Qing Yang Guan. During the last workshop in 2021 Yuansi Chen had already presented major progress on the hyperplane conjecture and related problems, based on Ronen Eldan's stochastic localization scheme, but the ultimate power of the method could only now be fully exploited. Topics covered in the course of the workshop included (but were not restricted to) valuations on convex functions and bodies, analytic, geometric and discrete inequalities and stability versions thereof, characterization results and extremal problems, new versions of classical problems (in the spirit of Busemann–Petty type problems), random and integral geometry, geometric inequalities in non-Euclidean spaces, Minkowski-type existence and uniqueness results and their relation to partial differential equations.

The workshop started with a survey by Monika Ludwig on recent work concerning the study of valuations on convex functions, including a comparison with valuations on convex bodies. In particular, Hadwiger type characterization theorems for convex functions, representation results for functional intrinsic volumes and a classification result involving a functional version of affine surface area were highlighted. In a similar spirit, the subsequent talk by Fabian Mußnig had characterization and integralgeometric results for moment vectors of convex bodies as its starting point and explained several intriguing functional analogues. The second talk in the morning and the presentations in the afternoon were all devoted to analytic, geometric and discrete inequalities in different settings. An insightful discussion of various versions of the Brascamp–Lieb inequality and Barthe's reverse inequality, methods of proof, equality cases and applications was given by Karoly Böröczky. In some rather specific cases, stability versions of these inequalities are known, but so far a general stability result seems to be out of reach. David Alonso-Gutiérrez explained how discrete versions of geometric inequalities can be used to derive the more classical continuous versions. The former are obtained via a discrete Brunn–Minkowski inequality. Dylan Langharst presented joint work with D. Cordero-Erausquin and M. Fradelizi concerning a study of monotonicity properties of the L^p-volume product of nonnegative convex functions along the Fokker–Planck heat flow. In the last talk on Monday, Rotem Assouline identified a general framework in which Brunn–Minkowski inequalities hold on manifolds (satisfying various requirements).

The second 50-minute lecture, delivered by Emanuel Milman on Tuesday, reports on his recent resolution (jointly with S. Shableman and A. Yehudayoff) of a thirty year old problem due to Erwin Lutwak. It asks whether a convex body whose (second) intersection body is a positive multiple of the body is necessarily an ellipsoid. Several partial answers had been obtained previously. The answer is yes, and in fact the method presented applies to a family of similar questions regarding different geometric operations. In the talk, essential ingredients of the mainly geometric proof were nicely communicated. The subsequent talk by Julián Haddad concerns another conjecture, which asks whether the volume of the convolution body of a convex body is maximized by ellipsoids. In this case, counterexamples exist. Various remaining open problems were stated in these two talks. Vlad Yaskin discussed recent contributions to the homothety conjecture, which asks whether a convex body K will be an ellipsoid provided some flotation body K_δ of K is proportional to K. In the plane, a counterexample is constructed (involving an asymmetric body), but in the planar and symmetric case a partial positive answer is obtained. The talk by Galyna Livshyts explored extensions of the functional Blaschke–Santaló inequality to a weighted setting, with a non-Gaussian measure appearing as an optimizer. If true, such an inequality would imply a certain strengthening of the Brascamp–Lieb inequality for a class of functions (analogously to the improved Gaussian Poincaré inequality for even functions). A collection of positive and negative results on the matter were presented in a talk at the black board. The afternoon session started with a talk by Artem Zvavitch. He revisited the famous Busemann–Petty problem (for which the solutions are known in all dimensions in case of Lebesgue measure), replacing Lebesgue measure by more general measures and also allowing dilates. In the particular case of measures with log-concave density, using a Large Deviation inequality, A. Zvavitch and PhD student M. Lafi are able to then give a positive answer to a corresponding Busemann–Petty problem (allowing dilates), under some additional assumptions. Jonas Knörr outlined his progress in describing the space of all continuous and dually epi-translation invariant valuations on convex functions via a new (constructive) approach. Instead of using tools from representation theory (as originally introduced into the field by Semyon Alesker and multiply used since then), the main idea is to directly obtain suitable integral representations of such functionals by applying a Paley–Wiener–Schwartz type regularity characterization for the Fourier–Laplace transform of certain (Goodey–Weil type) distributions associated to these valuations. The last contribution on Tuesday, made by Eli Putterman, dealt with the search for sharp bounds and extremizers for the volume of higher-order difference bodies of a given convex body. In dimension two, new relations are obtained (thus recovering a result by Rolf Schneider from 1970), in general dimensions several partial results and new perspectives are described.

The long talk on Wednesday morning was given by Anna Gusakova, based on her joined work with F. Besau and Ch. Thäle. The authors study the asymptotic behavior of the expected number of j-dimensional faces of a random polytope, i.e., the convex hull of n randomly chosen points in a given d dimensional convex body. The asymptotic behavior had been studied in the case when the convex body was smooth and in the case when the convex body was a polytope. The authors now look at a class of convex bodies that are neither and, interestingly, the expected number of j faces exhibits a completely new behavior, interpolating between the previously known ones. Minkowski type problems naturally connect convexity with discrete geometry and partial differential equations. In his talk, Dongmeng Xi reports on joint work with E. Lutwak, D. Yang, G. Zhang and Y. Zhao on Minkowski type problems that are related to chord measures and fractional affine surface area measures and originate from problems in integral geometry. Alexander Litvak explained the construction of examples showing that known bounds for the Rademacher projection (used to estimate $MM^*(K)$ for a convex body K) in the non-symmetric case are sharp.

Thursday's long talk was presented by Florian Besau (based on joint work with E. Werner). It dealt with analogs of classical centro-affine invariant isoperimetric inequalities, such as the Blaschke–Santaló inequality and the L_p-affine isoperimetric inequalities, for convex bodies in spherical space. Specifically, an isoperimetric inequality for the floating area, the spherical analog to the affine surface area, and a corresponding stability result are established. Additionally, an L_p-floating area is introduced and a novel curvature entropy functional for spherical convex bodies is proposed, based on the L_p-floating area. The talk by Thomas Wannerer featured non-standard variants for Busemann inequalities in the setting of complex and quaternionic convex bodies. Hiroshi Tsuji gave an account of the recent work he has obtained jointly with Shohei Nakamura, where they solved a conjecture of Kolesnikov and Werner on a general multi-function version for the Blaschke–Santaló inequality, of which only partial cases were known. The morning session ended with a talk by Tillmann Bühler on intersection processes of k-flats in hyperbolic space, where he illustrated which Central Limit Theorems are expected to hold in this context and how unexpected limit behavior arises, thereby verifying some relevant conjectures (joint work with D. Hug).

Matthieu Fradelizi opened the afternoon session discussing monotonicity of entropy, and how the known inequalities for discrete entropy can be adjusted so as to hold for differential entropy, up to factors which can be controlled as the entropy grows, thus not only verifying a conjecture of Terence Tao but in fact strengthening it with a better behavior as the number of variables averaged grows. There followed a very beautiful and clear board-talk by Kateryna Tatarko where she solved (presenting joint work with S. Myroshnychenko and V. Yaskin) a problem of B. Grunbaum from 1961, asking whether the centroid of a convex body must also be the centroid for at least $(n+1)$ of its hyperplane sections (or more generally, what is the minimal number of sections through the centroid for which the centroid remains the same, minimum taken over all convex bodies in dimension n). Grunbaum knew that through any point in the interior of a convex body, one can find a hyperplane section for which it is the centroid, but it turns out that taking the point to be the centroid, this number 1 is in fact, sometimes, the maximal number of hyperplane sections through the point for which it is the centroid. Tatarko carefully led us through the construction of a body of revolution in 5 dimensional space for which the answer to Grunbaum's problem is 1. The final talk on Thursday by Liran Rotem regarded his joint work with PhD student Tomer Falah. They solve the Minkowski problem for functional surface area measures, completing the picture of what was previously known and giving a full characterization of such measures.

The final long talk, given by Andreas Bernig on Friday morning, was based on joint work with J. Kotrbatý and T. Wannerer. The key part of their work is a deep structural result for smooth valuations, which they call the Kähler–Lefschetz package (Hard Lefschetz theorem and Hodge-Riemann relations), due to the occurrence of similar results in various branches of mathematics. As a consequence, they obtain higher-degree Alexandrov–Fenchel inequalities (which are restricted to specific classes of convex bodies though). Yiming Zhao reported on recent progress in the study of existence and uniqueness results in Minkowski type problems for C-asymptotic sets (noncompact convex sets with prescribed asymptotic cone). A fundamental problem due to Banach (1932) was the starting point for an investigation by Dmitry Faifman. It asks whether a convex body in $\mathbb{R}^n$ has to be an ellipsoid if all k-dimensional sections through a fixed interior point are linearly (affinely) equivalent. The answer is known to be positive for most values of n, k, but the problem is still not resolved in full generality. In joint work with G. Aishwary, Faifman explains how stability results for 2-dimensional sections can be established. It is commonly known that several variational functionals lead to Brunn–Minkowski type inequalities. In the last talk of the workshop, Lei Qin presented her joint results with Andrea Colesanti concerning a study of the first Dirichlet eigenvalue problem for the weighted p-Laplace operator. Using the method of viscosity solutions, they succeeded in removing regularity assumptions on the underlying domain.

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Workshop: Convex Geometry and its Applications

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Abstracts

Valuations on convex functions

Monika Ludwig
(joint work with Fernanda M. Baêta, Andrea Colesanti, Fabian Mussnig)

Let $\mathcal{W}(X)$ be a class of extended real-valued functions $w\colon X \to (-\infty, \infty]$ on a set X. A functional $\mathrm{Z}\colon \mathcal{W}(X) \to \mathbb{R}$ is called a *valuation* if

$$\mathrm{Z}(v) + \mathrm{Z}(w) = \mathrm{Z}(v \vee w) + \mathrm{Z}(v \wedge w)$$

whenever $v, w \in \mathcal{W}(X)$ and the pointwise maximum $v \vee w$ and the pointwise minimum $v \wedge w$ are also elements of $\mathcal{W}(X)$.

The classical examples are valuations on the space $\mathcal{K}^n$ of convex bodies (that is, compact convex sets) in $\mathbb{R}^n$, where we identify a convex body with its support function and $\mathcal{K}^n$ with the class of support functions of convex bodies. The following result is a cornerstone of geometric valuation theory.

Theorem 1 (Hadwiger 1952)**.** *A functional* $\mathrm{Z}\colon \mathcal{K}^n \to \mathbb{R}$ *is a continuous, translation and rotation invariant valuation if and only if there are* $c_0, \dots, c_n \in \mathbb{R}$ *such that*

$$\mathrm{Z}(K) = c_0 V_0(K) + \cdots + c_n V_n(K)$$

for every $K \in \mathcal{K}^n$*.*

Here, $\mathcal{K}^n$ is equipped with the topology induced by the Hausdorff metric, and $V_0, \dots, V_n$ are the intrinsic volumes. Hadwiger's celebrated result has numerous applications in integral geometry and geometric probability.

In the talk, results on valuations on convex functions are presented. Let $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ be the space of convex functions $v\colon \mathbb{R}^n \to \mathbb{R}$. We say that a functional $\mathrm{Z}\colon \mathrm{Conv}(\mathbb{R}^n; \mathbb{R}) \to \mathbb{R}$ is continuous if $\mathrm{Z}(v_k) \to \mathrm{Z}(v)$ when $v_k \to v$ pointwise. It is dually epi-translation invariant if

$$\mathrm{Z}(v + \ell) = \mathrm{Z}(v)$$

for all $v \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ and affine functions $\ell\colon \mathbb{R}^n \to \mathbb{R}$. The following result is a functional version of Theorem 1.

Theorem 2 ([4, 5])**.** *A functional* $\mathrm{Z}\colon \mathrm{Conv}(\mathbb{R}^n; \mathbb{R}) \to \mathbb{R}$ *is a continuous, dually epi-translation and rotation invariant valuation if and only if there are functions* $\alpha_0, \dots, \alpha_n \in C_c([0, \infty))$ *such that*

$$\mathrm{Z}(v) = \sum_{j=0}^{n} \int_{\mathbb{R}^n} \alpha_j(|y|) \, \mathrm{dMA}_j(v; y)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$*.*

Here, $\mathrm{MA}_j(v;\cdot) = \mathrm{MA}(v,\dots,v,h_B,\dots,h_B)$ is the mixed Monge–Ampère measure with j entries v and $(n-j)$ entries h_B, the support function of the Euclidean unit ball. A comparison with Theorem 1 shows that the functional

$$v \mapsto \int_{\mathbb{R}^n} \alpha_j(|y|)\,\mathrm{dMA}_j(v;y)$$

can be considered as a functional jth intrinsic volume. Applications of Theorem 2 are discussed in [6].

Blaschke's classification of continuous and equi-affine invariant (that is, translation and $\mathrm{SL}(n)$ invariant) valuations predates Hadwiger's result. It was extended to a classification of upper semicontinuous valuations on convex bodies and provides a characterization of the affine surface area,

$$\Omega(K) = \int_{\mathrm{bd}\,K} \kappa(K,x)^{\frac{1}{n+1}}\,\mathrm{d}x$$

for $K \in \mathcal{K}^n$, where $\kappa(K,x)$ is the generalized Gaussian curvature of K at x and integration is with respect to the $(n-1)$-dimensional Hausdorff measure on the boundary, $\mathrm{bd}\,K$, of K.

Theorem 3 ([7, 8]). *A functional $\mathrm{Z}\colon \mathcal{K}^n \to \mathbb{R}$ is an upper semicontinuous, equi-affine invariant valuation if and only if there are $c_0, c_1 \in \mathbb{R}$ and $c_2 \geq 0$ such that*

$$\mathrm{Z}(K) = c_0\,V_0(K) + c_1\,V_n(K) + c_2\,\Omega(K)$$

for every $K \in \mathcal{K}^n$.

Let $\mathrm{Conv}_{\mathrm{MA}}(\mathbb{R}^n;\mathbb{R})$ denote the space of convex functions $v\colon \mathbb{R}^n \to \mathbb{R}$ with compactly supported Monge–Ampère measure $\mathrm{MA}(v,\cdot)$. We say that a functional $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{MA}}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ is τ^*-upper semicontinuous if

$$\mathrm{Z}(v) \geq \limsup_{k\to\infty} \mathrm{Z}(v_k)$$

for every sequence $v_k \in \mathrm{Conv}_{\mathrm{MA}}(\mathbb{R}^n;\mathbb{R})$ with uniformly bounded support of the Monge–Ampère measures converging to $v \in \mathrm{Conv}_{\mathrm{MA}}(\mathbb{R}^n;\mathbb{R})$.

Theorem 4 ([1, 2, 3]). *A functional $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{MA}}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ is a τ^*-upper semicontinuous, equi-affine and dually epi-translation invariant valuation if and only if there are $c_0, c_1 \in \mathbb{R}$ and $\zeta \in \mathrm{Conc}(0,\infty)$ such that*

$$\mathrm{Z}(v) = c_0 + c_1\,\mathrm{MA}(v;\mathbb{R}^n) + \int_{\mathbb{R}^n} \zeta(\det \mathrm{D}^2 v(y))\,\mathrm{d}y$$

for every $v \in \mathrm{Conv}_{\mathrm{MA}}(\mathbb{R}^n;\mathbb{R})$.

Here, $\mathrm{D}^2 v$ is the Hessian matrix of v, and $\mathrm{Conc}(0,\infty)$ is the set of concave functions $\zeta\colon (0,\infty) \to [0,\infty)$ with $\lim_{t\to 0}\zeta(t) = 0$ and $\lim_{t\to\infty}\zeta(t)/t = 0$. A comparison with Theorem 3 shows that the functionals

$$v \mapsto \int_{\mathbb{R}^n} \zeta(\det \mathrm{D}^2 v(y))\,\mathrm{d}y$$

can be considered as functional affine surface areas.

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Vector-valued valuations on convex bodies and convex functions

Fabian Mussnig
(joint work with Mohamed A. Mouamine)

The celebrated theorem of Hadwiger [4] provides a complete characterization of (linear combinations of) intrinsic volumes as continuous, rigid motion invariant valuations on convex bodies. This result was later extended to the vector-valued setting by Hadwiger and Schneider [5]. For this, consider the vector-valued integral

$$m(K) = \int_K x \, \mathrm{d}\mathcal{H}^n(x),$$

which defines the *moment vector* of a convex body K (i.e., a non-empty, compact, convex subset of $\mathbb{R}^n$), where we integrate with respect to the n-dimensional Hausdorff measure. The *intrinsic moments* $z_{j+1}(K)$, $0 \leq j \leq n$, are now obtained via the Steiner-type formula

$$m(K + rB^n) = \sum_{j=0}^{n} r^{n-j} \operatorname{vol}_{n-j}(B^{n-j}) z_{j+1}(K)$$

for $r \geq 0$, where B^k denotes the Euclidean unit ball in $\mathbb{R}^k$ and where $K + rB^n$ denotes the set of points with distance at most r from K (which trivially includes K itself). Special cases include the *Steiner point* $s(K) = z_1(K)$ and the moment vector $m(K) = z_{n+1}(K)$. Up to a change of indices and renormalization, these quantities are also known as quermassvectors. Hadwiger and Schneider [5] characterized linear combinations as moment vectors as continuous, translation covariant, and rotation covariant vector-valued valuations on convex bodies. Notably, there are no (non-trivial) translation invariant valuations in this class.

Recently, together with Andrea Colesanti and Monika Ludwig, the author established a functional version of Hadwiger's theorem for convex functions, which characterizes *functional intrinsic volumes* [3]. These integral operators arise from Hessian measures and generalize intrinsic volumes of convex bodies. In a more recent project with Mohamed A. Mouamine, the corresponding problem for vector-valued operators is attacked. We start with a functional analog of the moment vector. For a continuous function α with compact support on $[0,\infty)$, we set

$$\mathrm{m}_\alpha^*(v) = \int_{\mathbb{R}^n} \alpha(|x|)\nabla v(x)\,\det(\mathrm{D}^2 v(x))\,\mathrm{d}x$$

for any convex function $v\colon \mathbb{R}^n \to \mathbb{R}$ which is twice continuously differentiable. By previous results on so-called *Hessian valuations* [2], this definition continuously extends to convex functions on $\mathbb{R}^n$ without any additional C^2 assumptions. The operator m_α^* generalizes the classical moment vector via $\mathrm{m}_\alpha^*(h_K) = \alpha(0)m(K)$ for every convex body K, where $h_K(x) = \max_{y\in K}\langle x,y\rangle$, $x \in \mathbb{R}^n$, is the *support function* of K. Furthermore, it can be characterized as a vector-valued valuation on convex functions similar to a characterization of the moment vector on convex bodies. Knoerr and Ulivelli independently obtained such a result in [8].

Surprisingly, a Steiner-type formula starting with the functional moment vector m_α^* gives rise to two families of functionals. One, as expected, generalizes the intrinsic moments. The second family, however, describes rotation covariant, vector-valued valuations that are not translation covariant (in an appropriate sense) but rather translation invariant. In particular, these operators do not have classical counterparts and vanish on support functions. This family can again be described by a Steiner-type formula. Let α be a continuous function with bounded support on $(0,\infty)$ such that $\lim_{s\to 0^+}\alpha(s)s = 0$. We define

$$\mathrm{t}_{n,\alpha}^*(v) = \int_{\mathbb{R}^n} \alpha(|x|)x\,\det(\mathrm{D}^2 v(x))\,\mathrm{d}x$$

for any convex function $v \in C^2(\mathbb{R}^n)$. Similar to m_α^*, this definition continuously extends to convex functions without additional C^2 regularity, using a *Monge–Ampère measure.* A new family of vector-valued operators is now obtained via the polynomial expansion

$$\mathrm{t}_{n,\alpha}^*(v + r h_{B^n}) = \sum_{j=1}^{n} r^{n-j}\,\mathrm{vol}_{n-j}(B^{n-j})\,\mathrm{t}_{j,\alpha}^*(v)$$

for $r \geq 0$. Let us point out that here, summation starts with $j = 1$ since the coefficient of r^n in the polynomial above vanishes. Alternative descriptions of the operators $\mathrm{t}_{j,\alpha}^*$ can be obtained using singular Hessian integrals or through Kubota-type formulas [6].

In our main result we show that every continuous (with respect to pointwise convergence of functions) vector-valued valuation z on convex functions such that $\mathrm{z}(v + a) = \mathrm{z}(v)$ for every affine function a and such that $\mathrm{z}(v \circ \vartheta^{-1}) = \vartheta\,\mathrm{z}(v)$ for every $\vartheta \in \mathrm{SO}(n)$ must be of the form

$$\mathrm{z} = \mathrm{t}_{1,\alpha_1}^* + \cdots + \mathrm{t}_{n,\alpha_n}^*\,.$$

Finally, let us remark that the setting of valuations described above has close connections to *zonal valuations* on convex bodies [9] by associating with a convex body in dimension n the convex function $x \mapsto h_K(x, -1)$ on $\mathbb{R}^{n-1}$. See [1, 7] for recent advances. Using this connection, the operators $\mathrm{t}^*_{j,\alpha}$ give rise to a new class of continuous, translation invariant, $\mathrm{SO}(n-1)$ covariant vector-valued valuations on convex bodies which we expect can be characterized similarly.

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The Brascamp–Lieb inequality, Barthe's reverse form and extremals

Károly J. Böröczky

For a proper linear subspace E of $\mathbb{R}^n$ ($E \neq \mathbb{R}^n$ and $E \neq \{0\}$), let P_E denote the orthogonal projection into E. We say that the subspaces $E_1, \ldots, E_k$ of $\mathbb{R}^n$ and $p_1, \ldots, p_k > 0$ form a Geometric Brascamp-Lieb datum if they satisfy

$$\sum_{i=1}^{k} p_i P_{E_i} = I_n. \tag{1}$$

The name "Geometric Brascamp-Lieb datum" coined by Bennett, Carbery, Christ, Tao [9] comes from the following theorem, originating in the work of Brascamp, Lieb [12] and Ball [2, 3] in the rank one case ($\dim E_i = 1$ for $i = 1, \ldots, k$), and Lieb [25] and Barthe [6] in the general case. In the rank one case, the Geometric Brascamp-Lieb datum is known by various names, like "John decomposition of the identity operator", or tight frame, or Parseval frame in coding theory and computer science (see for example Casazza, Tran, Tremain [16]).

Theorem 1 (Brascamp-Lieb, Ball, Barthe). *For the linear subspaces $E_1, \ldots, E_k$ of $\mathbb{R}^n$ and $p_1, \ldots, p_k > 0$ satisfying* (1)*, and for non-negative $f_i \in L_1(E_i)$, we have*

$$\int_{\mathbb{R}^n} \prod_{i=1}^{k} f_i(P_{E_i} x)^{p_i} \, dx \le \prod_{i=1}^{k} \left(\int_{E_i} f_i \right)^{p_i} \tag{2}$$

Remark 1. *This is Hölder's inequality if* $E_1 = \ldots = E_k = \mathbb{R}^n$ *and* $P_{E_i} = I_n$*, and hence* $\sum_{i=1}^k p_i = 1$.

We note that equality holds in Theorem 1 if $f_i(x) = e^{-\pi\|x\|^2}$ for $i = 1, \ldots, k$; and hence, each f_i is a Gaussian density. Actually, Theorem 1 is an important special case discovered by Ball [3, 4] in the rank one case and by Barthe [6] in the general case of the general Brascamp-Lieb inequality (cf. Theorem 3).

