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On the Serrin problem for ring-shaped domains

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Abstract. In this paper, we deal with the open problem of characterising rotationally symmetric solutions to $\Delta u = -2$, when Dirichlet boundary conditions are imposed on a *ring-shaped* planar domain. In contrast with Serrin's classical result, we show that the simplest possible set of overdetermining conditions, namely the prescription of *locally constant* Neumann boundary data, is not sufficient to obtain a complete characterisation of the solutions. A further requirement on the number of maximum points arises in our analysis as a necessary and sufficient condition for the rotational symmetry. Some new arguments are also introduced in the spirit of comparison geometry, that we believe of independent interest. In particular, the notion of *expected core radius* is defined and employed to achieve a qualitative description of the solutions, eventually leading to new classification results.

Keywords: overdetermined boundary value problems, free boundary problems, comparison geometry.

1. Introduction and statement of the main results

In this paper, we study pairs (Ω, u), where $\Omega \subset \mathbb{R}^2$ is an open bounded domain with smooth boundary and u is the unique solution to the Dirichlet problem

$$\begin{cases} \Delta u = -2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

A classical result due to Serrin [45] (see also [51]) states that, if a pair (Ω, u) solves (1.1), and the normal derivative of u at $\partial\Omega$ is constant, then necessarily Ω is a ball and u is rotationally symmetric. In this case, up to translations and rescaling, the solution is given by

$$\Omega_0 = B(0,1), \quad u_0(x) = \frac{1 - |x|^2}{2}. \tag{1.2}$$

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For future convenience, we observe that the function u_0 achieves its maximal value $(u_0)_{\max} = 1/2$ at the origin, and that

$$|\nabla u_0|^2 \equiv 1 \quad \text{on } \partial\Omega_0.$$

In the following, the couple (Ω_0, u_0) will be referred to as *Serrin's solution* to problem (1.1). As pointed out in Serrin's original paper [45], problem (1.1) has a nice and insightful fluid-dynamical interpretation. In fact, u can be thought as the velocity of a homogeneous incompressible fluid, flowing in steady laminar flow through a cylindrical pipe of cross section Ω, obeying the no-slip condition $u = 0$ at the pipe's wall. The normal derivative of u at $\partial\Omega$ is also relevant from this point of view, since it is related to the so-called *wall shear stress*, here denoted by τ, through the formula

$$\tau = \mu|\nabla u| \quad \text{on } \partial\Omega.$$

For the sake of simplicity, the dynamic viscosity μ of the fluid will be assumed to be constantly equal to 1 throughout the paper, so that the value of $|\nabla u|$ at $\partial\Omega$ will be identified with the wall shear stress (WSS for short) exerted by the fluid on the pipe's wall. In such a framework, Serrin's result says that

If the WSS assumes the same value at every point of the boundary, then the pipe's cross section must be a 2*-dimensional round ball, so that the pipe itself must have the geometry of a round cylinder.*

To introduce in more detail the problem of interest here and state our theorems, let us draw the reader's attention on a couple of remarkable features of Serrin's result. The first one is that the *connectedness* of the boundary is definitely not an assumption of the theorem. It is instead a consequence of the overdetermining condition

$$|\nabla u| \text{ is constant on } \partial\Omega. \tag{1.3}$$

In other words, the above requirement is strong enough to impose an extremely stringent prescription for the topology of $\partial\Omega$. The second feature that we would like to underline is that condition (1.3) also provides the solution u with an extra property. Indeed, it turns out that u is not only rotationally symmetric, but also *monotonically decreasing* with respect to the radial variable.

1.1. A multiplicity result

The main purpose of the present work is that of characterizing model solutions of problem (1.1) without assuming connectedness of the boundary of the domain and monotonicity of the function u. We will be particularly interested in domains Ω having two connected components, an outer one and an inner one. Problem (1.1) for such domains models the flow of a fluid along a hollow pipe. Such pipes have concrete interest in the field of hemodynamics, in particular in the analysis of blood pressure in arteries in presence of stents with flow divider [18] or of catheter probes [33, 50]. A second purely mathematical motivation for our study is that it allows us to introduce new techniques to deal with the lack of monotonicity of the model solutions and to demonstrate their effectiveness. In this respect, system (1.1) may be regarded as one of the most elementary PDE problem admitting nonmonotonic rotationally symmetric solutions defined on a compact annular domain. Indeed, it is elementary to observe that, whenever Ω is bounded, rotationally symmetric solutions to (1.1) are completely described, up to translations and rescaling, by the following family of *ring-shaped model solutions*, with $0 < R < 1$:

$$\Omega_R = \{r_{\mathrm{i}}(R) < |x| < 1\}, \quad u_R(x) = \frac{1-|x|^2}{2} + R^2 \log|x|, \tag{1.4}$$

where the *inner radius* $0 < r_{\mathrm{i}}(R) < R$ is the smallest positive zero of the function

$$(0, +\infty) \ni \rho \mapsto 1 - \rho^2 + 2R^2 \log \rho.$$

It is evident (see Figure 1) that the functions u_R's are not monotonically decreasing in $|x|$, and that $\partial\Omega_R$ is not connected. The parameter R, which we have chosen to describe the above family, will be referred to as the *core radius* of the ring-shaped model solution (Ω_R, u_R). It is a natural choice since the maximum points of u_R – here denoted by $\mathrm{MAX}(u_R)$ – are precisely located at the circle of radius R. In particular, we have that

$$(u_R)_{\max} = \frac{1-R^2}{2} + R^2 \log R \quad \text{and} \quad \mathrm{MAX}(u_R) = \{|x| = R\}, \tag{1.5}$$

so that $\Omega_R \setminus \mathrm{MAX}(u_R) = \Omega_{R,\mathrm{i}} \sqcup \Omega_{R,\mathrm{o}}$, where

$$\Omega_{R,\mathrm{i}} = \{r_{\mathrm{i}}(R) < |x| < R\} \quad \text{and} \quad \Omega_{R,\mathrm{o}} = \{R < |x| < 1\}. \tag{1.6}$$

To complete the description of this fundamental family of solutions, let us observe that

$$|\nabla u_R| \equiv \frac{R^2 - r_{\mathrm{i}}^2(R)}{r_{\mathrm{i}}(R)} \ \text{ on } \Gamma_{R,\mathrm{i}} \quad \text{and} \quad |\nabla u_R| \equiv 1 - R^2 \ \text{ on } \Gamma_{R,\mathrm{o}}, \tag{1.7}$$

where $\Gamma_{R,\mathrm{i}} = \{|x| = r_{\mathrm{i}}(R)\}$ and $\Gamma_{R,\mathrm{o}} = \{|x| = 1\}$ respectively denote the *inner* and the *outer boundary* of Ω_R, so that $\partial\Omega_R = \Gamma_{R,\mathrm{i}} \sqcup \Gamma_{R,\mathrm{o}}$. Notice that Serrin's solution can be recovered as the singular limit of the ring-shaped model solutions, as $R \to 0^+$. Indeed, it can be proved that the functions u_R's converge smoothly to u_0 on the compact subsets of $B(0,1) \setminus \{0\}$, so that both the sets $\mathrm{MAX}(u_R)$ and $\Gamma_{R,\mathrm{i}}$ are collapsing onto the origin, as $R \to 0^+$ (see Figure 1).

Having this picture in mind, it would be tempting to guess that a Serrin-type characterisation of ring-shaped model solutions (1.4) holds, provided the overdetermining condition (1.3) is replaced by the requirement that

$$|\nabla u| \text{ is locally constant on } \partial\Omega. \tag{1.8}$$

For the sake of exposition, we focus our attention on the case of *ring-shaped domains*, i.e., bounded domains whose boundary has precisely two connected components. To fix the notation, we agree that

$$\partial\Omega = \Gamma_{\mathrm{i}} \sqcup \Gamma_{\mathrm{o}}, \tag{1.9}$$

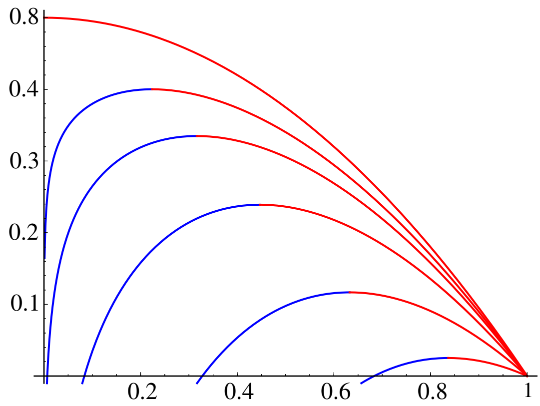

Fig. 1. The above diagram displays, according to our normalization, a comparative view of the profiles of the ring-shaped model solutions u_R's, with $0 < R < 1$, culminating in the profile of Serrin's solution for $R = 0$. The blue and the red parts of the graphs refer to the zones where the profiles are respectively monotonically increasing and decreasing, whereas the parameter R identifies the unique critical point of each profile. It is immediate to notice that for Serrin's solution only the decreasing regime is present, the only critical point being located at the origin.

where $\Gamma_{\rm i}$ and $\Gamma_{\rm o}$ denote the inner and the outer connected component of the boundary, respectively. If (Ω, u) is a solution to problem (1.1), note that $\{u = 0\} = \partial\Omega$. Also, for future convenience, we set

$$u_{\max} = \max_{\Omega} u \quad \text{and} \quad \mathrm{MAX}(u) = \{p \in \Omega : u(p) = u_{\max}\}.$$

Our first main result states that, even for ring-shaped domains, condition (1.8) is not strong enough to force the rotational symmetry of (Ω, u).

Theorem A. *There exist infinitely many solutions* (Ω, u) *to problem* (1.1)*, defined on a ring-shaped domain* Ω*, that are not rotationally symmetric and such that* $|\nabla u|$ *is locally constant on* $\partial\Omega$.

What in fact we are able to prove (see Section 6) is that there exist infinitely many one-parameter families of solutions to (1.1), which bifurcate from the family of ring-shaped model solutions (1.4). The proof of the above theorem essentially relies on the celebrated Crandall–Rabinowitz bifurcation theorem [15], and it is inspired by the very recent paper [29], which provides a multiplicity result for solutions to problem (1.10) below, obeying the further condition

$$\frac{\partial u}{\partial \nu} \equiv c \leq 0 \quad \text{on } \partial\Omega,$$

where ν is the outer unit normal to $\partial\Omega$. The construction of exotic solutions by means of bifurcation arguments is a topic that has registered a substantial developments in recent years. Among the pivotal results, we mention [36, 42, 47]. Other works more in line with ours are [21, 43], as well as the already mentioned [29]. We also mention the recent works [12, 19, 27], where nontrivial solutions to related problems have been obtained through the implicit function theorem.

Before introducing our next result, it is mandatory to put in evidence an important geometric feature of the solutions produced by Theorem A. Indeed, it turns out that each family of bifurcating solutions is – by construction – invariant under a symmetry group, whose cardinality can be chosen as large as desired. It follows that the number of maximum points displayed by the solutions in question is readily estimated from below by the cardinality of the symmetry group itself. At this point, one might wonder if it is possible to construct nonrotationally symmetric solutions to problem (1.1), as in Theorem A, with $|\nabla u|$ locally constant on $\partial\Omega$ and with an infinite number of maximum points. Our Theorem B below, shows that the latter construction is never possible on a ring shaped domain.

1.2. A partially overdetermined boundary value problem and a dichotomy theorem

The question arises whether it is possible to equip problem (1.1) with an overdetermining condition that is powerful enough to select all and only the rotationally symmetric solutions, avoiding on the one hand the overkill caused by condition (1.3), and on the other hand the multiplicity results allowed by condition (1.8). Answering this question would then provide a complete and satisfactory characterisation of the ring-shaped model solutions (1.4): a result that, to the best of the authors' knowledge, is so far missing in the literature. Such a gap might look surprising, if compared with the impressive amount of deep and beautiful works that have been inspired by Serrin's original paper. To give some examples, Serrin's moving plane method has been utilized to characterize rigidity of solutions to large families of elliptic overdetermined problems [17, 26, 34, 46] or nonlocal problems [4, 20] or to show nonexistence results [11]. The alternative integral method proposed by Weinberger [51], more in line with the approach pursued in the present paper, has also seen a lot of success [14, 22, 23, 25, 41]. Further alternative strategies and related results may be found in [9, 10, 13, 37].

As a possible explanation, we observe that the analysis of the case under consideration cannot uniquely rely on the *moving plane method*, since, whenever applicable, it has the drawback of providing the solutions with an undesirable monotonicity. More precisely, a solution fitting into that framework would turn out to be nonincreasing in the radial direction, which is false for ring-shaped model solutions (1.4), as depicted in Figure 1. Consequently, only partial classification results can be obtained in the present context through the moving plane method, for example when the model configuration is represented by the sole outer part $(\Omega_{R,\mathrm{o}}, u_{R|\Omega_{R,\mathrm{o}}})$ of a ring-shaped model solution (Ω_R, u_R). This is the content of the following beautiful theorem, firstly proved by Reichel [40, Theorem 3] and then refined by Syrakov [48, Theorem 2], that also represents so far the most advanced result of the literature along the directions indicated by the present paper.

Theorem 1.1 ([48, Theorem 2], see also [40, Theorem 3]). *Let* Ω *be a ring-shaped domain, let* $a > 0$ *be a positive real number, and let* (Ω, u) *be a solution to the problem*

$$\begin{cases} \Delta u = -2 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_{\mathrm{o}}, \\ u = a & \text{on } \Gamma_{\mathrm{i}}, \end{cases} \tag{1.10}$$

where, according to (1.9)*, we set* $\partial\Omega = \Gamma_{\mathrm{i}} \sqcup \Gamma_{\mathrm{o}}$. *Suppose that* $|\nabla u|$ *is locally constant on* $\partial\Omega$ *and that*

$$\frac{\partial u}{\partial \nu} \geq 0 \quad \text{on } \Gamma_{\mathrm{i}}, \tag{1.11}$$

where ν *is the outer unit normal to* Γ_{i}. *Then* (Ω, u) *is rotationally symmetric and, up to translations and rescaling, it corresponds to a portion of* $(\Omega_{R,\mathrm{o}}, u_{R|\Omega_{R,\mathrm{o}}})$ *for some* $0 < R < 1$. *In particular, u is nonincreasing in the radial direction.*

Remark 1.2. Observe that the requirement in (1.11) is never fulfilled by a solution to the Dirichlet problem (1.1), because of the Hopf lemma.

In contrast with the above result, we now state our second main theorem, in which a characterisation of the ring-shaped model solutions (1.4) on their entire domains is proposed, under the (implicit) assumption that the cardinality of the set of maximum points is not finite. A somehow unexpected feature of our theorem is that condition (1.8) is not needed in its full strength, making the problem only partially overdetermined, in the sense of [24].

Theorem B (Dichotomy theorem). *Let* (Ω, u) *be a solution to problem* (1.1) *defined on a ring-shaped domain* Ω. *Assume that* $|\nabla u|$ *is constant on either the inner or the outer boundary component. Then the following dichotomy holds true:*

(i) *either* (Ω, u) *is rotationally symmetric,*

(ii) *or else* (Ω, u) *has finitely many maximum points.*

In particular, the case of countably many maximum points is excluded.

This result can be deduced as a consequence of [48, Theorem 2], together with the regularity result on the set of the maximum points provided in [3], see Section 4.2 for more details. We also mention that in [5] an alternative strategy, based on the Pohozaev identity and the isoperimetric inequality, has been provided to prove Theorem B without relying on [48, Theorem 2].

To put our result into perspective, let us mention that the above theorem has a natural fluid-dynamical interpretation in the same framework as the one previously recalled for Serrin's result. The only obvious difference consists in considering a *hollow* cylindrical pipe with a ring-shaped 2-dimensional cross section Ω in place of a simply connected one. In particular, our theorem says that

> *If the WSS is constant on some connected component of the pipe's wall, then the velocity of the fluid may attain its maximal value only at finitely many streamlines, unless the hollow pipe itself consists of a couple of concentric cylindrical round tubes.*

Some comments are in order about the dichotomous condition on the cardinality of $\mathrm{MAX}(u)$. The first implication of Theorem B is that the multiple nonrotationally symmetric solutions produced by Theorem A must all have a finite number of maximum points, so that case (ii) turns out to be widely nonempty, and in fact should be thought of as the generic situation. On the contrary, case (i) corresponds to a rigidity statement, saying that if u has infinitely many maximum points, then (Ω, u) must belong to the family of ring-shaped model solutions (1.4), up to a suitable rescaling. As already observed, Theorem A (see also Theorem 6.1 for a more detailed statement of this multiplicity result) provides families of counterexamples to the conclusion (i) with an arbitrarily large – though finite – number of maximum points and with (locally) constant normal derivative on both the inner and the outer component of $\partial\Omega$. This leads us to conjecture that for any $k \in \mathbb{N}$, it should exist a nonrotationally symmetric solution to problem (1.1), defined on a ring-shaped domain, with exactly k maximum points, having constant normal derivative either at the inner or at the outer boundary. However, such a study would take us too far from the purposes of the present manuscript, and we defer it to future investigations.

1.3. A comparison algorithm

With the purpose of giving further qualitative descriptions of the solutions to problem (1.1), we develop a new approach, based on the comparison with the rotationally symmetric solutions (1.4). Such a comparison algorithm is meant to produce a number of sharp and rigid *a priori* bounds (e.g., Theorem 3.5 and Proposition 4.1), eventually leading to new classification results. A central role is played here by the concept of *expected core radius*, which is introduced in Definition 1.4 and readily employed to provide new geometric characterization of Serrin's solution (Theorem C) as well as of ring-shaped model solutions (Theorem D). Further applications of the same concept to the classical question of locating the hot spots, will then be discussed in Section 1.4 (Theorem E) and Section 5.

More concretely, given a solution (Ω, u) to problem (1.1), we are going to compare relevant analytic and geometric quantities (such as the gradient of u, or the lengths of the boundary components $|\Gamma_{\mathrm{i}}|$ and $|\Gamma_{\mathrm{o}}|$) with their corresponding counterparts on a ring-shaped model solution (Ω_R, u_R) (namely with $|\nabla u_R|$, $|\Gamma_{R,\mathrm{i}}| = 2\pi r_{\mathrm{i}}(R)$, and $|\Gamma_{R,\mathrm{o}}| = 2\pi$, to continue with the previous exemplification), bounding the former in terms of the latter.

