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An inverse Gauss curvature flow and its application to the p-capacitary Orlicz–Minkowski problem

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Abstract. This paper explores the p-capacitary Orlicz–Minkowski problem. Note that the p-capacitary Orlicz–Minkowski problem can be converted equivalently to a Monge–Ampère type equation in the smooth case:

$$f\phi(h_K)\,|\nabla\Psi|^p = \tau G \qquad (\star)$$

for $p \in (1,n)$ and some constant $\tau > 0$, where f is a positive function defined on the unit sphere $\mathcal{S}^{n-1}$, ϕ is a continuous positive function defined in $(0,+\infty)$, and G is the Gauss curvature.

We confirm for the first time the existence of smooth solutions to the p-capacitary Orlicz–Minkowski problem for $p \in (1,n)$ using a class of inverse Gauss curvature flows which converges smoothly to the solution of equation $(\star)$. Moreover, we prove uniqueness for equation $(\star)$ in a special case.

1. Introduction

The classical Brunn–Minkowski theory (abbreviated as BMT) of convex bodies (compact convex sets with nonempty interiors) in n-dimensional Euclidean space $\mathbb{R}^n$ plays an important role in the study of convex geometric analysis, and it has enjoyed a rapid development in recent years. The classical Minkowski problem is one of the cornerstones of the classical BMT (one can see [20,45] for details). Its aim is to find a convex body K in $\mathbb{R}^n$ with prescribed surface area measure $S(K,\cdot)$, which is induced by the volume variation, i.e., such that for each convex body L, there holds

$$\frac{d}{dt}V(K+tL)\Big|_{t=0^+} = \int_{\mathcal{S}^{n-1}} h(L,\cdot)\,dS(K,\cdot), \tag{1.1}$$

where $K+tL = \{x+ty : x\in K, y\in L\}$ is the Minkowski sum, $\mathcal{S}^{n-1}$ is the unit sphere, and $h(L,\cdot) = \{u\cdot y : y\in L, u\in\mathcal{S}^{n-1}\}$ is the support function of the convex body L in $\mathbb{R}^n$.

The development of the classical BMT has inspired many other theories of similar nature. Examples include the L_p BMT, Orlicz BMT and their dual theories. For the related Minkowski-type problems, see, e.g., [4, 9, 14, 18, 21, 25, 26, 30, 31, 36, 41–44, 48] and the references therein.

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The Minkowski-type problem for the measure associated with the solution to the boundary-value problem is an extremely important variant. As some typical examples, we refer to the seminal papers [15, 33] on *capacity* and on *torsional rigidity* by Jerison and Colesanti–Fimiani, and to subsequent progress, e.g., [2, 12, 16, 17, 27, 47, 49, 50].

In this paper, we will further study the p-capacitary Minkowski problem for the Orlicz case proposed by Hong–Ye–Zhang in [27]. To describe this type of problem, we first recall the definition of the p-capacity functional and its variational formula.

For $p \in (1, n)$, the electrostatic p-capacity of a convex body K in $\mathbb{R}^n$ is described by (see [16])

$$C_p(K) = \inf\left\{ \int_{\mathbb{R}^n} |\nabla\psi|^p dx : \psi \in C_c^\infty(\mathbb{R}^n), \ \ \psi \geq 1 \text{ on } K \right\},$$

where $C_c^\infty(\mathbb{R}^n)$ denotes the set of all infinitely differentiable functions with compact support in $\mathbb{R}^n$, and $\nabla\psi$ denotes the gradient of ψ. The geometric quantity $C_2(K)$ is the classical electrostatic (or Newtonian) capacity of K (see [33]).

Let K be a convex body and let $p \in (1, n)$. The p-equilibrium potential Ψ of K is the unique solution to the following boundary value problem (see [35]):

$$\begin{cases} \triangle_p \Psi = 0 & \text{in } \mathbb{R}^n \backslash K, \\ \Psi = 1 & \text{on } \partial K, \\ \Psi(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.2}$$

where

$$\triangle_p \Psi = \operatorname{div}(|\nabla\Psi|^{p-2} \nabla\Psi)$$

is the p-Laplace operator.

Similar to the volume variational formula (1.1), Colesanti et al. ([16]) established the variational formula for p-capacity as follows: let K and L be two convex bodies and let $p \in (1, n)$. Then we have

$$\frac{d}{dt} C_p(K + tL)\Big|_{t=0^+} = (p-1) \int_{\mathcal{S}^{n-1}} h(L, \xi)\, d\mu_p(K, \xi), \tag{1.3}$$

and the Poincaré p-capacity formula

$$C_p(K) = \frac{p-1}{n-p} \int_{\mathcal{S}^{n-1}} h(K, \xi)\, d\mu_p(K, \xi), \tag{1.4}$$

where $\mu_p(K, \cdot)$ is a finite Borel measure on $\mathcal{S}^{n-1}$, called the electrostatic p-capacitary measure of K, defined by

$$\mu_p(K, \eta) = \int_{g_K^{-1}(\eta)} |\nabla\Psi|^p \, d\mathcal{H}^{n-1} = \int_\eta |\nabla\Psi|^p \, dS(K, \cdot), \tag{1.5}$$

for each Borel set $\eta \subset \mathcal{S}^{n-1}$, where g_K^{-1} is the inverse Gauss map, and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

The *p-capacitary Minkowski problem* can be posed as follows: let μ be a finite Borel measure on $\mathcal{S}^{n-1}$ and let $p \in (1, n)$. Under what necessary and sufficient conditions is there a (unique) convex body K in $\mathbb{R}^n$ such that

$$d\mu_p(K, \cdot) = d\mu?$$

When $p = 2$, this problem was solved by Jerison in his seminal paper [33]. A convex solution of this problem for $p \in (1, 2)$ was obtained in [16]. The case of all $p \in (1, n)$ has been recently solved by Akman et al. in their groundbreaking work [1].

As an extension of the p-capacitary Minkowski problem, the Orlicz case was introduced by Hong et al. in [27]. It can be stated as follows: which are the necessary and sufficient conditions on a given function ϕ and a given finite Borel measure μ on $\mathcal{S}^{n-1}$, such that there exists a convex body K in $\mathbb{R}^n$ satisfying

$$\tau d\mu_p(K, \cdot) = \frac{d\mu}{\phi(h(K, \cdot))} \tag{1.6}$$

for some constant $\tau > 0$?

For this problem, Hong et al. proved the existence of solutions with $p \in (1, n)$ for both discrete and general measures under some mild conditions. When $\phi(h) = h^{1-\mathfrak{p}}$ in (1.6) for $\mathfrak{p} \in \mathbb{R}$, this is the $L_{\mathfrak{p}}$ p-capacitary Minkowski problem introduced in [50]. There are many results for different ranges of $\mathfrak{p}$ and p. For instance, when $p \in (1, n)$ and $\mathfrak{p} \in (1, \infty)$, the even convex solution was obtained in [50]. When $p \in (1, 2)$ and $\mathfrak{p} \in (0, 1)$, and when $p \geq n$ and $\mathfrak{p} \in (0, 1)$, the polytopal solutions where given in [49] and [39], respectively. Feng et al. [17] studied the case of $\mathfrak{p} \in (0, 1)$ and $p \in (1, n)$ for general measures. When $\mathfrak{p} = 0$ and $p \in (1, n)$ in (1.6), this is the logarithmic Minkowski problem for p-capacity, and its polytopal solution was obtained in [47].

It is worth noting that the smoothness of solutions to the Minkowski-type problems has always been an important issue. For the p-capacitary Minkowski problem, it is shown in [33] and [16], respectively for $p = 2$ and $p \neq 2$, that if μ has positive density in $C^{k,\alpha}(\mathcal{S}^{n-1})$, then the domain belongs to the class $C^{k+2,\alpha}$ using techniques of Caffarelli, see [6–8].

Motivated by the above mentioned works, this paper try to investigate and confirm the existence of non-symmetric smooth solutions to the p-capacitary Orlicz–Minkowski problem. One of the main methods used in this paper is the inverse Gauss curvature flow method.

The idea of using inverse Gauss curvature flow to solve the p-capacitary Orlicz–Minkowski problem (1.6) can be summarized as follows:

(1) The p-capacitary Orlicz–Minkowski problem (1.6) can be converted to a Monge–Ampère type equation equivalently in the smooth case (see [27]):

$$f\phi(h_K)|\nabla\Psi(g_K^{-1})|^p = \tau G, \tag{1.7}$$

for $p \in (1, n)$ and some constant $\tau > 0$. Here $f: \mathcal{S}^{n-1} \to (0, \infty)$ is the smooth data function, and G is the Gauss curvature (see Section 2 for details). In this case, the key of this paper is to find a convex body K in $\mathbb{R}^n$ with support function h_K satisfying (1.7).

(2) The solution to the Monge–Ampère type equation (1.7) is the limit of solutions of the inverse Gauss curvature flows (1.8) constructed below.

The Gauss curvature flow was first introduced and studied by Firey [19] to model the shape change of worn stones. Since then, the use of curvature flows has proven to be a very effective tool to solve Minkowski-type problems and geometric inequalities in convex geometric analysis, see [3, 5, 9, 11, 13, 23, 24, 28, 29, 37, 38] and the references therein.

Let Ω_0 be a smooth, closed, and strictly convex hypersurface in $\mathbb{R}^n$ enclosing the origin o in its interior, that is, with a sufficiently small positive constant δ_o such that the δ_o-neighbourhood of o is $U(o,\delta_o) \subset \Omega_0$. We consider an inverse Gauss curvature flow of the family of convex hypersurfaces $\{\Omega_t\}$ given by $\Omega_t = F(\mathcal{S}^{n-1}, t)$, where $F\colon \mathcal{S}^{n-1} \times [0,T) \to \mathbb{R}^n$ is a smooth map satisfying

$$\begin{cases} \dfrac{\partial F(\xi,t)}{\partial t} = f(\nu)(F\cdot\nu)\,\phi(F\cdot\nu)\,|\nabla\Psi(F,t)|^p\,\sigma_{n-1}\,\nu - \gamma(t)F(\xi,t), \\ F(\xi,0) = F_0(\xi), \end{cases} \tag{1.8}$$

where f is a given positive smooth function on $\mathcal{S}^{n-1}$, "$\cdot$" is the standard inner product in $\mathbb{R}^n$, $\sigma_{n-1}(\xi,t)$ is the product of the principal curvature radii with $\sigma_{n-1} = \det(\nabla_{ij}h + h\delta_{ij})$, ν is the out normal of Ω_t at $F(\xi,t)$, T is the maximal time for which the solution of (1.8) exists, and the scalar function $\gamma(t)$ is given by

$$\gamma(t) = \frac{n-p}{p-1}\,\frac{C_p(\Omega_t)}{\int_{\mathcal{S}^{n-1}} h/(f\phi(h))\,d\xi},$$

for $p \in (1,n)$.