After partial results by Barthe [6], Carlen, Lieb, Loss [14] and Bennett, Carbery, Christ, Tao [9], it was Valdimarsson [28] who characterized equality in the Geometric Brascamp-Lieb inequality.

A reverse form of the Geometric Brascamp-Lieb inequality was proved by Barthe [6]. We write $\int_{\mathbb{R}^n}^* \varphi$ to denote the outer integral for a possibly non-integrable function $\varphi : \mathbb{R}^n \to [0, \infty)$; namely, the infimum (actually minimum) of $\int_{\mathbb{R}^n} \psi$ where $\psi \geq \varphi$ is Lebesgue measurable.

Theorem 2 (Barthe)**.** *For the non-trivial linear subspaces* $E_1, \ldots, E_k$ *of* $\mathbb{R}^n$ *and* $p_1, \ldots, p_k > 0$ *satisfying* (1)*, and for non-negative* $f_i \in L_1(E_i)$*, we have*

$$\int_{\mathbb{R}^n}^* \sup_{x=\sum_{i=1}^k p_i x_i,\, x_i \in E_i} \prod_{i=1}^k f_i(x_i)^{p_i}\, dx \geq \prod_{i=1}^k \left(\int_{E_i} f_i \right)^{p_i}. \tag{3}$$

Remark 2. *This is the Prékopa-Leindler inequality if* $E_1 = \ldots = E_k = \mathbb{R}^n$ *and* $P_{E_i} = I_n$*, and hence* $\sum_{i=1}^k p_i = 1$.

Equality in Barthe's reverse Geometric Brascamp-Lieb (3) was characterized by Böröczky, Kalantzopoulos, Xi [10].

We note that Barthe's inequality (3) extends the celebrated Prékopa-Leindler inequality (proved in various forms by Prékopa [26, 27], Leindler [24] and Borell [11]) whose equality case was clarified by Dubuc [19] (see the survey Gardner [21]).

For completeness, let us state and discuss the general Brascamp-Lieb inequality and its reverse form due to Barthe. The following was proved by Brascamp, Lieb [12] in the rank one case and Lieb [25] in general.

Theorem 3 (Brascamp-Lieb Inequality)**.** *Let* $B_i : \mathbb{R}^n \to H_i$ *be surjective linear maps where* H_i *is* n_i*-dimensional Euclidean space,* $n_i \geq 1$*, for* $i = 1, \ldots, k$ *such that*

$$\cap_{i=1}^k \ker B_i = \{0\},$$

and let $p_1, \ldots, p_k > 0$ *satisfy* $\sum_{i=1}^k p_i n_i = n$*. Then for non-negative* $f_i \in L_1(H_i)$*, we have*

$$\int_{\mathbb{R}^n} \prod_{i=1}^k f_i(B_i x)^{p_i}\, dx \leq \mathrm{BL}(\mathbf{B}, \mathbf{p}) \cdot \prod_{i=1}^k \left(\int_{H_i} f_i \right)^{p_i} \tag{4}$$

where the optimal factor $\mathrm{BL}(\mathbf{B}, \mathbf{p}) \in (0, \infty]$ *depending on* $\mathbf{B} = (B_1, \ldots, B_k)$ *and* $\mathbf{p} = (p_1, \ldots, p_k)$ *(which we call a Brascamp-Lieb datum), and* $\mathrm{BL}(\mathbf{B}, \mathbf{p})$ *is determined by choosing centered Gaussians* $f_i(x) = e^{-\langle A_i x, x \rangle}$ *for some symmetric positive definite* $n_i \times n_i$ *matrix* A_i*,* $i = 1, \ldots, k$ *and* $x \in H_i$.

Remark 3. *The Geometric Brascamp-Lieb Inequality is readily a special case of* (4) *where* $\mathrm{BL}(\mathbf{B},\mathbf{p})=1$. *We note that* (4) *is Hölder's inequality if* $H_1=\ldots=H_k=\mathbb{R}^n$ *and each* $B_i=I_n$*, and hence* $\mathrm{BL}(\mathbf{B},\mathbf{p})=1$ *and* $\sum_{i=1}^k p_i=1$ *in that case.*

The condition $\sum_{i=1}^k p_in_i=n$ *makes sure that for any* $\lambda>0$*, the inequality* (4) *is invariant under replacing* $f_1(x_1),\ldots,f_k(x_k)$ *by* $f_1(\lambda x_1),\ldots,f_k(\lambda x_k)$*,* $x_i\in H_i$.

We say that two Brascamp-Lieb datum $\{(B_i,p_i)\}_{i=1,\ldots,k}$ and $\{(B_i',p_i')\}_{i=1,\ldots,k'}$ as in Theorem 3 are called equivalent if $k'=k$, $p_i'=p_i$, and there exists linear isomorphisms $\Psi:\mathbb{R}^n\to\mathbb{R}^n$ and $\Phi_i:H_i\to H_i'$, $i=1,\ldots,k$, such that $B_i'=\Phi_i\circ B_i\circ\Psi$. It was proved by Carlen, Lieb, Loss [14] in the rank one case, and by Bennett, Carbery, Christ, Tao [9] in general that there exists a set of extremizers $f_1,\ldots,f_k$ for (4) if and only if the Brascamp-Lieb datum $\{(B_i,p_i)\}_{i=1,\ldots,k}$ is equivalent to some Geometric Brascamp-Lieb datum.

The following reverse version of the Brascamp-Lieb inequality was proved by Barthe in [5] in the rank one case, and in [6] in general.

Theorem 4 (Barthe's Inequality)**.** *Let* $B_i:\mathbb{R}^n\to H_i$ *be surjective linear maps where* H_i *is* n_i*-dimensional Euclidean space,* $n_i\geq 1$*, for* $i=1,\ldots,k$ *such that*

$$\cap_{i=1}^k \ker B_i=\{0\},$$

and let $p_1,\ldots,p_k>0$ *satisfy* $\sum_{i=1}^k p_in_i=n$. *Then for non-negative* $f_i\in L_1(H_i)$*, we have*

$$\int_{\mathbb{R}^n}^{*}\sup_{x=\sum_{i=1}^k p_iB_i^*x_i,\,x_i\in H_i}\prod_{i=1}^k f_i(x_i)^{p_i}\,dx\geq \mathrm{RBL}(\mathbf{B},\mathbf{p})\cdot\prod_{i=1}^k\left(\int_{H_i}f_i\right)^{p_i}\tag{5}$$

where the optimal factor $\mathrm{RBL}(\mathbf{B},\mathbf{p})\in[0,\infty)$ *depends on the Brascamp-Lieb datum* $\mathbf{B}=(B_1,\ldots,B_k)$ *and* $\mathbf{p}=(p_1,\ldots,p_k)$*, and* $\mathrm{RBL}(\mathbf{B},\mathbf{p})$ *is determined by choosing centered Gaussians* $f_i(x)=e^{-\langle A_ix,x\rangle}$ *for some symmetric positive definite* $n_i\times n_i$ *matrix* A_i*,* $i=1,\ldots,k$ *and* $x\in H_i$.

Remark 4. *The Geometric Barthe's Inequality is readily a special case of* (5) *where* $\mathrm{RBL}(\mathbf{B},\mathbf{p})=1$. *We note that* (5) *is the Prékopa-Leindler inequality if* $H_1=\ldots=H_k=\mathbb{R}^n$ *and each* $B_i=I_n$*, and hence* $\mathrm{RBL}(\mathbf{B},\mathbf{p})=1$ *and* $\sum_{i=1}^k p_i=1$ *in that case.*

The condition $\sum_{i=1}^k p_in_i=n$ *makes sure that for any* $\lambda>0$*, the inequality* (5) *is invariant under replacing* $f_1(x_1),\ldots,f_k(x_k)$ *by* $f_1(\lambda x_1),\ldots,f_k(\lambda x_k)$*,* $x_i\in H_i$.

Remark 5 (The relation between $\mathrm{BL}(\mathbf{B},\mathbf{p})$ and $\mathrm{RBL}(\mathbf{B},\mathbf{p})$)**.** *For a Brascamp-Lieb datum* $\mathbf{B}=(B_1,\ldots,B_k)$ *and* $\mathbf{p}=(p_1,\ldots,p_k)$ *as in Theorem 3 and Theorem 4, possibly* $\mathrm{BL}(\mathbf{B},\mathbf{p})=\infty$ *and* $\mathrm{RBL}(\mathbf{B},\mathbf{p})=0$.

According to Barthe [6]*,* $\mathrm{BL}(\mathbf{B},\mathbf{p})<\infty$ *if and only if* $\mathrm{RBL}(\mathbf{B},\mathbf{p})>0$*, and in this case, we have*

$$\mathrm{BL}(\mathbf{B},\mathbf{p})\cdot\mathrm{RBL}(\mathbf{B},\mathbf{p})=1.\tag{6}$$

Concerning extremals in Theorem 4, Lehec [23] proved that if there exists some Gaussian extremizers for Barthe's Inequality (5), then the corresponding

Brascamp-Lieb datum $\{(B_i, p_i)\}_{i=1,\dots,k}$ is equivalent to some Geometric Brascamp-Lieb datum; therefore, the equality case of (5) can be understood via the chareacterization of equality case in the geometric case by Valdimarsson [28].

However, it is still not known whether having any extremizers in Barthe's Inequality (5) yields the existence of Gaussian extremizers. One possible approach is to use iterated convolutions and renormalizations as in Bennett, Carbery, Christ, Tao [9] in the case of Brascamp-Lieb inequality.

The importance of the Brascamp-Lieb inequality is shown by the fact that besides harmonic analysis and convex geometry, it has been also applied, for example,

- in discrete geometry, like about a quantitative fractional Helly theorem by Brazitikos [13],
- in combinatorics, like about exceptional sets by Gan [20],
- in number theory, like the paper by Guo, Zhang [22],
- to get central limit theorems in probability, like the paper by Avram, Taqqu [1].

We note the paper by Brazitikos [13] is especially interesting from the point of view that it does not simply consider the rank one Geometric Brascamp-Lieb inequality that is typically used for many inequalities in convex geometry, but an approximate version of it.

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Inequalities in convex geometry via discrete inequalities

David Alonso-Gutiérrez
(joint work with L.C. García Lirola, E. Lucas, J. Martín Goñi, J. Yepes Nicolás)

Brunn-Minkowski inequality plays a central role in convexity and it states that for any pair of compact sets, $K, L \subseteq \mathbb{R}^n$ and any $\lambda \in [0,1]$,

$$|(1-\lambda)K + \lambda L|^{\frac{1}{n}} \geq (1-\lambda)|K|^{\frac{1}{n}} + \lambda|L|^{\frac{1}{n}},$$

where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^n$.

In [3], the authors proved a discrete version of Brunn-Minkwoski inequality, in which the measure involved is the lattice point enumerator measure dG_n, defined for any $K \subseteq \mathbb{R}^n$ as the cardinality of $K \cap \mathbb{Z}^n$,

$$G_n(K) := \sharp(K \cap \mathbb{Z}^n),$$

instead of the Lebesgue measure. More precisely, the authors showed that for any non-empty bounded sets $K, L \subseteq \mathbb{R}^n$ and any $\lambda \in [0,1]$

$$G_n((1-\lambda)K + \lambda L + (-1,1)^n)^{\frac{1}{n}} \geq (1-\lambda)G_n(K)^{\frac{1}{n}} + \lambda G_n(L)^{\frac{1}{n}}.$$

In this talk we will consider several inequalities in convex geometry, in whose proof Brunn-Minkowski inequality plays an essential role, and discuss the discrete versions of such inequalities that can be obtained by means of the discrete version of Brunn-Minkowski inequality. The fact that the Minkowski addition of an open cube in the right-hand side of the discrete Brunn-Minkowski inequality is necessary for the inequality to be true will imply that some extra terms appear in the discrete versions of the geometric inequalities. However, the geometric inequalities can still obtained from their discrete counterparts.

Along the talk, we will focus in the following geometric and functional inequalities:

1. Rogers-Shephard inequality for the difference body [4]: For any convex body $K \subseteq \mathbb{R}^n$,

$$|K-K| \leq \binom{2n}{n}|K|.$$

2. Rogers-Shephard section/projection inequality [5], from which Rogers-Shephard inequality for the difference body can be obtained: For any convex body $K \subseteq \mathbb{R}^n$ and any k-dimensional linear subspace,

$$|P_{H^\perp}K| \max_{x_0 \in H} |K \cap (x_0+H)| \leq \binom{n}{k}|K|.$$

3. Berwald's inequality [1], from which the two previous inequalities can be obtained: Given a convex body $K \subseteq \mathbb{R}^n$ and $f : K \to [0,\infty)$ a concave function. Then, for every $0<p<q<\infty$

$$\left(\frac{\binom{n+q}{n}}{|K|}\int_K f^q(x)dx\right)^{\frac{1}{q}} \leq \left(\frac{\binom{n+p}{n}}{|K|}\int_K f^p(x)dx\right)^{\frac{1}{p}}.$$

4. Zhang's inequality [6], which can be obtained from Berwald's inequality: For any convex body $K \subseteq \mathbb{R}^n$

$$|K|^{n-1}|\Pi^*K| \geq \frac{1}{n^n}\binom{2n}{n},$$

where Π^*K denotes the polar projection body of K, which is the unit ball of the norm given by $\|x\| = |x||P_{x^\perp}K|$.

5. The following direct consequence of Borell's lemma [2]: There exists an absolute constant $C>0$ such that for any convex body $K \subseteq \mathbb{R}^n$ containing the origin in its interior, every $1<p<q<\infty$ and every $\theta \in S^{n-1}$, the Euclidean unit sphere,

$$\left(\mathbb{E}|\langle X,\theta\rangle|^q\right)^{\frac{1}{q}} \leq C\frac{q}{p}\left(\mathbb{E}|\langle X,\theta\rangle|^p\right)^{\frac{1}{p}},$$

where X is a random vector uniformly distributed on K.

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Functional volume product along the Fokker–Planck heat flow

Dylan Langharst
(joint work with Dario Cordero-Erausquin, Matthieu Fradelizi)

Recently, I have been primarily working on functional Santanló inequalities. We will denote by $\mathrm{Vol}_n(A)$ the volume (Lebesgue measure) of a Borel set $A \subset \mathbb{R}^n$. The polar of a convex body (compact, convex set with non-empty interior) $K \subset \mathbb{R}^n$ is precisely the set

$$K^\circ := \{x \in \mathbb{R}^n : x \cdot y \leq 1 \, \forall y \in K\}.$$

Santaló's inequality states that for a symmetric convex body K (i.e. $K = -K$) one has

$$\mathrm{Vol}_n(K)\mathrm{Vol}_n(K^\circ) \leq \mathrm{Vol}_n(B_2^n)^2,$$

with equality if and only if K is a centered ellipsoid. Here, B_2^n is the unit Euclidean ball. This was shown by Blaschke and Santaló [7]. I recommend the elegant proof by Meyer and Pajor [5] using Steiner symmetrization. The concept of polarity can be extended to functions. Given a non-identically zero function $f : \mathbb{R}^n \to [0, \infty)$, its polar is given by

$$f^\circ(x) = \inf_{y \in \mathbb{R}^n} \frac{e^{-x \cdot y}}{f(y)}.$$

With this definition in mind, we have Santaló's inequality for even functions:

$$M(f) := \int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} f^\circ \leq M(e^{-|x|^2}),$$

with equality if and only if f is Gaussian (i.e. $f(x) = e^{-Ax \cdot x + c}$ for some positive definite A matrix and $c \in \mathbb{R}$), which was proven by Ball, in his Ph.D thesis [2], and by Artstein-Klartag-Milman [1].

Terence Tao once suggested long along on a connection between polarity and the Laplace transform. We paraphrase him now [9]: "if f is log-concave and $p \in (0, 1)$, then the Laplace transform of $f^{\frac{1}{p}}(x)$ is essentially $(f^{\frac{1}{p}})^\circ(\frac{x}{p})$, ignoring lower-order contributions... in which case Klartag's formulation of Santaló's inequality begins to look quite a lot like Beckner's sharp Hausdorff-Young inequality as $p \to 0^+$."

We recall that the Laplace transform of a nonnegative function is

$$Lf(y) := \int_{\mathbb{R}^n} f(x)\, e^{x\cdot y}\, dx \qquad \forall y \in \mathbb{R}^n.$$

Note that Lf is always log-convex. We define, for $p \in (0,1)$, the L^p Laplace transform as

$$\mathcal{L}_p(f)(x) := (L(f^{\frac{1}{p}}))(x)^{\frac{p}{p-1}} = \Big(\int_{\mathbb{R}^n} f^{\frac{1}{p}}(y)\, e^{x\cdot y}\, dy\Big)^{\frac{p}{p-1}} \qquad \forall x \in \mathbb{R}^n.$$

Note that $\mathcal{L}_p(f)$ is always log-concave. The L^p Laplace transform converges to polarity in some sense:

$$\begin{aligned}\lim_{p\to 0^+} \mathcal{L}_p(f)(x/p) &= \lim_{p\to 0^+} \left(\int_{\mathbb{R}^n} (e^{x\cdot y} f(y))^{\frac{1}{p}}\, dy\right)^{\frac{p}{p-1}} \\ &= f^{\square}(x) = \operatorname*{ess\,inf}_{y\in\mathbb{R}^n} \frac{e^{-x\cdot y}}{f(y)} \geq \inf_{y\in\mathbb{R}^n} \frac{e^{-x\cdot y}}{f(y)} = f^{\circ}(x).\end{aligned}$$

The difference between $f^{\square}$ and f° can be quite dramatic. By considering $f(y) = e^{-|y|^2/2} + \delta(y)$, where $\delta(y)$ is the characteristic function of the origin, we see that

$$f^{\circ}(x) = \begin{cases} \frac{1}{2}, & \text{if } |x|^2 \leq 2\ln(2), \\ e^{-|x|^2/2}, & \text{otherwise.} \end{cases}$$

But, $f^{\square}(x) = (e^{-|y|^2/2})^{\circ}(x) = e^{-|x|^2/2}$. Nakamura-Tsuji [6] recently showed the Laplace-Santaló inequality for even functions: let f be an even function such that $\int_{\mathbb{R}^n} f \in (0,\infty)$. Then,

$$M_p(f) := \int_{\mathbb{R}^n} f \left(\int_{\mathbb{R}^n} \mathcal{L}_p(f)\left(\frac{x}{p}\right)\right)^{1-p} \leq M_p(e^{-|x|^2}),$$

with equality when f is Gaussian. And actually, they showed much more. We recall that, for a (nonnegative) integrable function f its Fokker-Planck flow is $P_0 f = f$, and, for $t > 0$,

$$\begin{aligned}P_t f(x) &= e^{nt/2} \int_{\mathbb{R}^n} f(y) e^{\frac{-|e^{t/2}x - y|^2}{2(e^t-1)}} \frac{dy}{(2\pi(e^t-1))^{\frac{n}{2}}} \\ &= \left(\int_{\mathbb{R}^n} f(y) e^{\frac{e^{t/2}}{e^t-1} x\cdot y - \frac{1}{2(e^t-1)}|y|^2}\, dy\right) \frac{e^{-\frac{1}{1-e^{-t}}|x|^2/2}}{(2\pi(1-e^{-t}))^{\frac{n}{2}}}.\end{aligned}$$

It verifies the equation $\partial_t P_t f = \mathcal{D}^{\star} P_t f$, where $\mathcal{D}^{\star} f = \frac{1}{2}\big(\Delta f + \operatorname{div}_x(xf)\big)$.

The full inequality by Nakamura-Tsuji for the functional volume product along the Fokker-Planck flow is that, if f is even and integrable, then

$$t \mapsto M_p(P_t f)$$

is increasing in t. This implies, by sending $p \to 0^+$, that $t \mapsto M(P_t f)$ is monotonically increasing when f is even, integrable and regular enough so that $f^{\square} = f^{\circ}$.

Myself, along with Dario Cordoero-Erausquin and Matthieu Fradelizi, were interested in expanding the inequality by Nakamura-Tsuji to non-even f. We first recall the convex setting. One can verify for a convex body K that

$$\mathrm{Vol}_n(K^\circ) < \infty \longleftrightarrow o \in \mathrm{int}(K).$$

Thus, we see that the placement of the origin, denoted o, is vital. Denote by

$$b(f) = \frac{1}{\|f\|_{L^1(\mathbb{R}^n)}} \int_{\mathbb{R}^n} x f(x) dx$$

the barycenter of an integrable function f, and $b(K) := b(\mathbf{1}_K)$ the barycetner of K. Then,

$$\mathrm{Vol}_n(K)\mathrm{Vol}_n((K - s(K))^\circ) \leq \mathrm{Vol}_n(K)\mathrm{Vol}_n((K - b(K))^\circ) \leq \mathrm{Vol}_n(B_2^n)^2,$$

as shown by Petty [8]. Here, $s(K)$, the Santaló point of K, is so that $b((K - s(K))^\circ) = o$. It has the property that $s(K) = \mathrm{argmin}_{z\in\mathbb{R}^n}\mathrm{Vol}_n((K - z)^\circ)$. The functional setting is very similar. Set $\tau_z f(x) = f(x - z)$. Then, the functional Santaló inequality stats that, for a nonnegative function f so that $\int_{\mathbb{R}^n} f \in (0, \infty)$, one has

$$M(f) := \left(\int_{\mathbb{R}^n} f\right) \inf_z \left(\int_{\mathbb{R}^n} (\tau_z f)^\circ\right) \leq \left(\int_{\mathbb{R}^n} f\right)\left(\int_{\mathbb{R}^n} (\tau_{-b(f)} f)^\circ\right) \leq M(e^{-|x|^2}),$$

with equality if and only if f is Gaussian. This was shown by Artstein-Klartag-Milman [1] and Lehec [4]. Furthermore, they established that the infimum is obtained at a unique point, the Santaló point of f. Since $(\tau_z f)^\circ(x) = f^\circ(x)e^{-x\cdot z}$, one has $\int_{\mathbb{R}^n} (\tau_z f)^\circ = L[f^\circ](-z)$.

Now that we have provided proper context, I can state some new results we proved [3].

Theorem 1 (L^p Santaló's inequality)**.** *Let f be a nonnegative function, $f \not\equiv 0$. Then,*

$$M_p(f) := \int_{\mathbb{R}^n} f \inf_{z\in\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \mathcal{L}_p(\tau_z f)\left(\frac{x}{p}\right)\right)^{1-p} \leq M_p(e^{-|x|^2}),$$

with equality for Gaussians.

An interesting phenomenon happens. Observe that $\mathcal{L}_p(\tau_z f)(x) = \mathcal{L}_p(f)(x)\, e^{\frac{p}{p-1}x\cdot z}$, and so the second integral is really $L\left(\mathcal{L}_p(f)\right)\left(-\frac{1}{1-p}z\right)^{1-p}$. If the infimum is obtained, then it is obtained at a unique point, which is precisely the barycenter of $\mathcal{L}_p(f)$.

Theorem 2 (Laplace-Santaló's inequality)**.** *Let f be a nonnegative function, $f \not\equiv 0$. Suppose either f or $\mathcal{L}_p(f)$ have barycenter at the origin. Then,*

$$\int_{\mathbb{R}^n} f \left(\int_{\mathbb{R}^n} \mathcal{L}_p(f)\left(\frac{x}{p}\right)\right)^{1-p} \leq M_p(e^{-|x|^2}),$$

with equality for Gaussians.

