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Ruelle–Taylor resonances of Anosov actions

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Abstract. Combining microlocal methods and a cohomological theory developed by J. Taylor, we define for Anosov $\mathbb{R}^\kappa$-actions a notion of joint Ruelle resonance spectrum. We prove that these Ruelle–Taylor resonances fit into a Fredholm theory, are intrinsic and form a discrete subset of $\mathbb{C}^\kappa$, with $\lambda = 0$ being always a leading resonance. The joint resonant states at 0 give rise to some new measures of SRB type and the mixing properties of these measures are related to the existence of purely imaginary resonances. The spectral theory developed in this article applies in particular to the case of Weyl chamber flows and provides a new way to study such flows.

Keywords: Ruelle resonances, Taylor joint spectrum, Anosov actions.

1. Introduction

If P is a differential operator on a manifold M that has purely discrete spectrum as an unbounded operator acting on $L^2(M)$ (e.g. an elliptic operator on a closed Riemannian manifold M), then the eigenvalues and eigenfunctions carry a huge amount of information about the dynamics generated by P. Furthermore, if P is a geometric differential operator (e.g. Laplace–Beltrami operator, Hodge Laplacian or Dirac operators), the discrete spectrum encodes important topological and geometric invariants of the manifold M.

Unfortunately, in many cases (e.g. if the manifold M is not compact or if P is non-elliptic) the L^2-spectrum of P is not discrete anymore but consists mainly of the essential spectrum. Still, there are certain cases where the essential spectrum of P is non-empty,

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but where there is a hidden intrinsic discrete spectrum attached to P, called the *resonance spectrum*. To be more concrete, let us give a couple of examples:

- Quantum resonances of Schrödinger operators $P = \Delta + V$ with $V \in C_c^\infty(\mathbb{R}^n)$ on $M = \mathbb{R}^n$ with n odd (see for example [18, Chapter 3] for a textbook account of this classical theory).
- Quantum resonances for the Laplacian on non-compact geometrically finite hyperbolic manifolds $M = \Gamma\backslash\mathbb{H}^{n+1}$: here $P = \Delta_M$ is the Laplace–Beltrami operator on M [32, 33, 44].
- Ruelle resonances for Anosov flows [8, 16, 21, 25]: here $P = iX$ with X being the vector field generating the Anosov flow.

The definition of the resonances can be stated in different ways (using meromorphically continued resolvents, scattering operators or discrete spectra on auxiliary function spaces), and also the mathematical techniques used to establish the existence of resonances in the above examples are quite diverse (ranging from asymptotics of special functions to microlocal analysis). Nevertheless, all three examples above share the common point that the existence of a discrete resonance spectrum can be proven via a parametrix construction, i.e. one constructs a meromorphic family of operators $Q(\lambda)$ (with $\lambda \in \mathbb{C}$) such that

$$(P - \lambda)Q(\lambda) = \mathrm{Id} + K(\lambda),$$

where $K(\lambda)$ is a meromorphic family of compact operators on a suitable Banach or Hilbert space. Once such a parametrix is established, the resonances are the λ where $\mathrm{Id} + K(\lambda)$ is not invertible, and the discreteness of the resonance spectrum follows directly from analytic Fredholm theory.

In general, being able to construct such a parametrix and define a theory of resonances involves non-trivial analysis and pretty strong assumptions, but they lead to powerful results on the long time dynamics of the propagator e^{itP}, for example in the study of dynamical systems [22, 43, 47] or for evolution equations in relativity [35]. Furthermore, resonances form an important spectral invariant that can be related to a large variety of other mathematical quantities such as geometric invariants [33, 53], topological invariants [10, 12, 17, 41] or arithmetic quantities [6]. They also appear in trace formulas and are the divisors of dynamical Ruelle and Selberg zeta functions [7, 16, 22, 25, 48].

The purpose of this work is to use analytic and microlocal methods to construct a theory of joint resonance spectrum for the generating vector fields of Anosov $\mathbb{R}^\kappa$-actions. In terms of PDE and spectral theory, this can be viewed as the construction of a good notion of joint spectrum for a family of κ commuting vector fields $X_1, \ldots, X_\kappa$, generating a rank κ subbundle $E_0 \subset TM$, when their flow is transversely hyperbolic with respect to that subbundle. These operators do not form an elliptic family and finding a good notion of joint spectrum is thus highly non-trivial. Our strategy is to work on anisotropic Sobolev spaces to make the non-elliptic region "small" and then obtain Fredholmness properties.

However, this involves working in a non-self-adjoint setting, even if the X_k's were to preserve a Lebesgue type measure. We are then using Koszul complexes and a cohomological theory developed by Taylor [55, 56] in order to define a proper notion of joint spectrum in these anisotropic spaces, and we will show that this spectrum is discrete.

We emphasize that, in terms of PDE and spectral theory, there are important new aspects to be considered and the results are far from being a direct extension of the $\kappa = 1$ case (the Anosov flows). But also outside the spectral theory of linear partial differential operators the theory we develop might be helpful: the classical examples of such Anosov $\mathbb{R}^\kappa$-actions are *Weyl chamber flows* for compact locally symmetric spaces of rank $\kappa \geq 2$, and it is conjectured by Katok–Spatzier [39] that essentially all non-product $\mathbb{R}^\kappa$-actions are smoothly conjugate to homogeneous cases. Despite important recent advances [54], the conjecture is still widely open and it is important to extract as much information as possible on a general Anosov $\mathbb{R}^\kappa$-action in order to address this conjecture: for example, having an ergodic invariant measure with full support plays an important role in this direction (see e.g. [37] where the existence of such a measure is a central assumption on which the results are based; see also the discussions in the recent preprint [54]). Based on the spectral theory developed in this article, we show in a follow-up paper [29] the existence of such ergodic measures of full support for any positively transitive[1] Anosov action.

Let us summarize the main novelties of this work and its first applications:

(1) We construct a new theory of joint resonance spectrum for a family of commuting differential operators by combining the theory of Taylor [56] with the use of anisotropic Sobolev spaces for the study of resonances; as far as we know, this is the first result on joint spectrum in the theory of classical or quantum resonances.

(2) All the Weyl chamber flows on locally symmetric spaces and the standard actions of Katok–Spatzier [39] are included in our setting, and our results are completely new in that setting where representation theory is usually one of the main tools. This gives a new, analytic way of studying homogeneous dynamics and spectral theory in higher rank.

(3) We show that the leading joint resonance provides a construction of a new Sinai–Ruelle–Bowen (SRB) invariant measure μ for all $\mathbb{R}^\kappa$-actions. In a companion paper [29] based on this work, we show that our measure μ has all the properties of SRB measures of Anosov flows (rank 1 case), and it has *full support* if the Weyl chamber is positively transitive, an important step in the direction of the rigidity conjecture.

(4) We show in [29] that the periodic tori of the $\mathbb{R}^\kappa$-action are equidistributed in the support of μ and that μ can be written as an infinite sum over Dirac measures on the periodic tori, in a way similar to Bowen's formula in rank 1. These results are new even in the case of locally symmetric spaces and give a new way to study periodic tori (also called flats) in higher rank.

(5) Based on the present paper, the last two authors of this paper together with L. Wolf [34] proved a *classical-quantum correspondence* between the joint resonant states of

[1] See [29, Definition 2.9].

Weyl chamber flows on compact locally symmetric spaces $\Gamma\backslash G/M$ of rank κ and the joint eigenfunctions of the commutative algebra of invariant differential operators on the locally symmetric space $\Gamma\backslash G/K$. This gives a higher rank version of [14].

Another expected consequence of this construction would be a proof of the exponential decay of correlations for the action and a gap of Ruelle–Taylor resonances under appropriate assumptions, with application to the local rigidity and the regularity of the invariant measure μ. These questions will be addressed in a forthcoming work.

1.1. Statement of the main results

Let us now introduce the setting in more detail and state the main results. Let $\mathcal{M}$ be a closed manifold, let $\mathbb{A} \cong \mathbb{R}^\kappa$ be an abelian group and let $\tau : \mathbb{A} \to \mathrm{Diffeo}(\mathcal{M})$ be a smooth locally free group action. If $\mathfrak{a} := \mathrm{Lie}(\mathbb{A}) \cong \mathbb{R}^\kappa$, we can define a *generating map*

$$X : \mathfrak{a} \to C^\infty(\mathcal{M}; T\mathcal{M}), \quad A \mapsto X_A := \frac{d}{dt}\Big|_{t=0} \tau(\exp(tA)),$$

so that for each basis $A_1, \dots, A_\kappa$ of $\mathfrak{a}$, $[X_{A_j}, X_{A_k}] = 0$ for all j, k. For $A \in \mathfrak{a}$ we denote by $\varphi_t^{X_A}$ the flow of the vector field X_A. Notice that, as a differential operator, we can view X as a map

$$X : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}; \mathfrak{a}^*), \quad (Xu)(A) := X_A u.$$

It is customary to call the action *Anosov* if there is an $A \in \mathfrak{a}$ such that there is a continuous $d\varphi_t^{X_A}$-invariant splitting

$$T\mathcal{M} = E_0 \oplus E_u \oplus E_s, \tag{1.1}$$

where $E_0 = \mathrm{span}(X_{A_1}, \dots, X_{A_\kappa})$, and there exist $C, \nu > 0$ such that for each $x \in \mathcal{M}$,

$$\begin{aligned} &\forall w \in E_s(x), \forall t \geq 0, \quad \|d\varphi_t^{X_A}(x)w\| \leq Ce^{-\nu|t|}\|w\|, \\ &\forall w \in E_u(x), \forall t \leq 0, \quad \|d\varphi_t^{X_A}(x)w\| \leq Ce^{-\nu|t|}\|w\|. \end{aligned}$$

Here the norm on $T\mathcal{M}$ is fixed by choosing any smooth Riemannian metric g on $\mathcal{M}$. We say that such an A is *transversely hyperbolic*. It can be easily proved that the splitting is invariant by the whole action. However, we do not assume that all $A \in \mathfrak{a}^*$ have this transversely hyperbolic behavior. In fact, there is a maximal open convex cone $\mathcal{W} \subset \mathfrak{a}$ containing A such that for all $A' \in \mathcal{W}$, $X_{A'}$ is also transversely hyperbolic with the same splitting as A (see Lemma 2.2); $\mathcal{W}$ is called a *positive Weyl chamber*. This name is motivated by the classical examples of such Anosov actions that are the Weyl chamber flows for locally symmetric spaces of rank κ (see Example 2.3). There are also several other classes of examples (see e.g. [39, 54]).

Since we now have a family of commuting vector fields, it is natural to consider a joint spectrum for the family $X_{A_1}, \dots, X_{A_\kappa}$ of first order operators if the A_j's are transversely hyperbolic with the same splitting. Guided by the case of a single Anosov flow (handled in [8, 16, 21]), we define $E_u^* \subset T^*\mathcal{M}$ to be the subbundle such that $E_u^*(E_u \oplus E_0) = 0$. We shall say that $\lambda = (\lambda_1 \dots, \lambda_\kappa) \in \mathbb{C}^\kappa$ is a *joint Ruelle resonance* for the Anosov action if there is a non-zero distribution $u \in C^{-\infty}(\mathcal{M})$ with wavefront set $\mathrm{WF}(u) \subset E_u^*$ such that[2]

$$\forall j = 1, \dots, \kappa, \quad (X_{A_j} + \lambda_j)u = 0. \tag{1.2}$$

The distribution u is called a *joint Ruelle resonant state* (from now on we will denote by $C^{-\infty}_{E_u^*}(\mathcal{M})$ the space of distributions u with $\mathrm{WF}(u) \subset E_u^*$). In an equivalent but more invariant way (i.e. independently of the choice of basis $(A_j)_j$ of $\mathfrak{a}$), we can define a *joint Ruelle resonance* as an element $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ of the complexified dual Lie algebra such that there is a non-zero $u \in C^{-\infty}_{E_u^*}(\mathcal{M})$ with

$$\forall A \in \mathfrak{a}, \quad (X_A + \lambda(A))u = 0.$$

We notice that we also define a notion of generalized joint Ruelle resonant states and Jordan blocks in our analysis (see Proposition 4.17). It is a priori not clear that the set of joint Ruelle resonances is discrete – or non-empty for that matter – nor that the dimension of joint resonant states is finite, but this is a consequence of our work.

Theorem 1. *Let τ be a smooth abelian Anosov action on a closed manifold $\mathcal{M}$ with positive Weyl chamber $\mathcal{W}$. Then the set of joint Ruelle resonances $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ is a discrete set contained in*

$$\bigcap_{A \in \mathcal{W}} \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid \mathrm{Re}(\lambda(A)) \le 0\}. \tag{1.3}$$

Moreover, for each joint Ruelle resonance $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ the space of joint Ruelle resonant states is finite-dimensional.

We remark that this spectrum always contains $\lambda = 0$ (with $u = 1$ being the joint eigenfunction) and that for locally symmetric spaces it contains infinitely many joint Ruelle resonances, as is shown in [34, Theorem 1.1].

We also emphasize that this theorem is definitely not a straightforward extension of the case of a single Anosov flow. It relies on a deeper result based on the theory of joint spectrum and joint functional calculus developed by Taylor [55, 56]. This theory allows us to set up a good Fredholm problem on certain functional spaces by using Koszul complexes, as we now explain.

Let us define $X + \lambda$, for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, as an operator

$$X + \lambda : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}; \mathfrak{a}^*_{\mathbb{C}}), \quad ((X + \lambda)u)(A) := (X_A + \lambda(A))u.$$

We can then define for each $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ the differential operators $d_{X+\lambda} : C^\infty(\mathcal{M}; \Lambda^j \mathfrak{a}^*_{\mathbb{C}}) \to C^\infty(\mathcal{M}; \Lambda^{j+1}\mathfrak{a}^*_{\mathbb{C}})$ by setting

$$d_{X+\lambda}(u \otimes \omega) := ((X + \lambda)u) \wedge \omega \quad \text{for } u \in C^\infty(\mathcal{M}),\ \omega \in \Lambda^j \mathfrak{a}^*_{\mathbb{C}}.$$

[2]See [36, Chapter VIII.1] for the definition and properties of the wavefront set.

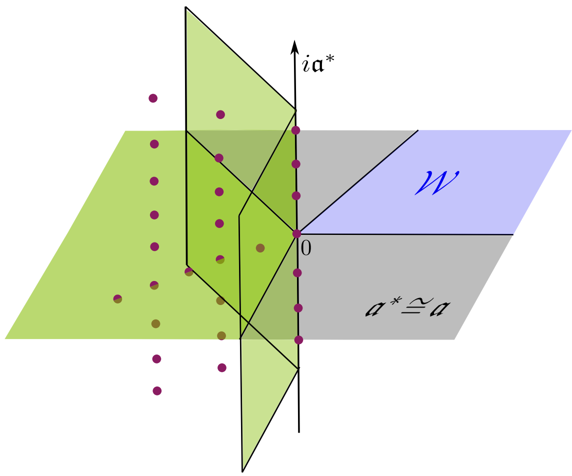

Fig. 1. Schematic sketch of the location of the resonances in $\mathfrak{a}^*_{\mathbb{C}}$. Note that in order to draw the at least four-dimensional space $\mathfrak{a}^*_{\mathbb{C}}$ the imaginary direction $i\mathfrak{a}^*$ has been reduced in the drawing to a one-dimensional line. The blue cone depicts the positive Weyl chamber $\mathcal{W}$ and the green region illustrates the region (1.3) in which the resonances can occur. The leading resonances discussed in Theorem 3 are located at the tip of this region.

Due to the commutativity of the family of vector fields X_A for $A \in \mathfrak{a}$, it can be easily checked that $d_{X+\lambda} \circ d_{X+\lambda} = 0$ (see Lemma 3.2). Moreover, as a differential operator, $d_{X+\lambda}$ extends to a continuous map

$$d_{X+\lambda} : C^{-\infty}_{E_u^*}(\mathcal{M}; \Lambda^j \mathfrak{a}^*_{\mathbb{C}}) \to C^{-\infty}_{E_u^*}(\mathcal{M}; \Lambda^{j+1} \mathfrak{a}^*_{\mathbb{C}})$$

and defines an *associated Koszul complex*

$$0 \to C^{-\infty}_{E_u^*}(\mathcal{M}) \xrightarrow{d_{X+\lambda}} C^{-\infty}_{E_u^*} \otimes \Lambda^1 \mathfrak{a}^*_{\mathbb{C}} \xrightarrow{d_{X+\lambda}} \cdots \xrightarrow{d_{X+\lambda}} C^{-\infty}_{E_u^*}(\mathcal{M}) \otimes \Lambda^\kappa \mathfrak{a}^*_{\mathbb{C}} \to 0. \tag{1.4}$$

We prove the following results on the cohomology of this complex:

Theorem 2. *Let τ be a smooth abelian Anosov action*[3] *on a closed manifold $\mathcal{M}$ with generating map X. Then for each $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $j = 0, \ldots, \kappa$, the cohomology*

$$\ker d_{X+\lambda}|_{C^{-\infty}_{E_u^*}(\mathcal{M}) \otimes \Lambda^j \mathfrak{a}^*_{\mathbb{C}}} / \operatorname{ran} d_{X+\lambda}|_{C^{-\infty}_{E_u^*}(\mathcal{M}) \otimes \Lambda^{j-1} \mathfrak{a}^*_{\mathbb{C}}}$$

is finite-dimensional, and non-trivial only at a discrete subset of $\{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid \mathrm{Re}(\lambda(A)) \leq 0, \forall A \in \mathcal{W}\}$.

We want to remark that not only is the statement about cohomology in Theorem 2 stronger than Theorem 1, but also the cohomological setting is in fact a fundamental

[3] We actually prove Theorems 1 and 2 in the more general setting of admissible lifts to vector bundles, as defined in Section 2.2.

ingredient in proving the discreteness of the resonance spectrum and its finite multiplicity. Our proof relies on the theory of joint *Taylor spectrum* (developed by J. Taylor [55, 56]), defined using such Koszul complexes carrying a suitable notion of Fredholmness. In our proof of Theorem 2 we show that the Koszul complex furthermore provides a good framework for a parametrix construction via microlocal methods. More precisely, the parametrix construction is not done on the topological vector spaces $C^{-\infty}_{E_u^*}(\mathcal{M})$ but on a scale of Hilbert spaces $\mathcal{H}_{NG}$, depending on the choice of an escape function $G \in C^\infty(T^*\mathcal{M})$ and a parameter $N \in \mathbb{R}^+$, by which one can in some sense approximate $C^{-\infty}_{E_u^*}(\mathcal{M})$. The spaces $\mathcal{H}_{NG}$ are *anisotropic Sobolev spaces*, which roughly speaking allow $H^N(\mathcal{M})$ Sobolev regularity in all directions except in E_u^* where we allow $H^{-N}(\mathcal{M})$ Sobolev regularity. They can be rigorously defined using microlocal analysis, following the techniques of Faure–Sjöstrand [21]. By further use of pseudodifferential and Fourier integral operator theory we can then construct a parametrix $Q(\lambda)$, which is a family of bounded operators on $\mathcal{H}_{NG} \otimes \Lambda\mathfrak{a}^*_{\mathbb{C}}$ depending holomorphically on $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and fulfilling

$$d_{X+\lambda}Q(\lambda) + Q(\lambda)d_{X+\lambda} = \mathrm{Id} + K(\lambda). \tag{1.5}$$

Here $K(\lambda)$ is a holomorphic family of compact operators on $\mathcal{H}_{NG} \otimes \Lambda\mathfrak{a}^*_{\mathbb{C}}$ for λ in a suitable domain of $\mathfrak{a}^*_{\mathbb{C}}$ that can be made arbitrarily large by letting $N \to \infty$. Even having this parametrix construction, the fact that the joint spectrum is discrete and intrinsic (i.e. independent of the precise construction of the Sobolev spaces) is more difficult than for an Anosov flow (the rank 1 case): this is because holomorphic functions in $\mathbb{C}^\kappa$ do not have discrete zeros when $\kappa \geq 2$ and we are lacking a good notion of resolvent, while for one operator the resolvent is an important tool. Due to the link with the theory of the Taylor spectrum, we call $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ a *Ruelle–Taylor resonance* for the Anosov action if for some $j = 0, \dots, \kappa$ the j-th cohomology is non-trivial,

$$\ker d_{X+\lambda}|_{C^{-\infty}_{E_u^*}(\mathcal{M})\otimes\Lambda^j\mathfrak{a}^*_{\mathbb{C}}} / \operatorname{ran} d_{X+\lambda}|_{C^{-\infty}_{E_u^*}\otimes\Lambda^{j-1}\mathfrak{a}^*_{\mathbb{C}}} \neq 0,$$

and we call the non-trivial cohomology classes *Ruelle–Taylor resonant states*. Note that the definition of joint Ruelle resonances precisely means that the 0-th cohomology is non-trivial. Thus, any joint Ruelle resonance is a Ruelle–Taylor resonance. The converse statement is not obvious but turns out to be true, as we will prove in Proposition 4.15: if the cohomology of degree $j > 0$ is not 0, then the cohomology of degree 0 is not trivial.

We continue with the discussion of the leading resonances. In view of (1.3) and Figure 1, a resonance is called a *leading resonance* when its real part vanishes. We show that this spectrum carries important information about the dynamics: it is related to a special type of invariant measures as well as to mixing properties of these measures.

First, let v_g be the Riemannian measure of a fixed metric g on $\mathcal{M}$. We call a τ-invariant probability measure μ on $\mathcal{M}$ a *physical measure* if there is $v \in C^\infty(\mathcal{M})$ non-negative such that for any continuous function f and any open cone $\mathcal{C} \subset \mathcal{W}$,

$$\mu(f) = \lim_{T\to\infty} \frac{1}{\mathrm{Vol}(\mathcal{C}_T)} \int_{A\in\mathcal{C}_T} \int_{\mathcal{M}} f(\varphi_1^{-X_A}(x))v(x)\,dv_g(x)\,dA, \tag{1.6}$$

where $\mathcal{C}_T := \{A \in \mathcal{C} \mid |A| \le T\}$, and $|\cdot|$ denotes a fixed Euclidean norm on $\mathfrak{a}$. In other words, μ is the weak Cesàro limit of a Lebesgue type measure under the dynamics. We prove the following result.

Theorem 3. *Let τ be a smooth abelian Anosov action with generating map X and let $\mathcal{W}$ be a positive Weyl chamber.*

(i) *The linear span over $\mathbb{C}$ of the physical measures is isomorphic (as a $\mathbb{C}$-vector space) to $\ker d_X|_{\mathcal{C}^{-\infty}_{E_u^*}}$, the space of joint Ruelle resonant states at $\lambda = 0 \in \mathfrak{a}^*_{\mathbb{C}}$; in particular, it is finite-dimensional.*[4]

(ii) *A probability measure μ is a physical measure if and only if it is τ-invariant and $\mathrm{WF}(\mu) \subset E_s^*$, where $E_s^* \subset T^*\mathcal{M}$ is defined by $E_s^*(E_s \oplus E_0) = 0$.*

(iii) *Assume that there is a unique physical measure μ (or by (i) equivalently that the space of joint resonant states at 0 is one-dimensional). Then the following are equivalent:*
- *The only Ruelle–Taylor resonance on $i\,\mathfrak{a}^*$ is zero.*
- *There exists $A \in \mathfrak{a}$ such that $\varphi_t^{X_A}$ is* weakly *mixing with respect to μ.*
- *For any $A \in \mathcal{W}$, $\varphi_t^{X_A}$ is* strongly *mixing with respect to μ.*

(iv) *$\lambda \in i\,\mathfrak{a}^*$ is a joint Ruelle resonance if and only if there is a complex measure μ_λ with $\mathrm{WF}(\mu_\lambda) \subset E_s^*$ satisfying for all $A \in \mathcal{W}$ and $t \in \mathbb{R}$ the following equivariance under push-forwards of the action: $(\varphi_t^{X_A})_*\mu_\lambda = e^{-\lambda(A)t}\mu_\lambda$. Moreover, such measures are absolutely continuous with respect to the physical measure obtained by taking $v = 1$ in (1.6).*

(v) *If $\mathcal{M}$ is connected and there exists a smooth invariant measure μ with $\operatorname{supp}\mu = \mathcal{M}$, then for any $j = 0, \dots, \kappa$,*

$$\dim\big(\ker d_X|_{\mathcal{C}^{-\infty}_{E_u^*}(\mathcal{M})\otimes\Lambda^j\mathfrak{a}^*_{\mathbb{C}}} / \operatorname{ran} d_X|_{\mathcal{C}^{-\infty}_{E_u^*}\otimes\Lambda^{j-1}\mathfrak{a}^*_{\mathbb{C}}}\big) = \binom{\kappa}{j}.$$

We show that the isomorphism stated in (i) and the complex measures in (iv) can be constructed explicitly in terms of spectral projectors built from the parametrix (1.5). We refer to Propositions 5.4 and 5.10 for these constructions and for slightly more complete statements.

In the case of a single Anosov flow, physical measures are known to coincide with SRB measures (see e.g. [59] and references therein). The latter are usually defined as invariant measures that can locally be disintegrated along the stable or unstable foliation of the flow with absolutely continuous conditional densities.

We prove in [29] that the microlocal characterization in Theorem 3 (ii) of physical measures via their wavefront set implies that the physical measures of an Anosov action are exactly those invariant measures that allow an absolutely continuous disintegration along the stable manifolds. We show in [29, Theorem 2] that for each physical/SRB mea-

[4] The dimension can be expressed more concretely in dynamical terms; see [29, Theorem 3].

sure, there is a basin $B \subset \mathcal{M}$ of positive Lebesgue measure such that for all $f \in C^0(\mathcal{M})$, all proper open subcones $\mathcal{C} \subset \mathcal{W}$ and all $x \in B$, we have the convergence

$$\mu(f) = \lim_{T\to\infty} \frac{1}{\mathrm{Vol}(\mathcal{C}_T)} \int_{A\in\mathcal{C}_T} f(\varphi_1^{-X_A}(x))\, dA. \tag{1.7}$$

Moreover, we prove in [29, Theorem 3] that the measure μ can be written as an infinite weighted sum over Dirac measures on the periodic tori of the action, showing an equidistribution of periodic tori in the support of μ. Finally, we show that this measure has full support in $\mathcal{M}$ if the action is positively transitive in the sense that there is a dense orbit $\bigcup_{A\in\mathcal{W}} \varphi_1^{X_A}(x)$ for some $x \in \mathcal{M}$. As mentioned before, the existence of such a measure is considered as an important step towards the resolution of the rigidity conjecture of [39].