Compared with the geometric flows in [5, 9, 11, 13, 38], the flow we construct in this paper is more complex because it contains the functions ϕ, $|\nabla\Psi|$ and $\gamma(t)$; thus, a priori estimates are more difficult to obtain.

Now we present the main results of this paper.

Theorem 1.1. *Let f be a positive smooth function on $\mathcal{S}^{n-1}$, and let Ω_0 be a smooth, closed and strictly convex hypersurface in $\mathbb{R}^n$ enclosing the origin in its interior. Suppose*

(1) $p \in (1,n)$;

(2) *the function* $\phi\colon (0,\infty) \to (0,\infty)$ *is smooth*;

(3) $\varphi(s) = \int_0^s 1/\phi(t)\,dt$ *exists for all* $s > 0$ *and* $\lim_{s\to\infty}\varphi(s) = \infty$.

Then, the flow (1.8) has a smooth solution Ω_t for all time $t > 0$. When $t \to \infty$, there is a subsequence of Ω_t that converges in C^∞ to a smooth, closed and strictly convex hypersurface Ω_∞ whose support function satisfies (1.7).

As an application, we have the following.

Corollary 1.2. *Under the assumptions of Theorem 1.1, there exists a smooth solution to the p-capacitary Orlicz–Minkowski problem (1.6) for $p \in (1,n)$.*

For general ϕ, the uniqueness of the solution to the p-capacitary Orlicz–Minkowski problem remains open. We consider here a special uniqueness result for (1.7) when $\tau = 1$.

Theorem 1.3. *Let $p \in (1, n-1]$ and $\delta \geq 1$. If*

$$\phi(\delta s) \leq \delta^{p+1-n}\,\phi(s) \tag{1.9}$$

holds for positive s, then the solution to (1.7) is unique.

Moreover, based on the parabolic approximation method, we also a weak solution to the p-capacitary Orlicz–Minkowski problem when $p \in (1, n)$; this has been obtained by Hong–Ye–Zhang in [27].

Theorem 1.4. *Let μ be a finite Borel measure on $\mathcal{S}^{n-1}$ whose support is not contained in any closed hemisphere and $p \in (1, n)$. Suppose that*

(1) $\phi\colon (0,\infty) \to (0,\infty)$ *is a continuous function*;

(2) $\varphi(s) = \int_0^s 1/\phi(t)\,dt$ *exists for all* $s > 0$ *and* $\lim_{s\to\infty} \varphi(s) = \infty$.

Then, there exists a convex body K such that (1.6) *holds.*

The paper is organized as follows. Section 2 presents the corresponding background material. Section 3 introduces the geometric flow and its correlation functional. In Section 4, we establish a priori estimates for the solution of the flow (1.8). Finally, we prove the main results in Section 5.

2. Preliminaries

In this section, we list some facts about convex hypersurfaces. We refer the readers to [46] and to the well-known book of Schneider [45] for details. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, let $\mathcal{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$, and let Ω be a smooth, closed and strictly convex hypersurface containing the origin in its interior. The support function of Ω is defined by

$$h_\Omega(\xi) = h(\Omega,\xi) = \max\{\xi \cdot Y : Y \in \Omega\}, \quad \text{for } \xi \in \mathcal{S}^{n-1}.$$

For $\pm v \in \mathcal{S}^{n-1}$, the support function of the line segment $\bar{v}$ joining the points $\pm v$ is defined as

$$h(\bar{v},\xi) = |\xi \cdot v|, \quad \text{for } \xi \in \mathcal{S}^{n-1}.$$

The radial function of Ω is defined by

$$r_\Omega(\upsilon) = r(\Omega,\upsilon) = \max\{c > 0 : c\upsilon \in \Omega\}, \quad \text{for } \upsilon \in \mathcal{S}^{n-1}.$$

Obviously, $r_\Omega(\upsilon)\,\upsilon \in \partial\Omega$.

Let $g\colon \partial\Omega \to \mathcal{S}^{n-1}$ be the Gauss map of Ω. For $\xi \in \mathcal{S}^{n-1}$, the inverse Gauss map, denoted by g^{-1}, is given by

$$g^{-1}(\xi) = F(\xi) = \{X \in \partial\Omega : g(X) \text{ is well defined and } g(X) \in \{\xi\}\}.$$

In particular, for a convex hypersurface Ω of class C^2_+ (Ω is C^2 smooth and has positive Gauss curvature), the support function of Ω can be written as

$$h(\Omega,\xi) = \xi \cdot g^{-1}(\xi) = g(X) \cdot X, \quad \text{for } X \in \Omega.$$

Furthermore, the gradient of $h(\Omega,\cdot)$ satisfies

$$\nabla h(\Omega,\xi) = g^{-1}(\xi). \tag{2.1}$$

Let $e = \{e_{ij}\}$ be the standard metric of $\mathcal{S}^{n-1}$. The reverse second fundamental form of Ω is defined as (see Section 2.5 in [45])

$$\Pi_{ij} = \nabla_{ij} h + h\, e_{ij}, \tag{2.2}$$

where ∇_{ij} is the second order covariant derivative with respect to e_{ij}. By the Weingarten formula and (2.2), the principal radii of Ω, under a smooth local orthonormal frame on $\mathcal{S}^{n-1}$, are the eigenvalues of the matrix

$$b_{ij} = \nabla_{ij} h + h\, \delta_{ij}. \tag{2.3}$$

In particular, the Gauss curvature of $F(\xi)$ can be expressed as

$$G(\xi) = \frac{1}{\det(\nabla_{ij} h + h\, \delta_{ij})}. \tag{2.4}$$

Next, we introduce the Orlicz norm, see [26] for details. Let $\varphi\colon [0,\infty) \to [0,\infty)$ be a continuous, strictly increasing, continuously differentiable on $(0,\infty)$ function with positive derivative, and that satisfies the assumption in Theorem 1.4, let μ be a finite Borel measure on $\mathcal{S}^{n-1}$, and let $\mathfrak{f}\colon \mathcal{S}^{n-1} \to [0,\infty)$ be a continuous function.

The Orlicz norm $\|\mathfrak{f}\|_{\varphi,\mu}$ is defined by

$$\|\mathfrak{f}\|_{\varphi,\mu} = \inf\left\{\lambda > 0 : \frac{1}{|\mu|}\int_{\mathcal{S}^{n-1}} \varphi\left(\frac{\mathfrak{f}}{\lambda}\right) d\mu \le \varphi(1)\right\}, \tag{2.5}$$

where $|\mu| = \mu(\mathcal{S}^{n-1})$. This norm satisfies the following properties:

$$\|c\mathfrak{f}\|_{\varphi,\mu} = c\|\mathfrak{f}\|_{\varphi,\mu}, \quad \text{for } c \ge 0,$$

and

$$\mathfrak{f} \le \mathfrak{g} \quad \Longrightarrow \quad \|\mathfrak{f}\|_{\varphi,\mu} \le \|\mathfrak{g}\|_{\varphi,\mu}. \tag{2.6}$$

If $\mu(\{\mathfrak{f} \ne 0\}) > 0$, the Orlicz norm $\|\mathfrak{f}\|_{\varphi,\mu} > 0$ and

$$\|\mathfrak{f}\|_{\varphi,\mu} = \lambda_0 \quad \Longleftrightarrow \quad \frac{1}{|\mu|}\int_{\mathcal{S}^{n-1}} \varphi\left(\frac{\mathfrak{f}}{\lambda_0}\right) d\mu = \varphi(1).$$

3. Inverse curvature flow and its associated functional

For convenience, the notion of curvature flow is restated here. Let Ω_0 be a smooth, closed, and strictly convex hypersurface in $\mathbb{R}^n$ enclosing the origin in its interior. We consider the following inverse Gauss curvature flow:

$$\begin{cases} \dfrac{\partial F(\xi,t)}{\partial t} = f(\nu)(F\cdot\nu)\,\phi(F\cdot\nu)\,|\nabla\Psi(F,t)|^p\,\sigma_{n-1}\,\nu - \gamma(t)\,F(\xi,t), \\ F(\xi,0) = F_0(\xi), \end{cases} \tag{3.1}$$

where the scalar function $\gamma(t)$ is given by

$$\gamma(t) = \frac{n-p}{p-1} \frac{C_p(\Omega_t)}{\int_{\mathcal{S}^{n-1}} h/(f\phi(h))\, d\xi}, \tag{3.2}$$

for $p \in (1,n)$. As discussed in Section 2, the support function of Ω_t can be expressed as $h(\xi,t) = \xi \cdot F(\xi,t)$. We thus derive the evolution equation for $h(\cdot,t)$ along the flow (3.1) as follows:

$$\begin{cases} \dfrac{\partial h(\xi,t)}{\partial t} = f(\xi)\, h\phi(h)\, |\nabla\Psi(F,t)|^p\, \sigma_{n-1} - \gamma(t)\, h(\xi,t), \\ h(\xi,0) = h_0(\xi). \end{cases} \tag{3.3}$$

Now we investigate the characteristics of two important geometric functionals that will be key in the proof of long-time existence of solutions to equation (3.3).