We also, of course, establish results along the Fokker-Planck heat semi-group.

Theorem 3 (Monotonicity of the functional L^p Volume product under heat flow)**.** *Let f be a nonnegative function, $f \not\equiv 0$. Then*

$$t \to M_p(P_t f)$$

is increasing in t. In particular, by sending $p \to 0^+$, we deduce that

$$t \to M(P_t f)$$

is increasing in t, if f is regular enough so that $f^{\square} = f^{\circ}$.

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Brunn–Minkowski inequalities for autonomous Hamiltonians on weighted manifolds

Rotem Assouline

The classical Brunn-Minkowski inequality asserts that for every pair of Borel measurable, nonempty subsets $A_0, A_1 \subseteq \mathbb{R}^n$ and every $0 < \lambda < 1$,

$$\mathrm{Vol}(A_\lambda) \geq \mathrm{Vol}(A_0)^{1-\lambda}\,\mathrm{Vol}(A_1)^{\lambda},$$

where Vol is the Lebesgue measure and

$$A_\lambda := \{(1-\lambda)a_0 + \lambda a_1 \mid a_0 \in A_0, a_1 \in A_1\}$$

is the *Minkowski λ-average* of A_0 and A_1. More generally, by the Prékopa-Leindler inequality, if μ is a measure on $\mathbb{R}^n$ with a log-concave density, then

$$\mu(A_\lambda) \geq \mu(A_0)^{1-\lambda}\,\mu(A_1)^{\lambda}.$$

A celebrated result of Cordero-Erausquin, McCann and Schmuckenschläger [12] and Sturm [23] generalizes the Prékopa-Leindler inequality to Riemannian manifolds: if (M,g) is a complete Riemannian manifold and $\mu = e^{-\psi}\mathrm{Vol}_g$, where Vol_g is the Riemannian measure and $\psi : M \to \mathbb{R}$ is a smooth function satisfying

$$\mathrm{Ric_g} + \mathrm{Hess}\psi \geq 0, \tag{1}$$

then for every pair of Borel measurable subsets $A_0, A_1 \subseteq M$ such that $\mu(A_0)\mu(A_1) > 0$ and every $0 < \lambda < 1$,

$$\mu(A_\lambda) \geq \mu(A_0)^{1-\lambda}\,\mu(A_1)^\lambda, \tag{2}$$

where A_λ is the set of λ-*midpoints*:

$$A_\lambda := \{\gamma(\lambda T) \mid \gamma \text{ is a unit-speed minimizing geodesic, } \gamma(0) \in A_0,\ \gamma(T) \in A_1\}.$$

In fact, the validity of inequality (2) for every data A_0, A_1, λ is *equivalent* to condition (1), see [20].

Consider now the case where the Riemannian metric is replaced by a more general *Lagrangian*, i.e. a function $L : TM \to \mathbb{R}$. Such a Lagrangian induces a *cost* function on the manifold, defined by

$$\mathrm{c}(x,y) := \inf\left\{ \int_0^T L(\dot\gamma(t))dt \,\middle|\, \gamma \in C^1([0,T],M),\ \gamma(0) = x,\ \gamma(T) = y,\ T > 0 \right\}.$$

A curve attaining this infimum is called a *minimizing extremal.* If L comes from a Riemannian metric g, i.e. when

$$L(v) = \frac{|v|_g^2}{2} + \frac{1}{2}, \qquad v \in TM,$$

then c is distance and minimizing extremals are unit-speed minimizing geodesics. We can similarly define

$$A_\lambda := \{\gamma(\lambda T) \mid \gamma \text{ is a minimizing extremal, } \gamma(0) \in A_0,\ \gamma(T) \in A_1\},$$

and, after fixing a smooth measure μ on M, we can ask whether the the Brunn-Minkowski inequality (2) holds. In the Riemannian case, the answer is given by [12, 23]. The more general case where $L = (F^2+1)/2$ where F is a *Finsler* metric was treated in [22]. Related results include Brunn-Minkowski inequalities on metric measure spaces [9], on sub-Riemannian manifolds [4, 5], and on Lorentzian spaces [7, 10]. The horocyclic Brunn-Minkowski proved in [2] and the magnetic Brunn-Minkowski proved in [1] can be interpreted as Brunn-Minkowski inequalities for the *magnetic* Lagrangian

$$L(v) := \frac{|v|_g^2}{2} + \frac{1}{2} - \eta(v), \qquad v \in TM,$$

where g is a Riemannian metric and η is a one-form.

Let M be a smooth manifold and let $H : T^*M \to \mathbb{R}$ be a function satisfying the following assumptions:

1. <u>Smoothness:</u> $H \in C^2(T^*M)$.
2. <u>Strong convexity:</u> $\nabla^2 H > 0$ on each fiber.

(3) Superlinearity: For every compact $A \subseteq M$ and every Riemannian metric g on M there exists $C_{A,g} > 0$ such that
$$H(p) \geq |p|_g - C_{A,g}, \qquad p \in T_x^*M,\, x \in A.$$
(4) Mañé supercriticality: For every closed curve γ, $\int L(\dot\gamma) > 0$.
(5) "Geodesic" convexity: every $x, y \in M$ are joined by a minimizing extremal.
(6) Properness: For every compact $A \subseteq M$ there exists a compact $\tilde{A} \subseteq M$ such that every minimizing extremal with endpoints in A lies in $\tilde{A}$.

The function H will be called a *Hamiltonian.* Assumptions (1)-(3) are classical and Hamiltonians satisfying them are called *Tonelli.* Assumption (4) guarantees that the cost is not identically $-\infty$, and assumptions (5)-(6) can be attained by requiring uniform growth with respect to a complete Riemannian metric [14, 11].

The Lagrangian L associated to the Hamiltonian H is
$$L(v) := \sup\left\{p(v) - H(p) \mid p \in T_x^*M\right\}, \qquad v \in T_xM,\, x \in M.$$
For a C^2 function $u : M \to \mathbb{R}$, we define its *gradient* with respect to H to be
$$\nabla^H u := \mathcal{L}du,$$
where $\mathcal{L} : T^*M \to TM$ is the *Legendre transform* associated to L:
$$\left.\frac{\partial L}{\partial v}\right|_{v=\mathcal{L}p} = p.$$
If, in addition, we have a smooth measure μ on M (i.e. a measure induced by a volume form or, if M is not oriented, by a density), we can also define a *Laplacian*:
$$\Delta_\mu^H u := \operatorname{div}_\mu(\nabla^H u).$$

In analogy to the curvature-dimension condition of Bakry-Émery [3] and Sturm-Lott-Villani [23, 19], we will say that μ satisfies $\mathrm{CD}_H(0,\infty)$ if for every solution u to the Hamilton-Jacobi equation
$$H(du) = 0,$$
defined on some open subset $U \subseteq M$, the *Bochner inequality* holds:
$$(d\Delta_\mu^H u)(\nabla^H u) \leq 0.$$
In the Riemannian case, this is equivalent to (1). A more general curvature-dimension condition $\mathrm{CD}_H(K,N)$ can be defined similarly for other $K \in \mathbb{R}, N \neq 1$.

Theorem (A. 24'+)**.** *Suppose that the measure μ satisfies $\mathrm{CD}_H(0,\infty)$. Then for every pair $A_0, A_1 \subseteq M$ of Borel sets with $\mu(A_0)\mu(A_1) > 0$ and every $0 < \lambda < 1$,*
$$\mu(A_\lambda) \geq \mu(A_0)^{1-\lambda}\mu(A_1)^\lambda.$$

The proof of the theorem uses the *needle decomposition technique.* By adapting the proof of Klartag's needle decomposition theorem [17], which is based on L^1 mass transport [13, 8, 15], to the more general Lagrangian setting, using results on optimal transport for Lagrangian costs found in [6, 16, 14], one obtains a needle decomposition theorem for Hamiltonians, which is the main ingredient of the proof.

Remark. A notion of Ricci curvature for Hamiltonians, and its relation to convexity of entropy along displacement interpolation, was studied in [21, 18]. Its precise relation to the condition formulated above remains to be explored. Note that our condition only depends on the behavior of H near the level set $\{H = 0\}$.

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Fixed and periodic points of the intersection body operator

Emanuel Milman
(joint work with Shahar Shableman and Amir Yehudayoff)

The intersection body IK of a star-body K in $\mathbb{R}^n$ was introduced by E. Lutwak following the work of H. Busemann, and plays a central role in the dual Brunn-Minkowski theory. It is defined as:

$$IK = \{r\theta \; ; \; r \in [0, \rho_{IK}(\theta)], \theta \in \mathbb{S}^{n-1}\} \; , \; \rho_{IK}(\theta) := |K \cap \theta^{\perp}|_{n-1},$$

where $|\cdot|_k$ denotes the k-dimensional Hausdorff measure.

We show that when $n \geq 3$, $I^2 K = cK$ for some $c > 0$ iff K is a centered ellipsoid, and hence $IK = cK$ iff K is a centered Euclidean ball, answering long-standing questions by Lutwak [4], Gardner [3], and Fish–Nazarov–Ryabogin–Zvavitch [2]. An equivalent formulation of the latter in terms of non-linear harmonic analysis states that a non-negative $\rho \in L^{\infty}(\mathbb{S}^{n-1})$ satisfies $\mathcal{R}(\rho^{n-1}) = c\rho$ for some $c > 0$ iff

ρ is constant, where $\mathcal{R}$ denotes the spherical Radon transform. Recall that $\mathcal{R}$ acts on continuous functions $f : \mathbb{S}^{n-1} \to \mathbb{R}$ via:

$$\mathcal{R}(f)(\theta) := \int_{\mathbb{S}^{n-1}\cap\theta^\perp} f(u)d\sigma(u),$$

where σ denotes the Haar probability measure on $\mathbb{S}^{n-1} \cap \theta^\perp$, and extends to $f \in L^2(\mathbb{S}^{n-1})$ by continuity.

Our proof is entirely geometrical. We first derive new formulas for the volume $|IK|$ of IK, which easily imply the monotonicity of $|IK|$ under Steiner symmetrization $S_u K$ (a result first established by Adamczak–Paouris–Pivovarov–Simanjuntak [1]), but also allow analysis of the equality case $|IK| = |I(S_u K)|$. We then recast the iterated intersection body equation $I^2K = cK$ as an Euler-Lagrange equation for the functional $\mathcal{F}(K) := |IK| - (n-1)c|K|$ under (admissible) radial perturbations. We introduce a continuous version of Steiner symmetrization $\{S_u^t K\}_{t\in[0,1]}$ of a Lipschitz star-body K in a.e. direction $u \in \mathbb{S}^{n-1}$, and show that it preserves the property of being a Lipschitz star-body for all $t \in [0,1]$ and serves as an admissible radial perturbation of K. Finally, we show that when $n \geq 3$, a Lipschitz star-body $K \subset \mathbb{R}^n$ satisfies that $\frac{d}{dt}|_{t=0^+}|I(S_u^t K)| = 0$ for a.e. $u \in \mathbb{S}^{n-1}$ iff K is a centered ellipsoid, thereby yielding a characterization of solutions to the corresponding Euler-Lagrange equation $I^2K = cK$.

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The volume of the convolution body is not maximized by ellipsoids

Julián Haddad

Let $K \subseteq \mathbb{R}^n$ be a convex body and let $g_K(x) = |K \cap (K + x)|_n$ denote the covariogram function, where $|\cdot|_n$ is the n-dimensional Lebesgue measure. For $\delta \in (0, 1)$, the convolution body of K of parameter δ is the set defined by

$$C_\delta K = \{x \in \mathbb{R}^n : g_K(x) \geq \delta |K|_n\}.$$

The set $C_\delta K$ is called the convolution body of K, due to the fact that g_K is the convolution of the indicator functions of K and $-K$.

When $\delta \to 1^-$ the set $C_\delta K$ collapses to the origin. The shape of $C_\delta K$, if scaled by a factor $(1-\delta)^{-1}$, approaches the polar projection body of K denoted by $\Pi^* K$, which is the unit ball of the norm defined by

$$\|v\|_{\Pi^* K} = |P_{v^\perp} K|_{n-1}$$

for every unit vector $v \in S^{n-1}$, where $P_{v^\perp}$ is the orthogonal projection to the hyperplane orthogonal to v. More precisely, (see [4, Theorem 2.2])

$$\lim_{\delta\to 1^-} \frac{|C_\delta K|_n}{(1-\delta)^n} = |\Pi^* K|_n. \tag{1}$$

The classical Petty projection inequality (see Section 10.9 of [5]) states that

$$|\Pi^* K|_n \leq |\Pi^* B_K|_n \tag{2}$$

where B_K is the Euclidean ball with same volume as K. Equality holds in (2) if and only if K is an ellipsoid.

At the opposite endpoint, $\delta \to 0^+$, the body $C_\delta K$ converges to the difference body of K, defined by

$$DK = \{x - y : x, y \in K\}.$$

By the Brunn-Minkowski inequality and its equality case one can show that

$$|DK|_n \geq |DB_K|_n, \tag{3}$$

which is reverse to the inequality (2). Nevertheless, (3) is an equality for all symmetric sets.

An extension of the Petty projection inequality to certain averages of volumes of $C_\delta K$ can be deduced from the results in [3].

Theorem 1. *For every non-decreasing function $\omega : [0,1] \to [0,\infty)$ and every convex body K,*

$$\int_0^1 \omega(\delta)|C_\delta K|_n d\delta \leq \int_0^1 \omega(\delta)|C_\delta B_K|_n d\delta.$$

The results in [3] follow from the well-known Riesz convolution inequality, and Theorem 1 recovers the Petty projection inequality (without the equality case) thanks to (1) and a limit argument.

A second application of the Riesz convolution inequality to convex bodies defined from $C_\delta K$, was given in [2]. For every convex body K and $p > -1, p \neq 0$, the p-radial mean body of K is the radial body defined by

$$\rho_{R_p K}(v) = \left(\int_0^1 \rho_{C_\delta K}(v)^p d\delta\right)^{1/p},$$

while $R_0 K$ is defined as a limit of the sets $R_p K$ when $p \to 0$.

Theorem 2 ([2, Theorem 20])**.** *For every convex body K and $p \in (-1, n)$,*

$$|R_p K|_n \leq |R_p B_K|_n.$$

For $p > n$ the inequality is reversed. Equality holds if and only if K is an ellipsoid.

It was proven in [1] that $R_p K$ approaches $\Pi^* K$ when $p \to -1^+$, so Theorem 2 is yet an other extension of the Petty projection inequality involving averages of $C_\delta K$.

Theorems 1 and 2 suggest the possibility that for a fixed $\delta \in (0,1)$, $|C_\delta K|_n$ is also maximized by ellipsoids, among sets of a fixed volume. Of course, due to (3) this is only possible if we restrict the problem to the symmetric case, or to some range of $\delta \in (0,1)$ far from 0. Let us formulate the weakest possible question:

Question 3. *Is there a value of $\delta \in (0,1)$ such that*

$$|C_\delta K|_n \leq |C_\delta B_K|_n \tag{4}$$

for every symmetric convex body K?

Proposition 4. *For every convex body $K \subseteq \mathbb{R}^n$ which is not an ellipsoid, $|C_\delta K|_n \leq |C_\delta B_K|_n$ for every $\delta > \varphi(d_{\mathrm{BM}}(K, \mathbb{B}))$, where $\varphi : [0, \infty) \to (0, 1]$ is a continuous function with $\varphi(t) = 1$ if and only if $t = 0$.*

Proposition 4 reduces the problem to a local question: If (4) is valid for every K sufficiently close to the Euclidean ball and δ close to 1, then thanks to Proposition 4, it is valid for every K and δ close to 1.

Definition 5. *For any positive continuous function ρ defined on S^{n-1} we will consider a one-parameter family of radial bodies K_t defined by*

$$\rho_{K_t}(v) = 1 + t\rho(v). \tag{5}$$

We also define

$$\overline{K_t} = K_t / |K_t|_n^{1/n}. \tag{6}$$

We will analyze $\left|C_\delta \overline{K_t}\right|_n$ as a function of t and δ, for t near 0. First we obtain:

Theorem 6. *For every C^1 radial set $K \subseteq \mathbb{R}^n$ and $\delta \in (0,1)$, the function $t \mapsto \left|C_\delta \overline{K_t}\right|_n$ is C^1 and we have*

$$\frac{\partial}{\partial t}\left|C_\delta \overline{K_t}\right|_n \Big|_{t=0} = 0.$$

Then it suffices to analyze the second derivative of $t \mapsto \left|C_\delta \overline{K_t}\right|_n$. We completely describe the limit when $\delta \to 1^-$ of this second derivative, and its sign is compatible with the fact that $t \mapsto \left|\Pi^* \overline{K_t}\right|_n$ has a maximum at $t = 0$.

Theorem 7. *For every C^2 smooth radial set $K \subseteq \mathbb{R}^2$ the function $t \mapsto \left|C_\delta \overline{K_t}\right|_n$ is C^2 for every $\delta \in (0,1)$ and*

$$\lim_{\delta \to 1^-} \frac{1}{(1-\delta)^2} \frac{\partial^2}{\partial t^2}\left|C_\delta \overline{K_t}\right|_2 \Big|_{t=0} \leq 0. \tag{7}$$

Equality holds if and only if ρ_K is the restriction of a polynomial of degree 2 to the unit circle.

The equality cases of Theorem 7 correspond to variations $\overline{K_t}$ that coincide up to first order with families of ellipsoids.

At this point it is natural to expect that Theorem 7 combined with an approximation argument and Proposition 4, could yield a positive answer to Question 3. However, for this argument to be complete we need the convergence of the second derivatives of the volume as $\delta \to 1^-$, to be uniform with respect to K. We were unable to show this uniform convergence, and the following counterexample shows why:

Theorem 8. *Let $K^m \subseteq \mathbb{R}^2$ be the (symmetric) radial set defined by $\rho_{K^m}(v) = \cos(2mv)^2$ with $v \in [0, 2\pi]$. Then for every $\delta \in (0,1)$ there exists $m \in \mathbb{N}$ such that*

$$\frac{\partial^2}{\partial t^2} |C_\delta \overline{K_t^m}|_2 \Big|_{t=0} > 0.$$

As a consequence, we get a negative answer to Question 3 in dimension 2, and every value of $\delta \in (0,1)$.

Theorem 9. *For every $\delta \in (0,1)$ there exists a symmetric convex body $K \subseteq \mathbb{R}^2$ such that $|C_\delta K|_n > |C_\delta B_K|_n$. Moreover, K can be chosen arbitrarily close to the Euclidean ball in the C^∞ topology.*

The following natural question remains open:

Question 10. *For each fixed $\delta \in (0,1)$, what convex bodies are maximizers of $C_\delta K$ when K runs among sets of the same volume?*

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Two results on the homothety conjecture for convex bodies of flotation on the plane

Vladyslav Yaskin

(joint work with M. Angeles Alfonseca, Fedor Nazarov, Dmitry Ryabogin, Alina Stancu)

Let K be a star body in $\mathbb{R}^2$. For every $\theta \in \mathbb{R}$ and the corresponding unit vector $e(\theta) = (\cos\theta, \sin\theta)$ and for every $t \in \mathbb{R}$, define the half-planes

$$W^+(\theta, t) = \{x : \langle x, e(\theta)\rangle \geq t\} \quad \text{and} \quad W^-(\theta, t) = \{x : \langle x, e(\theta)\rangle \leq t\}.$$

If $0 < \delta < \frac{1}{2}\mathrm{vol}_2(K)$, then for every $\theta \in \mathbb{R}$, there is a unique $t(\theta)$ such that

$$\mathrm{vol}_2(W^+(\theta, t(\theta)) \cap K) = \delta.$$

The corresponding *convex body of flotation* K_δ (also known as the convex floating body, introduced in the works of Bárány, Larman [1] and Schuett, Werner [2]) is defined as

$$K_\delta = \bigcap_{\theta \in \mathbb{R}} W^-(\theta, t(\theta)).$$

The *homothety conjecture* in $\mathbb{R}^2$ (see [3]) says that if a convex body K is homothetic to one of its convex bodies of flotation, i.e.,

$$K_\delta = \lambda K,$$

for some δ and $\lambda > 0$, then K is an ellipse.

We prove two theorems. The first one says that on the plane the homothety conjecture holds for origin symmetric convex bodies in a small neighborhood of the unit disk $B = \{x \in \mathbb{R}^2 : |x| \leq 1\}$.

Theorem 1. *For every $\mathcal{D} \in (0, \frac{1}{2})$, there is $\gamma > 0$ such that if*

$$K_\delta = \lambda K, \qquad \text{with } \delta = \mathcal{D}\mathrm{vol}_2(K),$$

for some $\lambda > 0$, $K \subset \mathbb{R}^2$ is origin symmetric, and $(1-\gamma)B \subset K \subset (1+\gamma)B$, then K is an ellipse.

The second theorem shows that in the asymmetric case, the homothety conjecture fails.

Theorem 2. *There exists an asymmetric convex body $K \subset \mathbb{R}^2$ such that*

$$K_\delta = \lambda K$$

for some $\delta > 0$ and $\lambda > 0$.

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On weighted Blaschke–Santaló inequalities and strong Brascamp–Lieb inequalities

Galyna V. Livshyts

Recall the classical Blaschke-Santalo inequality for a symmetric convex set K in $\mathbb{R}^n$:

$$|K| \cdot |K^\circ| \leq |B_2^n|^2.$$

Recall that if K is a symmetric convex body in $\mathbb{R}^n$ with support function h_K, then $\rho_{K^\circ} = 1/h_K$, and we have

$$|K^\circ| = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_K^{-n}(\theta)d\theta.$$

Therefore, using $h_{K^\circ} = \rho_K$, we get in dimension 2,

$$|K^o| = \frac{1}{2}\int_{-\pi}^{\pi} h_K^{-2}(\theta)d\theta.$$

Recall also Santaló's formula for the volume of a 2-dimensional convex body:

$$|K| = \frac{1}{2}\int_{\mathbb{S}^1} \left(h_K^2(\theta) - \dot{h}_K^2(\theta)\right) d\theta. \tag{1}$$

Note that the Blaschke-Santalo inequality in $\mathbb{R}^2$ implies $|K| \cdot |K^\circ| \leq \pi^2$, for a symmetric convex body K. Thus,

$$\left(\frac{1}{2}\int_{-\pi}^{\pi}\left(h_K^2(\theta)-\dot{h}_K^2(\theta)\right)d\theta\right)\cdot\left(\frac{1}{2}\int_{-\pi}^{\pi}h_K^{-2}(\theta)d\theta\right)\leq \pi^2,$$

which gives

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(h_K^2(\theta)-\dot{h}_K^2(\theta)\right)d\theta\right)\cdot\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}h_K^{-2}(\theta)d\theta\right)\leq 1.$$

In fact, we do not need to assume that h is a support function, since "convexifying" an arbitrary function would only make the LHS in the above expression greater. We therefore get that for any $f : [-\pi,\pi] \to \mathbb{R}$ such that $f \in C^1$, π-periodic,

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\dot{f}^2(\theta)-f^2(\theta)\right)d\theta\right)+\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}f^{-2}(\theta)d\theta\right)^{-1}\geq 0. \tag{2}$$

By making a change $\phi(\theta) = f(\frac{\theta}{2})$, we get that for all smooth $2\pi-$periodic functions ϕ,

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi^2\right)-\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi^{-2}\right)^{-1}\leq 4\cdot\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\dot{\phi}^2\right). \tag{3}$$

This result looks similar to the p-Beckner inequality. We recall it here for completeness: for $p \in [1,2)$,

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi^2\right)-\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi^{p}\right)^{\frac{2}{p}}\leq (2-p)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\dot{\phi}^2\right).$$

We have shown that the Blaschke-Santalo inequality in $\mathbb{R}^2$ implies the p-Beckner inequality on the circle for $p = -2$.