In order to obtain sharp and rigid inequalities, it is crucial to take care of two important and intimately related aspects. The first one is the choice of an appropriate *scale fixing*. Indeed, since our problem is invariant by translations and rescaling (see (3.1)) and since we have already chosen a scale to describe the family of ring-shaped model solutions (1.4), it is convenient to normalize in a consistent fashion also the generic solution (Ω, u) that we aim to analyse. The second aspect is the selection of a good *basis for comparison*. In other words, since in the chosen normalization each ring-shaped model solution is uniquely determined by the value of its *core radius*, we need to find a way to associate our generic solution with a number $0 < R < 1$. It actually turns out that the

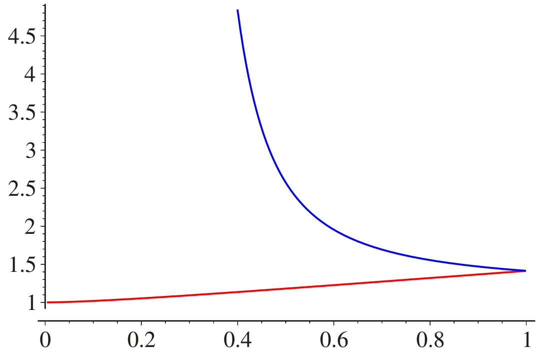

Fig. 2. The above figure represents the graphs of the normalized wall shear stress of the ring-shaped model solutions, as function of the core radius parameter $0 < R < 1$. In red, we have the graph of the *outer NWSS function* $\overline{\tau}_{\mathrm{o}}$, whereas in blue we have the graph of the *inner NWSS function* $\overline{\tau}_{\mathrm{i}}$.

resolution of the latter problem also gives an answer to the scale fixing question. Indeed, once a core radius $0 < R < 1$ is selected, it is sufficient to rescale the solution according to (3.1) in such a way that

$$u_{\max} = (u_R)_{\max}.$$

Let us focus then on the problem of finding the most convenient value of the parameter $0 < R < 1$. The heuristic idea, here, is to use a shooting paradigm to guess the value of the core radius from the slope of the solution at the boundary, i.e., from the measurement of its wall shear stress. A closer look shows that a more refined information is actually needed. Indeed, if two solutions are related to each other as in (3.1), the value of the expected core radius must coincide. It is then convenient to replace the wall shear stress of a boundary component with its scaling invariant version. Moreover, it is clear that, in the outlined scheme, every single boundary component might in principle give rise to a different guess for the core radius parameter. Let us take care of these two queries with a couple of definitions. Here and in the sequel, we make use of the shorthand notation $\pi_0(E)$ to indicate the collection of the connected components of a given set E.

Definition 1.3 (Normalized wall shear stress). Let (Ω, u) be a solution to problem (1.1), and let $\Gamma \in \pi_0(\partial\Omega)$ be a connected component of the boundary of Ω. We define the *normalized wall shear stress (NWSS)* of Γ as

$$\overline{\tau}(\Gamma) := \frac{\max_\Gamma |\nabla u|}{\sqrt{2u_{\max}}}.$$

More in general, if N is a connected component of $\Omega \setminus \mathrm{MAX}(u)$, we define the *normalized wall shear stress (NWSS)* of the region N as

$$\overline{\tau}(N) := \max\{\overline{\tau}(\Gamma) : \Gamma \in \pi_0(\partial\Omega \cap \overline{N})\}.$$

We finally define the NWSS of the whole domain Ω as

$$\overline{\tau}(\Omega) := \max\{\overline{\tau}(\Gamma) : \Gamma \in \pi_0(\partial\Omega)\}.$$

To introduce our second definition, it is important to observe that on ring-shaped model solutions the NWSS at either the inner or the outer boundary can be computed as a function of the *core radius*. More precisely, it is useful to consider the outer NWSS function $\overline{\tau}_{\mathrm{o}}$ and the inner NWSS function $\overline{\tau}_{\mathrm{i}}$, whose graphs are plotted in Figure 2 and which are defined as follows:

- The *outer NWSS function*

$$\overline{\tau}_{\mathrm{o}}\colon [0,1) \to [1,\sqrt{2})$$

is defined by

$$\overline{\tau}_{\mathrm{o}}(R) := \frac{|\nabla u_R|}{\sqrt{2(u_R)_{\max}}}\Big|_{\Gamma_{R,\mathrm{o}}} = \begin{cases} 1 & \text{if } R = 0, \\ \frac{1-R^2}{\sqrt{1-R^2+2R^2\log R}} & \text{if } 0 < R < 1. \end{cases} \tag{1.12}$$

Observe that $\overline{\tau}_{\mathrm{o}}$ is continuous, strictly increasing, and $\overline{\tau}_{\mathrm{o}}(R) \to \sqrt{2}$, as $R \to 1^-$.

- The *inner NWSS function*

$$\overline{\tau}_{\mathrm{i}}\colon (0,1] \to [\sqrt{2},+\infty)$$

is defined by

$$\overline{\tau}_{\mathrm{i}}(R) := \frac{|\nabla u_R|}{\sqrt{2(u_R)_{\max}}}\Big|_{\Gamma_{R,\mathrm{i}}} = \begin{cases} \frac{R^2-r_{\mathrm{i}}^2(R)}{r_{\mathrm{i}}(R)\sqrt{1-R^2+2R^2\log R}} & \text{if } 0 < R < 1, \\ \sqrt{2} & \text{if } R = 1. \end{cases} \tag{1.13}$$

Observe that $\overline{\tau}_{\mathrm{i}}$ is continuous, strictly decreasing, and $\overline{\tau}_{\mathrm{i}}(R) \to +\infty$, as $R \to 0^+$.

As pointed out, the key feature of $\overline{\tau}_{\mathrm{o}}$ and $\overline{\tau}_{\mathrm{i}}$ is that they are invertible. Building on this property, we are now ready to introduce the notion of *expected core radius*. In analogy with the NWSS, this invariant can be associated to either a boundary component of $\partial\Omega$ or, more in general, to a connected component of $\Omega \setminus \mathrm{MAX}(u)$ and even to the entire domain Ω.

Definition 1.4 (Expected core radius). Let (Ω, u) be a solution to problem (1.1), and let $\Gamma \in \pi_0(\partial\Omega)$ be a connected component of the boundary of Ω. We define the *expected core radius* of Γ as follows:

- if $1 \leq \overline{\tau}(\Gamma) < \sqrt{2}$, we set

$$R(\Gamma) = \overline{\tau}_{\mathrm{o}}^{-1}(\overline{\tau}(\Gamma)); \tag{1.14}$$

- if $\overline{\tau}(\Gamma) \geq \sqrt{2}$, we set

$$R(\Gamma) = \overline{\tau}_{\mathrm{i}}^{-1}(\overline{\tau}(\Gamma)). \tag{1.15}$$

More in general, if N is a connected component of $\Omega \setminus \mathrm{MAX}(u)$, we define the *expected core radius* of N as follows:

- if $\overline{\tau}(N) < \sqrt{2}$, we set

$$R(N) = \overline{\tau}_{\mathrm{o}}^{-1}(\overline{\tau}(N)); \tag{1.16}$$

- if $\overline{\tau}(N) \geq \sqrt{2}$, we set

$$R(N) = \overline{\tau}_{\mathrm{i}}^{-1}(\overline{\tau}(N)). \tag{1.17}$$

We finally define the *expected core radius* of the whole domain Ω as follows:

- if $\overline{\tau}(\Omega) < \sqrt{2}$, we set
$$R(\Omega) = \overline{\tau}_{\rm o}^{-1}(\overline{\tau}(\Omega)); \tag{1.18}$$
- if $\overline{\tau}(\Omega) \geq \sqrt{2}$, we set
$$R(\Omega) = \overline{\tau}_{\rm i}^{-1}(\overline{\tau}(\Omega)). \tag{1.19}$$

As it is immediate to check, the expected core radius of a boundary component of a ring-shaped model solution (Ω_R, u_R) coincides by construction with the value of its core radius parameter R. In other words, we have $R(\Gamma_{R,\rm o}) = R = R(\Gamma_{R,\rm i})$, and the same is true for the expected core radius of either the outer or the inner region $R(\Omega_{R,\rm o}) = R = R(\Omega_{R,\rm i})$. This picture also includes the extremal case of Serrin's solution (1.2), where the expected core radius is actually equal to 0.

It should be noticed that definition (1.14) differs from definition (1.16) (and subsequent (1.18)) in a subtle, though substantial way. In (1.14), the condition $1 \leq \overline{\tau}(\Gamma)$ has to be imposed in order to get a number in the range of $\overline{\tau}_{\rm o}$, that can be effectively used to define $R(\Gamma)$. Such a condition is not needed in (1.16), since it turns out to be always satisfied. In particular, the expected core radius $R(N)$ of a region $N \in \pi_0(\Omega \setminus \mathrm{MAX}(u))$ is always well defined, as such, it is obviously nonnegative and more remarkably it vanishes if and only if (Ω, u) is equivalent to Serrin's solution (1.2). This fact is stated in the following theorem, where no assumption is made *a priori* on either the topology of Ω, or the number of its boundary components.

Theorem C. *Let (Ω, u) be a solution to problem (1.1), and let N be a connected component of $\Omega \setminus \mathrm{MAX}(u)$. Then, the expected core radius of N is well defined and nonnegative*

$$R(N) \geq 0.$$

Moreover, the equality holds if and only if (Ω, u) is equivalent to Serrin's solution (1.2). The same conclusions hold a fortiori *for the expected core radius $R(\Omega)$ of the entire domain.*

The above theorem can be regarded as a first instance of how effective the notion of expected core radius might be for classification purposes. A second instance is contained in the following theorem. It says in particular that if the expected core radius that is guessed at the outer boundary $R(\Gamma_{\rm o})$ coincides with the one that is guessed at the inner boundary $R(\Gamma_{\rm i})$, then the solution is rotationally symmetric and coincides up to scaling with (Ω_R, u_R), where $R(\Gamma_{\rm o}) = R = R(\Gamma_{\rm i})$.

Theorem D. *Let (Ω, u) be a solution to problem (1.1) such that Ω is a ring-shaped domain and u has infinitely many maximum points. Assume that $\overline{\tau}(\Gamma_{\rm o}) < \sqrt{2}$. Then, the expected core radii of $\Gamma_{\rm o}$ and $\Gamma_{\rm i}$ are both well defined and positive. Moreover, they satisfy*

$$R(\Gamma_{\rm o}) \geq R(\Gamma_{\rm i}) > 0,$$

and the equality holds if and only if (Ω, u) is equivalent to the ring-shaped model solution whose core radius is given by the common value of the expected core radii.

It is important to observe that, in contrast with Theorem B, no constant Neumann data are imposed on $\partial\Omega$ in Theorem D. Concerning the condition $\overline{\tau}(\Gamma_o) < \sqrt{2}$, it must be noticed that it is always satisfied on the model solutions, as

$$\mathrm{Im}(\overline{\tau}_o) = [1, \sqrt{2}).$$

Building on this condition, we will deduce that

$$1 \leq \overline{\tau}(\Gamma_o) \quad \text{and} \quad \sqrt{2} \leq \overline{\tau}(\Gamma_i).$$

The first inequality implies that $R(\Gamma_o)$ is well defined, whereas the second says that the NWSS at Γ_i lies in the range of $\overline{\tau}_i$, as it is natural to expect, in view of the model situation.

1.4. Location of the hot spots

A classical topic in the qualitative study of the boundary value problems is the question of locating the hot spots, namely the maximum points of the solutions. In the very recent paper [35], Magnanini and Poggesi proved an estimate for the distance of a point $x \in \mathrm{MAX}(u)$ to the boundary $\partial\Omega$, where u is the solution to problem (1.1). In particular, in [35, Theorem 1.1] it is shown that, if $\partial\Omega$ is mean convex (meaning that the curvature $\kappa(p)$ of $\partial\Omega$ with respect to the normal pointing outside Ω is nonnegative for any point $p \in \partial\Omega$), then

$$\frac{d(x, \partial\Omega)}{r_\Omega} \geq \frac{1}{\sqrt{2}}, \tag{1.20}$$

where $r_\Omega = \max_{x\in\Omega} d(x, \partial\Omega)$ is the *inradius* of Ω. If $\partial\Omega$ is not mean convex, a similar, slightly more intricate, lower bound is shown in [35, Corollary 2.6]. Such results are obtained via comparison with Serrin's solution (1.2).

Our comparison technique, tailored on the more general annular solutions (1.4), allows us to obtain more precise – indeed sharp – lower bounds for the distance $d(x, \partial\Omega \cap \overline{N})$, where N is a connected component of $\Omega \setminus \mathrm{MAX}(u)$ and $x \in \mathrm{MAX}(u) \cap \overline{N}$, in terms of the expected core radius $R(N)$ (see Theorem 5.1). Such lower bounds, suitably rephrased in terms of the scaling-invariant ratio $d(x, \partial\Omega)/r_\Omega$ used in [35], reduces to the estimates object of Theorem 5.2. As a byproduct, we obtain in the mean convex case the following improvement of (1.20).

Theorem E. *Let* (Ω, u) *be a solution to problem* (1.1) *where* $\partial\Omega$ *is mean convex, and let* $R = R(\Omega) \in [0, 1)$ *be the expected core radius associated with* Ω*. Then*

$$\frac{d(x, \partial\Omega)}{r_\Omega} \geq \frac{1 - R}{\sqrt{1 - R^2 + 2R^2 \log R}}. \tag{1.21}$$

Moreover, the equality holds if and only if $R = 0$ *and* Ω *is a ball.*

It can be easily checked that the right-hand side of (1.21) is decreasing in R (see Figure 3) and tends to $1/\sqrt{2}$ when $R \uparrow 1$. Hence, formula (1.20) follows immediately from (1.21), with a strict inequality.

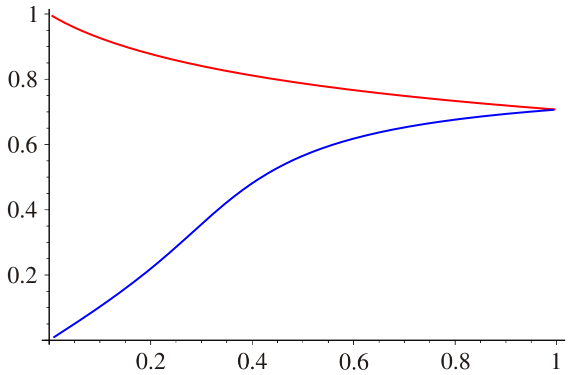

Fig. 3. The above figure represents the graphs of the right-hand sides of the formulae in Theorem 5.2, as functions of the expected core radius $R \in [0, 1)$. In red we have the plot of the right-hand side of (5.2), whereas in blue we have the plot of (5.3).

Remark 1.5. It should be mentioned that the results in [35] are more general in the sense that they work in any dimension and apply to a wider range of elliptic equations. We expect that it should be possible to develop our comparison technique to cover higher dimensions and more general PDEs, possibly recovering the results in [35] in more generality.

1.5. Comments and further directions

The results and techniques employed in this paper open the way to a number of natural questions and possibilities for further developments. In this subsection, we select a small number of them, among the ones that we consider more natural or stimulating.

Let us start with a basic, nonetheless fundamental, observation concerning our comparison algorithm. As it is clear from Section 1.3, in order to implement our method it is first crucial to select a rotationally symmetric solution to compare with. This is done by matching the NWSS of a region with the NWSS of the outer or inner region of a model solution. A common feature of all the rotationally symmetric solutions (Ω_R, u_R) is that the NWSS of the inner region $\Omega_{R,\mathrm{i}}$ is greater than $\sqrt{2}$, whereas the NWSS of the outer region $\Omega_{R,\mathrm{o}}$ is less than $\sqrt{2}$. It would be interesting to figure out whether this holds true for a general ring-shaped solution to (1.1), or whether there are counterexamples. In a broader sense, one would like to understand to what extent the fact that a region is outer or inner influences the NWSS of that region. A complete answer to this question could potentially allow us to remove the hypothesis $\overline{\tau}(\Gamma_\mathrm{o}) < \sqrt{2}$ from Theorem D.

Another natural question concerns the possibility of extending the results to higher dimensions. On this regard, we observe that the techniques employed in the paper do not depend deeply on the dimensional set-up. The choice of focussing on the 2-dimensional case was mostly due to conceptual and expository reasons. First of all, it is the most significant case from the point of view of the physical interpretation. Secondly, some of the computations, such as the ones in the proof of Theorem A, turn out to be considerably simpler in this framework. Finally, though the higher-dimensional case is conceptually very similar, it is basically impossible to find a unifying formalism to treat the two situations at the same time. Indeed, the model solutions in dimension $n \geq 3$ are given by

$$u(x) = \frac{1 - |x|^2}{2} - R^n \frac{|x|^{2-n}}{n-2},$$

which is formally different from (1.4). Since our strategy is based on comparison arguments referred to the model solutions, all computations and formulae in the proofs of Theorems C, D and E change significantly, when dealing with the higher-dimensional case.

We also mention that most of the results in the paper do not depend deeply on the special structure of the Euclidean space either. In fact, the techniques employed in the proofs of Theorems C, D and E are also available on general Riemannian manifolds.

With the purpose of testing the range of applicability of our technique, it would be interesting to understand whether it can be employed to study other kinds of overdetermined problems. There are already some positive answers in this direction. For instance, similar comparison techniques have been used in [7, 8] to study static spacetimes with positive cosmological constant. There are of course many other potential applications that deserve future investigation. For instance, a very natural direction would be the study of the problem $\Delta u = \delta_o$ inside a bounded domain containing the origin, since the rotationally symmetric solutions to this equation are very similar to the ones considered in this paper (namely, they are the same as (1.4) but with negative values of R^2). Another possible attempt would be to generalize the PDE considered in this paper, for instance, considering the equation $\Delta u = f(u)$ for a suitable family of functions f. It would also be interesting to see whether, beyond the ring-shaped model solutions, other (less) symmetric solutions to problem (1.1) can be characterized via suitable invariances of the WSS at $\partial\Omega$, and whether their stability can be analyzed, in the spirit of what have been done in [1, 2] for the Green's function in planar, simply connected, domains.

We conclude by mentioning a couple of other problems coming from physical models that are somehow related to ours. Remaining in the realm of fluid-dynamics, similar symmetry results on annular domains have been recently discussed for the Euler equation in [28]. Another problem that is worth mentioning is the so-called torsion problem, modeling the torsion of a bar with holes. From a mathematical viewpoint, this problem is very similar to the one discussed here, the only difference being the boundary condition in the inner boundary components. One of the main papers on this problem is the well-known [39], where Schwarz symmetrization [44] is used to prove that the rotationally symmetric solutions to the torsion problem are characterized as the ones maximizing torsional rigidity. Among the recent progresses on this problem, we mention [38], based again on the moving plane method, and [16].

1.6. Plan of the paper

In the rest of the paper, we will prove the results stated in this section. Section 2 is dedicated to the proof of Theorem C. This result is crucial as it grants us that Definition 1.4 is well posed, which in turn allows us to set up our comparison machinery. This is the topic of Section 3, where the important notion of pseudo-radial function is introduced and exploited to prove a sharp upper bound for the gradient of u. In Section 4, we will see that the gradient estimates lead to some curvature bounds for Γ_{o}, Γ_{i}, and ultimately to the proof of Theorem D. In the same section, we also provide a proof of Theorem B. In Section 5, the notion of expected core radius is employed to deduce some lower bounds for the distance of the hot spots to the boundary of Ω. Theorem E will be deduced as a corollary of these results. Finally, Section 6 is devoted to the proof of Theorem A.

2. Proof of Theorem C: The expected core radius of a region

This section is devoted to the proof of the following theorem, which implies at once Theorem C and, in turn, the fact that the expected core radius of a region $N \in \pi_0(M \setminus \mathrm{MAX}(u))$ is well defined, nonnegative and vanishes if and only if (Ω, u) is equivalent to Serrin's solution.

Theorem 2.1. *Let (Ω, u) be a solution to problem (1.1), where Ω is an arbitrary bounded domain with smooth boundary. Let N be a connected component of $\Omega \setminus \mathrm{MAX}(u)$, and suppose that $\overline{\tau}(N) \leq 1$, i.e.,*

$$\max_{\partial\Omega \cap \bar{N}} \frac{|\nabla u|}{\sqrt{2u_{\max}}} \leq 1.$$

Then, up to suitable dilations and translations, the solution (Ω, u) coincides with Serrin's solution (1.2).

The first step in the proof is to use a maximum principle argument to establish the following weaker version of Theorem 2.1.

Proposition 2.2. *Let (Ω, u) be a solution to problem (1.1), where Ω is an arbitrary bounded domain with smooth boundary, and suppose that*

$$\max_{\partial\Omega} \frac{|\nabla u|}{\sqrt{2u_{\max}}} \leq 1.$$

Then, up to suitable dilations and translations, the solution (Ω, u) coincides with Serrin's solution (1.2).