1.2. Relation to previous results

The notion of resonances for certain particular Anosov flows appeared in the work of Ruelle [50], and was later extended by Pollicott [49]. The introduction of a spectral approach based on anisotropic Banach and Hilbert spaces came later and allowed the definition of resonances in the general setting, first for Anosov/Axiom A diffeomorphisms [4, 5, 20, 26], then for general Anosov/Axiom A flows [8, 15, 16, 21, 25, 43, 45]. It was also applied to the case of pseudo-Anosov maps [19], Morse–Smale flows [11], geodesic flows for manifolds with cusps [30] and billiards [2]. This spectral approach has been used to study SRB measures [5, 8] but it also led to several important consequences on the dynamical zeta function [15, 16, 23, 25] of flows, and links with topological invariants [10, 12, 17].

Concerning the notion of joint spectrum in dynamics, there are several cases that have been considered but they correspond to a different context of systems with symmetries (e.g. [3]).

Higher rank Anosov $\mathbb{R}^\kappa$-actions have in particular been studied mostly for their rigidity: they are conjectured to be always smoothly conjugate to several models, mostly of algebraic nature (see e.g. the introduction of [54] for a precise statement and a state of the art on this question). The local rigidity of Anosov $\mathbb{R}^\kappa$-actions near *standard Anosov actions*[5] was proved in [39], and an important step of the proof relies on showing

$$\ker d_X|_{C^\infty(\mathcal{M})\otimes\Lambda^1\mathfrak{a}^*_{\mathbb{C}}} / \mathrm{ran}\, d_X|_{C^\infty(\mathcal{M})} = \mathbb{C}^\kappa.$$

The main tools are based on representation theory to prove fast mixing with respect to the canonical invariant (Haar) measure. It is also conjectured in [38] that, more generally, for such standard actions one has, for $j = 1, \dots, \kappa - 1$,

$$\ker d_X|_{C^\infty(\mathcal{M})\otimes\Lambda^j\mathfrak{a}^*_{\mathbb{C}}} / \mathrm{ran}\, d_X|_{C^\infty(\mathcal{M})\otimes\Lambda^{j-1}\mathfrak{a}^*_{\mathbb{C}}} = \mathbb{C}^{\binom{\kappa}{j}}.$$

This can be compared to (v) in Theorem 3, except that there the functional space is different. Having a notion of Ruelle–Taylor resonances provides an approach to obtain

[5] This class, defined in [39], consists of Weyl chamber flows associated to rank κ locally symmetric spaces and variations of those.

exponential mixing for more general Anosov actions by generalizing microlocal techniques for spectral gaps [47, 57] to a suitable class of higher rank Anosov actions, and by using the functional calculus of Taylor [55, 58]. We believe that such tools might be very useful to obtain new results on the rigidity conjecture.

We would like to conclude by pointing out a different direction: on rank $\kappa > 1$ locally symmetric spaces $\Gamma\backslash G/K$, there is a commuting algebra of invariant differential operators that can be considered as a quantum analogue of Weyl chamber flows. If the locally symmetric space is compact, this algebra always has a discrete joint spectrum of L^2-eigenvalues. Its joint spectrum and relations to trace formulae have been studied in [13]. In [34], it is shown that a subset of the Ruelle–Taylor resonances for the Weyl chamber flow are in correspondence with the joint discrete spectrum of the invariant differential operators on $\Gamma\backslash G/K$, giving a generalization of the classical/quantum correspondence of [14, 31, 42] to higher rank.

1.3. *Outline of the article*

In Section 2 we introduce the geometric setting of Anosov actions and the admissible lifts that we study. In Section 3 we explain how to define the Taylor spectrum for a certain class of unbounded operators and discuss some properties of this Taylor spectrum. In Section 4 we prove Theorems 1 and 2, using microlocal analysis. A sketch of the central techniques is given at the beginning of Section 4. The last Section 5 is devoted to the proof of Theorem 3. In Appendix A, we recall some classical results of microlocal analysis needed in this paper.

2. Geometric preliminaries

2.1. *Anosov actions*

We first want to explain the geometric setting of Anosov actions and the admissible lifts that we will study.

Let $(\mathcal{M}, g)$ be a closed, smooth Riemannian manifold (normalized with volume 1) equipped with a smooth locally free action $\tau : \mathbb{A} \to \mathrm{Diffeo}(\mathcal{M})$ of an abelian Lie group $\mathbb{A} \cong \mathbb{R}^\kappa$. Let $\mathfrak{a} := \mathrm{Lie}(\mathbb{A}) \cong \mathbb{R}^\kappa$ be the associated commutative Lie algebra and $\exp : \mathfrak{a} \to \mathbb{A}$ the Lie group exponential map. After identifying $\mathbb{A} \cong \mathfrak{a} \cong \mathbb{R}^\kappa$, this exponential map is simply the identity, but it will be quite useful to have a notation that distinguishes between transformations and infinitesimal transformations. Taking the derivative of the $\mathbb{A}$-action one obtains the infinitesimal action, called an $\mathfrak{a}$-*action*, which is an injective Lie algebra homomorphism

$$X : \mathfrak{a} \to C^\infty(\mathcal{M}; T\mathcal{M}), \quad A \mapsto X_A := \frac{d}{dt}\Big|_{t=0} \tau(\exp(At)). \tag{2.1}$$

Note that X can alternatively be seen as a Lie algebra morphism into the space $\mathrm{Diff}^1(\mathcal{M})$ of first order differential operators. By commutativity of $\mathfrak{a}$, $\mathrm{ran}(X) \subset C^\infty(\mathcal{M}; T\mathcal{M})$ is a

κ-dimensional subspace of commuting vector fields which span a κ-dimensional smooth subbundle which we call the *neutral subbundle* $E_0 \subset T\mathcal{M}$. Note that this subbundle is tangent to the $\mathbb{A}$-orbits on $\mathcal{M}$. It is often useful to study the one-parameter flow generated by a vector field X_A, which we denote by $\varphi_t^{X_A}$. One has the obvious identity $\varphi_t^{X_A} = \tau(\exp(At))$ for $t \in \mathbb{R}$. The Riemannian metric on $\mathcal{M}$ induces norms on $T\mathcal{M}$ and $T^*\mathcal{M}$, both denoted by $\|\cdot\|$.

Definition 2.1. An element $A \in \mathfrak{a}$ and its corresponding vector field X_A are called *transversely hyperbolic* if there is a continuous splitting

$$T\mathcal{M} = E_0 \oplus E_u \oplus E_s \tag{2.2}$$

that is invariant under the flow $\varphi_t^{X_A}$ and such that there are $\nu, C > 0$ with

$$\|d\varphi_t^{X_A} v\| \le Ce^{-\nu|t|}\|v\|, \quad \forall v \in E_s, \forall t \ge 0, \tag{2.3}$$

$$\|d\varphi_t^{X_A} v\| \le Ce^{-\nu|t|}\|v\|, \quad \forall v \in E_u, \forall t \le 0. \tag{2.4}$$

We say that the $\mathbb{A}$-action is *Anosov* if there exists an $A_0 \in \mathfrak{a}$ such that X_{A_0} is transversely hyperbolic.

Given a transversely hyperbolic element $A_0 \in \mathfrak{a}$ we define the *positive Weyl chamber* $\mathcal{W} \subset \mathfrak{a}$ to be the set of $A \in \mathfrak{a}$ which are transversely hyperbolic with the same stable/unstable bundle as A_0.

Lemma 2.2. *Given an Anosov action and a transversely hyperbolic element $A_0 \in \mathfrak{a}$, the positive Weyl chamber $\mathcal{W} \subset \mathfrak{a}$ is an open convex cone.*

Proof. Let us first take the $\varphi_t^{X_{A_0}}$-invariant splitting $E_0 \oplus E_u \oplus E_s$ and show that it is in fact invariant under the Anosov action τ. Let $v \in E_u$ and $A \in \mathfrak{a}$. Using $[X_{A_0}, X_A] = 0$, for each $t_0 \in \mathbb{R}$ fixed and all $t \in \mathbb{R}$ we find

$$d\varphi_{-t}^{X_{A_0}} d\varphi_{t_0}^{X_A} v = d\varphi_{t_0}^{X_A} d\varphi_{-t}^{X_{A_0}} v. \tag{2.5}$$

In particular, $\|d\varphi_{-t}^{X_{A_0}} d\varphi_{t_0}^{X_A} v\|$ decays exponentially fast as $t \to +\infty$. This implies that $d\varphi_{t_0}^{X_A} v \in E_u$ and the same argument works with E_s. Next, we choose an arbitrary norm on $\mathfrak{a}$. There exist $C, C' > 0$ such that for each $v \in E_u$ we have, for $t \ge 0$,

$$\|d\varphi_{-t}^{X_A} v\| \le \|d\varphi_{-t}^{X_{A-A_0}} d\varphi_{-t}^{X_{A_0}} v\| \le C\|v\|e^{-\nu t}\|d\varphi_{-t}^{X_{A-A_0}}\| \le C\|v\|e^{-\nu t}e^{C't\|A-A_0\|}.$$

This implies that by choosing $\|A - A_0\|$ small enough, E_u is an unstable bundle for A as well. The same construction works for E_s and we have thus shown that $\mathcal{W}$ is open.

By re-parametrization, it is clear that $\mathcal{W}$ is a cone, so that only the convexity is left to be proved. Now, take $A_1, A_2 \in \mathcal{W}$ and let C_1, ν_1, C_2, ν_2 be the corresponding constants for the transverse hyperbolicity estimates (2.3) and (2.4). Then for $s \in [0, 1]$ and $v \in E_u$ we can again use commutativity to obtain

$$\|d\varphi_{-t}^{X_{sA_1+(1-s)A_2}} v\| \le C_1C_2e^{-\nu_1 st-\nu_2(1-s)t}\|v\|, \tag{2.6}$$

and this shows that $sA_1 + (1-s)A_2 \in \mathcal{W}$. ∎

Here we emphasize that the Weyl chamber $\mathcal{W}$ only depends on the Anosov splitting associated to A_0 but not on A_0 itself. Notice also that in general there are other Weyl chambers $\mathcal{W}'$ associated to a different Anosov splitting. In the standard example of Weyl chamber flows they are images of $\mathcal{W}$ by the Weyl group of the higher rank locally symmetric space, explaining the Weyl chamber terminology (see for example [34] for details). In general the structure of Weyl chambers can be quite complicated (see for example the example of non-total Anosov actions given in [54, Section 6.3.4]). In that case, the Ruelle–Taylor spectrum that we shall define has no reason to be the same for $\mathcal{W}$ and for $\mathcal{W}'$.

There is an important class of examples given by the Weyl chamber flow on Riemannian locally symmetric spaces.

Example 2.3. Consider a real semisimple Lie group $\mathbb{G}$, connected and of non-compact type, and let $\mathbb{G} = \mathbb{K}\mathbb{A}\mathbb{N}$ be an Iwasawa decomposition with $\mathbb{A}$ abelian, $\mathbb{K}$ the maximal compact subgroup and $\mathbb{N}$ nilpotent. Then $\mathbb{A} \cong \mathbb{R}^\kappa$ and κ is called the *real rank* of $\mathbb{G}$. Let $\mathfrak{a}$ be the Lie algebra of $\mathbb{A}$ and consider the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$ which leads to the definition of a finite set of *restricted roots* $\Delta \subset \mathfrak{a}^*$. For $\alpha \in \Delta$ let $\mathfrak{g}_\alpha$ be the associated root space. It is then possible to choose a set $\Delta_+ \subset \Delta$ of positive roots and with respect to this choice there is an algebraic definition of a positive Weyl chamber:

$$\mathcal{W} := \{A \in \mathfrak{a} \mid \alpha(A) > 0 \text{ for all } \alpha \in \Delta_+\}.$$

If one now considers a torsion-free, discrete, cocompact subgroup $\Gamma < \mathbb{G}$, one can define the biquotient $\mathcal{M} := \Gamma\backslash\mathbb{G}/\mathbb{M}$ where $\mathbb{M} \subset \mathbb{K}$ is the centralizer of $\mathbb{A}$ in $\mathbb{K}$. As $\mathbb{A}$ commutes with $\mathbb{M}$, the space $\mathcal{M}$ carries a right $\mathbb{A}$-action. Using the definition of roots, it is direct to see that this is an Anosov action: all elements of the positive Weyl chamber $\mathcal{W}$ are transversely hyperbolic elements sharing the same stable/unstable distributions given by the associated vector bundles:

$$E_0 = \mathbb{G} \times_{\mathbb{M}} \mathfrak{a}, \quad E_s = \mathbb{G} \times_{\mathbb{M}} \mathfrak{n}, \quad E_u = \mathbb{G} \times_{\mathbb{M}} \overline{\mathfrak{n}}.$$

Here $\mathfrak{n} := \sum_{\alpha\in\Delta_+} \mathfrak{g}_\alpha$ and $\overline{\mathfrak{n}} := \sum_{-\alpha\in\Delta_+} \mathfrak{g}_\alpha$ are the sums of all positive, respectively negative, root spaces, and $\mathfrak{n}$ coincides with the Lie algebra of the nilpotent group $\mathbb{N}$.

Note that there are various other constructions of Anosov actions and we refer to [39, Section 2.2] for further examples.

2.2. *Admissible lifts*

We want to establish the spectral theory not only for the commuting vector fields X_A that act as first order differential operators on $C^\infty(\mathcal{M})$ but also for first order differential operators on Riemannian vector bundles $E \to \mathcal{M}$ which lift the Anosov action.

Definition 2.4. Let $\mathcal{M}$ be a closed manifold with an Anosov action of $\mathbb{A} \cong \mathbb{R}^\kappa$ and generating map X. Let $E \to \mathcal{M}$ be the complexification of a smooth Riemannian vector bundle over $\mathcal{M}$. Denote by $\mathrm{Diff}^1(\mathcal{M}; E)$ the Lie algebra of first order differential operators with smooth coefficients and scalar principal symbol, acting on sections of E. Then we call a Lie algebra homomorphism

$$\mathbf{X} : \mathfrak{a} \to \mathrm{Diff}^1(\mathcal{M}; E)$$

an *admissible lift* of the Anosov action if it satisfies the following Leibniz rule: for any section $s \in C^\infty(\mathcal{M}; E)$ and any function $f \in C^\infty(\mathcal{M})$ one has, for all $A \in \mathfrak{a}$,

$$\mathbf{X}_A(fs) = (X_A f)s + f\mathbf{X}_A s. \tag{2.7}$$

A typical example to have in mind would be when E is a tensor bundle (e.g. an exterior power $\Lambda^m T^*\mathcal{M}$ of the cotangent bundle or the bundle $\otimes_S^m T^*\mathcal{M}$ of symmetric tensors), and

$$\mathbf{X}_A s := \mathcal{L}_{X_A} s$$

where $\mathcal{L}$ denotes the Lie derivative. This admissible lift can be restricted to any subbundle that is invariant under the differentials $d\varphi_t^{X_A}$ for all $A \in \mathfrak{a}$ and $t > 0$. Another class of examples comes from flat connections. More generally, the above examples can be seen as a special case where the $\mathbb{A}$-action τ on $\mathcal{M}$ lifts to an action $\tilde{\tau}$ on E which is fiberwise linear. Then one can define an infinitesimal action

$$\mathbf{X}_A s(x) := \partial_t \tilde{\tau}(\exp(-At))s\big(\tau(\exp(At))x\big)\big|_{t=0} \tag{2.8}$$

which is an admissible lift.

3. Taylor spectrum and Fredholm complex

The Taylor spectrum was introduced by Taylor [55, 56] as a joint spectrum for commuting bounded operators, using the theory of Koszul complexes. While there are different competing notions of joint spectra (see e.g. the lecture notes [9]), the Taylor spectrum is from many perspectives the most natural notion. Its attractive feature is that it is defined in terms of operators acting on Hilbert spaces and does not depend on the choice of an ambient commutative Banach algebra. Furthermore, it comes with a satisfactory analytic functional calculus developed by Taylor and Vasilescu [55, 58].

3.1. Taylor spectrum for unbounded operators

Most references introduce the Taylor spectrum for tuples of bounded operators. In our case, we need to deal with unbounded operators. Additionally, working with a tuple implies choosing a basis, which should not be necessary. Let us thus explain how the notion of Taylor spectrum can easily be extended to an important class of abelian actions by unbounded operators.

We start with a smooth complex vector bundle $E \to \mathcal{M}$ over a smooth manifold $\mathcal{M}$ (not necessarily compact), an abelian Lie algebra $\mathfrak{a} \cong \mathbb{R}^\kappa$ and a Lie algebra morphism $\mathbf{X} : \mathfrak{a} \to \mathrm{Diff}^1(\mathcal{M}; E)$. For the moment we do not have to assume that $\mathcal{M}$ possesses an

Anosov action. Note that $\mathbf{X}$ extends by linearity to $\mathbf{X} : \mathfrak{a}_{\mathbb{C}} \to \mathrm{Diff}^1(\mathcal{M}; E)$ and for the definition of the spectra we will need to work with this complexified version. Using the map $\mathbf{X}$ we define

$$d_{\mathbf{X}} : C_c^\infty(\mathcal{M}; E) \to C_c^\infty(\mathcal{M}; E) \otimes \mathfrak{a}_{\mathbb{C}}^*, \quad u \mapsto \mathbf{X}u,$$

where we have set $(\mathbf{X}u)(A) := \mathbf{X}_A u$ for each $A \in \mathfrak{a}_{\mathbb{C}}$. This will be the central ingredient to define the Koszul complex which will lead to the definition of the Taylor spectrum. In order to do this we need some more notation: we denote by $\Lambda\mathfrak{a}_{\mathbb{C}}^* := \bigoplus_{\ell=0}^{\kappa} \Lambda^\ell \mathfrak{a}_{\mathbb{C}}^*$ the exterior algebra of $\mathfrak{a}_{\mathbb{C}}^*$ – this is just a coordinate-free version of $\Lambda\mathbb{C}^\kappa$. Given a topological vector space V we use the shorthand notation $V\Lambda^\ell := V \otimes \Lambda^\ell \mathfrak{a}_{\mathbb{C}}^*$ and $V\Lambda := V \otimes \Lambda\mathfrak{a}_{\mathbb{C}}^*$. As $\Lambda\mathfrak{a}_{\mathbb{C}}^*$ is finite-dimensional, $V\Lambda$ is again a topological vector space. We notice that since $\Lambda\mathfrak{a}_{\mathbb{C}}^*$ is a finite-dimensional vector space, we can view it as a trivial bundle $\mathcal{M} \times \Lambda\mathfrak{a}_{\mathbb{C}}^* \to \mathcal{M}$, and when $V = C_c^\infty(\mathcal{M}; E)$, $V = L^p(\mathcal{M}; E)$ or $V = \mathcal{D}'(\mathcal{M}; E)$, elements in $V \otimes \Lambda^\ell \mathfrak{a}_{\mathbb{C}}^*$ can be identified respectively with sections of $V = C_c^\infty(\mathcal{M}; E \otimes \Lambda\mathfrak{a}_{\mathbb{C}}^*)$, $L^p(\mathcal{M}; E \otimes \Lambda\mathfrak{a}_{\mathbb{C}}^*)$ or $\mathcal{D}'(\mathcal{M}; E \otimes \Lambda\mathfrak{a}_{\mathbb{C}}^*)$. We shall freely make this identification as this will sometimes be useful when we use pseudodifferential operators.

We have the *contraction* and *exterior product* maps

$$\iota : \mathfrak{a}_{\mathbb{C}} \times V\Lambda^\ell \to V\Lambda^{\ell-1}, \quad (A, v \otimes \omega) \mapsto \iota_A(v \otimes \omega) := v \otimes (\iota_A \omega),$$
$$\wedge : V\Lambda^\ell \times \Lambda^r \mathfrak{a}_{\mathbb{C}}^* \to V\Lambda^{\ell+r}, \quad (v \otimes \omega, \eta) \mapsto v \otimes (\omega \wedge \eta).$$

We can then extend $d_{\mathbf{X}}$ to a continuous map on $C_c^\infty\Lambda := C_c^\infty(\mathcal{M}; E) \otimes \Lambda\mathfrak{a}_{\mathbb{C}}^*$ (resp. $C^{-\infty}\Lambda := C^{-\infty}(\mathcal{M}; E) \otimes \Lambda\mathfrak{a}_{\mathbb{C}}^*$) by setting, for each $u \in C_c^\infty(\mathcal{M}; E)$ (resp. $u \in C^{-\infty}(\mathcal{M}; E)$) and $\omega \in \Lambda^\ell \mathfrak{a}_{\mathbb{C}}^*$,

$$d_{\mathbf{X}} : u \otimes \omega \mapsto (d_{\mathbf{X}} u) \wedge \omega.$$

Similarly, for each $A \in \mathfrak{a}$ we will also extend $\mathbf{X}_A$ on these spaces by setting

$$\mathbf{X}_A(u \otimes \omega) := \mathbf{X}_A u \otimes \omega = (\iota_A d_{\mathbf{X}} u) \otimes \omega.$$

Remark 3.1. Choosing a basis $A_1, \ldots, A_\kappa \in \mathfrak{a}$ provides an isomorphism $\Lambda\mathfrak{a}^* \cong \Lambda\mathbb{R}^\kappa$. One checks that under this isomorphism the coordinate free version $d_{\mathbf{X}} : V \otimes \Lambda^\ell \mathfrak{a}^* \to V \otimes \Lambda^{\ell+1}\mathfrak{a}^*$ of the Taylor differential transforms to the Taylor differential $d_X : V \otimes \Lambda^\ell \mathbb{R}^\kappa \to V \otimes \Lambda^{\ell+1}\mathbb{R}^\kappa$ of the operator tuple $X = (\mathbf{X}_{A_1}, \ldots, \mathbf{X}_{A_\kappa})$ defined as

$$d_X(u \otimes e_{i_1} \wedge \cdots \wedge e_{i_j}) := \sum_{k=1}^{\kappa} (\mathbf{X}_{A_k} u) \otimes e_k \wedge e_{i_1} \wedge \cdots \wedge e_{i_j} \tag{3.1}$$

if the basis $(e_j)_j$ of $\mathbb{R}^\kappa$ is identified with the dual basis of $(A_j)_j$ in $\mathfrak{a}^*$.

Lemma 3.2. *For each $A \in \mathfrak{a}_{\mathbb{C}}$ one has the following identities as continuous operators on $C_c^\infty\Lambda$ and $C^{-\infty}\Lambda$:*

(i) $\iota_A d_{\mathbf{X}} + d_{\mathbf{X}} \iota_A = \mathbf{X}_A$,

(ii) $\mathbf{X}_A d_{\mathbf{X}} = d_{\mathbf{X}} \mathbf{X}_A$,

(iii) $d_{\mathbf{X}} d_{\mathbf{X}} = 0$.

Proof. Let $u \otimes \omega \in C_c^\infty\Lambda$ or $u \otimes \omega \in C^{-\infty}\Lambda$. Then by definition

$$\iota_A d_{\mathbf{X}}(u \otimes \omega) = \iota_A((d_{\mathbf{X}}u) \wedge \omega) = (\mathbf{X}_A u) \otimes \omega - d_{\mathbf{X}}u \wedge (\iota_A\omega) = (\mathbf{X}_A - d_{\mathbf{X}}\iota_A)(u \otimes \omega),$$

which yields (i). In order to prove (ii) it suffices, by definition of $d_{\mathbf{X}}$, to prove the identity as a map $C_c^\infty(\mathcal{M}; E) \to C_c^\infty(\mathcal{M}; E) \otimes \mathfrak{a}_{\mathbb{C}}^*$. Take arbitrary $A, A' \in \mathfrak{a}_{\mathbb{C}}$ and note that by definition $\iota_{A'}\mathbf{X}_A = \mathbf{X}_A\iota_{A'}$. Then for $u \in C_c^\infty(\mathcal{M}; E)$ we get

$$\iota_{A'}(\mathbf{X}_A d_{\mathbf{X}} - d_{\mathbf{X}}\mathbf{X}_A)u = (\mathbf{X}_A\mathbf{X}_{A'} - \mathbf{X}_{A'}\mathbf{X}_A)u = 0,$$

which proves the statement. Note that we crucially use the commutativity of the differential operators $\mathbf{X}_A$ in this step.

For (iii) we first conclude from (i) and (ii) that $\iota_A d_{\mathbf{X}} d_{\mathbf{X}} = d_{\mathbf{X}} d_{\mathbf{X}} \iota_A$. Using this identity we deduce that for $u \in C_c^\infty\Lambda^\ell$ and arbitrary $A_1, \dots, A_{\ell+1} \in \mathfrak{a}_{\mathbb{C}}$,

$$\iota_{A_1} \dots \iota_{A_{\ell+1}} d_{\mathbf{X}} d_{\mathbf{X}} u = 0,$$

which implies $d_{\mathbf{X}} d_{\mathbf{X}} u = 0$. ∎

As a direct consequence of Lemma 3.2 (iii) we conclude that

$$0 \to C_c^\infty\Lambda^0 \xrightarrow{d_{\mathbf{X}}} C_c^\infty\Lambda^1 \xrightarrow{d_{\mathbf{X}}} \cdots \xrightarrow{d_{\mathbf{X}}} C_c^\infty\Lambda^\kappa \to 0 \tag{3.2}$$

and

$$0 \to C^{-\infty}\Lambda^0 \xrightarrow{d_{\mathbf{X}}} C^{-\infty}\Lambda^1 \xrightarrow{d_{\mathbf{X}}} \cdots \xrightarrow{d_{\mathbf{X}}} C^{-\infty}\Lambda^\kappa \to 0 \tag{3.3}$$

are complexes.