Next, let us switch gears and discuss the recent joint work with Colesanti, Kolesnikov and Rotem. A remarkable functional form of the Blaschke–Santaló inequality was discovered by K. Ball [3]. Let Φ be an arbitrary proper even function on $\mathbb{R}^n$ with values in $(-\infty,+\infty]$ and

$$\Phi^*(y)=\sup_{x\in\mathbb{R}^n}\left(\langle x,y\rangle-\Phi(x)\right)$$

be its Legendre transform. Then

$$\int e^{-\Phi(x)}dx\int e^{-\Phi^*(y)}dy\leq(2\pi)^n. \tag{4}$$

The equality holds if and only if $\Phi = a+\langle Ax, x\rangle$ for some symmetric non-degenerate matrix A. This result was later generalized by Artstein-Avidan, Klartag, Milman [1], and Fradelizi, Meyer [5]. Among many consequences and applications (see e.g. [2]), recall that (4) implies the sharp Gaussian Poincaré inequality for even functions: when f is even and smooth on $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n}f^2d\gamma-\left(\int_{\mathbb{R}^n}fd\gamma\right)^2\leq\frac{1}{2}\int_{\mathbb{R}^n}|\nabla f|^2d\gamma.$$

In order to see this, observe from (4) that

$$\frac{d}{dt^2}\left(\int e^{-(x^2/2+tf)}\int e^{-(x^2/2+tf)^*}\right)\le 0.$$

Moreover, Klartag [6] shown that for any symmetric convex set K,

$$\int_K e^{-\Phi(x)}dx\int_K e^{-\Phi^*(y)}dy\le\left(\int_K e^{-x^2/2}dx\right)^2. \tag{5}$$

Similarly, this implies, for any symmetric convex set K,

$$\frac{1}{\gamma(K)}\int_K f^2d\gamma-\left(\frac{1}{\gamma(K)}\int_K fd\gamma\right)^2\le\frac{1}{2\gamma(K)}\int_K|\nabla f|^2d\gamma.$$

The above is originally due to Cordero-Erausquin, Fradelizi, Maurey [4], and in the case of $f=x^2$ it boils down to the inequality

$$\frac{1}{\gamma(K)}\int_K x^4d\gamma-\left(\frac{1}{\gamma(K)}\int_K x^2d\gamma\right)^2\le\frac{2}{\gamma(K)}\int_K x^2d\gamma,$$

which implies that $\frac{d^2}{dt^2}\log\gamma(e^tK)\le 0$ at $t=0$, and by a homogeneity argument, this implies that $\gamma(e^tK)$ is log-concave in t – which is the content of their celebrated B-theorem.

More generally, the B-conjecture asks if $\mu(e^tK)$ is log-concave for an even log-concave measure μ and a symmetric convex K. Motivated by this circle of questions, we showed the following collection of results.

Theorem 1. *Let $p>1$ and let V be an even strictly convex p-homogeneous C^2 function on $\mathbb{R}^n$. Assume that V is an unconditional function, and that the function*

$$x=(x_1,\dots,x_n)\mapsto V\left(x_1^{\frac{1}{p}},...,x_n^{\frac{1}{p}}\right)$$

is concave in

$$\mathbb{R}^n_+=\{(x_1,\dots,x_n)\colon x_i\ge 0,\quad 1\le i\le n\}.$$

Then

$$\int e^{-\Phi(x)}dx\left(\int e^{-\frac{1}{p-1}\Phi^*(\nabla V(x))}dx\right)^{p-1}\le\left(\int e^{-V(x)}dx\right)^p, \tag{6}$$

holds for every unconditional convex Φ.

Assume, in addition, that for every coordinate hyperplane H, with unit normal e, and for every $x'\in H$, the function $\varphi\colon[0,+\infty)\to\mathbb{R}$ defined by

$$\varphi(t)=\det D^2V^*(x'+te)$$

is decreasing. Then inequality (6) holds for every even convex Φ.

As a consequence of this result, we get that under the full set of assumptions of Theorem 1, for any even f,

$$\int_{\mathbb{R}^n}f^2d\gamma-\left(\int_{\mathbb{R}^n}fd\gamma\right)^2\le\frac{p-1}{p}\int_{\mathbb{R}^n}\langle(\nabla V)^{-1}\nabla f,\nabla f\rangle d\gamma.$$

In the follow up works, we hope to be able to obtain a restriction of this inequality to a symmetric convex set K, for some measures $d\mu = e^{-V}dx$. B-conjecture for those measures would then follow by plugging $f = \langle \nabla V, x\rangle$.

We mention also that our results yield an interesting inequality about convex bodies, under the assumptions of the theorem:

$$|K| \cdot |\nabla V^*(K^o)|^{p-1} \leq \left|\left\{V \leq \frac{1}{p}\right\}\right|^p . \tag{7}$$

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Measure comparison problems for dilations of convex bodies

Artem Zvavitch

(joint work with Malak Lafi)

In 1956, Busemann and Petty posed the following volume comparison problem: Let K and L be symmetric convex bodies in $\mathbb{R}^n$ so that

$$\mathrm{Vol}_{n-1}\left(K \cap \theta^\perp\right) \leq \mathrm{Vol}_{n-1}\left(L \cap \theta^\perp\right), \forall \theta \in \mathbb{S}^{n-1},$$

where $\theta^\perp$ denotes the central hyperplane perpendicular to θ. Does it follow that

$$\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L)?$$

The answer is affirmative when $n \leq 4$, and negative whenever $n \geq 5$, we refer to [2, 4] for overviews of the problem, its history and solutions. It is natural to consider an analog of the Busemann-Petty problem for a more general class of measures. Consider an even, continuous function $f : \mathbb{R}^n \to (0, \infty)]$ and a measure μ with the density f, i.e.

$$\mu(K) = \int_K f(x)dx \text{ and } \mu(K \cap \theta^\perp) = \int\limits_{K\cap\theta^\perp} f(x)dx. \tag{1}$$

Fix $n \geq 2$. Given two convex origin-symmetric bodies K and L in $\mathbb{R}^n$ such that

$$\mu(K \cap \theta^\perp) \leq \mu(L \cap \theta^\perp),$$

for every $\theta \in \mathbb{S}^{n-1}$, does it follow that $\mu(K) \leq \mu(L)$?

It was proved in [6, 7] that the answer to the above question is independent from the "choice" of measure and depends only on the dimension n, i.e. affirmative when $n \leq 4$, and negative whenever $n \geq 5$.

A particularly interesting case of the Busemann-Petty problem for general measures is the case of Gaussian Measure. We remind Gaussian measure on $\mathbb{R}^n$ is defined by

$$\gamma_n(K) = \frac{1}{(\sqrt{2\pi})^n} \int_K e^{-\frac{|x|^2}{2}} dx.$$

V. Milman asked if the answer to the Busemann-Petty type problem for Gaussian Measure, would change in a positive direction if we compare not only the Gaussian measure of sections of the bodies but also the Gaussian measure of sections of their dilates. In our work with Malak Lafi [5] we consider a bit more general version of this problem for log-concave measure, i.e. we assume that the density f in (1) is a log-concave functions, in particular $f(x) = e^{-\phi(\|x\|_M)}dx$, where $\phi : [0, \infty) \to [0, \infty)$ is an increasing, convex function and $\|x\|_M$ is a norm corresponding to a convex, symmetric body M:

Question: *Consider two convex symmetric bodies $K, L \subset \mathbb{R}^n$, such that*

$$\mu(tK \cap \theta^\perp) \leq \mu(tL \cap \theta^\perp), \qquad \forall \theta \in \mathbb{S}^{n-1}, \qquad \forall t > 0.$$

Does it follow that $\mu(K) \leq \mu(L)$?

We were able to show a strongly positive result, proving that additional information on the dilates of convex bodies may lead to one of the bodies to be a subset of another. More precisely, we proved that

Let $K, L \subset \mathbb{R}^n$ be convex, symmetric bodies, and μ be a log-*concave probability measure, with density $e^{-\phi(\|x\|_K)}$, where $\phi : [0, \infty) \to [0, \infty)$ is an increasing, convex function. If for every t large enough and some $R > 0$*

$$\mu(tRK) \leq \mu(tL),$$

then $RK \subseteq L$.

This fact follows from an analog of the Large Deviation inequality, which we hope To prove the Theorem we have to prove yet another Large Deviation Principe which we hope this will be of independent interest. Recall that $r(K, L) = \max\{R > 0 : RK \subset L\}$.

Consider a convex symmetric body $K \subset \mathbb{R}^n$. Let μ be a probability log-*concave measure on $\mathbb{R}^n$ with non-constant density $e^{-\phi(\|x\|_K)}$, where $\phi : [0, \infty) \to [0, \infty)$ is a convex, increasing function. Let $L \subset \mathbb{R}^n$ be convex and symmetric body. Then*

$$\limsup_{t \to \infty} \frac{\ln \mu((tL)^c)}{\phi(r(K, L)t)} = -1$$

Unfortunately, we were also able to show that in a general case the solution to the Busemann-Petty problem with dilates is still negative in dimension 5 and higher:

For $n \geq 5$, there are convex symmetric bodies $K, L \subset \mathbb{R}^n$ and log-*concave measure μ with density $e^{-\phi(\|x\|_L)}$, such that*

$$\mu(tK \cap \theta^\perp) \leq \mu(tL \cap \theta^\perp), \quad \forall \theta \in \mathbb{S}^{n-1}, \quad \forall t > 0,$$

but $\mu(K) > \mu(L)$.

The idea of the proof of the above fact is to relate μ with the volume to use the same bodies provided to solve the classical Busemann-Petty problem. In particular, we used that in dimension 5 and higher there exists a convex body L which is not an intersection body [2, 4]. We note that this approach does not resolve V. Milman's question for Gaussian Measures, where the body M defining the density is an Euclidean ball. We solved this issue and proved a counterexample in $\mathbb{R}^n$ for $n \geq 7$. The construction is based counterexamples to the classical Busemann-Petty problem constructed by Giannopoulos [3] and Bourgain [1] in $\mathbb{R}^n$ for $n \geq 7$ of convex body $K \subset \mathbb{R}^n$, that satisfies

$$\mathrm{Vol}_{n-1}(K \cap \theta^\perp) \leq \mathrm{Vol}_{n-1}(B_2^n \cap \theta^\perp), \quad \forall \theta \in \mathbb{S}^{n-1},$$

but $\mathrm{Vol}_n(K) > \mathrm{Vol}_n(B_2^n)$.

We would like to finish with two open problems:

Open Problem: *Fix $n = 5$ or $n = 6$. Let μ be a* log-*concave, rotation invariant measure on with strictly decreasing density. Consider two convex symmetric bodies $K, L \subset \mathbb{R}^n$, such that $\mu(tK \cap \theta^\perp) \leq \mu(tL \cap \theta^\perp)$, $\forall \theta \in \mathbb{S}^{n-1}$, $\forall t > 0$. Does it follow that $\mu(K) \leq \mu(L)$?*

One approach is to create a counterexample to original Busemann-Petty problem where L, the body with the larger volume of sections is an euclidean ball. Another approach could be to find a construction independent of the original Busemann-Petty.

Open Problem: *Let μ be a* log-*concave, measure, so that $d\mu(x) = e^{-\phi(\|x\|_M)}$, where M is an intersection body. Consider two convex symmetric bodies $K, L \subset \mathbb{R}^n$, such that $\mu(tK \cap \theta^\perp) \leq \mu(tL \cap \theta^\perp)$, $\forall \theta \in \mathbb{S}^{n-1}$, $\forall t > 0$. Does it follow that $\mu(K) \leq \mu(L)$?*

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A Paley–Wiener–Schwartz Theorem for valuations on convex functions

Jonas Knörr

The classical Paley–Wiener–Schwartz Theorem relates the regularity properties of compactly supported distributions to the decay properties of their Fourier–Laplace transform. The subject of this talk was a corresponding result for distributions associated to certain valuations on convex functions, which can be used to obtain integral representations of these functionals under suitable regularity assumptions.

Denote by $\mathrm{Conv}(\mathbb{R}^n,\mathbb{R})$ the space of all convex functions $f:\mathbb{R}^n\to\mathbb{R}$. A functional $\mu:\mathrm{Conv}(\mathbb{R}^n,\mathbb{R})\to\mathbb{C}$ is called a valuation if

$$\mu(f)+\mu(h)=\mu(f\vee h)+\mu(f\wedge h)$$

for all $f,h\in\mathrm{Conv}(\mathbb{R}^n,\mathbb{R})$ such that their pointwise minimum $f\wedge h$ is convex (where $f\vee h$ denotes the pointwise maximum). We equip $\mathrm{Conv}(\mathbb{R}^n,\mathbb{R})$ with the topology induced by epi-convergence (which coincides with the topology induced by pointwise convergence and locally uniform convergence) and denote by $\mathrm{VConv}(\mathbb{R}^n)$ the space of all continuous valuations $\mu:\mathrm{Conv}(\mathbb{R}^n,\mathbb{R})\to\mathbb{C}$ that are dually epi-translation invariant, i.e. that satisfy

$$\mu(f+\ell)=\mu(f)\quad\text{for all } f\in\mathrm{Conv}(\mathbb{R}^n,\mathbb{R}),\ \ell:\mathbb{R}^n\to\mathbb{R}\text{ affine}.$$

Valuations of this type are closely related to translation invariant valuations on convex bodies, and consequently, one can ask whether classical results for translation invariant valuations on convex bodies admit a corresponding version for $\mathrm{VConv}(\mathbb{R}^n)$. One if these results is the homogeneous decomposition obtained by Colesanti, Ludwig, and Mussnig [2]:

$$\mathrm{VConv}(\mathbb{R}^n)=\bigoplus_{k=0}^{n}\mathrm{VConv}_k(\mathbb{R}^n),$$

where $\mu\in\mathrm{VConv}_k(\mathbb{R}^n)$ if and only if $\mu(tf)=t^k\mu(f)$ for all $f\in\mathrm{Conv}(\mathbb{R}^n,\mathbb{R})$, $t\geq 0$. For $k=0$ and $k=n$, these functionals can be described explicitly, whereas in the intermediate degrees only descriptions of certain dense subspaces are available [6], which are based on Alesker's Irreducibilty Theorem [1].

A helpful tool in the study of these functionals are the so called Goodey–Weil distributions introduced in [4] based on ideas by Goodey and Weil from [3]: Due to the homogeneous decomposition, we may define the polarization of $\mu\in\mathrm{VConv}_k(\mathbb{R}^n)$ for $f_1,\dots,f_k\in\mathrm{Conv}(\mathbb{R}^n,\mathbb{R})$ by

$$\bar{\mu}(f_1,\dots,f_k):=\frac{1}{k!}\frac{\partial^k}{\partial\lambda_1\dots\partial\lambda_k}\Big|_0\mu\left(\sum_{j=1}^k\lambda_jf_j\right).$$

It was shown in [4] that $\bar{\mu}$ can be lifted uniquely to a distribution $\mathrm{GW}(\mu)$ on $(\mathbb{R}^n)^k$ with compact support such that

$$\mathrm{GW}(\mu)[f_1\otimes\cdots\otimes f_k]=\bar{\mu}(f_1,\dots,f_k)$$

for $f_1, \dots, f_k \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \cap C^\infty(\mathbb{R}^n)$. In particular, the Fourier-Laplace transform of such a distribution defines an entire function on $(\mathbb{C}^n)^k$, which we denote by $\mathcal{F}(\mathrm{GW}(\mu))$. The main point of this talk was to sketch how the Fourier-Laplace transform of these distributions can be used to transfer regularity properties of the underlying valuations into integral representations, or more precisely, to show that the following are equivalent for $\mu \in \mathrm{VConv}_k(\mathbb{R}^n)$:

(1) μ is smooth in the following sense: The map

$$\begin{aligned} \mathbb{R}^n &\to \mathrm{VConv}_k(\mathbb{R}^n) \\ x &\mapsto [f \mapsto \mu(f(\cdot + x))] \end{aligned}$$

is a smooth map between locally convex vector spaces, where $\mathrm{VConv}_k(\mathbb{R}^n)$ is equipped with the topology of uniform convergence on compact subsets of $\mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$.

(2) There exists a compact convex set $K \subset \mathbb{R}^n$ such that the Fourier-Laplace transform of $\mathrm{GW}(\mu)$ satisfies for every $N \in \mathbb{N}$ an inequality of the form

$$\begin{aligned} &|\mathcal{F}(\mathrm{GW}(\mu))[w_1, \dots, w_k]| \\ &\leq C_N |w - \mathrm{diag}(w)|^{2(k-1)} (1 + |\mathrm{diag}(w)|)^{-N} \exp\left(\sum_{j=1}^k h_K(\mathrm{Im}(w_j))\right) \end{aligned}$$

for $w = (w_1, \dots, w_k) \in (\mathbb{C}^n)^k$ for constants $C_N > 0$, where $\mathrm{diag}(w) = \left(\frac{1}{k}\sum_{j=1}^k w_j, \dots, \frac{1}{k}\sum_{j=1}^k w_j\right)$.

(3) There exist tuples $(Q_1^j, \dots, Q_{n-k}^j)$, $1 \leq j \leq N_{n_k} := \binom{n}{k}^2 - \binom{n}{k-1}\binom{n}{n-(k+1)}$, of positive definite quadratic forms and functions $\phi_j \in C_c^\infty(\mathbb{R}^n)$ such that

$$\mu(f) = \sum_{j=1}^{N_{n,k}} \int_{\mathbb{R}^n} \phi_j(x) d\mathrm{MA}(f[k], Q_1^j, \dots, Q_{n-k}^j),$$

where MA denotes the mixed Monge-Ampère operator.

The main difficulty lies in the step (2) $\Rightarrow$ (3), i.e. in extracting a suitable representation from the entire function $\mathcal{F}(\mathrm{GW}(\mu))$. The proof relies on the observation that any such entire function belongs to a certain module of entire functions generated by quadratic products of k-minors of a matrix in $(\mathbb{C}^n)^k$. As observed in [5], these polynomials correspond to the Fourier-Laplace transforms of certain Monge-Ampère operators. Using the classical Paley–Wiener–Schwartz Theorem and a global version of the Weierstrass Division Theorem to find a suitable presentation of the given entire function, this can be used to provide a constructive way to recover the desired representation from a given function.

This approach allows us in particular to recover one of the main results of [6]: Valuations of the type considered in (3) form a dense subspace of $\mathrm{VConv}(\mathbb{R}^n)$ with respect to the topology of locally uniform convergence on compact subsets. This follows from a simple convolution argument using the equivalent characterization of these valuations given in (1).

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On Schneider's higher-order difference body

Eli Putterman

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex sets with non-empty interior) in $\mathbb{R}^n$. For $K \in \mathcal{K}^n$, one has the following classical inequalities:

$$2^n \leq \frac{|DK|}{|K|} \leq \binom{2n}{n}, \tag{1}$$

where $DK = \{x \in \mathbb{R}^n : K \cap (K + x) \neq \emptyset\}$ is the difference body of K, equivalently defined as

$$DK = K - K = \{x - y : x, y \in K\},$$

and $|K|$ denotes the volume of K.

The lower bound follows from the Brunn-Minkowski inequality; equality holds if, and only if, K is centrally symmetric. (We write $\mathcal{K}^n_e$ for the set of centrally-symmetric convex bodies.) The upper bound is the Rogers-Shephard inequality, and equality holds if, and only if, K is an n-dimensional simplex [2].

In [3], Schneider introduced a higher-order analogue of this inequality. He defined the mth order difference body $D^m(K) \subset (\mathbb{R}^n)^m$ as

$$D^m(K) = \left\{(x_1, \ldots, x_m) \in (\mathbb{R}^n)^m : K \cap \bigcap_{i=1}^{m} (K + x_i) \neq \emptyset\right\}. \tag{2}$$

Schneider showed [3, Satz 2] the following generalization of the Rogers-Shephard inequality:

$$\delta_m(K) := \frac{|D^m(K)|}{|K|^m} \leq \binom{nm + m}{n},$$

again with equality if, and only if, K is an n-dimensional simplex.

Corresponding lower bounds for $|D^m K|$ are, apart from a few special cases, wide open. The case $m = 1$ is inequality (1) above, and so the minimum possible value for $\delta_1(K)$ is attained precisely when $K \in \mathcal{K}^n_e$. Schneider also obtained a sharp lower bound on $\delta_m(K)$ for the case $n = 2$ (convex bodies in the plane) and arbitrary m [3, Satz 1]:

Theorem 1. *Let $K \subset \mathbb{R}^2$ be a convex body. Then* $\delta_m(K) = \frac{m(m+1)}{2}\delta_1(K)+1-m^2$.

In particular, as $\delta_1(K) \geq 4$ by (1), one obtains $\delta_m(K) \geq (m+1)^2$, with equality if and only if $K \in \mathcal{K}_e^n$. That is, if K is planar, then the equality conditions in the lower bound on $\delta_m(K)$ are precisely the same as for $\delta_1(K)$.

In higher dimensions, however, this is no longer the case. Schneider showed [3, §4] that $\delta_2(B_2^3) = 21 + \left(\frac{3\pi}{4}\right)^2 < 27 = \delta_2(B_\infty^3)$, where B_2^3 is the unit Euclidean ball in $\mathbb{R}^3$ and $B_\infty^3 = [-1,1]^3$, and hence central symmetry cannot be a sufficient condition to attain the lower bound. Schneider conjectured that for $n \geq 3, m \geq 2$, the minimizers of $\delta_m(K)$ are precisely ellipsoids.

The starting point of the present work is the observation that the mth-order difference body may equivalently be defined in the following way:

$$D^m(K) = K^m + (-\Delta(K)), \tag{3}$$

where $K^m = \overbrace{K \times K \times \cdots \times K}^{m \text{ times}}$ is the m-fold Cartesian product of K, $\Delta : \mathbb{R}^n \to (R^n)^m$ is the diagonal map $\Delta(x) = \overbrace{(x, x, \ldots, x)}^{m \text{ times}}$, and the sum is the Minkowski sum

$$A + B = \{a + b : a \in A, b \in B\}.$$

Note also that K^m is itself a Minkowski sum of linear embeddings $\imath_1(K), \ldots, \imath_m(K)$, where $\imath_j : \mathbb{R}^n \to (\mathbb{R}^n)^m$ sends x to the block vector with x in the jth position and 0 elsewhere. Letting $\imath_0 = -\Delta$, by Minkowski's theorem, we have

$$|D^m(K)| = \sum_{s_0+\cdots+s_m=mn} \binom{mn}{s_0, \ldots, s_m} V(\imath_0(K)[s_0], \ldots, \imath_m(K)[s_m]) \tag{4}$$

where $V(L_1[s_1], \ldots, L_k[s_k])$ denotes the mixed volume of the sets L_i, each taken s_i times. This means that to understand the volume of $D^m(K)$, we need to understand mixed volumes of different linear embeddings of K.