Proof. The Bochner formula coupled with the first equation in (1.1) gives

$$\Delta(|\nabla u|^2 + 2u) = 2\Big[|\nabla^2 u|^2 - \frac{(\Delta u)^2}{2}\Big] \geq 0. \tag{2.1}$$

Since $|\nabla u|^2 \leq 2u_{\max}$ on $\partial\Omega$ by hypothesis, the maximum principle implies that

$$\max_{\Omega}(|\nabla u|^2 + 2u) = \max_{\partial\Omega} |\nabla u|^2 \leq 2u_{\max}.$$

On the other hand, at any maximum point for u in Ω it holds $|\nabla u|^2 + 2u = 2u_{\max}$. Hence, by the strong maximum principle we obtain that

$$|\nabla u|^2 + 2u \equiv 2u_{\max}$$

in Ω. In turn, the equality holds in (2.1), which yields $\nabla^2 u = -g_{\mathbb{R}^2}$ and hence the conclusion. ∎

Having fixed a connected component N of $\Omega \setminus \mathrm{MAX}(u)$, we now introduce the function $U\colon [0, u_{\max}) \to \mathbb{R}$ given by

$$t \mapsto U(t) = \frac{1}{(u_{\max} - t)} \int_{\{u=t\}\cap\bar{N}} |\nabla u| \, d\sigma. \tag{2.2}$$

This function is well defined, because the integrand is globally bounded and classical results ensure that the level sets of u have finite $\mathcal{H}^1$-measure. The second step in the proof of Theorem 2.1 consists in showing that the function U is nonincreasing.

Proposition 2.3. *Let* (Ω, u) *be a solution to problem* (1.1)*, where* Ω *is an arbitrary bounded domain with smooth boundary. Let* N *be a connected component of* $\Omega \setminus \mathrm{MAX}(u)$ *and suppose that* $\overline{\tau}(N) \leq 1$*, i.e.,*

$$\max_{\partial\Omega\cap\bar{N}} \frac{|\nabla u|}{\sqrt{2u_{\max}}} \leq 1.$$

Then the function U *defined in* (2.2) *is continuous and nonincreasing.*

Proof. Given a region $N \in \pi_0(\Omega \setminus \mathrm{MAX}(u))$, let us consider the domain $N_\eta = N \cap \{u < u_{\max} - \eta\}$, where $\eta \in \mathbb{R}$ is small enough so that the level set $\{u = \eta\}$ is regular. Notice that, since the critical level sets of u are discrete (see [49]), the parameter η introduced above can be chosen as small as desired. Applying the maximum principle to (2.1), we obtain that

$$\max_{N_\eta}(|\nabla u|^2 + 2u) \leq \max_{\partial N_\eta}(|\nabla u|^2 + 2u).$$

On the other hand, one has that

$$\lim_{\eta\to 0^+} \max_{\partial N_\eta}(|\nabla u|^2 + 2u) \leq 2u_{\max}.$$

In fact, $|\nabla u|^2 + 2u = |\nabla u|^2 \leq 2u_{\max}$ on $\partial\Omega \cap \bar{N}$ by hypothesis and

$$\lim_{\eta\to 0^+} \max_{\{u=u_{\max}-\eta\}} (|\nabla u|^2 + 2u) = 2u_{\max},$$

since $|\nabla u| \to 0$ as we approach $\mathrm{MAX}(u)$. It follows that

$$|\nabla u|^2 + 2u \leq 2u_{\max},$$

on the whole N. In particular, using that $\Delta u = -2$, we obtain

$$\operatorname{div}\left(\frac{\nabla u}{u_{\max} - u}\right) = \frac{|\nabla u|^2 + 2u - 2u_{\max}}{(u_{\max} - u)^2} \le 0. \tag{2.3}$$

We now integrate by parts inequality (2.3) on the finite perimeter set $\{t_1 \le u \le t_2\} \cap \overline{N}$, for some $0 \le t_1 < t_2 < u_{\max}$. Applying the divergence theorem, we deduce that

$$\begin{aligned}\int_{\{u=t_1\}\cap\overline{N}} \left\langle \frac{\nabla u}{u_{\max} - u} \,\middle|\, \mathrm{n} \right\rangle \mathrm{d}\sigma + \int_{\{u=t_2\}\cap\overline{N}} \left\langle \frac{\nabla u}{u_{\max} - u} \,\middle|\, \mathrm{n} \right\rangle \mathrm{d}\sigma \\ = \int_{\{t_1 \le u \le t_2\}\cap\overline{N}} \frac{|\nabla u|^2 + 2u - 2u_{\max}}{(u_{\max} - u)^2} \, \mathrm{d}\mu \le 0.\end{aligned} \tag{2.4}$$

Notice that the (measure theoretic) unit normal n is well defined $\mathcal{H}^1$-a.e. on $\{u = t_1\}$ and on $\{u = t_2\}$. The thesis follows noticing that

$$\begin{aligned} U(t) &= \frac{1}{(u_{\max} - t)} \int_{(\{u=t\}\cap\overline{N})\setminus \mathrm{Crit}(u)} |\nabla u| \, \mathrm{d}\sigma \\ &= \int_{(\{u=t\}\cap\overline{N})\setminus \mathrm{Crit}(u)} \left\langle \frac{\nabla u}{u_{\max} - u} \,\middle|\, \pm\mathrm{n} \right\rangle \mathrm{d}\sigma \\ &= \int_{(\{u=t\}\cap\overline{N})} \left\langle \frac{\nabla u}{u_{\max} - u} \,\middle|\, \pm\mathrm{n} \right\rangle \mathrm{d}\sigma, \end{aligned}$$

since the outer unit normal n coincides with $\pm\nabla u/|\nabla u|$ on $(\{u = t\} \cap \overline{N}) \setminus \mathrm{Crit}(u)$. The continuity of U is a straightforward consequence of the absolute continuity of the Lebesgue integral on the right-hand side of the equality in (2.4). ■

Combining Propositions 2.2 and 2.3, we are finally able to prove the main result of this section.

Proof of Theorem 2.1. We claim that if $\overline{\tau}(N) \le 1$, then $\pi_0(\Omega \setminus \mathrm{MAX}(u)) = \{N\}$. In other words, N is the only connected component of $\Omega \setminus \mathrm{MAX}(u)$, under our assumptions. Observe that such a claim implies that $\partial\Omega \cap \overline{N} = \partial\Omega$, and thus the thesis follows from Proposition 2.2. To prove the claim, we argue by contradiction. If N is not the unique connected component of $\Omega \setminus \mathrm{MAX}(u)$, then it must be separated by the other ones, and thus $\mathcal{H}^1\big(\mathrm{MAX}(u) \cap \overline{N}\big) > 0$. On the other hand, according to the classical Łojasiewicz inequality (see, e.g., [31]) and the compactness of $\mathrm{MAX}(u)$, there exist a neighborhood V of $\mathrm{MAX}(u)$ and two constants $c > 0$ and $1/2 \le \theta < 1$ such that $u_{\max} - u < 1$ in V and

$$|\nabla u(x)| \ge c(u_{\max} - u(x))^{\theta}$$

for every $x \in V$. In particular, since $\{u = t\} \cap \overline{N} \subseteq V$ for every t sufficiently close to $u_{\max}$, we have that

$$\frac{1}{u_{\max}} \int_{\partial\Omega\cap\overline{N}} |\nabla u| \, \mathrm{d}\sigma = U(0) \ge U(t) \ge \frac{1}{(u_{\max} - t)^{1-\theta}} |\{u = t\} \cap \overline{N}|,$$

where we also used the monotonicity formula proved in Proposition 2.3. It is now easy to see that if $\mathcal{H}^1(\mathrm{MAX}(u) \cap \overline{N}) > 0$, the rightmost-hand side is unbounded, as $t \to u_{\max}^-$. This gives the desired contradiction. ■

For the sake of completeness, we now show how Theorem C can be easily deduced from Theorem 2.1.

Proof of Theorem C. To check that $R(N)$ is well defined, it is sufficient to exclude that $\overline{\tau}(N) < 1$, for some $N \in \pi_0(\Omega \setminus \mathrm{MAX}(u))$. Indeed, Theorem 2.1 says that if $\overline{\tau}(N) \leq 1$, then $\overline{\tau}(N) = 1$. To prove the rigidity part, it is sufficient to observe that if $R(N) = 0$, then $\overline{\tau}(N) = 1$, and one can apply again Theorem 2.1. The statement for $R(\Omega)$ follows at once. ■

3. Gradient estimates

The aim of this section is to compare the gradient $|\nabla u|$ of a generic solution (Ω, u) to problem (1.1) with the gradient $|\nabla u_R|$ of a ring-shaped model solution (Ω_R, u_R). In order for the comparison to make sense, we need to consider suitable normalizations, as described in Section 1.3. A crucial role in this procedure will be played by the concept of *expected core radius* $R = R(N)$ of a region $N \subseteq \Omega \setminus \mathrm{MAX}(u)$, whose existence is now guaranteed by Theorem C. To be more precise, we notice that if (Ω, u) is a solution to problem (1.1), then $(\Omega_\lambda, u_\lambda)$, with

$$\Omega_\lambda = \{\lambda x : x \in \Omega\} \quad \text{and} \quad u_\lambda(x) = \lambda^2 u\left(\frac{x}{\lambda}\right), \tag{3.1}$$

is also a solution for every $\lambda > 0$. This means that we are allowed to rescale u, provided that we also apply a suitable homothety to the domain Ω. With the notation introduced in (1.5), it will be convenient to adopt the following normalization.

Normalization 3.1. *Let (Ω, u) be a solution to problem (1.1), let N be a connected component of $\Omega \setminus \mathrm{MAX}(u)$, and let $R = R(N) \in [0, 1)$ be the expected core radius associated with the region N. Up to rescaling the domain and the function as in (3.1), we assume that*

$$u_{\max} = (u_R)_{\max} = \frac{1 - R^2}{2} + R^2 \log R.$$

With this normalization in force, we are going to compare, in Theorem 3.5 below, the squared gradient of the solution $|\nabla u|^2$ on N to the squared gradient $|\nabla u_R|^2$ of the ring-shaped model solution (Ω_R, u_R) that satisfies $R = R(N)$. We agree that if $\overline{\tau}(N) < \sqrt{2}$, then the comparison will be drawn with the restriction of $|\nabla u_R|^2$ to the outer region $\Omega_{R,\mathrm{o}}$ of the model solution, otherwise the comparison will be drawn with the restriction of $|\nabla u_R|^2$ to the inner region $\Omega_{R,\mathrm{i}}$ (see (1.6)). More precisely, if p is a point in N, we are going to bound $|\nabla u|^2(p)$ with the value of $|\nabla u_R|^2$ at a point where $u_R = u(p)$, belonging to either the outer or the inner region of the model solution (Ω_R, u_R), according to what the NWSS of N dictates. To make the computations affordable, we are going to introduce, in the next subsection, the notion of *pseudo-radial function*.

3.1. The pseudo-radial functions

This subsection is aimed at defining *pseudo-radial functions*, that is, functions that mimic the behavior of the radial coordinate $|x|$ in the rotationally symmetric solutions (1.4). As above, let $N \in \pi_0(\Omega \setminus \mathrm{MAX}(u))$ and let $R = R(N)$ be its expected core radius. As in (1.4), we let $r_\mathrm{i} = r_\mathrm{i}(R) \in (0, R)$ be the smallest positive root of the function $\rho \mapsto 1 - \rho^2 + 2R^2 \log\rho$, and we define the function

$$F_R\colon [0, (u_R)_{\max}] \times [r_\mathrm{i}(R), 1] \to \mathbb{R},$$
$$(u, \psi) \mapsto F_R(u, \psi) := 2u - 1 + \psi^2 - 2R^2 \log\psi.$$

A simple computation shows that $\partial F_R/\partial\psi = 0$ if and only if $\psi = R$. As a consequence of the implicit function theorem, we have that there exist two smooth functions

$$\psi_-\colon [0, (u_R)_{\max}] \to [r_\mathrm{i}(R), R] \quad \text{and} \quad \psi_+\colon [0, (u_R)_{\max}] \to [R, 1]$$

such that

$$F_R(u, \psi_-(u)) = 0 = F_R(u, \psi_+(u))$$

for all $u \in [0, (u_R)_{\max}]$. For future convenience, let us list some elementary properties of ψ_+ and ψ_-, which can be derived easily from their definition.

- First of all, we can compute ψ_+, ψ_- and their derivatives using the following formulae:

$$u = \frac{1 - \psi_\pm^2 + 2R^2 \log\psi_\pm}{2}, \tag{3.2}$$

$$\dot\psi_\pm = -\frac{\psi_\pm}{\psi_\pm^2 - R^2}, \qquad \ddot\psi_\pm = 2\,\dot\psi_\pm^3 + \frac{\dot\psi_\pm^2}{\psi_\pm}. \tag{3.3}$$

- The function ψ_- takes values in $[r_\mathrm{i}(R), R]$, hence $\psi_-^2 \leq R^2$. Thus, from (3.3) we deduce

$$\dot\psi_- \geq 0, \qquad \ddot\psi_- \geq 0, \qquad \lim_{u \to (u_R)_{\max}^-} \dot\psi_- = +\infty.$$

- The function ψ_+ takes values in $[R, 1]$, hence $\psi_+^2 \geq R^2$. Thus, from the first formula in (3.3) we deduce that $\dot\psi_+$ is nonpositive and diverges as u approaches $(u_R)_{\max}$. Moreover, the second formula in (3.3) can be rewritten as

$$\ddot\psi_+ = \dot\psi_+^3(1 + R^2\psi_+^{-2}),$$

from which it follows $\ddot\psi_+ \leq 0$. Summing up, we have

$$\dot\psi_+ \leq 0, \qquad \ddot\psi_+ \leq 0, \qquad \lim_{u \to (u_R)_{\max}^-} \dot\psi_+ = -\infty.$$

Coming back to our case of interest, we are now going to use the functions $\psi_\pm$ in order to define a *pseudo-radial function* on N. To this end, we distinguish the cases where the NWSS of N is either above or below the threshold value $\sqrt{2}$.

Definition 3.2 (Pseudo-radial functions). Let (Ω, u) be a solution to problem (1.1), let N be a connected component of $\Omega \setminus \mathrm{MAX}(u)$, and let $R = R(N) \in [0, 1)$ be the expected core radius associated with the region N. Also, assume that Normalization 3.1 is in force.

(i) If $\overline{\tau}(N) < \sqrt{2}$, then the pseudo-radial function Ψ_+ is defined as

$$\Psi_+ \colon N \to [R, 1], \quad p \mapsto \Psi_+(p) := \psi_+(u(p)). \tag{3.4}$$

Notice that if N is the outer region $\Omega_{R,\mathrm{o}}$ of the rotationally symmetric solution (1.4) with core radius R, then for every $p \in \Omega_{R,\mathrm{o}}$, the value of $\Psi_+(p) = \psi_+(u_R(p))$ is equal to the value of the radial coordinate $|x|$ at p.

(ii) If $\overline{\tau}(N) > \sqrt{2}$, then the pseudo-radial function Ψ_- is defined as

$$\Psi_- \colon N \to [r_{\mathrm{i}}(R), R], \quad p \mapsto \Psi_-(p) := \psi_-(u(p)). \tag{3.5}$$

Notice that if N is the inner region $\Omega_{R,\mathrm{i}}$ of the rotationally symmetric solution (1.4) with core radius R, then for every $p \in \Omega_{R,\mathrm{i}}$, the value of $\Psi_-(p) = \psi_-(u_R(p))$ is equal to the value of the radial coordinate $|x|$ at p.

Remark 3.3. The threshold case $\overline{\tau}(N) = \sqrt{2}$ is not considered in the above definition. In fact, for that value one has $r_{\mathrm{i}}(R) = R = 1$, so either (3.4) or (3.5) would give us a pseudo-radial function that is just constant on the whole N and as such not interesting. The reason for this issue should be traced back to the fact that no rotationally symmetric solution (Ω_R, u_R) has a boundary component with NWSS equal to $\sqrt{2}$, hence when $\overline{\tau}(N) = \sqrt{2}$, we do not have a model to compare with. For this reason, in the future our analysis will be mostly focused on the cases $\overline{\tau}(N) < \sqrt{2}$ and $\overline{\tau}(N) > \sqrt{2}$, whereas the case $\overline{\tau}(N) = \sqrt{2}$ will be treated separately with ad hoc arguments.

In analogy with (3.3) and for future convenience, we point out that the following relationships hold true between the derivatives of the pseudo-radial function Ψ and the potential u:

$$\nabla \Psi_\pm = (\dot{\psi}_\pm \circ u) \nabla u, \quad \nabla^2 \Psi_\pm = (\dot{\psi}_\pm \circ u) \nabla^2 u + (\ddot{\psi}_\pm \circ u)\, du \otimes du. \tag{3.6}$$

Notation 3.4. In the following sections, we will perform several formal computations. In order to simplify the notations, we will avoid to indicate the subscript $\pm$, and we will simply denote by $\Psi = \psi \circ u$ the pseudo-radial function on a region N of $\Omega \setminus \mathrm{MAX}(u)$, where we understand that Ψ is defined by (3.4) if we are in an outer region and by (3.5) if we are in an inner region. When there is no risk of confusion, we will also avoid to explicitate the composition with u, namely, we will write ψ, $\dot{\psi}$ or $\ddot{\psi}$ instead of $\psi \circ u$, $\dot{\psi} \circ u$ or $\ddot{\psi} \circ u$, respectively. For instance, the formulae in (3.6) will be simply written as

$$\nabla \Psi = \dot{\psi} \nabla u, \quad \nabla^2 \Psi = \dot{\psi}\, \nabla^2 u + \ddot{\psi}\, du \otimes du.$$

From the definition of Ψ given in (3.4), (3.5), we can also easily recover an explicit formula for the comparison function $W_R = |\nabla u_R|^2 \circ \Psi$ as a function of the pseudo-radial function

$$W_R = \Big(\frac{\Psi^2 - R^2}{\Psi}\Big)^2. \tag{3.7}$$

3.2. Gradient estimates

We are now ready to state the main result of this section, in which we prove that the function $W = |\nabla u|^2$ is bounded from above by

$$W_R = |\nabla u_R|^2 \circ \Psi,$$

where $R = R(N)$ is the expected core radius of the region N, which we are considering.

