JEMS

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Consistent random vertex-orderings of graphs

Abstract. Given a hereditary graph property $\mathcal{P}$, consider distributions of random orderings of vertices of graphs $G \in \mathcal{P}$ that are preserved under isomorphisms and under taking induced subgraphs. We show that for many properties $\mathcal{P}$ the only such random orderings are uniform, and give some examples of non-uniform orderings when they exist.

Keywords: random orderings, graphs, exchangeability.

1. Introduction

For any (finite or countably infinite) graph G, write $\mathcal{O}_G$ for the set of possible total orderings of the vertex set $V(G)$, and write $\mathcal{D}_G$ for the set of all probability distributions on $\mathcal{O}_G$. (For countably infinite graphs, we use the σ-algebra generated by all events of the form $u < v$, $u, v \in V(G)$.) Recall that H is an *induced* subgraph of G if the vertex set $V(H)$ is a subset of $V(G)$ and an edge xy lies in H if and only if $x, y \in V(H)$ and xy is an edge of G. Note that an induced subgraph is determined by the subset $V(H) \subseteq V(G)$. We shall write $G[S]$ for the induced subgraph of G with vertex set S.

We call a distribution $\mathbb{P}_G \in \mathcal{D}_G$ *consistent* if for any two finite isomorphic induced subgraphs H_1, H_2 of G, and any isomorphism $\phi: H_1 \to H_2$, the induced orders on H_1 and H_2 have distributions that are mapped to each other by ϕ, i.e., for all $v_1, \ldots, v_k \in H_1$,

$$\mathbb{P}_G(v_1 < \cdots < v_k) = \mathbb{P}_G(\phi(v_1) < \cdots < \phi(v_k)).$$

(In fact, this then implies the same result even for infinite induced subgraphs.)

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Example 1.1. Define the *uniform* random ordering on G by assigning to the vertices i.i.d. uniform $U(0, 1)$ random variables X_v and declaring that $v_1 < v_2$ if and only if $X_{v_1} < X_{v_2}$. This almost surely gives a total ordering of $V(G)$, and the resulting distribution of orderings is clearly consistent. For a finite graph of order n, the uniform random ordering is just the natural uniform probability distribution on all $|\mathcal{O}_G| = n!$ orderings of $V(G)$.

There are some cases when the uniform random ordering is the only consistent random ordering. In this case we shall call the graph G itself *uniform*. As an example, consider a *homogeneous* graph G, namely a graph that is either a complete graph or an empty graph. As every induced subgraph of order k is isomorphic to itself by any permutation, we must have that the ordering on any k vertices is uniformly chosen from the $k!$ possible orderings. It thus agrees with the uniform model defined above on any finite subset of vertices, and hence on the whole graph. The converse is false in general – there exist infinite non-homogeneous graphs which are uniform. Indeed, we shall see many examples below. However, for finite non-homogeneous graphs there are always non-uniform consistent random orderings (see for example Theorem 2.2 and Lemma 2.3 below). Hence for finite graphs, G is uniform if and only if it is homogeneous. We remark that Lemma 2.3 in particular is useful in constructing several instructive examples. The proof of this lemma however relies on some detailed calculations that we defer to the Appendix.

A *graph property* $\mathcal{P}$ is a collection of finite labelled graphs (typically on vertex sets of the form $[n] = \{1, \ldots, n\}$), which is closed under isomorphism, so if the labelled graph G is isomorphic to G' then $G \in \mathcal{P}$ if and only if $G' \in \mathcal{P}$. A graph property is called *hereditary* if whenever $G \in \mathcal{P}$ and H is an induced subgraph of G then $H \in \mathcal{P}$. Hereditary properties of graphs have been studied for over two decades, and there is a huge family of results concerning the structure of graphs, hypergraphs, and other combinatorial structures having a certain hereditary property, the number of graphs of order n in a property, the difficulty of approximation by graphs in the property, etc. For a sample of results, see [1, 3–5, 7–10, 13–18, 28–31].

There are two obvious ways of defining a hereditary property of graphs. First, let $\mathcal{H}$ be a collection of graphs, and write $\mathcal{F}_{\mathcal{H}}$ for the hereditary property consisting of all finite graphs G that do not contain any induced subgraph isomorphic to some graph in $\mathcal{H}$. We call the graphs in this property $\mathcal{H}$-*free*. Second, the collection $\mathcal{P}_G$ of all finite graphs isomorphic to some induced subgraph of a (finite or countably infinite) graph G is also a hereditary property.

Given a hereditary property $\mathcal{P}$, consider probability models that assign to each graph $G \in \mathcal{P}$ an element $\mathbb{P}_G \in \mathcal{D}_G$, i.e., a random total ordering of its vertex set $V(G)$. We call this model *consistent* if, whenever $H, G \in \mathcal{P}$ and H is isomorphic to an induced subgraph H' of G, by say $\phi \colon H \to H'$, then the random order $\mathbb{P}_H$ has the same distribution as the random order induced on H' by $\mathbb{P}_G$. In other words, for all $x_1, \ldots, x_k \in V(H)$,

$$\mathbb{P}_H(x_1 < \cdots < x_k) = \mathbb{P}_G(\phi(x_1) < \cdots < \phi(x_k)). \tag{1.1}$$

Note that it follows that each $\mathbb{P}_G$ must be consistent. For any hereditary property $\mathcal{P}$, the *uniform* model, defined by choosing the uniform distribution on all orderings of $V(G)$ for

each $G \in \mathcal{P}$, is clearly consistent. We call the property $\mathcal{P}$ *uniform* if the only consistent ordering model on $\mathcal{P}$ is the uniform one. This terminology is justified by the following observation.

Lemma 1.2. *Let G be a finite or countably infinite graph. Then any consistent random ordering on G induces a consistent random ordering model on $\mathcal{P}_G$. Conversely, any consistent random ordering model on $\mathcal{P}_G$ is induced from a unique and consistent random ordering on G. In particular, G is uniform iff $\mathcal{P}_G$ is uniform.*

Proof. Given a consistent ordering $\mathbb{P}_G$ on G, we define for each $H \in \mathcal{P}_G$ the random ordering given by (1.1), where $\phi \colon H \to H'$ is any identification of H with an induced subgraph H' of G. The fact that $\mathbb{P}_G$ is consistent implies that the distribution of this ordering is independent of the choice of ϕ, and the collection $\{\mathbb{P}_H\}_{H \in \mathcal{P}_G}$ is clearly a consistent random ordering model on $\mathcal{P}_G$. Conversely, suppose we have a consistent random ordering model $\{\mathbb{P}_H\}_{H \in \mathcal{P}_G}$ on $\mathcal{P}_G$. Define a random ordering on G so that for any finite set of vertices $x_1, \ldots, x_k \in V(G)$,

$$\mathbb{P}(x_1 < \cdots < x_k) = \mathbb{P}_H(x_1 < \cdots < x_k), \tag{1.2}$$

where $H = G[\{x_1, \ldots, x_k\}]$. Consistency of $\{\mathbb{P}_H\}_{H \in \mathcal{P}_G}$ implies that this produces a well defined probability distribution in $\mathcal{D}_G$, which is clearly itself consistent. Moreover, any distribution in $\mathcal{D}_G$ that induces $\{\mathbb{P}_H\}_{H \in \mathcal{P}_G}$ must satisfy (1.2), so this distribution on $\mathcal{O}_G$ is unique. The last statement also follows as the random ordering on G is uniform iff it is uniform when restricted to any finite subgraph. ∎

The study of consistent ordering models on families of graphs was started by Angel, Kechris, and Lyons [6], who showed that the class of all graphs is uniform, as also is, for example, the class of K_n-free graphs. In fact, they studied not only graphs, but also hypergraphs and metric spaces, and gave several applications of their results to uniquely ergodic groups. Russ Lyons suggested to the authors that they continue the study of consistent ordering models on hereditary properties of graphs. Some related results on uniform consistent orderings can be found in [26].

The main aim of this paper is to show that for many natural choices of hereditary property $\mathcal{P}$, the only consistent ordering model is uniform, thus greatly extending the result just mentioned in [6]. In particular, we shall prove the following result in Section 4.

Theorem 1.3. *Suppose that $\mathcal{P}$ is a hereditary property such that for any graph $G \in \mathcal{P}$ and any vertex $v \in G$ there exists a graph $G' \in \mathcal{P}$ which is obtained from G by replacing v by two twin vertices v_1, v_2 with the same neighbourhoods as v in $G \setminus \{v\}$. Suppose also that there exists a graph $G \in \mathcal{P}$ that is not a disjoint union of cliques or a complete multipartite graph. Then $\mathcal{P}$ is uniform.*

Recall that vertices $v_1, v_2 \in G$ are called *twins* if the neighbourhoods of v_1 and v_2 are the same in $G \setminus \{v_1, v_2\}$. Twin vertices may be either adjacent or non-adjacent, and either possibility is allowed for the vertices v_1 and v_2 in Theorem 1.3.

Remark 1.4. The hereditary properties satisfying the assumption of Theorem 1.3 have an equivalent characterization using the theory of graph limits (see [23]). Each graph limit (or graphon) W defines a hereditary property $\mathcal{P}_W$ consisting of all graphs G such that the induced subgraph density satisfies $t_{\mathrm{ind}}(G, W) > 0$. Lovász and Szegedy [24, Proposition 4.10] have shown that $\mathcal{P}$ equals a union $\bigcup_{W \in \mathcal{W}} \mathcal{P}_W$ for some set $\mathcal{W}$ of graph limits if and only if the first condition in Theorem 1.3 holds.

The next result concerns $\mathcal{H}$-free graphs introduced earlier; it follows from Theorem 1.3 (see Section 4).

Theorem 1.5. *Suppose that $\mathcal{H}$ is a set of finite graphs such that either no $H \in \mathcal{H}$ contains a pair of adjacent twins, or no $H \in \mathcal{H}$ contains a pair of non-adjacent twins. Suppose also that $\mathcal{H}$ does not contain the path P_3 on three vertices, or its complement $\bar{P}_3$. Then $\mathcal{F}_\mathcal{H}$ is uniform.*

For example, Theorem 1.5 applies to triangle-free graphs (as a triangle does not contain a pair of non-adjacent twins), claw-free graphs (the claw $K_{1,3}$ does not contain adjacent twins), and chordal graphs ($\{C_4, C_5, C_6, \dots\}$-free graphs) as cycles of length at least 4 do not contain adjacent twins. However it cannot be applied to, for example, the hereditary property consisting of all graphs of girth at least 5 ($\{C_3, C_4\}$-free graphs) as C_3 contains a pair of adjacent twins and C_4 contains a pair of non-adjacent twins. We can however deduce that the class of all graphs with girth at least g is uniform from the following more general result, proved in Section 5.

Theorem 1.6. *Assume $\mathcal{P}$ is a hereditary property such that for any $G_1, G_2 \in \mathcal{P}$ and any vertices $v_1 \in V(G_1)$, $v_2 \in V(G_2)$, the graph obtained from the disjoint union $G_1 \cup G_2$ by identifying the vertices v_1 and v_2 also lies in $\mathcal{P}$. Then $\mathcal{P}$ is uniform.*

Remark 1.7. The condition of Theorem 1.6 is equivalent to the condition that a graph G lies in $\mathcal{P}$ if and only if all its 2-connected induced subgraphs do (assuming $\mathcal{P}$ contains some graph with at least two vertices). Indeed, it is not hard to see that $\mathcal{P}$ is also closed under disjoint unions. In particular, Theorem 1.6 applies to the class of all bipartite graphs, the class of all forests, and the class of all planar graphs, thus answering Question 3.4 of [6]. It also generalises Theorem 5.1 of [6]. Indeed, it shows that the class of all $\mathcal{H}$-free graphs is uniform whenever $\mathcal{H}$ consists only of 2-connected graphs.

We actually derive Theorem 1.6 from the more general, but technical, Theorem 5.1 given in Section 5.

Although Theorem 1.6 applies to the class of all forests, in the case of hereditary properties of forests we can say much more. Recall that a *leaf* is a vertex of degree 1.

Theorem 1.8. *Suppose $\mathcal{P}$ is a hereditary property of forests and suppose that for every non-empty forest $F \in \mathcal{P}$, at least one of the following holds:*

(i) *There exists a leaf u of F such that any forest obtained from F by replacing u by an arbitrary number of (non-adjacent) twins and then adding an arbitrary number of independent vertices lies in $\mathcal{P}$.*

(ii) *There exist two leaves u_1, u_2 of F adjacent to distinct vertices v_1, $v_2 \in V(F) \setminus \{u_1, u_2\}$ such that the forest obtained by replacing both u_1 and u_2 by arbitrary numbers of (non-adjacent) twins lies in $\mathcal{P}$.*

Then $\mathcal{P}$ is uniform.