We now want to construct a complex of bounded operators on Hilbert spaces which lies between the complexes on $C_c^\infty\Lambda$ and $C^{-\infty}\Lambda$. For this, we consider a Hilbert space $\mathcal{H}$ with continuous embeddings $C_c^\infty(\mathcal{M}; E) \subset \mathcal{H} \subset C^{-\infty}(\mathcal{M}; E)$ such that $C_c^\infty(\mathcal{M}; E)$ is a dense subspace of $\mathcal{H}$. If we fix a non-degenerate Hermitian inner product $\langle\cdot,\cdot\rangle_{\mathfrak{a}_{\mathbb{C}}^*}$, then this induces a scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}\Lambda}$ and gives a Hilbert space structure on $\mathcal{H}\Lambda$. While the precise value of $\langle\cdot,\cdot\rangle_{\mathcal{H}\Lambda}$ obviously depends on the choice of the Hermitian product on $\mathfrak{a}_{\mathbb{C}}^*$, the finite-dimensionality of $\mathfrak{a}_{\mathbb{C}}^*$ implies that all Hilbert space structures on $\mathcal{H}\Lambda$ obtained in this way are equivalent. Note that on the Hilbert spaces $\mathcal{H}\Lambda^\ell$ the operators $d_{\mathbf{X}}$ will in general be unbounded operators. However, we have the following result.

Lemma 3.3. *For any choice of a non-degenerate Hermitian product on $\mathfrak{a}_{\mathbb{C}}^*$, the vector space $\mathcal{D}(d_{\mathbf{X}}) := \{u \in \mathcal{H}\Lambda \mid d_{\mathbf{X}}u \in \mathcal{H}\Lambda\}$ becomes a Hilbert space when endowed with the scalar product*

$$\langle\cdot,\cdot\rangle_{\mathcal{D}(d_{\mathbf{X}})} := \langle\cdot,\cdot\rangle_{\mathcal{H}\Lambda} + \langle d_{\mathbf{X}}\cdot, d_{\mathbf{X}}\cdot\rangle_{\mathcal{H}\Lambda}. \tag{3.4}$$

Furthermore, all scalar products obtained this way are equivalent and induce the same topology on $\mathcal{D}(d_{\mathbf{X}})$. Finally, $d_{\mathbf{X}}$ is bounded *on $\mathcal{D}(d_{\mathbf{X}})$.*

Proof. We have to check that $\mathcal{D}(d_{\mathbf{X}})$ is complete with respect to the topology of $\langle\cdot,\cdot\rangle_{\mathcal{D}(d_{\mathbf{X}})}$. Suppose u_n is a Cauchy sequence in $\mathcal{D}(d_{\mathbf{X}})$. Then u_n and $d_{\mathbf{X}}u_n$ are Cauchy sequences in $\mathcal{H}\Lambda$ and we denote by $v_0, v_1 \in \mathcal{H}\Lambda$ their respective limits. By the continuous embedding $\mathcal{H} \subset C^{-\infty}(\mathcal{M}; E)$ and the continuity of $d_{\mathbf{X}}$ on $C^{-\infty}(\mathcal{M}; E)$ we deduce

$$v_1 = \lim_{n\to\infty} d_{\mathbf{X}} u_n = d_{\mathbf{X}} \lim_{n\to\infty} u_n = d_{\mathbf{X}} v_0$$

in $C^{-\infty}(\mathcal{M}; E)$, which proves the completeness. For the boundedness, we take $u \in \mathcal{D}(d_{\mathbf{X}})$ and we compute

$$\|d_{\mathbf{X}} u\|^2_{\mathcal{D}(d_{\mathbf{X}})} = \|d_{\mathbf{X}} u\|^2_{\mathcal{H}\Lambda} + \|d_{\mathbf{X}} d_{\mathbf{X}} u\|^2_{\mathcal{H}\Lambda} \leq \|u\|^2_{\mathcal{D}(d_{\mathbf{X}})}. \qquad \blacksquare$$

To be able to use the usual techniques, it is crucial that $C_c^\infty(\mathcal{M}; E)$ is not only dense in $\mathcal{H}$ but also in $\mathcal{D}(d_{\mathbf{X}})$ – on this level of generality, this is not a priori guaranteed. For this reason, we say the $\mathfrak{a}$-action $\mathbf{X}$ has a *unique extension* to $\mathcal{H}$ if

$$\overline{C_c^\infty(\mathcal{M}; E)\Lambda}^{\mathcal{D}(d_{\mathbf{X}})} = \mathcal{D}(d_{\mathbf{X}}). \tag{3.5}$$

We note that by [21, Lemma A.1], if $\mathcal{M}$ is a closed manifold and $\mathcal{H} = \mathcal{A}(L^2(\mathcal{M}; E)\Lambda)$ for some invertible pseudodifferential operator $\mathcal{A}$ on $\mathcal{M}$ with $\mathcal{A}^{-1} d_{\mathbf{X}} \mathcal{A} \in \Psi^1(\mathcal{M}; E \otimes \Lambda)$ (see Appendix A for the notation), then $C^\infty(\mathcal{M}; E\Lambda)\Lambda$ is dense in $\mathcal{D}(d_{\mathbf{X}})$ and there is only one closed extension for $d_{\mathbf{X}}$.

In order to finally define the Taylor spectrum in an invariant way, we consider $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ as a Lie algebra morphism

$$\lambda : \mathfrak{a}_{\mathbb{C}} \to \mathrm{Diff}^0(\mathcal{M}; E) \subset \mathrm{Diff}^1(\mathcal{M}; E), \quad \lambda(A)(u) := \lambda(A)u.$$

In this way we can define $\mathbf{X} - \lambda : \mathfrak{a}_{\mathbb{C}} \to \mathrm{Diff}^1(\mathcal{M}; E)$ and the associated operator $d_{\mathbf{X}-\lambda}$ on $C_c^\infty\Lambda$ and $C^{-\infty}\Lambda$. Since $d_{\mathbf{X}-\lambda} = d_{\mathbf{X}} - d_\lambda$, and d_λ is bounded on $\mathcal{H}\Lambda$, $\mathcal{D}(d_{\mathbf{X}-\lambda})$ does not depend on λ. Furthermore, note that from Lemma 3.2 we know that $d^2_{\mathbf{X}-\lambda} = 0$ on $C_c^\infty\Lambda$ and by density of $C_c^\infty(\mathcal{M}, E)\Lambda \subset \mathcal{D}(d_{\mathbf{X}})$ and boundedness of $d_{\mathbf{X}-\lambda} : \mathcal{D}(d_{\mathbf{X}}) \to \mathcal{D}(d_{\mathbf{X}})$ we deduce $d^2_{\mathbf{X}-\lambda} = 0$ on $\mathcal{D}(d_{\mathbf{X}})$. For $k = 0, \ldots, \kappa$, we write $\mathcal{D}^k(d_{\mathbf{X}}) := \mathcal{D}(d_{\mathbf{X}}) \cap \mathcal{H}\Lambda^k$ and we gather the results above in the following lemma.

Lemma 3.4. *For an $\mathfrak{a}$-action $\mathbf{X}$ with a unique closed extension to $\mathcal{H}$, for any $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$,*

$$0 \to \mathcal{D}^0(d_{\mathbf{X}}) \xrightarrow{d_{\mathbf{X}-\lambda}} \mathcal{D}^1(d_{\mathbf{X}}) \xrightarrow{d_{\mathbf{X}-\lambda}} \cdots \xrightarrow{d_{\mathbf{X}-\lambda}} \mathcal{D}^\kappa(d_{\mathbf{X}}) \to 0 \tag{3.6}$$

defines a complex of bounded operators, and the operators $d_{\mathbf{X}-\lambda}$ depend holomorphically on $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.

Recall from the discussion above that the unique extension property was crucially used to get $d_{\mathbf{X}-\lambda} \circ d_{\mathbf{X}-\lambda} = 0$, thus to have a well-defined complex of bounded operators.

We introduce the notation

$$\begin{aligned}
\ker_{\mathcal{H}\Lambda} d_{\mathbf{X}-\lambda} &:= \ker_{\mathcal{D}(d_{\mathbf{X}})\to\mathcal{D}(d_{\mathbf{X}})} d_{\mathbf{X}-\lambda}, \\
\mathrm{ran}_{\mathcal{H}\Lambda} d_{\mathbf{X}-\lambda} &:= \mathrm{ran}_{\mathcal{D}(d_{\mathbf{X}})\to\mathcal{D}(d_{\mathbf{X}})} d_{\mathbf{X}-\lambda}, \\
\ker_{\mathcal{H}\Lambda^j} d_{\mathbf{X}-\lambda} &:= \ker_{\mathcal{D}^j(d_{\mathbf{X}})\to\mathcal{D}^{j+1}(d_{\mathbf{X}})} d_{\mathbf{X}-\lambda}, \\
\mathrm{ran}_{\mathcal{H}\Lambda^j} d_{\mathbf{X}-\lambda} &:= \mathrm{ran}_{\mathcal{D}^{j-1}(d_{\mathbf{X}})\to\mathcal{D}^j(d_{\mathbf{X}})} d_{\mathbf{X}-\lambda}.
\end{aligned} \tag{3.7}$$

Now, following the previous discussion of the Taylor spectrum, we can make the following definition.

Definition 3.5. Let $\mathbf{X} : \mathfrak{a} \to \mathrm{Diff}^1(\mathcal{M}; E)$ be a Lie algebra morphism and $\mathcal{H}$ a Hilbert space such that the $\mathfrak{a}$-action $\mathbf{X}$ has a unique extension to $\mathcal{H}$. Then we define the *Taylor spectrum* $\sigma_{\mathrm{T},\mathcal{H}}(\mathbf{X}) \subset \mathfrak{a}^*_{\mathbb{C}}$ by

$$\lambda \in \sigma_{\mathrm{T},\mathcal{H}}(\mathbf{X}) \iff \mathrm{ran}_{\mathcal{H}\Lambda}\, d_{\mathbf{X}-\lambda} \neq \ker_{\mathcal{H}\Lambda} d_{\mathbf{X}-\lambda}.$$

This is equivalent to saying that (3.6) is not an exact sequence. The complex is said to be *Fredholm* if $\mathrm{ran}_{\mathcal{H}\Lambda}\, d_{\mathbf{X}-\lambda}$ is closed and the cohomology $\ker_{\mathcal{H}\Lambda} d_{\mathbf{X}-\lambda}/\,\mathrm{ran}_{\mathcal{H}\Lambda}\, d_{\mathbf{X}-\lambda}$ has finite dimension. In this case we say that λ is not in the *essential Taylor spectrum* $\sigma^{\mathrm{ess}}_{\mathrm{T},\mathcal{H}}(\mathbf{X})$ of $\mathbf{X}$ and define the *index* by

$$\mathrm{index}(\mathbf{X}-\lambda) := \sum_{\ell=0}^{\kappa} (-1)^{\ell} \dim(\ker_{\mathcal{H}\Lambda^{\ell}} d_{\mathbf{X}-\lambda}/\mathrm{ran}_{\mathcal{H}\Lambda^{\ell}}\, d_{\mathbf{X}-\lambda}). \tag{3.8}$$

As the usual Fredholm index for a single operator, the Fredholm index in the Taylor complex is also a locally constant function of λ (see [9, Theorem 6.6]).

Note that the non-vanishing of the zeroth cohomology $\ker_{\mathcal{H}\Lambda^0} d_{\mathbf{X}-\lambda}$ of the complex is equivalent to

$$\exists u \in \mathcal{D}^0(d_{\mathbf{X}}) \setminus \{0\}, \quad (\mathbf{X}_{A_j} - \lambda_j)u = 0,$$

which corresponds to $(\lambda_1, \ldots, \lambda_\kappa)$ being a joint eigenvalue of $(\mathbf{X}_{A_1}, \ldots, \mathbf{X}_{A_\kappa})$. Obviously, on infinite-dimensional vector spaces the joint eigenvalues do not provide a satisfactory notion of joint spectrum. Recall that for a single operator, $\lambda \in \mathbb{C}$ is in its spectrum if $\mathbf{X} - \lambda$ is either not injective or not surjective. In terms of the Taylor complex for a single operator ($\kappa = 1$) the non-injectivity corresponds to the vanishing of the zeroth cohomology group whereas the surjectivity corresponds to the vanishing of the first cohomology group.

Remark 3.6. So far we always started with a Lie algebra morphism $\mathbf{X} : \mathfrak{a} \to \mathrm{Diff}^1(\mathcal{M}; E)$, then considered the action of $\mathrm{Diff}^1(\mathcal{M}; E)$ on some topological vector space V (e.g. $C^\infty_c(\mathcal{M})$) in order to define the Taylor complex and the Taylor spectrum. This will also be our main case of interest. However, we notice that the construction of the operator $d_{\mathbf{X}}$ and the complex associated to $d_{\mathbf{X}}$ works exactly the same if we take instead any Lie algebra morphism

$$\mathbf{X} : \mathfrak{a} \to \mathcal{L}(V),$$

where V is a topological vector space and $\mathcal{L}(V)$ denotes the Lie algebra of continuous linear operators on V with Lie bracket $[A, B] := AB - BA$. We shall call the complex induced by $d_{\mathbf{X}}$ on $V\Lambda$ the *Taylor complex* of $\mathbf{X}$ on V. If V is a Hilbert space, we define the Taylor spectrum of $\mathbf{X}$ on V by

$$\lambda \in \sigma_{\mathrm{T},V}(\mathbf{X}) \iff \mathrm{ran}_{V\Lambda}\, d_{\mathbf{X}-\lambda} \neq \ker_{V\Lambda} d_{\mathbf{X}-\lambda}.$$

Such Lie algebra morphisms that do not directly come from differential operators will occasionally show up within the parametrix constructions in Sections 4 and 5.

3.2. Useful observations

For the reader not familiar with the Taylor spectrum, and for our own use, we have gathered in this section several observations that are helpful when manipulating these objects. First, we shall say that an operator $P : C^{-\infty}(\mathcal{M}; E) \otimes \Lambda\mathfrak{a}^*_{\mathbb{C}} \to C^{-\infty}(\mathcal{M}; E) \otimes \Lambda\mathfrak{a}^*_{\mathbb{C}}$ is Λ-*scalar* if there is an operator $P' : C^{-\infty}(\mathcal{M}; E) \to C^{-\infty}(\mathcal{M}; E)$ such that

$$\forall u \in C^{-\infty}(\mathcal{M}; E), \omega \in \Lambda\mathfrak{a}^*_{\mathbb{C}}, \quad P(u \otimes \omega) = (P'u) \otimes \omega.$$

As usual with differential complexes, we have a dual notion of divergence complex. For this, we need a way to identify $\mathfrak{a}$ with $\mathfrak{a}^*$, i.e. a scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{a}$, extended to a $\mathbb{C}$-bilinear two-form. If one chooses a basis, the implicit scalar product is given by the standard one in that basis. In any case, $A \mapsto A' := \langle A, \cdot\rangle$ is an isomorphism between $\mathfrak{a}$ and $\mathfrak{a}^*$. If

$$\mathbf{Y} : \mathfrak{a} \to \mathcal{L}(C^\infty_c(\mathcal{M}; E))$$

satisfies $[\mathbf{Y}_{A_1}, \mathbf{Y}_{A_2}] = 0$ for any $A_1, A_2 \in \mathfrak{a}$, then we can define the action $\mathbf{Y}' : \mathfrak{a}^* \mapsto \mathcal{L}(C^\infty_c(\mathcal{M}; E))$ by setting, for $u \in C^\infty_c(\mathcal{M}; E)$ and $A' \in \mathfrak{a}^*$, if A is dual to A',

$$\mathbf{Y}'_{A'} u := \mathbf{Y}_A u.$$

In this fashion, $d_{\mathbf{Y}'} u := \mathbf{Y}' u$ is an element of $C^\infty_c(\mathcal{M}; E) \otimes \mathfrak{a}$, while $d_{\mathbf{Y}} u$ is an element of $C^\infty_c(\mathcal{M}; E) \otimes \mathfrak{a}^*$. We can thus define the divergence operator associated to $\mathbf{Y}$ by

$$\delta_{\mathbf{Y}} : C^\infty_c(\mathcal{M}; E) \otimes \Lambda^j \mathfrak{a}^*_{\mathbb{C}} \to C^\infty_c(\mathcal{M}; E) \otimes \Lambda^{j-1}\mathfrak{a}^*_{\mathbb{C}}, \quad u \otimes \omega \mapsto -\iota_{\mathbf{Y}' u}\omega. \tag{3.9}$$

In an orthonormal basis $(e_j)_j$ of $\mathfrak{a}$ for $\langle\cdot,\cdot\rangle$ and $(e'_j)_j$ the dual basis in $\mathfrak{a}^*$, we get, for $u \in C^\infty_c(\mathcal{M}; E)$ and $\omega = e'_{i_1} \wedge \cdots \wedge e'_{i_\ell}$,

$$\delta_{\mathbf{Y}}(u \otimes \omega) = \sum_{j=1}^{\ell} (-1)^j (\mathbf{Y}_{e_{i_j}} u) e'_{i_1} \wedge \cdots \wedge \widehat{e'_{i_j}} \wedge \cdots \wedge e'_{i_\ell}.$$

We see directly that for $A' \in \mathfrak{a}^*$,

$$A' \wedge \delta_{\mathbf{Y}}(u \otimes \omega) + \delta_{\mathbf{Y}}(A' \wedge (u \otimes \omega)) = -A' \wedge \iota_{\mathbf{Y}'u}\omega - \iota_{\mathbf{Y}'u}(A' \wedge \omega) = -(A'(\mathbf{Y}'u)) \otimes \omega.$$

It follows from similar arguments as before that

$$\mathbf{Y}_A \delta_{\mathbf{Y}} = \delta_{\mathbf{Y}} \mathbf{Y}_A, \quad \delta_{\mathbf{Y}}\delta_{\mathbf{Y}} = 0.$$

We have the following result.

Lemma 3.7. *Let* $\mathbf{X} : \mathfrak{a} \to \mathrm{Diff}^1(\mathcal{M}; E)$ *be an admissible lift and* $\mathbf{Y} : \mathfrak{a} \to \mathcal{L}(C^\infty_c(\mathcal{M}; E))$ *satisfying* $\mathbf{Y}_{B_1}\mathbf{Y}_{B_2} = \mathbf{Y}_{B_2}\mathbf{Y}_{B_1}$ *for any* $B_1, B_2 \in \mathfrak{a}$, *such that* $\mathbf{X}_A\mathbf{Y}_B = \mathbf{Y}_B\mathbf{X}_A$ *for any* $A, B \in \mathfrak{a}$. *If we fix an inner product* $\langle\cdot,\cdot\rangle$ *on* $\mathfrak{a}$ *and a corresponding orthonormal basis* $(e_j)_j$ *and if* $\mathbf{X}_i := \mathbf{X}_{e_i}$ *and* $\mathbf{Y}_j := \mathbf{Y}_{e_j}$, *then we have, as continuous operators on* $C^\infty_c(\mathcal{M}; E)\Lambda$,

$$\delta_{\mathbf{Y}} d_{\mathbf{X}} + d_{\mathbf{X}} \delta_{\mathbf{Y}} = -\Big(\sum_{k=1}^{\kappa} \mathbf{X}_k \mathbf{Y}_k\Big) \otimes \mathrm{Id}\,.$$

The sum does not depend on the choice of basis, because it is the trace of the matrix representing $\mathbf{XY}$ *under* $\langle\cdot,\cdot\rangle$.

Proof. Let e_i' be the dual basis to the chosen orthogonal basis e_i. For $I = (i_1, \ldots, i_\ell)$ let $e_I' := e_{i_1}' \wedge \cdots \wedge e_{i_\ell}'$. Then for $u \in C_c^\infty(\mathcal{M}; E)$ we compute

$$\begin{aligned}
&d_{\mathbf{X}}\delta_{\mathbf{Y}}(u \otimes e_I')\\
&\quad = -\Big(\sum_{k\in I} (\mathbf{X}_k\mathbf{Y}_k u) \otimes e_I' + \sum_{k\notin I,\, j} (-1)^{j-1}(\mathbf{X}_k\mathbf{Y}_{i_j} u) \otimes e_k' \wedge e_{i_1}' \wedge \cdots \wedge \widehat{e_{i_j}'} \wedge \cdots \wedge e_{i_\ell}'\Big),\\
&\delta_{\mathbf{Y}} d_{\mathbf{X}}(u \otimes e_I')\\
&\quad = -\Big(\sum_{k\notin I} (\mathbf{Y}_k\mathbf{X}_k u) \otimes e_I' + \sum_{k\notin I,\, j} (-1)^{j}(\mathbf{Y}_{i_j}\mathbf{X}_k u) \otimes e_k' \wedge e_{i_1}' \wedge \cdots \wedge \widehat{e}_{i_j}' \wedge \cdots \wedge e_{i_\ell}'\Big).
\end{aligned}$$

Using the commutation $[\mathbf{X}_i, \mathbf{Y}_j] = 0$, we obtain the result. ∎

As an illustration, let us recall the following classical fact.

Lemma 3.8. *Let* $X_1, \ldots, X_\kappa$ *be commuting operators on a finite-dimensional vector space* V*. Then* $\sigma_{\mathrm{T},V}(X) = \{\text{joint eigenvalues of } X_1, \ldots, X_\kappa\} \subset \mathbb{C}^\kappa$.

Proof. By the basic theory of weight spaces (see e.g. [40, Proposition 2.4]) V can be decomposed into generalized weight spaces, i.e. there are finitely many $\lambda^{(j)} = (\lambda_1^{(j)}, \ldots \lambda_\kappa^{(j)}) \in \mathbb{C}^\kappa$ and a direct sum decomposition $V = \bigoplus_j V_j$ which is invariant under all $X_1, \ldots, X_\kappa$ and there are n_j such that

$$(X_i - \lambda_i^{(j)})^{n_j}|_{V_j} = 0, \quad \forall i = 1, \ldots, \kappa,\ \forall j.$$

Commutativity and the Jordan normal form then imply that the $\lambda^{(j)}$ are precisely the joint eigenvalues of the tuple X. Now let $\mu \neq \lambda^{(j)}$ for all j. We have to prove that $\mu \notin \sigma_{\mathrm{T},V}(X)$. Since $\mu \neq \lambda^{(j)}$ we deduce that for any j there is at least one $1 \le k_j \le \kappa$ such that $\mu_{k_j} \neq \lambda_{k_j}^{(j)}$ and again by Jordan normal form, $X_{k_j} - \mu_{k_j} : V_j \to V_j$ is invertible. Setting $\widetilde{V}_k := \bigoplus_{k_j = k} V_j$, we can thus find an X-invariant decomposition $V = \bigoplus_{k=1}^\kappa \widetilde{V}_k$ such that $X_k - \mu_k : \widetilde{V}_k \to \widetilde{V}_k$ is invertible. Let Π_k be the projection onto $\widetilde{V}_k$ with respect to the above direct sum decomposition. Now set $Y_k := (X_k - \mu_k)^{-1}\Pi_k : V \to V$. Then the Y_k satisfy all the assumptions of Lemma 3.7 and

$$\delta_Y d_{X-\mu} + d_{X-\mu}\delta_Y = -\,\mathrm{Id}\,.$$

Consequently, the Taylor complex (3.6) is exact. ∎

In the particular case that $X = (X_1, \ldots, X_\kappa)$ are symmetric matrices, using the spectral theorem we can reduce the problem to the case that $X_1, \ldots, X_\kappa$ are scalars acting on some $\mathbb{R}^m$. From this we deduce that for $\lambda \in \sigma_{\mathrm{T},\mathbb{R}^m}(X)$,

$$\dim(\ker_{\Lambda^j} d_{X-\lambda} / \operatorname{ran}_{\Lambda^j} d_{X-\lambda}) = \dim(\mathbb{R}^m \otimes \Lambda^j \mathbb{R}^\kappa) = m\binom{\kappa}{j},$$

and we check that

$$\operatorname{index}(X-\lambda) = m\sum_{j=1}^{\kappa}(-1)^j\binom{\kappa}{j} = 0.$$

Our next step is to give a criterion for $d_{\mathbf{X}-\lambda}$ to be Fredholm. We first notice that since $\operatorname{ran} d_{\mathbf{X}-\lambda} \subset \ker d_{\mathbf{X}-\lambda}$, the closedness of $\operatorname{ran} d_{\mathbf{X}-\lambda}$ in $\mathcal{D}(d_{\mathbf{X}})$ and in $\mathcal{H}\Lambda$ is equivalent. Below, if $F \subset C^{-\infty}(\mathcal{M}; E)\Lambda$ is a vector subspace, we shall denote $\operatorname{ran}_F d_{\mathbf{X}} := \{d_{\mathbf{X}}u \mid u \in F\}$ and $\ker_F d_{\mathbf{X}} := \{u \in F \mid d_{\mathbf{X}}u = 0\}$. We shall use the following criterion for the $d_{\mathbf{X}}$-complex to be Fredholm.