Theorem 3.5 (Gradient estimates). *Let* (Ω, u) *be a solution to problem* (1.1)*, let* N *be a connected component of* $\Omega \setminus \mathrm{MAX}(u)$*, and let* $R = R(N) \in [0, 1)$ *be the expected core radius associated with the region* N*. Also assume that Normalization* 3.1 *is in force. Then it holds*

$$W \leq W_R, \quad \textit{i.e.,}\ |\nabla u| \leq |\nabla u_R| \circ \Psi$$

on the whole N*. Moreover, if* $W = W_R$ *at some point in the interior of* N*, then* (Ω, u) *coincides with the ring-shaped model solution with core radius* R*.*

Proof. We start by using the Bochner formula to compute the following:

$$\begin{aligned}\Delta(W - W_R) &= \Delta|\nabla u|^2 + 2\Big(1 + \frac{R^2}{\Psi^2}\Big)\Delta u - \frac{4R^2}{\Psi^2(R^2 - \Psi^2)}|\nabla u|^2 \\ &= 2|\nabla^2 u|^2 - 4\Big(1 + \frac{R^2}{\Psi^2}\Big) - \frac{4R^2}{\Psi^2(R^2 - \Psi^2)}|\nabla u|^2. \end{aligned} \tag{3.8}$$

The next step is to find a suitable estimate for $|\nabla^2 u|^2$. It turns out that the one we need is obtained from the following quantity:

$$\Big|\nabla^2 u + \frac{2R^2}{(R^2 - \Psi^2)^2}\, \mathrm{d}u \otimes \mathrm{d}u + \Big(1 - \frac{R^2}{(R^2 - \Psi^2)^2}|\nabla u|^2\Big) g_{\mathbb{R}^2}\Big|^2 \geq 0.$$

Computing explicitly the squared norm and isolating the term $|\nabla^2 u|^2$, we obtain the estimate

$$\begin{aligned}|\nabla^2 u|^2 \geq &- \frac{2R^2}{(R^2 - \Psi^2)^2}\langle \nabla(W - W_R) | \nabla u\rangle \\ &+ 2\Big[1 + \frac{2R^4}{\Psi^2(R^2 - \Psi^2)^2}|\nabla u|^2 - \frac{R^4}{(R^2 - \Psi^2)^4}|\nabla u|^4\Big].\end{aligned}$$

We can see why this estimate is the appropriate one for our purposes by plugging it in (3.8), as by doing so we obtain an elliptic inequality for the quantity $W - W_R$:

$$\begin{aligned}\Delta(W - W_R) \geq &- \frac{4R^2}{(R^2 - \Psi^2)^2}\langle \nabla(W - W_R) | \nabla u\rangle \\ &+ \frac{4R^2}{(R^2 - \Psi^2)^2}\Big[1 - \frac{R^2}{(R^2 - \Psi^2)^2}|\nabla u|^2\Big](W - W_R).\end{aligned}$$

Unfortunately, this is not yet enough, as we do not have a sign for the coefficient of the zero order term. For this reason, we then consider the new function $F_\beta = \beta\,(W - W_R)$, where $\beta = \beta(\Psi) > 0$. A simple computation gives

$$\begin{aligned}\Delta F_\beta \geq &-\frac{2\Psi}{R^2-\Psi^2}\Big[\frac{\beta'}{\beta} - \frac{2R^2}{\Psi(R^2-\Psi^2)}\Big]\langle \nabla F_\beta | \nabla u\rangle \\ &-\frac{2\Psi}{R^2-\Psi^2}\Big[\frac{\beta'}{\beta} - \frac{2R^2}{\Psi(R^2-\Psi^2)}\Big] F_\beta \\ &+\frac{\Psi^2|\nabla u|^2}{(R^2-\Psi^2)^2}\Big[\Big(\frac{\beta'}{\beta}\Big)' - \Big(\frac{\beta'}{\beta}\Big)^2 + \frac{\Psi^2+5R^2}{\Psi(R^2-\Psi^2)}\frac{\beta'}{\beta} - \frac{4R^4}{\Psi^2(R^2-\Psi^2)^2}\Big]F_\beta,\end{aligned}$$

where we have used $'$ to denote the differentiation with respect to Ψ. We now need to find a function β such that the coefficients of the zero order terms have the right sign. A good choice is to set

$$\frac{\beta'}{\beta} = \frac{2R^2}{\Psi(R^2-\Psi^2)},$$

which corresponds to choosing

$$\beta = \frac{\Psi}{\sqrt{W_R}} = \frac{\Psi^2}{|R^2-\Psi^2|}.$$

With this choice, F_β satisfies

$$\Delta F_\beta - \frac{8R^2\Psi^2}{(R^2-\Psi^2)^4}|\nabla u|^2 F_\beta \geq 0.$$

It follows that F_β satisfies the maximum principle in N. Since $W \leq W_R$ on the horizon with maximum surface gravity, we have

$$F_\beta \leq 0 \quad \text{on } \Gamma_N.$$

To see the behaviour of F_β near $\mathrm{MAX}(u)$, we rewrite that quantity as

$$F_\beta = \beta(W - W_R) = \Psi\frac{W}{\sqrt{W_R}} - \Psi\sqrt{W_R}.$$

Notice that W and W_R go to zero as we approach $\mathrm{MAX}(u)$. Furthermore, using the expansion proved in Lemma A.1 below, we have that for every $p \in \mathrm{MAX}(u) \cap \overline{N}$,

$$\lim_{x\to p,\, x\in N} \frac{W}{\sqrt{W_R}} = \lim_{x\to p,\, x\in N} \frac{|\nabla u|^2}{2\sqrt{u_{\max} - u}}.$$

We now apply the reverse Łojasiewicz inequality [6, Theorem 2.2] to conclude that the limit on the right-hand side is zero. Therefore, F_β tends to zero as we approach $\mathrm{MAX}(u)$. The maximum principle then implies that $F_\beta \leq 0$ (equivalently, $W \leq W_R$) on the whole N. Furthermore, if the equality $W = W_R$ holds at one point p in the interior of N, then, applying the strong maximum principle in a neighborhood of p, we deduce that $W = W_R$ on the whole N, providing the desired rigidity statement. ∎

4. Curvature bounds and comparison geometry

In this section, we exploit the analysis of the previous sections to prove Theorems D and B.

4.1. Proof of Theorem D

In this section, we are going to exploit the gradient estimates proved in Theorem 3.5 to deduce in Proposition 4.1 some geometric *a priori* bounds on the curvature of the boundary components of Ω. Analogous results are then obtained in Proposition 4.2 for the curvature of the top stratum of $\mathrm{MAX}(u)$, whenever it is present. Building on the latter, we will then give a proof of Theorem D. We start with the curvature bounds that are taking place at the boundary components of Ω.

Proposition 4.1. *Let (Ω, u) be a solution to problem (1.1), let N be a connected component of $\Omega \setminus \mathrm{MAX}(u)$, and let $R = R(N) \in [0, 1)$ be the expected core radius associated with the region N. Also assume that Normalization 3.1 is in force. Then, at any point $p \in \partial\Omega \cap \overline{N}$ where*

$$|\nabla u|(p) = \max_{\partial\Omega\cap\overline{N}} |\nabla u|,$$

it holds

$$\kappa(p) \leq 1 \ \text{ if } \overline{\tau}(N) < \sqrt{2} \quad \text{and} \quad \kappa(p) \leq -\frac{1}{r_{\mathrm{i}}(R)} \ \text{ if } \overline{\tau}(N) > \sqrt{2}.$$

Here, $\kappa(p)$ is the curvature of $\partial\Omega$ at p, computed with respect to the exterior unit normal.

Proof. Let $p \in \partial\Omega \cap \overline{N}$ be a point as in the statement, and let $\gamma(s) = p + (\nabla u/|\nabla u|)(p)s$. In other words, γ is a unit speed straight segment starting at p and pointing towards the interior of N. If W and W_R are the functions defined as in the *incipit* of Section 3.2, it is readily checked that $W(p) = W_R(p)$. The Taylor expansion of W along γ gives

$$W(\gamma(s)) = W(p) - 2[(2 - \kappa(p)\sqrt{W(p)})\sqrt{W(p)}]s + o(s),$$

where we used the identity

$$\kappa(p) = \frac{\nabla^2 u|_p(\nabla u, \nabla u) - |\nabla u|^2 \Delta u(p)}{|\nabla u|^3(p)} = \frac{2|\nabla u|^2 + \nabla^2 u|_p(\nabla u, \nabla u)}{|\nabla u|^3(p)},$$

the curvature κ being computed with respect to the exterior unit normal $-\nabla u/|\nabla u|$. To obtain the expansion of W_R along γ, it is convenient to make use of (3.7). This leads to

$$\begin{aligned} W_R(\gamma(s)) &= W_R(p) + \Big\langle \nabla W_R(p) \,\Big|\, \frac{\nabla u}{|\nabla u|}(p) \Big\rangle s + o(s) \\ &= W_R(p) + \Big[\frac{\partial}{\partial \Psi}\Big(\frac{\Psi^2 - R^2}{\Psi}\Big)^2(p)\dot{\psi}(0)|\nabla u|(p)\Big]s + o(s) \\ &= W_R(p) - 2\Big[\Big(\frac{\Psi^2(p) + R^2}{\Psi^2(p)}\Big)\sqrt{W(p)}\Big]s + o(s). \end{aligned}$$

To compare the two expansions, we recall that $W(p) = W_R(p)$ and $W \leq W_R$ in N by Theorem 3.5. It follows that

$$2 - \kappa(p)\sqrt{W_R(p)} \geq \frac{\Psi^2(p) + R^2}{\Psi^2(p)},$$

which can be rewritten as

$$\kappa(p)\frac{|\Psi^2(p) - R^2|}{\Psi(p)} \leq \frac{\Psi^2(p) - R^2}{\Psi^2(p)}.$$

Now, according to definition (3.4), if $\overline{\tau}(N) < \sqrt{2}$, then $\Psi(p) = \Psi_+(p) = \psi_+(0) = 1$ and thus $\kappa(p) \leq 1$. On the other hand, according to definition (3.5), if $\overline{\tau}(N) > \sqrt{2}$, then $\Psi(p) = \Psi_-(p) = \psi_-(0) = r_{\mathrm{i}}(R)$, so that $\kappa(p) \leq -1/r_{\mathrm{i}}(R)$. This completes the proof of the proposition. ∎

We now pass to the curvature bounds that are taking place at the top stratum of $\mathrm{MAX}(u) \cap \overline{N}$. Notice that, while the previous results (Proposition 4.1, as well as Theorem 3.5) required $R(N) \neq 1$ in order to rule out the critical case $\overline{\tau}(N) = \sqrt{2}$, in the next result we will be able to address that special case as well with an argument based on a limiting procedure in the final part of the proof.

Proposition 4.2. *Let* (Ω, u) *be a solution to problem* (1.1)*, let* N *be a connected component of* $\Omega \setminus \mathrm{MAX}(u)$*, and let* $R = R(N) \in [0, 1]$ *be the expected core radius associated with the region* N*. Also assume that Normalization 3.1 is in force and denote by* Σ *the* 1*-dimensional top stratum of* $\mathrm{MAX}(u) \cap \overline{N}$*. Then, at any point* $p \in \Sigma$*, it holds*

$$\kappa(p) \leq -\frac{1}{R} \ \text{ if } \overline{\tau}(N) < \sqrt{2} \quad \text{and} \quad \kappa(p) \leq \frac{1}{R} \ \text{ if } \overline{\tau}(N) \geq \sqrt{2},$$

where $\kappa(p)$ *is the curvature of* Σ *at* p*, computed with respect to the exterior*[1] *unit normal.*

Proof. Lemma A.2 provide us with the following Taylor expansions:

$$W = 4r^2[1 + \kappa(p)r] + \mathcal{O}(r^4),$$

$$W_R = 4r^2\Big[1 + \Big(\frac{\kappa(p)}{3} - \frac{2}{3R}\Big)r\Big] + \mathcal{O}(r^4) \quad \text{if } \overline{\tau}(N) < \sqrt{2},$$

$$W_R = 4r^2\Big[1 + \Big(\frac{\kappa(p)}{3} + \frac{2}{3R}\Big)r\Big] + \mathcal{O}(r^4) \quad \text{if } \overline{\tau}(N) > \sqrt{2},$$

where $\kappa(p)$ is the curvature of Σ at p computed with respect to the exterior unit normal, and $r(x) = \mathrm{dist}(x, \Sigma)$ denotes the distance from Σ.[2] Combining the above expansions

[1] Exterior to N.

[2] We agree that $r(x) > 0$ for every $x \in N$.

with the gradient estimate $W \le W_R$ obtained in Theorem 3.5, it is immediate to deduce that

$$\kappa(p) \le \frac{\kappa(p)}{3} - \frac{2}{3R} \ \text{ if } \overline{\tau}(N) < \sqrt{2} \quad \text{and} \quad \kappa(p) \le \frac{\kappa(p)}{3} + \frac{2}{3R} \ \text{ if } \overline{\tau}(N) > \sqrt{2}.$$

This concludes the proof for $\overline{\tau}(N) \neq \sqrt{2}$. The case $\overline{\tau}(N) = \sqrt{2}$ can be obtained by a limiting procedure: one just treats N as if it were $\overline{\tau}(N) > \sqrt{2}$, defining the pseudo-radial function as in (3.5), with respect to an expected core radius $R_\varepsilon = 1 - \varepsilon$. By construction, we have $W < W_{R_\varepsilon}$ on $\partial\Omega \cap \overline{N}$. Retracing then the proof of Theorem 3.5, one can easily check that the gradient estimate $W \le W_{R_\varepsilon}$ is still in force in the whole N. Hence, proceeding as above, we obtain the inequality $\kappa(p) \le 1/R_\varepsilon$. Letting $\varepsilon \to 0$, we deduce the desired bound. ∎

The following theorem can be seen as the *prelude* to the proof of Theorem D, and it shows as, in a ring-shaped domain, the notions of expected core radii can be fruitfully employed to deduce a sharp and rigid pinching estimate on the curvature of the top stratum of MAX(u) (see Figure 4).

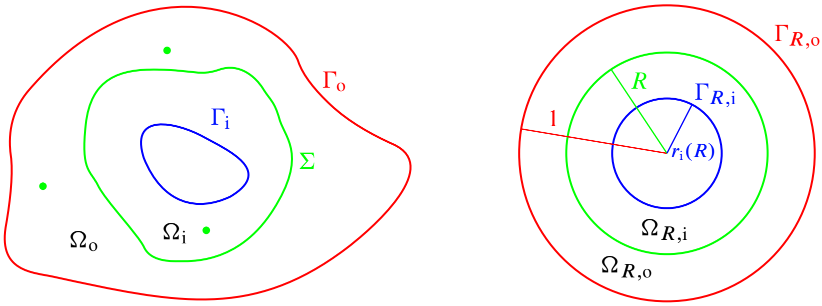

Fig. 4. On the left is a generic example of a ring shaped domain satisfying the hypotheses of Theorem 4.3. The statement of Theorem 4.3 is that under the additional assumption $R_o = R_i$, then necessarily the domain is rotationally symmetric, shaped as the picture on the right.

Theorem 4.3. *Let Ω be a ring-shaped domain, let (Ω, u) be a solution to problem (1.1), and, according to (1.9), let $\partial\Omega = \Gamma_i \sqcup \Gamma_o$, where Γ_i and Γ_o denote the inner and the outer connected components of the boundary of Ω, respectively. Assume that there exists a simple closed curve $\Sigma \subseteq$ MAX(u) separating Ω into two regions Ω_i and Ω_o, with $\partial\Omega \cap \overline{\Omega}_i = \Gamma_i$ and $\partial\Omega \cap \overline{\Omega}_o = \Gamma_o$. Also assume that $\overline{\tau}(\Omega_o) < \sqrt{2}$. Then, at any point $p \in \Sigma$, it holds*

$$\frac{\sqrt{(u_{R_o})_{\max}}}{R_o} \le \kappa(p)\sqrt{u_{\max}} \le \frac{\sqrt{(u_{R_i})_{\max}}}{R_i}, \tag{4.1}$$

where $\kappa(p)$ is the curvature of Σ computed with respect to the unit normal pointing outside Ω_i (equiv. inside Ω_o), and, according to Definition 1.4, $R_i = R(\Gamma_i)$ and $R_o = R(\Gamma_o)$

are the expected core radii of Γ_{i} *and* Γ_{o}*, respectively. In particular, we have that* $R_{\mathrm{o}} \geq R_{\mathrm{i}}$*, and the equality holds if and only if* (Ω, u) *is equivalent to the ring-shaped model solution whose core radius is given by the common value of the two expected core radii.*

Proof. Let us start from the analysis of the inner region Ω_{i}. As in the statement, let $R_{\mathrm{i}} = R(\Gamma_{\mathrm{i}}) = R(\Omega_{\mathrm{i}}) > 0$ be the expected core radius of this region. Up to considering an equivalent pair $(\Omega_{\lambda_{\mathrm{i}}}, u_{\lambda_{\mathrm{i}}})$ as in (3.1), with $\lambda_{\mathrm{i}} = \sqrt{(u_{R_{\mathrm{i}}})_{\max}/u_{\max}}$, we may assume that Normalization 3.1 is in force on the region Ω_{i}. Hence, applying Proposition 4.2 in Ω_{i}, we obtain the upper bound

$$\kappa_{\mathrm{i}} \leq -\frac{1}{R_{\mathrm{i}}} \ \text{ if } \overline{\tau}(\Gamma_{\mathrm{i}}) < \sqrt{2} \quad \text{and} \quad \kappa_{\mathrm{i}} \leq \frac{1}{R_{\mathrm{i}}} \ \text{ if } \overline{\tau}(\Gamma_{\mathrm{i}}) \geq \sqrt{2},$$

where κ_{i} is the curvature of $\lambda_{\mathrm{i}}\Sigma$, computed with respect to the unit normal pointing outside Ω_{i}. It is immediate to realize that only the second case is allowed, for if $\overline{\tau}(\Gamma_{\mathrm{i}}) < \sqrt{2}$, then the curvature of the simple closed curve $\lambda_{\mathrm{i}}\Sigma$ would be negative at each point. On the other hand, our choice of the unit normal implies that $\lambda_{\mathrm{i}}\Sigma$ is oriented in the counterclockwise direction, so that the integral of κ_{i} along $\lambda_{\mathrm{i}}\Sigma$ must be equal to $2\pi > 0$. Therefore, $\overline{\tau}(\Gamma_{\mathrm{i}}) \geq \sqrt{2}$, so that, in terms of the unnormalized quantities, the valid upper bound reads

$$\kappa(p)\sqrt{u_{\max}} \leq \frac{\sqrt{(u_{R_{\mathrm{i}}})_{\max}}}{R_{\mathrm{i}}}. \tag{4.2}$$

A similar argument, based on Proposition 4.2, leads to the desired lower bound

$$\frac{\sqrt{(u_{R_{\mathrm{o}}})_{\max}}}{R_{\mathrm{o}}} \leq \kappa(p)\sqrt{u_{\max}}. \tag{4.3}$$

The only relevant difference is that, when working in the outer region Ω_{o}, one cannot exclude a priori the case $\overline{\tau}(\Omega_{\mathrm{o}}) \geq \sqrt{2}$. This motivates the assumption $\overline{\tau}(\Omega_{\mathrm{o}}) < \sqrt{2}$ in the statement of the theorem. Combining (4.2) and (4.3), we obtain (4.1) and the fact that $R_{\mathrm{o}} \geq R_{\mathrm{i}}$ follows immediately from the fact that $R \mapsto \sqrt{(u_R)_{\max}}/R$ is nonincreasing.

If $R_{\mathrm{i}} = R = R_{\mathrm{o}}$ for some $0 < R < 1$, then the curvature of Σ is necessarily constant, and up to normalizing everything so that $\sqrt{u_{\max}} = \sqrt{(u_R)_{\max}}$, we have that $\kappa \equiv 1/R$ on the whole Σ. It follows that Σ is a round circle of radius R. Now we observe that, in a neighborhood U of Σ, our function u solves the following initial value problem:

$$\begin{cases} \Delta u = -2 & \text{in } U, \\ u = (u_R)_{\max} & \text{on } \Sigma, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Sigma. \end{cases} \tag{4.4}$$

The coefficients appearing in the above problem are clearly analytic, and Σ is a noncharacteristic curve for $\Delta u = -2$, as there are no characteristic hypersurfaces for an elliptic PDE. Therefore, we can invoke the uniqueness statement in the Cauchy–Kovalevskaya theorem, applied to the initial value problem (4.4). On the other hand, since Σ is a circle of radius R, we immediately check that the ring-shaped model solution (1.4) of core radius R also satisfies (4.4). The uniqueness of the solution then implies that (Ω, u) must coincide with the model solution (1.4) in a neighborhood of Σ. From the analyticity of u, it follows that the two solutions coincide everywhere. ∎

We are now ready to prove our main comparison result, namely Theorem D.