Theorem 1.8 too is proved in Section 5. Note that the conditions of Theorem 1.8 imply that either $\mathcal{P}$ consists entirely of empty graphs, or $\mathcal{P}$ contains all graphs of the form $K_{1,n} \cup \bar{K}_m$. (Consider the case when F is a single edge.) Indeed, $\{K_{1,n} \cup \bar{K}_m\}_{n,m \geqslant 0}$ is an example where Theorem 1.8 applies. By comparison, the class of all induced subgraphs of stars $K_{1,n}$, $n \geqslant 1$, (i.e., the class of all stars and empty graphs) is not uniform (see Example 2.1 below).

2. Some non-uniform consistent orderings

Before we prove that many properties $\mathcal{P}$ are uniform, we first give some examples of properties and graphs with non-uniform consistent orderings.

Example 2.1. Suppose that every graph $G \in \mathcal{P}$ is a disjoint union of cliques, and that some $G \in \mathcal{P}$ is non-homogeneous. We can construct a non-uniform consistent order by first taking a uniform random order of the cliques, and then a uniform random order of the vertices within each clique. By taking graph complements we can similarly construct an example when every $G \in \mathcal{P}$ is a complete multipartite graph. We take a uniform random order of the partite classes, and then a uniform random order of the vertices within each partite class.

The following results give constructions of non-uniform consistent orderings for large classes of graphs and properties. The first construction was suggested by Leonard Schulman and proved by Angel, Kechris and Lyons [6]; the alternative proof we give below was sketched to us by Lyons.

Theorem 2.2. *Suppose that there exists $\Delta < \infty$ such that for every graph $G \in \mathcal{P}$, the maximum degree of G is at most Δ. Then there exists a consistent random order model on $\mathcal{P}$ that is non-uniform on any non-homogeneous graph in $\mathcal{P}$.*

Proof. Let $G \in \mathcal{P}$ be a graph with n vertices. We first show that we can embed G into Euclidean space $\mathbb{R}^n$ in such a way that the distance between vertices $x, y \in V(G)$ is c_0 if x and y are not adjacent, and $c_1 \neq c_0$ if x and y are adjacent in G. Indeed, let $A = (a_{xy})$ be the adjacency matrix of G, defined by $a_{xy} = 1$ if $xy \in E(G)$ and $a_{xy} = 0$ otherwise. Then A is symmetric and all its eigenvalues are real and lie between $-\Delta$ and Δ. Thus if $\varepsilon < 1/\Delta$, the matrix $I_n + \varepsilon A$ is positive definite, and so there exists a symmetric matrix $B = (b_{ij})$ such that $B^T B = B^2 = I_n + \varepsilon A$. Place each vertex $x \in V(G)$ at the point $p_x = (b_{ix})_{i=1}^n \in \mathbb{R}^n$. Then the distance between any two distinct vertices $x, y \in V(G)$ is given by $\|p_x - p_y\|^2 = p_x \cdot p_x - 2 p_x \cdot p_y + p_y \cdot p_y = 2 - 2\varepsilon a_{xy}$. Thus non-adjacent vertices are at distance $c_0 = \sqrt{2}$ and adjacent vertices are at distance $c_1 = \sqrt{2 - 2\varepsilon}$.

Now construct a random ordering of the vertices of G by taking a unit vector $u \in \mathbb{R}^n$ uniformly at random, and setting $x < y$ if $p_x \cdot u < p_y \cdot u$. This almost surely gives a total ordering on $V(G)$ and it is clear that it is consistent. Indeed, any induced subgraph H is mapped to a set of points that is isometric to the set of points produced by the same construction applied to H. We also note that this ordering is non-uniform on G, provided that G is not homogeneous. Indeed, any non-homogeneous graph contains a subgraph isomorphic to either the path P_3 or its complement $\overline{P}_3$, and so it is enough to show that the ordering is non-uniform on any such subgraph. On such a subgraph, the ordering is given by a random projection of a non-equilateral triangle, which it is easy to see is non-uniform. For example, the probability that a vertex v is in the middle of the ordering is proportional to the angle at the corresponding vertex of the triangle. ∎

Lemma 2.3. *Let G be a non-homogeneous graph with n vertices. Then there exists a non-uniform consistent random ordering that is uniform on any subset of $n - 1$ vertices. Moreover, it can be realised by assigning uniform (dependent) random variables $X_v \in [0, 1]$ to vertices $v \in V(G)$ in such a way that any set of $n - 1$ variables X_v are independent.*

Proof. Fix an $\alpha \in [0, 1]$ and a $v_0 \in V(G)$ and define a random ordering on G by giving each vertex $v \neq v_0$ an i.i.d. $U(0, 1)$ random variable $X_v \in [0, 1]$. Pick an edge xy uniformly at random from G (independently of the X_v, $v \neq v_0$), and define $X_{v_0} \in [0, 1]$ so that

$$\sum_{v \in V(H)} \varepsilon_v X_v \equiv \alpha \bmod 1, \tag{2.1}$$

where $\varepsilon_v = -1$ if $v \in \{x, y\}$ and $\varepsilon_v = 1$ otherwise. Note that for any choice of edge $xy \in E(G)$ this is essentially equivalent to assigning i.i.d. $U(0, 1)$ random variables to *all* vertices and conditioning on the event that (2.1) holds. Hence the resulting distribution is independent of the choice of v_0, and is uniform on any subset of $n - 1$ vertices. Moreover, the overall probability distribution on orderings is obtained by averaging the distributions for each choice of edge $xy \in E(G)$, and is therefore invariant under any automorphism of G. Consistency follows as the distribution is uniform on any proper induced subgraph.

We now show that, for suitable α, this ordering is *not* uniform on G itself. Let the vertices of G be $\{1, \ldots, n\}$ and define $P_{j_1, \ldots, j_r}$ to be the probability that

$$X_{j_1} < \cdots < X_{j_r} < \min \{X_s : s \notin \{j_1, \ldots, j_r\}\}, \tag{2.2}$$

i.e., $X_{j_1}, \ldots, X_{j_r}$ are the smallest r values of the X_v, and in that order. Define $P^{(x,y)}_{j_1, \ldots, j_r}$ to be the probability that (2.2) holds conditioned on the chosen edge being $xy \in E(G)$. Then

$$P_{j_1, \ldots, j_r} = \frac{1}{|E(G)|} \sum_{xy \in E(G)} P^{(x,y)}_{j_1, \ldots, j_r}.$$

Assume first that G is not regular and label the vertices so that the degree d_1 of vertex 1 is not equal to the degree d_2 of vertex 2. Consider

$$\delta = P_{1,2} - P_{2,1} = \frac{1}{|E(G)|} \sum_{xy \in E(G)} (P^{(x,y)}_{1,2} - P^{(x,y)}_{2,1}).$$

By symmetry, $P_{1,2}^{(x,y)} = P_{2,1}^{(x,y)}$ unless $|\{x, y\} \cap \{1, 2\}| = 1$. Hence, again by symmetry, letting d_j' be the number of neighbours of j in $V(G) \setminus \{1, 2\}$,

$$
\begin{aligned}
|E(G)|\delta &= d_1'(P_{1,2}^{(1,3)} - P_{2,1}^{(1,3)}) + d_2'(P_{1,2}^{(2,3)} - P_{2,1}^{(2,3)}) \\
&= (d_1' - d_2')(P_{1,2}^{(1,3)} - P_{2,1}^{(1,3)}) \\
&= (d_1 - d_2)\frac{(-1)^n}{(n-1)!}\binom{n}{2}B_{n-1}(\alpha),
\end{aligned}
$$

where the last line follows from Lemma A.2 and $B_n(x)$ denotes the nth Bernoulli polynomial. In particular, $\delta \neq 0$ unless α is one of the zeros of the polynomial $B_{n-1}(x)$.

Now assume G is regular with vertex degree d. As G is not homogeneous, $n \geqslant 4$ and we can order the vertices so that $\{1, 3\} \in E(G)$ but $\{2, 3\} \notin E(G)$. Consider

$$
\delta' = P_{1,2,3} - P_{2,1,3} = \frac{1}{|E(G)|} \sum_{xy \in E(G)} (P_{1,2,3}^{(x,y)} - P_{2,1,3}^{(x,y)}).
$$

Once again by symmetry, $P_{1,2,3}^{(x,y)} = P_{2,1,3}^{(x,y)}$ unless $|\{x, y\} \cap \{1, 2\}| = 1$. Hence, again by symmetry,

$$
\begin{aligned}
|E(G)|\delta' &= (P_{1,2,3}^{(1,3)} - P_{2,1,3}^{(1,3)}) + (d-1)(P_{1,2,3}^{(1,4)} - P_{2,1,3}^{(1,4)}) + d(P_{1,2,3}^{(2,4)} - P_{2,1,3}^{(2,4)}) \\
&= (P_{1,2,3}^{(1,3)} - P_{2,1,3}^{(1,3)}) - (P_{1,2,3}^{(1,4)} - P_{2,1,3}^{(1,4)}).
\end{aligned}
$$

Now

$$
P_{1,2}^{(1,3)} = \sum_{i>2} P_{1,2,i}^{(1,3)} = P_{1,2,3}^{(1,3)} + (n-3)P_{1,2,3}^{(1,4)},
$$

and similarly for $P_{2,1}^{(1,3)}$. Hence by Lemma A.2 (noting that $n \geqslant 4$),

$$
\begin{aligned}
|E(G)|\delta' &= (P_{1,2}^{(1,3)} - P_{2,1}^{(1,3)}) - (n-2)(P_{1,2,3}^{(1,4)} - P_{2,1,3}^{(1,4)}) \\
&= \frac{(-1)^n}{(n-1)!}\left(\binom{n}{2} - (n-2)(n-3+2H_{n-3})\right)B_{n-1}(\alpha) - \frac{(-1)^n}{(n-3)!}B_{n-2}(\alpha),
\end{aligned}
$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. As $B_{n-1}(\alpha)$ and $B_{n-2}(\alpha)$ are linearly independent, this is non-zero for all but a finite number of $\alpha \in [0, 1]$.

Thus in all cases the distribution is non-uniform on $V(G)$ for suitable α. ∎

Theorem 2.4. *Suppose $\mathcal{P}$ is a hereditary property and H is a graph on at least two vertices such that for every $G \in \mathcal{P}$, all induced subgraphs of G isomorphic to H are vertex-disjoint. Then there is a consistent random ordering model on $\mathcal{P}$ that is uniform on all graphs $G \in \mathcal{P}$ without an induced subgraph isomorphic to H, and is non-uniform on all non-homogeneous graphs $G \in \mathcal{P}$ containing H as an induced subgraph.*

Note that H itself may be either homogeneous or non-homogeneous.

Proof. Assume first that H is homogeneous. Fix $\alpha \in [0, 1]$ and define the following random order for each $G \in \mathcal{P}$. Each vertex $v \in V(G)$ is assigned an i.i.d. $U(0, 1)$ random

variable X_v, except that if G contains induced subgraphs $H_1, \ldots, H_k$ isomorphic to H, a fixed vertex v_i is chosen from each $V(H_i)$, and $X_{v_i} \in [0, 1]$ is redefined so that

$$\sum_{v \in V(H_i)} X_v \equiv \alpha \bmod 1. \tag{2.3}$$

This is essentially equivalent to conditioning on the event that (2.3) occurs for each i. The ordering on G is then obtained from the ordering of the X_v in $\mathbb{R}$. Note that the joint distribution of the X_v, $v \in V(G)$, and hence the distribution on the ordering, is independent of the choices of the v_i, and hence is symmetric under all permutations of $V(H_i)$. Let G' be an induced subgraph of G and assume G' contains H_i only for $i \in S \subseteq \{1, \ldots, k\}$. By independence of the choice of v_i we may assume $v_i \notin V(G')$ for $i \notin S$. Hence the induced ordering on G' is given by exactly the same model. By independence on the v_i, the distribution is clearly invariant under automorphisms of G', so the random ordering model described is consistent on $\mathcal{P}$. It is also clearly uniform on any $G \in \mathcal{P}$ that does not contain H as an induced subgraph. It remains to show that if $G \in \mathcal{P}$ does contain H as a proper induced subgraph then the ordering on G is non-uniform. (Note that in this case G is necessarily non-homogeneous as otherwise it would contain non-vertex-disjoint copies of H.) Let $v \in V(G) \setminus V(H)$ and assume $V(H) = \{1, \ldots, n\}$. Then by Lemma A.1,

$$\mathbb{P}(X_v < X_1 < \cdots < X_n) = \frac{1}{(n+1)!} + \frac{(-1)^{n-1}}{n!^2} B_n(\alpha).$$

Hence, for all but a finite number of choices of α, this probability is not $1/(n+1)!$ as it would be in the case of the uniform distribution. Thus, for a suitable choice of α, the distribution is not uniform on G.