Lemma 3.9. *Let* $\mathbf{X}$ *be an* $\mathfrak{a}$*-action with a unique extension to* $\mathcal{H}$*. Assume that there are bounded operators* Q*,* R *and* K *on* $\mathcal{H}\Lambda$*, acting continuously on* $C^{-\infty}(\mathcal{M}; E)\Lambda$*, such that* K *is compact,* $\|R\|_{\mathcal{L}(\mathcal{H}\Lambda)} < 1$*, and*

$$Qd_{\mathbf{X}} + d_{\mathbf{X}}Q = \mathrm{Id} + R + K.$$

Then the complex defined by $d_{\mathbf{X}}$ *is Fredholm. Denote by* Π_0 *the projector on the eigenvalue* 0 *of the Fredholm operator* $\mathrm{Id} + R + K$*; it is bounded on* $\mathcal{D}(d_{\mathbf{X}})$ *and commutes with* $d_{\mathbf{X}}$*. Then the map* $u \mapsto \Pi_0 u$ *from* $\ker d_{\mathbf{X}} \cap \mathcal{D}(d_{\mathbf{X}})$ *to* $\ker d_{\mathbf{X}} \cap \operatorname{ran}\Pi_0$ *descends to an isomorphism*

$$\Pi_0 : \ker_{\mathcal{D}(d_{\mathbf{X}})} d_{\mathbf{X}}/\operatorname{ran}_{\mathcal{D}(d_{\mathbf{X}})} d_{\mathbf{X}} \to \ker_{\operatorname{ran}\Pi_0} d_{\mathbf{X}}/\operatorname{ran}_{\operatorname{ran}\Pi_0} d_{\mathbf{X}}. \tag{3.10}$$

Proof. First, since Q, R and K are continuous on distributions, it makes sense to write $d_{\mathbf{X}}Q + Qd_{\mathbf{X}} = \mathrm{Id} + R + K$ in the distribution sense. Further, from this relation, we deduce that Q is bounded on $\mathcal{D}(d_{\mathbf{X}})$. Additionally, without loss of generality (by modifying R) we can assume that K is a finite rank operator.

Let us prove that the range of $d_{\mathbf{X}}$ is closed. Consider $u \in (\ker d_{\mathbf{X}})^{\perp} \cap \mathcal{D}(d_{\mathbf{X}})$. Since $d_{\mathbf{X}}Qu \in \operatorname{ran}(d_{\mathbf{X}}) \subset \ker d_{\mathbf{X}}$, we have

$$\langle(\mathrm{Id} + R + K)u, u\rangle_{\mathcal{H}\Lambda} = \langle Qd_{\mathbf{X}}u, u\rangle_{\mathcal{H}\Lambda}. \tag{3.11}$$

It follows that there is $C > 0$ such that for each $u \in (\ker d_{\mathbf{X}})^{\perp} \cap \mathcal{D}(d_{\mathbf{X}})$ we have

$$(1 - \|R\|)\|u\|_{\mathcal{H}\Lambda} - \|Ku\|_{\mathcal{H}\Lambda} \le C\|d_{\mathbf{X}}u\|_{\mathcal{H}\Lambda}. \tag{3.12}$$

Since K is of finite rank, we deduce by a standard argument that $d_{\mathbf{X}}$ has closed range (both in $\mathcal{H}\Lambda$ and in $\mathcal{D}(d_{\mathbf{X}})$).

The operator $F := \mathrm{Id} + R + K$ is Fredholm of index 0, and since $Fd_{\mathbf{X}} = d_{\mathbf{X}}Qd_{\mathbf{X}} = d_{\mathbf{X}}F$ *on distributions*, we deduce that $F : \mathcal{D}(d_{\mathbf{X}}) \to \mathcal{D}(d_{\mathbf{X}})$ is bounded. Since F is Fredholm of index 0, we know that $s \mapsto (F - s)^{-1} \in \mathcal{L}(\mathcal{H}\Lambda)$ is meromorphic in $\mathbb{C} \setminus B(1, \|R\|_{\mathcal{L}(\mathcal{H}\Lambda)})$ and for $s \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ close to 0 it is analytic. Note that

$$\begin{aligned}\|Fu\|^2_{\mathcal{D}(d_{\mathbf{X}})} &= \|Fu\|^2_{\mathcal{H}\Lambda} + \|d_{\mathbf{X}}Fu\|^2_{\mathcal{H}\Lambda}\\ &= \|Fu\|^2_{\mathcal{H}\Lambda} + \|Fd_{\mathbf{X}}u\|^2_{\mathcal{H}\Lambda} \le \|F\|^2_{\mathcal{L}(\mathcal{H}\Lambda)}\|u\|^2_{\mathcal{D}(d_{\mathbf{X}})}\end{aligned}$$

and thus $F - s$ is invertible on $\mathcal{D}(d_{\mathbf{X}})$ for $|s| > \|F\|_{\mathcal{L}(\mathcal{H}\Lambda)}$. This implies that $(F-s)^{-1} : \mathcal{D}(d_{\mathbf{X}}) \to \mathcal{D}(d_{\mathbf{X}})$ is itself bounded in $\{|s| > \|F\|_{\mathcal{L}(\mathcal{H}\Lambda)}\}$ with $d_{\mathbf{X}}(F-s)^{-1} = (F-s)^{-1} d_{\mathbf{X}}$, and it extends meromorphically to $s \in \mathbb{C} \setminus B(1, \|R\|_{\mathcal{L}(\mathcal{H}\Lambda)})$ as an operator $(F-s)^{-1} : \mathcal{D}(d_{\mathbf{X}}) \to \mathcal{H}\Lambda$. By meromorphic continuation we have, for all $u \in \mathcal{D}(d_{\mathbf{X}})$ and $s \in \mathbb{C} \setminus B(1, \|R\|_{\mathcal{L}(\mathcal{H}\Lambda)})$ not a pole of $(F-s)^{-1}$,

$$d_{\mathbf{X}}(F-s)^{-1}u = (F-s)^{-1} d_{\mathbf{X}} u \quad \text{in } C^{-\infty}(\mathcal{M}; E)\Lambda.$$

In particular, for all s close to 0 we get $d_{\mathbf{X}}(F-s)^{-1}u \in \mathcal{H}\Lambda$ with $\|d_{\mathbf{X}}(F-s)^{-1}u\|_{\mathcal{H}\Lambda} \leq \|(F-s)^{-1}\|_{\mathcal{L}(\mathcal{H}\Lambda)} \|u\|_{\mathcal{D}(d_{\mathbf{X}})}$, i.e. $(F-s)^{-1} : \mathcal{D}(d_{\mathbf{X}}) \to \mathcal{D}(d_{\mathbf{X}})$ is bounded, and $d_{\mathbf{X}}(F-s)^{-1} = (F-s)^{-1} d_{\mathbf{X}}$ on $\mathcal{D}(d_{\mathbf{X}})$.

In that case, the spectral projector Π_0 of F for the eigenvalue 0 commutes with $d_{\mathbf{X}}$, is bounded on $\mathcal{D}(d_{\mathbf{X}})$, and since $\mathcal{D}(d_{\mathbf{X}})$ is dense in $\mathcal{H}\Lambda$ and Π_0 has finite rank, its image is contained in $\mathcal{D}(d_{\mathbf{X}})$. Further, we can write $F = (F + \Pi_0)(\mathrm{Id} - \Pi_0)$, and $\widetilde{F} := F + \Pi_0$ is invertible on $\mathcal{H}\Lambda$ and $\mathcal{D}(d_{\mathbf{X}})$, and commuting with $d_{\mathbf{X}}$, so that on $\mathcal{D}(d_{\mathbf{X}})$,

$$d_{\mathbf{X}} \widetilde{F}^{-1} Q + \widetilde{F}^{-1} Q d_{\mathbf{X}} = \mathrm{Id} - \Pi_0. \tag{3.13}$$

In particular, for $u \in \ker d_{\mathbf{X}} \cap \mathcal{D}(d_{\mathbf{X}})$, we have

$$u = d_{\mathbf{X}} \widetilde{F}^{-1} Q u + \Pi_0 u. \tag{3.14}$$

Since Π_0 and $d_{\mathbf{X}}$ commute, $u \mapsto \Pi_0 u$ descends to a homomorphism (3.10) in cohomology. This map in cohomology is obviously surjective since $\operatorname{ran} \Pi_0 \subset \mathcal{D}(d_{\mathbf{X}})$. To prove that it is injective, we need to prove that if $\Pi_0 u \in d_{\mathbf{X}} \operatorname{ran} \Pi_0$ for $u \in \ker d_{\mathbf{X}} \cap \mathcal{D}(d_{\mathbf{X}})$, then $u \in d_{\mathbf{X}} \mathcal{D}(d_{\mathbf{X}})$. This actually follows directly from (3.14) by using the fact that both $\widetilde{F}^{-1}$ and Q are bounded on $\mathcal{D}(d_{\mathbf{X}})$. ∎

We can also deduce the following.

Lemma 3.10. *Under the assumptions of Lemma 3.9, if $F := \mathrm{Id} + K + R$ is of the form $F = F' \otimes \mathrm{Id}$ where F' is an operator on $\mathcal{H}$ (i.e. F is Λ-scalar), then $0 \in \sigma_{\mathrm{T},\mathcal{H}}(\mathbf{X})$ if and only if there exists a non-zero $u \in \mathcal{D}(d_{\mathbf{X}}) \cap \mathcal{H}$ such that $\mathbf{X}u = 0$.*

Proof. From Lemma 3.9, we deduce that $0 \in \sigma_{\mathrm{T},\mathcal{H}}(\mathbf{X})$ if and only if the complex given by $d_{\mathbf{X}}$ is not exact on $\operatorname{ran} \Pi_0$ (recall that $d_{\mathbf{X}}$ commutes with Π_0). However, if F is Λ-scalar, then $\Pi_0 = \Pi'_0 \otimes \mathrm{Id}$ with Π'_0 the spectral projector at 0 of F' on $\mathcal{H}$, and $\operatorname{ran} \Pi_0 = (\operatorname{ran} \Pi'_0) \otimes \Lambda \mathfrak{a}^*_{\mathbb{C}}$. It follows that $d_{\mathbf{X}}$ restricted to $\operatorname{ran} \Pi_0$ is exactly the Taylor complex of the operator $\mathbf{X}$ on $\operatorname{ran} \Pi'_0$ in the sense of Remark 3.6. We are thus reduced to finite dimension and we can apply Lemma 3.8. ∎

The version of the Analytic Fredholm Theorem for the Taylor spectrum is the following statement.

Proposition 3.11. *Let $\mathbf{X}$ be an $\mathfrak{a}$-action with a unique extension to $\mathcal{H}$. Then the set $\sigma_{\mathrm{T},\mathcal{H}}(\mathbf{X}) \setminus \sigma^{\mathrm{ess}}_{\mathrm{T},\mathcal{H}}(\mathbf{X})$ is a complex analytic submanifold of $\mathbb{C}^\kappa \setminus \sigma^{\mathrm{ess}}_{\mathrm{T},\mathcal{H}}(\mathbf{X})$.*

Proof. As the complex (3.6) is an analytic Fredholm complex of bounded operators on $\mathbb{C}^\kappa \setminus \sigma^{\mathrm{ess}}_{\mathrm{T},\mathcal{H}}(\mathbf{X})$, the statement is classical and a proof can be found in [46, Theorem 2.9]. ■

In general, the question of whether the spectrum is discrete does not seem to have a very simple answer. For example, a characterization can be found in [1, Corollary 2.6 and Lemma 2.7]. Such a criterion is particularly adapted to microlocal methods and it can actually be used in our setting. However, it turns out that an even simpler criterion is sufficient for us.

Lemma 3.12. *Under the assumptions of Lemma 3.9, assume in addition that $Q = \delta_{\mathbf{Q}}$ for some Lie algebra morphism $\mathbf{Q} : \mathfrak{a} \to \mathcal{L}(\mathcal{H})$ such that $\mathbf{Q}_A$ acts continuously on $C^{-\infty}(\mathcal{M}; E)$ and $[\mathbf{Q}_A, \mathbf{X}_B] = 0$ for all $A, B \in \mathfrak{a}$. Then Lemma 3.10 applies, and the Taylor spectrum of $\mathbf{X}$ on $\mathcal{H}$ is discrete in a neighborhood of* 0.

Proof. Let $A_1, \dots, A_\kappa \in \mathfrak{a}$ be an orthonormal basis for $\langle \cdot, \cdot \rangle$ and let $Q_j := \mathbf{Q}_{A_j}$. We observe from Lemma 3.7 that for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ the following identity holds on $\mathcal{D}(d_{\mathbf{X}})$:

$$d_{\mathbf{X}-\lambda} Q + Q d_{\mathbf{X}-\lambda} = (\underbrace{-\mathbf{X}_{A_1} Q_1 - \cdots - \mathbf{X}_{A_\kappa} Q_\kappa}_{=F'} + \underbrace{\lambda_1 Q_1 + \cdots + \lambda_\kappa Q_\kappa}_{=\lambda \cdot Q}) \otimes \mathrm{Id}\,.$$

Thus, denoting $F'(\lambda) := F' + \lambda \cdot Q$ on $\mathcal{H}$ and $F(\lambda) := F'(\lambda) \otimes \mathrm{Id}$ on $\mathcal{H}\Lambda$, we see that Lemma 3.10 indeed applies.

Next, we observe two things. The first is that F' and $\lambda \cdot Q$ commute. The second is that for λ small enough, $F'(\lambda)$ can still be decomposed in the form $\mathrm{Id} + R(\lambda) + K(\lambda)$ with $\|R(\lambda)\|_{\mathcal{L}(\mathcal{H})} < 1$ and $K(\lambda)$ compact, because Q is bounded. It follows that $d_{\mathbf{X}-\lambda}$ is Fredholm for λ close enough to 0.

From Lemma 3.9, we know that the cohomology of $d_{\mathbf{X}-\lambda}$ on $\mathcal{D}(d_{\mathbf{X}})$ is isomorphic to

$$\ker_{\operatorname{ran}\Pi_0(\lambda)} d_{\mathbf{X}-\lambda} / \operatorname{ran}_{\operatorname{ran}\Pi_0(\lambda)} d_{\mathbf{X}-\lambda},$$

and the isomorphism is given by $[u] \mapsto [\Pi_0(\lambda)u]$ if $\Pi_0(\lambda)$ denotes the spectral projector of $F(\lambda)$ at 0 and $[\cdot]$ denotes cohomology class. Let us now describe a sort of *sandwiching procedure*. Assume that we have a projector Π_2 bounded on $\mathcal{D}(d_{\mathbf{X}})$, commuting with $d_{\mathbf{X}-\lambda}$. Then the mapping $[u] \mapsto [\Pi_2 u]$ is well-defined and surjective as a map

$$\ker_{\mathcal{D}(d_{\mathbf{X}})} d_{\mathbf{X}-\lambda} / \operatorname{ran}_{\mathcal{D}(d_{\mathbf{X}})} d_{\mathbf{X}-\lambda} \to \ker_{\operatorname{ran}\Pi_2} d_{\mathbf{X}-\lambda} / \operatorname{ran}_{\operatorname{ran}\Pi_2} d_{\mathbf{X}-\lambda}. \tag{3.15}$$

In general, there is no reason for this map to be *injective*. However, if we further assume that Π_2 and $\Pi_0(\lambda)$ commute, and $\operatorname{ran}\Pi_0(\lambda) \subset \operatorname{ran}\Pi_2$, then we can see $\Pi_0(\lambda)$ as a projector on $\operatorname{ran}\Pi_2$. The mapping $[\Pi_2 u] \mapsto [\Pi_0(\lambda)u]$ for $u \in \ker_{\mathcal{D}(d_{\mathbf{X}})} d_{\mathbf{X}-\lambda} / \operatorname{ran}_{\mathcal{D}(d_{\mathbf{X}})} d_{\mathbf{X}-\lambda}$ is well-defined as a map

$$\ker_{\operatorname{ran}\Pi_2} d_{\mathbf{X}-\lambda} / \operatorname{ran}_{\operatorname{ran}\Pi_2} d_{\mathbf{X}-\lambda} \to \ker_{\operatorname{ran}\Pi_0(\lambda)} d_{\mathbf{X}-\lambda} / \operatorname{ran}_{\operatorname{ran}\Pi_0(\lambda)} d_{\mathbf{X}-\lambda}$$

by using $\ker \Pi_2 \subset \ker \Pi_0(\lambda)$, and it has to be surjective. Using this and the surjectivity of (3.15) we deduce the bounds

$$\begin{aligned}&\dim(\ker_{\mathcal{D}(d_{\mathbf{X}})} d_{\mathbf{X}-\lambda}/\operatorname{ran}_{\mathcal{D}(d_{\mathbf{X}})} d_{\mathbf{X}-\lambda})\\&\quad\leq \dim(\ker_{\operatorname{ran}\Pi_2} d_{\mathbf{X}-\lambda}/\operatorname{ran}_{\operatorname{ran}\Pi_2} d_{\mathbf{X}-\lambda}) \leq \dim(\ker_{\operatorname{ran}\Pi_0(\lambda)} d_{\mathbf{X}-\lambda}/\operatorname{ran}_{\operatorname{ran}\Pi_0(\lambda)} d_{\mathbf{X}-\lambda}).\end{aligned}$$

Since we have proved above that the lower and upper bounds are equal, (3.15) is actually an isomorphism.

Let us write $\widetilde{F}' := F' + \Pi_0'$ where Π_0' is the spectral projector of F' at 0. We have the following identity on $\mathcal{H}$:

$$\widetilde{F}'^{-1}F'(\lambda) = \mathrm{Id} - \Pi_0' + \widetilde{F}'^{-1}\lambda\cdot Q.$$

For $u \in \ker F'(\lambda)$, we have $(\mathrm{Id}-\Pi_0')u = -\widetilde{F}'^{-1}\lambda\cdot Qu$. Since $\widetilde{F}'^{-1}\lambda\cdot Q$ commutes with Π_0' (as F' commutes with $\lambda\cdot Q$), we deduce that, for $u\in\ker F'(\lambda)\subset\mathcal{H}$,

$$(\mathrm{Id}-\Pi_0')u = (\mathrm{Id}-\Pi_0')^2u = -\widetilde{F}'^{-1}\lambda\cdot Q(\mathrm{Id}-\Pi_0')u.$$

For λ small enough $\mathrm{Id}+\widetilde{F}'^{-1}\lambda\cdot Q$ is invertible on $\mathcal{H}$, which implies that $(\mathrm{Id}-\Pi_0')u=0$. In particular, $u\in\operatorname{ran}\Pi_0'$, so that $\ker F'(\lambda)\subset\ker F'$ and $\operatorname{ran}\Pi_0'(\lambda)\subset\operatorname{ran}\Pi_0'$. But certainly Π_0' and $\Pi_0'(\lambda)$ commute. So we can apply the argument above with Π_0' playing the role of Π_2, and deduce that for λ sufficiently small,

$$\ker_{\mathcal{D}(d_{\mathbf{X}})} d_{\mathbf{X}-\lambda}/\operatorname{ran}_{\mathcal{D}(d_{\mathbf{X}})} d_{\mathbf{X}-\lambda} \simeq \ker_{\operatorname{ran}\Pi_0'} d_{\mathbf{X}-\lambda}/\operatorname{ran}_{\operatorname{ran}\Pi_0'} d_{\mathbf{X}-\lambda}.$$

Since $\operatorname{ran}\Pi_0'$ is a fixed finite-dimensional space, the Taylor spectrum of $\mathbf{X}$ is discrete near zero by Lemma 3.8. ∎

4. Discrete Ruelle–Taylor resonances via microlocal analysis

In this section, $\mathcal{M}$ is a compact manifold, equipped with a vector bundle $E\to\mathcal{M}$ and an admissible lift $\mathbf{X}$ of an Anosov action (see Definition 2.4). We have seen in Section 3.1 how to define the Taylor differential $d_{\mathbf{X}}$ which acts in its coordinate free form on $C^\infty(\mathcal{M};E)\otimes\Lambda\mathfrak{a}^*$. We have furthermore seen how $d_{\mathbf{X}}$ can be used to define a Taylor spectrum $\sigma_{\mathrm{T},\mathcal{H}}(\mathbf{X})\subset\mathfrak{a}^*_{\mathbb{C}}$. We take coordinates whenever it is convenient. In that case, we will use the notation d_X to avoid confusion. In what follows, it will be convenient to pass back and forth between these versions and we will mostly use the shorthand notation $C^\infty\Lambda$, leaving open which version we currently consider.

The Ruelle–Taylor resonances that we will introduce will correspond to a discrete spectrum of $-\mathbf{X}$ on some anisotropic Sobolev spaces. From a spectral-theoretic point of view this sign convention might seem unnatural. However, from a dynamical point of view this convention is very natural: given the flow φ_t^X of a vector field X, the one-parameter group that propagates probability densities with respect to an invariant measure is given by $(\varphi_{-t}^X)^*$ and thus is generated by the differential operator $-X$. We will therefore from now on consider the holomorphic family of complexes generated by $d_{\mathbf{X}+\lambda}$ for

$\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ (respectively $\lambda \in \mathbb{C}^\kappa$ after a choice of coordinates). Let us denote by $e^{-t\mathbf{X}_A}$ the 1-parameter family generated by $\mathbf{X}_A$, solving $\partial_t e^{-t\mathbf{X}_A} f = -\mathbf{X}_A e^{-t\mathbf{X}_A} f$ with $e^{-t\mathbf{X}_A} f|_{t=0} = f$. Since we work with spaces that are deformations of $L^2(\mathcal{M})$, we will compare our results with the growth rate of the action on $L^2(\mathcal{M})$, defined for $A \in \mathfrak{a}$ as

$$C_{L^2}(A) := \limsup \frac{1}{t} \log \|e^{-t\mathbf{X}_A}\|_{\mathcal{L}(L^2)}. \tag{4.1}$$

Naturally, the spectrum of $\mathbf{X}_A$ on L^2 is contained in $\{s \in \mathbb{C} \mid \operatorname{Re} s \leq C_{L^2}(A)\}$.

The goal of this section is to show the following:

Theorem 4. *Let τ be a smooth abelian Anosov action with generating map X and $\mathbf{X}$ an admissible lift. Let $A_0 \in \mathcal{W}$ be in the positive Weyl chamber. There exists $c > 0$, locally uniform with respect to A_0, such that for each $N > 0$, there is a Hilbert space $\mathcal{H}_N$ containing $C^\infty(\mathcal{M})$ and contained in $C^{-\infty}(\mathcal{M})$ such that the following holds true:*

(1) *$-X$ has no essential Taylor spectrum on the Hilbert space $\mathcal{H}_N$ in the region*

$$\mathcal{F}_N := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \operatorname{Re}(\lambda(A_0)) > -cN + C_{L^2}(A_0)\}.$$

(2) *For each $\lambda \in \mathcal{F}_N$ one has an isomorphism of finite-dimensional spaces*

$$\begin{aligned}\ker d_{\mathbf{X}+\lambda}|_{\mathcal{D}^j_N(d_{\mathbf{X}})} / \operatorname{ran} d_{\mathbf{X}+\lambda}|_{\mathcal{D}^{j-1}_N(d_{\mathbf{X}})} \\ = \ker d_{\mathbf{X}+\lambda}|_{C^{-\infty}_{E_u^*}(\mathcal{M})\otimes\Lambda^j\mathfrak{a}^*_{\mathbb{C}}} / \operatorname{ran} d_{\mathbf{X}+\lambda}|_{C^{-\infty}_{E_u^*}(\mathcal{M})\otimes\Lambda^{j-1}\mathfrak{a}^*_{\mathbb{C}}}\end{aligned}$$

with $\mathcal{D}^j_N(d_{\mathbf{X}}) := \{u \in \mathcal{H}_N \otimes \Lambda^j \mathfrak{a}^*_{\mathbb{C}} \mid d_{\mathbf{X}} u \in \mathcal{H}_N \otimes \Lambda^{j+1}\mathfrak{a}^*_{\mathbb{C}}\}$, showing that the cohomology dimension is independent of N and A_0.

(3) *The Taylor spectrum of $-\mathbf{X}$ contained in $\mathcal{F}_N$ is discrete and contained in*

$$\bigcap_{A\in\mathcal{W}} \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid \operatorname{Re}(\lambda(A)) \leq C_{L^2}(A)\}.$$

(4) *An element $\lambda \in \mathcal{F}_N$ is in the Taylor spectrum of $-\mathbf{X}$ on $\mathcal{H}_N$ if and only if λ is a joint Ruelle resonance of $\mathbf{X}$.*

The Hilbert space $\mathcal{H}_N$ will be rather written $\mathcal{H}_{NG}$ below, where G is a certain weight function on $T^*\mathcal{M}$ giving the rate of Sobolev differentiability in phase space. We use this notation in order to emphasize the dependence of the space on G.

The central point of the proof will be a parametrix construction for the exterior differential $d_{\mathbf{X}+\lambda}$. We will prove in Proposition 4.7 that there are holomorphic families of operators $Q(\lambda), F(\lambda) : C^{-\infty}\Lambda \to C^{-\infty}\Lambda$ such that

$$Q(\lambda) d_{\mathbf{X}+\lambda} + d_{\mathbf{X}+\lambda} Q(\lambda) = F(\lambda).$$

The operators $Q(\lambda)$ and $F(\lambda)$ will be Fourier integral operators and independent of any Hilbert space on which the operators act. However, the crucial fact is that for these operators there exists a scale of Hilbert spaces $C^\infty \subset \mathcal{H}_{NG} \subset C^{-\infty}$ (with $N \geq 0$ and

$G \in C^\infty(T^*\mathcal{M})$ a weight function) and domains $\mathcal{F}_{NG} \subset \mathfrak{a}^*_{\mathbb{C}}$ with $\mathfrak{a}^*_{\mathbb{C}} = \bigcup_{N>0} \mathcal{F}_{NG}$ such that for $\lambda \in \mathcal{F}_{NG}$ the operators $Q(\lambda) : \mathcal{H}_{NG} \to \mathcal{H}_{NG}$ are bounded and the operators $F(\lambda) : \mathcal{H}_{NG} \to \mathcal{H}_{NG}$ are Fredholm and can be decomposed as $F(\lambda) = \mathrm{Id} + R(\lambda) + K(\lambda)$ with $K(\lambda)$ compact and $\|R(\lambda)\|_{\mathcal{L}(\mathcal{H}_{NG}\Lambda)} < 1/2$. Then by Lemma 3.9 we directly conclude that the Taylor complex on $\mathcal{H}_{NG}\Lambda$ is Fredholm at $\lambda \in \mathcal{F}_{NG}$. The fact that the construction of the operator family $F(\lambda) : C^\infty\Lambda \to C^{-\infty}\Lambda$ is independent of the specific Hilbert spaces on which they act will be the key for proving in Section 4.3 that the Taylor spectrum of $d_{\mathbf{X}+\lambda}$ is intrinsic to the Anosov action, i.e. independent of the spaces $\mathcal{H}_{NG}$ constructed. The flexibility which we will have in the construction of the escape function G will furthermore allow us to identify this intrinsic spectrum with the spectrum of $d_{\mathbf{X}+\lambda}$ on the space $C^{-\infty}_{E_u^*}\Lambda$ of distributions with wavefront set contained in the annihilator $E_u^* \subset T^*\mathcal{M}$ of $E_u \oplus E_0$ (see Proposition 4.10). Finally, we will see that the choice of $Q(\lambda)$ can be made more *geometric*, to enable the use of Lemma 3.12 and prove that this intrinsic spectrum is discrete in $\mathfrak{a}^*_{\mathbb{C}}$.