Proof of Theorem D. We only need to argue that if Ω is a ring-shaped domain and u has infinitely many maximum points, then a simple closed curve $\Sigma \subseteq \mathrm{MAX}(u)$ as in the statement of Theorem 4.3 does actually exist. To see this, we first recall from the Łojasiewicz structure theorem [32] (see also [30, Theorem 6.3.3]) that locally the set $\mathrm{Crit}(u)$ of the critical points of a nonconstant real analytic function $u\colon \mathbb{R}^n \to \mathbb{R}$ has the structure of a real analytic sub-variety, whose strata may in principle be of any integer dimension between 0 (points) and $(n-1)$ (top stratum). Moreover, the zero-dimensional stratum is discrete and all the lower-dimensional strata lie in the topological closure of the top stratum, whenever the latter is nonempty. In particular, in the case under consideration, we have that the zero-dimensional stratum of $\mathrm{MAX}(u) \subset \mathrm{Crit}(u)$ must be finite, so that the (1-dimensional) top stratum is necessarily nonempty, as u has infinitely many maximum points. It turns out that the top stratum of $\mathrm{MAX}(u)$ enjoys further regularity properties in the present context. In fact, as proved in [6, Corollary 3.4], if Σ is a connected component of the top stratum and $\nabla^2 u$ is nowhere vanishing in $\overline{\Sigma}$, then $\Sigma = \overline{\Sigma}$, and Σ is a real analytic simple closed curve. In particular, $\mathrm{MAX}(u)$ is given by a finite number of isolated points and a finite number of isolated simple closed curves. An elementary application of the strong maximum principle shows that the only possibility is that the top stratum of $\mathrm{MAX}(u)$ is given by one single simple closed curve Σ dividing the ring-shaped domain Ω into two region Ω_{i} and Ω_{o} as the ones described in the statement of Theorem 4.3. This latter can now be invoked to complete the proof of Theorem D. ∎

4.2. *Proof of Theorem* B

We conclude this section by discussing the proof of Theorem B.

Proof. Suppose we are not in the case (ii), so that the function u has infinitely many maximum points. The first step is that of showing that, in this situation, the set $\mathrm{MAX}(u)$ consists of a finite number of points and a single closed curve separating Ω in two regions Ω_{i} and Ω_{o}, with $\partial\Omega \cap \overline{\Omega}_{\mathrm{i}} = \Gamma_{\mathrm{i}}$ and $\partial\Omega \cap \overline{\Omega}_{\mathrm{o}} = \Gamma_{\mathrm{o}}$. This can be achieved by invoking [6, Corollary 3.4], as done in the proof of Theorem D above. However, we decided to employ here a different argument, which has been suggested to us by one of the referees and that we found particularly direct and elegant. It is based on [3, Corollary 3.4].

If $\mathrm{MAX}(u)$ has infinite points, then, since Ω is compact, there must be an accumulation point p for $\mathrm{MAX}(u)$. From the continuity of u, it follows immediately that $p \in \mathrm{MAX}(u)$. Furthermore, since $\Delta u = -2 \neq 0$, we can suppose without loss of generality that $\partial^2_{xx} u \neq 0$ at the point p. We can then exploit the real analytic implicit function theorem (see [30, Theorem 2.3.5]) to deduce that the set $\{\partial_x u = 0\}$ is a simple real analytic curve in a neighborhood of p. The restriction of u to the curve $\{\partial_x u = 0\}$ is also real analytic. Furthermore, since $\mathrm{MAX}(u)$ is contained in $\{\partial_x u = 0\}$ and p is an accumulation point for $\mathrm{MAX}(u)$, it follows that there is a sequence of points $p_{\mathrm{i}} \to p$ belonging to the curve $\{\partial_x u = 0\}$ with $u(p_{\mathrm{i}}) = u(p) = u_{\max}$. By analytic continuation, we conclude that u is constant on the curve, hence all points in the curve belong to $\mathrm{MAX}(u)$. This proves that accumulation points are necessarily part of a curve of maximum points. This is an alternative simple and direct way to show that necessarily $\mathrm{MAX}(u)$ is the union of a finite number of points and a finite number of closed curves. It is then easy (as mentioned in the proof of Theorem D) to see that if $|\mathrm{MAX}(u)| = +\infty$, there must exist a closed curve Σ separating Ω in two regions Ω_{i} and Ω_{o}, with $\partial\Omega \cap \overline{\Omega}_{\mathrm{i}} = \Gamma_{\mathrm{i}}$ and $\partial\Omega \cap \overline{\Omega}_{\mathrm{o}} = \Gamma_{\mathrm{o}}$, as wished.

If $|\nabla u|$ is constant on Γ_{o}, then u solves the problem

$$\begin{cases} \Delta u = -2 & & \text{in } \Omega_{\mathrm{o}}, \\ u = 0, & \dfrac{\partial u}{\partial \nu} = c & \text{on } \Gamma_{\mathrm{o}}, \\ u = u_{\max}, & \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Sigma. \end{cases}$$

We can then invoke Theorem 1.1, which tells us that u is rotationally symmetric in Ω_{o}. In particular, Σ is a round circle, and we can conclude as in the proof of Theorem 4.3 that (Ω, u) is rotationally symmetric. If instead $|\nabla u|$ is constant on Γ_{i}, then u solves

$$\begin{cases} \Delta u = -2 & & \text{in } \Omega_{\mathrm{i}}, \\ u = 0, & \dfrac{\partial u}{\partial \nu} = c & \text{on } \Gamma_{\mathrm{i}}, \\ u = u_{\max}, & \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Sigma. \end{cases}$$

The solution to the above system does not fit the hypothesis of Theorem 1.1 straight away. However, $v = u_{\max} - u$ fits the assumptions of [48, Theorem 2], where a wider class of equations is considered. Hence, we deduce that Σ is a round circle and in turn that (Ω, u) is rotationally symmetric. ■

We conclude with a couple of comments. First of all, the arguments above generalize easily to higher dimensions $n \geq 3$. Indeed, one can give an argument in the spirit of [3] or exploit [6, Corollary 3.4] to check that again the set $\mathrm{MAX}(u)$ contains a hypersurface separating Γ_{o} and Γ_{i}, provided

$$\mathcal{H}^{n-1}(\mathrm{MAX}(u)) > 0.$$

One can then apply Sirakov's result, that holds in every dimension, to prove rotational symmetry.

As mentioned in the introduction, Theorem 1.1, and hence the proposed proof of Theorem B, uses the moving plane method. An alternative proof of Theorem B relying on the Pohozaev identity and the isoperimetric inequality is given instead in [5]. This alternative argument may be of interest when trying to generalize Theorem B to ambient spaces that are less symmetric than the Euclidean space.

5. Location of the hot spots

In this section, we first prove a sharp estimate for the distance of a hot spot (i.e., a maximum point of u) to $\partial\Omega \cap \overline{N}$, where N is a given connected component of $\Omega \setminus \mathrm{MAX}(u)$, and u is a solution to problem (1.1). This is the content of Theorem 5.1 below. Building on this result, we deduce, in Theorem 5.2, an estimate for the scaling-invariant ratio between the distance of the hot spot to $\partial\Omega$ and the inradius r_Ω of Ω, in the spirit of [35]. Combining the latter result with Proposition 4.1, we finally obtain, in the mean convex case, the lower bound object of Theorem E, that is,

$$\frac{d(x, \partial\Omega)}{r_\Omega} \geq \frac{1 - R}{\sqrt{1 - R^2 + 2R^2 \log R}},$$

where $R = R(\Omega)$ is the expected core radius of Ω introduced in Definition 1.4.

Let us start with an observation which will be useful in the proof of Theorem 5.1. For a given connected component N of $\Omega \setminus \mathrm{MAX}(u)$, let us fix $x, y \in \overline{N}$, and let $\gamma\colon [0,1] \to \overline{N}$ be a smooth constant-speed curve starting at $x = \gamma(0)$ and ending at $y = \gamma(1)$, with $\gamma((0,1)) \subset N$. Then

$$\Psi(y) - \Psi(x) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\Psi(\gamma(t))\,\mathrm{d}t = \int_0^1 \langle \nabla\Psi \mid \dot\gamma\rangle|_{\gamma(t)}\,\mathrm{d}t = \int_0^1 \dot\psi\langle \nabla u \mid \dot\gamma\rangle|_{\gamma(t)}\,\mathrm{d}t,$$

where $\Psi(p) = \psi(u(p))$ is the pseudo-radial function of Definition 3.2. From the Cauchy–Schwarz inequality and the gradient estimate $W \leq W_R$ proved in Theorem 3.5, we deduce

$$|\langle \nabla u \mid \dot\gamma\rangle| \leq |\dot\gamma||\nabla u| = L(\gamma)\sqrt{W} \leq L(\gamma)\sqrt{W_R}.$$

Since by construction $|\dot\psi| = 1/\sqrt{W_R}$, we get

$$|\Psi(y) - \Psi(x)| \leq L(\gamma). \tag{5.1}$$

Furthermore, if the equality holds, then $W = W_R$ along γ. In this case, the solution must be rotationally symmetric by the rigidity statement of Theorem 3.5.

Theorem 5.1. *Let* (Ω, u) *be a solution to problem* (1.1)*, let* N *be a connected component of* $\Omega \setminus \mathrm{MAX}(u)$*, and let* $\overline{\tau}(N)$ *and* $R = R(N) \in [0, 1)$ *be the NWSS and expected core radius associated with* N*, respectively. Also, assume that Normalization* 3.1 *is in force. Then, for any* $x \in \mathrm{MAX}(u) \cap \overline{N}$*, the following holds:*

- *if* $\overline{\tau}(N) < \sqrt{2}$*, then* $d(x, \partial\Omega \cap \overline{N}) \geq 1 - R$*; moreover, the equality holds if and only if* (Ω, u) *coincides with the ring-shaped model solution* (1.4) *with core radius* R*;*
- *if* $\overline{\tau}(N) > \sqrt{2}$*, then* $d(x, \partial\Omega \cap \overline{N}) \geq R - r_{\mathrm{i}}(R)$*; moreover, the equality holds if and only if* (Ω, u) *coincides with the ring-shaped model solution* (1.4) *with core radius* R*.*

Proof. Let y be a point realizing the distance between x and $\partial\Omega \cap \overline{N}$. Let us first assume that the segment γ joining x to y is entirely contained in N, except for $x = \gamma(0)$ and

$y = \gamma(1)$. We can then apply the above formula (5.1), where $L(\gamma) = d(x, \partial\Omega \cap \overline{N})$. Moreover, by definition, we have $\Psi(x) = R$ and either $\Psi(y) = 1$ or $\Psi(y) = r_{\mathrm{i}}(R)$, depending on whether $\overline{\tau}(N) < \sqrt{2}$ or $\overline{\tau}(N) > \sqrt{2}$. Hence, the thesis follows.

It remains to discuss the case where the segment $\gamma\colon [0,1] \to \mathbb{R}^2$ from x to y is such that $\gamma((0,1))$ is not entirely contained in N. In this case, the segment $\gamma((0,1))$ must intersect $\mathrm{MAX}(u)$ at some point. It is easy to argue that there must exist a maximal value $\bar{t} \in (0,1)$ such that $\overline{x} = \gamma(\bar{t}) \in \mathrm{MAX}(u)$. Since clearly $d(\partial\Omega \cap \overline{N}, x) \geq d(\partial\Omega \cap \overline{N}, \overline{x})$, it is sufficient to prove our result for $\overline{x}$. Let us then consider a new constant speed curve $\overline{\gamma}$ such that $\overline{\gamma}(0) = \overline{x}$ and $\overline{\gamma}(1) = y$. Moreover, in view of the previous discussion, we can assume that $\overline{\gamma}((0,1))$ does not intersect $\mathrm{MAX}(u)$. Since $\overline{\gamma}((0,1))$ is connected, the only possibility is that it is all contained in a connected component of $\Omega \setminus \mathrm{MAX}(u)$. As $\overline{\gamma}(1) = y \in \partial\Omega \cap \overline{N}$, we must have $\overline{\gamma}((0,1)) \subset N$. The thesis now follows from the case previously discussed. ■

For the reader's convenience, we recall that the normalized wall shear stress NWSS of Ω is defined as

$$\overline{\tau}(\Omega) := \max\{\overline{\tau}(\Gamma) : \Gamma \in \pi_0(\partial\Omega)\}.$$

Moreover, the expected core radius of Ω is defined as

$$R(\Omega) = \begin{cases} \overline{\tau}_{\mathrm{o}}^{-1}(\overline{\tau}(\Omega)) & \text{if } \overline{\tau}(\Omega) < \sqrt{2}, \\ \overline{\tau}_{\mathrm{i}}^{-1}(\overline{\tau}(\Omega)) & \text{if } \overline{\tau}(\Omega) \geq \sqrt{2}. \end{cases}$$

Theorem 5.2. *Let (Ω, u) be a solution to problem (1.1), and let $R = R(\Omega) \in [0,1)$ and $\overline{\tau}(\Omega)$ be the expected core radius and NWSS associated with the domain Ω, respectively. Then, for any $x \in \mathrm{MAX}(u)$, the following holds:*

(i) *if $\overline{\tau}(\Omega) < \sqrt{2}$, then*

$$\frac{d(x, \partial\Omega)}{r_\Omega} \geq \frac{1 - R}{\sqrt{1 - R^2 + 2R^2 \log R}} = \frac{1}{1 + R}\,\overline{\tau}(\Omega); \tag{5.2}$$

moreover, the equality holds if and only if $R = 0$ and Ω is a ball;

(ii) *if $\overline{\tau}(\Omega) > \sqrt{2}$, then*

$$\frac{d(x, \partial\Omega)}{r_\Omega} > \frac{R - r_{\mathrm{i}}(R)}{\sqrt{1 - R^2 + 2R^2 \log R}} = \frac{r_{\mathrm{i}}(R)}{r_{\mathrm{i}}(R) + R}\,\overline{\tau}(\Omega). \tag{5.3}$$

Proof. First, we consider a region N and estimate $d(x, \partial\Omega \cap \overline{N})/r_\Omega$. Notice that the distance $d(x, \partial\Omega \cap \overline{N})/r_\Omega$ is invariant under rescaling of the domain, hence, without loss of generality we can assume that Normalization 3.1 is in force. A simple comparison argument involving Serrin's solution (1.2), as done in [35, Lemma 2.1], yields

$$u(x) \geq \frac{1}{2} d(x, \partial\Omega)^2,$$

for any $x \in \Omega$. In particular, if we choose $\overline{x}$ as one of the points realizing r_Ω, then

$$r_\Omega^2 = d(\overline{x}, \partial\Omega)^2 \leq 2\,u(\overline{x}) \leq 2\,u_{\max}.$$

In turn, using also Theorem 5.1, we get

$$\frac{d(x,\partial\Omega\cap\overline{N})}{r_\Omega} \geq \frac{d(x,\partial\Omega\cap\overline{N})}{\sqrt{2u_{\max}}} \geq \frac{1-R(N)}{\sqrt{2u_{\max}}}. \tag{5.4}$$

Now, rewriting the right-hand side of the latter inequality via relations (1.12), (1.13), (1.16), and (1.17), we obtain

$$\begin{aligned}
\frac{d(x,\partial\Omega\cap\overline{N})}{r_\Omega} &\geq \frac{1-R(N)}{\sqrt{1-R(N)^2+2R(N)^2\log R(N)}} \\
&= \frac{\overline{\tau}(N)}{1+R(N)} \quad \text{if } \overline{\tau}(N) < \sqrt{2}, \\
\frac{d(x,\partial\Omega\cap\overline{N})}{r_\Omega} &\geq \frac{R(N)-r_{\mathrm{i}}(R(N))}{\sqrt{1-R(N)^2+2R(N)^2\log R(N)}} \\
&= \frac{r_{\mathrm{i}}(R(N))\overline{\tau}(N)}{r_{\mathrm{i}}(R(N))+R(N)} \quad \text{if } \overline{\tau}(N) > \sqrt{2}.
\end{aligned} \tag{5.5}$$

If there is only one region (that is, $\Omega \setminus \mathrm{MAX}(u)$ is connected), then estimates (5.5) are the ones we were looking for. Let us now discuss the case where there is more than one region, in which case

$$\frac{d(x,\partial\Omega)}{r_\Omega} = \min_{N\in\pi_0(\Omega\setminus\mathrm{MAX}(u))} \frac{d(x,\partial\Omega\cap\overline{N})}{r_\Omega}. \tag{5.6}$$

Assume first that $\overline{\tau}(\Omega) < \sqrt{2}$. Therefore, all regions must have NWSS $< \sqrt{2}$ and from the first inequality in (5.5) we get

$$\frac{d(x,\partial\Omega)}{r_\Omega} \geq \min_{N\in\pi_0(\Omega\setminus\mathrm{MAX}(u))} \frac{1-R(N)}{\sqrt{1-R(N)^2+2R(N)^2\log R(N)}}.$$

The function on the right-hand side is decreasing in $R(N)$ (see Figure 3), hence the minimum is realized in correspondence of the maximum among the expected core radii. In turn, since in the regime NWSS $< \sqrt{2}$ the maximum expected core radius is realized by the regions with maximum NWSS, and since $\overline{\tau}(\Omega) = \max_N \overline{\tau}(N)$ by definition, estimate (5.2) is proved. Let us now discuss the rigidity case and suppose that the equality holds in (5.2). Then (5.4) and the above discussion yield

$$\frac{d(x,\partial\Omega\cap\overline{N})}{r_\Omega} = \frac{d(x,\partial\Omega\cap\overline{N})}{\sqrt{2u_{\max}}} = \frac{1-R(N)}{\sqrt{2u_{\max}}},$$

where N is the region realizing $\overline{\tau}(\Omega)$. In particular, the second identity tells us that the rigidity case in Theorem 5.1 applies, so that (Ω, u) must be rotationally symmetric. At the same time, the first identity says that $r_\Omega = \sqrt{2u_{\max}}$. It is now easy to check that the only rotationally symmetric solution with $r_\Omega = \sqrt{2u_{\max}}$, up to translation and rescaling, is Serrin's solution (1.2).

Turning to point (ii), suppose now that there is at least one region with $\overline{\tau}(N) > \sqrt{2}$. Our estimate for regions with $\overline{\tau}(N) > \sqrt{2}$ is worse (see Figure 3), therefore from (5.6) we have

$$\frac{d(x, \partial\Omega)}{r_\Omega} \geq \min_{\substack{N \in \pi_0(\Omega \setminus \mathrm{MAX}(u)) \\ \overline{\tau}(N) > \sqrt{2}}} \frac{R(N) - r_{\mathrm{i}}(R(N))}{\sqrt{1 - R(N)^2 + 2R(N)^2 \log R(N)}}.$$

The function on the right-hand side is increasing in $R(N)$ (see again Figure 3), hence the minimum is realized in correspondence of the minimum among the expected core radii. At the same time, in the regime NWSS $\geq \sqrt{2}$, the minimum expected core radius is realized by the regions with maximum NWSS. Since again

$$\overline{\tau}(\Omega) = \max_N \overline{\tau}(N),$$

we obtain that

$$\frac{d(x, \partial\Omega)}{r_\Omega} \geq \frac{r_{\mathrm{i}}(R)}{r_{\mathrm{i}}(R) + R} \overline{\tau}(\Omega) = \frac{R - r_{\mathrm{i}}(R)}{\sqrt{1 - R^2 + 2R^2 \log R}}.$$

Finally, observe that the equality cannot be achieved in the above inequality. Indeed, if the equality holds, arguing as done in point (i) one would deduce that u coincides with Serrin's solution (1.2), up to translation and rescaling. But this solution has NWWS equal to 1, which contradicts $\overline{\tau}(\Omega) > \sqrt{2}$. This completes the proof of estimate (5.3). ∎

Proof of Theorem E. The key observation here is that the mean convexity of $\partial\Omega$ coupled with the estimates on the curvature of Ω in Proposition 4.1 implies that $\overline{\tau}(\Omega) < \sqrt{2}$. It then suffices to apply Theorem 5.2 (i). ∎

6. Proof of Theorem A

This section is dedicated to the proof of Theorem A, which, we recall, tells us that there are infinitely many solutions (Ω, u) to problem (1.1) that are not rotationally symmetric and such that Ω is a ring-shaped domain and $|\nabla u|$ is locally constant on $\partial\Omega$. In fact, we will be able to prove a far more precise statement, namely Theorem 6.1, that will give us a better picture of such exotic solutions. On top of that, it will be clear from the proof that these solutions are symmetric with respect to a group of rotations whose cardinality can be chosen to be arbitrarily large. In particular, as already mentioned in the introduction, this allows to produce solutions with an arbitrarily large number of maximum points.