Assume now that H itself is not homogeneous. Fix a non-uniform distribution on H as given by Lemma 2.3. Fix $G \in \mathcal{P}$ and suppose G contains (vertex-disjoint) copies $H_1, \ldots, H_k$ of H. Define a random ordering on the vertices of $G \in \mathcal{P}$ by giving each vertex $v \in V(G)$ an independent uniform random variable $X_v \in [0, 1]$, except that on each H_i we apply the construction of Lemma 2.3, independently for each H_i. In other words, we fix a choice of vertex $v_i \in V(H_i)$ and then uniformly and independently choose one edge from each H_i. The random variable X_{v_i} is then redefined so that (2.1) holds on each H_i. Once again, if G' is an induced subgraph of G containing only the copies H_i, $i \in S \subseteq \{1, \ldots, k\}$, then we can without loss of generality assume that $v_i \notin V(G')$ for each $i \notin S$. Then the induced ordering on G' is given by exactly the same model. Hence the ordering model on $\mathcal{P}$ is consistent and has the stated properties. ∎

Remark 2.5. We note that it is important in Theorem 2.4 that the copies of H be *vertex*-disjoint. For example, taking H as a single edge and $\mathcal{P}$ as any of the uniform properties mentioned above gives examples with each copy of H being *edge*-disjoint but the conclusion of Theorem 2.4 failing. Another instructive example is given in Example 5.3 below, where the copies of H intersect in at most one vertex and each copy has "private" vertices not included in any other copy of H. Nevertheless $\mathcal{P}$ is still uniform.

Despite Remark 2.5, a construction similar to that in Theorem 2.4 is occasionally possible even when not all copies of H are vertex-disjoint. The following gives an example.

Example 2.6. Let $n \geqslant 3$ and define G to be the infinite *double broom* consisting of a path P_n on n vertices with an infinite number of leaves added to the end-vertices of P_n (so that the longest path in G is P_{n+2}). Let the vertices of the central path be $u_1, \ldots, u_n$. Assign i.i.d. $U(0, 1)$ random variables X_v to all $v \in V(G)$ except that $X_{u_n} \in [0, 1]$ is redefined so that $\sum_{i=1}^{n} X_{u_i} \equiv \alpha \bmod 1$, where $\alpha \in [0, 1]$ is a zero of the Bernoulli polynomial $B_n(x)$. Any induced subgraph of G that does not contain all vertices of the central path P_n receives a uniform ordering, as does P_n itself (by symmetry). The only remaining induced subgraphs are P_{n+1}, single brooms containing P_n and at least two leaves attached at one end-vertex, and double brooms with one or more leaves at each end. Any pair of such single brooms or double brooms are isomorphic only by an isomorphism which either fixes P_n or reverses its direction, and hence receive the same distribution of orderings. Any copy of P_{n+1} consists of the central P_n with one leaf at either end. Such a graph has the uniform random ordering by Lemma A.1 as $B_n(\alpha) = 0$. (The X_{u_i} are exchangeable, so it is enough to check the distribution of the rank of the leaf v in the ordering of $v, u_1, \ldots, u_n$.) Thus the ordering is consistent. On the other hand, $B_{n+1}(\alpha) \neq 0$ by Lemma A.3, so the second formula in Lemma A.1 implies that the random ordering is *not* uniform on any P_{n+2} subgraph.

Remark 2.7. Note that the ordering in Example 2.6 is not consistent for $n = 2$ (the infinite double star) as the single brooms obtained by adding leaves to one end-vertex of a P_2 are in fact stars, and have many automorphisms which do not preserve the distribution of the given random order. This is to be expected as the infinite double star is in fact uniform by Theorem 1.8. Moreover, the class of all induced subgraphs of double brooms with central path of length $\leqslant n$ is also uniform by Theorem 1.8. Example 2.6 demonstrates that Theorem 1.8 does not however apply when the central path length is required to be *exactly* n. Indeed, the single broom subgraphs of the double brooms do not satisfy the conditions of Theorem 1.8.

3. Templates and infinite blow-ups

Consider a (finite) *template* G, i.e., a graph with a set $\mathcal{V}$ of vertices, each vertex labelled as either *full* or *empty*. Define the *infinite blow-up* G_∞ of G as an infinite graph with vertex set $\bigcup_{v \in \mathcal{V}} \mathcal{W}_v$ where $\mathcal{W}_v = \{v_i\}_{i=1}^{\infty}$, such that $\mathcal{W}_v$ induces an empty or complete graph according to whether v is empty or full respectively, and for any distinct $v, w \in \mathcal{V}$ and all $i, j \geqslant 1$, $v_i w_j$ is an edge in G_∞ if and only if vw is an edge in G. Define the hereditary property $\mathcal{P}_G$ as the set of all finite induced subgraphs of G_∞, i.e., $\mathcal{P}_G = \mathcal{P}_{G_\infty}$. We shall call a template G *uniform* if G_∞ (or equivalently $\mathcal{P}_G$) is uniform, i.e., if the only consistent random ordering is the uniform one. Our aim is to prove that most templates are uniform. This is, however, not always the case.

Example 3.1. Suppose that the template has no edges and at least two vertices with at least one of the vertices full. Thus G_∞ is a disjoint union of some infinite cliques and (perhaps) some infinite empty graphs, and thus a disjoint union of at least two cliques (infinite or singletons). Any induced subgraph is thus also a disjoint union of cliques. We can construct a non-uniform consistent order as in Example 2.1 by first taking a uniform random order of the cliques, and then a uniform random order of the vertices within each clique.

Suppose we are given an arbitrary template G and consider first each 'blob' $\mathcal{W}_v$ separately. Fix $v \in \mathcal{V}$ and $v_i \in \mathcal{W}_v$. Since any permutation of $\mathcal{W}_v$ is an automorphism of G_∞, and thus preserves the distribution of the order, the random variables $\{\mathbf{1}\{v_k < v_i\}\}_{k \neq i}$ are exchangeable. Thus, by de Finetti's theorem (see e.g., [22, Theorem 1.1 and Proposition 1.4]), a.s. the limit

$$U_{v_i} := \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}\{v_k < v_i\} \tag{3.1}$$

exists. Then each U_{v_i} is a random variable with $U_{v_i} \in [0, 1]$. Moreover, if $v_i < v_j$, then $\mathbf{1}\{v_k < v_i\} \leqslant \mathbf{1}\{v_k < v_j\}$ for every k, and thus $U_{v_i} \leqslant U_{v_j}$.

Lemma 3.2. *For each v, $\{U_{v_i}\}_{i=1}^{\infty}$ is a sequence of i.i.d. uniformly distributed random variables: $U_{v_i} \sim U(0, 1)$.*

Proof. The order restricted to $\mathcal{W}_v$ has a distribution invariant under all permutations, and thus it is the uniform random order. We may thus assume that the random order on $\mathcal{W}_v$ is defined by a collection of i.i.d. uniform random variables X_{v_i} as in Example 1.1. But then (3.1) and the law of large numbers a.s. yield

$$U_{v_i} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}\{X_{v_k} < X_{v_i}\} = X_{v_i}. \qquad \blacksquare$$

Moreover, this extends to all blobs, jointly.

Lemma 3.3. *The random variables U_{v_i}, $v \in \mathcal{V}$ and $i \geqslant 1$, are i.i.d. and uniform on $[0, 1]$.*

Proof. Consider a finite subset A_v of each $\mathcal{W}_v$. Any permutation of A_v is an automorphism of G_∞, and thus the induced order on A_v is the uniform random order, and this also holds even if we condition on the induced orders on all A_w, $w \neq v$. Hence the induced orders on the subsets A_v are independent (and uniform). Since the sets A_v are arbitrary finite subsets of the $\mathcal{W}_v$, this means that the induced orders on the sets $\mathcal{W}_v$, $v \in \mathcal{V}$, are independent, and thus the families $\{U_{v_i}\}_{i=1}^{\infty}$, $v \in \mathcal{V}$, are independent. $\blacksquare$

Next, take two vertices $v, u \in \mathcal{V}$ and compare vertices in the two blobs $\mathcal{W}_v$ and $\mathcal{W}_u$. For every $v_i \in \mathcal{W}_v$, we see in analogy with (3.1), again by de Finetti's theorem, that a.s. the limit

$$V_{u,v_i} := \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}\{u_k < v_i\} \tag{3.2}$$

exists. Note that $V_{v,v_i} = U_{v_i}$. Each V_{u,v_i} is a random variable with values in $[0, 1]$ and gives the 'rank' of vertex v_i with respect to $\mathcal{W}_u$, i.e., the proportion of vertices in $\mathcal{W}_u$ that it exceeds. Note that these random variables are in general neither independent nor uniform.

Example 3.4. Let the template consist of two full vertices and no edge; thus $\mathcal{V} = \{1, 2\}$ and G_∞ consists of two disjoint infinite cliques. For the random order described in Example 3.1, we have $V_{1,2_i} = V_{1,2_j} \in \{0, 1\}$ for all $i, j \geqslant 1$, and $V_{1,2_i} \sim \mathrm{Be}(1/2)$.

Lemma 3.5. *For each pair $u, v \in \mathcal{V}$, there exists a random distribution function $F_{u,v}$ on $[0, 1]$ such that, a.s., for every $x \in [0, 1]$,*

$$F_{u,v}(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\{V_{u,v_i} \leqslant x\}. \tag{3.3}$$

Furthermore, conditioned on $F_{u,v}$, the random variables V_{u,v_i}, $i \geqslant 1$, are i.i.d. with cumulative distribution function $F_{u,v}$.

Remark 3.6. When $v = u$, this holds by Lemma 3.3 with $F_{u,u}(x) = x$ a.s., so $F_{u,u}$ is non-random.

Proof of Lemma 3.5. We may assume that $v \neq u$. Since any permutation of $\mathcal{W}_v$ is an automorphism of G_∞, it follows from (3.2) that the random variables $\{V_{u,v_i}\}_{i=1}^{\infty}$ are exchangeable. The result follows from another application of de Finetti's theorem. ∎

It follows immediately from the definition (3.2) that, for any $v, w \in \mathcal{V}$ and $i, j \geqslant 1$,

$$v_i < w_j \implies V_{u,v_i} \leqslant V_{u,w_j}. \tag{3.4}$$

Equivalently, interchanging v_i and w_j,

$$V_{u,v_i} < V_{u,w_j} \implies v_i < w_j. \tag{3.5}$$

Remark 3.7. The order is thus described by the variables V_{u,v_i}, for any fixed $u \in \mathcal{V}$, in the case when these random variables are a.s. distinct. (This is not necessarily the case, as is seen in Example 3.4; in that example the variables $V_{1,2_i}$ do not identify the order on $\mathcal{W}_2$. See also Remark 3.12 below.)

Let $\mathcal{F}_N$ be the σ-algebra generated by all events $v_i < w_j$ for $v, w \in \mathcal{V}$ and $i, j > N$, and let $\mathcal{F}_\infty := \bigcap_{N=0}^{\infty} \mathcal{F}_N$ be the tail σ-algebra.

Lemma 3.8. *Each $F_{u,v}$ is $\mathcal{F}_\infty$-measurable.*

Proof. As the limits (3.2) and (3.3) do not depend on the first N terms in the sums, V_{u,v_i}, $i > N$, and hence $F_{u,v}$ are $\mathcal{F}_N$-measurable for all N. ∎

Lemma 3.9. *The i.i.d. uniform random variables U_{v_i}, $v \in \mathcal{V}$ and $i \geqslant 1$, are (jointly) independent of $\mathcal{F}_\infty$. Thus the two families $\{U_{v_i}\}_{v,i}$ and $\{F_{u,v}\}_{u,v \in \mathcal{V}}$ are independent.*

Note that the random variables $\{F_{u,v}\}_{u,v \in \mathcal{V}}$ may be dependent on each other.

Proof. The induced orders on the subsets $\mathcal{W}_{v,N} := \{v_i\}_{i=1}^N$, $v \in \mathcal{V}$, are independent and uniform, even conditioned on $\mathcal{F}_N$, since permutations of $\mathcal{W}_{v,N}$ are automorphisms of G_∞. Hence these induced orders are independent of $\mathcal{F}_\infty$, and letting $N \to \infty$, we find that the induced orders on the blobs $\mathcal{W}_v$, $v \in \mathcal{V}$, are (jointly) independent of $\mathcal{F}_\infty$. The random variables U_{v_i} depend on these induced orders only. The result now follows by Lemma 3.8. ∎

We note some useful formulae.