The construction of the parametrix $Q(\lambda)$ and the Hilbert spaces $\mathcal{H}_{NG}$ will be done using microlocal analysis. Appendix A contains a brief summary of the necessary microlocal tools. Section 4.1 will be devoted to the construction of the anisotropic Sobolev spaces. With these tools at hand we will construct the parametrix (Section 4.2), and prove that the spectrum is intrinsic (Section 4.3) as well as discrete (Section 4.4).

4.1. *Escape function and anisotropic Sobolev spaces*

In this section we define the anisotropic Sobolev spaces. Their construction will be based on the choice of a so-called escape function for the given Anosov action. We first give the definition for such a function and then prove the existence of escape functions with additional useful properties.

Given any smooth vector field $X \in C^\infty(\mathcal{M}; T\mathcal{M})$ with flow φ_t^X we define the *symplectic lift* of the flow and the corresponding vector field by

$$\begin{aligned} &\Phi_t^X : T^*\mathcal{M} \to T^*\mathcal{M}, \quad (x,\xi) \mapsto (\varphi_t^X(x), ((d\varphi_t^X)^{-1})^T\xi), \\ &X^H := \frac{d}{dt}\Big|_{t=0} \Phi_t^X \in C^\infty(T^*\mathcal{M}; T(T^*\mathcal{M})), \end{aligned} \tag{4.2}$$

where $((d\varphi_t^X)^{-1})^T$ is the transpose of the inverted differential $(d\varphi_t^X)^{-1}$. The notation X^H is chosen because it is the Hamilton vector field of the principal symbol $\sigma_p^1(X)(x,\xi) = i\xi(X(x)) \in C^\infty(T^*\mathcal{M})$ of X (see Example A.2). Recall from Example A.2 that for an admissible lift of an Anosov action, the principal symbols of the lifted differential operator $\mathbf{X}_A$ and that of the vector field X_A tensorized with Id_E coincide. This will turn out to be the reason why we do not have to care about the admissible lifts for the construction of the escape function. We will denote by $\{0\} := \{(x,0) \in T^*\mathcal{M} \mid x \in M\}$ the zero section.

Definition 4.1. Let $c_X > 0$, $A \in \mathcal{W}$, and let $\Gamma_{E_0^*} \subset T^*\mathcal{M}$ be an open cone containing E_0^* satisfying $\overline{\Gamma}_{E_0^*} \cap (E_u^* \oplus E_s^*) = \{0\}$. Then a function $G \in C^\infty(T^*\mathcal{M}; \mathbb{R})$ is called an *escape function* for A compatible with c_X and $\Gamma_{E_0^*}$ if there is $R > 0$ with the following properties:

(1) $G(x,\xi) = 1$ for $|\xi| \leq R/2$, and for $|\xi| > 1$ one can write $G(x,\xi) = m(x,\xi)\log(1 + f(x,\xi))$. Here $m \in C^\infty(T^*\mathcal{M}; [-1/2, 8])$ and for $|\xi| > R$, m is positively homogeneous of degree 0, with $m \leq -1/4$ in a conic neighborhood of E_u^* and $m \geq 4$ in a conic neighborhood of E_s^*. Furthermore, $f \in C^\infty(T^*\mathcal{M}; \mathbb{R}^+)$ is positively homogeneous of degree 1 for $|\xi| > R$. We call m the *order function* of G.

(2) $X_A^H m(x,\xi) \leq 0$ for all $|\xi| > R$.

(3) $X_A^H G(x,\xi) \leq -c_X$ for $\xi \notin \Gamma_{E_0^*}$ with $|\xi| > R$.

Below (see Proposition 4.3), we will prove the existence of escape functions for Anosov actions. First, let us explain how we can build anisotropic Sobolev spaces based on an escape function. Given an escape function G, property (1) of Definition 4.1 implies that $m \in S_1^0(\mathcal{M})$ and for any $N > 0$, $e^{NG} \in S_{1-}^{Nm}(\mathcal{M})$ is a real elliptic symbol. According to [20, Lemma 12 and Corollary 4] there exists a pseudodifferential operator

$$\hat{\mathcal{A}}_{NG} \in \Psi_{1-}^{Nm}(\mathcal{M}; E) \tag{4.3}$$

such that

(1) $\sigma_p^{Nm}(\hat{\mathcal{A}}_{NG}) = e^{NG}\,\mathrm{Id}_E \mod S_{1-}^{Nm-1+}$,

(2) $\hat{\mathcal{A}}_{NG} : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E)$ is invertible,

(3) $\hat{\mathcal{A}}_{NG}^{-1} \in \Psi_{1-}^{-Nm}(\mathcal{M}; E)$ and $\sigma_p^{-Nm}(\hat{\mathcal{A}}_{NG}^{-1}) = e^{-NG}\,\mathrm{Id}_E \mod S_{1-}^{-Nm-1+}$.

We can now define the *anisotropic Sobolev spaces*

$$\mathcal{H}_{NG} := \hat{\mathcal{A}}_{NG}^{-1} L^2(\mathcal{M}; E) \quad \text{with scalar product} \quad \langle u, v\rangle_{\mathcal{H}_{NG}} := \langle \hat{\mathcal{A}}_{NG} u, \hat{\mathcal{A}}_{NG} v\rangle_{L^2}.$$

Note that the scalar product $\langle u, v\rangle_{\mathcal{H}_{NG}}$ depends not only on the choice of the escape function but also on the choice of its quantization $\hat{\mathcal{A}}_{NG}$. However, by L^2-continuity (Proposition A.9), these different choices all yield equivalent scalar products on the given vector space $\mathcal{H}_{NG}$. For that reason we can suppress this dependence in our notation.

We want to study the Taylor spectrum of the admissible lift of the Anosov action on these anisotropic Sobolev spaces. Recall from Section 3.1 that due to the unboundedness of the differential operators we have to verify the unique extension property:

Lemma 4.2. *For any escape function G the $\mathfrak{a}$-action of an admissible lift has a unique extension (in the sense that (3.5) holds) to the anisotropic Hilbert space $\mathcal{H}_{NG}$.*

Proof. Let us consider the Taylor differential $d_{\mathbf{X}}$ as an unbounded operator on $\mathcal{H}_{NG}\Lambda$ with domain $C^\infty(\mathcal{M}; E \otimes \Lambda)$. Then, in the language of closed extensions, the desired equality (3.5) corresponds to the uniqueness of possible closed extensions. By unitary equivalence we can instead study the conjugate operator $P := \hat{\mathcal{A}}_{NG}\, d_{\mathbf{X}} \hat{\mathcal{A}}_{NG}^{-1}$ acting as an unbounded operator on $L^2(\mathcal{M}; E \otimes \Lambda)$. We want to apply [21, Lemma A.1] which states that any operator in $\Psi_1^1(\mathcal{M}; E \otimes \Lambda)$ has a unique closed extension as an unbounded operator on L^2 with domain C^∞. By Proposition A.3 and since $\hat{A}_{NG}$ has scalar principal symbol we can write $P = d_{\mathbf{X}} + [\hat{\mathcal{A}}_{NG}, d_{\mathbf{X}}]\hat{\mathcal{A}}_{NG}^{-1}$, where the first summand is obviously in $\Psi^1_1(\mathcal{M}; E \otimes \Lambda)$ and the second one in $\Psi^{0+}_{1-}(\mathcal{M}; E \otimes \Lambda)$. Now, by Definition A.1 of symbol spaces, one checks that $S^{0+}_{1-}(\mathcal{M}; E \otimes \Lambda) \subset S^1_1(\mathcal{M}; E \otimes \Lambda)$. We conclude that $P \in \Psi^1_1(\mathcal{M}; E \otimes \Lambda)$ and we can apply [21, Lemma A.1], which completes the proof. ■

Let us now turn to the existence of escape functions.

Proposition 4.3. *Fix an arbitrary $A_0 \in \mathcal{W} \subset \mathfrak{a}$, an open cone $\Gamma_{\mathrm{reg}} \subset T^*\mathcal{M}$ which is disjoint from E_u^*, and a small conic neighborhood Γ_0 of E_0^* such that $\overline{\Gamma}_0 \cap (E_s^* \oplus E_u^*) = \{0\}$. Then there is a $c_X > 0$, an open conic neighborhood $\Gamma_{E_0^*} \subset \Gamma_0$ of E_0^*, and $R > 0$ such that there is an escape function G for A_0 compatible with c_X and $\Gamma_{E_0^*}$ with the additional property that the order function satisfies*

$$m(x, \xi) \geq 1/2 \quad \text{for } (x, \xi) \in \Gamma_{\mathrm{reg}} \text{ and } |\xi| > R. \tag{4.4}$$

Proof. This follows from [10, Lemma 3.2]: indeed, first we note that the proof there only uses the continuity of the decomposition $T^*\mathcal{M} = E_0^* \oplus E_u^* \oplus E_s^*$ and the contracting/expanding properties of E_s^*, E_u^* but not the fact that E_0^* is one-dimensional. It suffices to take, in the notations of [10], $N_1 = 4$, $N_0 = 1/4$ and $\Gamma_{\mathrm{reg}} = T^*\mathcal{M} \setminus C^{uu}(\alpha_0)$ with $\alpha_0 > 0$ small enough. Although it is not explicitly written in the statement of [10, Lemma 3.3], the order function m constructed there satisfies $X^H_{A_0} m(x, \xi) \leq 0$ for $|\xi|$ large enough and $\Gamma_{E_0^*}$ is arbitrarily small if $\alpha_0 > 0$ is small (see [10, Section A.2]). ■

In the proof of the fact that the Ruelle–Taylor spectrum is discrete, we shall also need an escape function that works for all A in a neighborhood $\mathcal{U} \subset \mathcal{W}$ of a fixed element $A_1 \in \mathcal{W}$.

Lemma 4.4. *Let $A_1 \in \mathcal{W}$ be fixed. Then there is an escape function G for A_1, a conic neighborhood $\Gamma^*_{E_0} \subset T^*\mathcal{M}$ of E_0^* such that $\overline{\Gamma}_{E_0^*} \cap (E_u^* \oplus E_s^*) = \{0\}$, a constant $c_X > 0$ and a neighborhood $\mathcal{U} \subset \mathcal{W}$ of A_1 such that G is an escape function for all $A \in \mathcal{U}$ compatible with $c_X > 0$ and $\Gamma_{E_0^*}$. Moreover, G can be chosen to satisfy $X^H_A G \leq 0$ in $\{|\xi| \geq R\}$ for some $R \geq 1$.*

Proof. In a first step we need to construct an order function m that has all properties of Definition 4.11 and additionally $X^H_A m \leq 0$ for $|\xi| \geq R$ for all A close enough to A_1. To obtain it, we can follow exactly the construction for Anosov flows given in [28, Section 2]. It works mutatis mutandis in our case as the proof simply uses the continuity of the decomposition $T^*\mathcal{M} = E_0^* \oplus E_s^* \oplus E_u^*$ and the expanding/contracting properties of E_s^* and E_u^*, but not the fact that $\dim E_0^* = 1$.

We can then define the function G as in [21, Lemma 1.2] by setting $G(x, \xi) = m(x, \xi) \log(1 + f(x, \xi))$, where $f > 0$ is positively homogeneous of degree 1 in ξ for $|\xi| > R$, satisfies $f(x, \xi) = |\xi(X_{A_1})|$ near $E_0^* \cap \{|\xi| \geq 1\}$, and

$$X^H_A f < -c_1(1 + f) \quad (\text{resp. } X^H_A f > c_1/(1 + f)) \tag{4.5}$$

in a conic neighborhood of E_s^* (resp. of E_u^*) for some $c_1 > 0$. To construct such f

near E_s^*, we can use the construction from [16, Lemma C.1]: for (x,ξ) in a conic neighborhood N_s of E_s^*, set

$$f(x,\xi) := \int_0^T |e^{-tX_{A_1}^H}(x,\xi)|\,dt, \quad T>0,$$

so that, if $A = A_1 + \varepsilon A'$ with $|A'|_{\mathfrak{a}} \le 1$, one has $X_A^H = X_{A_1}^H + \varepsilon X_{A'}^H$,

$$\begin{aligned} X_A^H f(x,\xi) &= |\xi| - |e^{-TX_{A_1}^H}(x,\xi)| + \varepsilon \int_0^T X_{A'}^H |e^{-tX_{A_1}^H}(x,\xi)|\,dt \\ &= |\xi| - |e^{-TX_{A_1}^H}(x,\xi)| + \mathcal{O}(\varepsilon e^{CT}|\xi|) \end{aligned}$$

for some $C>0$ uniform with respect to A' as above, the last term following from the classical estimate $\max_{|\alpha|+|\beta|\le 1} \sup_{(x,\xi),\,|\xi|=1} \partial_x^\alpha \partial_\xi^\beta |e^{-tX_A^H}(x,\xi)| \le Ce^{C|t|}$ and the homogeneity in ξ. Fix T large enough that $|\xi| - |e^{-TX_A^H}(x,\xi)| \le -2|\xi|$ for all $|\xi|>1$ in N_s. Once T has been fixed, one can choose $0<\varepsilon<e^{-CT}$ so that $X_A^H f(x,\xi) \le -|\xi|$ in $N_s \cap \{|\xi|>1\}$. Since $1 + f(x,\xi) > c_1^{-1}|\xi|$ in $N_s \cap \{|\xi|>1\}$ for some $c_1>0$, we obtain (4.5). The same construction applies near E_u^*. We then extend f to a positively homogeneous function of degree 1 in $\{|\xi| \ge R\}$ in a smooth fashion (its value far from $E_u^* \cup E_s^* \cup E_0^*$ will not matter). The proof of [21, Lemma 1.2] (using the fact that $X_A^H|\xi(X_{A_1})| = 0$ as $[X_A, X_{A_1}] = 0$) shows that $X_A^H G \le 0$ for all $|\xi| \ge R$ if R is large enough and that G is an escape function for all $A \in \mathcal{U} := A_1 + \{\varepsilon A' \in \mathfrak{a} \mid |A'|_{\mathfrak{a}} \le 1\}$ compatible with some $c_X>0$ and some $\Gamma_{E_0^*}$. ∎

4.2. Parametrix construction

The goal of this section is to construct an operator $Q(\lambda)$ as in Lemma 3.9 for the complex $d_{\mathbf{X}+\lambda}$, and so that Q will be bounded on the anisotropic Sobolev spaces $\mathcal{H}_{NG}\Lambda$. The construction will be microlocal in the elliptic region and dynamical near the characteristic set. In Section 4.4 we will provide an alternative construction of a $Q(\lambda)$ which is purely dynamical, i.e. which is a function of the operators $\mathbf{X}_{A_j}$.

Recall the notation $E\otimes\Lambda = E\otimes\Lambda\mathfrak{a}^*$. We will also freely identify operators $P: C^\infty(\mathcal{M};E) \to C^{-\infty}(\mathcal{M};E)$ with their Λ-scalar extensions on sections of $E\otimes\Lambda$.

Lemma 4.5. *Let $P \in \Psi^0(\mathcal{M};E)$ be such that* $\mathrm{WF}(\mathrm{Id}-P)$ *does not intersect a conic neighborhood of $E_u^* \oplus E_s^*$, and we make it act as a Λ-scalar operator. There exists a holomorphic family of pseudodifferential operators $Q_{\mathrm{ell}}(\lambda) \in \Psi^{-1}(\mathcal{M}; E\otimes\Lambda\mathfrak{a}_{\mathbb{C}}^*)$ for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ such that $Q_{\mathrm{ell}}(\lambda): C^\infty(\mathcal{M}; E\otimes\Lambda^k\mathfrak{a}_{\mathbb{C}}^*) \to C^\infty(\mathcal{M}; E\otimes\Lambda^{k-1}\mathfrak{a}_{\mathbb{C}}^*)$ for all k and*

$$d_{\mathbf{X}+\lambda} Q_{\mathrm{ell}}(\lambda) + Q_{\mathrm{ell}}(\lambda) d_{\mathbf{X}+\lambda} = (\mathrm{Id}-P) + S_1(\lambda) + S_2(\lambda) \tag{4.6}$$

with $S_1(\lambda) \in \Psi^{-1}(\mathcal{M}, E\otimes\Lambda)$ holomorphic in λ satisfying $\mathrm{WF}(S_1(\lambda)) \subset \mathrm{WF}(P) \cap \mathrm{WF}(\mathrm{Id}-P)$ *and $S_2(\lambda) \in \Psi^{-\infty}(\mathcal{M}; E\otimes\Lambda)$, also holomorphic in λ.*

Proof. We will use an arbitrary choice of basis $A_1, \dots, A_\kappa$ in $\mathfrak{a}$ and consider the commuting differential operators $\mathbf{X}_{A_1}, \dots, \mathbf{X}_{A_\kappa}$. Recall that the corresponding divergence operator $\delta_{\mathbf{X}+\lambda}$ on $C^\infty(\mathcal{M}; E) \otimes \Lambda\mathfrak{a}^*$ is defined by

$$\delta_{\mathbf{X}+\lambda}(u \otimes e_{i_1} \wedge \dots \wedge e_{i_\ell}) = \sum_{j=1}^{\ell} (-1)^j (\mathbf{X}_{A_{i_j}} + \lambda_{i_j}) u \otimes e_{i_1} \wedge \dots \wedge \widehat{e}_{i_j} \wedge \dots \wedge e_{i_\ell},$$

where $\lambda_j := \lambda(A_j) \in \mathbb{C}$ (here $(e_j)_j$ is a dual basis to A_j in $\mathfrak{a}^*$). Thus, using the commutations $[\mathbf{X}_{A_j} + \lambda_j, \mathbf{X}_{A_k} + \lambda_k] = 0$ and Lemma 3.7 with $\mathbf{Y}_j = \mathbf{X}_{A_j} + \lambda_j$, we find that the operator $\Delta_{\mathbf{X}+\lambda} := d_{\mathbf{X}+\lambda}\delta_{\mathbf{X}+\lambda} + \delta_{\mathbf{X}+\lambda} d_{\mathbf{X}+\lambda}$ is Λ-scalar and given for each $\omega \in \Lambda\mathfrak{a}^*$ by the expression

$$\Delta_{\mathbf{X}+\lambda}(u \otimes \omega) = -\Big(\sum_{k=1}^{\kappa} (\mathbf{X}_{A_k} + \lambda_k)^2 u\Big) \otimes \omega.$$

This shows that $\Delta_{\mathbf{X}+\lambda} \in \Psi^2(\mathcal{M}; E \otimes \Lambda)$ with principal symbol given by (see Example A.2)

$$\sigma_p^2(\Delta_{\mathbf{X}+\lambda})(x,\xi) = \|\xi_{E_0}\|^2 \, \mathrm{Id}_{E\otimes\Lambda} \quad \text{with} \quad \|\xi_{E_0}\|^2 := \sum_{k=1}^{\kappa} \xi(X_{A_k})^2.$$

It is an operator which is microlocally elliptic outside $E_u^* \oplus E_s^*$ (i.e. $\mathrm{ell}^2(\Delta_{\mathbf{X}+\lambda}) = T^*\mathcal{M} \setminus (E_u^* \oplus E_s^*)$). Thus, by Proposition A.7, if $P' \in \Psi^0(\mathcal{M}, E \otimes \Lambda)$ is Λ-scalar and has $\mathrm{WF}(P')$ contained in a conic open subset of $T^*\mathcal{M}$ not intersecting $E_u^* \oplus E_s^*$, then there exists a Λ-scalar pseudodifferential operator $Q_\Delta(\lambda) \in \Psi^{-2}(\mathcal{M}; E \otimes \Lambda)$ holomorphic in λ with $\mathrm{WF}(Q_\Delta(\lambda)) \subset \mathrm{WF}(P')$ such that

$$\Delta_{\mathbf{X}+\lambda} Q_\Delta(\lambda) = P' + S'(\lambda)$$

with $S'(\lambda) \in \Psi^{-\infty}(\mathcal{M}; E \otimes \Lambda)$ holomorphic in λ and Λ-scalar. We now choose P' so that $\mathrm{WF}(P') \cap (E_u^* \oplus E_s^*) = \emptyset$ and $\mathrm{WF}(\mathrm{Id} - P') \cap \mathrm{WF}(\mathrm{Id} - P) = \emptyset$; in other words, $P' = \mathrm{Id}$ microlocally on $\mathrm{WF}(\mathrm{Id} - P)$. Note that $d_{\mathbf{X}+\lambda}\Delta_{\mathbf{X}+\lambda} = \Delta_{\mathbf{X}+\lambda} d_{\mathbf{X}+\lambda}$ implies that

$$\Delta_{\mathbf{X}+\lambda}\big(Q_\Delta(\lambda) d_{\mathbf{X}+\lambda} - d_{\mathbf{X}+\lambda} Q_\Delta(\lambda)\big) = [P', d_{\mathbf{X}+\lambda}] + [S'(\lambda), d_{\mathbf{X}+\lambda}].$$

Using microlocal ellipticity of $\Delta_{\mathbf{X}+\lambda}$ outside $E_u^* \oplus E_s^*$ and the fact that

$$\begin{aligned} \mathrm{WF}([P', d_{\mathbf{X}+\lambda}]) &= \mathrm{WF}([\mathrm{Id} - P', d_{\mathbf{X}+\lambda}]) \subset \mathrm{WF}(P') \cap \mathrm{WF}(\mathrm{Id} - P'), \\ \mathrm{WF}([S'(\lambda), d_{\mathbf{X}+\lambda}]) &= \emptyset, \end{aligned}$$

we deduce from (A.1) that $\mathrm{WF}([Q_\Delta(\lambda), d_{\mathbf{X}+\lambda}]) \subset \mathrm{WF}(P') \cap \mathrm{WF}(\mathrm{Id} - P')$. In particular, since $P' = \mathrm{Id}$ microlocally on $\mathrm{WF}(\mathrm{Id} - P)$, this implies that $[Q_\Delta(\lambda), d_{\mathbf{X}+\lambda}](\mathrm{Id} - P) \in \Psi^{-\infty}(\mathcal{M}; E \otimes \Lambda)$. Thus, with $Q_{\mathrm{ell}}(\lambda) := \delta_{\mathbf{X}+\lambda} Q_\Delta(\lambda)(\mathrm{Id} - P)$ (mapping $C^\infty\Lambda^k$ to $C^\infty\Lambda^{k-1}$) we obtain

$$\begin{aligned} &d_{\mathbf{X}+\lambda} Q_{\mathrm{ell}}(\lambda) + Q_{\mathrm{ell}}(\lambda) d_{\mathbf{X}+\lambda} \\ &\quad = \Delta_{\mathbf{X}+\lambda} Q_\Delta(\lambda)(\mathrm{Id} - P) + \delta_{\mathbf{X}+\lambda}[Q_\Delta(\lambda), d_{\mathbf{X}+\lambda}](\mathrm{Id} - P) + \delta_{\mathbf{X}+\lambda} Q_\Delta(\lambda)[d_{\mathbf{X}+\lambda}, P] \\ &\quad = (\mathrm{Id} - P) + S_1(\lambda) + S_2(\lambda) \end{aligned}$$

with $S_2(\lambda) \in \Psi^{-\infty}(\mathcal{M}; E \otimes \Lambda)$ and

$$S_1(\lambda) := \delta_{\mathbf{X}+\lambda} Q_\Delta(\lambda)[d_{\mathbf{X}+\lambda}, P] = -\delta_{\mathbf{X}+\lambda} Q_\Delta(\lambda)[d_{\mathbf{X}+\lambda}, \mathrm{Id} - P] \in \Psi^{-1}(\mathcal{M}; E \otimes \Lambda)$$

has wavefront set contained in $\mathrm{WF}(P) \cap \mathrm{WF}(\mathrm{Id} - P)$. ∎

A second ingredient for the construction of the parametrix will be the following estimates of the essential spectral radius of the propagator on the anisotropic Sobolev spaces. We recall that if Y is a bounded operator on a Hilbert space $\mathcal{H}$,

$$r_{\mathrm{ess}}(Y) := \max\{|\lambda| \mid \lambda \in \sigma_{\mathrm{ess}}(Y)\}.$$

The proof of the following lemma is inspired by the argument in [20] for Anosov diffeomorphisms.