Lemma 3.1. *Let $F(\cdot,t)$ be a smooth solution to the flow (3.1), with $t \in [0,T)$, and let $\Omega_t = F(\mathcal{S}^{n-1},t)$ be a smooth, closed and uniformly convex hypersurface enclosing the origin in its interior. If $p \in (1,n)$, then the p-capacity $C_p(\Omega_t)$ is monotone non-decreasing along the flow (3.1).*

Proof. Let $\Psi(F,t)$ be the p-equilibrium potential of K_t. Theorem 3.5 in [16] shows that

$$\begin{aligned} \partial_t C_p(\Omega_t) &= \frac{p-1}{n-p}\, \partial_t\Big(\int_{\mathcal{S}^{n-1}} h(\xi,t)\, |\nabla\Psi(F(\xi,t),t)|^p\, \sigma_{n-1}\, d\xi\Big) \\ &= (p-1)\int_{\mathcal{S}^{n-1}} |\nabla\Psi(F,t)|^p\, \sigma_{n-1}\, \partial_t h(\xi,t)\, d\xi. \end{aligned}$$

From (3.3), and by Hölder's inequality, we obtain

$$\begin{aligned} \partial_t C_p(\Omega_t) &= (p-1)\int_{\mathcal{S}^{n-1}} |\nabla\Psi(F,t)|^p\, \sigma_{n-1}\partial_t h\, d\xi \\ &= (p-1)\Big(\int_{\mathcal{S}^{n-1}} f h\phi(h)\, |\nabla\Psi|^{2p}\, \sigma_{n-1}^2\, d\xi - \gamma(t)\int_{\mathcal{S}^{n-1}} h|\nabla\Psi|^p\sigma_{n-1}\, d\xi\Big) \\ &= \frac{p-1}{\int_{\mathcal{S}^{n-1}} \frac{h}{f\phi(h)}\, d\xi}\Big[\int_{\mathcal{S}^{n-1}} f h\phi(h)\, |\nabla\Psi|^{2p}\, \sigma_{n-1}^2\, d\xi \int_{\mathcal{S}^{n-1}} \frac{h}{f\phi(h)}\, d\xi \\ &\quad - \Big(\int_{\mathcal{S}^{n-1}} h|\nabla\Psi|^p\sigma_{n-1}\, d\xi\Big)^2\Big] \\ &\geq \frac{p-1}{\int_{\mathcal{S}^{n-1}} \frac{h}{f\phi(h)}\, d\xi}\Big[\Big(\int_{\mathcal{S}^{n-1}} h\, |\nabla\Psi|^p\sigma_{n-1}\, d\xi\Big)^2 - \Big(\int_{\mathcal{S}^{n-1}} h\, |\nabla\Psi|^p\sigma_{n-1}\, d\xi\Big)^2\Big] = 0. \end{aligned}$$

Using the equality condition in Hölder's inequality, it can be seen that equality holds if and only if $h(\cdot,t)$ solves the equation $f\phi(h)|\nabla\Psi|^p\, \sigma_{n-1} = \tau$ for some constant $\tau > 0$. ∎

Lemma 3.2. *Suppose that the function $\varphi(\cdot)$ satisfies the assumption in Theorem 1.1, and let $p \in (1,n)$. Define the functional*

$$\Phi(t) := \Phi(\Omega_t) = \int_{\mathcal{S}^{n-1}} \frac{\varphi(h)}{f(\xi)}\, d\xi.$$

Then, along the flow (3.3)*, the functional* $\Phi(t)$ *remains unchanged, i.e.,* $\Phi(t) \equiv R$ *for some positive constant* R*.*

Proof. From (3.2), (3.3) and the definition of $\varphi(\cdot)$, we obtain

$$\begin{aligned}
\partial_t \Phi(t) &= \int_{\mathcal{S}^{n-1}} \frac{\partial_t h}{f\phi(h)}\, d\xi \\
&= \int_{\mathcal{S}^{n-1}} \Big(f\phi(h) h |\nabla\Psi|^p \sigma_{n-1} - \gamma(t) h \Big) \frac{1}{f\phi(h)}\, d\xi \\
&= \int_{\mathcal{S}^{n-1}} h |\nabla\Psi|^p \sigma_{n-1}\, d\xi - \frac{(n-p) C_p(\Omega_t)}{(p-1)\int_{\mathcal{S}^{n-1}} \frac{h}{f\phi(h)}\, d\xi} \int_{\mathcal{S}^{n-1}} \frac{h}{f\phi(h)}\, d\xi = 0. \quad \blacksquare
\end{aligned}$$

Next, we give the evolution equation of $\Psi(F,t)$.

Lemma 3.3. *Let* $F(\cdot,t)$ *be a smooth solution to the flow* 3.1 *with* $t \in [0,T)$*, let* $\Omega_t = F(\mathcal{S}^{n-1},t)$ *be a smooth, closed and uniformly convex hypersurface enclosing the origin in its interior, and let* $\Psi(F,t)$ *be the* p*-equilibrium potential of* Ω_t*. Then*

$$\partial_t \Psi(F(\xi,t),t) = |\nabla\Psi(F(\xi,t),t)|\, \partial_t h(\xi,t).$$

Proof. Let $h(\xi,t)$ be the support function of Ω_t. From Lemma 3.1 in [16], one can see that $\Psi(F,t)$ is differentiable with respect to t. As $\Psi(F,t) = 1$ in Ω_t, taking the derivative of both sides with respect to t, we have

$$\partial_t \Psi + \nabla\Psi \cdot \partial_t F(\xi,t) = 0.$$

Here $\partial_t F = \nabla_i(\partial_t h) e_i + \partial_t h \xi$. Further,

$$\partial_t \Psi = -\nabla\Psi \cdot (\nabla_i(\partial_t h) e_i + \partial_t h \xi).$$

Recall, from [16], that $|\nabla\Psi(F,t)| = -\nabla\Psi(F,t)\cdot\xi$. Thus,

$$\partial_t \Psi = |\nabla\Psi| \xi \cdot (\nabla_i(\partial_t h) e_i + \partial_t h \xi) = |\nabla\Psi| \partial_t h. \quad \blacksquare$$

4. A priori estimates

In this section, we establish the a priori estimates for the solution to equation (3.3).

4.1. C^0 and C^1-estimates

Lemma 4.1. *Let* $h(\cdot,t)$*,* $t \in [0,T)$*, be a non-symmetric smooth solution to equation* (3.3)*, and let* T *be the maximal time for which the smooth solution of* (3.3) *exists. Under the assumptions of Theorem* 1.1*, there exist positive constants* l *and* L*, independent of* t*, such that*

$$l \le h(\cdot,t) \le L, \tag{4.1}$$

and

$$l \le r(\cdot,t) \le L. \tag{4.2}$$

Proof. From the definitions of support function and radial function, we have

$$r(\upsilon,t)\,\upsilon=\nabla h(\xi,t)+h(\xi,t)\,\xi,\quad \text{for } \upsilon,\xi\in\mathcal{S}^{n-1}. \tag{4.3}$$

So we only need to prove (4.1) (or (4.2)).

We first deal with the right-hand side of (4.1). Let T be the maximal time for which the smooth solution of equation (3.3) exists. For fixed $t_0\in[0,T)$, suppose that the maximum of $h(\cdot,t_0)$ is attained at (ξ_{t_0},t_0) for $\xi_{t_0}\in\mathcal{S}^{n-1}$, that is,

$$\max_{\xi\in\mathcal{S}^{n-1}} h(\xi,t_0)=h(\xi_{t_0},t_0).$$

Let

$$\hbar=\sup_{t_0\in[0,T)} h(\xi_{t_0},t_0).$$

By the convexity of Ω_t and the definition of support function, one has (see, for instance, Lemma 2.6 in [10])

$$h(\xi,t_0)\geq \hbar\,\xi_{t_0}\cdot\xi,\quad \text{for all } \xi\in\mathcal{S}^{n-1}, \tag{4.4}$$

where h and $\hbar$ are on the same convex hypersurface.

Let $S_{\xi_{t_0}}=\{\xi\in\mathcal{S}^{n-1}:\xi_{t_0}\cdot\xi>0\}$ be the hemisphere containing ξ_{t_0}. From the definition of φ, we know that $\varphi'(h)>0$. By Lemma 3.2 and (4.4), we have

$$\Phi(t)=\Phi(0)\geq\int_{S_{\xi_{t_0}}}\frac{\varphi(h)}{f}\,d\xi\geq\int_{S_{\xi_{t_0}}}\frac{1}{f}\,\varphi(\hbar\,\xi_{t_0}\cdot\xi)\,d\xi=\int_S\frac{1}{f}\,\varphi(\hbar\hat{\xi})\,d\xi, \tag{4.5}$$

where $S=\{\xi\in\mathcal{S}^{n-1}:\hat{\xi}>0\}$. Further, let $S_{1/2}=\{\xi\in\mathcal{S}^{n-1}:\hat{\xi}\geq 1/2\}$. Since f is a positive smooth function on $\mathcal{S}^{n-1}$, it follows that $\int_{S_{1/2}}(1/f)\,d\xi=c_0$ for some positive constant c_0. Thus, from (4.5) it can be deduced that

$$\Phi(0)\geq c_0\,\varphi\Big(\frac{\hbar}{2}\Big),$$

which means that $\varphi(\hbar/2)\leq R/c_0$, that is, $\varphi(\hbar/2)$ is uniformly bounded. Since φ is strictly increasing, we infer that $h(\cdot,t)$ has a uniform positive upper bound.

Now we deal with the left-hand side of (4.1). Let Ω_0 be a smooth, closed, and strictly convex hypersurface in $\mathbb{R}^n$ enclosing the origin in its interior. By Lemma 3.1, we have

$$C_p(\Omega_t)\geq C_p(\Omega_0)\geq c>0, \tag{4.6}$$

for some positive constant c. Let $h(\cdot,t)\to 0$. By (1.4), since $h(\cdot,t)$ has a uniform positive upper bound, it follows from the dominated convergence theorem that

$$C_p(\Omega_t)\to 0,$$

which contradicts (4.6). Thus $h(\cdot,t)$ has a uniform positive lower bound. ∎

Combining Lemma 4.1 with the convexity of Ω_t yields the following C^1-estimates.

Lemma 4.2. *Under the assumptions of Lemma 4.1, we obtain*

$$|\nabla h(\cdot,t)| \leq L_0 \quad \text{and} \quad |\nabla r(\cdot,t)| \leq L_0,$$

where L_0 is a positive constant depending on the constants of Lemma 4.1.