Lemma 3.10. *Let $v, u \in \mathcal{V}$. Then the following hold a.s.:*

(i) *For every $i \geqslant 1$,*
$$V_{u,v_i} = \sup_k \{U_{u_k} : u_k < v_i\}. \tag{3.6}$$

(ii) *For every $i \geqslant 1$,*
$$V_{u,v_i} = F_{v,u}(U_{v_i}). \tag{3.7}$$

(iii) *For $x \in [0,1]$,*
$$F_{u,v}(x) = \sup\{s : F_{v,u}(s) \leqslant x\}. \tag{3.8}$$

Hence, $F_{u,v}$ is the right-continuous inverse of $F_{v,u}$.

Proof. (i) Let $x := \sup_k \{U_{u_k} : u_k < v_i\}$. Then
$$U_{u_j} < x \implies u_j < v_i \implies U_{u_j} \leqslant x.$$

Hence (3.6) follows from definition (3.2) and the law of large numbers.

(ii) By (3.4)–(3.5), recalling that $U_{v_i} = V_{v,v_i}$,
$$V_{v,u_k} < U_{v_i} \implies u_k < v_i \implies V_{v,u_k} \leqslant U_{v_i}.$$

Hence, the definitions (3.2) and (3.3) imply a.s. that for all $\varepsilon > 0$,
$$F_{v,u}(U_{v_i} - \varepsilon) \leqslant V_{u,v_i} \leqslant F_{v,u}(U_{v_i}).$$

Since U_{v_i} is a continuous random variable, and independent of $F_{u,v}$ by Lemma 3.9, U_{v_i} is a.s. a continuity point of $F_{u,v}$, and the result follows.

(iii) By (3.3), (3.7) and the fact that $\{U_{v_i}\}_i$ are i.i.d. and uniform, a.s.,
$$F_{u,v}(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{F_{v,u}(U_{v_i}) \leqslant x\} = \sup\{s : F_{v,u}(s) \leqslant x\}.$$

(This holds a.s., e.g., for all rational $x \in [0,1]$, and thus for all x simultaneously.) ∎

Theorem 3.11. *Fix any $u \in \mathcal{V}$. Then the following are equivalent:*

(i) *The random order on G_∞ is uniform.*

(ii) *The random variables V_{u,v_i}, $v \in \mathcal{V}$ and $i \geqslant 1$, are all i.i.d. and uniform.*

(iii) *The random distribution functions $F_{u,v}$, $u, v \in \mathcal{V}$, are a.s. all equal to the identity:*
$$F_{u,v}(x) = x, \; x \in [0, 1].$$

We may assume $u \neq v$ in (iii) as this always holds for $u = v$ (see Remark 3.6).

Proof. (i)$\Rightarrow$(ii): We may assume that the random order is given by i.i.d. uniform random variables X_{v_i} as in Example 1.1, and then $V_{u,v_i} = X_{v_i}$ a.s. by (3.2) and the law of large numbers.

(ii)$\Rightarrow$(i): Immediate by (3.4)–(3.5).

(ii)$\Rightarrow$(iii): By (3.3) and the law of large numbers.

(iii)$\Rightarrow$(ii): By Lemma 3.10 (iii), $F_{v,u} = F_{u,v}^{-1}$ is the identity. Thus Lemma 3.10 (ii) yields $V_{u,v_i} = U_{v_i}$, and (ii) follows by Lemma 3.3. ∎

Remark 3.12. Consider again any consistent order on G_∞. It follows from Lemmas 3.3 and 3.9 together with (3.7) that, for any pair $u, v \in \mathcal{V}$ and $i, j \geqslant 1$, the random variables $V_{u,u_i} = U_{u_i}$ and $V_{u,v_j} = F_{v,u}(U_{v_j})$ are independent, with U_{u_i} uniform. In particular, these two random variables are a.s. distinct, and thus they determine the order between u_i and v_j by (3.4)–(3.5). Hence, the order is a.s. determined by the collection of all V_{u,v_i} ($u, v \in \mathcal{V}$, $i \geqslant 1$). (As remarked in Remark 3.7, it is sometimes, but not always, possible to use just a single u.)

Note also that Lemmas 3.3 and 3.9 together with (3.7) show that the random V_{u,v_i} may be constructed by randomly selecting first $\{F_{u,v}\}_{u,v}$ with the right distribution and then i.i.d. uniform U_{u_i}, and defining $V_{u,v_i} := F_{v,u}(U_{v_i})$. The conditional distribution of V_{u,v_i} given $\{F_{w,z}\}_{w,z\in\mathcal{V}}$ is thus $F_{u,v}$ (see Lemma 3.5).

Remark 3.13. This section only uses automorphisms of G_∞ that preserve each $\mathcal{W}_v$ (and thus are permutations of each $\mathcal{W}_v$). Remark 3.12 thus gives a description of all random orders that are invariant under this group of permutations of the vertices of G_∞. Conversely, the construction above yields such a random order. In particular, if we fix u and any distribution of $\{F_{u,v}\}_v$ such that each $F_{u,v}$ is continuous, this defines a random order of this type on G_∞. If some $F_{u,v}$ have atoms, we may have to further specify the order.

4. Uniformity of templates

Recall that a template G is uniform if the only consistent random order on G_∞ is the uniform random order.

Remark 4.1. If G is uniform, then so is its complement $\bar{G}$ (with the labels full and empty interchanged), since the corresponding graphs G_∞ and $\bar{G}_\infty$ are complements of each other, and thus have the same isomorphisms between subgraphs.

Lemma 4.2. *A template with a single vertex is uniform. More generally, any template consisting only of empty vertices and no edges is uniform, and so is any complete template consisting only of full vertices.*

Proof. In the cases described, G_∞ is homogeneous, and thus any permutation of the vertices is an isomorphism. Hence any consistent random order is uniform. (Cf. the proof of Lemma 3.2.) ∎

Given a consistent random order on G_∞, we define a relation $\equiv$ on $\mathcal{V}$ by letting $v \equiv w$ if the induced random order on $\mathcal{W}_v \cup \mathcal{W}_w$ is uniform. This relation is clearly symmetric, and it is reflexive by Lemma 4.2. We shall soon see that it also is transitive.

Lemma 4.3. *Suppose that $v, w \in \mathcal{V}$. Then the following are equivalent:*

(i) $v \equiv w$.

(ii) $V_{v,u_i} = V_{w,u_i}$ *a.s., for every $u \in \mathcal{V}$ and $i \geqslant 1$.*

(iii) $F_{v,u} = F_{w,u}$ *a.s., for every $u \in \mathcal{V}$.*

(iv) $F_{u,v} = F_{u,w}$ *a.s., for every $u \in \mathcal{V}$.*

(v) $F_{w,v}(x) = x$ *a.s., for every $x \in [0, 1]$.*

Proof. (i)⇒(ii): Suppose $v \equiv w$. By Theorem 3.11 applied to $\mathcal{W}_v \cup \mathcal{W}_w$, $F_{w,v}(x) = F_{v,w}(x) = x$ a.s. Hence, Lemma 3.10 (ii) yields $V_{v,w_i} = U_{w_i}$.

Fix u and i. Let $\varepsilon > 0$ and choose first a $j \geqslant 1$ such that $U_{v_j} \in (V_{v,u_i} - \varepsilon, V_{v,u_i})$ and then a $k \geqslant 1$ such that $U_{w_k} \in (V_{v,u_i} - \varepsilon, U_{v_j})$. Then $V_{v,w_k} = U_{w_k} < U_{v_j} < V_{v,u_i}$, so $w_k < u_i$ by (3.5). Hence, (3.6) yields

$$V_{w,u_i} \geqslant U_{w_k} > V_{v,u_i} - \varepsilon.$$

Since ε is arbitrary, this yields $V_{w,u_i} \geqslant V_{v,u_i}$, Interchanging v and w we obtain (ii).

(ii)⇒(iii): By definition (3.3).

(iii)⇒(iv): By Lemma 3.10 (iii).

(iv)⇒(v): Taking $u = w$ we have $F_{w,v}(x) = F_{w,w}(x) = x$.

(v)⇒(i): Theorem 3.11 shows that the induced random order on $\mathcal{W}_v \cup \mathcal{W}_w$ is uniform. ∎

Corollary 4.4. *The relation $\equiv$ is an equivalence relation on $\mathcal{V}$.*

Proof. By Lemma 4.3, since (for example) (ii) defines an equivalence relation. ∎

Corollary 4.5. *If $v \equiv w$, then $V_{w,v_i} = U_{v_i}$ a.s. for every $i \geqslant 1$.*

Proof. By Lemma 4.3, $V_{w,v_i} = V_{v,v_i} = U_{v_i}$. ∎

Lemma 4.6. *The random order on G_∞ is uniform if and only if $v \equiv w$ for any two vertices $v, w \in \mathcal{V}$.*

Proof. A consequence of Lemma 4.3 and Theorem 3.11. ∎

Lemma 4.7. *Suppose that the template G contains two (not necessarily disjoint) pairs u, v and w, z such that the induced subtemplates with vertices $\{u, v\}$ and $\{w, z\}$ are isomorphic. If $u \equiv v$, then $w \equiv z$.*

Proof. The induced subgraphs of G_∞ on $\mathcal{W}_u \cup \mathcal{W}_v$ and $\mathcal{W}_w \cup \mathcal{W}_z$ are isomorphic, and thus the induced random orders on these subgraphs have distributions that are mapped to each other by the isomorphism mapping $u_i \mapsto w_i$ and $v_i \mapsto z_i$. Hence, if the random order induced on $\mathcal{W}_u \cup \mathcal{W}_v$ is uniform, then so is the random order induced on $\mathcal{W}_w \cup \mathcal{W}_z$. ∎

Lemma 4.8. *Suppose that the template G contains an induced subtemplate H such that any consistent ordering on G_∞ induces a uniform ordering on H_∞. Moreover, suppose H contains two (not necessarily disjoint) pairs of vertices u, v and u', v' such that u and u' are full, v and v' are empty, and furthermore $uv \in E(G)$ and $u'v' \notin E(G)$. Then G is uniform.*

Proof. Since the ordering on H_∞ is uniform, we have $u \equiv v \equiv u' \equiv v'$.

If $z \in \mathcal{V}$ is empty and $zu \in E(G)$, then the subtemplates $\{z, u\}$ and $\{v, u\}$ are isomorphic. Since $v \equiv u$, we have $z \equiv u$ by Lemma 4.7.

If $z \in \mathcal{V}$ is empty and $zu \notin E(G)$, we argue similarly using the isomorphic subtemplates $\{z, u\}$ and $\{v', u'\}$ and obtain $z \equiv u$.

If $z \in \mathcal{V}$ is full we argue similarly using the pairs $\{z, v\}$ and $\{u, v\}$, or $\{z, v\}$ and $\{u', v'\}$ to obtain $z \equiv v$.

Hence $z \equiv u \equiv v$ for every $z \in \mathcal{V}$, and Lemma 4.6 shows that the random order on G_∞ is uniform. ∎

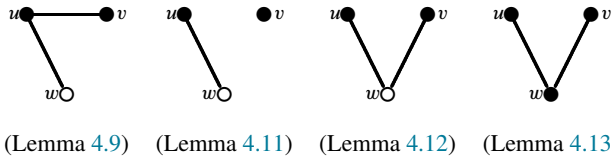We now show that any template G containing certain 3-vertex subtemplates is necessarily uniform (see Figure 1).

(Lemma 4.9)   (Lemma 4.11)   (Lemma 4.12)   (Lemma 4.13)

Fig. 1. Subtemplates implying uniformity of G.

Lemma 4.9. *Suppose that the template G contains two full vertices u and v and an empty vertex w, with $uv, uw \in E(G)$ and $vw \notin E(G)$. Then G is uniform.*

Proof. First, $u \equiv v$ by Lemma 4.2 applied to the subtemplate $\{u, v\}$.

The two subgraphs induced by $\mathcal{W}_u \cup \{w_1, w_2\}$ and $\mathcal{W}_u \cup \{w_1, v_1\}$ are isomorphic, by an isomorphism mapping $w_2 \mapsto v_1$ and fixing everything else; thus the distributions of their induced random orders are mapped to each other by this isomorphism. Hence, by (3.2) and Corollary 4.5,

$$(V_{u,w_1}, V_{u,w_2}) \overset{d}{=} (V_{u,w_1}, V_{u,v_1}) = (V_{u,w_1}, U_{v_1}).$$

Let $x, y \in [0, 1]$. By Lemma 3.5, $\mathbb{P}(V_{u,w_1} \leqslant x, V_{u,w_2} \leqslant y) = \mathbb{E}(F_{u,w}(x) F_{u,w}(y))$. Similarly, and also using Lemma 3.9, $\mathbb{P}(V_{u,w_1} \leqslant x, U_{v_1} \leqslant y) = \mathbb{E}(F_{u,w}(x)) y$. Hence,

$$\mathbb{E}(F_{u,w}(x) F_{u,w}(y)) = \mathbb{E}(F_{u,w}(x)) y, \quad x, y \in [0, 1]. \tag{4.1}$$

Taking $x = 1$ in (4.1) yields $\mathbb{E}\, F_{u,w}(y) = y$, and then taking $x = y$ yields

$$\mathbb{E}(F_{u,w}(x)^2) = \mathbb{E}(F_{u,w}(x))x = (\mathbb{E}\, F_{u,w}(x))^2.$$

Hence $\mathrm{Var}(F_{u,w}(x)) = 0$, and thus $F_{u,w}(x) = \mathbb{E}\, F_{u,w}(x) = x$ a.s. Consequently, $w \equiv u$ by Lemma 4.3.