Lemma 4.6. *Let $P \in \Psi^0(\mathcal{M}; E)$ be such that $\mathrm{WF}(P)$ is disjoint from E_0^*, and choose an arbitrary constant $C_P' > C_P := \limsup_{|\xi|\to\infty} \|\sigma_P^0(P)(x, \xi)\|$ and some $T > 0$. Let $A \in \mathcal{W} \subset \mathfrak{a}$, let Γ_{reg} be an open cone disjoint from $E_u^* \subset T^*\mathcal{M}$, and let $\Gamma_0 \subset T^*\mathcal{M}$ be a small conic neighborhood of E_0^*. By Proposition 4.3, associated to Γ_{reg} there exists an escape function G for $A_0 := A$ and an open conic set $\Gamma_{E_0^*} \subset \Gamma_0$ such that G is compatible with c_X and $\Gamma_{E_0^*}$ in the sense of Definition 4.1. If in addition $\overline{\Gamma}_{E_0^*} \cap \Phi_t^{X_A}(\mathrm{WF}(P)) = \emptyset$ for all $0 \le t \le T$, then for all $0 \le t \le T$ the operator*

$$e^{-t\mathbf{X}_A} P : \mathcal{H}_{NG} \to \mathcal{H}_{NG}$$

is bounded and can be decomposed as

$$e^{-t\mathbf{X}_A} P = R_{N,G}(t) + K_{N,G}(t)$$

with $\|R_{N,G}(t)\|_{\mathcal{L}(\mathcal{H}_{NG})} \le C_P' e^{-c_X N t} \|e^{-t\mathbf{X}_A}\|_{\mathcal{L}(L^2)}$ and $K_{N,G}(t)$ compact on $\mathcal{H}_{NG}$. Both $R_{N,G}(t)$, $K_{N,G}(t)$ depend on N, G. As a consequence, the essential spectral radius of $e^{-t\mathbf{X}_A} P : \mathcal{H}_{NG} \to \mathcal{H}_{NG}$ is bounded by $C_P' e^{-c_X N t} \|e^{-t\mathbf{X}_A}\|_{\mathcal{L}(L^2)}$.

Proof. Let $m \in C^\infty(T^*\mathcal{M}; \mathbb{R})$ be the order function of the escape function G (see Definition 4.1 (1)). Instead of $e^{-t\mathbf{X}_A} P$ on $\mathcal{H}_{NG}$ we consider the operator $\hat{\mathcal{A}}_{NG} e^{-t\mathbf{X}_A} P \hat{\mathcal{A}}_{NG}^{-1}$ on $L^2(\mathcal{M}; E)$ which is a Fourier integral operator. We write this operator as

$$\hat{\mathcal{A}}_{NG} e^{-t\mathbf{X}_A} P \hat{\mathcal{A}}_{NG}^{-1} = e^{-t\mathbf{X}_A} \underbrace{e^{t\mathbf{X}_A} \hat{\mathcal{A}}_{NG} e^{-t\mathbf{X}_A}}_{=:B_t} P \hat{\mathcal{A}}_{NG}^{-1}. \tag{4.7}$$

For the newly introduced operator B_t we apply Egorov's lemma (Lemma A.8) and deduce that it is a pseudodifferential operator $B_t \in \Psi_{1-}^{N(m\circ\Phi_t^{X_A})}(\mathcal{M}; E)$ with principal symbol

$$\sigma_P^{N(m\circ\Phi_t^{X_A})}(B_t) = e^{N(G\circ\Phi_t^{X_A})} \bmod S_{1-}^{N(m\circ\Phi_t^{X_A})-1+}.$$

Consequently, $B_t P \hat{\mathcal{A}}_{NG}^{-1} \in \Psi_{1-}^{N(m\circ\Phi_t^{X_A}-m)}$ and by Definition 4.1 (2), $m \circ \Phi_t^{X_A}(x, \xi) - m(x, \xi) \le 0$ for $|\xi|$ large enough. Thus $B_t P \hat{\mathcal{A}}_{NG}^{-1} \in \Psi_{1-}^0(\mathcal{M}; E)$ is bounded on L^2, and we can apply Proposition A.9 to this operator. We calculate its principal symbol:

$$\sigma_p^0(B_t P \hat{\mathcal{A}}_{NG}^{-1}) = e^{N(G\circ\Phi_t^{X_A}-G)}\sigma_p^0(P).$$

Now, using Definition 4.1 (3), our assumption that $\overline{\Gamma}_{E_0^*} \cap \Phi_t^{X_A}(\mathrm{WF}(P)) = \emptyset$ for $0 \leq t \leq T$ ensures that, for any $(x,\xi) \in \mathrm{WF}(P)$ and $|\xi|$ sufficiently large, $\partial_t(G \circ \Phi_t^{X_A}) \leq -c_X$ for all $0 \leq t \leq T$. Thus

$$\limsup_{R\to\infty} \sup_{(x,\xi)\in \mathrm{WF}(P),\,|\xi|>R} \|e^{N(G\circ\Phi_t^{X_A}(x,\xi)-G(x,\xi))}\sigma_p^0(P)(x,\xi)\| \leq C_P e^{-Nc_X t}.$$

By closedness of $\overline{\Gamma}_{E_0^*}$ and $\mathrm{WF}(P)$ this estimate can also be extended to a small conical neighborhood of $\mathrm{WF}(P)$. On the complement of this neighborhood, by the definition of the wavefront set, we deduce $\limsup_{|\xi|\to\infty} \|\sigma_p^0(P)(x,\xi)\| = 0$. We have seen above that $e^{N(G\circ\Phi_t^{X_A}-G)} \in S^0_{1-}$. In particular, this factor is uniformly bounded. Putting everything together we get

$$\limsup_{|\xi|\to\infty} \|\sigma_p^0(B_t P \hat{\mathcal{A}}_{NG}^{-1})(x,\xi)\| \leq C_P e^{-Nc_X t}.$$

Using Proposition A.9 we can write $B_t P \hat{\mathcal{A}}_{NG}^{-1} = \widetilde{R}_N(t) + \widetilde{K}_N(t)$ with $\widetilde{K}_N(t) \in \Psi^{-\infty}(\mathcal{M}; E)$ and $\|\widetilde{R}_N(t)\|_{\mathcal{L}(L^2)} \leq C'_P e^{-Nc_X t}$. Now, by (4.7), our operator of interest can be written as

$$\hat{\mathcal{A}}_{NG} e^{-t\mathbf{X}_A} P \hat{\mathcal{A}}_{NG}^{-1} = e^{-t\mathbf{X}_A}(\widetilde{R}_N(t) + \widetilde{K}_N(t)),$$

and we get the desired property by setting $R_{N,G}(t) := e^{-t\mathbf{X}_A}\widetilde{R}_N(t)$ and $K_{N,G}(t) := e^{-t\mathbf{X}_A}\widetilde{K}_N(t)$. ∎

Recall that $C_{L^2}(A)$ was defined in (4.1). We can now turn to the construction of our full parametrix for the Taylor complex.

Proposition 4.7. *For any $A_0 \in \mathcal{W}$, any open cone $\Gamma_0 \subset T^*\mathcal{M}$ containing E_0^* and satisfying $\overline{\Gamma}_0 \cap (E_u^* \oplus E_s^*) = \{0\}$, there are families of operators $Q(\lambda), F(\lambda) : C^\infty(\mathcal{M}; E \otimes \Lambda) \to C^{-\infty}(\mathcal{M}; E \otimes \Lambda)$ depending holomorphically on $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ such that*

$$Q(\lambda)d_{\mathbf{X}+\lambda} + d_{\mathbf{X}+\lambda}Q(\lambda) = F(\lambda).$$

Furthermore, for any escape function G for A_0 compatible with $c_X > 0$ and $\Gamma_{E_0^*} \subset \Gamma_0$, and any $N > 0$ and $\delta > 0$, the following properties hold:

1. *$Q(\lambda) : \mathcal{H}_{NG}\Lambda^j \to \mathcal{H}_{NG}\Lambda^{j-1}$ is bounded for any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $0 \leq j \leq \kappa$.*
2. *$F(\lambda)$ can be decomposed as $F(\lambda) = \mathrm{Id} + R_{N,G}(\lambda) + K_{N,G}(\lambda)$, where $K_{N,G}(\lambda)$ is a compact operator on $\mathcal{H}_{NG}\Lambda$ and $R_{N,G}(\lambda) : \mathcal{H}_{NG}\Lambda \to \mathcal{H}_{NG}\Lambda$ is bounded with $\|R_{N,G}(\lambda)\|_{\mathcal{L}(\mathcal{H}_{NG})} < 1/2$, for*

$$\lambda \in \mathcal{F}_{NG,A_0,\delta} := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \mathrm{Re}(\lambda(A_0)) > -Nc_X + C_{L^2}(A_0) + \delta\} \subset \mathfrak{a}_{\mathbb{C}}^*.$$

Both operators $R_{N,G}(\lambda), K_{N,G}(\lambda)$ depend on N, G, while $Q(\lambda)$ and $F(\lambda)$ do not.

Remark 4.8. (1) If there is a smooth volume density μ preserved by the Anosov action (e.g. the Haar measure for Weyl chamber flows), and if we consider the scalar case $\mathbf{X}_A = X_A$, then e^{tX_A} is unitary on $L^2(\mathcal{M}, \mu)$ and the constant $C_{L^2}(A)$ vanishes.

(2) To prove that the Ruelle–Taylor spectrum is independent of the choice of $\mathcal{H}_{NG}$ it will be essential that the operators $Q(\lambda)$, $F(\lambda)$ only depend on the choice of A_0 and $\Gamma_{E_0^*}$ but not on the choice of the anisotropic Sobolev space $\mathcal{H}_{NG}$ as long as the escape function G satisfies the required compatibility conditions.

Proof of Proposition 4.7. From the definition of $C_{L^2}(A)$, we deduce that there exists $T_0 > 0$ such that $\|e^{-T\mathbf{X}_{A_0}}\| \le e^{T(C_{L^2}(A_0)+\delta/2)}$ for $T \ge T_0$; we fix T so that both $T > T_0$ and $T \ge 2\log(3)/\delta$. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we define the operators $\mathbf{X}_{A_0}(\lambda) := \mathbf{X}_{A_0} + \lambda(A_0)$ and let

$$Q'_T(\lambda) := \int_0^T e^{-t\mathbf{X}_{A_0}(\lambda)}\, dt : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E).$$

We have the relations

$$\begin{aligned} &\mathbf{X}_{A_0}(\lambda) Q'_T(\lambda) = Q'_T(\lambda)\mathbf{X}_{A_0}(\lambda) = 1 - e^{-T\mathbf{X}_{A_0}(\lambda)}, \\ &[\mathbf{X}_A, Q'_T(\lambda)] = 0 \quad \text{for all } A \in \mathfrak{a}. \end{aligned} \tag{4.8}$$

Now we extend $Q'_T(\lambda)$ to an operator $C^\infty(\mathcal{M}; E) \otimes \Lambda^\ell \mathfrak{a}^* \to C^\infty(\mathcal{M}; E) \otimes \Lambda^{\ell-1}\mathfrak{a}^*$ for each ℓ as follows: define the linear map $\mathbf{Q}_T(\lambda) : \mathfrak{a} \to \mathcal{L}(C^\infty(\mathcal{M}; E))$ by $\mathbf{Q}_T(\lambda)A_0 = Q'_T(\lambda)$ and $\mathbf{Q}'_T(\lambda)A = 0$ if $\langle A, A_0\rangle = 0$ (recall $\langle\cdot,\cdot\rangle$ is a fixed scalar product on $\mathfrak{a}$), and let

$$Q_T(\lambda)(u \otimes \omega) := -\delta_{\mathbf{Q}_T(\lambda)}(u \otimes \omega) = (Q'_T(\lambda)u) \otimes \iota_{A_0}\omega$$

for $u \in C^\infty(\mathcal{M}; E)$ and $\omega \in \Lambda^\ell \mathfrak{a}^*$. Using the relations (4.8) and Lemma 3.7 we get

$$\big(Q_T(\lambda) d_{\mathbf{X}+\lambda} + d_{\mathbf{X}+\lambda} Q_T(\lambda)\big)(u \otimes \omega) = ((1 - e^{-T\mathbf{X}_{A_0}(\lambda)})u) \otimes \omega. \tag{4.9}$$

We observe that by the commutativity of the Anosov action $[\mathbf{X}_A, e^{-T\mathbf{X}_{A_0}(\lambda)}] = 0$, and therefore on $C^\infty(\mathcal{M}; E \otimes \Lambda)$ we have

$$[d_{\mathbf{X}+\lambda}, e^{-T\mathbf{X}_{A_0}(\lambda)}] = 0. \tag{4.10}$$

Next, we use the microlocal parametrix in the elliptic region from Lemma 4.5 with a carefully chosen microlocal cutoff function. By our assumption that $\overline{\Gamma}_0 \cap (E_u^* \oplus E_s^*) = \{0\}$ and the fact that $E_u^* \oplus E_s^*$ is a $\Phi_t^{X_{A_0}}$-invariant subset, there exists a conic neighborhood $\Gamma_1 \subset T^*\mathcal{M}$ of $E_u^* \oplus E_s^*$ such that $\Phi_t^{X_{A_0}}(\Gamma_1) \cap \overline{\Gamma}_0 = \emptyset$ of $0 \le t \le T$. Let us choose a second, smaller conical neighborhood $E_u^* \oplus E_s^* \subset \Gamma_2 \Subset \Gamma_1$. Now we fix a microlocal cutoff $P = \mathrm{Op}(p) \in \Psi^0(\mathcal{M}, \mathbb{C})$ which is microsupported in Γ_1 (i.e. $\mathrm{WF}(P) \subset \Gamma_1$) and microlocally equal to 1 on Γ_2 (i.e. $\mathrm{WF}(\mathrm{Id} - P) \cap \Gamma_2 = \emptyset$) and which furthermore has globally bounded symbol, $\sup_{(x,\xi)} |p(x,\xi)| \le 1$. We apply Lemma 4.5 with this choice of P and multiply (4.6) on the left with $e^{-T\mathbf{X}_{A_0}(\lambda)}$. Using (4.10), we get

$$\begin{aligned} d_{\mathbf{X}+\lambda} e^{-T\mathbf{X}_{A_0}(\lambda)} Q_{\mathrm{ell}}(\lambda) &+ e^{-T\mathbf{X}_{A_0}(\lambda)} Q_{\mathrm{ell}}(\lambda) d_{\mathbf{X}+\lambda} \\ &= e^{-T\mathbf{X}_{A_0}(\lambda)}\big(\mathrm{Id} - P + S_1(\lambda) + S_2(\lambda)\big). \end{aligned} \tag{4.11}$$

We define $Q(\lambda) := Q_T(\lambda) + e^{-T\mathbf{X}_{A_0}(\lambda)} Q_{\mathrm{ell}}(\lambda)$ and obtain, by adding up (4.9) and (4.11),

$$d_{\mathbf{X}+\lambda} Q(\lambda) + Q(\lambda) d_{\mathbf{X}+\lambda} = F(\lambda) \quad \text{with} \quad F(\lambda) := \mathrm{Id} - e^{-T\mathbf{X}_{A_0}(\lambda)}\big(P - S_1(\lambda) - S_2(\lambda)\big).$$

Let us now show that $Q(\lambda)$ and $F(\lambda)$ have the required properties. By precisely the same argument as in Lemma 4.6 (using $X^H_{A_0} m(x,\xi) \le 0$ for $|\xi|$ large enough) we deduce that $e^{-t\mathbf{X}_{A_0}}$ is bounded on $\mathcal{H}_{NG}$ uniformly for $t \in [0,T]$ for any escape function G associated to A_0 compatible with $c_X > 0$ and $\Gamma_{E_0^*} \subset \Gamma_0$. Consequently, $Q_T(\lambda)$ and $e^{-T\mathbf{X}_{A_0}(\lambda)}$ are bounded operators on $\mathcal{H}_{NG}\Lambda$. As $\hat{\mathcal{A}}_{NG} Q_{\mathrm{ell}}(\lambda) \hat{\mathcal{A}}_{NG}^{-1} \in \Psi^{-2}(\mathcal{M}; E \otimes \Lambda)$, this is a bounded operator on L^2, thus $Q_{\mathrm{ell}}(\lambda)$ is bounded on $\mathcal{H}_{NG}\Lambda$ as well. Putting everything together we deduce that $Q(\lambda)$ is bounded on $\mathcal{H}_{NG}\Lambda$ for any $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. As $Q_T(\lambda)$ and $Q_{\mathrm{ell}}(\lambda)$ decrease the order in $\Lambda\mathfrak{a}^*$ by 1, $Q(\lambda)$ has this property as well.

Let us deal with $F(\lambda)$: by our choice of Γ_1 we can apply Lemma 4.6 to $e^{-T\mathbf{X}_{A_0}(\lambda)} P = e^{-T\lambda(A_0)} e^{-T\mathbf{X}_{A_0}} P$ and deduce that $e^{-T\mathbf{X}_{A_0}(\lambda)} P = R'_N(\lambda) + K'_N(\lambda)$ for some $R'_N(\lambda)$ bounded on $\mathcal{H}_{NG}$ and $K'_N(\lambda)$ compact on that space, with

$$\|R'_N(\lambda)\|_{\mathcal{L}(\mathcal{H}_{NG})} \le (1+\varepsilon) e^{T(-Nc_X - \mathrm{Re}(\lambda(A_0)) + C_{L^2}(A_0) + \delta/2)}$$

for some $\varepsilon > 0$. Consequently, by our choice of $T > 2\log(3)/\delta$ and for $\lambda \in \mathcal{F}_{NG,A_0,\delta}$ we get $\|R'_N(\lambda)\|_{\mathcal{L}(\mathcal{H}_{NG})} \le (1+\varepsilon)/3$. Note that $S_1(\lambda) + S_2(\lambda) \in \Psi^{-1}(\mathcal{M}; E \otimes \Lambda)$ is compact on $\mathcal{H}_{NG}$ (this can be easily checked by conjugating it with $\hat{\mathcal{A}}_{NG}$ to obtain an operator in $\Psi^{-1}(\mathcal{M}; E \otimes \Lambda)$, thus compact on L^2). This completes the proof of Proposition 4.7 by setting $R_N(\lambda) := -R'_N(\lambda)$ and $K_N(\lambda) := -K'_N(\lambda) + e^{-T\mathbf{X}_{A_0}(\lambda)}(S_1(\lambda) + S_2(\lambda))$. ∎

As a consequence we get the following.

Proposition 4.9. *For $A_0 \in \mathcal{W}$ there exists an escape function G such that for any $N > 0$ the operator $d_{\mathbf{X}+\lambda}$ on $\mathcal{H}_{NG}\Lambda$ defines a Fredholm complex for $\lambda \in \mathcal{F}_{NG,A_0,0}$, i.e.*

$$\sigma^{\mathrm{ess}}_{\mathrm{T},\mathcal{H}_{NG}}(-\mathbf{X}) \cap \mathcal{F}_{NG,A_0,0} = \emptyset.$$

Proof. By Proposition 4.3, there is an escape function G that allows us to apply Proposition 4.7. Then we can use Lemma 3.9 applied to $\mathbf{X} + \lambda$ to deduce Fredholmness. ∎

4.3. Ruelle–Taylor resonances are intrinsic

So far we have shown that the admissible lift of an Anosov action X acting as differential operators on $\mathcal{H}_{NG}$ has a Fredholm Taylor spectrum on $\mathcal{F}_{NG,A} := \mathcal{F}_{NG,A,0} \subset \mathfrak{a}^*_{\mathbb{C}}$, where $A \in \mathcal{W}$ and G is an escape function associated to A. Further, we have seen that $\mathcal{F}_{NG,A}$ can be made arbitrarily large by letting $N \to \infty$. However, it is not yet clear whether this Fredholm spectrum is intrinsic to $\mathbf{X}$ or whether it depends on the choice of the anisotropic Sobolev spaces $\mathcal{H}_{NG}$, i.e. in particular on the choices of N or G.

Let us denote by $C^{-\infty}_{E_u^*}(\mathcal{M}; E)$ the space of distributions in $C^{-\infty}(\mathcal{M}; E)$ with wavefront set contained in E_u^*. Equipped with a suitable topology, this space becomes a topological vector space [36, Chapter 8], and the lift $\mathbf{X}$ acts continuously on $C^{-\infty}_{E_u^*}(\mathcal{M}; E)$.

In particular, it makes sense to consider the complex generated by the operator $d_{\mathbf{X}+\lambda}$ on $C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$. The main result of this section is the following.

Proposition 4.10. *Let $A_0 \in \mathcal{W}$ and $N \geq 0$ and let G be an escape function for A_0. Then for any $\lambda \in \mathcal{F}_{NG,A_0}$ one has vector space isomorphisms*

$$\ker_{\mathcal{H}_{NG}\Lambda^j} d_{\mathbf{X}+\lambda}/\operatorname{ran}_{\mathcal{H}_{NG}\Lambda^j} d_{\mathbf{X}+\lambda} \cong \ker_{C^{-\infty}_{E_u^*}\Lambda^j} d_{\mathbf{X}+\lambda}/\operatorname{ran}_{C^{-\infty}_{E_u^*}\Lambda^j} d_{\mathbf{X}+\lambda}.$$

Using this result, we see that the Ruelle–Taylor spectrum is independent of A_0 and of the anisotropic space $\mathcal{H}_{NG}\Lambda$ in the region $\mathcal{F}_{NG,A_0}$ of $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ where the Taylor complex $d_{\mathbf{X}+\lambda}$ is Fredholm on $\mathcal{H}_{NG}\Lambda$. We can then define the notion of Ruelle–Taylor resonance as follows:

Definition 4.11. We define the *Ruelle–Taylor resonances* of $\mathbf{X}$ to be the set

$$\operatorname{Res}_{\mathbf{X}} := \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid \exists j,\ \ker_{C^{-\infty}_{E_u^*}\Lambda^j} d_{\mathbf{X}+\lambda}/\operatorname{ran}_{C^{-\infty}_{E_u^*}\Lambda^j} d_{\mathbf{X}+\lambda} \neq 0\},$$

and the *Ruelle–Taylor resonant cohomology space* of degree j of $\lambda \in \operatorname{Res}_{\mathbf{X}}$ to be

$$\operatorname{Res}_{\mathbf{X},\Lambda^j}(\lambda) := \ker_{C^{-\infty}_{E_u^*}\Lambda^j} d_{\mathbf{X}+\lambda}/\operatorname{ran}_{C^{-\infty}_{E_u^*}\Lambda^j} d_{\mathbf{X}+\lambda}.$$

Another consequence of Proposition 4.10 is the following.

Corollary 4.12 (Location of Ruelle–Taylor resonances). *One has*

$$\operatorname{Res}_{\mathbf{X}} \subset \bigcap_{A \in \mathcal{W}} \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid \operatorname{Re}(\lambda(A)) \leq C_{L^2}(A)\}.$$

Proof. Assume that there exists an $A \in \mathcal{W}$ such that $\operatorname{Re}(\lambda(A)) > C_{L^2}(A)$. Then for some $\delta > 0$, $\lambda \in \mathcal{F}_{0G,A,\delta}$ and consequently $\lambda \in \operatorname{Res}_{\mathbf{X}}$ iff $\ker_{L^2\Lambda} d_{\mathbf{X}+\lambda}/\operatorname{ran}_{L^2\Lambda} d_{\mathbf{X}+\lambda} \neq 0$. However, by (4.9) there is a bounded operator $Q_T(\lambda) : L^2(\mathcal{M}; E \otimes \Lambda) \to L^2(\mathcal{M}; E \otimes \Lambda)$ such that

$$d_{\mathbf{X}+\lambda} Q_T(\lambda) + Q_T(\lambda) d_{\mathbf{X}+\lambda} = \operatorname{Id} + e^{-T\mathbf{X}_A} e^{-T\lambda(A)}.$$

Since $\operatorname{Re}(\lambda(A)) > C_{L^2}(A)$, the right hand side is invertible on $L^2(\mathcal{M}; E \otimes \Lambda)$ provided $T > 0$ is large enough. As furthermore $\operatorname{Id} + e^{-T\mathbf{X}_A} e^{-T\lambda(A)}$ and its inverse commute with $d_{\mathbf{X}+\lambda}$, we conclude that $\ker_{L^2\Lambda} d_{\mathbf{X}+\lambda}/\operatorname{ran}_{L^2\Lambda} d_{\mathbf{X}+\lambda} = 0$. ∎

The strategy to prove Proposition 4.10 is to show that in each cohomology class in $\ker_{\mathcal{H}_{NG}\Lambda} d_{\mathbf{X}+\lambda}/\operatorname{ran}_{\mathcal{H}_{NG}\Lambda} d_{\mathbf{X}+\lambda}$ one can find a representative that lies already in $\ker_{C^{-\infty}_{E_u^*}} d_{\mathbf{X}+\lambda}$. To this end we will construct for fixed λ a projector $\Pi_0(\lambda)$ of finite rank such that we can find in each cohomology class a representative in the range of $\Pi_0(\lambda)$. The fact that the range of $\Pi_0(\lambda)$ is independent of the anisotropic Sobolev spaces and contained in $C^{-\infty}_{E_u^*}$ then follows very similarly to the corresponding characterization of Anosov flows [21, Theorem 1.7] by the flexibility in the choice of the escape function.

Proof of Proposition 4.10. Given A_0, N, G and $\lambda \in \mathcal{F}_{NG,A_0}$, let us first fix $\delta > 0$ such that $\lambda \in \mathcal{F}_{NG,A_0,\delta}$ and an open cone $\Gamma_0 \subset T^*\mathcal{M}$ containing $\Gamma_{E_0^*}$ (the conic set in Proposition 4.3) and such that $\overline{\Gamma}_0 \cap (E_s^* \oplus E_u^*) = \{0\}$. Then Proposition 4.7 provides operators $Q(\lambda), F(\lambda) : C^{-\infty}(\mathcal{M}; E \otimes \Lambda) \to C^{-\infty}(\mathcal{M}; E \otimes \Lambda)$ which only depend on $\delta, \lambda, A_0, \Gamma_0$ and satisfy

$$d_{\mathbf{X}+\lambda} Q(\lambda) + Q(\lambda) d_{\mathbf{X}+\lambda} = F(\lambda). \tag{4.12}$$

We can thus apply Lemma 3.9, and deduce that if $\Pi_0(\lambda)$ is the spectral projector of $F(\lambda)$ on its kernel, then

$$\Pi_0(\lambda) : \ker_{\mathcal{H}_{NG}\Lambda} d_{\mathbf{X}+\lambda} / \operatorname{ran}_{\mathcal{H}_{NG}\Lambda} d_{\mathbf{X}+\lambda} \to \ker_{\operatorname{ran}\Pi_0(\lambda)} d_{\mathbf{X}+\lambda} / \operatorname{ran}_{\operatorname{ran}\Pi_0(\lambda)} d_{\mathbf{X}+\lambda} \tag{4.13}$$

is an isomorphism. Here, $\operatorname{ran}\Pi_0(\lambda) = \Pi_0(\lambda)\mathcal{H}_{NG}$. But since C^∞ is dense in $\mathcal{H}_{NG}$, and $\Pi_0(\lambda)$ has finite rank, this range is equal to $\Pi_0(\lambda) C^\infty(\mathcal{M}; E \otimes \Lambda)$. We now need the following lemma.