Proof. From (4.3), one has that

$$r^2 = h^2 + |\nabla h|^2.$$

We conclude the proof using Lemma 4.1. ■

Lemma 4.3. *Let the convex body K_t contain the origin in its interior, let $\Omega_t = \partial K_t$, and let $\Psi(F,t)$ be the p-equilibrium potential of K_t. Under the assumptions of Lemma 4.1, there are positive constants $\hat{l}$, $\tilde{l}$ and $\bar{l}$, independent of t, such that*

$$\hat{l} \leq |\nabla \Psi(\cdot,t)| \leq \tilde{l} \quad \text{and} \quad |\nabla^k \Psi(\cdot,t)| \leq \bar{l},$$

for any positive integer $k \geq 2$.

Proof. Since we have already obtained a uniform upper bound of $h(\cdot,t)$, and Ω_t is smooth, it follows from Lemma 2.18 in [16] that there exists a positive constant $\hat{l}$, depending only on n, q and the uniform upper bound of $h(\cdot,t)$, such that

$$|\nabla \Psi| \geq \hat{l}.$$

Next, we will prove that $|\nabla\Psi| \leq \tilde{l}$. This can be found in the proof of Lemma 3.1 in [16], but, for completeness, we list the main steps of the proof here. Let $\xi \in \Omega_t$ and note that there exists a ball B, included in Ω_t and internally tangent to Ω_t at ξ, with radius r which can be chosen to be independent of t and ξ. Let $\bar{\Psi}$ be the p-equilibrium potential of B. By the comparison principle, $\Psi(\cdot,t) \geq \bar{\Psi}(\cdot)$ in $\mathbb{R}^n \backslash \Omega_t$, and, since $\Psi(\xi,t) = \bar{\Psi}(\xi)$, we have

$$|\nabla \Psi(\xi,t)| \leq |\nabla \bar{\Psi}(\xi)|.$$

On the other hand, the value $|\nabla\bar{\Psi}(\xi)|$ can be explicitly computed, and it is a positive constant depending on r and n only. Combined with (2.1), it is easy to conclude that

$$|\nabla \Psi(\nabla h(\xi,t),t)| \leq \tilde{l}, \quad \text{for all } (\xi,t) \in \mathcal{S}^{n-1} \times [0,T).$$

In addition, by virtue of Schauder's theory (see, e.g., Lemmas 6.4 and 6.17 in [22]), there is a positive constant $\bar{l}$, independent of t, satisfying that

$$|\nabla^k \Psi(\nabla h(\xi,t),t)| \leq \bar{l}, \quad \text{for all } (\xi,t) \in \mathcal{S}^{n-1} \times [0,T),$$

for any positive integer $k \geq 2$. ■

As a result of Lemmas 4.1 and 4.3, we can obtain the following corollary.

Corollary 4.4. *Under the assumptions of Lemma 4.1, the scalar function $\gamma(t)$ has uniform positive upper and lower bounds.*

Proof. From Lemma 4.1, we know that $h(\cdot,t)$ has a uniform positive upper bound L, so that the convex hypersurface Ω_t generated by $h(\cdot,t)=L$ is enclosed by a sphere with radius L. By Lemma 3.1 and the homogeneity of p-capacity, we have

$$C_p(\Omega_t)\le C_p(B_L)=\omega_n\Big(\frac{n-p}{p-1}\Big)^{p-1}L^{n-p},$$

where ω_n denotes the surface area of the unit sphere in $\mathbb{R}^n$. This means that $C_p(\Omega_t)$ has a positive upper bound independent of t.

Similarly, since $h(\cdot,t)$ has a uniform positive lower bound l, then the convex hypersurface Ω_t generated by $h(\cdot,t)=l$ contains a sphere with radius l. By Lemma 3.1, we have

$$\omega_n\Big(\frac{n-p}{p-1}\Big)^{p-1}l^{n-p}=C_p(B_l)\le C_p(\Omega_t).$$

Obviously, this implies that $C_p(\Omega_t)$ has a positive lower bound independent of t.

Lemma 4.1 concludes the proof. ■

4.2. C^2-Estimates

We first obtain a lower bound for the Gauss curvature, which is equivalent to getting an upper bound of $\sigma_{n-1}(\cdot,t)=\det(\nabla_{ij}h+h\delta_{ij})$. This estimate can be obtained by considering a proper auxiliary function, see [32] for similar techniques. Let $\alpha=fh\phi(h)|\nabla\Psi|^p$. In order to deal with $\partial_t|\nabla\Psi|$ and simplify the calculation process, the auxiliary function in [32] is obviously no longer effective. Therefore, we need to create the following auxiliary functions:

$$\Theta(\xi,t)=\frac{1}{1-\lambda r^2/2}\frac{\alpha\sigma_{n-1}}{h},$$

for $\lambda>0$ sufficiently small.

Lemma 4.5. *Under the assumptions of Lemma 4.1, we have*

$$\sigma_{n-1}(\cdot,t)\le L_2,$$

where L_2 is a positive constant independent of t.

Proof. Let c_{ij} be the cofactor matrix of $(h_{ij}+h\delta_{ij})$, with

$$\sum_{i,j}c_{ij}(h_{ij}+h\delta_{ij})=(n-1)\sigma_{n-1}.$$

Suppose the spatial maximum of Θ is obtained at $(\hat{\xi}_{\hat{t}},\hat{t})$. Then we have

$$\nabla_i\Theta=0,\quad \text{i.e.,}\quad \nabla_i\Big(\frac{\alpha\sigma_{n-1}}{h}\Big)+\frac{\alpha\sigma_{n-1}}{h}\frac{\lambda}{1-\lambda r^2/2}\nabla_i\Big(\frac{r^2}{2}\Big)=0, \tag{4.7}$$

and

$$\nabla_{ij}\Theta\le 0. \tag{4.8}$$

Now we estimate Θ. By (4.8), we have

$$\begin{aligned}
\partial_t\Theta &\le \partial_t\Theta - \alpha c_{ij}\nabla_{ij}\Theta\\
&= \partial_t\Big(\frac{1}{1-\lambda r^2/2}\frac{\alpha\sigma_{n-1}}{h}\Big) - \alpha c_{ij}\nabla_{ij}\Big(\frac{1}{1-\lambda r^2/2}\frac{\alpha\sigma_{n-1}}{h}\Big)\\
&= \frac{1}{1-\lambda r^2/2}\Big[\partial_t\Big(\frac{\alpha\sigma_{n-1}}{h}\Big) - \alpha c_{ij}\nabla_{ij}\Big(\frac{\alpha\sigma_{n-1}}{h}\Big)\Big]\\
&\quad + \frac{\lambda}{(1-\lambda r^2/2)^2}\frac{\alpha\sigma_{n-1}}{h}\Big[\partial_t\Big(\frac{r^2}{2}\Big) - \alpha c_{ij}\nabla_{ij}\Big(\frac{r^2}{2}\Big)\Big]\\
&\quad - 2\alpha c_{ij}\frac{\lambda}{(1-\lambda r^2/2)^2}\nabla_i\Big(\frac{\alpha\sigma_{n-1}}{h}\Big)\nabla_j\Big(\frac{r^2}{2}\Big)\\
&\quad - 2\alpha c_{ij}\frac{\lambda^2}{(1-\lambda r^2/2)^3}\frac{\alpha\sigma_{n-1}}{h}\nabla_i\Big(\frac{r^2}{2}\Big)\nabla_j\Big(\frac{r^2}{2}\Big).
\end{aligned}\tag{4.9}$$

Substituting (4.7) into (4.9), we have

$$\begin{aligned}
\partial_t\Theta \le{}& \frac{1}{1-\lambda r^2/2}\Big[\partial_t\Big(\frac{\alpha\sigma_{n-1}}{h}\Big) - \alpha c_{ij}\nabla_{ij}\Big(\frac{\alpha\sigma_{n-1}}{h}\Big)\Big]\\
&+ \frac{\lambda}{(1-\lambda r^2/2)^2}\frac{\alpha\sigma_{n-1}}{h}\Big[\partial_t\Big(\frac{r^2}{2}\Big) - \alpha c_{ij}\nabla_{ij}\Big(\frac{r^2}{2}\Big)\Big].
\end{aligned}\tag{4.10}$$

Next, we need to calculate

$$\partial_t\Big(\frac{\alpha\sigma_{n-1}}{h}\Big) - \alpha c_{ij}\nabla_{ij}\Big(\frac{\alpha\sigma_{n-1}}{h}\Big)$$

and

$$\partial_t\Big(\frac{r^2}{2}\Big) - \alpha c_{ij}\nabla_{ij}\Big(\frac{r^2}{2}\Big).$$

We start with the following:

$$\begin{aligned}
&\sigma_{n-1}\,\partial_t\alpha + \alpha\,\partial_t\sigma_{n-1}\\
&\quad= \sigma_{n-1}(f\phi(h)|\nabla\Psi|^p\partial_t h + fh|\nabla\Psi|^p\phi'(h)\partial_t h + fh\phi(h)\partial_t|\nabla\Psi|^p)\\
&\qquad + \alpha c_{ij}(\nabla_{ij}(\partial_t h) + \delta_{ij}\partial_t h)\\
&\quad= f\phi(h)|\nabla\Psi|^p\sigma_{n-1}(\alpha\sigma_{n-1} - \gamma h) + fh|\nabla\Psi|^p\phi'(h)\sigma_{n-1}(\alpha\sigma_{n-1} - \gamma h)\\
&\qquad + pfh\phi(h)\sigma_{n-1}|\nabla\Psi|^{p-1}\partial_t|\nabla\Psi| + \alpha c_{ij}\nabla_{ij}(\alpha\sigma_{n-1}) + \alpha^2 c_{ij}\delta_{ij}\sigma_{n-1}\\
&\qquad - \gamma\alpha c_{ij}(\nabla_{ij}h + h\delta_{ij}).
\end{aligned}$$

Notice that

$$\begin{aligned}
\partial_t|\nabla\Psi| &= -[(\nabla^2\Psi)\xi\cdot(\nabla_i(\partial_t h)e_i + \partial_t h\xi)] - \nabla(\partial_t\Psi)\cdot\xi\\
&= -(\nabla^2\Psi)\xi\cdot[\nabla_i(\alpha\sigma_{n-1} - \gamma h)e_i + (\alpha\sigma_{n-1} - \gamma h)\xi] - \nabla(\partial_t\Psi)\cdot\xi.
\end{aligned}\tag{4.11}$$