We have shown that $w \equiv u \equiv v$. In other words, the ordering is uniform on the sub-template induced by $\{u, v, w\}$. The result follows from Lemma 4.8, using the pairs u, w and v, w. ∎

Lemma 4.10. *Let* $F\colon [0, 1] \to [0, 1]$ *be a distribution function on* $[0, 1]$, *and let* $F^{-1}\colon [0, 1] \to [0, 1]$ *be its right-continuous inverse. If* X *and* Y *are random variables such that* X *has distribution* F *and* Y *has distribution* F^{-1}, *then*

$$\mathbb{E}(X^2) + \mathbb{E}(Y^2) \geqslant 2/3,$$

with equality if and only if F *is the uniform distribution* $F(x) = x$.

Proof. Note first the well-known formula

$$\mathbb{E}(X^2) = \mathbb{E} \int_0^1 2x\, \mathbf{1}\{x < X\}\, dx = \int_0^1 2x(1 - F(x))\, dx.$$

Next, if $U \sim U(0, 1)$, then $F(U)$ has the distribution function F^{-1}, so $Y \overset{\mathrm{d}}{=} F(U)$ and thus

$$\mathbb{E}(Y^2) = \mathbb{E}(F(U)^2) = \int_0^1 F(x)^2\, dx.$$

Hence,

$$\begin{aligned}
\mathbb{E}(X^2) + \mathbb{E}(Y^2) &= \int_0^1 \left(2x(1 - F(x)) + F(x)^2\right) dx \\
&= \int_0^1 (F(x) - x)^2\, dx + \int_0^1 (2x - x^2)\, dx \\
&= \int_0^1 (F(x) - x)^2\, dx + \frac{2}{3}.
\end{aligned}$$

The result follows. ∎

Lemma 4.11. *Suppose that the template* G *contains two full vertices* u *and* v *and an empty vertex* w, *with* $uw \in E(G)$, $uv, vw \notin E(G)$. *Then* G *is uniform.*

Proof. Let $\mathcal{W}'_u := \mathcal{W}_u \setminus \{u_1\}$. There is an isomorphism between $\mathcal{W}'_u \cup \{w_1\} \cup \mathcal{W}_v$ and $\mathcal{W}_u \cup \mathcal{W}_v$ fixing $\mathcal{W}'_u \cup \mathcal{W}_v$ and sending w_1 to u_1. It follows that $V_{u,w_1} \overset{\mathrm{d}}{=} U_{u_1}$, even when conditioned on the order in $\mathcal{W}'_u \cup \mathcal{W}_v$. Since $F_{v,u}$ is determined by the order in $\mathcal{W}'_u \cup \mathcal{W}_v$, it follows for any $x \in [0, 1]$, using also Lemma 3.9 and Remark 3.12, that

$$\begin{aligned}
\mathbb{E}(F_{u,w}(x) \mid F_{v,u}) &= \mathbb{P}(V_{u,w_1} \leqslant x \mid F_{v,u}) \\
&= \mathbb{P}(U_{u_1} \leqslant x \mid F_{v,u}) = \mathbb{P}(U_{u_1} \leqslant x) = x.
\end{aligned}$$

Since $V_{u,v_1} = F_{v,u}(U_{v_1})$ by (3.7), and U_{v_1} is independent of $\{F_{u,w}, F_{u,v}\}$, it follows that

$$\mathbb{E}(F_{u,w}(V_{u,v_1}) \mid F_{v,u}, U_{v_1}) = \mathbb{E}(F_{u,w}(F_{v,u}(U_{v_1}))) \mid F_{v,u}, U_{v_1})$$
$$= F_{v,u}(U_{v_1}) = V_{u,v_1}. \tag{4.2}$$

Next, note that by the same isomorphism, $\mathbb{P}(V_{u,v_1} = V_{u,w_1}) = \mathbb{P}(V_{u,v_1} = U_{u_1}) = 0$, since U_{u_1} is continuous and independent of V_{u,v_1}. By symmetry, a.s. $V_{u,v_1} \neq V_{u,w_j}$ for every j, and thus by Remark 3.12, these random variables determine the order between v_1 and w_j. It follows that, a.s., using (3.3),

$$F_{u,w}(V_{u,v_1}) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\{V_{u,w_i} \leqslant V_{u,v_1}\} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\{w_i < v_1\}. \tag{4.3}$$

However, $\mathcal{W}_w \cup \{v_1\}$ is an infinite empty graph, isomorphic to $\mathcal{W}_w$, and by Lemma 3.2 and (3.1), the r.h.s. has a uniform distribution. Thus $\tilde{U} := F_{u,w}(V_{u,v_1}) \sim U(0,1)$, and by (4.2), $\mathbb{E}(\tilde{U} \mid F_{v,u}, U_{v_1}) = V_{u,v_1}$. Consequently,

$$1/3 = \mathbb{E}(\tilde{U}^2) = \mathbb{E}((\tilde{U} - V_{u,v_1})^2) + \mathbb{E}(V_{u,v_1}^2) \geqslant \mathbb{E}(V_{u,v_1}^2). \tag{4.4}$$

By the obvious isomorphism of $\mathcal{W}_u \cup \mathcal{W}_v$ interchanging $\mathcal{W}_u$ and $\mathcal{W}_v$, $V_{v,u_1} \overset{\mathrm{d}}{=} V_{u,v_1}$, so $\mathbb{E}\, V_{v,u_1}^2 \leqslant 1/3$ too.

Conditioned on $F_{u,v}$ and $F_{v,u} = F_{u,v}^{-1}$, V_{u,v_1} and V_{v,u_1} have distributions $F_{u,v}$ and $F_{v,u}$, and thus Lemma 4.10 applies and yields

$$\mathbb{E}(V_{u,v_1}^2 + V_{v,u_1}^2 \mid F_{u,v}) \geqslant 2/3. \tag{4.5}$$

Thus, taking the expectation,

$$\mathbb{E}(V_{u,v_1}^2 + V_{v,u_1}^2) \geqslant 2/3. \tag{4.6}$$

Consequently, there must be equality in both (4.4) and (4.6), and thus a.s. in (4.5). By Lemma 4.10, this implies that $F_{v,u}(x) = F_{u,v}(x) = x$ a.s. Furthermore, by (4.4), $F_{u,w}(V_{u,v_1}) = \tilde{U} = V_{u,v_1}$ a.s., where $V_{u,v_1} = F_{v,u}(U_{v_1}) = U_{v_1}$ is independent of $F_{u,w}$, and thus $F_{u,w}(x) = x$. Thus $v \equiv u \equiv w$ by Lemma 4.3.

This shows that the ordering is uniform on the subgraph of G induced by $\{u, v, w\}$. Finally, G is uniform by Lemma 4.8 applied to the pairs u, w and v, w. ∎

Lemma 4.12. *Suppose that the template G contains two full vertices u and v, and one empty vertex w, with $uw, vw \in E(G)$ and $uv \notin E(G)$. Then G is uniform.*

Proof. The induced subgraph of G_∞ with vertex set $\{w_1, w_2, u_1, v_1\}$ has an isomorphism $w_1 \leftrightarrow u_1$, $w_2 \leftrightarrow v_1$. Hence, the assumption that the random order of G_∞ is consistent implies

$$\mathbb{P}(w_1, w_2 < u_1) = \mathbb{P}(u_1, v_1 < w_1). \tag{4.7}$$

By (3.4)–(3.5), and since $\{V_{u,w_i}\}_i$ are independent of U_{u_1} by Lemma 3.9 and (3.7) (or

Remark 3.12),

$$\mathbb{P}(w_1, w_2 < u_1) = \mathbb{P}(V_{u,w_1}, V_{u,w_2} < U_{u_1})$$
$$= \mathbb{E}(F_{u,w}(U_{u_1})^2) = \mathbb{E}\int_0^1 F_{u,w}(x)^2\,dx. \tag{4.8}$$

Furthermore, $\{w_1\} \cup \mathcal{W}_u$ is an infinite complete graph, and thus $V_{u,w_1} \stackrel{d}{=} U_{u_i} \sim U(0,1)$ (see Lemma 3.2). Thus, for $x \in [0,1]$,

$$x = \mathbb{P}(V_{u,w_1} \leqslant x) = \mathbb{E}\,F_{u,w}(x).$$

Consequently, by (4.8),

$$\mathbb{P}(w_1, w_2 < u_1) = \int_0^1 \mathbb{E}(F_{u,w}(x)^2)\,dx$$
$$\geqslant \int_0^1 (\mathbb{E}\,F_{u,w}(x))^2\,dx = \int_0^1 x^2\,dx = 1/3. \tag{4.9}$$

On the other hand, using the Cauchy–Schwarz inequality,

$$\mathbb{P}(u_1, v_1 < w_1) = \mathbb{P}(V_{w,u_1}, V_{w,v_1} < U_{w_1}) = \mathbb{E}(F_{w,u}(U_{w_1})F_{w,v}(U_{w_1}))$$
$$\leqslant \big(\mathbb{E}(F_{w,u}(U_{w_1})^2)\big)^{1/2}\big(\mathbb{E}(F_{w,v}(U_{w_1})^2)\big)^{1/2}. \tag{4.10}$$

By (3.7), $F_{w,u}(U_{w_1}) = V_{u,w_1}$ and, as noted above, $V_{u,w_1} \sim U(0,1)$. Hence we deduce that $\mathbb{E}(F_{w,u}(U_{w_1})^2) = 1/3$. Similarly, by symmetry, $\mathbb{E}(F_{w,v}(U_{w_1})^2) = 1/3$. Consequently, (4.10) yields

$$\mathbb{P}(u_1, v_1 < w_1) \leqslant 1/3. \tag{4.11}$$

By (4.7), we thus must have equality in both (4.9) and (4.10). The equality in (4.9) implies that for a.e. x, $F_{u,w}(x) = \mathbb{E}\,F_{u,w}(x) = x$ a.s., which implies that a.s. $F_{u,w}(x) = x$ for all $x \in [0,1]$. Hence $w \equiv u$ by Lemma 4.3. By symmetry, $w \equiv v$ also.

Suppose z is any full vertex of G. If $uz \in E(G)$ then $z \equiv u$ by Lemma 4.2, while if $uz \notin E(G)$ then $z \equiv u$ by applying Lemma 4.7 to $\{u, v\}$ and $\{u, z\}$. Now suppose z is an empty vertex of G. If $zu \in E(G)$ then $z \equiv w \equiv u$ by applying Lemma 4.7 to $\{u, w\}$ and $\{u, z\}$. If $zw \notin E(G)$ then $z \equiv w \equiv u$ by applying Lemma 4.2 to $\{z, w\}$. Finally, if $zw \in E(G)$ and $zu \notin E(G)$ then we deduce that $\bar{G}$, and hence G, is uniform by applying Lemma 4.11 to $\{z, u, w\}$ in $\bar{G}$. In all cases we see that $z \equiv u$. Hence G is uniform by Lemma 4.6. ∎

Lemma 4.13. *Suppose that the subgraph of G induced by the set of full vertices has a component that is not a clique. Then G is uniform.*

Proof. The assumption implies that there exist three full vertices u, v, w in G with $uw, vw \in E(G)$, but $uv \notin E(G)$. By Lemma 4.2, $w \equiv u$ and $w \equiv v$. For any full vertex $z \neq v, w$, either the template induced by $\{w, z\}$ is isomorphic to that induced by $\{u, v\}$ or that induced by $\{u, w\}$. Hence $z \equiv w$ for every full vertex z.

Now let z be an empty vertex. We consider two cases.

Case 1: Either $uz \in E(G)$ or $vz \in E(G)$. In this case G is uniform by either Lemma 4.11 or Lemma 4.12 applied to the subtemplate $\{u, v, z\}$.