The proof is a modification of the arguments given in [29], where the same result is proved for a similar problem. It will however be quite clear from our proof that there are several technical complications that make the computations much more delicate in our case. In an attempt to keep the presentation as clear-cut as possible, we will try to refer to [29] whenever possible, stressing only the main differences. To this aim, in order to have a notation as similar as possible to the one in [29], we will use the notation $\lambda = r_{\mathrm{i}}(R) \in (0, 1)$, and we will consider R as a function of λ. Notice in fact that λ, R are related by

$$1 - \lambda^2 + 2R^2 \log \lambda = 0, \quad \text{or equivalently} \quad R^2 = \frac{1-\lambda^2}{-2\log\lambda}. \tag{6.1}$$

For any $\lambda \in (0, 1)$, we denote by

$$\Omega_\lambda = \{\lambda < |x| < 1\}, \quad u_\lambda = \frac{1-|x|^2}{2} + R^2 \log |x|$$

the rotationally symmetric solution with core radius R, by $\Gamma_\lambda = \{|x| = \lambda\}$, $\Gamma_1 = \{|x| = 1\}$ the two connected components of $\partial\Omega$, and by $c^{\mathrm{i}}_\lambda = R^2/\lambda - \lambda$, $c^{\mathrm{o}}_\lambda = 1 - R^2$ the constant value of $|\nabla u_\lambda|$ on Γ_λ and Γ_1, respectively.

Let now $\alpha \in (0, 1)$ and $\mathbf{v} = (v_1, v_2) \in (\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2$. Consider the domain

$$\Omega^{\mathbf{v}}_\lambda = \left\{\lambda + v_1\Big(\frac{x}{|x|}\Big) < |x| < 1 - v_2\Big(\frac{x}{|x|}\Big)\right\},$$

and let $u^{\mathbf{v}}_\lambda \colon \Omega^{\mathbf{v}}_\lambda \to \mathbb{R}$ be the solution to the problem

$$\begin{cases} \Delta u = -2, & \text{in } \Omega^{\mathbf{v}}_\lambda, \\ u = 0, & \text{on } \partial\Omega^{\mathbf{v}}_\lambda. \end{cases} \tag{6.2}$$

Let $\Gamma^{\mathbf{v}}_\lambda$ and $\Gamma^{\mathbf{v}}_1$ be the interior and exterior connected components of $\partial\Omega^{\mathbf{v}}_\lambda$, respectively. We can now state the main result of this section.

Theorem 6.1. *Let $\alpha \in (0, 1)$. There is a strictly increasing sequence $\{\lambda_\sigma\}_{\sigma=1}^\infty$ of positive real numbers with $\lim_{\sigma\to+\infty} \lambda_\sigma = 1$, such that for every $\sigma \in \mathbb{N}$, there exist $\varepsilon > 0$ and a smooth curve*

$$(-\varepsilon, \varepsilon) \to (\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2 \times (0, 1), \quad s \mapsto (\mathbf{v}(s), \lambda(s))$$

with $\mathbf{v}(0) = (0, 0)$, $\lambda(0) = \lambda_\sigma$, such that for every $s \in (-\varepsilon, \varepsilon)$, in the notations introduced above, the solution $u^{\mathbf{v}(s)}_{\lambda(s)}$ to (6.2) in $\Omega^{\mathbf{v}(s)}_{\lambda(s)}$ satisfies

$$|\nabla u^{\mathbf{v}(s)}_{\lambda(s)}| = c^{\mathrm{i}}_{\lambda(s)} \text{ on } \Gamma^{\mathbf{v}(s)}_{\lambda(s)} \quad \text{and} \quad |\nabla u^{\mathbf{v}(s)}_{\lambda(s)}| = c^{\mathrm{o}}_{\lambda(s)} \text{ on } \Gamma^{\mathbf{v}(s)}_1,$$

and $\Omega^{\mathbf{v}(s)}_{\lambda(s)}$ is not rotationally symmetric for any $s \neq 0$.

It is clear that this result implies Theorem A at once. The rest of the section is therefore dedicated to the proof of Theorem 6.1. Let $U \subseteq (\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2$ be a small neighborhood of $\mathbf{0} = (0, 0)$, and define the function

$$F_\lambda \colon U \to (\mathcal{C}^{1,\alpha}(\mathbb{S}^1))^2, \quad \mathbf{v} \mapsto \left(\frac{\partial u^{\mathbf{v}}_\lambda}{\partial \nu}\Big|_{\Gamma^{\mathbf{v}}_\lambda} - c^{\mathrm{i}}_\lambda, \frac{\partial u^{\mathbf{v}}_\lambda}{\partial \nu}\Big|_{\Gamma^{\mathbf{v}}_1} - c^{\mathrm{o}}_\lambda\right), \tag{6.3}$$

where ν is the inward unit normal to $\partial\Omega^{\mathbf{v}}_\lambda$.

Our aim is *to find* $\mathbf{v}$ *that is not trivial (meaning that* $\mathbf{v} \neq \mathbf{0}$ *and it is not a simple translation) and such that* $F_\lambda(\mathbf{v}) = \mathbf{0}$. We start by linearizing F_λ, that is, we consider the function

$$L_\lambda(\mathbf{w}) = \lim_{t\to 0} \frac{F_\lambda(t\,\mathbf{w})}{t}.$$

Proceeding in the same way as in [29, Proposition 3.1], we obtain the following expression for L_λ:

$$L_\lambda(\mathbf{w}) = \Big(w_1 \frac{\partial^2 u_\lambda}{\partial r^2}\Big|_{\Gamma_\lambda} - \frac{\partial \varphi_\lambda^{\mathbf{w}}}{\partial \nu}\Big|_{\Gamma_\lambda}, w_2 \frac{\partial^2 u_\lambda}{\partial r^2}\Big|_{\Gamma_1} - \frac{\partial \varphi_\lambda^{\mathbf{w}}}{\partial \nu}\Big|_{\Gamma_1} \Big),$$

where $\varphi_\lambda^{\mathbf{w}}$ is the solution to

$$\begin{cases} \Delta\varphi = 0 & \text{in } \Omega_\lambda, \\ \varphi = c_\lambda^{\mathrm{i}} w_1 & \text{on } \Gamma_\lambda, \\ \varphi = c_\lambda^{\mathrm{o}} w_2 & \text{on } \Gamma_1. \end{cases}$$

In order to obtain a more explicit formula for L_λ, it is convenient to restrict the attention to spherical harmonics. For a given integer $k \in \mathbb{N}_0$, let then $Y \in \mathcal{C}^\infty(\mathbb{S}^1)$ be a nontrivial solution to

$$\Delta_{\mathbb{S}^1} Y + k^2 Y = 0,$$

and let $W \subseteq (\mathcal{C}^\infty(\mathbb{S}^1))^2$ be the subspace generated by $(Y, 0)$ and $(0, Y)$. Finally, for a fixed $\lambda \in (0,1)$, let

$$\mathbf{e}_1 = \Big(\frac{1}{\sqrt{\lambda}} Y, 0\Big), \quad \mathbf{e}_2 = (0, Y)$$

be a base of W, orthonormal with respect to the scalar product

$$\langle \mathbf{w}, \mathbf{z}\rangle_\lambda = \lambda \int_{\mathbb{S}^1} w_1 z_1 \mathrm{d}\sigma + \int_{\mathbb{S}^1} w_2 z_2 \mathrm{d}\sigma. \tag{6.4}$$

For any element $\mathbf{w} = a\mathbf{e}_1 + b\mathbf{e}_2 \in W$, one can compute

$$\varphi_\lambda^{\mathbf{w}} = \begin{cases} \big(a \frac{c_\lambda^{\mathrm{i}}}{\sqrt{\lambda}} \frac{|x|^k - |x|^{-k}}{\lambda^k - \lambda^{-k}} + b c_\lambda^{\mathrm{o}} \frac{\lambda^k |x|^{-k} - \lambda^{-k}|x|^k}{\lambda^k - \lambda^{-k}}\big) Y & \text{if } k > 0, \\ \big(a \frac{c_\lambda^{\mathrm{i}}}{\sqrt{\lambda}} \frac{\log|x|}{\log\lambda} + b c_\lambda^{\mathrm{o}} \frac{\log\lambda - \log|x|}{\log\lambda}\big) Y & \text{if } k = 0. \end{cases}$$

In particular, $\mathrm{Im}(L_\lambda|_W) \subseteq W$ and the matrix associated to the restriction $L_\lambda|_W$ with respect to the basis $\mathbf{e}_1, \mathbf{e}_2$ can be computed as

$$M_{\lambda,k} = \widetilde{M}_{\lambda,k} - 2\,\mathrm{Id},$$

where

$$\widetilde{M}_{\lambda,k} = \begin{cases} \begin{bmatrix} \frac{R^2-\lambda^2}{\lambda^2}(k\coth\omega - 1) & -\frac{k}{\sqrt{\lambda}}(1-R^2)\frac{1}{\sinh\omega} \\ -\frac{k}{\sqrt{\lambda}} \frac{R^2-\lambda^2}{\lambda} \frac{1}{\sinh\omega} & (1-R^2)(k\coth\omega + 1) \end{bmatrix} & \text{if } k > 0, \\ \begin{bmatrix} \frac{R^2-\lambda^2}{\lambda^2}(-\frac{1}{\log\lambda} - 1) & \frac{1}{\sqrt{\lambda}}(1-R^2)\frac{1}{\log\lambda} \\ \frac{1}{\sqrt{\lambda}} \frac{R^2-\lambda^2}{\lambda} \frac{1}{\log\lambda} & (1-R^2)(-\frac{1}{\log\lambda} + 1) \end{bmatrix} & \text{if } k = 0. \end{cases}$$

Here we have denoted by ω the function satisfying $e^{\omega} = \lambda^{-k}$. While we are of course interested only in integer values of k, the matrix $M_{\lambda,k}$ makes sense for any real value of $k \geq 0$. Notice that $M_{\lambda,k}$ is analytic in both variables $(\lambda, k) \in (0,1) \times (0, +\infty)$, and it can be checked easily that $\lim_{k\to 0^+} M_{\lambda,k} = M_{\lambda,0}$, which implies that $M_{\lambda,k}$ is continuous up to $(\lambda, k) \in (0,1) \times \{0\}$.

If we denote by

$$\begin{aligned} T_{\lambda,k} &= R^2\Big(\frac{1}{\lambda^2} - 1\Big) k \coth\omega + 2 - R^2 - \frac{R^2}{\lambda^2}, \\ D_{\lambda,k} &= \Big(\frac{R^2}{\lambda^2} - 1\Big)(1 - R^2)(k^2 - 1) \end{aligned} \tag{6.5}$$

the trace and determinant of $\widetilde{M}_{\lambda,k}$, following a computation that is completely analogous to the one leading to estimate [29, (4.17)], we obtain

$$T_{\lambda,k}^2 - 4D_{\lambda,k} = \Big\{k\Big(\frac{c_\lambda^{\mathrm{i}}}{\lambda} - c_\lambda^{\mathrm{o}}\Big)\coth\omega + \Big(\frac{c_\lambda^{\mathrm{i}}}{\lambda} + c_\lambda^{\mathrm{o}}\Big)\Big\}^2 + 4k^2\frac{c_\lambda^{\mathrm{i}} c_\lambda^{\mathrm{o}}}{\lambda}\frac{1}{\sinh^2\omega} > 0. \tag{6.6}$$

As a consequence, the eigenvalues of $M_{\lambda,k}$, given by

$$\begin{aligned} \mu_1(\lambda,k) &= \frac{T_{\lambda,k} - \sqrt{T_{\lambda,k}^2 - 4D_{\lambda,k}}}{2} - 2, \\ \mu_2(\lambda,k) &= \frac{T_{\lambda,k} + \sqrt{T_{\lambda,k}^2 - 4D_{\lambda,k}}}{2} - 2, \end{aligned} \tag{6.7}$$

are distinct real numbers. Furthermore, $\mu_1(\lambda,k)$ and $\mu_2(\lambda,k)$ have the same regularity as $M_{\lambda,k}$, namely they are analytic for $(\lambda,k) \in (0,1) \times (0,+\infty)$ and continuous up to $(\lambda,k) \in (0,1) \times \{0\}$.

For $k = 1$, we can easily compute them explicitly

$$\mu_1(\lambda,1) = -2, \quad \mu_2(\lambda,1) = 0.$$

Another simple computation shows that for any $k > 1$ it holds

$$\lim_{\lambda\to 1}\mu_1(\lambda,k) = -2, \quad \lim_{\lambda\to 1}\mu_2(\lambda,k) = 0.$$

Concerning the limit when $\lambda \to 0$, we first observe that, from relation (6.1) between R and λ, it easily follows that R/λ diverges to $+\infty$ as $\lambda \to 0$. Recalling the explicit expressions (6.5) of $T_{\lambda,k}$ and $D_{\lambda,k}$, for any $k > 1$, we easily obtain the following behavior for λ close to zero:

$$\begin{aligned} T_{\lambda,k} &= \frac{R^2}{\lambda^2}(k-1) + o\Big(\frac{R^2}{\lambda^2}\Big), \\ D_{\lambda,k} &= \frac{R^2}{\lambda^2}(k^2-1) + o\Big(\frac{R^2}{\lambda^2}\Big). \end{aligned}$$

As a consequence, both $T_{\lambda,k}$ and $D_{\lambda,k}$ diverge to $+\infty$ as $\lambda \to 0$, so that in particular $\mu_2(\lambda,k) \to +\infty$ when $\lambda \to 0$. Concerning the first eigenvalue, with some easy computations we find that

$$\begin{aligned}
\lim_{\lambda\to 0} \mu_1(\lambda,k) &= \frac{1}{2}\lim_{\lambda\to 0}\left\{T_{\lambda,k}\left[1-\sqrt{1-4\frac{D_{\lambda,k}}{T^2_{\lambda,k}}}\right]\right\}-2\\
&= \frac{1}{2}\lim_{\lambda\to 0}\left\{T_{\lambda,k}\frac{4\frac{D_{\lambda,k}}{T^2_{\lambda,k}}}{1+\sqrt{1-4\frac{D_{\lambda,k}}{T^2_{\lambda,k}}}}\right\}-2\\
&= \lim_{\lambda\to 0}\frac{D_{\lambda,k}}{T_{\lambda,k}}-2 = \frac{k^2-1}{k-1}-2 = k-1>0.
\end{aligned}$$

Notice in particular that the limits of $\mu_1(\lambda,k)$ as $\lambda \to 0$ and $\lambda \to 1$ have different signs, from which it follows that, for any positive $k \in \mathbb{N}$, *there is at least one value λ_k such that*

$$\mu_1(\lambda_k,k)=0.$$

We will come back to this point later, in Proposition 6.4, where we will show that such points possess a number of crucial properties. Before stating that proposition and dealing with its proof, we need a couple of preparatory results, concerning the monotonicity of the eigenvalues with respect to λ and k.

Lemma 6.2. *The eigenvalues $\mu_1(\lambda,k)$, $\mu_2(\lambda,k)$ are monotonically increasing in k.*

Proof. We want to follow the same strategy used in [29, Lemma 4.4]. In order to do that, we need to work with a symmetric matrix. While it is true that $M_{\lambda,k}$ is not symmetric, we can easily find a symmetric matrix with the same eigenvalues, namely

$$M^S_{\lambda,k} = \widetilde{M}^S_{\lambda,k} - 2\,\mathrm{Id},$$

where

$$\widetilde{M}^S_{\lambda,k} = \begin{bmatrix} \frac{R^2-\lambda^2}{\lambda^2}(k\coth\omega - 1) & -\frac{k}{\lambda}\frac{\sqrt{1-R^2}\sqrt{R^2-\lambda^2}}{\sinh\omega} \\ -\frac{k}{\lambda}\frac{\sqrt{1-R^2}\sqrt{R^2-\lambda^2}}{\sinh\omega} & (1-R^2)(k\coth\omega+1)\end{bmatrix}.$$

It is clear that $M^S_{\lambda,k}$ has the same trace and determinant, and thus the same eigenvalues, of $M_{\lambda,k}$. Arguing as in [29, Lemma 4.4], in order to prove that $\mu_1(\lambda,k)$ and $\mu_2(\lambda,k)$ are monotonically increasing in k, it is sufficient to show that the matrix

$$\partial_k M^S_{\lambda,k} = \begin{bmatrix} \frac{R^2-\lambda^2}{\lambda^2}(\coth\omega - \frac{\omega}{\sinh^2\omega}) & \frac{\sqrt{1-R^2}\sqrt{R^2-\lambda^2}}{\lambda}(\frac{\omega\cosh\omega}{\sinh^2\omega}-\frac{1}{\sinh\omega}) \\ \frac{\sqrt{1-R^2}\sqrt{R^2-\lambda^2}}{\lambda}(\frac{\omega\cosh\omega}{\sinh^2\omega}-\frac{1}{\sinh\omega}) & (1-R^2)(\coth\omega-\frac{\omega}{\sinh^2\omega})\end{bmatrix}$$

is positive definite. This is done exactly as in [29], so we avoid to give the details. ∎

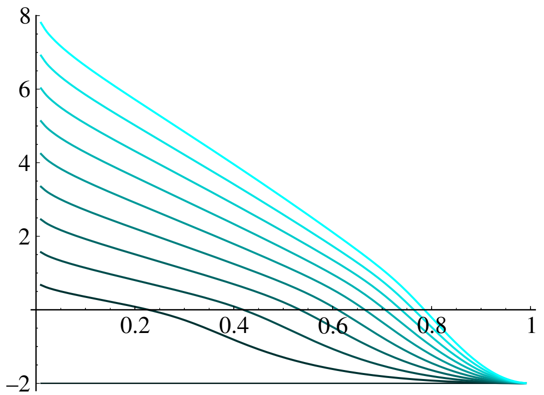

Fig. 5. Plot of the eigenvalue $\mu_1(\lambda, k)$ as a function of λ for the integers k ranging from 1 (dark blue) to 10 (light blue).

In [29], it is also proved that the first eigenvalue $\mu_1(\lambda, k)$ is monotonically decreasing in λ. This seems to hold true in our framework as well (see Figure 5), but it appears to be harder to prove, as the explicit expression for $\partial\mu_1/\partial\lambda$ is more complicated. Luckily, we do not need such a strong result, but it will be enough to show that the derivative $\partial\mu_1/\partial\lambda$ is negative at the points where μ_1 vanishes. Namely, we will need the following lemma.