From Lemma 3.3,

$$\begin{aligned}\nabla(\partial_t\Psi)\cdot\xi &= \nabla(|\nabla\Psi|\partial_t h)\cdot\xi\\ &= (|\nabla\Psi|^{-1}\,\nabla\Psi\,\nabla^2\Psi\cdot\xi)(\alpha\,\sigma_{n-1}-\gamma h)+|\nabla\Psi|\,\partial_t(\nabla h)\cdot\xi\\ &= (|\nabla\Psi|^{-1}\,\nabla\Psi\,\nabla^2\Psi\cdot\xi)(\alpha\,\sigma_{n-1}-\gamma h)+|\nabla\Psi|\,\partial_t(\nabla_i h e_i+h\xi)\cdot\xi\\ &= (|\nabla\Psi|^{-1}\,\nabla\Psi\,\nabla^2\Psi\cdot\xi)(\alpha\,\sigma_{n-1}-\gamma h)+|\nabla\Psi|\,(\alpha\,\sigma_{n-1}-\gamma h).\end{aligned} \tag{4.12}$$

Substituting (4.12) into (4.11), we have

$$\begin{aligned}\partial_t|\nabla\Psi| = &-(\nabla^2\Psi)\xi\cdot\nabla_i(\alpha\,\sigma_{n-1}-\gamma h)e_i-(\alpha\,\sigma_{n-1}-\gamma h)(\nabla^2\Psi)\xi\cdot\xi\\ &-(|\nabla\Psi|^{-1}\nabla\Psi\nabla^2\Psi\cdot\xi)(\alpha\,\sigma_{n-1}-\gamma h)-|\nabla\Psi|(\alpha\,\sigma_{n-1}-\gamma h).\end{aligned}$$

This, together with the fact $\sum_{i,j}c_{ij}(\nabla_{ij}h+h\delta_{ij})=(n-1)\,\sigma_{n-1}$, yields

$$\begin{aligned}&\sigma_{n-1}\,\partial_t\alpha+\alpha\,\partial_t\sigma_{n-1}\\ &\quad=\Big(\frac{1}{h}+\frac{\phi'(h)}{\phi(h)}\Big)(\alpha\,\sigma_{n-1})^2-\Big(n+h\,\frac{\phi'(h)}{\phi(h)}\Big)\,\gamma\alpha\,\sigma_{n-1}\\ &\qquad+\alpha^2\,\sigma_{n-1}\,c_{ij}\,\delta_{ij}+\alpha c_{ij}\,\nabla_{ij}(\alpha\,\sigma_{n-1})\\ &\qquad-p\alpha\,\sigma_{n-1}|\nabla\Psi|^{-1}\,[(\nabla^2\Psi)\xi\cdot(\beta\alpha\,\sigma_{n-1}r\,\nabla_i r)e_i+(\alpha\,\sigma_{n-1}-\gamma h)(\nabla^2\Psi)\xi\cdot\xi]\\ &\qquad-p\alpha\,\sigma_{n-1}\,[(|\nabla\Psi|^{-2}\,\nabla\Psi\nabla^2\Psi\cdot\xi)(\alpha\,\sigma_{n-1}-\gamma h)+(\alpha\,\sigma_{n-1}-\gamma h)],\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}\nabla_{ij}\Big(\frac{\alpha\,\sigma_{n-1}}{h}\Big)=&\,\frac{1}{h}\,\nabla_{ij}(\alpha\,\sigma_{n-1})-\frac{1}{h^2}\,(\alpha\,\sigma_{n-1})\,\nabla_{ij}h\\ &-\frac{2}{h^2}\,\nabla_i(\alpha\,\sigma_{n-1})\,\nabla_j h+\frac{2}{h^3}\,(\alpha\,\sigma_{n-1})\nabla_i h\,\nabla_j h.\end{aligned} \tag{4.14}$$

From (4.13) and (4.14), we have

$$\begin{aligned}&\partial_t\Big(\frac{\alpha\,\sigma_{n-1}}{h}\Big)-\alpha c_{ij}\,\nabla_{ij}\Big(\frac{\alpha\,\sigma_{n-1}}{h}\Big)\\ &\quad=\frac{1}{h}\,\partial_t(\alpha\,\sigma_{n-1})-\frac{1}{h^2}\,\alpha\,\sigma_{n-1}\,\partial_t h-\alpha c_{ij}\,\nabla_{ij}\Big(\frac{\alpha\,\sigma_{n-1}}{h}\Big)\\ &\quad=\frac{1}{h}\,\big(\partial_t(\alpha\,\sigma_{n-1})-\alpha c_{ij}\,\nabla_{ij}(\alpha\,\sigma_{n-1})\big)-\Big(\frac{\alpha\,\sigma_{n-1}}{h}\Big)^2+\frac{\gamma}{h}\,\alpha\,\sigma_{n-1}\\ &\qquad+\frac{1}{h^2}\alpha c_{ij}(\alpha\,\sigma_{n-1})\nabla_{ij}h+\frac{2}{h^2}\alpha c_{ij}\,\nabla_i(\alpha\,\sigma_{n-1})\nabla_j h-\frac{2}{h^3}\alpha c_{ij}(\alpha\,\sigma_{n-1})\nabla_i h\,\nabla_j h\\ &\quad=\Big(\frac{n-1}{h}+\frac{\phi'(h)}{\phi(h)}\Big)\,\frac{(\alpha\,\sigma_{n-1})^2}{h}-\Big(n-1+h\,\frac{\phi'(h)}{\phi(h)}\Big)\,\frac{\gamma\alpha\,\sigma_{n-1}}{h}\\ &\qquad+2\alpha c_{ij}\,\frac{\nabla_j h}{h}\,\nabla_i\Big(\frac{\alpha\,\sigma_{n-1}}{h}\Big)\\ &\qquad-pf\phi\sigma_{n-1}|\nabla\Psi|^{p-1}[(\nabla^2\Psi)\xi\cdot(\alpha\,\sigma_{n-1}-\gamma h)_i e_i+(\alpha\,\sigma_{n-1}-\gamma h)(\nabla^2\Psi)\xi\cdot\xi]\\ &\qquad-pf\phi\sigma_{n-1}|\nabla\Psi|^{p}[(|\nabla\Psi|^{-2}\nabla\Psi\nabla^2\Psi\cdot\xi)(\alpha\,\sigma_{n-1}-\gamma h)+(\alpha\,\sigma_{n-1}-\gamma h)].\end{aligned} \tag{4.15}$$

Moreover, since $r^2 = h^2 + |\nabla h|^2$, then

$$\begin{aligned}\partial_t\Big(\frac{r^2}{2}\Big) &= \partial_t\Big(\frac{h^2}{2}\Big) + \partial_t\Big(\frac{|\nabla h|^2}{2}\Big) = h\partial_t h + \sum \nabla_i h\, \nabla_i(\partial_t h)\\ &= h\alpha\sigma_{n-1} + \sigma_{n-1}\sum \nabla_i h\,\nabla_i\alpha + \alpha\sum\nabla_i h\,\nabla_i\sigma_{n-1} - \gamma r^2,\end{aligned}$$

and

$$\begin{aligned}c_{ij}\nabla_{ij}\Big(\frac{r^2}{2}\Big) &= c_{ij}\Big(h\nabla_{ij}h + \nabla_i h\,\nabla_j h + \sum h_k\nabla_i h_{kj} + \sum h_{ik}h_{jk}\Big)\\ &= h[(n-1)\,\sigma_{n-1} - c_{ij}\delta_{ij}h] + c_{ij}\nabla_i h\,\nabla_j h + \sum h_k\nabla_k\sigma_{n-1}\\ &\quad - c_{ij}\nabla_i h\,\nabla_j h + \sum c_{ij}\, b_{ik}\, b_{jk} + c_{ij}\,\delta_{ij}h^2 - 2(n-1)\,h\sigma_{n-1}\\ &= -(n-1)\,h\sigma_{n-1} + \sum h_k\nabla_k\sigma_{n-1} + \sum c_{ij}b_{ik}b_{jk}.\end{aligned}$$

Thus

$$\begin{aligned}&\partial_t\Big(\frac{r^2}{2}\Big) - \alpha c_{ij}\nabla_{ij}\Big(\frac{r^2}{2}\Big)\\ &\qquad = nh\alpha\sigma_{n-1} + \sigma_{n-1}\sum\nabla_i h\,\nabla_i\alpha - \alpha\sum c_{ij}b_{ik}b_{jk} - \gamma r^2.\end{aligned} \tag{4.16}$$

Using the arithmetic-geometric mean inequality, we have

$$\sum c_{ij}\, b_{ik}\, b_{jk} \ge l_0\,\sigma_{n-1}^{1+1/(n-1)}$$

for some positive constant l_0.