Case 2: $uz, vz \notin E(G)$. Let $\mathcal{W}'_z := \mathcal{W}_z \setminus \{z_1, z_2\}$. In this case, the subgraphs of G_∞ induced by $\mathcal{W}_z$ and $\mathcal{W}'_z \cup \{u_1, v_1\}$ are isomorphic, by an isomorphism fixing $\mathcal{W}'_z$. Again, since the random order is consistent, it follows by (3.2) that

$$(V_{z,u_1}, V_{z,v_1}) \stackrel{\mathrm{d}}{=} (V_{z,z_1}, V_{z,z_2}) = (U_{z_1}, U_{z_2}).$$

Hence, arguing as in the proof of Lemma 4.9, for $x \in [0, 1]$, $\mathbb{E}\, F_{z,u}(x) = \mathbb{P}(V_{z,u_1} \leqslant x) = \mathbb{P}(U_{z_1} \leqslant x) = x$ and

$$\mathbb{E}(F_{z,u}(x) F_{z,v}(x)) = \mathbb{P}(V_{z,u_1} \leqslant x,\, V_{z,v_1} \leqslant x) = \mathbb{P}(U_{z_1} \leqslant x,\, U_{z_2} \leqslant x) = x^2.$$

Furthermore, $F_{z,u} = F_{z,v}$ a.s., by Lemma 4.3 since $u \equiv v$. Consequently, $\mathbb{E}(F_{z,u}(x)^2) = (\mathbb{E}\, F_{z,u}(x))^2$, and thus $F_{z,u}(x) = \mathbb{E}\, F_{z,u}(x) = x$ a.s. Hence $z \equiv u$ by Lemma 4.3.

In both cases we see that $z \equiv u$. Hence G is uniform by Lemma 4.6. ∎

We call a template G *reduced* if if contains no adjacent twin full vertices, and no non-adjacent twin empty vertices. Clearly any adjacent twin full vertices or non-adjacent twin empty vertices can be merged in a non-reduced template G without affecting G_∞ and hence without affecting whether or not G is uniform. Merging all such twins results in a reduced template, so it is enough to consider just these.

Theorem 4.14. *If G is a non-uniform reduced template, then G is either an empty graph (with at most one empty vertex) or complete (with at most one full vertex). In particular, for any non-uniform template G, G is either complete or empty, and so G_∞ is either a disjoint union of cliques or a complete multipartite graph.*

Proof. By Lemmas 4.9 and 4.11, any empty vertex must be joined to either all the full vertices, or none of them. By taking complements we also see that each full vertex is either joined to all empty vertices or none of them. Thus either all full vertices are joined to all empty vertices, or no full vertex is joined to any empty vertex. Without loss of generality (taking complements if necessary), we may assume that every full vertex is joined to every empty vertex.

By Lemma 4.13, the subgraph of G induced by the full vertices consists of a disjoint union of cliques. Since we assume G is reduced and any two full vertices in a clique of full vertices would be adjacent twins, we deduce that no two full vertices are adjacent. Similarly, applying Lemma 4.13 to the complement of G, we may assume any two empty vertices are adjacent.

If G contained at least two full vertices and at least one empty vertex, then G would be uniform by Lemma 4.12. Hence we deduce that either there is no empty vertex, and G is an empty graph of full vertices; or there is at most one full vertex and G is a complete graph consisting of empty vertices and at most one full vertex. ∎

Lemma 4.15. *Suppose that G is a template and that G_∞ has a consistent random order such that for any three vertices $u, v, w \in V(G_\infty)$, the induced random ordering on $\{u, v, w\}$ is uniform. Then the ordering is uniform.*

Proof. Pick any two vertices $u, v \in V(G)$, and consider the three vertices u_1, u_2, v_1 in G_∞. By Remark 3.12 (and the argument there), U_{u_1}, U_{u_2} and $V_{u,v_1} = F_{v,u}(U_{v_1})$ are independent, with U_{u_i} uniform, and these three random variables determine the order between u_1, u_2 and v_1. By assumption, this order is uniform, and thus

$$1/3 = \mathbb{P}(u_1, u_2 < v_1) = \mathbb{P}(U_{u_1}, U_{u_2} < V_{u,v_1}) = \mathbb{E}(V_{u,v_1}^2).$$

Similarly, $\mathbb{E}(V_{v,u_1}^2) = 1/3$. As in the proof of Lemma 4.11, it follows from Lemma 4.10 that $F_{u,v}(x) = x$ a.s., and thus $u \equiv v$ by Lemma 4.3. As u and v were arbitrary, the ordering on G_∞ is uniform by Lemma 4.6. ∎

Proof of Theorem 1.3. Consider a consistent ordering model on $\mathcal{P}$.

Suppose $G \in \mathcal{P}$. By repeatedly replacing vertices by twins and using Ramsey's theorem on each subgraph corresponding to one of the original vertices of G, we see that for all $N > 0$ there exists a $G_N \in \mathcal{P}$ which is obtained from G by replacing each vertex with either a complete graph or an empty graph on N vertices. By the infinite pigeonhole principle, there must be a template G' with underlying graph G such that for infinitely many N, G_N is an induced subgraph of G'_∞ (with N copies of each vertex in G). But then $\mathcal{P}_{G'} \subseteq \bigcup_{N=1}^\infty \mathcal{P}_{G_N} \subseteq \mathcal{P}$. Hence the random ordering model on $\mathcal{P}$ induces a random ordering model on $\mathcal{P}_{G'}$.

Suppose first that G is not a disjoint union of cliques or a complete multipartite graph. Since G'_∞ contains G as an induced subgraph, Theorem 4.14 shows that the template G' is uniform. In particular, the random ordering on $G \in \mathcal{P}_{G'}$ is uniform.

As G is not a disjoint union of cliques, it contains an induced subgraph isomorphic to the path P_3 on three vertices. Similarly, as G is not complete multipartite, G contains the graph $\bar{P}_3$ consisting of an edge and an isolated vertex. Thus $P_3, \bar{P}_3 \in \mathcal{P}$ and receive the uniform ordering on their vertices. The only other graphs on three vertices are homogeneous, so we deduce that for any graph $H \in \mathcal{P}$ and any three vertices $u, v, w \in V(H)$, the induced random ordering on $\{u, v, w\}$ is uniform.

Now suppose G is any graph in $\mathcal{P}$. Let, as above, G' be a template with underlying graph G and $\mathcal{P}_{G'} \subseteq \mathcal{P}$. By what we just have shown, any set of three vertices in G'_∞ receives the uniform ordering, and thus the ordering of G'_∞ is uniform by Lemma 4.15. Hence the ordering of G is uniform. ∎

Proof of Theorem 1.5. The hereditary property $\mathcal{F}_{\mathcal{H}}$ has the property that for any $G \in \mathcal{F}_{\mathcal{H}}$ and $v \in V(G)$, some graph G' obtained by replacing v by twins v_1, v_2 is also in $\mathcal{F}_{\mathcal{H}}$. Indeed, we can take the twins to be adjacent if there is no graph $H \in \mathcal{H}$ with adjacent twins, and we can take v_1, v_2 to be non-adjacent if there is no graph $H \in \mathcal{H}$ with non-adjacent twins. In both cases no copy of $H \in \mathcal{H}$ in G' could use both vertices v_1, v_2, and hence H would have to be an induced subgraph of G. Without loss of generality (by

taking complements if necessary), assume we are in the first case, so that any vertex can be replaced by adjacent twins. If $P_3 \in \mathcal{F}_{\mathcal{H}}$ then we are done by Theorem 1.3 as $\mathcal{F}_{\mathcal{H}}$ contains blow-ups of P_3 that are neither a disjoint union of cliques nor complete multipartite (for example, a triangle with a pendant edge). If $P_3 \notin \mathcal{F}_{\mathcal{H}}$, then $\mathcal{H}$ must contain an induced subgraph of P_3. As $P_3 \notin \mathcal{H}$, $\mathcal{H}$ must then contain a graph with two (or fewer) vertices. But then $\mathcal{F}_{\mathcal{H}}$ consists only of homogeneous graphs, and is therefore uniform. ∎

5. Gluing graphs

In this section we show in particular that hereditary properties that are closed under joining graphs at a single vertex, and many hereditary properties of forests, are uniform. We start by proving the result for any hereditary property that satisfies a certain technical condition.

Denote the disjoint union of two graphs G_1 and G_2 by $G_1 \cup G_2$. Suppose G is a graph and H is an induced subgraph. Define the graph $[G]_H^n$ to be the graph obtained by taking n copies of G (i.e., $G \cup \cdots \cup G$, n times) and identifying the corresponding subgraphs H from each copy. Thus, for example, $|V([G]_H^n)| = n|V(G) \setminus V(H)| + |V(H)|$. Let $\bar{K}_n$ denote the empty graph on n vertices. We also extend these notations in the obvious way to the case when $n = \infty$.

Theorem 5.1. *Suppose $\mathcal{P}$ is a hereditary property such that for any $G \in \mathcal{P}$ with at least two vertices, there exists a proper induced subgraph $H \neq \emptyset$ of G such that for all $n \geqslant 1$, $[G]_H^n \cup [G]_H^n \cup \bar{K}_n \in \mathcal{P}$. Then $\mathcal{P}$ is uniform.*

Proof. We may assume $\mathcal{P}$ contains some non-empty graph as otherwise $\mathcal{P}$ is clearly uniform. Note that, by taking an induced subgraph, for any $G \in \mathcal{P}$, $G \cup G \cup \bar{K}_n \in \mathcal{P}$. (For $|V(G)| < 2$ take an induced subgraph of $G' \cup G' \cup \bar{K}_n$ with $|V(G')| \geqslant 2$.) We shall prove by induction on $|V(G)|$ that if $G \in \mathcal{P}$ then the ordering on $G \cup G \cup \bar{K}_n$ is uniform for any n. This clearly implies the result. As $G \cup G \cup \bar{K}_n$ is homogeneous for $|V(G)| < 2$, we may assume $|V(G)| \geqslant 2$. Thus by assumption there exists a proper induced subgraph $H \neq \emptyset$ of G such that for all $n \geqslant 1$, $[G]_H^n \cup [G]_H^n \cup \bar{K}_n \in \mathcal{P}$. Let $\tilde{G} = [G]_H^\infty \cup [G]_H^\infty \cup \bar{K}_\infty$. Then $\mathcal{P}_{\tilde{G}} \subseteq \mathcal{P}$, and so the consistent ordering on $\mathcal{P}$ induces a consistent ordering on $\mathcal{P}_{\tilde{G}}$, and hence on $\tilde{G}$ (see Lemma 1.2). Denote the vertices of $\bar{K}_\infty$ as $\{u_i\}_{i=1}^\infty$, and the copies of H as H_i, $i = 1, 2$, with vertices $V(H_i) = \{v_{i,1}, \ldots, v_{i,r}\}$. Denote the remaining vertices in the jth copy of $G' := G \setminus H$ associated to H_i as $w_{i,j,k}$, $k = 1, \ldots, s$. Let $\tilde{G}' = \tilde{G} \setminus (H_1 \cup H_2)$ be the graph $\tilde{G}$ with the two copies of H removed, so that $\tilde{G}'$ consists of an infinite number of disjoint copies of G' together with $\bar{K}_\infty$. We first consider the induced random ordering on $\tilde{G}'$. One can define random variables

$$V_{i,j,k} = V_{u,w_{i,j,k}} := \lim_{n \to \infty} \frac{1}{n} \sum_{\ell=1}^{n} \mathbf{1}\{u_\ell < w_{i,j,k}\}$$

as in Section 3 giving the order of the $w_{i,j,k}$ relative to the vertices in the $\bar{K}_\infty$ subgraph. As the copies of G' can be permuted in $\tilde{G}'$, the random variables $V_{i,j} := (V_{i,j,1}, \ldots, V_{i,j,s})$

are exchangeable for $i \in \{1, 2\}$, $j \geqslant 1$. Hence, de Finetti's theorem implies that there is a random distribution μ on $[0, 1]^s$ such that, conditioned on μ, the $V_{i,j}$ are i.i.d. with distribution μ. However, we know that the joint distribution of $V_{1,1}$ and $V_{2,1}$, say, is uniform in $[0, 1]^s$ as by induction the induced subgraph $G' \cup G' \cup \bar{K}_n$ has a uniform random order for all n, and hence $G' \cup G' \cup \bar{K}_\infty$ receives a uniform random ordering. Thus for any measurable subset $S \subseteq [0, 1]^s$, $\mathbb{E}(\mu(S)\mu(S)) = |S|^2 = \mathbb{E}(\mu(S))\,\mathbb{E}(\mu(S))$. Thus μ is a.s. constant and uniform. Thus all $V_{i,j}$ are i.i.d. uniform random variables in $[0, 1]^s$, i.e., all $V_{i,j,k}$ are i.i.d. $U(0, 1)$ random variables.