Lemma 4.13. *The projector $\Pi_0(\lambda)$ maps $C^\infty(\mathcal{M}; E \otimes \Lambda)$ into $C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$. Additionally, it has a continuous extension to $C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$.*

Proof. Recall that $\Pi_0(\lambda) : \mathcal{H}_{NG}\Lambda \to \mathcal{H}_{NG}\Lambda$ has been defined as the spectral projector at $z = 0$ of $F(\lambda) : \mathcal{H}_{NG}\Lambda \to \mathcal{H}_{NG}\Lambda$, it has finite rank. Since $F(\lambda)$ and its Fredholmness do not depend on the choice of N, G as long as $\lambda \in \mathcal{F}_{NG,A_0}$, neither does its spectral projector at 0. The image of $\Pi_0(\lambda)$ is thus contained in the intersection of the $\mathcal{H}_{N'G'}\Lambda$ such that $\lambda \in \mathcal{F}_{N'G',A_0}$.

Let us show that this intersection is contained in $C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$. We thus take u in all the $\mathcal{H}_{N'G'}$ such that $\lambda \in \mathcal{F}_{N'G',A_0}$. By Proposition 4.3 for an arbitrary cone Γ'_{reg} disjoint from E_u^*, there exists an escape function G' for A_0 compatible with c'_X and $\Gamma'_{E_0^*} \subset \Gamma_0$ such that microlocally on Γ'_{reg}, $\mathcal{H}_{N'G'}$ is contained in the standard Sobolev space $H^{N'/2}(\mathcal{M}; E)$. In particular, taking N' arbitrarily large, we have $\lambda \in \mathcal{F}_{N'G',A_0}$ and $\mathrm{WF}(u) \cap \Gamma'_{\mathrm{reg}} = \emptyset$. Since Γ'_{reg} was arbitrary, $\mathrm{WF}(u) \subset E_u^*$.

To prove that $\Pi_0(\lambda)$ has a continuous extension to $C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$, it suffices to observe that $C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$ is also contained in the union of all the $\mathcal{H}_{N'G'}$ such that $\lambda \in \mathcal{F}_{N'G',A_0}$. This follows from Definition 4.1 (1), since we know that in a conic neighborhood around E_u^* we have $m(x, \xi) \leq -1/4$. As a consequence, $\Pi_0(\lambda)$ is a linear operator from $C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$ to $C^{-\infty}(\mathcal{M}, E \otimes \Lambda)$. It is continuous as it has finite rank. ■

To finish the proof of Proposition 4.10, it suffices to apply a variation of the sandwiching trick presented in the proof of Lemma 3.12. Indeed, since $\Pi_0(\lambda)$ is a bounded projector on $C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$, commuting with $d_{\mathbf{X}+\lambda}$, the map $u \mapsto \Pi_0(\lambda)u$ descends to a surjective map

$$\ker_{C^{-\infty}_{E_u^*}\Lambda} d_{\mathbf{X}+\lambda} / \operatorname{ran}_{C^{-\infty}_{E_u^*}\Lambda} d_{\mathbf{X}+\lambda} \to \ker_{\operatorname{ran}\Pi_0(\lambda)} d_{\mathbf{X}+\lambda} / \operatorname{ran}_{\operatorname{ran}\Pi_0(\lambda)} d_{\mathbf{X}+\lambda}. \tag{4.14}$$

We need to show the injectivity of this map. This will follow from the fact that $C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$ is contained in the union of the $\mathcal{H}_{N'G'}\Lambda$ such that $\lambda \in \mathcal{F}_{N'G',A_0}$. We consider $u \in C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$ such that $d_{\mathbf{X}+\lambda}u = 0$, and $[\Pi_0(\lambda)u] = 0$, i.e. $\Pi_0(\lambda)u = d_{\mathbf{X}+\lambda}\Pi_0(\lambda)v$

for some $v \in C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$. Since u belongs to some $\mathcal{H}_{N'G'}\Lambda$, we then write $\tilde{F}(\lambda) = F(\lambda) + \Pi_0(\lambda)$, and observe, just as in (3.13), that

$$\tilde{F}(\lambda)^{-1} Q(\lambda) d_{\mathbf{X}+\lambda} + d_{\mathbf{X}+\lambda} \tilde{F}(\lambda)^{-1} Q(\lambda) = \mathrm{Id} - \Pi_0(\lambda),$$

so that

$$u = d_{\mathbf{X}+\lambda}\big(\tilde{F}(\lambda)^{-1} Q(\lambda) u + \Pi_0(\lambda) v\big).$$

It remains to check that $\tilde{F}^{-1}(\lambda) Q(\lambda) u \in C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$. But, since $Q(\lambda)$ and $\tilde{F}^{-1}(\lambda)$ are bounded on each $\mathcal{H}_{N'G'}\Lambda$ such that $\lambda \in \mathcal{F}_{N'G',A_0}$, this is an element of each such $\mathcal{H}_{N'G'}\Lambda$, so it is contained in the intersection thereof. We have seen in the proof of Lemma 4.13 that this intersection is contained in $C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$.

Finally, the operator $F(\lambda) : \mathcal{H}_{NG}\Lambda \to \mathcal{H}_{NG}\Lambda$ preserves the order in the Koszul complex, i.e. $F(\lambda) : \mathcal{H}_{NG}\Lambda^j \to \mathcal{H}_{NG}\Lambda^j$, and so do all the subsequent constructions such as $\Pi_0(\lambda)$ as well. The isomorphism $\Pi_0(\lambda)$ can thus be restricted to the individual cohomology $\ker_{C^{-\infty}_{E_u^*}\Lambda^j} d_{\mathbf{X}+\lambda} / \mathrm{ran}_{C^{-\infty}_{E_u^*}\Lambda^j} d_{\mathbf{X}+\lambda}$, and we have completed the proof of Proposition 4.10. ∎

4.4. Discrete Ruelle–Taylor spectrum

In this section we show that the Ruelle–Taylor resonance spectrum of the admissible lift $\mathbf{X} : \mathfrak{a} \to \mathrm{Diff}^1(\mathcal{M}; E)$ of the Anosov action, for E a Riemannian vector bundle, is discrete in $\mathfrak{a}^*_{\mathbb{C}}$. Our goal is to use Lemma 3.12. In contrast to just obtaining the Fredholm property of the Taylor complex, this section requires using a parametrix $Q(\lambda)$ in Proposition 4.7 that is more intrinsically related to the $\mathbf{X}$ action, in particular we shall construct $Q(\lambda)$ as a function of $(X_1, \dots, X_\kappa) = (\mathbf{X}_{A_1}, \dots, \mathbf{X}_{A_\kappa})$ if $A_j \in \mathcal{W}$ is an orthonormal basis for some scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{a}$. This requires a slightly better escape function of Lemma 4.4 that provides decay not only in a fixed direction $A_1 \in \mathcal{W}$, but also for all other elements in a small neighborhood $\mathcal{U}$ of A_1.

Let us now fix an orthonormal basis $A_1, \dots, A_\kappa \in \mathcal{U} \subset \mathcal{W}$ of $\mathfrak{a}$ in the positive Weyl chamber, and denote the associated scalar product in $\mathfrak{a}$ by $\langle\cdot,\cdot\rangle$. In order to be able to use Lemma 3.12, we will prove the following.

Lemma 4.14. *For each fixed $\lambda \in \mathcal{F}_{NG,A_0,\delta}$ there is a Lie algebra morphism $\mathbf{Q}(\lambda) : \mathfrak{a} \to \mathcal{L}(\mathcal{H}_{NG}) \cap \mathcal{L}(C^\infty(\mathcal{M}; E))$ commuting with $\mathbf{X}(\lambda) := \mathbf{X} + \lambda$ in the sense that $[\mathbf{X}_{A_j}(\lambda), \mathbf{Q}_{A_k}(\lambda)] = 0$ for all j, k, such that*

$$d_{\mathbf{X}+\lambda} \delta_{\mathbf{Q}(\lambda)} + \delta_{\mathbf{Q}(\lambda)} d_{\mathbf{X}+\lambda} = \mathrm{Id} + R(\lambda) + K(\lambda)$$

with $R(\lambda), K(\lambda) \in \mathcal{L}(\mathcal{H}_{NG}\Lambda)$, $\|R(\lambda)\|_{\mathcal{L}(\mathcal{H}_{NG})} < 1/2$ and $K(\lambda)$ compact on $\mathcal{H}_{NG}\Lambda$. Moreover, $R(\lambda), K(\lambda)$ are Λ-scalar.

Proof. Let $T_j > 0$ for $j = 1, \dots, \kappa$, and consider $\chi_j \in C_c^\infty([0, \infty[; [0, 1])$ non-increasing with $\chi_j = 1$ on $[0, T_j]$ and $\mathrm{supp}\, \chi_j \subset [0, T_j + 1]$. Then we set

$$Q_j'(\lambda) := \int_0^\infty e^{-t_j \mathbf{X}_{A_j}(\lambda)} \chi_j(t_j)\, dt_j$$

and we make it act on $C^\infty(\mathcal{M}; E) \otimes \Lambda\mathfrak{a}^*$ by $\widetilde{Q}_j(\lambda) : u \otimes w \mapsto (Q_j'(\lambda)u) \otimes \iota_{A_j}\omega$. As in Proposition 4.7, we compute

$$\begin{aligned} d_{\mathbf{X}(\lambda)}\widetilde{Q}_j(\lambda) + \widetilde{Q}_j(\lambda)d_{\mathbf{X}(\lambda)} &= \mathrm{Id} + R_j(\lambda), \\ R_j(\lambda)(u \otimes \omega) &:= \left(\int_0^\infty e^{-t_j X_j(\lambda)} u \chi_j'(t_j)\, dt_j\right) \otimes w, \end{aligned}$$

nd note that $R_j(\lambda) = R_j'(\lambda) \otimes \mathrm{Id}$ is scalar. We thus have

$$d_{\mathbf{X}(\lambda)}Q(\lambda) + Q(\lambda)d_{\mathbf{X}(\lambda)} = F(\lambda), \quad F(\lambda) := \mathrm{Id} - (-1)^\kappa \prod_{j=1}^{\kappa} R_j(\lambda), \tag{4.15}$$

with $Q(\lambda) := \sum_{j=1}^{\kappa}(-1)^{j-1}\widetilde{Q}_j(\lambda)\prod_{k=1}^{j-1} R_k(\lambda)$. First we observe that $Q(\lambda) = \delta_{\mathbf{Q}(\lambda)}$ is the divergence associated to the Lie algebra morphism $\mathbf{Q}(\lambda) : \mathfrak{a} \to \mathcal{L}(C^\infty(\mathcal{M}; E))$ defined by

$$\mathbf{Q}_{A_j}(\lambda) = (-1)^j Q_j'(\lambda) \prod_{k=1}^{j-1} R_k(\lambda).$$

We notice that $\mathbf{Q}_{A_j}(\lambda)$ commutes with $\mathbf{X}_{A_i}(\lambda)$ for all i, j. As in the proof of Proposition 4.7, $\mathbf{Q}(\lambda)$ maps to $\mathcal{L}(\mathcal{H}_{NG})$ and $Q(\lambda)$ is bounded on $\mathcal{H}_{NG}\Lambda$; here we use Lemma 4.4 as it is important that the order function m satisfies $X_{A_j}^H m \le 0$ for $|\xi|$ large enough and all $j = 1, \dots, \kappa$. We take P microsupported in a neighborhood of $E_u^* \oplus E_s^*$ and $\mathrm{WF}(P)$ in a sufficiently close conical neighborhood of $E_u^* \oplus E_s^*$, as in the proof of Proposition 4.7, and follow the arguments given there, which were based on Lemma 4.6: if $T_j := T$ is chosen large enough (as in the proof of Proposition 4.7), then

$$\begin{aligned} \prod_{k=1}^{\kappa} R_k(\lambda)P &= \int_{[T,T+1]^\kappa} e^{-\sum_{j=1}^{\kappa} t_j \mathbf{X}_{A_j}(\lambda)} P \prod_{j=1}^{\kappa} \chi_j'(t_j)\, dt \\ &= \int_{[T,T+1]^\kappa} (R(t,\lambda) + K(t,\lambda)) \prod_{j=1}^{\kappa} \chi_j'(t_j)\, dt, \end{aligned}$$

where $\|R(t,\lambda)\|_{\mathcal{L}(\mathcal{H}_{NG})} \prod_j \|\chi_j'\|_{L^\infty} \le 1/2$ and $K(t,\lambda)$ is compact on $\mathcal{H}_{NG}$ for all $t \in [T, T+1]^\kappa$ (both depend on N, G). This shows that the operator $(F(\lambda) - \mathrm{Id})P$ decomposes as $(F(\lambda) - \mathrm{Id})P = R(\lambda) + K_1(\lambda)$ with $\|R(\lambda)\|_{\mathcal{L}(\mathcal{H}_{NG}\Lambda)} < 1/2$ and $K_1(\lambda)$ compact on $\mathcal{H}_{NG}\Lambda$. Next, we claim that using the fact that $P \in \Psi^0(\mathcal{M})$ is scalar with $\mathrm{WF}(\mathrm{Id} - P)$ not intersecting a conic neighborhood of $E_u^* \oplus E_s^*$, we can see that $K_2(\lambda) := (F(\lambda) - \mathrm{Id})(\mathrm{Id} - P)$ is a compact operator on $\mathcal{H}_{NG}\Lambda$. Indeed, let us first take a microlocal partition of $\mathrm{Id} - P$ such that $(\mathrm{Id} - P) - \sum_{k=1}^{\kappa} P_k \in \Psi^{-\infty}(\mathcal{M})$ with $P_k \in \Psi^0(\mathcal{M})$ and $\mathrm{WF}(P_k)$ not intersecting a conic neighborhood of the characteristic set $\{(x, \xi) \in T^*\mathcal{M} \mid \xi(X_{A_k}) = 0\}$. Let us show that $R_k(\lambda)P_k$ is compact on $\mathcal{H}_{NG}$. First,

$$R_k(\lambda)P_k\mathbf{X}_{A_k}(\lambda) = \int_T^{T+1} e^{-t_k\mathbf{X}_{A_k}(\lambda)} P_k \chi_k''(t_k)\, dt_k + R_k(\lambda)[P_k, \mathbf{X}_{A_k}] \in \mathcal{L}(\mathcal{H}_{NG}), \tag{4.16}$$

where we use the fact that $[P_k, \mathbf{X}_{A_k}] \in \Psi^0(\mathcal{M})$ and $e^{-t_k \mathbf{X}_{A_k}(\lambda)}$ is bounded on $\mathcal{H}_{NG}$. Since $\mathbf{X}_k(\lambda)$ is elliptic near $\mathrm{WF}(P_k)$, we can construct a parametrix $Z_k(\lambda) \in \Psi^{-1}(\mathcal{M})$ so that $\mathbf{X}_{A_k}(\lambda) Z_k(\lambda) - P_k' \in \Psi^{-\infty}(\mathcal{M})$ for some $P_k' \in \Psi^0(\mathcal{M})$ with $P_k' P_k - P_k \in \Psi^{-\infty}(\mathcal{M})$. We thus obtain

$$R_k(\lambda) P_k \mathbf{X}_{A_k}(\lambda) Z_k(\lambda) - R_k(\lambda) P_k \in \Psi^{-\infty}(\mathcal{M}),$$

but $Z_k(\lambda)$ being compact on $\mathcal{H}_{NG}$, we find that $R_k(\lambda) P_k$ is compact on $\mathcal{H}_{NG}$ using (4.16). Next, we write

$$\Big(\prod_{k=1}^{\kappa} R_k(\lambda)\Big)(\mathrm{Id} - P) - \sum_{j=1}^{\kappa}\Big(\prod_{k=1}^{\kappa} R_k(\lambda)\Big) P_j \in \Psi^{-\infty}(\mathcal{M}).$$

This operator is compact since all the $R_k(\lambda)$ are bounded on $\mathcal{H}_{NG}$ and commute with each other and $R_k(\lambda) P_k$ is compact. Putting everything together we deduce that $F(\lambda)$ has the desired properties by setting $K(\lambda) := K_1(\lambda) + K_2(\lambda)$. ∎

Remark. We notice that in the proof above, it is sufficient to take only one of the T_j to be large while the others can be small, as this is sufficient to get the norm estimate $\|R(\lambda)\|_{\mathcal{L}(\mathcal{H}_{NG})} < 1/2$.

As a corollary, using Lemmas 3.12 and 3.10, we deduce the following.

Proposition 4.15. *For an admissible lift of an Anosov action* $\mathbf{X}$*, the Ruelle–Taylor resonance spectrum is a discrete subset of* $\mathfrak{a}_{\mathbb{C}}^*$*. Moreover,* $\lambda \in \mathcal{F}_{NG,A_0} \cap \sigma_{\mathrm{T},\mathcal{H}_{NG}}(-\mathbf{X})$ *if and only if there is* $u \in \mathcal{H}_{NG}$ *such that*

$$(\mathbf{X} + \lambda) u = 0.$$

This completes the proof of Theorem 4. In the scalar case (i.e. when E is the trivial bundle) we will show in Corollary 4.16 below that part (3) of Theorem 4 can be sharpened using the dynamical parametrix $Q(\lambda)$ in Lemma 4.14 (the same argument also works for admissible lifts under the condition $\|e^{-t\mathbf{X}_A} f\|_{\mathcal{L}(L^\infty)} \leq C$ for all $t \in \mathbb{R}$).

Corollary 4.16. *Let* X *be an Anosov action. Then*

$$\mathrm{Res}_X \subset \bigcap_{A \in \mathcal{W}} \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \mathrm{Re}(\lambda(A)) \leq 0\}.$$

Proof. Let $A \in \mathcal{W}$ and assume that $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ satisfies $\mathrm{Re}(\lambda(A)) > 0$. We will show that λ cannot be a Ruelle–Taylor resonance. We use the parametrix $Q(\lambda)$ of Lemma 4.14 with $A_1 := A$ and $(A_j)_j \in \mathcal{W}^\kappa$ forming a basis of $\mathfrak{a}$ with A_j in an arbitrarily small neighborhood of A_1 so that $\mathrm{Re}(\lambda(A_j)) > 0$ for all j. We see that (4.15) holds with $F(\lambda)$ having discrete spectrum near $z = 0$. Let $\Pi_0(\lambda)$ be the spectral projector of the Fredholm operator $F(\lambda)$ at $z = 0$, which can be written

$$\Pi_0(\lambda) = \frac{1}{2\pi i} \int_{|z|=\varepsilon} (z\mathrm{Id} - F(\lambda))^{-1} \, dz \tag{4.17}$$

for some small enough $\varepsilon > 0$. We notice that for $f \in L^\infty(\mathcal{M})$,

$$\begin{aligned}\|(\mathrm{Id} - F(\lambda))f\|_{L^\infty} &\leq \int_{(\mathbb{R}+)^\kappa} \|e^{-\sum_{j=1}^\kappa t_j \mathbf{X}_{A_j}(\lambda)} f\|_{L^\infty} \prod_{j=1}^\kappa (-\chi_j'(t_j))\, dt_1 \dots dt_\kappa \\ &\leq \|f\|_{L^\infty} e^{-\sum_{j=1}^\kappa T_j \lambda(A_j)} \int_{(\mathbb{R}+)^\kappa} \prod_{j=1}^\kappa (-\chi_j'(t_j))\, dt_1 \dots dt_\kappa \\ &= \|f\|_{L^\infty} e^{-\sum_{j=1}^\kappa T_j \lambda(A_j)}.\end{aligned}$$

This shows that by choosing the $T_j > 0$ (introduced in Lemma 4.14) large enough, $\|(\mathrm{Id} - F(\lambda))\|_{\mathcal{L}(L^\infty)} < 1/2$. In particular, $F(\lambda)$ is invertible on L^∞ and therefore $\Pi_0(\lambda) = 0$ since the expression (4.17) holds also as a map $C^\infty(\mathcal{M}) \to C^{-\infty}(\mathcal{M})$. This ends the proof. ∎

Let us end the section with a statement about joint Jordan blocks for an admissible lift $\mathbf{X}$. Given $\alpha \in \mathbb{N}^\kappa$ we define $\mathbf{X}^\alpha(\lambda) := \prod_{j=1}^\kappa (\mathbf{X}_{A_j} + \lambda_j)^{\alpha_j}$.

Proposition 4.17. *For any Ruelle–Taylor resonance $\lambda \in \mathrm{Res}_\mathbf{X}$ there is $J \in \mathbb{N}^*$ which is the minimal integer such that, whenever for some $u \in C^{-\infty}_{E_u^*}(\mathcal{M})$ and $k \in \mathbb{N}^*$ one has $\mathbf{X}^\beta(\lambda)u = 0$ for all $|\beta| = k$ then $\mathbf{X}^\alpha(\lambda)u = 0$ for all $|\alpha| = J$. Moreover, the space of* generalized joint resonant states *is the finite-dimensional space given by*

$$\{u \in C^{-\infty}_{E_u^*}(\mathcal{M}) \mid \mathbf{X}^\alpha(\lambda)u = 0 \text{ for all } \alpha \text{ with } |\alpha| = J\} \subset \mathrm{ran}\, \Pi_0(\lambda), \tag{4.18}$$

where $\Pi_0(\lambda)$ is the spectral projector of $F(\lambda)$ at $z = 0$, defined in (4.17).

Proof. Let $\mathcal{H}_{NG}$ be an anisotropic Sobolev space such that $\lambda \in \mathcal{F}_{NG,A_0}$. We construct the parametrix from (4.15),

$$Q(\lambda)d_{\mathbf{X}+\lambda} + d_{\mathbf{X}+\lambda}Q(\lambda) = \mathrm{Id} - R(\lambda) \otimes \mathrm{Id},$$

for some appropriate choice of basis $A_1, \dots, A_j \in \mathcal{W} \subset \mathfrak{a}$, and writing $\psi_j := -\chi_j' \in C_c^\infty((0,\infty))$, $\mathbf{X}_j := \mathbf{X}_{A_j}$ and $\lambda_j := \lambda(A_j)$ we set

$$R(\lambda) = \prod_{j=1}^\kappa \int e^{-t_j(\mathbf{X}_j + \lambda_j)} \psi_j(t_j)\, dt_j. \tag{4.19}$$

We denote by $\Pi_0(\lambda) : \mathcal{H}_{NG} \to \mathcal{H}_{NG}$ the spectral projector on the generalized eigenspace of $R(\lambda)$ for the eigenvalue 1. Note that it commutes with $\mathbf{X}_j$ for all j, since $R(\lambda)$ does.

We now show by induction that for any $u \in C^{-\infty}_{E_u^*}(\mathcal{M})$ with $\mathbf{X}^\alpha(\lambda)u = 0$ for all $|\alpha| = k$ we have $u \in \mathrm{ran}\, \Pi_0(\lambda) \subset \mathcal{H}_{NG}$. The base case is easily deduced from (4.19): for $u \in \mathcal{H}_{NG}$ with $(X_j + \lambda_j)u = 0$ we deduce

$$R(\lambda)u = \prod_{j=1}^\kappa \int \psi_j(t_j) e^{-t_j(X_j + \lambda_j)} u\, dt_j = u,$$

thus $u \in \operatorname{ran} \Pi_0(\lambda)$. Next we show that if the property is satisfied at step k then it also is at step $k+1$: if for $u \in C^{-\infty}_{E_u^*}(\mathcal{M})$, $\mathbf{X}^\alpha(\lambda)u = 0$ for $|\alpha| = k+1$ then for $|\beta| = k$ we have $\mathbf{X}_{A_j}(\lambda)(\mathbf{X}^\beta(\lambda)u) = 0$ for $j = 1, \dots, \kappa$. Thus, by the the same argument as above we know $\mathbf{X}^\beta(\lambda)u \in \operatorname{ran} \Pi_0(\lambda)$. As $[\Pi_0(\lambda), \mathbf{X}^\beta(\lambda)] = 0$ we conclude $\mathbf{X}^\beta(\lambda)(\Pi_0(\lambda)u - u) = 0$. Consequently, by induction hypothesis, $\Pi_0(\lambda)u - u \in \operatorname{ran} \Pi_0(\lambda)$ and the claim follows. The statement of the proposition follows because $\operatorname{ran} \Pi_0(\lambda)$ is a finite-dimensional $\mathbf{X}_{A_i}(\lambda)$-invariant subspace. ∎

We also notice that the non-triviality of the space (4.18) (with J minimal) implies that λ is a Ruelle–Taylor resonance, since for u in this space, there is an α with $|\alpha| = J-1$ such that $v := \mathbf{X}^\alpha(\lambda)u \neq 0$ satisfies $v \in \operatorname{Res}_{\mathbf{X},\Lambda^0}(\lambda)$. We also note that equality in (4.18) does not hold in general. One rather has the following result.