Since

$$c_{ij}\,\frac{\nabla_j h}{h}\,\nabla_i\Big(\frac{\alpha\,\sigma_{n-1}}{h}\Big) \le 0$$

from (4.7), and

$$\nabla_i(\alpha\,\sigma_{n-1} - \gamma h) = \frac{1}{h}\,\alpha\,\sigma_{n-1}\nabla_i h - \frac{\lambda\alpha\,\sigma_{n-1}}{1-\lambda r^2/2}\,r\nabla_i r - \gamma\nabla_i h,$$

it follows from (4.10), (4.15) and (4.16) that

$$\begin{aligned}\partial_t\Theta \le{}& \frac{1}{1-\lambda r^2/2}\Big\{\Big(\frac{n-1}{h} + \frac{\phi'}{\phi}\Big)\frac{(\alpha\,\sigma_{n-1})^2}{h} - \Big(n-1+h\,\frac{\phi'}{\phi}\Big)\,\frac{\gamma\alpha\,\sigma_{n-1}}{h}\\ &- \frac{p\alpha\,\sigma_{n-1}}{h|\nabla\Psi|}\Big[\Big(\frac{\nabla_i h}{h}\alpha\,\sigma_{n-1} - \frac{\lambda r\alpha\,\sigma_{n-1}\nabla_i r}{1-\lambda r^2/2} - \gamma\nabla_i h\Big)\\ &\times(\nabla^2\Psi)\,\xi\cdot e_i + (\alpha\,\sigma_{n-1} - \gamma h)(\nabla^2\Psi)\,\xi\cdot\xi\Big]\\ &- \frac{p\alpha\,\sigma_{n-1}}{h}\Big((|\nabla\Psi|^{-2}\nabla\Psi\nabla^2\Psi\cdot\xi)(\alpha\,\sigma_{n-1}-\gamma h) + (\alpha\,\sigma_{n-1}-\gamma h)\Big)\Big\}\\ &+ \frac{\lambda}{(1-\lambda r^2/2)^2}\,\frac{\alpha\,\sigma_{n-1}}{h}\Big[nh\alpha\,\sigma_{n-1} - \alpha\sum\nabla_k h\nabla_k\sigma_{n-1} - \alpha\sum c_{ij}b_{ik}b_{ik} - \gamma r^2\Big],\end{aligned}$$

i.e.,

$$\begin{aligned}\partial_t\Theta \le& \Big[\frac{(\nabla_i h+2ph)p\lambda|\nabla^2\Psi|}{|\nabla\Psi|}+p\gamma h\Big]\frac{\alpha\sigma_{n-1}}{h(1-\lambda r^2/2)}\\ &+\Big[n+h\,\frac{\phi'}{\phi}+\frac{\lambda h(pr\nabla_i r|\nabla^2\Psi|+nh|\nabla\Psi|)}{|\nabla\Psi|}\Big]\Big(\frac{\alpha\sigma_{n-1}}{h(1-\lambda r^2/2)}\Big)^2\\ &-l_0\,\lambda\Big(\frac{h^n}{\alpha}\Big)^{1/(n-1)}\Big(\frac{\alpha\sigma_{n-1}}{h(1-\lambda r^2/2)}\Big)^{2+1/(n-1)}.\end{aligned}\tag{4.17}$$

Since $\varphi'(h)=1/\phi(h)>0$, we have that $\varphi(h)$ is strictly increasing. In conjunction with Lemma 4.1, we find that $\varphi(h)$ has positive upper and lower bounds. This also shows that $\phi(h)$ has positive upper and lower bounds. Using the previous estimates in Section 4.1, the definition of λ, and $p\in(1,n)$, it can be found that there exists a positive constant l_1, depending on the constants of Lemmas 4.1, 4.2 and 4.3, as well as Corollary 4.4, such that

$$\frac{(\nabla_i h+2ph)p\lambda|\nabla^2\Psi|}{|\nabla\Psi|}+p\gamma h\le l_1,$$

and that there exists a positive constant l_2, depending on Lemmas 4.1, 4.2 and 4.3, such that

$$n+h\,\frac{\phi'}{\phi}+\frac{\lambda h(pr\nabla_i r|\nabla^2\Psi|+nh|\nabla\Psi|)}{|\nabla\Psi|}\le l_2,$$

and that there exists a positive constant l_3 depending on Lemmas 4.1 and 4.3 such that

$$l_0\,\lambda\Big(\frac{h^n}{\alpha}\Big)^{1/(n-1)}\ge l_3.$$

Therefore, (4.17) can be further estimated as

$$\partial_t\Theta\le l_1\,\Theta+l_2\,\Theta^2-l_3\,\Theta^{2+1/(n-1)}.$$

By the maximum principle, we have

$$\Theta(\hat{\xi}_{\hat{t}},\hat{t})\le L_2,$$

for some t-independent positive constant L_2. Since $\sigma_{n-1}=G^{-1}$, we obtain a uniform positive lower bound for the Gauss curvature. ∎

From Lemma 4.1, as discussed in Section 2 (or see [46]), we know that the eigenvalues of matrix $\{b_{ij}\}$ are positive, i.e., $\{b_{ij}\}$ is positive definite, and that the principal curvatures are the eigenvalues of $\{b^{ij}\}$. Therefore, deriving a positive upper bound of the principal curvatures of $F(\cdot,t)$ is equivalent to obtaining an upper bound of the eigenvalues of $\{b^{ij}\}$.

Lemma 4.6. *Under the assumptions of Lemma 4.1, the principal curvatures satisfy*

$$\kappa_i(\cdot,t)\le L_3,\quad \textit{for } i=1,\dots,n-1,$$

where L_3 is a positive constant independent of t.

Proof. We study the following auxiliary function:

$$\overline{\mathcal{E}}(\xi,t)=\log\zeta_{\max}(\{b^{ij}\})-a\log h+\frac{s}{2}r^2, \tag{4.18}$$

where a and s are positive constants to be specified later, and $\zeta_{\max}$ is the maximal eigenvalue of $\{b^{ij}\}$. We suppose that the spatial maximum of $\overline{\mathcal{E}}(\xi,t)$ is attained at $\xi_0\in\mathcal{S}^{n-1}$ for $t>0$. By a rotation of coordinates, we can suppose that $\{b^{ij}(\xi_0,t)\}$ is diagonal, and that $\zeta_{\max}=b^{11}(\xi_0,t)$. Then, (4.18) can be rewritten as

$$\mathcal{E}(\xi,t)=\log b^{11}-a\log h+\frac{s}{2}r^2.$$

It is sufficient to prove that $\mathcal{E}(\cdot,t)$ has a positive upper bound. For convenience, we write $\nabla_{ij}h=h_{ij}$ and $\nabla_{ij}r=r_{ij}$.

At the point ξ_0, we have

$$\begin{aligned} 0=\nabla_i\mathcal{E}&=-b^{11}\nabla_i b_{11}-a\frac{h_i}{h}+srr_i\\ &=-b^{11}\nabla_i(h_{11}+h\delta_{11})-a\frac{h_i}{h}+srr_i. \end{aligned} \tag{4.19}$$

and

$$0\ge\nabla_{ij}\mathcal{E}=-b^{11}\nabla_{ij}b_{11}+(b^{11})^2(\nabla_i b_{11})^2-a\left(\frac{h_{ij}}{h}-\frac{h_i^2}{h^2}\right)+sr_i^2+srr_{ij}. \tag{4.20}$$

Furthermore, for $t>0$,

$$\begin{aligned} \partial_t\mathcal{E}&=-b^{11}\partial_t b_{11}-a\frac{\partial_t h}{h}+sr\partial_t r\\ &=-b^{11}((\partial_t h)_{11}+\partial_t h)-a\frac{\partial_t h}{h}+sr\partial_t r. \end{aligned} \tag{4.21}$$

From equation (3.3), we write

$$\log(\partial_t h+\gamma h)=\log\sigma_{n-1}+\Lambda(\xi,t), \tag{4.22}$$

where

$$\Lambda(\xi,t)=\log(fh\phi(h)|\nabla\Psi|^p).$$

Differentiating (4.22),

$$\frac{(\partial_t h)_j+\gamma h_j}{\partial_t h+\gamma h}=\sum b^{ik}\nabla_j b_{ik}+\nabla_j\Lambda, \tag{4.23}$$

and

$$\frac{(\partial_t h)_{11}+\gamma h_{11}}{\partial_t h+\gamma h}-\frac{(\gamma h_1+(\partial_t h)_1)^2}{(\partial_t h+\gamma h)^2}=\sum b^{ii}\nabla_{11}b_{ii}-\sum b^{ii}b^{jj}(\nabla_1 b_{ij})^2+\nabla_{11}\Lambda. \tag{4.24}$$

By the Ricci identity on $\mathcal{S}^{n-1}$,

$$\nabla_{11}b_{ij}=\nabla_{ij}b_{11}-\delta_{ij}b_{11}+\delta_{11}b_{ij}-\delta_{1i}b_{1j}+\delta_{1j}b_{1i},$$

and (4.20), (4.21), (4.23) and (4.24), we have at ξ_0 that

$$\begin{aligned}
\frac{\partial_t \mathcal{E}}{\partial_t h+\gamma h} &= \frac{-b^{11}((\partial_t h)_{11}+\partial_t h)}{\partial_t h+\gamma h} - a\,\frac{\partial_t h}{h(\partial_t h+\gamma h)} + s\,\frac{r\partial_t r}{\partial_t h+\gamma h}\\
&= -b^{11}\Big(\frac{(\partial_t h)_{11}+\gamma h_{11}-\gamma h_{11}-\gamma h+\gamma h+\partial_t h}{\partial_t h+\gamma h}\Big)\\
&\quad - a\,\frac{\partial_t h}{h(\partial_t h+\gamma h)} + s\,\frac{r\partial_t r}{\partial_t h+\gamma h}\\
&= -b^{11}\,\frac{(\partial_t h)_{11}+\gamma h_{11}}{\partial_t h+\gamma h} + \frac{\gamma}{\partial_t h+\gamma h} - b^{11} - \frac{a}{h} + \frac{a\gamma}{\partial_t h+\gamma h} + s\,\frac{r\partial_t r}{\partial_t h+\gamma h}\\
&\le -b^{11}\sum b^{ii}\,\nabla_{11}b_{ii} + b^{11}\sum b^{ii}b^{jj}(\nabla_1 b_{ij})^2 - b^{11}\,\nabla_{11}\Lambda\\
&\quad + \frac{\gamma(1+a)}{\partial_t h+\gamma h} + s\,\frac{r\partial_t r}{\partial_t h+\gamma h}\\
&= -b^{11}\sum b^{ii}(\nabla_{ii}b_{11}-b_{11}+b_{ii}) + b^{11}\sum b^{ii}b^{jj}(\nabla_1 b_{ij})^2 - b^{11}\,\nabla_{11}\Lambda\\
&\quad + \frac{\gamma(1+a)}{\partial_t h+\gamma h} + s\,\frac{r\partial_t r}{\partial_t h+\gamma h}\\
&\le -\sum b^{ii}(b^{11})^2(\nabla_i b_{11})^2 + \sum b^{ii}a\Big(\frac{h_{ii}}{h}-\frac{h_i^2}{h^2}\Big) - \sum b^{ii}sr_i^2\\
&\quad - \sum b^{ii}srr_{ii} + b^{11}\sum b^{ii}b^{jj}(\nabla_1 b_{ij})^2 - b^{11}\,\nabla_{11}\Lambda\\
&\quad + \frac{\gamma(1+a)}{\partial_t h+\gamma h} + s\,\frac{r\partial_t r}{\partial_t h+\gamma h} + \sum b^{ii} - (n-1)b^{11}\\
&\le -a\sum b^{ii} + \frac{(n-1)a}{h} - b^{11}\,\nabla_{11}\Lambda + \frac{\gamma(1+a)}{\partial_t h+\gamma h}\\
&\quad + s\Big(\frac{r\partial_t r}{\partial_t h+\gamma h} - \sum b^{ii}(r_i^2+rr_{ii})\Big),
\end{aligned} \tag{4.25}$$

where

$$\begin{aligned}
\partial_t r &= \frac{h\,\partial_t h+\sum h_k(\partial_t h)_k}{r},\\
r_i &= \frac{hh_i+\sum h_k h_{ki}}{r} = \frac{h_i b_{ii}}{r},\\
r_{ij} &= \frac{hh_{ij}+h_i h_j+\sum h_k h_{kij}+\sum h_{ki}h_{kj}}{r} - \frac{h_i h_j b_{ii} b_{jj}}{r^3}.
\end{aligned} \tag{4.26}$$