Lemma 6.3. *For any $k > 1$, if λ_k is such that $\mu_1(\lambda_k, k) = 0$, then*

$$\frac{\partial\mu_1}{\partial\lambda}(\lambda_k, k) < 0.$$

Proof. Denote by $D'_{\lambda,k}$, $T'_{\lambda,k}$ the derivatives of $D_{\lambda,k}$, $T_{\lambda,k}$ with respect to λ. Recalling the explicit expression (6.7) of $\mu_1(\lambda, k)$ and differentiating, we find that $\partial\mu_1/\partial\lambda < 0$ is equivalent to

$$T'_{\lambda,k} - \frac{T_{\lambda,k}T'_{\lambda,k} - 2D'_{\lambda,k}}{\sqrt{T^2_{\lambda,k} - 4D_{\lambda,k}}} < 0,$$

which can be rewritten as

$$\left(\sqrt{T^2_{\lambda,k} - 4D_{\lambda,k}} - T_{\lambda,k}\right)T'_{\lambda,k} + 2D'_{\lambda,k} < 0. \tag{6.8}$$

Using expression (6.7) of μ_1 in terms of $T_{\lambda,k}$, $D_{\lambda,k}$, we find out that at the points where $\mu_1(\lambda, k)$ vanishes, it holds

$$\sqrt{T^2_{\lambda,k} - 4D_{\lambda,k}} = T_{\lambda,k} - 4, \quad \text{or equivalently,} \quad \frac{D_{\lambda,k}}{2} = T_{\lambda,k} - 2. \tag{6.9}$$

Therefore, when $\lambda = \lambda_k$, condition (6.8) can be rewritten as

$$2\,T'_{\lambda_k,k} - D'_{\lambda_k,k} > 0. \tag{6.10}$$

In order to write condition (6.10) more explicitly, we compute $T'_{\lambda,k}$ and $D'_{\lambda,k}$,

$$\begin{aligned} T'_{\lambda,k} &= R^2\frac{1-\lambda^2}{\lambda^3}\frac{k^2}{\sinh^2\omega} - 2\frac{k}{\lambda}\coth\omega + 2\frac{1-2R^2+\lambda^2}{\lambda(1-\lambda^2)} \\ &= \frac{R^2}{\lambda^3(1-\lambda^2)}\Big[(1-\lambda^2)^2\frac{k^2}{\sinh^2\omega} + 2(R^2-\lambda^2-1)(1-\lambda^2)k\coth\omega \\ &\qquad + 2(1-R^2-R^2\lambda^2+\lambda^4)\Big], \\ D'_{\lambda,k} &= -2R^2\frac{k^2-1}{\lambda^3(1-\lambda^2)}[(R^2-\lambda^2)^2+(1-R^2)^2]. \end{aligned}$$

We now specialize these formulae at the points where μ_1 vanishes. To this end, from (6.9) and the explicit expressions (6.5) for $T_{\lambda,k}$ and $D_{\lambda,k}$, we get

$$2R_k^2(1-\lambda_k^2)k\coth\omega_k = (R_k^2-\lambda_k^2)(1-R_k^2)k^2 + (R_k^2+\lambda_k^2)(1+R_k^2), \tag{6.11}$$

where we have used the notation $R_k = R(\lambda_k)$ and $\omega_k = \omega(\lambda_k)$ (recall that both R and ω are functions of λ). Starting from the above formulae for $T'_{\lambda,k}$ and $D'_{\lambda,k}$, recalling that $1/\sinh^2\omega = \coth^2\omega - 1$ and plugging in identity (6.11), with some computations we can rewrite condition (6.10) as

$$\begin{aligned} &4R_k^2(1-\lambda_k^2)k^3\coth\omega_k + 2(\lambda_k^2-3R_k^4-3R_k^2\lambda_k^2-3R_k^2)k^2 + 3R_k^4 \\ &\quad + R_k^2\lambda_k^2 + R_k^2 - \lambda_k^2 > 0. \end{aligned} \tag{6.12}$$

Since $R_k > \lambda_k$, the 0-th order term of (6.12) is positive. It follows that, in order for (6.12) to hold, it is sufficient to prove

$$2R_k^2(1-\lambda_k^2)k\coth\omega_k + \lambda_k^2 - 3R_k^4 - 3R_k^2\lambda_k^2 - 3R_k^2 > 0. \tag{6.13}$$

In order to prove this inequality, we first need an estimate for $k\coth\omega_k$. To obtain it, we start from (6.9) and recall (6.6) to get

$$T_{\lambda_k,k} = 4 + \sqrt{T^2_{\lambda_k,k} - 4D_{\lambda_k,k}} > 4 + \left(\frac{c^{\mathrm{i}}_{\lambda_k}}{\lambda_k} - c^{\mathrm{o}}_{\lambda_k}\right)k\coth\omega_k + \left(\frac{c^{\mathrm{i}}_{\lambda_k}}{\lambda_k} + c^{\mathrm{o}}_{\lambda_k}\right).$$

We can then exploit the explicit expression (6.5) for $T_{\lambda,k}$ to obtain, with some computations, the following estimate for $k\coth\omega_k$:

$$k\coth\omega_k > \frac{R_k^2+\lambda_k^2}{\lambda_k^2(1-R_k^2)}.$$

We are now ready to prove (6.13). Using the inequality for $k\coth\omega_k$ that we just found, we can estimate the left-hand side of (6.13) as follows:

$$\begin{aligned} &2R_k^2(1-\lambda_k^2)k\coth\omega_k + \lambda_k^2 - 3R_k^4 - 3R_k^2\lambda_k^2 - 3R_k^2 \\ &\quad > \frac{2R_k^2(1-\lambda_k^2)(R_k^2+\lambda_k^2)}{\lambda_k^2(1-R_k^2)} + \lambda_k^2 - 3R_k^4 - 3R_k^2\lambda_k^2 - 3R_k^2 \\ &\quad = \frac{2R_k^4 - R_k^2\lambda_k^2 + \lambda_k^4 - 2R_k^4\lambda_k^2 - 6R_k^2\lambda_k^4 + 3R_k^6\lambda_k^2 + 3R_k^4\lambda_k^4}{\lambda_k^2(1-R_k^2)}. \end{aligned}$$

Notice that

$$\begin{aligned} 2R_k^4 - R_k^2\lambda_k^2 + \lambda_k^4 - 2R_k^4\lambda_k^2 - 6R_k^2\lambda_k^4 + 3R_k^6\lambda_k^2 + 3R_k^4\lambda_k^4 \\ = R_k^2(1+3\lambda_k^2)(R_k^2 + R_k^2\lambda_k^2 - 2\lambda_k^2) + 3R_k^2\lambda_k^2(1-R_k^2)^2 + (R_k^2 - \lambda_k^2)^2, \end{aligned}$$

so that, since

$$R_k^2 + R_k^2\lambda_k^2 - 2\lambda_k^2 = \lambda_k^2\Big(\frac{1}{\lambda_k}c^{\mathrm{i}}_{\lambda_k} - c^{\mathrm{o}}_{\lambda_k}\Big) > \lambda_k^2(c^{\mathrm{i}}_{\lambda_k} - c^{\mathrm{o}}_{\lambda_k}) > 0,$$

inequality (6.13) holds true. ■

We are finally ready to state the main proposition, that collects all the properties of the eigenvalues $\mu_1(\lambda,k)$, $\mu_2(\lambda,k)$ that we need.

Proposition 6.4. *The following properties hold:*

(i) *For any* $\lambda \in (0,1)$, *we have* $\mu_1(\lambda,0) < \mu_2(\lambda,0) < 0$, $\mu_1(\lambda,1) = -2$ *and* $\mu_2(\lambda,1) = 0$.

(ii) $\mu_2(\lambda,k) > 0$ *for any* $\lambda \in (0,1)$ *and any integer* $k \geq 2$.

(iii) *For any integer* $k \geq 2$, *there exists a unique value* $\lambda_k \in (0,1)$ *such that* $\mu_1(\lambda_k,k) = 0$. *Furthermore,* $\partial\mu_1/\partial\lambda(\lambda_k,k) < 0$.

(iv) *The sequence* $\{\lambda_k\}_{k\geq 2}$ *is monotonically increasing with* $\lim_{k\to+\infty}\lambda_k = 1$.

(v) $\lim_{k\to+\infty}\mu_{\mathrm{i}}(\lambda,k)/k$ *is finite and positive for all* $\lambda \in (0,1)$, *for* $i = 1,2$.

Proof. We have already noticed that $\mu_1(\lambda,1) = -2$ and $\mu_2(\lambda,1) = 0$ for all $\lambda \in (0,1)$. Since we also know from Lemma 6.2 that μ_1 and μ_2 are strictly increasing in k, properties (i) and (ii) follow at once.

Concerning the first eigenvalue μ_1, we have already shown that for any $k \geq 2$ it holds

$$\lim_{\lambda\to 0}\mu_1(\lambda,k) = k-1 > 0, \qquad \text{and} \qquad \lim_{\lambda\to 1}\mu_1(\lambda,k) = -2 < 0.$$

Since μ_1 is a continuous function of λ for any fixed $k \geq 2$, it follows that there exists at least one value $\lambda_k \in (0,1)$ such that $\mu_1(\lambda_k,k) = 0$. On the other hand, we know from Lemma 6.3 that the derivative of μ_1 with respect to λ has to be strictly negative at each point where μ_1 vanishes. It follows that, once the function μ_1 becomes negative, it cannot become positive again for larger values of λ. In other words, there can only be a single value $\lambda = \lambda_k$ at which μ_1 vanishes. This proves property (iii) of the proposition.

The fact that $\{\lambda_k\}_{k\geq 2}$ is monotonically increasing follows immediately from Lemma 6.2, thus, in order to prove property (iv), it is enough to show that λ_k goes to 1 as $k \to \infty$. To do this, we argue by contradiction. Suppose that $\lim_{k\to\infty}\lambda_k = \lambda_\infty < 1$ (notice that the limit exists because $\{\lambda_k\}$ is a monotone sequence). Recalling that formula (6.11) is in force at the point $\lambda = \lambda_k$, recalling also that R and ω are both functions of λ and setting $\omega_k = \omega(\lambda_k)$ and $R_k = R(\lambda_k)$, we have that it must hold

$$2R_k^2(1-\lambda_k^2)\coth\omega_k = (R_k^2 - \lambda_k^2)(1-R_k^2)k + (R_k^2+\lambda_k^2)(1+R_k^2)\frac{1}{k}. \tag{6.14}$$

Since λ_k converges to a value $\lambda_\infty < 1$, it follows that R_k also converges to a value R_∞ such that $\lambda_\infty < R_\infty < 1$, whereas $\coth \omega_k = (1+\lambda_k^{2k})/(1-\lambda_k^{2k}) \to 1$ as $k \to +\infty$. As a consequence, the left-hand side of (6.14) converges to a finite value as $k \to \infty$, whereas the right-hand side goes to infinity. This is a contradiction, as wished.

Finally, it remains to prove property (v). Recalling (6.7), we have

$$\lim_{k\to+\infty} \frac{\mu_1(\lambda,k)}{k} = \frac{1}{2} \lim_{k\to+\infty} \left\{ \frac{T_{\lambda,k}}{k} \left[1 - \sqrt{1 - 4\frac{D_{\lambda,k}}{T_{\lambda,k}^2}} \right] \right\},$$

$$\lim_{k\to+\infty} \frac{\mu_2(\lambda,k)}{k} = \frac{1}{2} \lim_{k\to+\infty} \left\{ \frac{T_{\lambda,k}}{k} \left[1 + \sqrt{1 - 4\frac{D_{\lambda,k}}{T_{\lambda,k}^2}} \right] \right\}.$$

We now notice that for any fixed λ, it holds $\coth \omega = (1+\lambda^{2k})/(1-\lambda^{2k}) \to 1$ as $k \to +\infty$. As a consequence, from the explicit expressions (6.5) for $T_{\lambda,k}$ and $D_{\lambda,k}$, we easily compute

$$\lim_{k\to+\infty} \frac{T_{\lambda,k}}{k} = \frac{R^2(1-\lambda^2)}{\lambda^2} \quad \text{and} \quad \lim_{k\to+\infty} \frac{D_{\lambda,k}}{T_{\lambda,k}^2} = \frac{\lambda^2(R^2-\lambda^2)(1-R^2)}{R^4(1-\lambda^2)^2}.$$

The desired result then follows. ∎

The properties described in Proposition 6.4 are the ones needed in order to be able to invoke the Crandall–Rabinowitz bifurcation theorem [15] to prove Theorem A. Now the proof follows exactly the same strategy highlighted in [29, Section 5]. Let us just recall briefly the main steps to show how the properties (i)–(v) of the eigenvalues come into play.

The first step is to restrict the functional F_λ to functions invariant under a suitable group G of isometries of $\mathbb{R}^2$. We can choose as G any subgroup of the orthogonal group $O(2)$ such that the eigenvalues $\{\sigma_{\mathrm{i}}\}_{i\in\mathbb{N}_0}$ of $-\Delta_{\mathbb{S}^1}$ have multiplicity 1 and satisfy $\sigma_0 = 0$, $\sigma_1 > 1$. For instance, one possible choice is the group $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ acting on $\mathbb{R}^2$ by reflections along the two coordinate axes, in which case the eigenvalues form the sequence $\{(2i)^2\}_{i\in\mathbb{N}_0}$, corresponding to the eigenfunctions $\cos(2i\theta)$. Notice that any other choice of a group of rotations containing $\mathbb{Z}_2 \times \mathbb{Z}_2$ is also clearly admissible. In particular, we have the freedom to choose our G as a group with an arbitrarily large cardinality.

Let us fix such a G once and for all, and let $\{\sigma_{\mathrm{i}}\}_{i\in\mathbb{N}_0}$ be the corresponding sequence of eigenvalues. For every $i \in \mathbb{N}_0$, let Y_{i} be the unique G-invariant unit $L^2(\mathbb{S}^1)$-norm eigenfunction of $-\Delta_{\mathbb{S}^1}$ corresponding to the eigenvalue σ_{i}, and let $W_{\mathrm{i}} = \operatorname{Span}\{(Y_{\mathrm{i}},0),(0,Y_{\mathrm{i}})\}$.

Fix now $k \in \mathbb{N}$. Recall from Proposition 6.4 (iii) that there exists one value λ_{σ_k} such that $\mu_1(\lambda_{\sigma_k},\sigma_k) = 0$. Let $F_k := F_{\lambda_{\sigma_k}}$ be the operator defined as in (6.3) and let $L_k := L_{\lambda_{\sigma_k}}$ be its linearization. Let us also denote by $\mathcal{C}_G^{\kappa,\alpha}(\mathbb{S}^1)$ the Hölder space of $\mathcal{C}^{\kappa,\alpha}(\mathbb{S}^1)$-functions that are G-invariant. It is easily seen that the image of G-invariant functions via F_k and L_k is still G-invariant. We can then consider the restrictions

$$F_k\colon U \to (\mathcal{C}_G^{1,\alpha}(\mathbb{S}^1))^2, \quad L_k\colon (\mathcal{C}_G^{2,\alpha}(\mathbb{S}^1))^2 \to (\mathcal{C}_G^{1,\alpha}(\mathbb{S}^1))^2,$$

where $U \subseteq (\mathcal{C}^{2,\alpha}_G(\mathbb{S}^1))^2$ is a small neighborhood of $\mathbf{0}$. In order to apply the bifurcation theorem to F_k, it is sufficient to show that $\mathrm{Ker}(L_k)$ has dimension 1, that $\mathrm{Im}(L_k)$ is closed with codimension 1 and that $\partial L_\lambda/\partial\lambda|_{\lambda=\lambda_{\sigma_k}}(\mathbf{z}) \notin \mathrm{Im}(L_k)$, where $\mathbf{z}$ is an element that spans the kernel of L_k.

For $i \in \mathbb{N}_0$, let $\mathbf{z}_{i,1}$ and $\mathbf{z}_{i,2}$ be eigenvectors of $L_k|_{W_i}$ relative to the eigenvalues $\mu_1(\lambda_{\sigma_k}, \sigma_i)$, $\mu_2(\lambda_{\sigma_k}, \sigma_i)$ and orthonormal with respect to the scalar product $\langle\cdot,\cdot\rangle_{\lambda_{\sigma_k}}$ defined as in (6.4). Denote $H^s_G(\mathbb{S}^1) := H^s(\mathbb{S}^1) \cap L^2_G(\mathbb{S}^1)$, where $L^2_G(\mathbb{S}^1)$ is the space of G-invariant $L^2(\mathbb{S}^1)$-integrable functions, and consider the map from $(H^2_G(\mathbb{S}^1))^2$ to $(H^1_G(\mathbb{S}^1))^2$ defined as

$$\sum_{\ell=0}^{\infty}(a_{\ell,1}\mathbf{z}_{\ell,1} + a_{\ell,2}\mathbf{z}_{\ell,2}) \mapsto \sum_{\ell=0}^{\infty}(a_{\ell,1}\mu_1(\lambda_{\sigma_k}, \sigma_\ell)\mathbf{z}_{\ell,1} + a_{\ell,2}\mu_2(\lambda_{\sigma_k}, \sigma_\ell)\mathbf{z}_{\ell,2}). \tag{6.15}$$

This map coincides with L_k on its domain, thus extending it. Recall that the Sobolev norm on $H^s(\mathbb{S}^1)$ is equivalent to the norm

$$\|f\| := \sum_{j=0}^{\infty}(1+j^2)^s\|P_j(f)\|^2_{L^2},$$

where P_j is the L^2-orthogonal projection on the subspace generated by spherical harmonics of degree j. It follows then easily from the asymptotic behaviour of μ_1 and μ_2 proved in Proposition 6.4 (v) that (6.15) is a continuous mapping. Furthermore, from Proposition 6.4 (i), (ii), (iv), recalling that $\sigma_0 = 0$ and $\sigma_1 > 1$, it is clear that both $\mu_1(\lambda_{\sigma_k}, \sigma_i)$ and $\mu_2(\lambda_{\sigma_k}, \sigma_i)$ are different from zero for every $i \in \mathbb{N}_0$, with the only exception of $\mu_1(\lambda_{\sigma_k}, \sigma_k)$. As a consequence, we can write down the right inverse of L_k as

$$\begin{aligned}&\sum_{\ell=0}^{\infty}(b_{\ell,1}\mathbf{z}_{\ell,1} + b_{\ell,2}\mathbf{z}_{\ell,2})\\ &\qquad\mapsto \sum_{\ell=0,\,\ell\neq k}^{\infty}\left(\frac{b_{\ell,1}}{\mu_1(\lambda_{\sigma_k}, \sigma_\ell)}\mathbf{z}_{\ell,1} + \frac{b_{\ell,2}}{\mu_2(\lambda_{\sigma_k}, \sigma_\ell)}\mathbf{z}_{\ell,2}\right) + \frac{b_{k,2}}{\mu_2(\lambda_{\sigma_k}, \sigma_k)}\mathbf{z}_{k,2}.\end{aligned}$$

Again from Proposition 6.4 (v), we deduce that this inverse is also continuous. It follows that map (6.15), restricted to elements $\mathbf{v}$ satisfying $\langle \mathbf{v}, \mathbf{z}_{k,1}\rangle_{\lambda_{\sigma_k}} = 0$, is an isomorphism. Since (6.15) is an extension of L_k, we can then expect, and indeed it can be proved with some work, that L_k is an isomorphism as well when we restrict to elements orthogonal to $\mathbf{z}_{k,1}$. It follows immediately that $\mathrm{Ker}(L_k)$ has dimension 1, generated by $\mathbf{z}_{k,1}$, whereas $\mathrm{Im}(L_k)$ is the space orthogonal to $\mathbf{z}_{k,1}$ with respect to the scalar product $\langle\cdot,\cdot\rangle_{\lambda_{\sigma_k}}$, and thus it is closed with codimension 1. Finally, we have

$$\frac{\partial L_\lambda}{\partial\lambda}\Big|_{\lambda=\lambda_{\sigma_k}}(\mathbf{z}_{k,1}) = \frac{\partial\mu_1}{\partial\lambda}(\lambda_{\sigma_k}, \sigma_k)\mathbf{z}_{k,1},$$

which does not belong to $\mathrm{Im}(L_{\lambda_k})$ thanks to Proposition 6.4 (iii). These are the properties that we needed in order to invoke the Crandall–Rabinowitz bifurcation theorem [15] (see also [29, Theorem 7.1]), which tells us that, for sufficiently small $\varepsilon > 0$, $\delta > 0$, there exists a smooth curve

$$(-\varepsilon, \varepsilon) \to (\mathcal{C}^{2,\alpha}_G(\mathbb{S}^1))^2 \times (\lambda_{\sigma_k} - \delta, \lambda_{\sigma_k} + \delta), \quad s \mapsto (\mathbf{w}(s), \lambda(s))$$

with $\mathbf{w}(0) = \mathbf{0}$, $\lambda(0) = \lambda_{\sigma_k}$, $\langle \mathbf{w}(s), \mathbf{z}_{k,1}\rangle_{\lambda_{\sigma_k}} = 0$ and $F_{\lambda(s)}(\mathbf{v}(s)) = 0$, where $\mathbf{v}(s) = s(\mathbf{w}(s) + \mathbf{z}_{k,1})$. It follows that the corresponding functions $u^{\mathbf{v}(s)}_{\lambda(s)} \colon \Omega^{\mathbf{v}(s)}_{\lambda(s)} \to \mathbb{R}$ form a 1-parameter family of solutions to problem (1.1) having gradient constantly equal to $c^{\mathrm{o}}_{\lambda(s)}$ on the outer boundary component and constantly equal to $c^{\mathrm{i}}_{\lambda(s)}$ on the inner boundary component. This proves Theorem 6.1.