Schneider [4], studying the more general mixed diffference body

$$D(K_1, K_2, \ldots, K_k) = \left\{(x_1, \ldots, x_{k-1}) \in (\mathbb{R}^n)^{k-1} : K_k \cap \bigcap_{i=1}^{k-1}(K_i + x_i) \neq \emptyset\right\},$$

using the techniques of translative integral geometry, obtained the decomposition of $D(K_1, K_2, \ldots, K_k)$ into multilinear translation-invariant functionals, which he denoted $V^{(0)}_{m_1,\ldots,m_k}(K_1, \ldots, K_k)$. These functionals coincide with the mixed volumes $V(\imath_1(K)[m_1], \ldots, \imath_{k-1}(K)[m_{k-1}], -\Delta(K_k)[m_k])$, giving a different perspective.

Armed with the decomposition (4), we prove several results relating to extremizers of $|D^m(K)|$ under different conditions, though we have not been able to resolve Schneider's conjecture. First, we use mixed volumes to provide a new, simple proof of Schneider's result on the volume of the mth order difference body of a planar convex body (Theorem 1).

Next, we solve an optimization problem similar to that introduced by Schneider, but under a different constraint. An n-zonoid Z is a convex body in $\mathbb{R}^n$ whose support function is given by

$$h_Z(u) = \int_{S^{n-1}} |\langle u, v\rangle| \, d\mu(v) \tag{5}$$

for some measure μ on S^{n-1}; we call μ the generating measure of Z, and write $Z = Z_\mu$. Similarly, if μ is a signed measure on S^{n-1} and Z is a convex body with h_Z given by (5) then we call Z a generalized zonoid. If we restrict to even measures, then a generalized zonoid is determined by its generating measure [1].

We say Z_μ is isotropic if μ is an isotropic measure, i.e., $\int \langle u, v\rangle^2 \, d\mu(v) = |u|^2$ for all $u \in \mathbb{R}^n$. (Note that in particular this implies that $\int |v|^2 \, d\mu(v) = n$.) Any zonoid has a linear image – unique up to isometry – which is isotropic, so this is a natural normalization. For example, the cube $B^n_\infty = [-1, 1]^n$ has generating measure $\sum_{i=1}^n \delta_{e_i}$ and thus is isotropic, and there exists a constant $c_n \simeq \sqrt{n}$ such that the scaled Euclidean ball $c_n B^n_2$ is isotropic. We write $\mathcal{IZ}_n$ for the space of isotropic zonoids in dimension n.

Our goal in this talk is to investigate extremizers of the functional $f(Z) = |D^m(Z)|$ on $\mathcal{IZ}_n$, rather than on the class of zonoids with fixed volume as in Schneider's original conjecture. Finding the maximizer of f is easily done:

Proposition 2. *For any $n \geq 2$, $m \geq 2$, $\max\{|D^m(Z)| : Z \in \mathcal{IZ}_n\}$ is attained precisely when Z is a Euclidean ball.*

The proof is quite simple: one just combines the fact that, under Minkowski addition, the volume is strictly $\frac{1}{n}$-concave on $\mathcal{K}^n$ and $\mathcal{IZ}_n \subset \mathcal{K}^n$ is a compact convex set with the rotation-invariance of $|D^m(Z)|$. (In the talk, we give a slightly more detailed sketch of the proof.)

Our main result is a solution of the corresponding minimization problem:

Theorem 3. *For any $n \geq 2$, $m \geq 2$, $\min\{|D^m(Z)| : Z \in \mathcal{IZ}_n\}$ is attained precisely when Z is a cube.*

To prove Theorem 3 in arbitrary dimension requires a bit of multilinear algebra which may be intimidating; hence, in the talk we prove only the case $n = 3$, which requires no more than the usual properties of the cross product.

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Random polytopes in convex bodies: Bridging the gap between extremal containers

Anna Gusakova

(joint work with Florian Besau, Christoph Thäle)

Let $K \subset \mathbb{R}^d$ be a convex body and let $X_1, \ldots, X_n$ be independent random points chosen uniformly in K. Consider a random polytope

$$K(n) := \mathrm{co}\,\{X_1, \ldots, X_n\},$$

which is a convex hull of points $X_1, \ldots, X_n$. In this talk we will be interested in asymptotic behaviour of the expected number of j-dimensional faces $\mathbb{E}[f_j(K(n))]$, $j \in \{0, \ldots, d-1\}$ of $K(n)$ as $n \to \infty$. We also note that by Efron's identity the expected number of vertices $\mathbb{E}[f_0(K(n))]$ is asymptotically equivalent to the expected missed volume $\mathbb{E}[\mathrm{vol}(K \setminus K(n))]$ as $n \to \infty$.

At the moment there are only two types of container bodies K, for which the behaviour of random polytope $K(n)$ is well-understood. In particular, if $K \subset \mathbb{R}^d$ with $\mathrm{vol}(K) = 1$ is a convex body with smooth boundary, then by [3, Thm. 4] we have

$$\mathbb{E}[f_j(K(n))] = c_{d,j}\,\Omega(K)\,n^{\frac{d-1}{d+1}} + o(n^{\frac{d-1}{d+1}}), \qquad j \in \{0, \ldots, d-1\},$$

where $c_{d,j} \in (0, \infty)$ is a constant only depending on d and j and $\Omega(K)$ is affine surface area of K. In contrast to this, if $P \subset \mathbb{R}^d$ is a polytope with $\mathrm{vol}(P) = 1$, then

$$\mathbb{E}[f_j(P(n))] = \tilde{c}_{d,j}\,F(P)\,(\ln n)^{d-1} + o((\ln n)^{d-1}), \qquad j \in \{0, \ldots, d-1\},$$

by [3, Thm. 8], where $\tilde{c}_{d,j} \in (0, \infty)$ is another constant only depending on d and j, and $F(P)$ is the number of complete flags of the polytope P.

The above two cases are extremal in a sense that for any convex body K we have

$$c(\ln n)^{d-1} \le \mathbb{E}[f_j(K(n))] \le Cn^{\frac{d-1}{d+1}},$$

where $c, C \in (0, \infty)$ are some constants, independent on n. At the same time Bárány and Larman [1, Thm. 5] demonstrated that for most convex bodies $K \subset \mathbb{R}^d$ (in the sense of Baire category) the asymptotic behaviour of $\mathbb{E}[f_j(K(n))]$ is unpredictable, namely it oscillates infinitely often between the extremal asymptotics. This raises a major open question in the field: are there 'natural' classes of convex bodies which do not exhibit such a chaotic behaviour and interpolate between extremal containers?

In this talk we introduce such class of convex bodies. Let $d \ge 2$ and $m \in \{1, \ldots, d\}$ be integers, and consider an m-tuple $\mathbb{d} := (d_1, \ldots, d_m) \in \mathbb{N}^m$, such that $d_1 + \ldots + d_m = d$. We define the origin symmetric convex body $Z_{\mathbb{d}}$ as the product

$$Z_{\mathbb{d}} := \prod_{i=1}^{m} B_2^{d_i} \subset \mathbb{R}^d,$$

of d_i-dimensional centered Euclidean unit balls $B_2^{d_i}$, $i \in \{1, \ldots, m\}$. In particular the case $m = 1$ reduces to the d-dimensional Euclidean unit ball B_2^d, while for

$m = d$ the body $Z_{\mathbb{d}}$ is the centered cube $B_\infty^d = [-1,1]^d$, corresponding to polytopal container. In the recent preprint [2] we have shown the following result.

Theorem 1. *There are constants* $c, C \in (0,\infty)$ *only depending on* $\mathbb{d}$ *and* j *such that for any* $j \in \{0,\dots,d-1\}$ *we have*

$$c\, n^{\frac{d_{\max}-1}{d_{\max}+1}} (\ln n)^{\# d_{\max}-1} \le \mathbb{E}[f_j(Z_{\mathbb{d}}(n))] \le C\, n^{\frac{d_{\max}-1}{d_{\max}+1}} (\ln n)^{\# d_{\max}-1},$$

with $d_{\max} := \max_{i=1,\dots,m} d_i$ *and* $\# d_{\max} := \# \{ i : d_i = d_{\max} \}$.

The above theorem and its proof reveal that if the product body $Z_{\mathbb{d}}$ has precisely one factor whose dimension dominates all the others, the faces of $Z_{\mathbb{d}}(n)$ will eventually cluster in this part of $Z_{\mathbb{d}}$, whose local behaviour is that of a smooth convex body. The number of faces lying in other parts of $Z_{\mathbb{d}}$ are of lower order. Conversely, if there is more than one factor with maximal dimension, the order of $\mathbb{E}[f_j(Z_{\mathbb{d}}(n))]$ increases by an additional logarithmic factor. Intuitively, this logarithmic factor corresponds to the number of surplus faces that are generated by connecting points of two (or more) of different clusters of points, concentrating in the parts of $Z_{\mathbb{d}}$ of maximal dimension. The power of logarithm reflects the number of relevant clusters, or, equivalently, the number of maximal dimensions of $Z_{\mathbb{d}}$.

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Minkowski problems arising from integral geometry

Dongmeng Xi

(joint work with Erwin Lutwak, Deane Yang, Gaoyong Zhang, Yiming Zhao)

The classical Minkowski problem. Given a finite Borel measure μ on S^{n-1}, find the necessary and sufficient conditions on μ so that

$$\mu = S_{n-1}(K,\cdot)$$

for some convex body $K \subset \mathbb{R}^n$. Here $S_{n-1}(K,\cdot)$ is the fundamental *surface area measure*, which is a Borel measure on the unit sphere S^{n-1}:

$$S(K,\eta) = \mathcal{H}^{n-1}(\nu_K^{-1}(\eta)), \qquad \text{Borel } \eta \subset S^{n-1}, \tag{1}$$

where $\mathcal{H}^i$ is i-dimensional Hausdorff measure and ν_K is the generalized Gauss map. It was studied throughout the entire last century, as demonstrated by the works of Minkowski, Aleksandrov, Cheng-Yau, Pogorelov, and Caffarelli.

Let h_K be the support function and f be a continuous function on S^{n-1}. Define

$$K_t = \{x \in \mathbb{R}^n : x\cdot v \le h_K(v) + tf(v),\ \text{for all } v \in S^{n-1}\}. \tag{2}$$

Then, Aleksandrov's variation formula stated that the surface area measure can be viewd as the differential of the volume operator:

$$\frac{d}{dt}\bigg|_{t=0} V(K_t) = \int_{S^{n-1}} f(v) dS_{n-1}(K, v). \tag{3}$$

The other classical geometric invariant, surface area, turned out to be much more *ill-behaved* under perturbations of the body. Write $K+L$ to be the Minkowski sum of two convex bodies K and L. The surface area is one-sided differentiable:

$$\frac{d}{dt}\bigg|_{t=0^+} S(K + tL) = (n-1) \int_{S^{n-1}} h_L(v) dS_{n-2}(K, \cdot), \tag{4}$$

where $S_{n-2}(K, \cdot)$ is known as the $(n-2)$-area measure.

The Christoffel-Minkowski problem of order $(n-2)$ asks to solve the measure equation

$$S_{n-2}(K, \cdot) = \mu.$$

See Guan-Ma [1] for a regular case.

Chord Minkowski problem in Integral Geometry. In [2], we (Lutwak-Xi-Yang-Zhang) paved an alternative route to $S_{n-2}(K, \cdot)$ by investigating geometric measures in integral geometry. For $q > 0$, the *chord integral* $I_q(K)$ is the L_q mean of the length of intersection of the body K with a random line in $\mathbb{R}^n$:

$$I_q(K) = \int_{\mathbb{L}^n} |K \cap \ell|^q \, d\ell \tag{5}$$

where the integration is with respect to the (appropriately normalized) Haar measure on the affine Grassmannian $\mathbb{L}^n$ of lines in $\mathbb{R}^n$. It includes the volume and surface area as two important special cases:

$$I_1(K) = V(K), \quad I_0(K) = \frac{\omega_{n-1}}{n\omega_n} S(K), \quad I_{n+1}(K) = \frac{n+1}{\omega_n} V(K)^2. \tag{6}$$

We established a family of *robust* variation formulas for *chord integrals* $I_q(K)$, leading to *chord measures* $F_q(K, \cdot)$.

Theorem 1 (Lutwak-Xi-Yang-Zhang)**.** *Let K_t be given by* (2)*.*

$$\frac{d}{dt}\bigg|_{t=0^+} I_q(K_t) = \int_{S^{n-1}} f(v) \, dF_q(K, v), \tag{7}$$

where $F_q(K, \eta)$ is the so-called chord measures.

As $q \to 0$, it is surprising that, not only the chord integral $I_q(K)$ converges to the surface area $S(K)$, but also the chord measure $F_q(K, \cdot)$ converges to $S_{n-2}(K, \cdot)$, under regularity assumptions. If uniform (in q) estimates can be obtained for the measure equation $\mu = F_q(K, \cdot)$, then a limiting argument will lead to a solution to the Christoffel-Minkowski problem for $S_{n-2}(K, \cdot)$. We solved the chord Minkowski problem for all the $q > 0$ case.

Theorem 2 (Lutwak-Xi-Yang-Zhang)**.** *Suppose real $q > 0$ is fixed. If μ is a finite Borel measure on S^{n-1} that is not concentrated on a closed hemisphere, then there exists a convex body $K \subset \mathbb{R}^n$ so that $F_q(K, \cdot) = \mu$, if and only if,*

$$\int_{S^{n-1}} v \, d\mu(v) = 0.$$

Affine Minkowski problem in Integral Geometry. From an integral geometry viewpoint, *Cauchy's integral formula* tells us that surface area can be represented, up to a constant, as the average of areas of projections:

$$S(K) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \mathrm{vol}_{n-1}(K|u^\perp) du. \tag{8}$$

Here $K|u^\perp$ denotes the image of the orthogonal projection of K onto $u^\perp$. While volume is *affine invariant* ($SL(n)$), the surface area $S(K)$ is not. Instead, up to a constant, by taking the L_{-n} mean, one obtains the *integral affine surface area*

$$\Phi(K) = \left(\frac{1}{n} \int_{S^{n-1}} \mathrm{vol}_{n-1}(K|u^\perp)^{-n} du \right)^{-1/n}, \tag{9}$$

which is affine invariant. The celebrated affine isoperimetric inequality (*Petty projection inequality*), which is stronger than the classical one, states

$$n\omega_n^{\frac{1}{n}} V(K)^{\frac{n-1}{n}} \le \frac{n\omega_n^{(n+1)/n}}{\omega_{n-1}} \Phi(K) \le S(K). \tag{10}$$

As an affine invariant counterpart of the chord integral, the fractional affine surface area ($q > -1$) is defined by

$$\Phi_q(K) = \left(\frac{1}{n} \int_{S^{n-1}} \left(\int_{u^\perp} X_K(y, u)^{q+1} dy \right)^{\frac{n}{q}} du \right)^{\frac{q}{n}}.$$

One of the main results (by Xi-Zhao [3]) in this part is to show that it is possible to differentiate $\Phi_q(K)$ for *any* $-1 < q \neq 0$.

Theorem 3 (Xi-Zhao)**.** *Let $0 \neq q > -1$, and K_t be defined by* (2)*. Then,*

$$\left.\frac{d}{dt}\right|_{t=0} \Phi_q(K_t) = 2\Phi_q^{\frac{q-n}{q}}(K) \int_{S^{n-1}} f(v) dF_q^{(\mathrm{a})}(K, v). \tag{11}$$

Here $F_q^{(\mathrm{a})}(K, \cdot)$ is called the fractional affine area measre.

We also present the following solution to the Minkowski problem for $F_q^{(\mathrm{a})}(K, \cdot)$.

Theorem 4 (Xi-Zhao)**.** *Let $0 \neq q > -1$ and μ be a finite Borel measure on S^{n-1}. There is a convex body K in $\mathbb{R}^n$ such that $\mu = F_q^{(\mathrm{a})}(K, \cdot)$ if and only if μ is not concentrated in any subspheres and*

$$\int_{S^{n-1}} v d\mu(v) = o. \tag{12}$$

Finally, we provided three equivalent ways to define the so-called *affine* $(n-2)$-*area measure*. We discovered the one-sided variation formula of $\Phi(\cdot)$:

$$\frac{d}{dt}\bigg|_{t=0^+}\Phi(K+tL)=2\Phi(K)^{n+1}\int_{S^{n-1}}h_L(v)dS^{(\mathrm{a})}_{n-2}(K,v).$$

Here $S^{(\mathrm{a})}_{n-2}(K,\cdot)$ means the *affine* $(n-2)$-*area measure*, defined as a mixed surface area measure

$$S^{(\mathrm{a})}_{n-2}(K,\omega)=\frac{n-1}{4n}S_{n-2}(K,\mathbb{P}K,\omega),$$

where

$$h_{\mathbb{P}K}(v)=\int_{S^{n-1}}|v\cdot u|\rho^{n+1}_{\Pi^*K}(u)du,\quad \text{for all } v\in S^{n-1},$$

which is equivalent to the centroid body of the projection body Π^*K up to a suitable normalization. The other two equivalent definitions are based on studies on the interpretation of $F^{\mathrm{a}}_q(K,\cdot)$ in differential geometry and, on the convergence $F^{\mathrm{a}}_q(K,\cdot)\to S^{(\mathrm{a})}_{n-2}(K,\cdot)$ under regularity assumptions.

When $q\in[-1,0]$, the Brunn-Minkowski inequalities of Φ_q (and hence related uniqueness results) are also established.

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The Rademacher projection and MM^*-estimate in the non-symmetric case

Alexander E. Litvak

(joint work with F. L. Nazorov)

For basic notions, definitions and statements mentioned below we refer to [1, 4, 5, 6]. As usual, the letters C, C_0, C_1, ..., c, c_0, c_1, ... always denote absolute positive constants, whose values may change from line to line.

When dealing with non-symmetric convex bodies, there is no natural choice of the center, so we will be working with a shift of K by a point $a\in \mathrm{int}K$ (which will be playing the role of the center) and denote it by $K_a:=K-a$. Arguably, one of the most important parameters in asymptotic geometric analysis, which played a crucial role in several proofs, is the following quantity

$$MM^*(K)=\inf_{T,a}\int_{S^{n-1}}\|x\|_{TK_a}\,d\sigma(x)\int_{S^{n-1}}\|x\|_{(TK_a)^\circ}\,d\sigma(x),$$

where σ denotes the normalized Lebesgue measure on the sphere, $\|\cdot\|_L$ denotes the Minkowski functional of a convex body L and the infimum is taken over all non-degenerate linear operators $T:\mathbb{R}^n\to\mathbb{R}^n$ and $a\in\mathrm{int}K$.

Let K and L be two convex bodies in $\mathbb{R}^n$. We define the geometric distance, the linear Banach–Mazur distance and the (affine) Banach-Mazur distance between them as

$$d_g(K,L) = \inf\{\beta/\alpha \mid \alpha > 0,\, \beta > 0, \quad \alpha K \subset L \subset \beta K\};$$
$$d_\ell(K,L) = \inf\{\lambda > 0 \mid K \subset TL \subset \lambda K\} = \inf d_g(K,TL),$$

where the infimum is taken over all non-degenerate linear operators $T : \mathbb{R}^n \to \mathbb{R}^n$ (note that we don't allow shifts, the center is fixed at the origin); and

$$d(K,L) = \inf\{\lambda > 0 \mid K_a \subset TL_b \subset \lambda K_a\} = \inf d_r(K_a, L_b),$$

where infimum is taken over all non-degenerated linear operators $T : \mathbb{R}^n \to \mathbb{R}^n$ and all $a \in \mathrm{int}K$, $b \in \mathrm{int}L$.

Fix $m > 1$. Let $\Omega = \{\pm 1\}^m$ and let μ be the normalized counting measure on Ω. Define Rademacher functions $r_i : \Omega \to \mathbb{R}$ by $r_i(\varepsilon) = \varepsilon_i$. Given $A \subset \{1, 2, ..., m\}$, the Walsh function $w_A : \Omega \to \mathbb{R}$ is defined as $w_\emptyset \equiv 1$ and for $A \neq \emptyset$, $w_A(\varepsilon) = \prod_{i\in A} r_i(\varepsilon)$. Note that $w_{\{i\}} = r_i$. It is known (and easy to check) that the Walsh system forms an orthonormal basis of $L_2 = L_2(\Omega, \mathbb{R})$. Furthermore, consider a convex body $K \subset \mathbb{R}^n$ with the origin in its interior and, given $p \geq 1$, the space

$$L_p(K) := L_p(\Omega, (\mathbb{R}^n, K)) = \left\{ F : \Omega \to \mathbb{R}^n \mid \|F\|_{L_p(K)} = \left(\int_\Omega \|F(\varepsilon)\|_K^p \, d\mu \right)^{1/p} \right\}$$

(for $p = \infty$ we use the sup norm, $\sup_{\varepsilon\in\Omega} \|F(\varepsilon)\|_K$). Then each $F \in L_p(K)$ can be decomposed as

$$F = \sum_{k=0}^{m} \sum_{\substack{A\subset\{1,2,\dots,m\}\\ |A|=k}} \widehat{F}_A w_A, \qquad \text{where} \qquad \widehat{F}_A = \int_\Omega F w_A \, d\mu.$$

Then the Rademacher projection of F is defined as $RF = \sum_{i=1}^m \widehat{F}_{\{i\}} r_i$.

The importance of the norm of the Rademacher projection in asymptotic geometric analysis comes from its relation to the so-called K-convexity and bounds on $MM^*(K)$. We also would like to mention that the standard and more direct way to obtain estimates for MM^* is through the Gaussian projection, which is a Gaussian analogue of the Rademacher projection. It is known that their norms are equivalent and here we will deal with the Rademacher projection for the sake of clarity. A combination of results by Lewis and by Figiel–Tomczak-Jaegermann implies in the centrally symmetric case ($K = -K$) that

$$MM^*(K) \leq C_1 \|R : L_2(K) \to L_2(K)\|$$

while a result of Pisier, also for $K = -K$, asserts that

$$\|R : L_2(K) \to L_2(K)\| \leq C_2 \log(2d(K, B_2^n)) \leq C_2 \log(2n)$$

(rigorously speaking, in both inequalities we have to take an additional supremum over all $m \geq 1$ in the definition of Ω, however for the purpose of MM^* it is enough to consider $m \leq n$ only). The Pisier bound shows that up to a logarithmic factor the norm of the Rademacher projection behaves as in the Euclidean case. Since in high-dimensional convex geometry the condition of central symmetry is not very natural, there is a demand to extend the theory to the non-symmetric case. It turns out that many results can be extended to the non-symmetric case. In order to estimate MM^*, one may try to extend the approach through the Rademacher projection. However, easy examples show that the direct extensions of the norm of Rademacher projection will lead to bad bounds, as such a norm can be of the order of the dimension. Thus one needs to substitute the norm of the Rademacher projection with a more appropriate functional. The following functional was suggested by Gluskin,

$$\varphi_K(R) = \sup_{F\,:\,\Omega\to K} \inf_{a\in K} \|RF\|_{L_2(K_a)}$$

(again, we have to take an additional supremum over all $m \geq 1$ in the definition of Ω). We also consider R restricted to mean zero functions, namely let

$$\phi_K(R) = \sup_{\int_\Omega F\,d\mu=0} \frac{\|RF\|_{L_2(K)}}{\|F\|_{L_2(K)}}$$

(so $\phi_K(R)$ is the direct extension of the norm of the Rademacher projection on the set of mean-zero functions). In [2] it was shown that

$$MM^*(K) \leq C\varphi_K(R)\log(2d(K,B_2^n))$$

(the log factor here is not needed if one deals with the Gaussian projection) and

$$\varphi_K(R) \leq C\sqrt{d(K,B_2^n)}.$$

One of the main steps in proving the latter inequality was the following estimate, obtained by adapting the Pisier proof to the non-symmetric case,

$$\forall a \in \mathrm{int}K, \qquad \phi_{K_a}(R) \leq C\sqrt{d_\ell(K_a,B_2^n)}.$$

We show that these bounds are sharp, namely following holds.