Thus,

$$\begin{aligned}
&\frac{r\partial_t r}{\partial_t h+\gamma h} - \sum b^{ii}(r_i^2+rr_{ii})\\
&\qquad = \frac{h\partial_t h}{\partial_t h+\gamma h} - h\sum b^{ii}h_{ii} - b^{ii}\sum h_{ii}^2 - \frac{\gamma|\nabla h|^2}{\partial_t h+\gamma h} + \sum h_k\nabla_k\Lambda\\
&\qquad = nh - \frac{\gamma r^2}{\partial_t h+\gamma h} - \sum b_{ii} + \sum h_k\nabla_k\Lambda.
\end{aligned} \tag{4.27}$$

Substituting (4.27) into (4.25),

$$\begin{aligned}\frac{\partial_t \mathcal{E}}{\partial_t h+\gamma h} \leq & -a\sum b^{ii}+nh(a+s)+\frac{\gamma(1+a-sr^2)}{\partial_t h+\gamma h}-s\sum b_{ii}\\ & -b^{11}\,\nabla_{11}\Lambda+s\sum h_k\nabla_k\Lambda. \end{aligned} \tag{4.28}$$

Next we calculate $-b^{11}\,\nabla_{11}\Lambda$ and $s\sum h_k\nabla_k\Lambda$. From the expression for $\Lambda(\xi,t)$, we have

$$\nabla_k\Lambda=\frac{f_k}{f}+\frac{h_k}{h}+\frac{\phi'(h)}{\phi(h)}h_k+p\,\frac{|\nabla\Psi|_k}{|\nabla\Psi|},$$

and

$$\begin{aligned}\nabla_{kl}\Lambda= & \frac{ff_{kl}-f_kf_l}{f^2}+\frac{hh_{kl}-h_kh_l}{h^2}+\frac{\phi''h_kh_l+\phi'h_{kl}}{\phi}-\frac{(\phi')^2h_kh_l}{\phi^2}\\ & +p\,\frac{|\nabla\Psi|_{kl}}{|\nabla\Psi|}-p\,\frac{|\nabla\Psi|_k|\nabla\Psi|_l}{|\nabla\Psi|^2}.\end{aligned}$$

Recall that

$$|\nabla\Psi(F,t)|=-\nabla\Psi(F,t)\cdot\xi.$$

Let $e^1,\dots,e^{n-1}$ be an orthonormal frame on $\mathcal{S}^{n-1}$. By the Gauss formula on $\mathcal{S}^{n-1}$, we deduce that

$$|\nabla\Psi|_i=(-\nabla\Psi\cdot\xi)_i=-\nabla\Psi\cdot e^i-\nabla^2\Psi[\xi\cdot(h_i\cdot e^i+h\xi)_i]=-\nabla^2\Psi\xi\cdot e^kb_{ki},$$

and

$$|\nabla\Psi|_{ij}=-\nabla^3\Psi e^ke^l\cdot\xi\, b_{ki}b_{ij}-\nabla^2\Psi e^j\cdot e^kb_{ki}+\nabla^2\Psi\xi\cdot\xi b_{ji}-\nabla^2\Psi\xi\cdot e^kb_{kj;j}. \tag{4.29}$$

It follows that

$$\begin{aligned}s\sum h_k\nabla_k\Lambda= & s\sum h_k\Big(\frac{f_k}{f}+\frac{h_k}{h}+\frac{\phi'}{\phi}h_k\Big)-ps\,\frac{h_k}{|\nabla\Psi|}((\nabla^2\Psi)e^k\cdot\xi)b_{kk}\\ \leq & c_1s-ps\,\frac{h_k}{|\nabla\Psi|}((\nabla^2\Psi)e^k\cdot\xi)b_{kk},\end{aligned} \tag{4.30}$$

and

$$\begin{aligned}-b^{11}\,\nabla_{11}\Lambda= & -b^{11}\Big[\frac{ff_{11}-f_1^2}{f^2}+\frac{hh_{11}-h_1^2}{h^2}+\frac{\phi''h_1^2+\phi'h_{11}}{\phi}-\frac{(\phi')^2h_1^2}{\phi^2}\Big]\\ & -pb^{11}\frac{|\nabla\Psi|_{11}}{|\nabla\Psi|}+pb^{11}\frac{(|\nabla\Psi|_1)^2}{|\nabla\Psi|^2}\\ \leq & c_2b^{11}+c_3+c_4b_{11}+pb^{11}b_{i11}\,\frac{(\nabla^2\Psi)e^i\cdot\xi}{|\nabla\Psi|},\end{aligned} \tag{4.31}$$

where c_1, c_2, c_3 and c_4 are positive constants independent of t.

From (4.19) and (4.26), we have

$$b^{11} b_{i11} = -a \frac{h_i}{h} + s r r_i = -a \frac{h_i}{h} + s h_i b_{ii}.$$

This, together with (4.31), yields

$$-b^{11} \nabla_{11} \Lambda \leq c_2 b^{11} + c_3 + c_4 b_{11} + c_5 s \sum b_{ii} + c_6 a,$$

where c_5 and c_6 are positive constants independent of t. Hence

$$-b^{11} \nabla_{11} \Lambda + s \sum h_k \nabla_k \Lambda \leq \hat{c}_1 s + \hat{c}_2 a + \hat{c}_3 b^{11} + \hat{c}_4 b_{11} + \hat{c}_5 s \sum b_{ii} + \hat{c}_6. \tag{4.32}$$

Substituting (4.32) into (4.28), if we choose $s \gg a$, then

$$\frac{\partial_t \mathcal{E}}{\partial_t h + \gamma h} \leq -a \sum b^{ii} + nh(a+s) - s \sum b_{ii} + \hat{c}_3 b^{11} + \hat{c}_4 b_{11} + \hat{c}_5 s \sum b_{ii} + \hat{c}_6.$$

Furthermore, let $a > \hat{c}_3$, and let b^{ii} be large enough. Then,

$$\frac{\partial_t \mathcal{E}}{\partial_t h + \gamma h} < 0.$$

Therefore

$$\mathcal{E}(\xi_0, t) = \overline{\mathcal{E}}(\xi_0, t) \leq L_3,$$

for some positive constant L_3 independent of t. The proof is completed. ∎

As a consequence of Lemmas 4.5 and 4.6, we obtain the following corollary.

Corollary 4.7. *Under the assumptions of Lemma 4.1, the principal curvatures satisfy*

$$L_4 \leq \kappa_i(\cdot, t) \leq L_3, \quad \textit{for } i = 1, \ldots, n-1,$$

for all $(\cdot, t) \in \mathcal{S}^{n-1} \times (0, T)$. *Here,* L_4 *is a positive constant independent of* t.

5. Proofs of main theorems

5.1. Proof of Theorem 1.1

From the C^2-estimates obtained in Corollary 4.7, we know that equation (3.3) is uniformly parabolic on any finite time interval and has short time existence. By the C^0, C^1 and C^2-estimates (Lemmas 4.1 and 4.2, and Corollary 4.7), and Krylov's theory [34], we get the Hölder continuity of $\nabla^2 h$ and $\partial_t h$. Then we get estimates for higher order derivatives by the regularity theory of uniformly parabolic equations. Therefore, we obtain the long-time existence and regularity of the solution to equation (3.3). Moreover, we have

$$\|h\|_{C^{i,j}_{\xi,t}(\mathcal{S}^{n-1} \times [0,T))} \leq C \tag{5.1}$$

for some $C > 0$, independent of t, and for each pair of nonnegative integers i and j.

With the aid of the Arzelà–Ascoli theorem and a diagonal argument, we deduce that there exist a sequence of t, denoted by $\{t_k\}_{k\in\mathbb{N}} \subset (0,\infty)$, and a smooth function $h(\xi)$ such that

$$\|h(\xi,t_k)-h(\xi)\|_{C^i(\mathbb{S}^{n-1})} \to 0 \tag{5.2}$$

uniformly for any nonnegative integer i as $t_k \to \infty$. This shows that $h(\xi)$ is a support function of a convex hypersurface. If Ω is the convex body determined by $h(\xi)$, we conclude that Ω is smooth and strictly convex with the origin in its interior.

We prove now that (1.7) has a non-symmetric smooth solution. From Lemma 3.1, we see that

$$\partial_t C_p(\Omega_t) \geq 0. \tag{5.3}$$

If there exists a time $\tilde{t}$ such that

$$\partial_t C_p(\Omega_t)\big|_{t=\tilde{t}} = 0,$$

then, by the equality condition in Lemma 3.1, we have

$$f\phi(h)\,|\nabla\Psi(F,\tilde{t})|^p\,\sigma_{n-1} = \tau,$$

for some constant $\tau > 0$, that is, the support function $h(\xi,\tilde{t})$ of $\Omega_{\tilde{t}}$ satisfies equation (1.7).

Next we analyze the case of $\partial_t C_p(\Omega_t) > 0$. From the proof of Corollary 4.4, we infer that there exists a positive constant $\mathcal{L}$, independent of t, such that

$$C_p(\Omega_t) \leq \mathcal{L}, \tag{5.4}$$

and such that $\partial_t C_p(\Omega_t)$ is uniformly continuous.