Let $\mathcal{E}$ be any event determined by the ordering on $H_1 \cup H_2 \cup \bar{K}_\infty$, and assume $\mathbb{P}(\mathcal{E}) = p > 0$. The pairs $(V_{1,j}, V_{2,j})$, $j \geqslant 1$, are exchangeable, even conditioned on $\mathcal{E}$. Hence, there is a random measure $\mu_{\mathcal{E}}$ on $[0, 1]^{2s}$ such that conditioned on $\mathcal{E}$ and $\mu_{\mathcal{E}}$, $(V_{1,j}, V_{2,j})$ are i.i.d. with distribution $\mu_{\mathcal{E}}$. However, for any measurable subset $S \subseteq [0, 1]^{2s}$, a.s. on $\mathcal{E}$,

$$\mu_{\mathcal{E}}(S) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \mathbf{1}\{(V_{1,j}, V_{2,j}) \in S\} = |S|.$$

Hence, the pairs $(V_{1,j}, V_{2,j})$, $j \geqslant 1$, are i.i.d. and uniform, even conditioned on $\mathcal{E}$. In other words, all $V_{i,j,k}$ are i.i.d. and uniform, and independent of the ordering on $H_1 \cup H_2 \cup \bar{K}_\infty$. However, by induction, the random ordering on $H_1 \cup H_2 \cup \bar{K}_\infty$ is also uniform as it is uniform on every subgraph $H_1 \cup H_2 \cup \bar{K}_n$. The ordering on $\tilde{G}$ is a.s. determined by the ordering on $H_1 \cup H_2 \cup \bar{K}_\infty$ and the variables $V_{i,j,k}$ as the $V_{i,j,k}$ are continuous. Clearly this distribution is uniform. The result follows as $G \cup G \cup K_n$ is an induced subgraph of $\tilde{G}$. ∎

Example 5.2. We note that the requirement that we have two copies of $[G]_H^n$ in Theorem 5.1 is essential. For example, let $\mathcal{P}$ be the set of all graphs that are induced subgraphs of some $[C_4]_{\{u\}}^n$ (i.e., a collection of 4-cycles with a single vertex identified). Then for any $G \in \mathcal{P}$, $|V(G)| \geqslant 2$, the graph $[G]_H^n \cup \bar{K}_n$ lies in $\mathcal{P}$ for some $H \subset G$, $H \neq \emptyset$. Indeed, we can take $H = \{u\}$ if $u \in V(G)$, H a component of G if G is disconnected, the central vertex if $G = P_3$ and either vertex if $G = P_2$. However, the following gives a non-uniform model on $\mathcal{P}$. Fix $\alpha \in [0, 1]$ and assign to each vertex v an i.i.d. $U(0, 1)$ random variable X_v, conditioned so that the sum of X_v round any 4-cycle is α mod 1. It is not hard to see that any induced subgraph H that does not contain either a $P_3 \cup P_3$ or a proper C_4 subgraph is uniform, and any H that is of this form is embedded in $[C_4]_{\{u\}}^n$ in an essentially unique manner. In particular, if $H = P_3 \cup P_3$ then it must occur as a subgraph of the form $[C_4]_{\{u\}}^2 \setminus \{u\}$. If we denote the vertices of the two P_3s as $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ then $\sum_i X_{v_i} \equiv \sum_i X_{w_i} \equiv \alpha - X_u$ mod 1, which implies that the ordering $v_1 < w_1 < v_2 < w_2 < v_3 < w_3$ is impossible as $X_{v_1} + X_{v_2} + X_{v_3} < X_{w_1} + X_{w_2} + X_{w_3} < X_{v_2} + X_{v_3} + (X_{v_1} + 1)$. Hence the ordering is not uniform.

Example 5.3. In contrast to Example 5.2, let $\mathcal{P}'$ be the set of all graphs that are disjoint unions of induced subgraphs of some $[C_4]_{\{u\}}^n$. Then $\mathcal{P}'$ satisfies the conditions of Theorem 5.1. Hence $\mathcal{P}'$ is uniform. The ordering described in Example 5.2 is not

consistent on $\mathcal{P}'$ due to the fact that there are two distinct induced distributions on subgraphs isomorphic to $P_3 \cup P_3$; the one on $[C_4]^2_{\{u\}} \setminus \{u\}$ is not uniform, while the one on $([C_4]^1_{\{u\}} \setminus \{u\}) \cup ([C_4]^1_{\{u'\}} \setminus \{u'\})$ is uniform. The class $\mathcal{P}'$ also has the property that all C_4 subgraphs are edge-disjoint, and indeed also have private vertices that do not belong to any other C_4 (cf. Remark 2.5).

Proof of Theorem 1.6. If $\mathcal{P}$ consists only of empty graphs then it is uniform and we are done, so assume $\mathcal{P}$ contains some non-empty graph. Then $K_2 \in \mathcal{P}$, and so by assumption on $\mathcal{P}$, $P_3 \in \mathcal{P}$. Take any graph $G \in \mathcal{P}$ and any vertex $v \in V(G)$. We can attach multiple copies of G together at v to obtain $[G]^n_{\{v\}} \in \mathcal{P}$. Joining two of these to the end-vertices of a P_3 and then removing the central vertex gives $[G]^n_{\{v\}} \cup [G]^n_{\{v\}} \in \mathcal{P}$. Now repeatedly attaching this graph to an end-vertex of P_3 and removing the central vertex of the P_3 gives $[G]^n_{\{v\}} \cup [G]^n_{\{v\}} \cup \bar{K}_n \in \mathcal{P}$. Hence $\mathcal{P}$ satisfies the conditions of Theorem 5.1, so is uniform. ∎

In the case when $G \setminus H$ always is a set of isolated vertices, one can weaken the conditions of Theorem 5.1 so that only one copy of $[G]^n_H$ is required. Indeed, in this case we can prove by induction that $G \cup K_n$ is uniform and, in the proof, note that $\tilde{G}'$ is an empty graph, so is automatically uniform. This implies Theorem 1.8 in the case when (i) always holds as we can take H to be $G \setminus \{u\}$. We modify the proof slightly to obtain Theorem 1.8 in its entirety.

Proof of Theorem 1.8. Given any forest F, write S_F for the set of vertices of F that are adjacent to a leaf of F. Write F_n^* for the forest obtained by adding (or deleting) isolated vertices so that F_n^* has exactly n isolated vertices. For $u \in S_F$, write F_n^u for the forest obtained by adding (or deleting) leaves attached to u so that F_n^u has exactly n leaves attached to u.

Consider a consistent random ordering on $\mathcal{P}$. We prove that for every forest $F \in \mathcal{P}$ and every $u \in S_F \cup \{*\}$, the random ordering on F_n^u is uniform, provided these graphs lie in $\mathcal{P}$ for every $n \geqslant 2$. The proof is by induction on $|V(F)|$. If F is empty then $S_F = \emptyset$ and F_n^* is empty, so uniform. Thus we may assume F is non-empty. As no F_n^u is empty, either (i) or (ii) holds for F_n^u. This implies there exists $v_n \in S_{F_n^u} \cup \{*\}$ with $v_n \neq u$, such that the graph $F_{n,m}^{u,v_n} := (F_n^u)_m^{v_n}$ lies in $\mathcal{P}$. As $S_{F_n^u} \subseteq S_F$ is finite, this implies that there is a single $v \in S_F \cup \{*\}$, $v \neq u$, such that $F_{n,m}^{u,v} \in \mathcal{P}$ for all n, m. Let $F_{\infty,\infty}^{u,v}$ be the infinite graph with infinitely many leaves or isolated vertices associated with u and v. Let the leaves or isolated vertices associated to u be $\{u_i\}_{i \geqslant 1}$ and let the leaves or isolated vertices associated to v be $\{v_i\}_{i \geqslant 1}$.

Any finite subgraph of $F_{\infty,\infty}^{u,v}$ belongs to $\mathcal{P}$, so the ordering on $\mathcal{P}$ induces a random ordering on $F_{\infty,\infty}^{u,v}$. Assume first that $v \neq *$. As in Section 3, we can define random variables

$$V_i = V_{u,v_i} := \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}\{v_k < v_i\}.$$

As $\{u_i\}_{i \geqslant 1} \cup \{v_i\}_{i \geqslant 1}$ is a homogeneous set, V_i are i.i.d. $U(0, 1)$ random variables. Moreover, as in the proof of Theorem 5.1, these random variables are independent of the

ordering on $F_{\infty,0}^{u,v}$. But the random ordering on $F_{\infty,0}^{u,v}$ is also uniform as it is uniform on all subgraphs $F_{n,0}^{u,v}$ by induction applied to the proper subgraph $F_{1,0}^{u,v}$ (or $F_{0,0}^{u,v}$ if $u = *$) of F. Also, a.s. the ordering on $F_{\infty,\infty}^{u,v}$ is determined by the ordering on $F_{\infty,0}^{u,v}$ and the V_i as the V_i are continuous, and this random ordering is clearly uniform. If $v = *$ then, interchanging u and v, we again find that the ordering on $F_{\infty,\infty}^{u,v}$ is uniform. Hence in both cases the ordering on F_n^u is uniform for all n.

Finally, we note that for any non-empty $F \in \mathcal{P}$ conditions (i) or (ii) imply that there is a $u \in S_F$ such that $F_n^u \in \mathcal{P}$ for all n. Hence the ordering on F is also uniform. ∎

Appendix A. Non-uniformity of some explicit distributions

We recall the *Bernoulli polynomials* $B_n(x)$, which can be defined by the generating function

$$\frac{t e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

(see e.g. [27, Section 24.2]). The first few values are $B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}$, and $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$. The most important property for our purposes is the Fourier series representation of $B_n(x)$ [27, (24.8.3)],

$$B_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{k \neq 0} \frac{1}{k^n} e^{2\pi i k x}, \tag{A.1}$$

which is valid for $x \in [0, 1]$ when $n \geq 2$ and for $x \in (0, 1)$ when $n = 1$.

Lemma A.1. *Let $n \geq 2$ and $\alpha \in [0, 1]$, and let $X_1, \ldots, X_{n-1}$ and X, X' be i.i.d. $U(0, 1)$ random variables. Define $X_n \in [0, 1]$ so that*

$$\sum_{i=1}^{n} X_i \equiv \alpha \bmod 1. \tag{A.2}$$

Then for $1 \leq k \leq n$,

$$\mathbb{P}[X < X_k \text{ and } X_1 < \cdots < X_n] = \frac{k}{(n+1)!} + \frac{(-1)^{n-k}}{n!^2} \binom{n-1}{k-1} B_n(\alpha),$$

and

$$\mathbb{P}[X, X' < X_k \text{ and } X_1 < \cdots < X_n]$$
$$= \frac{k(k+1)}{(n+2)!} + \frac{(-1)^{n-k}}{n!(n+1)!} \binom{n-1}{k-1} \left((n+1)B_n(\alpha) + 2H_n B_{n+1}(\alpha)\right),$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

Proof. Let $P_k^1(\alpha) = \mathbb{P}[X < X_k \text{ and } X_1 < \cdots < X_n]$ and $P_k^2(\alpha) = \mathbb{P}[X, X' < X_k \text{ and } X_1 < \cdots < X_n]$. If α is replaced by a uniform random variable on $[0, 1]$, independent of $X_1, \ldots, X_{n-1}, X, X'$, then $X_1, \ldots, X_n, X, X'$ are i.i.d. $U(0, 1)$ random variables and α satisfies (A.2). Thus the Fourier transform

$$\hat{P}_k^j(t) := \int_0^1 P_k^j(\alpha) e^{-2\pi i t \alpha} \, d\alpha, \quad t \in \mathbb{Z},$$

can be represented as

$$\hat{P}_k^j(t) = \mathbb{E}(P_k^j(\alpha) e^{\omega X_1 + \cdots + \omega X_n}) = \int_{X_1 < \cdots < X_n} X_k^j e^{\omega X_1 + \cdots + \omega X_n} \, dX_1 \cdots dX_n,$$

where $\omega = -2\pi i t$. If $t = 0$ then $\hat{P}_k^1(0) = k/(n + 1)!$ and $\hat{P}_k^2(0) = k(k + 1)/(n + 2)!$ because there are k (respectively $k(k + 1)$) orderings of $X, X_1, \ldots, X_n$ (respectively $X, X', X_1, \ldots, X_n$) contributing to $\hat{P}_k^j$ and the $X, X', X_1, \ldots, X_n$ are i.i.d. Hence we may now assume $t \neq 0$. By symmetry,

$$\begin{aligned}
\hat{P}_k^j(t) &= \frac{1}{(k - 1)! \, (n - k)!} \int_{X_1, \ldots, X_{k-1} < X_k < X_{k+1}, \ldots, X_n} X_k^j e^{\omega X_1 + \cdots + \omega X_n} \, dX_1 \cdots dX_n \\
&= \frac{1}{(k - 1)! \, (n - k)!} \int_0^1 \left(\int_0^x e^{\omega y} \, dy \right)^{k-1} \left(\int_x^1 e^{\omega y} \, dy \right)^{n-k} x^j e^{\omega x} \, dx \\
&= \frac{1}{\omega^{n-1}(k - 1)! \, (n - k)!} \int_0^1 (e^{\omega x} - 1)^{k-1} (1 - e^{\omega x})^{n-k} x^j e^{\omega x} \, dx \\
&= \frac{(-1)^{n-k}}{\omega^{n-1}(k - 1)! \, (n - k)!} \int_0^1 x^j (e^{\omega x} - 1)^{n-1} e^{\omega x} \, dx.
\end{aligned}$$