Theorem 1. *For every $n \geq 1$ and every $1 \leq d \leq \sqrt{n}$, there exists a convex body $K \subset \mathbb{R}^n$ such that*

$$c_1 d \leq d(K,B_2^n) \leq d_g(K,B_2^n) \leq Cd, \quad \varphi_K(R) \geq c_2\sqrt{d} \quad \text{and} \quad \phi_K(R) \geq c_2\sqrt{d}.$$

In particular, there exists a body K such that

$$c_3\sqrt{n} \leq d(K,B_2^n) \leq C\sqrt{n}, \qquad \varphi_K(R) \geq c_3 n^{1/4} \qquad \text{and} \qquad \phi_K(R) \geq c_3 n^{1/4}.$$

First we note that it is enough to prove the theorem for $d = \sqrt{n}$. Indeed, if proved, then for smaller values of d, we can construct such a body K' in $\mathbb{R}^\ell$ with $\ell \approx d^2$ and take $K = K' \oplus B_2^k$ where $k = n - \ell$. Next we describe our construction.

Fix a positive integer $m > 1$ and let $n = m(m+1)/2$. Let u_k, $1 \leq k \leq m$ and v_{ij}, $1 \leq i < j \leq m$, be the canonical orthonormal basis of $\mathbb{R}^n$. Define the function $f : \Omega \to \mathbb{R}^n$ by

$$f(\varepsilon) = \sum_{k=1}^m \varepsilon_k u_k + \sum_{1\leq i<j\leq m} \varepsilon_i\varepsilon_j v_{ij} \qquad \text{and let} \qquad K = \text{ conv } \{f(\varepsilon)\}_{\varepsilon\in\Omega} \subset \mathbb{R}^n.$$

We would like to mention that our construction is somewhat similar (although much less involved) to the one used by J. Bourgain in [3].

We would also like to mention that in addition to establishing Theorem 1, our body is quite regular. We summarize its properties in the following theorem.

Theorem 2. *Let $m > 1$, $n = m(m+1)/2$, $f : \Omega \to \mathbb{R}^n$ and $K \subset \mathbb{R}^n$ be as above. Then $\sqrt{n}B_2^n$ is the ellipsoid of the minimal volume for K,*

$$\frac{1}{24} B_2^n \subset K \subset B_\infty^n \subset \sqrt{n} B_2^n,$$

$$M_K \leq C \frac{\log n}{n^{1/4}} \qquad \text{and} \qquad c n^{1/4} \leq M_K^* \leq C n^{1/4},$$

and for every $\varepsilon \in \Omega$, $\| - f(\varepsilon)\|_K \geq \sqrt{n}$.

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Banaszczyk's 5K-theorem, the Gaussian way

Maud Szusterman

(joint work with Piotr Nayar)

A well-known theorem of Banaszczyk (1998) states that given any convex body $K \subset \mathbb{R}^n$ of gaussian measure at least 1/2, and any sequence of vectors from B_2^n, one may find signs $\epsilon_i = \pm 1$ (for each vector) so that the signed sum lies in $5K$. The proof relies on the monotonicity of the gaussian measure of large convex bodies under certain transformations $T_u(K) \subset (K-u) \cup (K+u)$, $u \in \mathbb{R}^n$. We replace T_u with a smaller transform $G_u(K) \subset T_u(K)$, which simplifies significantly the proof of the monotonicity, at the cost of loosing $5K$ and only concluding "lies in $7K$" in the end. Our proof is purely geometric and uses successive Ehrhard symmetrizations, and gaussian isoperimetric inequality in $\mathbb{R}$ and $\mathbb{R}^2$. We remark that monotonicity of the gaussian measure (under $K \mapsto G_u(K)$) also holds when weakening the assumption $\gamma_n(K) \geq 1/2$ to $\gamma_n(K) \geq p$ (for $0 < p < \frac{1}{2}$), at the cost of asking $||u||_2$ to be smaller (for centrally symmetric convex bodies, one must replace $\frac{1}{7} = f(1/2)$ with $f(p) \sim cp$ $(0 < p < \frac{1}{2})$, whereas if considering all convex bodies K, $f(p)$ is exponentially small in p^{-1}).

The L_p-floating area and new inequalities on the sphere

Florian Besau

(joint work with Elisabeth M. Werner)

1. Isoperimetric inequality for the floating area

The affine surface area of a convex body $\overline{K} \subset \mathbb{R}^n$ can be defined using centro-affine notions by

$$\mathrm{as}_1(\overline{K}) = n \int_{\partial \overline{K}} \kappa_0(\overline{K}, \mathbf{x})^{\frac{1}{n+1}} \, V_{\overline{K}}(\mathrm{d}\mathbf{x}),$$

where $\kappa_0(\overline{K}, \cdot)$ is the centro-affine curvature and $V_{\overline{K}}$ denotes the cone volume measure of $\overline{K}$. In this way, for a centered convex body $\overline{K}$, the affine isoperimetric inequality can be derived using the Blaschke–Santaló inequality by

$$\begin{aligned}\mathrm{as}_1(\overline{K}) &\leq n \left(\int_{\partial \overline{K}} \kappa_0(\overline{K}, \mathbf{x}) \; V_{\overline{K}}(\mathrm{d}\mathbf{x}) \right)^{\frac{1}{n+1}} \mathrm{vol}(\overline{K})^{\frac{n}{n+1}} \\ &\leq n \mathrm{vol}(\overline{K}^\circ)^{\frac{1}{n+1}} \mathrm{vol}(\overline{K})^{\frac{n}{n+1}} \\ &\leq n \mathrm{vol}(B_n^2)^{\frac{2}{n+1}} \mathrm{vol}(\overline{K})^{\frac{n-1}{n+1}} = \mathrm{as}_1(B_{\overline{K}}),\end{aligned}$$

where $\overline{K}^\circ = \bigcap_{\mathbf{x} \in \overline{K}} \{\mathbf{y} : \mathbf{x} \cdot \mathbf{y} \leq 1\}$ is the polar body of $\overline{K}$ and $B_{\overline{K}}$ is a centered Euclidean ball with the same volume as $\overline{K}$.

The floating area Ω_1^s of a spherical convex body K on the unit sphere $\mathbb{S}^n := \{\mathbf{u} \in \mathbb{R}^{n+1} : \mathbf{u} \cdot \mathbf{u} = 1\}$ was introduced in [3] as a natural non-Euclidean analog the the affine surface area, see also [2, 4]. It is a curvature measure that was derived via the volume derivative of spherical floating bodies and can be defined as

$$\Omega_1^s(K) = \int_{\partial K} H_{n-1}^s(K, \mathbf{u})^{\frac{1}{n+1}} \, \mathrm{vol}_{\partial K}^s(\mathrm{d}\mathbf{u}),$$

where $H_{n-1}^s(K, \cdot)$ denotes the spherical Gauss–Kronecker curvature and $\mathrm{vol}_{\partial K}^s$ is the spherical surface area measure of ∂K.

Gao, Hug & Schneider [6] have show that for a spherical convex body K that is contained in an open half-sphere, one can find a uniquely determined centered convex body $\overline{K} \subset \mathbb{R}^n$. Using this notion, in [5] we now establish a spherical analog to the affine isoperimetric inequality, that is,

$$\Omega_1^s(K) \leq \Omega_1^s(C_K),$$

for convex bodies $K \subset \mathbb{S}^n$, $n \geq 3$, when $\overline{K} \subset \sqrt{n(n-2)} B_2^n$, where B_2^n is the centered Euclidean unit ball in $\mathbb{R}^n$.

2. L_p-floating area

The L_p-affine surface area as_p was introduced by Lutwak [7] and is a family of centro-affine invariant surface area measures. Using again centro-affine invariant notions for a convex body of $\overline{K}$ of class C^2_+, we may define it as

$$\mathrm{as}_p(\overline{K}) = \int_{\partial\overline{K}} \kappa_0(\overline{K}, \mathbf{x})^{\frac{p}{n+p}}\, V_{\overline{K}}(\mathrm{d}\mathbf{x}),$$

for $p \neq -n$. Extensions for general convex bodies are available in the literature.

We propose to consider the family of L_p-floating areas, defined by

$$\Omega^s_p(K) = \int_{\partial K} H^s_{n-1}(K, \mathbf{u})^{\frac{p}{n+p}}\, \mathrm{vol}^s_{\partial K}(\mathrm{d}\mathbf{u}),$$

for $p \neq -n$ and a spherical convex body of $K \subset \mathbb{S}^n$ of class C^2_+.

Similar to the L_p-affine surface, the L_p-floating area is Ω^s_p a semi-continuous valuation. We derive an isoperimetric inequality

$$\Omega_p(K) \leq P^s(K)^{\frac{n}{n+p}} P^s(K^*)^{\frac{p}{n+p}},$$

with equality if and only if K is a geodesic ball. Here $P^s(K) := \mathrm{vol}^s_{\partial K}(\partial K)$ denotes the total spherical surface area of K and $K^* = \bigcap_{\mathbf{u}\in K}\{\mathbf{v} \in \mathbb{S}^n : d_s(\mathbf{u}, \mathbf{v}) \leq \frac{\pi}{2}\}$ is the spherical dual body. Furthermore, we show that the functional

$$p \mapsto \left(\frac{\Omega^s_p(K)}{P^s(K^*)}\right)^{1+\frac{p}{n}}$$

is monotone decreasing for $p \geq 0$.

Similar observations were obtained for the L_p-affine surface area by Paouris & Werner [8].

3. Curvature Entropy

Paouris & Werner [8] introduced the centro-affine entropy of a convex body $\overline{K} \subset \mathbb{R}^n$, which can be defined by

$$E_{PW}(\overline{K}) = \frac{1}{\mathrm{vol}(\overline{K}^\circ)} \int_{\partial\overline{K}} \kappa_0(K, \mathbf{x})\, \log \kappa_0(K, \mathbf{x})\, V_{\overline{K}}(\mathrm{d}\mathbf{x}).$$

A rigid-motion invariant entropy measure, the Gaussian entropy,

$$E_C(\overline{K}) = \frac{1}{\mathrm{vol}(\mathbb{S}^{n-1})} \int_{\partial\overline{K}} H_{n-1}(K, \mathbf{x})\, \log H_{n-1}(K, \mathbf{x})\, \mathrm{vol}_{\partial K}(\mathrm{d}\mathbf{x}),$$

was introduced by Hamilton and Chow. Here $H_{n-1}(\overline{K}, \cdot)$ is the Gauss–Kronecker curvature and $\mathrm{vol}_{\partial K}$ is the surface area measure.

Following ideas from the centro-affine setting introduced by Paouris & Werner [8], we derive a spherical curvature entropy for spherical convex bodies $K \subset \mathbb{S}^n$ of class C^2_+ by

$$E^s(K) := \lim_{p\to\infty} \left(1 + \frac{p}{n}\right) \log \frac{P^s(K^*)}{\Omega^s_p(K)}.$$

In [5] we are able to show that

$$E^s(K) = \frac{1}{P^s(K^*)} \int_{\partial K} H_{n-1}^s(K, \mathbf{u}) \log H_{n-1}^s(K, \mathbf{u}) \, \mathrm{vol}_{\partial K}^s(\mathrm{d}\mathbf{x}).$$

We believe that $E^s(K)$ can be seen as a non-Euclidean analog to both, the centro-affine entropy, as well as, the Gaussian entropy. We also obtain two inequalities

- $E^s(K) \geq \log \dfrac{P^s(K^*)}{P^s(K)}$, and
- $E^s(K^*) \geq E^s(C_K^*)$.

4. Open Questions

We believe that the isoperimetric inequality for the floating area, that is,

$$\Omega_1(K) \overset{?}{\leq} \Omega_1(C_K),$$

should be true in general for $n \geq 2$.

Furthermore, for the Gaussian entropy we have the isoperimetric inequality

$$E_C(\overline{K}) \geq E_C(B_{\overline{K}}),$$

which follows using the affine isoperimetric inequality. We conjecture that a similar inequality is also true for the centro-affine entropy, that is,

$$E_{PW}(\overline{K}) \overset{?}{\geq} E_{PW}(B_{\overline{K}}).$$

A similar question also remains open for the spherical analog: Do we have

$$E^s(K) \overset{?}{\geq} E^s(C_K)?$$

We are able to show that for $n = 2$ the isoperimetric inequality for the floating area implies the isoperimetric inequality for the curvature entropy. Thus, by a recent result of Basit, Hoehner, Lángi & Ledford [1] the curvature entropy inequality is true for symmetric spherical convex bodies on $\mathbb{S}^2$ of class C_+^2.

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Complex and quaternionic analogues of Busemann's random simplex and intersection inequalities

Thomas Wannerer
(joint work with Christos Saroglou)

The Busemann random simplex inequality provides a sharp lower bound on the expected volume of a random simplex in $\mathbb{R}^n$ formed by the origin and n vertices sampled uniformly from convex bodies $K_1, \ldots, K_n \subseteq \mathbb{R}^n$. It is a cornerstone of a beautiful theory of affine isoperimetric inequalities for convex bodies. From this inequality several other important inequalities such as the Petty projection inequality, the Busemann–Petty centroid inequality, and the Busemann intersection inequality can be deduced. For further information, see the books by Gardner [3], Schneider [8], and Schneider–Weil [9].

The first main result of the paper [10] is an analogue of the Busemann random simplex inequality for complex and quaternionic vector spaces. By restricting the scalar field, these can be viewed as real vector spaces and hence they possess convex sets in the usual sense. However, the complex and quaternionic structures give rise to additional geometric objects, such as complex or quaternionic subspaces. The underlying theme of [10] is the interaction of these geometric structures with classical convexity.

We denote the scalar field by $\mathbb{F}$, which is allowed to be either the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, or the quaternions $\mathbb{H}$. We denote by p the dimension of $\mathbb{F}$ over $\mathbb{R}$ (hence $p \in \{1, 2, 4\}$) and by $\det(x_1, \ldots, x_n)$ the determinant of the matrix with columns $x_1, \ldots, x_n \in \mathbb{F}^n$. Since we also consider matrices with quaternionic entries, it is important to note that the term "determinant" always refers to the Dieudonné determinant. A (real, complex, quaternionic) ellipsoid in $\mathbb{F}^n$ is by definition the image of the euclidean unit ball under an $\mathbb{F}$-affine transformation. In what follows, we will write $\mathbb{R}_+$ for the interval $[0, \infty)$.

Theorem 1. *Let $\Phi\colon \mathbb{R}_+ \to \mathbb{R}$ be a fixed strictly increasing function. For convex bodies $K_1, \ldots, K_n \subseteq \mathbb{F}^n$, set*

$$\mathscr{B}(K_1, \ldots, K_n) = \int_{x_1 \in K_1} \cdots \int_{x_n \in K_n} \Phi(|\det(x_1, \ldots, x_n)|) dx_1 \cdots dx_n.$$

Then

$$\mathscr{B}(K_1, \ldots, K_n) \geq \mathscr{B}(B_1, \ldots, B_n), \tag{1}$$

where B_i is the euclidean ball with center at the origin and of volume equal to the volume of K_i.

Moreover, if the bodies $K_1, \ldots, K_n$ have non-empty interior, then equality holds in (1) if and only if the K_i are homothetic (real, complex, or quaternionic) ellipsoids centered at the origin.

Remark 1. *As expected, the convexity can be dropped in Theorem 1 without any significant changes in the proof (see [7]). If one assumes $K_1, \ldots, K_n$ to be merely compact (or even just measurable) and of positive volume, then (1) is still true,* *with equality if and only if $K_1, \dots, K_n$ are, up to sets of measure zero, homothetic ellipsoids centered at the origin.*

In the real case, the above theorem is known as the Busemann random simplex inequality. The complex version of Theorem 1 is discussed in Grinberg's work [4]. However, there is an issue with the proof presented therein, as we will elaborate.

The classical proof of the Busemann random simplex inequality is based on the property that Steiner symmetrization does not increase $\mathscr{B}$. More precisely,

$$\mathscr{B}(K_1, \dots, K_n) \geq \mathscr{B}(\mathrm{S}_H K_1, \dots, \mathrm{S}_H K_n) \tag{2}$$

holds for every real hyperplane $H \subseteq \mathbb{R}^n$. Grinberg's paper suggests that in the complex case the same property holds with the same proof as in the real case. This is problematic, as it contradicts the characterization of the equality case as described in [4, Theorem 11] and Theorem 1.

Indeed, consider $K_1 = \cdots = K_n = E$, where E is a complex ellipsoid but not a Euclidean ball. In this scenario, equality holds in (1). However, there exist real hyperplanes H in $\mathbb{C}^n$ such that $\mathrm{S}_H E$ is not a complex ellipsoid. Consequently, the inequality (2) cannot hold true for such H.

Saroglou and Wannerer demonstrated in [10], using a different argument than the one used in the real case, that (2) holds for symmetrization in *complex* and *quaternionic* hyperplanes. In the quaternionic case, this required first establishing a weak form of the Laplace expansion for the Dieudonné determinant.

The second main result of [10] is an analogue of the Busemann intersection inequality for complex or quaternionic vector spaces. In the following theorem, $\mathrm{Gr}_m(n, \mathbb{F})$ denotes the Grassmannian of m-dimensional $\mathbb{F}$-linear subspaces of $\mathbb{F}^n$ and dE denotes integration with respect to the unique Haar probability measure on the Grassmannian. Let κ_n denote the volume of the n-dimensional euclidean unit ball in $\mathbb{R}^n$. The Lebesgue measure of a set $A \subseteq \mathbb{R}^n$ is denoted by $|A|$.

Theorem 2. *Let $m \in \{1, \dots, n-1\}$ and let $K \subseteq \mathbb{F}^n$ be a convex body with non-empty interior. Then*

$$|K|^m \geq \frac{(\kappa_{np})^m}{(\kappa_{mp})^n} \int_{\mathrm{Gr}_m(n,\mathbb{F})} |K \cap E|^n dE, \tag{3}$$

where $|K \cap E|$ denotes the Lebesgue measure of $K \cap E$ in E.

For $m = 1$ equality holds if and only if K is invariant under multiplication by scalars of unit norm. If $m \geq 2$, then equality holds in (3) *if and only if K is a (real, complex, or quaternionic) ellipsoid centered at the origin.*

The Busemann intersection inequality was initially proven by Busemann in [1] for $m = n-1$ and later in [2, Equation (9.4)] for general m. Grinberg rediscovered the general case in [4]. The paper also introduces the complex version of the inequality.

In the real case, the integral on the right-hand side of (3) is called the mth *dual affine quermassintegral*. It is well known to be invariant under volume-preserving linear transformations. This was proved by Grinberg [4], who also observed affine invariance over the complex numbers. We reproved these results and established the analogous property over the quaternions in [10], along with invariance of the mth *affine quermassintegral* $\int_{\mathrm{Gr}_m(n,\mathbb{F})} |P_E K|^{-n} dE$. Here $P_E \colon \mathbb{F}^n \to E$ denotes the orthogonal projection and $|P_E K|$ denotes the (euclidean) volume of $P_E K$ in E.

The results of our work [10] suggest to formulate a conjecture by Lutwak [5], which was recently confirmed by Milman–Yehudayoff [6], also over the complex numbers and the quaternions:

Conjecture 1. *For every convex body K in $\mathbb{F}^n$ with non-empty interior*

$$|K|^{-m} \geq \frac{(\kappa_{mp})^n}{(\kappa_{np})^m} \int_{\mathrm{Gr}_m(n,\mathbb{F})} |P_E K|^{-n} dE \tag{4}$$

with equality if and only if K is a (real, complex, or quaternionic) ellipsoid.

It is not difficult to see that Conjecture 1 is true in the specific case where $m = 1$ and K is the unit ball of a (complex or quaternionic) norm.

As in the real case, the conjecture directly implies the isoperimetric inequalities

$$\frac{\kappa_{np}}{\kappa_{mp}} \int_{\mathrm{Gr}_m(n,\mathbb{F})} |P_E K| dE \geq |K|^{m/n}. \tag{5}$$

While these inequalities are highly compelling, they remain open over the complex numbers and the quaternions.

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A generalized Blaschke–Santaló inequality for multiple even functions

Hiroshi Tsuji
(joint work with Shohei Nakamura)

Let $K \subset \mathbb{R}^n$ be a symmetric convex body, namely K is a compact and convex set with nonempty interior and satisfies $K = -K := \{-x\,;\, x \in K\}$. The polar body of K is defined by $K^\circ := \{x \in \mathbb{R}^n\,;\, \langle x, y\rangle \le 1, \forall y \in K\}$. For instance, let $\mathbf{B}_p^n := \{x \in \mathbb{R}^n\,;\, (\sum_{i=1}^n x_i^p)^{\frac{1}{p}} \le 1\}$ for $p \in [1,\infty]$ be the closed unit ball in ℓ^p-space. Then we see that the polar body of $\mathbf{B}_p^n$ is $\mathbf{B}_q^n$, where $q^{-1} + p^{-1} = 1$. For a symmetric convex body $K \subset \mathbb{R}^n$, the volume product is given by $v(K) := |K||K^\circ|$, where $|\cdot|$ is the n-dimensional Lebesgue measure. Regarding the volume product, the Blaschke–Santaló inequality [3, 10] provides the optimal upper bound of the volume product, which states that $v(K) \le v(\mathbf{B}_2^n)$ for any symmetric convex body $K \subset \mathbb{R}^n$.

Furthermore regarding the Blaschke–Santaló inequality, its functional version was discovered by Ball [2], Artstein-Avidan–Klartag–Milman [1], see also [4, 7, 8]. Let $f_1, f_2 \in L^1_+(\mathbb{R}^n) := \{f \in L^1(\mathbb{R}^n)\,;\, f \ge 0,\;\; 0 < \int_{\mathbb{R}^n} f\,dx < +\infty\}$ be even and satisfy

$$f_1(x_1)f_2(x_2) \le e^{-\langle x_1, x_2\rangle}, \quad x = (x_1, x_2) \in (\mathbb{R}^n)^2.$$

Then the functional Blaschke–Santaló inequality states that

$$\int_{\mathbb{R}^n} f_1\,dx_1 \int_{\mathbb{R}^n} f_2\,dx_2 \le (2\pi)^n$$

holds true.