Combining (5.3) and (5.4), and applying the fundamental theorem of calculus, we obtain

$$\int_0^t C_p'(\Omega_t)\,dt = C_p(\Omega_t) - C_p(\Omega_0) \leq C_p(\Omega_t) \leq \mathcal{L},$$

which leads to

$$\int_0^\infty C_p'(\Omega_t)\,dt < \mathcal{L}.$$

This implies that there exists a subsequence of times $t_j \to \infty$ such that

$$\lim_{t_j\to\infty} \partial_t C_p(\Omega_{t_j}) = 0.$$

From the proof of Lemma 3.1, we have

$$\partial_t C_p(\Omega_t)\big|_{t=t_j} = (p-1)\int_{\mathbb{S}^{n-1}} |\nabla\Psi(F,t)|^p\,\sigma_{n-1}\,\partial_t h\,d\xi\big|_{t=t_j}.$$

Passing to the limit, we have

$$
\begin{aligned}
0 &= \lim_{t_j\to\infty} \partial_t C_p(\Omega_t)\Big|_{t=t_j} \\
&= \frac{p-1}{\int_{\mathbb{S}^{n-1}} \frac{h_\infty}{f\phi(h_\infty)}\, d\xi} \Big[\int_{\mathbb{S}^{n-1}} f h_\infty\, \phi(h_\infty)\, |\nabla\Psi|^{2p}\, \tilde{\sigma}_{n-1}^2\, d\xi \int_{\mathbb{S}^{n-1}} \frac{h_\infty}{f\phi(h_\infty)}\, d\xi \\
&\quad - \Big(\int_{\mathbb{S}^{n-1}} h_\infty |\nabla\Psi|^p\, \tilde{\sigma}_{n-1}\, d\xi\Big)^2\Big] \\
&\geq \frac{p-1}{\int_{\mathbb{S}^{n-1}} \frac{h_\infty}{f\phi(h_\infty)}\, d\xi} \Big[\Big(\int_{\mathbb{S}^{n-1}} h_\infty |\nabla\Psi|^p\, \tilde{\sigma}_{n-1}\, d\xi\Big)^2 - \Big(\int_{\mathbb{S}^{n-1}} h_\infty |\nabla\Psi|^p\, \tilde{\sigma}_{n-1}\, d\xi\Big)^2\Big] = 0.
\end{aligned}
$$

This means that

$$f\phi(h_\infty)\,|\nabla\Psi|^p \tilde{\sigma}_{n-1} = \tau,$$

for some constant $\tau > 0$, where h_∞ and $\tilde{\sigma}_{n-1}$ are the support function and the product of the principal curvature radii of the limit convex domain Ω_∞, respectively. The proof of Theorem 1.1 is completed. ■

Proof of Theorem 1.3

Let h_1 and h_2 be two solutions of equation (1.7). We first prove the following fact:

$$\max \frac{h_1}{h_2} \leq 1. \tag{5.5}$$

We use proof by contradiction. Suppose (5.5) is not true, namely, $\max h_1/h_2 > 1$. Suppose that $\max h_1/h_2$ is attained at $z_0 \in \mathbb{S}^{n-1}$.Then $h_1(z_0) > h_2(z_0)$. Let $\mathcal{P} = \log(h_1/h_2)$. At z_0, we have that

$$0 = \nabla\mathcal{P} = \frac{\nabla h_1}{h_1} - \frac{\nabla h_2}{h_2} \quad \text{and} \quad 0 \geq \nabla^2\mathcal{P} = \frac{\nabla^2 h_1}{h_1} - \frac{\nabla^2 h_2}{h_2}.$$

By (1.7) and the homogeneity of p-capacitary measure (see [16]), one has

$$
\begin{aligned}
1 &= \frac{\phi(h_2)\,|\nabla\Psi(\nabla h_2)|^p \det(\nabla^2 h_2 + h_2 I)}{\phi(h_1)\,|\nabla\Psi(\nabla h_1)|^p \det(\nabla^2 h_1 + h_1 I)} = \frac{\phi(h_2)\, h_2^{n-p-1} \det(\frac{\nabla^2 h_2}{h_2} + I)}{\phi(h_1)\, h_1^{n-p-1} \det(\frac{\nabla^2 h_1}{h_1} + I)} \\
&\geq \frac{\phi(h_2)\, h_2^{n-p-1} \det(\frac{\nabla^2 h_1}{h_1} + I)}{\phi(h_1)\, h_1^{n-p-1} \det(\frac{\nabla^2 h_1}{h_1} + I)} = \frac{\phi(h_2)\, h_2^{n-p-1}}{\phi(h_1)\, h_1^{n-p-1}}.
\end{aligned}
$$

Let $h_2(z_0) = \delta h_1(z_0)$. Then we have

$$\phi(\delta h_1) \leq \delta^{p+1-n}\, \phi(h_1).$$

Since $\delta \geq 1$, it follows that $h_2(z_0) \geq h_1(z_0)$. This is a contradiction. Thus (5.5) holds.

Interchanging h_1 and h_2, (5.5) implies

$$\max \frac{h_2}{h_1} \leq 1.$$

Combining this with (5.5), we have $h_1 \equiv h_2$. The proof of Theorem 1.3 is completed. ■

5.2. Proof of Theorem 1.4

Let φ be as in Theorem 1.4, and let μ be a finite Borel measure on $\mathcal{S}^{n-1}$. Given a function $f: \mathcal{S}^{n-1} \to (0,\infty)$, we define the following measure:

$$d\mu_f = \frac{1}{f}\, d\xi.$$

Suppose ϕ is smooth. By the proof of Lemma 3.7 in [10], there exists a family $\{f_k\} \subset C^\infty(\mathcal{S}^{n-1})$ of positive functions so that $\mu_{f_k} \to \mu$ as $k \to \infty$, weakly.

Let $\Omega_{0,k} = B$ be the unit ball in $\mathbb{R}^n$. For a smooth, closed, and strictly convex hypersurface $\Omega_{t,k}$, its support function satisfies the flow (3.3) and $h(\cdot,0) = 1$. From Theorem 1.1, we know that the domain $\Omega_{t,k}$ converges in C^∞ to a smooth, closed, and strictly convex hypersurface $\Omega_{\infty,k}$ as $t \to \infty$, and satisfies

$$\phi(h_{\Omega_{\infty,k}})\, d\mu_p(\Omega_{\infty,k},\cdot) = \frac{n-p}{p-1} \frac{C_p(\Omega_{\infty,k})}{\int_{\mathcal{S}^{n-1}} [h_{\Omega_{\infty,k}}/\phi(h_{\Omega_{\infty,k}})]\, d\mu_{f_k}}\, d\mu_{f_k}. \tag{5.6}$$

We shall obtain now uniform upper and lower bounds for $h_{\Omega_{\infty,k}}$. Choose $v \in \mathcal{S}^{n-1}$, and let $h_{\bar{v}}$ be the support function of the line segment joining $\pm v$. It follows from Lemma 3.6 and Corollary 3.7 in [30] that there exists a constant $d > 0$ such that

$$\min_{v\in\mathcal{S}^{n-1}} \|h_{\bar{v}}\|_{\varphi,\mu_{f_k}} \geq d, \tag{5.7}$$

for all k. For any $v \in \mathcal{S}^{n-1}$, let R_k be the maximal distance from the origin to $\Omega_{\infty,k}$. We have that $\pm R_k v \in \Omega_{\infty,k}$, thus $R_k h_{\bar{v}}(\xi) \leq h(\Omega_{\infty,k},\xi)$ for all $\xi \in \mathcal{S}^{n-1}$.

Furthermore, we define

$$\Phi_k(t) = \frac{1}{|\mu_{f_k}|} \int_{\mathcal{S}^{n-1}} \varphi(h_{\Omega_{t,k}})\, d\mu_{f_k}.$$

By Lemma 3.2, we have $\frac{d}{dt}\Phi_k(t) = 0$, it follows that $\Phi_k(t) = \Phi_k(0) = \varphi(1)$. From (2.5), it suffices to have

$$\|h_{\Omega_{t,k}}\|_{\varphi,\mu_{f_k}} \leq 1.$$

Combining (2.6) with Lemma 4 in [26], one has

$$R_k \min_{v\in\mathcal{S}^{n-1}} \|h_{\bar{v}}\|_{\varphi,\mu_{f_k}} \leq R_k \|h_{\bar{v}}\|_{\varphi,\mu_{f_k}} \leq \|h_{\Omega_{\infty,k}}\|_{\varphi,\mu_{f_k}} \leq 1. \tag{5.8}$$

The uniform upper bound of $h_{\Omega_{\infty,k}}$ follows from (5.7) and (5.8).

By Lemma 1 in [40], Lemma 3.1, and the upper bound of $h_{\Omega_{\infty,k}}$, we get, for $p \in (1,n)$,

$$S_p(\Omega_{\infty,k}) \geq \left(\frac{p-1}{n-p}\right)^{p-1} C_p(\Omega_{\infty,k}) \geq \left(\frac{p-1}{n-p}\right)^{p-1} C_p(\Omega_{0,k}) = c_0,$$

where $\Omega_{0,k}$ is the unit ball B, and c_0 is a positive constant depending on p and $C_p(B)$. This means that $h_{\Omega_{\infty,k}}$ has a uniform lower bound. Hence, we can find some positive constants c_1 and c_2, independent of k, such that

$$c_1 \leq h_{\Omega_{\infty,k}} \leq c_2.$$

Therefore, there are positive numbers c and C, depending on c_1 and c_2, such that

$$c \leq \int_{\mathbb{S}^{n-1}} \frac{h_{\Omega_{\infty,k}}}{\phi(h_{\Omega_{\infty,k}})} \, d\mu_{f_k} \leq C,$$

for large enough k.

By the Blaschke selection theorem, we deduce that $\Omega_{\infty,k}$ converges to a convex hypersurface $\Omega_{\infty,0}$. Taking the limit $k \to \infty$ in (5.6), combining the positive homogeneity and weak convergence of p-capacitary measure (see [16]), we can find a convex body Ω, generated by $\Omega_{\infty,0}$, such that

$$\lambda \phi(h_\Omega) \, d\mu_p(\Omega, \cdot) = d\mu$$

for some positive constant λ. Thus Ω is the desired solution. A further approximation allows us to confirm that ϕ is merely continuous. ■

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