Appendix A. Expansions of W and W_R about MAX(u)

In this appendix, we collect some basic, though very important expansions of the functions $W = |\nabla u|^2$ and $W_R = |\nabla u_R|^2 \circ \Psi$ about $\mathrm{MAX}(u)$, that have been invoked in the proof of Theorem 3.5.

Lemma A.1. *Let (Ω, u) be a solution to problem (1.1), let N be a connected component of $\Omega \setminus \mathrm{MAX}(u)$, and let $R = R(N) \in [0, 1)$ be the expected core radius associated with the region N. Also assume that Normalization 3.1 is in force. Then for every $p \in \mathrm{MAX}(u)$, it holds*

$$\lim_{x \to p,\, x \in N} \frac{W_R}{u_{\max} - u} = 4,$$

where W_R is the function defined in N by (3.7).

Proof. Let us remember that W_R, $u_{\max}$ and u can be rewritten explicitly in terms of R and Ψ via the following formulae:

$$W_R = \Big(\frac{\Psi^2 - R^2}{\Psi}\Big)^2, \quad 2u_{\max} = 1 - R^2 + R^2 \log R^2, \quad 2u = 1 - \Psi^2 + 2R^2 \log \Psi.$$

Therefore, we have

$$\lim_{x \to p,\, x \in N} \frac{W_R}{u_{\max} - u} = \lim_{\Psi \to R} \frac{2(\Psi^2 - R^2)^2}{\Psi^2(\Psi^2 - R^2 + R^2 \log R^2 - 2R^2 \log \Psi)}.$$

Setting $z = \Psi^2 - R^2$, this limit can be easily computed with the following Taylor expansion:

$$\begin{aligned} \frac{W_R}{u_{\max} - u} &= \frac{2z^2}{(R^2 + z)[z - R^2 \log(1 + \frac{z}{R^2})]} \\ &= \frac{2z}{(R^2 + z)[1 - \frac{R^2}{z}(\frac{z}{R^2} - \frac{1}{2}\frac{z^2}{R^4} + \frac{1}{3}\frac{z^3}{R^6} + \mathcal{O}(z^4))]} \\ &= \frac{2z}{\frac{1}{2}z + \frac{1}{6}\frac{z^2}{R^2} + \mathcal{O}(z^3)} = 4 - \frac{4}{3}\frac{z}{R^2} + \mathcal{O}(z^2). \end{aligned} \tag{A.1}$$

The desired statement then follows at once. ∎

Notice that in the above proof we have actually shown a more precise estimate. Let us rephrase it in a more convenient way, as it will be helpful in the proof of the next lemma. Expanding $u_{\max} - u$ in terms of z, we have

$$u_{\max} - u = \frac{1}{4}\frac{z^2}{R^2} + \mathcal{O}(z^3).$$

Inverting this relationship yields

$$z = \pm 2R\sqrt{u_{\max} - u} + \mathcal{O}(u_{\max} - u).$$

The sign $\pm$ appears here because $z = \Psi^2 - R^2$ is positive if $\overline{\tau}(N) < \sqrt{2}$ and it is negative if $\overline{\tau}(N) > \sqrt{2}$. Therefore, expansion (A.1) above can be rewritten as

$$\begin{aligned} W_R &= 4(u_{\max} - u) - \frac{4}{3}(u_{\max} - u)\frac{z}{R^2} + \mathcal{O}((u_{\max} - u)z^2) \\ &= 4(u_{\max} - u) \mp \frac{8}{3R}(u_{\max} - u)^{3/2} + \mathcal{O}((u_{\max} - u)^2), \end{aligned} \tag{A.2}$$

where the $-$ sign holds on regions where $\overline{\tau}(N) < \sqrt{2}$, and the $+$ sign holds on regions where $\overline{\tau}(N) > \sqrt{2}$.

In the following lemma, we provide more refined expansions for both W and W_R in a neighborhood of the top stratum of $\mathrm{MAX}(u)$.

Lemma A.2. *Let (Ω, u) be a solution to problem (1.1), let N be a connected component of $\Omega \setminus \mathrm{MAX}(u)$, and let $R = R(N) \in [0, 1)$ be the expected core radius associated with the region N. Also assume that Normalization 3.1 is in force and denote by Σ_N the 1-dimensional top stratum of $\mathrm{MAX}(u) \cap \overline{N}$. Then, at any point $p \in \Sigma_N$, it holds*

$$\begin{aligned} W &= 4r^2[1 + \kappa(p)r] + \mathcal{O}(r^4), \\ W_R &= 4r^2\Big[1 + \Big(\frac{\kappa(p)}{3} - \frac{2}{3R}\Big)r\Big] + \mathcal{O}(r^4) \qquad \text{if } \overline{\tau}(N) < \sqrt{2}, \\ W_R &= 4r^2\Big[1 + \Big(\frac{\kappa(p)}{3} + \frac{2}{3R}\Big)r\Big] + \mathcal{O}(r^4) \qquad \text{if } \overline{\tau}(N) > \sqrt{2}, \end{aligned}$$

where $\kappa(p)$ is the curvature of Σ_N at p, computed with respect to the exterior unit normal, and $r(x) = \mathrm{dist}(x, \Sigma_N)$ denotes the distance from Σ_N.

Proof. From [6, Theorem 3.1], we have the following expansion:

$$u = u_{\max} - r^2 - \frac{\kappa(p)}{3}r^3 + \mathcal{O}(r^4).$$

In particular, ∇u satisfies

$$\nabla u = -r[2 + \kappa(p)r]\frac{\partial}{\partial r} + \mathcal{O}(r^3),$$

so that

$$W = |\nabla u|^2 = 4r^2 + 4\kappa(p)r^3 + \mathcal{O}(r^4) = 4r^2[1 + \kappa(p)r] + \mathcal{O}(r^4).$$

Concerning W_R, let us observe that formula (A.2) provides us with an expansion of W_R in terms of $u_{\max} - u$. It follows immediately that

$$\begin{aligned} W_R &= 4(u_{\max} - u) \mp \frac{8}{3R}(u_{\max} - u)^{3/2} + \mathcal{O}((u_{\max} - u)^2) \\ &= 4r^2 \mp \frac{8}{3R}r^3 + \frac{4}{3}\kappa(p)r^3 + \mathcal{O}(r^4) \\ &= 4r^2\Big[1 + \Big(\frac{\kappa(p)}{3} \mp \frac{2}{3R}\Big)r\Big] + \mathcal{O}(r^4), \end{aligned}$$

where the sign ambiguity is the one specified in the statement. ∎

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References

[1] Agostiniani, V., Magnanini, R.: Stability in an overdetermined problem for the Green's function. Ann. Mat. Pura Appl. (4) **190**, 21–31 (2011) Zbl 1213.35324 MR 2747460

[2] Agostiniani, V., Magnanini, R.: Symmetries in an overdetermined problem for the Green's function. Discrete Contin. Dyn. Syst. Ser. S **4**, 791–800 (2011) Zbl 1210.35169 MR 2746441

[3] Alessandrini, G., Magnanini, R.: The index of isolated critical points and solutions of elliptic equations in the plane. Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) **19**, 567–589 (1992) Zbl 0793.35021 MR 1205884

[4] Barrios, B., Montoro, L., Sciunzi, B.: On the moving plane method for nonlocal problems in bounded domains. J. Anal. Math. **135**, 37–57 (2018) Zbl 1394.35545 MR 3827344

[5] Borghini, S.: Symmetry results for Serrin-type problems in doubly connected domains. Math. Eng. **5**, article no. 027 (2023) MR 4431667

[6] Borghini, S., Chruściel, P. T., Mazzieri, L.: On the uniqueness of Schwarzschild–de Sitter spacetime. Arch. Ration. Mech. Anal. **247**, article no. 22 (2023) Zbl 1520.83023 MR 4558991

[7] Borghini, S., Mazzieri, L.: On the mass of static metrics with positive cosmological constant: I. Classical Quantum Gravity **35**, article no. 125001 (2018) Zbl 1391.83037 MR 3826161

[8] Borghini, S., Mazzieri, L.: On the mass of static metrics with positive cosmological constant: II. Comm. Math. Phys. **377**, 2079–2158 (2020) Zbl 1443.83019 MR 4121617

[9] Brandolini, B., Nitsch, C., Salani, P., Trombetti, C.: On the stability of the Serrin problem. J. Differential Equations **245**, 1566–1583 (2008) Zbl 1173.35019 MR 2436453

[10] Brandolini, B., Nitsch, C., Salani, P., Trombetti, C.: Serrin-type overdetermined problems: an alternative proof. Arch. Ration. Mech. Anal. **190**, 267–280 (2008) Zbl 1161.35025 MR 2448319

[11] Busca, J., Manásevich, R.: A Liouville-type theorem for Lane–Emden systems. Indiana Univ. Math. J. **51**, 37–51 (2002) Zbl 1033.35032 MR 1896155

[12] Cavallina, L., Magnanini, R., Sakaguchi, S.: Two-phase heat conductors with a surface of the constant flow property. J. Geom. Anal. **31**, 312–345 (2021) Zbl 1461.35166 MR 4203648

[13] Ciraolo, G., Figalli, A., Roncoroni, A.: Symmetry results for critical anisotropic p-Laplacian equations in convex cones. Geom. Funct. Anal. **30**, 770–803 (2020) Zbl 1453.35104 MR 4135671

[14] Ciraolo, G., Vezzoni, L.: On Serrin's overdetermined problem in space forms. Manuscripta Math. **159**, 445–452 (2019) Zbl 1414.35251 MR 3959271

[15] Crandall, M. G., Rabinowitz, P. H.: Bifurcation from simple eigenvalues. J. Funct. Anal. **8**, 321–340 (1971) Zbl 0219.46015 MR 288640

[16] Crasta, G., Fragalà, I., Gazzola, F.: A sharp upper bound for the torsional rigidity of rods by means of web functions. Arch. Ration. Mech. Anal. **164**, 189–211 (2002) Zbl 1021.74020 MR 1930391

[17] Damascelli, L., Pacella, F., Ramaswamy, M.: Symmetry of ground states of p-Laplace equations via the moving plane method. Arch. Ration. Mech. Anal. **148**, 291–308 (1999) Zbl 0937.35050 MR 1716666

[18] Dehlaghi, V., Shadpoor, M. T., Najarian, S.: Analysis of wall shear stress in stented coronary artery using 3d computational fluid dynamics modeling. J. Mater. Process. Tech. **197**, 174–181 (2008)

[19] Domínguez-Vázquez, M., Enciso, A., Peralta-Salas, D.: Solutions to the overdetermined boundary problem for semilinear equations with position-dependent nonlinearities. Adv. Math. **351**, 718–760 (2019) Zbl 1418.35280 MR 3954957

[20] Fall, M. M., Jarohs, S.: Overdetermined problems with fractional Laplacian. ESAIM Control Optim. Calc. Var. **21**, 924–938 (2015) Zbl 1329.35223 MR 3395749

[21] Fall, M. M., Minlend, I. A., Weth, T.: Unbounded periodic solutions to Serrin's overdetermined boundary value problem. Arch. Ration. Mech. Anal. **223**, 737–759 (2017) Zbl 1364.35207 MR 3590664

[22] Farina, A., Kawohl, B.: Remarks on an overdetermined boundary value problem. Calc. Var. Partial Differential Equations **31**, 351–357 (2008) Zbl 1166.35353 MR 2366129

[23] Farina, A., Valdinoci, E.: A pointwise gradient estimate in possibly unbounded domains with nonnegative mean curvature. Adv. Math. **225**, 2808–2827 (2010) Zbl 1200.35131 MR 2680184

[24] Fragalà, I., Gazzola, F.: Partially overdetermined elliptic boundary value problems. J. Differential Equations **245**, 1299–1322 (2008) Zbl 1156.35049 MR 2436831

[25] Garofalo, N., Lewis, J. L.: A symmetry result related to some overdetermined boundary value problems. Amer. J. Math. **111**, 9–33 (1989) Zbl 0681.35016 MR 980297

[26] Gidas, B., Ni, W.-M., Nirenberg, L.: Symmetry and related properties via the maximum principle. Comm. Math. Phys. **68**, 209–243 (1979) Zbl 0425.35020 MR 544879

[27] Gilsbach, A., Onodera, M.: Linear stability estimates for Serrin's problem via a modified implicit function theorem. Calc. Var. Partial Differential Equations **60**, article no. 241 (2021) Zbl 1478.35153 MR 4328462

[28] Hamel, F., Nadirashvili, N.: Circular flows for the Euler equations in two-dimensional annular domains, and related free boundary problems. J. Eur. Math. Soc. (JEMS) **25**, 323–368 (2023) Zbl 1510.35221 MR 4556785

[29] Kamburov, N., Sciaraffia, L.: Nontrivial solutions to Serrin's problem in annular domains. Ann. Inst. H. Poincaré C Anal. Non Linéaire **38**, 1–22 (2021) Zbl 1462.35211 MR 4200475

[30] Krantz, S. G., Parks, H. R.: A primer of real analytic functions. 2nd ed., Birkhäuser Adv. Texts Basler Lehrbücher, Birkhäuser, Boston, MA (2002) Zbl 1015.26030 MR 1916029

[31] Łojasiewicz, S.: Une propriété topologique des sous-ensembles analytiques réels. In: Les Équations aux Dérivées Partielles (Paris, 1962), Colloq. Internat. CNRS 117, Éditions du Centre National de la Recherche Scientifique (CNRS), Paris, 87–89 (1963) Zbl 0234.57007 MR 160856

[32] Łojasiewicz, S.: Introduction to complex analytic geometry. Birkhäuser, Basel (1991) Zbl 0773.32007 MR 1131081

[33] Lorenzini, G.: Blood velocity field numerical assessment using a gpl code in case of intravascular doppler catheter affections: comparative analysis of different rheological models. J. Biomech. **38**, 2058–2069 (2005)

[34] Ma, L., Liu, B.: Symmetry results for decay solutions of elliptic systems in the whole space. Adv. Math. **225**, 3052–3063 (2010) Zbl 1202.35080 MR 2729001

[35] Magnanini, R., Poggesi, G.: The location of hot spots and other extremal points. Math. Ann. **384**, 511–549 (2022) Zbl 1496.35120 MR 4476237

[36] Pacard, F., Sicbaldi, P.: Extremal domains for the first eigenvalue of the Laplace–Beltrami operator. Ann. Inst. Fourier (Grenoble) **59**, 515–542 (2009) Zbl 1166.53029 MR 2521426

[37] Pacella, F., Tralli, G.: Overdetermined problems and constant mean curvature surfaces in cones. Rev. Mat. Iberoam. **36**, 841–867 (2020) Zbl 1445.35258 MR 4109828

[38] Payne, L. E., Philippin, G. A.: On two free boundary problems in potential theory. J. Math. Anal. Appl. **161**, 332–342 (1991) Zbl 0760.31003 MR 1132110

[39] Polya, G., Weinstein, A.: On the torsional rigidity of multiply connected cross-sections. Ann. of Math. (2) **52**, 154–163 (1950) Zbl 0038.37502 MR 40159

[40] Reichel, W.: Radial symmetry by moving planes for semilinear elliptic BVPs on annuli and other non-convex domains. In: Elliptic and parabolic problems (Pont-à-Mousson, 1994), Pitman Res. Notes Math. 325, Longman Scientific & Technical, Harlow, 164–182 (1995) Zbl 0839.35047 MR 1416582

[41] Roncoroni, A.: A Serrin-type symmetry result on model manifolds: An extension of the Weinberger argument. C. R. Math. Acad. Sci. Paris **356**, 648–656 (2018) Zbl 1391.35380 MR 3806894

[42] Ros, A., Ruiz, D., Sicbaldi, P.: Solutions to overdetermined elliptic problems in nontrivial exterior domains. J. Eur. Math. Soc. (JEMS) **22**, 253–281 (2020) Zbl 1440.35167 MR 4046014

[43] Schlenk, F., Sicbaldi, P.: Bifurcating extremal domains for the first eigenvalue of the Laplacian. Adv. Math. **229**, 602–632 (2012) Zbl 1233.35147 MR 2854185

[44] Schwarz, H. A.: Gesammelte mathematische Abhandlungen. Band I, II. Chelsea Publishing Co., Bronx, NY (1972) MR 392470

[45] Serrin, J.: A symmetry problem in potential theory. Arch. Ration. Mech. Anal. **43**, 304–318 (1971) Zbl 0222.31007 MR 333220

[46] Serrin, J., Zou, H.: Symmetry of ground states of quasilinear elliptic equations. Arch. Ration. Mech. Anal. **148**, 265–290 (1999) Zbl 0940.35079 MR 1716665

[47] Sicbaldi, P.: New extremal domains for the first eigenvalue of the Laplacian in flat tori. Calc. Var. Partial Differential Equations **37**, 329–344 (2010) Zbl 1188.35122 MR 2592974

[48] Sirakov, B.: Symmetry for exterior elliptic problems and two conjectures in potential theory. Ann. Inst. H. Poincaré C Anal. Non Linéaire **18**, 135–156 (2001) Zbl 0997.35014 MR 1808026

[49] Souček, J., Souček, V.: Morse–Sard theorem for real-analytic functions. Comment. Math. Univ. Carolin. **13**, 45–51 (1972) Zbl 0235.26012 MR 308345

[50] Torii, R., Wood, N. B., Hughes, A. D., Thom, S. A., Aguado-Sierra, J., Davies, J. E., Francis, D. P., Parker, K. H., Xu, X. Y.: A computational study on the influence of catheter-delivered intravascular probes on blood flow in a coronary artery model. J. Biomech. **40**, 2501–2509 (2007)

[51] Weinberger, H. F.: Remark on the preceding paper of Serrin. Arch. Ration. Mech. Anal. **43**, 319–320 (1971) Zbl 0222.31008 MR 333221