Recently, a further generalized form of the functional Blaschke–Santaló inequality was proposed by Kolesnikov and Werner [6] as follows. Let $m \in \mathbb{N}$ be $m \ge 2$, and take unconditional functions $f_1, \dots, f_m \in L^1_+(\mathbb{R}^n)$ satisfying

$$\prod_{i=1}^m f_i(x_i) \le e^{-\frac{1}{m-1}\sum_{i<j}\langle x_i, x_j\rangle}, \quad x = (x_1, \dots, x_m) \in (\mathbb{R}^n)^m.$$

Then it holds that

$$\prod_{i=1}^m \int_{\mathbb{R}^n} f_i\,dx_i \le (2\pi)^{\frac{nm}{2}}.$$

In particular, we can recover the functional Blaschke–Santaló inequality for unconditional functions when $m = 2$. In the work by Kolesnikov and Werner, they also conjectured the same conclusion for any **even** functions. Toward this conjecture, Kalantzopoulos–Saroglou [5] have obtained some progress. The main goal in this talk is to give an affirmative answer to the conjecture by Kolesnikov and Werner. More precisely, we obtained a further generalization of the inequality proposed by Kolesnikov and Werner.

The following is our main result and proved in [9]: Let $m \in \mathbb{N}$ be $m \ge 2$, $n_1, \dots, n_m \in \mathbb{N}$, $N := \sum_{i=1}^m n_i$, $c_1, \dots, c_m > 0$ and $\mathcal{Q}$ be a $N \times N$ symmetric matrix, and take an **even** function $f_i \in L^1_+(\mathbb{R}^{n_i})$ for $i = 1, \dots, m$ satisfying

$$\prod_{i=1}^m f_i(x_i)^{c_i} \le e^{-\langle x, \mathcal{Q}x\rangle}, \quad x = (x_1, \dots, x_m) \in \bigoplus_{i=1}^m \mathbb{R}^{n_i} = \mathbb{R}^N.$$

Then it holds that

$$\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i \, dx_i \right)^{c_i} \le \sup_{A_1, \dots, A_m} \left(\int_{\mathbb{R}^{n_i}} g_{A_i} \, dx_i \right)^{c_i},$$

where $g_A(x) := e^{-\frac{1}{2}\langle x, Ax\rangle}$ for a symmetric matrix A, and sup is taken over all $A_i \in \mathbb{R}^{n_i \times n_i}$ satisfying $A_i > 0$ and

$$\prod_{i=1}^m g_{A_i}(x_i)^{c_i} \le e^{-\langle x, \mathcal{Q}x\rangle}, \quad x = (x_1, \dots, x_m) \in \bigoplus_{i=1}^m \mathbb{R}^{n_i} = \mathbb{R}^N.$$

Especially, we may obtain a positive answer to the conjecture by Kolesnikov and Werner when $n_i = n$, $c_i = 1$ for all $i = 1, \dots, m$ and

$$\mathcal{Q} = \frac{1}{2(m-1)} \begin{pmatrix} 0 & \mathrm{id}_n & \cdots & \mathrm{id}_n \\ \mathrm{id}_n & 0 & \cdots & \vdots \\ \vdots & & \ddots & \mathrm{id}_n \\ \mathrm{id}_n & \dots & \mathrm{id}_n & 0 \end{pmatrix}.$$

We finally remark that our argument does not provide any equality condition immediately since our methods rely heavily on a limiting argument.

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Intersection processes of k-flats in hyperbolic space

Tillmann Bühler
(joint work with Daniel Hug)

We consider an isometry invariant Poisson process η of k-flats, $0 \leq k \leq d-1$, in d-dimensional hyperbolic space $\mathbb{H}^d$. For $d-m(d-k) \geq 0$, the m-th order intersection process of η consists of all intersections of distinct flats $E_1, \dots, E_m \in \eta$ satisfying $\dim(E_1 \cap \ldots \cap E_m) = d - m(d-k)$. (For more details and precise definitions, we refer the reader to [2]). While the study of such processes in euclidean space has a long history (see, e.g., [4, 10, 9]), the investigation in the hyperbolic setting was initiated only a few years ago in [5].

Let $F_r^{(m)}$ denote the volume of the intersection process inside a ball of radius $r > 0$. The central question is whether $F_r^{(m)}$ satisfies a standard central limit theorem (CLT) as $r \to \infty$, that is, whether the standardized random variable

$$\frac{F_r^{(m)} - \mathbb{E}[F_r^{(m)}]}{\sqrt{\mathbb{V}(F_r^{(m)})}}$$

converges in law to a unit normal distribution for $r \to \infty$.

The history of this problem can be summarized as follows: In [5], which considers only processes of hyperplanes (that is, $k = d-1$), a CLT is established for dimensions $d = 2, 3$. It is conjectured and partially verified that no CLT holds in dimensions $d \geq 4$. In [7] (which again considers only hyperplanes), the asymptotic limit distribution is established for $d \geq 4$ and $m = 1$. This is possible since for $m = 1$ one can explicitly calculate the characteristic function of $F_r^{(1)}$. The more general problem of intersection processes of k-flats was treated in [2], where it is revealed that the asymptotic distribution of $F_r^{(m)}$ does not only depend on the dimension d of the ambient space, but rather on the relation of d and the dimension k of the flats: a standard CLT holds precisely when $2k \leq d+1$. For $2k > d+1$ and $m = 1$, the limit distribution is determined using the same approach as in [7]. For $m \geq 2$, the distribution remained unknown, and even the fact that it is not Gaussian was only conjectured (except for the cases already verified in [5]). The paper [1] (on which this talk is based) verifies this conjecture and establishes the limit distribution in all previously unknown cases.

1. Proof sketch

For k-flats $E_1, \dots, E_m$, we define

$$f_r^{(m)}(E_1, \dots, E_m) := \mathcal{H}^{d-m(d-k)}(E_1 \cap \cdots \cap E_m \cap B_r)\, \mathbb{1}\{\dim(E_1 \cap \cdots \cap E_m) = d - m(d-k)\}.$$

The functional $F_r^{(m)}$ can then be written as

$$F_r^{(m)} = \sum_{(E_1,\dots,E_m) \in \eta^m_{\neq}} f^{(m)}(E_1, \dots, E_m),$$

where the sum extends over all tuples of distinct flats $E_1, \dots, E_m \in \eta$. A random variable of this form is called a *Poisson U-statistic*. It is a classical result in the theory of Poisson processes, that such a U-statistic admits a *chaos decomposition*, that is, it can be written as a sum of compensated Wiener-Itô integrals

$$F_r^{(m)} = \mathbb{E}[F_r^{(m)}] + I_1(f_1^{(m)}) + \cdots + I_m(f_m^{(m)}). \tag{1}$$

For further details and the definition of the integrals we refer the reader to [8] or [6]. The functions $f_1^{(m)}, \dots, f_m^{(m)}$ are defined by

$$f_{r,i}^{(m)}(E_1, \dots, E_i) := \binom{m}{i} \frac{1}{m!} \int_{A(d,k)^{m-i}} f_r^{(m)}(E_1, \dots, E_i, E_{i+1}, \dots, E_m)\, \mu^{m-i}(d\,(E_{i+1}, \dots, E_m)),$$

where $A(d,k)$ denotes the space of k-flats in $\mathbb{H}^d$ and μ is the intensity measure of η. Using the *Crofton formula* from integral geometry, on finds that

$$f_r^{(m)}(E) = C \cdot f_r^{(1)}(E)$$

for some constant $C > 0$ that depends only on d, k and m. As a consequence, one gets that

$$I_1(f_1^{(m)}) = C \cdot (F_r^{(1)} - \mathbb{E}[F_r^{(1)}]). \tag{2}$$

A variance analysis shows that for $2k > d+1$, the terms $I_2(f_r^{(2)}), \dots, I_m(f_r^{(m)})$ are asymptotically negligible. This reduces the problem to the known case $m = 1$.

To be more precise, [2] determines random variables $Y_{d,k}$ for $2k > d+1$ so that

$$\frac{F_r^{(1)} - \mathbb{E}[F_r^{(1)}]}{e^{r(k-1)}} \xrightarrow{D} Y_{d,k}, \qquad r \to \infty.$$

Combining (1), (2) and the variance analysis then yields

$$\frac{F_r^{(m)} - \mathbb{E}[F_r^{(m)}]}{e^{r(k-1)}} = C \cdot \frac{F_r^{(1)} - \mathbb{E}[F_r^{(1)}]}{e^{r(k-1)}} + \text{lower order terms} \xrightarrow{D} C \cdot Y_{d,k}.$$

This qualitative result is further complemented by a *quantitative limit theorem*, more specifically, a bound on the Kolmogorov distance d_K of the left hand-side of the above expression and $C \cdot Y_{d,k}$. (The Kolmogorov distance of two distributions with cumulative distribution functions F, G is $||F - G||_\infty$.) To derive the bound, we use the following result which is due to Esseen (cf. [3]):

Theorem 1 (Esseen, 1945). *Let F, G be distribution functions with characteristic functions φ_F, φ_G respectively. If G has bounded density $g \leq M$, then*

$$|F(x) - G(x)| \leq \frac{1}{\pi} \int_{-T}^{T} \frac{|\varphi_F(t) - \varphi_G(t)|}{|t|}\, dt + \frac{24M}{\pi T}$$

for all $x \in \mathbb{R}$ and $T > 0$.

This allows us to tackle the problem on the level of characteristic functions, which can be explicitly calculated for $m = 1$.

2. Properties of the limit distribution

For $2k > d+1$, let

$$Y_{d,k}^* := \frac{Y_{d,k} - \mathbb{E}[Y_{d,k}]}{\sqrt{\mathbb{V}(Y_{d,k})}}$$

be the standardized limit distribution of $F_r^{(m)}$. While its characteristic function $\psi_{d,k}$ can be explicitly calculated, there are still open questions concerning this distribution.

We managed to numerically approximate and plot its density $f_{d,k}$. Those plots seemed to suggest that, for $d \to \infty$ and $k \approx d/2$, the distribution converges to $\mathcal{N}(0,1)$. An analysis of the cumulants $\mathrm{cum}_\ell(Y_{d,k}^*)$, $\ell \in \mathbb{N}$, however painted a different picture: While all cumulants (except for the second) of a unit normal distribution vanish, $\mathrm{cum}_\ell(Y_{d,k}^*)$ diverges for $\ell \geq 3$. This leaves us with the following open problem:

Question. *Let $(d_j), (k_j)$ be sequences of integers satisfying $0 \leq k_j \leq d_j - 1$, $2k_j > d_j + 1$ and $d_j \to \infty$, $j \to \infty$. What is the asymptotic behavior of Y_{d_j,k_j}^* as $j \to \infty$? In particular, are there any conditions on d_j and k_j (such as $k_j/d_j \to 1/2$) which ensure that Y_{d_j,k_j}^* converges in distribution, and if so, is the limit distribution Gaussian?*

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On the monotonicity of discrete entropy for log-concave random vectors in $\mathbb{Z}^d$

Matthieu Fradelizi
(joint work with Lampros Gavalakis and Martin Rapaport)

We prove the following type of discrete entropy monotonicity for sums of isotropic, log-concave, independent and identically distributed random vectors $X_1, \dots, X_{n+1}$ on $\mathbb{Z}^d$:

$$H(X_1 + \cdots + X_{n+1}) \geq H(X_1 + \cdots + X_n) + \frac{d}{2} \log\left(\frac{n+1}{n}\right) + o(1),$$

where $o(1)$ vanishes as $H(X_1) \to \infty$. The optimality of the constant can be seen by using discrete Gaussians random variables and thus, also with Binomial distributions $X \sim B(m, 1/2)$. Contrary to the continuous case, in the discrete case, one cannot hope for the same inequality without a correction term $o(1)$ as can be seen with the example of a Dirac distribution. Moreover, for the $o(1)$-term, we obtain a rate of convergence $O(H(X_1)e^{-\frac{1}{d}H(X_1)})$, where the implied constants depend on d and n. This generalizes to $\mathbb{Z}^d$ the one-dimensional result of the second named author [2] and this was conjectured by Tao in [4].

As in dimension one, our strategy is to establish that the discrete entropy $H(X_1 + \cdots + X_n)$ is close to the differential (continuous) entropy $h(X_1 + U_1 + \cdots + X_n + U_n)$, where $U_1, \dots, U_n$ are independent and identically distributed uniform random vectors on $[0,1]^d$ and to apply the theorem of Artstein, Ball, Barthe and Naor [1] on the monotonicity of differential entropy.

There are multiple definitions of log-concavity in $\mathbb{Z}^d$, see [3]. For our purpose, we choose the following one: we say that a probability mass function p on $\mathbb{Z}^d$ is log-concave if there exists a log-concave function f on $\mathbb{R}^d$ such that its restriction on $\mathbb{Z}^d$ coincide with p. This is an open question to know if self-convolution preserves this definition.

In fact, we show this result under more general assumptions than log-concavity, which are preserved up to constants under convolution. Namely, we consider families of random variables for which, as the determinant of the covariance matrix increases, the probability mass function:

i) is bounded above in terms of the the determinant of the covariance matrix,
ii) has subexponential tails,
iii) has (discrete) bounded variation.

In order to show that log-concave distributions satisfy our assumptions in dimension $d \geq 2$, more involved tools from convex geometry are needed because a suitable position is required. We show that, for a log-concave function on $\mathbb{R}^d$ in isotropic position, its integral, barycenter and covariance matrix are close to their discrete counterparts. Moreover, in the log-concave case, we weaken the isotropicity assumption to what we call *almost isotropicity.* One of our technical tools is a discrete analogue to the upper bound on the isotropic constant of a log-concave function, which extends to dimensions $d \geq 1$ a result of Bobkov, Marsiglietti and Melbourne (2022) in dimension one and may be of independent interest.

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How often do centroids of sections coincide with the centroid of a convex body?

Kateryna Tatarko
(joint work with Sergii Myroshnychenko, Vladyslav Yaskin)

We say that K is a convex body in $\mathbb{R}^n$ if K is a compact convex subset with non-empty interior. The centroid of K (also known as the center of mass or the barycenter) is defined as

$$c(K) = \frac{1}{|K|} \int_K x\, dx$$

where $|K|$ denotes the volume of K and the integration is with respect to Lebesgue measure.

In 1961, Grunbaum [1] asked the following question:

Problem 1. *Is the centroid $c(K)$ of $K \subset \mathbb{R}^n$ the centroid of at least $n+1$ different $(n-1)$-dimensional sections of K through $c(K)$?*

A few years later, the more general question was asked by Loewner [3, Problem 28].

Problem 2. *What is the minimum number $\mu(n)$ of $(n-1)$-dimensional sections of $K \subset \mathbb{R}^n$ passing through $c(K)$ whose centroid coincide with $c(K)$, where the minimum is taken over all convex bodies K in $\mathbb{R}^n$?*

Grunbaum [1] (see also [5]) showed that through any point p in the interior of K, there exists a hyperplane H containing p such that p is the centroid of a convex body $K \cap H$. The above problems are concerned whether there can be more than one of such hyperplanes.

It is known that Problem 1 has an affirmative answer in dimension $n = 2$, so every planar convex body K has at least three different chords that are bisected by the centorid $c(K)$ (see [6] for a proof). For example, the centroid of any triangle bisects three chords that are parallel to the sides.

The main goal of this talk is to present answers to Problems 1 and 2 in dimensions $n \geq 5$. Our main result is

Theorem 3. *There exists a convex body of revolution $K \subset \mathbb{R}^n$, $n \geq 5$ with the property that the centroid $c(K)$ is at the origin, and there is exactly one hyperplane H passing through $c(K)$ such that the centroid of $K \cap H$ coincides with $c(K)$.*

Thus, $\mu(n) = 1$ in Problem 2 and Problem 1 has a negative answer for $n \geq 5$. Our proof of Theorem 3 uses Fourier analytic tools, and relies on the fact that there exist origin-symmetric convex bodies that are non-intersection bodies in $\mathbb{R}^n$, $n \geq 5$. The notion of the intersection body of a star body was first introduced by Lutwak [4] and since then has played an important role in convex geometry, and in particular, in the solution of the celebrated Busemann–Petty problem.

We remark that both Problems 1 and 2 remain to be open in dimensions $n = 3$ and $n = 4$.

We also would like to point out that Grunbaum in [2, Section 6.2] claimed that for every convex body $K \subset \mathbb{R}^n$ there exists an interior point through which there are at least $n+1$ distinct $(n-1)$-dimensional sections of K whose centroid coincide with $c(K)$. However, Patakova, Tancer and Wagner [7] recently discovered that one of the auxiliary statements in Grunbaum's argument is incorrect. They also showed that every convex body $K \subset \mathbb{R}^n, n \geq 3$ contains a point p that is the centroid of at least four hyperplane sections passing through p. This confirms Grunbaum's claim in dimension $n = 3$, but it leaves open the question about existence of such a point in dimension $n \geq 4$.

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Existence and continuity for the functional Minkowski problem

Liran Rotem

(joint work with Tomer Falah)

We work with the following class of convex functions

$$\mathrm{Cvx}_n = \left\{ \phi : \mathbb{R}^n \to (-\infty, \infty] : \begin{array}{l} \phi \text{ is convex, lower-semicontinuous} \\ \text{and } 0 < \int e^{-\phi} < \infty \end{array} \right\},$$

and the corresponding space of log-concave functions

$$\mathrm{LC}_n = \left\{ e^{-\varphi} : \ \varphi \in \mathrm{Cvx}_n \right\}.$$

Addition of log-concave functions is given by the *sup-convolution*: For $f, g \in \mathrm{LC}_n$ we set

$$(f \star g)(x) = \sup_{y \in \mathbb{R}^n} (f(y)g(x-y)).$$

The associated dilation operation is $(t \cdot f)(x) = f\left(\frac{x}{t}\right)^t$ ("associated" means that $(t \cdot f) \star (s \cdot f) = (t+s) \cdot f$). Finally, the support function h_f of a $f = e^{-\varphi} \in \mathrm{LC}_n$ if given by the *Legendre transform*

$$h_f(y) = \varphi^*(y) = \sup_{x \in \mathbb{R}^n} [\langle x, y \rangle - \varphi(x)].$$

Our goal is to study the first variation of the integral of log-concave functions. The fundamental result in this direction is:

Theorem 1 ([3] and [4], following [1] and [2])**.** *For all $f, g \in \mathrm{LC}_n$ we have*

$$\lim_{t \to 0^+} \frac{\int f \star (t \cdot g) - \int f}{t} = \int_{\mathbb{R}^n} h_g \mathrm{d}\mu_f + \int_{\mathbb{S}^{n-1}} h_{\mathrm{supp}(g)} \mathrm{d}\nu_f.$$

Here the measures (μ_f, ν_f) are two measures associated with f that together constitute the *surface area measures* of f. The measure μ_f is a Borel measure on $\mathbb{R}^n$ defined by $\mu_f = (\nabla\varphi)_\sharp (f \mathrm{d}x)$ (where $f = e^{-\varphi}$). The measure ν_f is a Borel measure on the unit sphere $\mathbb{S}^{n-1}$ defined by $\nu_f = \left(n_{\mathrm{supp}(f)}\right)_\sharp \left(f \mathrm{d}\mathcal{H}|_{\partial\, \mathrm{supp}(f)}\right)$.

Our first main theorem is a solution to the Minkowski problem for the pair (μ_f, ν_f):

Theorem 2. *Let μ be a finite Borel measure on $\mathbb{R}^n$ and ν be a finite Borel measure on $\mathbb{S}^{n-1}$. Then there exists $f \in \mathrm{LC}_n$ such that $\mu_f = \mu$ and $\nu_f = \nu$ if and only if (μ, ν) satisfy the following properties:*

(1) $\mu \not\equiv 0$.

(2) *μ has a finite first moment and $\mu + \nu$ is centered in the sense that*

$$\int_{\mathbb{R}^n} x \mathrm{d}\mu + \int_{\mathbb{S}^{n-1}} x \mathrm{d}\nu = 0.$$

(3) *μ and ν are not supported on the same hyperplane.*

In this case the function f is unique of the translations (i.e. up to replacing it by functions of the form $f_v(x) = f(x+v)$).

In the case $\nu \equiv 0$ this theorem was proved by Cordero-Erausquin–Klartag ([2]), with another proof by Santambrogio ([5]).

Our second theorem regards continuity of the surface area measures and requires two definitions:

Definition 1. We say that a function $g : \mathbb{R}^n \to \mathbb{R}$ is cosmically continuous if:

(1) g is continuous.

(2) The horizon function $\overline{g}(\theta) = \lim_{t \to \infty} \frac{g(t\theta)}{t}$ exists and is finite, and the limit is uniform in $\theta \in \mathbb{S}^{n-1}$.

Definition 2. We say that $(\mu_i, \nu_i) \to (\mu, \nu)$ cosmically if for all cosmically continuous functions g we have

$$\int_{\mathbb{R}^n} g \mathrm{d}\mu_i + \int_{\mathbb{S}^{n-1}} \overline{g} \mathrm{d}\nu_i \to \int_{\mathbb{R}^n} g \mathrm{d}\mu + \int_{\mathbb{S}^{n-1}} \overline{g} \mathrm{d}\nu.$$

The motivation for the same "cosmic" comes from the fact that cosmic convergence can be viewed as a standard weak convergence on a compactification of $\mathbb{R}^n$ known as "cosmic $\mathbb{R}^n$".

We also need a suitable notation of convergence for log-concave functions. If $f_i = e^{-\varphi_i}$ and $f = e^{-\varphi}$ we say that $f_i \to f$ if $\varphi_i \to \varphi$ is the sense of epi-convergence. Under these definitions our continuity result reads as follows:

Theorem 3. *Fix* $\{f_i\}_{i=1}^{\infty}, f \in \mathrm{LC}_n$.

(1) *If $f_i \to f$ then $(\mu_{f_i}, \nu_{f_i}) \to (\mu_f, \nu_f)$ cosmically.*
(2) *If $(\mu_{f_i}, \nu_{f_i}) \to (\mu_f, \nu_f)$ cosmically then up to translating the functions we have $f_i \to f$.*

The proof of the last theorem is partially inspired by results of Ulivelli ([6]) though his theorems are not used directly.

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Hard Lefschetz Theorem and Hodge–Riemann relations for convex valuations

Andreas Bernig

(joint work with Jan Kotrbatý, Thomas Wannerer)

Let V denote the mixed volume of n compact convex bodies in $\mathbb{R}^n$ and $\mathcal{K}_+^{\infty}$ the space of smooth convex bodies with positive curvature.