Integrating by parts gives

$$\begin{aligned}
\hat{P}_k^j(t) &= \frac{(-1)^{n-k}}{\omega^n n(k - 1)!(n - k)!} \left(x^j (e^{\omega x} - 1)^n \big|_0^1 - \int_0^1 j x^{j-1} (e^{\omega x} - 1)^n \, dx \right) \\
&= \frac{(-1)^{k+1}}{\omega^n n!} \binom{n - 1}{k - 1} \int_0^1 j x^{j-1} (1 - e^{\omega x})^n \, dx.
\end{aligned}$$

For $j = 1$, expand $(1 - e^{\omega x})^n$ using the binomial theorem and note that $\int_0^1 e^{s \omega x} \, dx = 0$ for $s \in \mathbb{Z} \setminus \{0\}$. This gives

$$\hat{P}_k^1(t) = \frac{(-1)^{k+1}}{\omega^n n!} \binom{n - 1}{k - 1}.$$

For $j = 2$, we note that

$$I_n := \int_0^1 x(1 - e^{\omega x})^n \, dx = \frac{1}{2} - \frac{1}{\omega} H_n, \tag{A.3}$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Indeed, $I_0 = \frac{1}{2}$ and, for $n \geq 1$,

$$
\begin{aligned}
I_n - I_{n-1} &= \int_0^1 x(-e^{\omega x})(1 - e^{\omega x})^{n-1} \, dx \\
&= \frac{1}{n\omega} x(1 - e^{\omega x})^n \big|_0^1 - \int_0^1 \frac{1}{n\omega}(1 - e^{\omega x})^n \, dx = -\frac{1}{n\omega}.
\end{aligned}
$$

Hence

$$
\hat{P}_k^2(t) = \frac{(-1)^{k+1}}{\omega^n n!} \binom{n-1}{k-1}\left(1 - \frac{2}{\omega} H_n\right).
$$

Now we take inverse Fourier transforms, noting that by (A.1) the inverse Fourier transform of ω^{-n} is

$$
\sum_{t \neq 0} \frac{1}{\omega^n} e^{2\pi i \alpha t} = \frac{1}{(-2\pi i)^n} \sum_{t \neq 0} \frac{1}{t^n} e^{2\pi i \alpha t} = -\frac{(-1)^n}{n!} B_n(\alpha). \tag{A.4}
$$

We obtain

$$
\begin{aligned}
P_k^1(\alpha) &= \frac{k}{(n+1)!} + \sum_{t \neq 0} \frac{(-1)^{k+1}}{\omega^n n!} \binom{n-1}{k-1} e^{2\pi i t \alpha} \\
&= \frac{k}{(n+1)!} + \frac{(-1)^{n-k}}{n!^2} \binom{n-1}{k-1} B_n(\alpha),
\end{aligned}
$$

and

$$
\begin{aligned}
P_k^2(\alpha) &= \frac{k(k+1)}{(n+2)!} + \sum_{t \neq 0} \frac{(-1)^{k+1}}{\omega^n n!} \binom{n-1}{k-1}\left(1 - \frac{2}{\omega} H_n\right) e^{2\pi i t \alpha} \\
&= \frac{k(k+1)}{(n+2)!} + \frac{(-1)^{n-k}}{n!(n+1)!} \binom{n-1}{k-1}\big((n+1)B_n(\alpha) + 2H_n B_{n+1}(\alpha)\big)
\end{aligned}
$$

for almost all $\alpha \in [0, 1]$. As in both cases both sides are continuous in α, these in fact hold for all $\alpha \in [0, 1]$. ∎

Lemma A.2. *Let $X_1, \ldots, X_{n-1}$ be i.i.d. $U(0, 1)$ random variables. Fix $\alpha \in [0, 1]$ and $1 \leq i < \ell \leq n$ and define $X_n \in [0, 1]$ so that*

$$
\sum_{\substack{j \neq i, \ell}}^n X_j - X_i - X_\ell \equiv \alpha \bmod 1. \tag{A.5}
$$

Define $P_{j_1, j_2, \ldots, j_r}^{(i,\ell)}$ to be the probability that

$$
X_{j_1} < \cdots < X_{j_r} < \min(X_s : s \notin \{j_1, \ldots, j_r\}), \tag{A.6}
$$

i.e., the smallest r values of X_k are $X_{j_1}, \ldots, X_{j_r}$ in that order. Then for distinct i, j, k, ℓ,

$$
P_{i,j}^{(i,\ell)} - P_{j,i}^{(i,\ell)} = \frac{(-1)^n}{(n-1)!} \binom{n}{2} B_{n-1}(\alpha) \qquad (n \geq 3),
$$

$$
P_{i,j,k}^{(i,\ell)} - P_{j,i,k}^{(i,\ell)} = \frac{(-1)^n}{(n-1)!}(n - 3 + 2H_{n-3}) B_{n-1}(\alpha) + \frac{(-1)^n}{(n-2)!} B_{n-2}(\alpha) \quad (n \geq 4),
$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

Proof. Consider the Fourier transform

$$\hat{P}^{(i,\ell)}_{j_1,\ldots,j_r}(t) = \int_0^1 P^{(i,\ell)}_{j_1,\ldots,j_r}(\alpha)e^{-2\pi i t \alpha}\, d\alpha, \quad t \in \mathbb{Z}.$$

If we consider α to be a uniform random variable in $[0, 1]$ independent of $X_1, \ldots, X_{n-1}$, then $X_1, \ldots, X_n$ are now i.i.d. $U(0, 1)$ random variables and α satisfies (A.5). Thus

$$\hat{P}^{(i,\ell)}_{j_1,\ldots,j_r}(t) = \int_D e^{\varepsilon_1 \omega X_1 + \cdots + \varepsilon_n \omega X_n}\, dX_1 \cdots dX_n,$$

where $\omega = -2\pi i t$, $\varepsilon_s = 1$ if $s \neq i, \ell$ and $\varepsilon_i = \varepsilon_\ell = -1$, and $D \subseteq [0, 1]^n$ is the domain given by (A.6). For the first statement we can by symmetry assume $(i, j, \ell) = (1, 2, 3)$. Then

$$\hat{P}^{(1,3)}_{1,2}(t) - \hat{P}^{(1,3)}_{2,1}(t)$$
$$= \hat{P}^{(1,3)}_{1,2}(t) - \hat{P}^{(2,3)}_{1,2}(t)$$
$$= \int_{X_1 < X_2 < X_3, \ldots, X_n} (e^{-\omega X_1 + \omega X_2} - e^{\omega X_1 - \omega X_2})e^{-\omega X_3 + \omega X_4 + \cdots + \omega X_n}\, dX_1 \cdots dX_n.$$

For $t = 0$ (i.e., $\omega = 0$) this is clearly zero, so assume now that $t \neq 0$. Then

$$\int_{X_2}^1 e^{\varepsilon \omega x}\, dx = \frac{1}{\varepsilon\omega}(1 - e^{\varepsilon\omega X_2})$$

for $\varepsilon \in \{-1, 1\}$, and

$$\int_0^{X_2} (e^{-\omega X_1 + \omega X_2} - e^{\omega X_1 - \omega X_2})\, dX_1 = \frac{1}{\omega}(e^{\omega X_2} + e^{-\omega X_2} - 2).$$

Hence integrating over all X_s, $s \neq 2$, gives

$$\hat{P}^{(1,3)}_{1,2}(t) - \hat{P}^{(1,3)}_{2,1}(t) = \frac{1}{\omega^{n-1}}\int_0^1 (e^{\omega x} + e^{-\omega x} - 2)(-1 + e^{-\omega x})(1 - e^{\omega x})^{n-3}\, dx$$
$$= \frac{1}{\omega^{n-1}}\int_0^1 (e^{\omega x} - 1)^2(1 - e^{\omega x})^{n-2}e^{-2\omega x}\, dx$$
$$= \frac{1}{\omega^{n-1}}\int_0^1 (1 - e^{\omega x})^n e^{-2\omega x}\, dx$$
$$= \frac{1}{\omega^{n-1}}\binom{n}{2},$$

where in the last line we have expanded $(1 - e^{\omega x})^n$ using the binomial theorem and used $\int_0^1 e^{s\omega x}\, dx = 0$ for $s \in \mathbb{Z} \setminus \{0\}$. Now take the inverse Fourier transform using (A.4) to get

$$P^{(1,3)}_{1,2}(\alpha) - P^{(1,3)}_{2,1}(\alpha) = \frac{(-1)^n}{(n-1)!}\binom{n}{2}B_{n-1}(\alpha)$$

for almost all $\alpha \in [0, 1]$. However, as both sides are continuous in α, this holds for all $\alpha \in [0, 1]$.

For the second statement we can assume without loss of generality that $(i, j, k, \ell) = (1, 2, 3, 4)$. Then, performing the integration over $X_1, X_4, \ldots, X_n$, and finally over X_2, we have

$$
\begin{aligned}
\hat{P}_{1,2,3}^{(1,4)}&(t) - \hat{P}_{2,1,3}^{(1,4)}(t) \\
&= \hat{P}_{1,2,3}^{(1,4)}(t) - \hat{P}_{1,2,3}^{(2,4)}(t) \\
&= \int_{X_1 < X_2 < X_3 < X_4, \ldots, X_n} (e^{-\omega X_1 + \omega X_2} - e^{\omega X_1 - \omega X_2}) e^{\omega X_3} e^{-\omega X_4 + \omega X_5 + \cdots} \, dX_1 \cdots dX_n \\
&= \frac{1}{\omega^{n-2}} \int_{X_2 < X_3} (e^{\omega X_2} + e^{-\omega X_2} - 2) e^{\omega X_3} (-1 + e^{-\omega X_3})(1 - e^{\omega X_3})^{n-4} \, dX_2 \, dX_3 \\
&= \frac{1}{\omega^{n-2}} \int_{X_2 < X_3} (e^{\omega X_2} + e^{-\omega X_2} - 2)(1 - e^{\omega X_3})^{n-3} \, dX_2 \, dX_3 \\
&= \frac{1}{\omega^{n-1}} \int_0^1 (e^{\omega x} - e^{-\omega x} - 2\omega x)(1 - e^{\omega x})^{n-3} \, dx.
\end{aligned}
$$

Hence, using (A.3),

$$
\begin{aligned}
\hat{P}_{1,2,3}^{(1,4)}(t) - \hat{P}_{2,1,3}^{(1,4)}(t) &= \frac{1}{\omega^{n-1}} \int_0^1 (e^{\omega x} - e^{-\omega x} - 2\omega x)(1 - e^{\omega x})^{n-3} \, dx \\
&= \frac{1}{\omega^{n-1}} ((n - 3) - \omega + 2H_{n-3}).
\end{aligned}
$$

Taking inverse Fourier transforms, again using (A.4), gives

$$
P_{1,2,3}^{(1,4)}(\alpha) - P_{2,1,3}^{(1,4)}(\alpha) = \frac{(-1)^n}{(n-1)!} (n - 3 + 2H_{n-3}) B_{n-1}(\alpha) + \frac{(-1)^n}{(n-2)!} B_{n-2}(\alpha)
$$

for almost all $\alpha \in [0, 1]$, and hence for all $\alpha \in [0, 1]$ by continuity. ∎

Finally, we record a well-known fact, easily shown by induction using symmetry and $B_n'(x) = n B_{n-1}(x)$.

Lemma A.3. *The only zeros of $B_n(x)$ in $[0, 1]$ are $0, \frac{1}{2}, 1$ for odd $n \geq 3$, and exactly two values, one in $(0, \frac{1}{2})$ and one in $(\frac{1}{2}, 1)$, for even $n \geq 2$.* ∎

In particular, $B_n(x)$ and $B_{n+1}(x)$ have no common zeros in $[0, 1]$. (In fact, this extends to all complex zeros; equivalently, all zeros are simple, see [19, 20].)

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