Theorem 1. [5]*: Let* $0 \leq k \leq \frac{n}{2}, x_i \in \mathbb{R}$, $A_j^i, C_l \in \mathcal{K}_+^{\infty}$. *If*

$$\sum_{i=1}^{N} x_i V(A_1^i, \ldots, A_k^i, C_0, \ldots, C_{n-2k}, \bullet) = 0,$$

then

$$(-1)^k \sum_{i,j} x_i x_j V(A_1^i,\dots,A_k^i,A_1^j,\dots,A_k^j,C_1,\dots,C_{n-2k}) \geq 0$$

with equality if and only if

$$\sum_i x_i V(A_1^i,\dots,A_k^i,\bullet) = 0.$$

Remarks:

- For $k=0$, we obtain that mixed volumes are positive. This is well-known but non-trivial.
- For $k=1$ we obtain the classical Alexandrov–Fenchel inequality from the theorem as follows. First we have

  $$V(A,C_0,C_1,\dots,C_{n-2}) + cV(C_0,C_0,C_1,\dots,C_{n-2}) = 0,$$

  where $c = -\frac{V(A,C_0,C_1,\dots,C_{n-2})}{V(C_0,C_0,C_1,\dots,C_{n-2})}$. It follows from the theorem that

  $$\begin{aligned} V(A,A,C_1,\dots,C_{n-2}) &+ 2cV(A,C_0,\dots,C_{n-2}) \\ &+ c^2 V(C_0,C_0,\dots,C_{n-2}) \leq 0, \end{aligned}$$

  which is equivalent to

  $$V(A,C_0,C_1,\dots,C_{n-2})^2 \geq V(A,A,C_1,\dots,C_{n-2})V(C_0,C_0,C_1,\dots,C_{n-2}).$$

  By approximation, this inequality still holds true if the bodies $A, C_0, \dots, C_{n-2}$ are arbitrary compact convex bodies, which is the Alexandrov–Fenchel inequality.
- The case $k \geq 2$ can be seen as a higher degree version of the Alexandrov–Fenchel inequality.
- An example due to van Handel [13] shows that the assumption that the bodies are smooth with positive curvature can not be omitted.

The theorem can be reformulated in terms of a Kähler–Lefschetz package for valuations. Let Val denote the space of continuous translation invariant valuations on $\mathbb{R}^n$. By [11], there is a grading

$$\mathrm{Val} = \bigoplus_{k=0}^n \mathrm{Val}_k,$$

where

$$\mathrm{Val}_k = \{\mu \in \mathrm{Val} : \mu(tK) = t^k \mu(K), t > 0\}.$$

Alesker [2] has shown that valuations of the form $K \mapsto V(K,\dots,K,A_1,\dots,A_{n-k})$ span a dense subspace in Val_k. We denote by Val_k^∞ the linear combinations of such valuations where $A_1,\dots,A_{n-k} \in \mathcal{K}_+^\infty$ and $\mathrm{Val}^\infty = \bigoplus_{k=0}^n \mathrm{Val}_k^\infty$. By [4, 10] there exists a convolution product on Val^∞ such that

$$\mathrm{vol}(\bullet + A) * \mathrm{vol}(\bullet + B) = \mathrm{vol}(\bullet + A + B), \quad A, B \in \mathcal{K}_+^\infty.$$

Theorem 2. [5] *Let* $0 \leq k \leq \frac{n}{2}$, $C_0, C_1, \ldots, C_{n-2k} \in \mathcal{K}^\infty_+$.

(1) *Hard Lefschetz theorem: convolution with* $V(\bullet, C_1, \ldots, C_{n-2k}) \in \mathrm{Val}^\infty_{2k}$ *defines an isomorphism of topological vector spaces*
$$\mathrm{Val}^\infty_{n-k} \to \mathrm{Val}^\infty_k.$$

(2) *Hodge-Riemann relations: if* $\phi \in \mathrm{Val}^\infty_{n-k}$ *satisfies*
$$V(\bullet, C_0, C_1, \ldots, C_{n-2k}) * \phi = 0,$$
then
$$(-1)^k \phi * \phi * V(\bullet, C_1, \ldots, C_{n-2k}) > 0$$
with equality if and only if $\phi = 0$.

In the case where all the reference bodies $C_0, \ldots, C_{n-2k}$ are euclidean unit balls, the hard Lefschetz theorem was shown earlier in [3], while the Hodge–Riemann relations have been obtained more recently in [9].

A similar set of theorems holds in various other areas of mathematics:

- Cohomology algebra of a compact Kähler manifold. The (mixed) hard Lefschetz theorem was shown by Dinh-Nguyen [6].
- McMullen's polytope algebra of a simple polytope [12], which implies the quadratic inequalities for strongly isomorphic polytopes.
- Combinatorial intersection theory of a convex polytope [8].
- Chow ring of a matroid [1], with many applications to combinatorial problems.
- Grothendieck's standard conjectures on algebraic cycles [7], still open.

Remark: These algebras are finite-dimensional, while Val^∞ is infinite-dimensional.

Our proof of the Hard Lefschetz theorem and the Hodge–Riemann relations uses tools from differential geometry and functional analysis. More precisely, we study unbounded linear operators on certain Hilbert space completions of the space of valuations and apply perturbation theory.

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The growth rate of surface area measure for C-asymptotic sets

Yiming Zhao
(joint work with Vadim Semenov)

Our motivation is to study the shape of a closed, convex, but potentially unbounded set in $\mathbb{R}^n$. We say such a set K is a *pseudo cone* if $\lambda x \in K$ for all $x \in K$ and $\lambda \geq 1$. Its *recession cone* is given by $\operatorname{rec} K = \{y \in \mathbb{R}^n : x + \lambda y \in K, \forall x \in K, \lambda \geq 1\}$. It can be shown that a pseudo cone K is always a subset of its recession cone. We call K a C-pseudo set if K is a pseudo cone and $\operatorname{rec} K = C$.

Depending on what the interests are, different subclasses of C-pseudo sets are appropriate. For example, Khovanskiĭ-Timorin [5] stated that *C-full sets* appear naturally in singularity theory and commutative algebra. Schneider [7] showed that various elements of the Brunn-Minkowski theory of convex bodies can be extended to *C-close* sets (those K such that $V(C \setminus K)$ is finite). Copolarity can be defined for C-pseudo sets and appeared in Artstein-Avidan, Sadovsky, Wyczesany [1] and Xu-Li-Leng [4].

This talk is focused on C-asymptotic sets. A C-pseudo set K is called C-asymptotic if for all $x \in \partial K$ as $|x| \to \infty$, we have $\operatorname{dist}(x, \partial C) \to 0$. Let $\Omega = \{v \in S^{n-1} : v \cdot u < 0, \forall u \in C \cap S^{n-1}\}$. Define the support function of K, denoted by $h_K : \overline{\Omega} \to \mathbb{R}$, by $h_K(v) = \sup_{x \in K} v \cdot x$. It can be shown that a set K is C-asymptotic if and only if $h_K = 0$ on $\partial\Omega$. Therefore, requiring a set to be C-asymptotic prescribes a Dirichlet-type boundary condition and such sets are natural if PDE is to be studied.

Define the *surface area measure* of a C-asymptotic set K to be the Borel measure on Ω given by

$$S_K(\omega) = \mathcal{H}^{n-1}(\{x \in \partial K : x \cdot v = h_K(v) \text{ for some } v \in \omega\}).$$

Contrary to the classical setting, the surface area measure of a generic C-asymptotic set might be an *infinite* measure. There is also no Aleksandrov-type variational formula that leads to S_K since $V(C \setminus K)$ might be infinite!

It is natural and makes sense to ask the following Minkowski-type problem:

The Minkowski problem for C-asymptotic sets. Given a (potentially *infinite*) Borel measure μ on Ω, what are the necessary and sufficient conditions on μ

so that there exists a C-asymptotic set K with $\mu = S_K$? If such a K exists, is it unique?

We emphasize that the given measure μ might be infinite and that is in fact the most challenging case. Indeed, one can show that the measure equation is equivalent to solving the following Monge-Ampère equation on an open bounded convex domain $\widetilde{\Omega}$ of $\mathbb{R}^{n-1}$:

$$\begin{cases} \det(\nabla^2 u) = \widetilde{\mu} & \text{in } \widetilde{\Omega}, \\ u|_{\partial\widetilde{\Omega}} = 0. \end{cases} \tag{1}$$

Here, the measure $\widetilde{\mu}$ is finite if and only if μ is finite. The PDE (1) is well-studied *if $\widetilde{\mu}$ is finite.* In particular, one can use this and a (relative) isoperimetric inequality to show

Theorem 1. *Let μ be a finite Borel measure on Ω. Then there exists a unique C-asymptotic set $K \subset C$ such that $S_K = \mu$. Moreover, K is C-close.*

Schneider in [7, 8] provided a separate (and direct) proof using a variational scheme. It is important to point out that in the finite case, the essential ingredient is the Alesksandrov maximum principle which states

$$|u(x)|^n \leq_c \operatorname{diam}(\widetilde{\Omega})^{n-1} \operatorname{dist}(x, \partial\widetilde{\Omega}) |\widetilde{\mu}|.$$

Note that this estimate becomes ineffective if $\widetilde{\mu}$ is an infinite measure.

Uniqueness of solution also follows immediately from the comparison principle for Monge-Ampère equations.

Theorem 2. *Let K, L be C-asymptotic sets. If $S_K = S_L$, then $K = L$.*

In the special case where K and L are C-close, Schneider [7] provided a new proof via the Brunn-Minkowski inequality for C-close sets.

It is also worth pointing out that as an application of the comparison principle, one can define the Blaschke sum of two C-asymptotic sets.

Since "μ has its centroid at the origin" was the only non-trivial condition required in the classical Minkowski problem for convex bodies and that this condition intuitively guarantees the solution convex body "closes up", Aleksandrov in his book "Convex Polyhedra" claimed that the Minkowski problem for C-asymptotic sets can be solved without any non-trivial conditions and that the solution can be obtained by passing to the limit from the discrete case. However, this is not the case by looking even at the 2-dimensional case. In fact, the growth of S_K near $\partial\Omega$ is paramount here and is what needed to be estimated before successful resolution of the problem can happen.

Indeed, when μ (or equivalently $\widetilde{\mu}$) is an infinite measure, existence results are scarce. Pogorelov [6] and later improved by Bakelman [2] and Chou-Wang [3], gave a sufficient condition in the special case when μ is absolutely continuous. Their method involves constructing explicitly a convex hypersurface as a "barrier function". The existence of such a surface was guaranteed by the hypotheses of their result.

In this talk, a necessary and sufficient condition in dimension 2 is presented. See Semenov-Zhao [9].

Theorem 3. *Let μ be a Borel measure on Ω. There exists a unique C-asymptotic set $K \subset C$ such that*

$$\mu = S_K \text{ on } \Omega$$

if and only if

$$\int_0^{\frac{\pi}{2}} \mu(\omega_\alpha) d\alpha < \infty.$$

In higher dimensions, while necessary (but not sufficient) and sufficient (but not necessary) conditions exist, the problem is still widely open. In Semenov-Zhao [9], the growth rate of surface area measures for C-asymptotic sets is estimated in higher dimensions. Schneider [8] showed that for any C-asymptotic set K and $\omega_\alpha = \{v \in \Omega : \operatorname{dist}(v, \partial\Omega) > \alpha\}$, one has $S_K(\omega_\alpha) \leq_c 1/\alpha^{n-1}$. We formulate the following conjecture.

Conjecture 1. *If K is C-asymptotic, then*

$$\int_0^{\pi/2} S_K(\omega_\alpha)^{\frac{1}{n-1}} d\alpha < \infty. \tag{2}$$

Note that this is indeed the case in dimension 2 and is slightly stronger than Schneider's necessary condition. It was also shown (through rather complicated constructions) in Semenov-Zhao [9] that the power in (2) is the best possible in dimension 3. That is, if the power were to be replaced by any bigger number, then there exists a C-asymptotic set K such that the integral is not finite. As a matter of fact, Semenov-Zhao [9] showed that in dimension 3, conditions of the type (2) for any power cannot be both necessary and sufficient. While the computation was done in dimension 3, it can be adapted to general dimensions (but with much messier computations and notations). We believe that global growth conditions like (2) might be only one of the many pieces of the puzzle to the Minkowski problem for C-asymptotic sets.

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The stable Banach isometric problem

Dmitry Faifman
(joint work with Gautam Aishwarya)

In 1932 Banach [1] proposed the following question.

Conjecture. Let V be a real or complex normed space, and $2 \leq k < \dim V$ an integer. Suppose that all linear k-dimensional subspaces of V are isometric to each other. Then V must be Euclidean.

Equivalently, if the k-sections of a convex body through a fixed interior point are linearly equivalent, it is an Ellipsoid? Banach's conjecture is known to be equivalent to the seemingly stronger one, where affine equivalence replaces linear [2].

This conjecture has been proven in various cases. The earliest proof for $k = 2$ by Auerbach–Mazur–Ulam [3] was rooted in the topology of the sphere. Topological obstructions remained the main tool in Gromov [4] who settled all even k, as well some cases of the complex Banach problem. Recently, Montejano et al [5] combined topological and convex-geometric ideas to prove the conjecture for $k = 4m+1$, with the possible exception of $k = 133$. Some further cases were settled for complex normed spaces in [6]. For $k = 3$, the conjecture has recently been confirmed by Ivanov et al [7].

In a recent joint work with G. Aishwarya, we considered the stable version of the Banach conjecture. Roughly put, it asks whether a normed spaces all of whose k-sections are nearly-isometric, is nearly Euclidean? For a precise statement, we let d_{BM} denote the Banach-Mazur distance between normed spaces, or between convex bodies.

Conjecture. Let V be a real or complex n-dimensional normed space, and $2 \leq k < n$ an integer. Suppose that all linear k-dimensional subspaces $E, E' \subset V$ satisfy $d_{BM}(E, E') < \delta$. Then $d_{BM}(V, \mathbb{E}^n) < \epsilon(n, k, \delta)$.

We were able to resolve the conjecture in the positive for $k = 2$.

Theorem [Aishwarya-F.] Let $V = \mathbb{R}^n$ be a normed space, not necessarily symmetric. Assume that $d_{BM}(E, F) < 1+\delta$ for all 2-dimensional subspaces $E, F \subset V$. Then $d_{BM}(V, \mathbb{E}^n) < 1 + cn^2\delta^{1/3}$. We proved stability also for the affine variant of the problem.; unlike the original question, stability in the affine setting is not equivalent to the linear setting, and enjoys worse constants.

Theorem [Aishwarya-F.] Let $K \subset \mathbb{R}^n$ be a convex body with 0 in its interior. Assume that $d_{BM}(K \cap E, K \cap F) < 1 + \delta$ for all linear 2-dimensional subspaces $E, F \subset \mathbb{R}^n$. Then $d_{BM}(K, B^n) < 1 + cn^{2n^2}\delta^{1/6}$.

Both results follow from the non-integrable stable Banach conjecture, which reads as follows.

Theorem [Aishwarya-F.] Let Σ be a closed surface which is not a torus or a Klein bottle. If a continuous field of convex bodies $(K_x)_{x\in\Sigma}$ satisfies $d_{BM}(K_x, K_y) < 1+\delta$ for all $x, y \in \Sigma$, then for all $x \in \Sigma$, $d_{BM}(K_x, B^2) < 1 + c\delta^{1/3}$.

The basic idea of the proof is to mimic Auerbach-Mazur-Ulam. In the original statement, one constructs a covering space of the surface out of isometries of the tangent convex sets, and then choose a continuous section of such maps to arrive at a contradiction to the Poincaré-Hopf theorem.

In the stable setting, we utilize approximate isometries, and find a scale for which the approximate isometries form clusters, out of which a covering space can again be constructed. Those clusters then furnish a "roughly" continuous section of maps between the tangent spaces, which can subsequently be smoothed out.

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Geometric properties of solutions to elliptic PDE's in Gauss space and related Brunn–Minkowski type inequalities

Lei Qin

(joint work with Andrea Colesanti, Paolo Salani)

I will introduce my work about the first eigenvalue problem to the weighted p-Laplace operator in Florence. I will use two different methods to consider the same problem.

- The method of PDE, it needs regularity of domain.
- The method of viscosity solution, we can remove the regularity of domain (a joint work with Andrea Colesanti and Paolo Salani).

We prove a Brunn-Minkowski type inequality for the first (nontrivial) Dirichlet eigenvalue of bounded Lipschitz domains in $\mathbb{R}^n$ for the weighted p-operator

$$-\Delta_{p,\gamma} u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + (x, \nabla u)|\nabla u|^{p-2},$$

where $p > 1$. We also prove that the corresponding (positive) first eigenfunction is log-concave if the domain is convex.

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Reporter: Tillmann Bühler

Participants

Prof. Dr. David Alonso-Gutiérrez
Departamento de Matemáticas
Universidad de Zaragoza
Facultad de Ciencias
C/Pedro Cerbuna 12
50009 Zaragoza
SPAIN

Prof. Dr. Shiri Artstein-Avidan
Department of Mathematics
School of Mathematical Sciences
Tel Aviv University
P.O. Box 39040
Ramat Aviv, Tel Aviv 69978
ISRAEL

Rotem Assouline
Department of Mathematics
The Weizmann Institute of Science
P. O. Box 26
Rehovot 76 100
ISRAEL

Prof. Dr. Andreas Bernig
Institut für Mathematik
Goethe-Universität Frankfurt
Robert-Mayer-Straße 6-10
60325 Frankfurt am Main
GERMANY

Dr. Florian Besau
Institut für Diskrete Mathematik
und Geometrie
Technische Universität Wien
Wiedner Hauptstraße 8-10
1040 Wien
AUSTRIA

Prof. Dr. Karoly Böröczky
Alfred Renyi Institute of Mathematics
Realtanoda utca 13-15
1053 Budapest
HUNGARY

Tillmann Bühler
Fakultät für Mathematik
Karlsruher Institut für Technologie
(KIT)
76128 Karlsruhe
GERMANY

Xiaxing Cai
Institut für Diskrete Mathematik
und Geometrie
Technische Universität Wien
Wiedner Hauptstraße 8-10/104
1040 Wien
AUSTRIA

Prof. Dr. Andrea Colesanti
Dipartimento di Matematica e
Informatica
"U.Dini"
Università di Firenze
Viale Morgagni, 67/A
50134 Firenze
ITALY

Dr. Dmitry Faifman
Université de Montréal
Pavillon André-Aisenstadt
2920 chemin de la Tour
Montréal H3T1N8
CANADA

Dr. Matthieu Fradelizi
Laboratoire d'Analyse et de
Mathématiques Appliquées, UMR 8050
Université Gustave Eiffel
5, Boulevard Descartes,
Champs-sur-Marne
77454 Marne-la-Vallée Cedex 2
FRANCE

JProf. Dr. Anna Gusakova
Institut für Mathematische Stochastik
Universität Münster
Orléans-Ring 10
48149 Münster
GERMANY

Dr. Julian Haddad
Universidad de Sevilla
Avenida Reina Mercedes S/N
Sevilla 31270-320
SPAIN

Dr. Orli Herscovici
Dept. of Mathematics
St. John's University
8000 Utopia Parkway
Queens, NY 11439
UNITED STATES

Prof. Dr. Daniel Hug
Institut für Stochastik
Karlsruher Institut für Technologie (KIT)
Englerstraße 2
76131 Karlsruhe
GERMANY

Luca Iffland
Institut für Mathematik
Goethe-Universität Frankfurt
Postfach 111932
60054 Frankfurt am Main
GERMANY

Dr. Jonas Knörr
Technische Universität Wien
Wiedner Hauptstr. 8 - 10
1040 Wien
AUSTRIA

Dr. Dylan Langharst
Sorbonne Université, Institut de Mathématiques de Jussieu, Equipe Analyse Fonctionnelle
4 Place Jussieu
75252 Paris
FRANCE

Prof. Dr. Alexander Litvak
Department of Mathematical and Statistical Sciences,
University of Alberta
Edmonton T6G 2G1
CANADA

Dr. Galyna Livshyts
School of Mathematics
Georgia Institute of Technology
Skiles 228
686 Cherry Street
Atlanta, GA 30332-0160
UNITED STATES

Prof. Dr. Monika Ludwig
Institut für Diskrete Mathematik und Geometrie
Technische Universität Wien
Wiedner Hauptstraße 8-10/104
1040 Wien
AUSTRIA

Prof. Dr. Emanuel Milman
Department of Mathematics
TECHNION
Israel Institute of Technology
Amado 630
Haifa 32000
ISRAEL

Dr. Stephanie Mui
School of Mathematics
Georgia Institute of Technology
686 Cherry Street
Atlanta, GA 30332-0160
UNITED STATES

Dr. Fabian Mußnig
Institut für Diskrete Mathematik und Geometrie
Technische Universität Wien
Wiedner Hauptstraße 8-10
1040 Wien
AUSTRIA

Dr. Peter Pivovarov
Department of Mathematics
University of Missouri-Columbia
202 Mathematical Science Bldg.
Columbia, MO 65211-4100
UNITED STATES

Prof. Dr. Joscha Prochno
Fakultät für Mathematik
und Informatik
Universität Passau
Dr.-Hans-Kapfinger-Str. 30
94032 Passau
GERMANY

Eli Putterman
Mathematics Department
Tel Aviv University
69978 Tel Aviv
ISRAEL

Lei Qin
Hunan University
School of Mathematics
YueLu street, Changsha
410082 Hunan 410082
CHINA

Prof. Dr. Matthias Reitzner
Fachbereich Mathematik/Informatik
Universität Osnabrück
Albrechtstraße 28a
49076 Osnabrück
GERMANY

Dr. Liran Rotem
Technion – Israel Institute of Technology
Haifa 3200003
ISRAEL

Prof. Dr. Mark Rudelson
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Shay Sadovsky
Department of Mathematics
School of Mathematical Sciences
Tel Aviv University
P.O. Box 39040
Ramat Aviv, Tel Aviv 69978
ISRAEL

Prof. Dr. Rolf Schneider
Mathematisches Institut
Universität Freiburg
Ernst-Zermelo-Straße 1
79104 Freiburg i. Br.
GERMANY

Prof. Dr. Franz Schuster
Institut für Diskrete Mathematik
und Geometrie
Technische Universität Wien
Wiedner Hauptstrasse 8 - 10
1040 Wien
AUSTRIA

Dr. Maud Szusterman
Institut de Mathématiques de Jussieu
Paris-Rives Gauches
Université de Paris VII
(Université Paris Cité)
8 place Aurélie Nemours
75252 Paris Cedex 013
FRANCE

Kateryna Tatarko
Dept. of Pure Mathematics
University of Waterloo
200 University Avenue West
Waterloo ON N2L 3G1
CANADA

Prof. Dr. Christoph Thäle
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstraße 150
44801 Bochum
GERMANY

Hiroshi Tsuji
Department of Mathematics
Faculty of Science
Saitama University
255 Shimo-Okubo
Saitama 338-8570
JAPAN

Dr. Beatrice-Helen Vritsiou
Department of Mathematical and
Statistical Sciences, CAB 632
University of Alberta
Edmonton AB T6G 2G1
CANADA

Prof. Dr. Thomas Wannerer
Fakultät für Mathematik und Informatik
Friedrich-Schiller-Universität Jena
Ernst-Abbe-Platz 2
07743 Jena
GERMANY

Prof. Dr. Elisabeth M. Werner
Department of Mathematics
Case Western Reserve University
Yost Hall 325
10900 Euclid Avenue
Cleveland OH 44106
UNITED STATES

Dr. Katarzyna Wyczesany
School of Mathematics
University of Leeds
Leeds LS2 9JT
UNITED KINGDOM

Prof. Dr. Dongmeng Xi
Department of Mathematics
Shanghai University
Shanghai Shi 200 444
CHINA

Prof. Dr. Vladyslav Yaskin
Department of Mathematics and
Statistical Sciences
University of Alberta
632 Central Academic Building
Edmonton AB T6G 2G1
CANADA

Dr. Yiming Zhao
Department of Mathematics
Syracuse University
Syracuse, NY 13244-1150
UNITED STATES

Prof. Dr. Artem Zvavitch
Department of Mathematical Sciences
Kent State University
P.O. Box 5190
Kent OH 44242-0001
UNITED STATES