Proposition 4.18. *If $\Pi_0(\lambda)$ is the spectral projector of $R(\lambda)$ from (4.19) then*

$$\operatorname{ran} \Pi_0(\lambda) = \bigoplus_{\eta \in \operatorname{Res}_{\mathbf{X}},\, \prod_j \hat{\psi}_j(-i(\lambda_j - \eta_j)) = 1} \{u \in C^{-\infty}_{E_u^*}(\mathcal{M}) \mid \mathbf{X}^\alpha(\eta)u = 0 \text{ for } |\alpha| = J\},$$

where $J \in \mathbb{N}$ is the integer from Proposition 4.17.

Proof. First note that $\operatorname{ran} \Pi_0(\lambda)$ is finite-dimensional and $\mathbf{X}_j$-invariant, and thus we can decompose the space into joint generalized eigenstates. If η is such a joint eigenvalue then Proposition 4.15 implies that η is also a Ruelle–Taylor resonance. Now let $u \in \operatorname{ran} \Pi_0(\lambda)$ be a joint eigenstate of $\mathbf{X}$ with eigenvalue η. Then by (4.19),

$$R(\lambda)u = \prod_{j=1}^{\kappa} \int e^{-t_j(\lambda_j - \eta_j)} \psi_j(t_j)\, dt_j = \prod_{j=1}^{\kappa} \hat{\psi}_j(-i(\lambda_j - \eta_j))u.$$

Thus u is an eigenstate of $R(\lambda)$; but as u is also required to be in the generalized eigenspace of eigenvalue 1, we deduce that $\prod_j \hat{\psi}_j(-i(\lambda_j - \eta_j)) = 1$. This shows that the left hand side of the formula in the statement is contained in the right hand side.

For the converse inclusion we note that any joint resonant state u with $(X_j + \eta_j)u = 0$ whose joint resonance fulfills $\prod_j \hat{\psi}_j(-i(\lambda_j - \eta_j)) = 1$ is an eigenstate of $R(\lambda)$ with eigenvalue 1 and thus is contained in $\operatorname{ran} \Pi_0(\lambda)$. For the generalized eigenstates of higher order we argue as above in Proposition 4.18 by induction. ∎

5. The leading resonance spectrum

In this section we study the leading resonance spectrum, i.e. those resonances with vanishing real part, and show that they give rise to particular measures and are related to mixing properties of the Anosov action. In this section the bundle E will be trivial.

5.1. *Imaginary Ruelle–Taylor resonances in the non-volume-preserving case*

In this section, we investigate the purely imaginary Ruelle–Taylor resonances and in particular the resonance at 0 for the action on functions. We assume that the Anosov action X

does not necessarily preserve a smooth invariant measure. We choose a basis $A_1, \dots, A_\kappa$ of $\mathfrak{a}$, with dual basis $(e_j)_j$ in $\mathfrak{a}^*$, we set $X_j := X_{A_j}$, and we use dv_g, the smooth Riemannian probability measure on $\mathcal{M}$. Let us choose $\lambda \in \mathfrak{a}^*$ and fix non-negative functions $\chi_j \in C_c^\infty(\mathbb{R}^+)$, equal to 1 on a large interval $[0, T_j]$, with $\chi_j' \leq 0$, and use the parametrix $Q(i\lambda)$ in the divergence form from Lemma 4.14, so that by (4.15),

$$Q(i\lambda)d_{X+i\lambda} + d_{X+i\lambda}Q(i\lambda) = \mathrm{Id} - R(i\lambda) \otimes \mathrm{Id},$$

and writing $\psi_j := -\chi_j' \in C_c^\infty((0,\infty))$ and $\lambda_j := \lambda(A_j)$ we get

$$R(i\lambda) = \prod_{j=1}^{\kappa} \int e^{-t_j(X_j + i\lambda_j)} \psi_j(t_j)\, dt_j. \tag{5.1}$$

We proved that $R(i\lambda)$ has essential spectral radius < 1 in the anisotropic space $\mathcal{H}_{NG}$, and the resolvent $(R(i\lambda) - z)^{-1}$ is meromorphic outside $|z| < 1 - \varepsilon$ for some ε, and the poles in $|z| > 1 - \varepsilon$ are the eigenvalues of $R(i\lambda)$. Moreover, for $f \in L^\infty$,

$$\|R(i\lambda)f\|_{L^\infty} \leq \|f\|_{L^\infty} \prod_{j=1}^{\kappa} \int_{\mathbb{R}} \psi_j(t_j)\, dt_j = \|f\|_{L^\infty}. \tag{5.2}$$

Since $R(i\lambda)$ is bounded, for $|z|$ large enough one has, on $\mathcal{H}_{NG}$,

$$(z - R(i\lambda))^{-1} = z^{-1} \sum_{k \geq 0} z^{-k} R(i\lambda)^k, \tag{5.3}$$

but the L^∞ estimate (5.2) shows that this series converges in $\mathcal{L}(L^\infty)$ and is analytic for $|z| > 1$. Therefore, using the density of $C^\infty(\mathcal{M})$ in $\mathcal{H}_{NG}$, we deduce that $R(i\lambda)$ has no eigenvalues in $|z| > 1$. We will use the notation $\langle u, v\rangle$ for the distributional pairing associated to the Riemannian measure dv_g fixed on $\mathcal{M}$, which also extends to a complex bilinear pairing $\mathcal{H}_{NG} \times \mathcal{H}_{-NG} \to \mathbb{C}$; in particular if $u, v \in L^2(\mathcal{M})$, this is simply $\int_{\mathcal{M}} uv\, dv_g$. Accordingly, we also write $\langle u, v\rangle_{L^2}$ for the pairing $\int_{\mathcal{M}} u\bar{v}\, dv_g$ and its sesquilinear extension to the pairing $\mathcal{H}_{NG} \times \mathcal{H}_{-NG} \to \mathbb{C}$.

The next three lemmas (Lemmas 5.1–5.3) characterize the spectral projector of $R(i\lambda)$ onto the possible eigenvalue 1. Keep in mind that by Lemma 3.9 this spectral projector is closely related to the Ruelle–Taylor resonant states. Finally, in Proposition 5.4 we will use the knowledge about this spectral projector to characterize the leading resonance spectrum and to define physical measures.

Lemma 5.1. *Let $\lambda \in \mathfrak{a}^*$. If τ is an eigenvalue of $R(i\lambda)$ with modulus* 1, *it has no associated Jordan block, i.e. $(z - R(i\lambda))^{-1}$ has at most a pole of order* 1 *at $z = \tau$.*

Proof. We take $u \in \mathcal{H}_{NG}$ such that $(R(i\lambda) - z)^{-1}u$ has a pole of order > 1 at $z = \tau$. By density of C^∞ in $\mathcal{H}_{NG}$, we can always assume that u is smooth. Denoting by $\psi^{(k)} = \psi * \cdots * \psi$ (k-th convolution power), we can write

$$R(i\lambda)^k = \prod_{j=1}^{\kappa} \int_{\mathbb{R}} e^{-t_j X_j(i\lambda)} \psi_j^{(k)}(t_j)\, dt_j.$$

Note that $R(0)1 = 1$. If v is another smooth function, then

$$|\langle R(i\lambda)^k u, v\rangle| = \left| \int_{\mathbb{R}^\kappa} \prod_{j=1}^{\kappa} \psi_j^{(k)}(t_j) e^{-i\sum t_j\lambda_j} \left(\int_{\mathcal{M}} v e^{-\sum t_j X_j} u \, dv_g \right) dt_1 \dots dt_\kappa \right|$$
$$\leq |v|_{L^\infty} |u|_{L^\infty} R(0)^k 1 = |v|_{L^\infty} |u|_{L^\infty}.$$

We deduce that for $|z| > 1$,

$$|\langle z(z - R(i\lambda))^{-1} u, v\rangle| \leq \sum_{k=0}^{\infty} |z|^{-k} |v|_{L^\infty} |u|_{L^\infty} = |u|_{L^\infty} |v|_{L^\infty} (1 - |z|^{-1})^{-1}.$$

This is in contradiction with the assumption that τ is a pole of order > 1. ∎

Then we can prove the following.

Lemma 5.2. *For $\lambda = \sum_{j=1}^{\kappa} \lambda_j e_j \in \mathfrak{a}^*$, $R(i\lambda)$ has an eigenvalue of modulus 1 on $\mathcal{H}_{NG}$ if and only if $i\lambda$ is a Ruelle–Taylor resonance. In that case, the only eigenvalue of modulus 1 of $R(i\lambda)$ in $\mathcal{H}_{NG}$ is $\tau = 1$ and the eigenfunctions of $R(i\lambda)$ at $\tau = 1$ are the joint Ruelle resonant states of X at λ. Moreover, if $\Pi(i\lambda)$ is the spectral projector of $R(i\lambda)$ at $\tau = 1$, one has, as bounded operators in $\mathcal{H}_{NG}$,*

$$\lim_{k\to\infty} R(i\lambda)^k = \Pi(i\lambda). \tag{5.4}$$

Proof. First, if $i\lambda$ is a Ruelle–Taylor resonance, Proposition 4.17 implies that $R(i\lambda)$ has 1 as an eigenvalue and the resonant states are included in the range of the spectral projector of $R(i\lambda)$ at 1.

Conversely, let $\Pi(i\lambda)$ be the spectral projector of $R(i\lambda)$ at $\tau \in \mathbb{S}^1$; it commutes with the X_j, so we can use Lemma 3.8 to decompose ran $\Pi(i\lambda)$ in terms of joint eigenspaces for X_j. Let u be a joint eigenfunction of X_j in ran $\Pi(i\lambda)$, with $X_j u = \zeta_j u$. By Lemma 5.1, $R(i\lambda)$ has no Jordan block at τ, and thus $u \in \mathcal{H}_{NG}$ is a non-zero eigenfunction of $R(i\lambda)$ with eigenvalue $\tau \in \mathbb{S}^1$. Then

$$\tau u = R(i\lambda)u = u \int_{\mathbb{R}^\kappa} \prod_{j=1}^{\kappa} e^{-t_j(\zeta_j + i\lambda_j)} \psi_j(t_j) \, dt_j = u \prod_{j=1}^{\kappa} \hat{\psi}_j(\lambda_j - i\zeta_j).$$

For τ to have modulus 1, we need $\prod_{j=1}^{\kappa} |\hat{\psi}_j(\lambda_j - i\zeta_j)| = 1$. But since $\int_{\mathbb{R}} \psi_j = 1$ and the ζ_j's have non-negative real part,

$$|\hat{\psi}_j(\lambda_j - i\zeta_j)| \leq \int_{\mathbb{R}} e^{-t\mathrm{Re}(\zeta_j)} \psi_j(t) \, dt \leq 1,$$

so $\mathrm{Re}(\zeta_j) = 0$ and $|\hat{\psi}_j(\lambda_j - i\zeta_j)| = 1$ for all j. But then there is $\alpha \in \mathbb{R}$ such that $1 = \int_{\mathbb{R}} \psi_j(t) = \int_{\mathbb{R}} \cos(t(\lambda_j + \mathrm{Im}(\zeta_j)) + \alpha)\psi_j(t) \, dt$ and thus $\cos(t(\lambda_j + \mathrm{Im}(\zeta_j)) + \alpha) = 1$ on supp ψ_j since $\psi_j \geq 0$. This implies that $\zeta_j = -i\lambda_j$ and $\alpha \in 2\pi\mathbb{Z}$. Then we get $\tau = 1$. In particular,

$$R(i\lambda) = \Pi(i\lambda) + K(i\lambda)$$

with $K(i\lambda)\Pi(i\lambda) = \Pi(i\lambda)K(i\lambda) = 0$, and $K(i\lambda)$ having spectral radius $r < 1$ on $\mathcal{H}_{NG}$, so for all $\varepsilon > 0$, there is n_0 large such that for all $n \geq n_0$,

$$\|K(i\lambda)^n\|_{\mathcal{L}(\mathcal{H}_{NG})} \leq (r+\varepsilon)^n.$$

We can choose $r + \varepsilon < 1$, which implies that

$$\forall n \geq n_0, \quad R(i\lambda)^n = \Pi(i\lambda) + K(i\lambda)^n \to \Pi(i\lambda) \quad \text{in } \mathcal{L}(\mathcal{H}_{NG}), \tag{5.5}$$

proving (5.4).

To conclude the proof, we want to prove that $(X_j + i\lambda_j)\Pi(i\lambda) = 0$ for all $j = 1, \dots, \kappa$. By the discussion above, 0 is the only joint eigenvalue of $(X_1 + i\lambda_1, \dots, X_\kappa + i\lambda_\kappa)$ on $\operatorname{ran}\Pi(i\lambda)$, i.e. there is $J > 0$ such that $\prod_{j=1}^{\kappa}(X_j + i\lambda_j)^{\alpha_j}\Pi(i\lambda) = 0$ for all multi-indices $\alpha \in \mathbb{N}^\kappa$ with length $|\alpha| = J$. We already know that R has no Jordan block, and we want to deduce that this is also true for the X_j's. By Proposition 4.17, we get

$$\operatorname{ran}\Pi(i\lambda) = \Big\{u \in C^{-\infty}_{E_u^*}(\mathcal{M}) \;\Big|\; \prod_{j=1}^{\kappa}(X_j + i\lambda_j)^{\alpha_j}u = 0,\ \forall \alpha \in \mathbb{N}^\kappa,\ |\alpha| = J\Big\}.$$

In particular, this space does not depend on the choice of the χ_j (and thus ψ_j). The operator $e^{-\sum_j t_j(X_j + i\lambda_j)} : \operatorname{ran}\Pi(i\lambda) \to \operatorname{ran}\Pi(i\lambda)$ is represented by a finite-dimensional matrix $M(t)$ with $t = (t_1, \dots, t_\kappa)$, and $R(i\lambda)|_{\operatorname{ran}\Pi(i\lambda)} = \mathrm{Id}$ (since $R(i\lambda)$ has no Jordan block), thus

$$\mathrm{Id} = \int_{\mathbb{R}^\kappa} M(t)\psi_j(t_j)\,dt_1\dots dt_\kappa$$

for all choices of χ_j (and $\psi_j = -\chi_j'$). We can thus take, for $T = (T_1, \dots, T_\kappa)$, the family ψ_j converging to the Dirac mass δ_{T_j} and we obtain $M(T) = \mathrm{Id}$. This shows that $M(t) = \mathrm{Id}$ for all $t \in \mathbb{R}^\kappa_+$ large enough such that Lemma 4.14 can be applied, and therefore $(X_j + i\lambda_j)\Pi(i\lambda) = 0$ for all j. This implies that $\operatorname{ran}\Pi(i\lambda)$ is exactly the space of Ruelle resonant states for X at $i\lambda$. ∎

From what we have shown in Lemma 5.2, we deduce that we can write the spectral projector as $\Pi(i\lambda)f = \sum_{k=1}^{J} v_k\langle f, w_k\rangle_{L^2}$ with $v_k \in \mathcal{H}_{NG}$ spanning the space of joint Ruelle resonant states of resonance $i\lambda$ and $w_k \in \mathcal{H}_{NG}^* \simeq \mathcal{H}_{-NG}$. Recall that we have shown that the space of joint Ruelle resonant states (i.e. the range of $\Pi(i\lambda)$) is intrinsic, i.e. does not depend on the precise form of the parametrix. But surely the operator $R(i\lambda)$ depends on the choice of the cutoff functions ψ_j (see (5.1)) and thus also $\Pi(i\lambda)$ might depend on that choice. In order to see that this is not the case, let us consider $X_j^* = -X_j + \operatorname{div}_{v_g}(X_j)$, which are the adjoints with respect to the fixed measure v_g. Note that by the commutativity of the X_j, the operators X_j^* also commute and are admissible operators (in the sense of Definition 2.4) for the inverted Anosov action $\tau^-(a) := \tau(-a)$, which is obviously again an Anosov action (with the same positive Weyl chamber after swapping the stable and unstable bundles). Therefore we can apply the results of Section 4 to the admissible operators X_j^*, in particular they have discrete joint spectrum on the spaces $\mathcal{H}_{-NG}$. Using $(X_j + i\lambda_j)\Pi(i\lambda) = 0$ and the fact that $[X_j, \Pi(i\lambda)] = 0$ we deduce that $(X_j^* - i\lambda_j)w_k = 0$ and thus all w_k, $k = 1, \dots, J$, are joint resonant states of the X_j^*. We can even see that they span the space of joint resonant states: one can apply the same parametrix construction Lemma 4.14 to X_j^*,

$$Q_{X^*}(i\lambda)d_{X^*+i\lambda} + d_{X^*+i\lambda}Q_{X^*}(i\lambda) = \mathrm{Id} - R_{X^*}(i\lambda) \otimes \mathrm{Id},$$

and if we choose the same cutoff functions as in the parametrix for X_j at the beginning of this section, we find

$$R_{X^*}(i\lambda) = \prod_{j=1}^{\kappa} \int e^{-t_j(X_j^*+i\lambda_j)}\psi_j(t_j)\,dt_j. \tag{5.6}$$

In particular, $R_{X^*}(-i\lambda) = (R_X(i\lambda))^*$ as bounded operators on $\mathcal{H}_{-NG}$, where the adjoint is defined by $\langle R_X(i\lambda)f, f'\rangle_{L^2} = \langle f, (R_X(i\lambda))^* f'\rangle_{L^2}$ for all $f \in \mathcal{H}_{NG}$, $f' \in \mathcal{H}_{-NG}$. If $\Pi_{X^*}(i\lambda)$ is the spectral projector of $R_{X^*}(i\lambda)$ onto the eigenvalue 1 then we obtain $\Pi_{X^*}(-i\lambda)f = \Pi_X(i\lambda)^* f = \sum_{k=1}^{J} w_k\langle f, v_k\rangle_{L^2}$ with adjoint defined as above. By Lemma 3.9 the space $\{w \in \mathcal{H}_{-NG} \mid (X_j^* - i\lambda_j)w = 0 \text{ for all } j\}$ of joint resonant states is in the range of $\Pi_{X^*}(-i\lambda)$, and consequently the w_j span the space of joint resonant states of X^* with joint resonance $-i\lambda$. Putting everything together, we have the following.

Lemma 5.3. *Let $\lambda \in \mathfrak{a}^*$ be such that $i\lambda$ is a Ruelle–Taylor resonance of X. Then $-i\lambda$ is also a Ruelle–Taylor resonance of X^* and the spaces of joint resonant states have the same dimension. If $v_1, \dots, v_J \in C^{-\infty}_{E_u^*}(\mathcal{M})$ and $w_1, \dots, w_J \in C^{-\infty}_{E_s^*}(\mathcal{M})$ are such that they span the space of joint resonant states of X at $i\lambda$ and of X^* at $-i\lambda$ respectively and fulfill $\langle v_j, w_k\rangle_{L^2} = \delta_{jk}$, then we can write $\Pi(i\lambda) = \sum_{k=1}^{J} v_k\langle\cdot, w_k\rangle_{L^2}$. In particular, $\Pi(i\lambda)$ depends only on the X_j but not on the choice of $R(i\lambda)$.*

We can now identify resonant states on the imaginary axis with some particular invariant measures.

Proposition 5.4. (1) *For each $v \in C^\infty(\mathcal{M}; \mathbb{R}^+)$, the map*

$$\mu_v : C^\infty(\mathcal{M}) \ni u \mapsto \langle \Pi(0)u, v\rangle$$

is a non-negative Radon measure with mass $\mu_v(\mathcal{M}) = \int_{\mathcal{M}} v\,dv_g$, invariant by X_j for all $j = 1, \dots, \kappa$ in the sense $\mu_v(X_j u) = 0$ for all $u \in C^\infty(\mathcal{M})$.

(2) *The space*

$$\mathrm{span}\,\{\mu_v \mid v \in C^\infty(\mathcal{M}; \mathbb{R}^+)\} = \Pi(0)^*(C^\infty(\mathcal{M}))$$

is a finite-dimensional subspace of $C^{-\infty}_{E_s^*}(\mathcal{M})$ and it is precisely the space spanned by all finite measures μ with $\mathrm{WF}(\mu) \subset E_s^*$ that are invariant under the Anosov action. Here $\Pi(0)^* : \mathcal{H}_{-NG} \to \mathcal{H}_{-NG}$ is a bounded projector for all $N \gg 1$.

(3) *Let $f \in L^1(\mathcal{W}; [0, 1])$ with compact support contained in $\overline{\mathcal{W}}$ and $\int_{\mathcal{W}} f > 0$. Then for any $u, v \in C^\infty(\mathcal{M})$,*

$$\mu_v(u) = \lim_{T\to\infty} \frac{1}{T^\kappa \int_{\mathcal{W}} f} \int_{A\in\mathcal{W}} f\left(\frac{A}{T}\right)\langle e^{-X_A}u, v\rangle\,dA, \tag{5.7}$$

where dA is the Lebesgue–Haar measure on $\mathfrak{a}$.

(4) *Similarly, for* $\lambda \in \mathfrak{a}^*$ *and* $v \in C^\infty(\mathcal{M})$ *the map*

$$\mu_v^\lambda : C^\infty(\mathcal{M}) \ni u \mapsto \langle \Pi(i\lambda)u, v\rangle$$

is a complex-valued measure. These measures are flow-equivariant in the sense that $\mu_v^\lambda(X_j u) = -i\lambda_j \mu_v^\lambda(u)$ *and the set* $\{\mu_v^\lambda \mid v \in C^\infty(\mathcal{M})\}$ *is finite-dimensional and coincides with the space of finite complex measures* μ *with* $\mathrm{WF}(\mu) \subset E_s^*$ *which are equivariant in the above sense.*

(5) *Let* $v_1, v_2 \in C^\infty(\mathcal{M}; \mathbb{R}^+)$ *with* $v_1 \le C v_2$ *for some* $C > 0$ *and* $\lambda \in \mathfrak{a}^*$ *such that* $i\lambda$ *is a Ruelle–Taylor resonance. Then* $\mu_{v_1}^\lambda$ *is absolutely continuous with bounded density with respect to* $\mu_{v_2} = \mu_{v_2}^0$*. In particular, any* μ_v^λ *is absolutely continuous with respect to* μ_1.

Proof. First $R(0)1 = 1$ is clear and X has a Ruelle–Taylor resonance at $\lambda = 0$ by Lemma 3.10. If $u, v \in C^\infty(\mathcal{M})$ are non-negative, we have $a_k := \langle R(0)^k u, v\rangle \ge 0$ and

$$\lim_{k\to\infty} \langle R(0)^k u, v\rangle = \langle \Pi(0)u, v\rangle \ge 0.$$

Note also that for each k, and each $u \in C^\infty(\mathcal{M})$ non-negative,

$$\forall x \in \mathcal{M}, \quad 0 \le (R(0)^k u)(x) \le (R(0)^k 1)\|u\|_{C^0} \le \|u\|_{C^0}.$$

This implies that for each $v \in C^\infty$ with $v \ge 0$, $\mu_v^k : u \mapsto \langle R(0)^k u, v\rangle$ is a Radon measure with finite mass $\mu_v^k(\mathcal{M}) = \int_{\mathcal{M}} v\, dv_g$ and thus so is μ_v as well. The invariance of μ_v is a direct consequence of Lemma 5.3. The same holds for property (2). The invariance of the space spanned by these measures with respect to X_j follows from $\Pi(0)X_j = X_j\Pi(0) = 0$, obtained from Lemma 5.2.

Let us next show that for an arbitrary Ruelle–Taylor resonance $i\lambda \in i\mathfrak{a}^*$ we get complex measures μ_v^λ, and at the same time prove the absolute continuity statement (5). We consider $u \in C^\infty(\mathcal{M})$ and $v_1, v_2 \in C^\infty(\mathcal{M}; \mathbb{R}^+)$ with $v_1 \le v_2$, and get, for all k,

$$|\langle R(i\lambda)^k u, v_1\rangle| \le \langle R(0)^k |u|, v_1\rangle \le \langle R(0)^k |u|, v_2\rangle,$$

thus $|\mu_{v_1}^\lambda(u)| \le \mu_{v_2}(|u|)$. This proves that $\mu_{v_1}^\lambda$ is a complex measure. A priori, $\mu_{v_1}^\lambda$ is absolutely continuous with respect to μ_{v_2}, so it has an L^1 density f with respect to μ_{v_2}. This density is actually bounded by 1, or equivalently

$$|\mu_{v_1}^\lambda(\mathcal{A})| \le \mu_{v_2}(\mathcal{A}) \tag{5.8}$$

for every Borel set $\mathcal{A}$. If $\mathcal{A}$ is a closed set, we can find a sequence of smooth functions g_n, valued in [0, 1], which converges pointwise to the characteristic function of $\mathcal{A}$. By dominated convergence $\mu_{v_1}^\lambda(g_n) \to \mu_{v_1}^\lambda(\mathcal{A})$, and likewise $\mu_{v_2}(g_n) \to \mu_{v_2}(\mathcal{A})$. Taking products of sequences of functions, or sequences $1 - g_n$, and using a diagonal argument, we see that the set of Borel sets $\mathcal{A}$ for which (5.8) holds contains closed sets, and is stable by countable intersection and complement. It is thus equal to the whole tribe of Borel sets, and the proof of $\|f\|_{L^\infty} \le 1$ is complete.

Let us finally show (5.7). For each $f \in L^1(\mathfrak{a}; [0,1])$ with $\operatorname{supp} f \subset \overline{\mathcal{W}}$ being compact and $\int f > 0$, we want to prove that

$$C_f(T) := \frac{1}{T^\kappa \int_{\mathcal{W}} f} \int_{A \in \mathcal{C}_T} f\left(\frac{A}{T}\right) \langle e^{-X_A} u, v\rangle \, dA \to \mu_v(u) \quad \text{as } T \to +\infty. \tag{5.9}$$

Assume that (5.9) is satisfied for a dense set in $L^1(\mathcal{W})$ of compactly supported functions, $F := (f_i)_{i \in I} \subset C_c(\mathcal{W})$. Then, for all $\delta > 0$ small, one can find a sequence $f_{i(n)} \in F$ such that $\int_{\mathcal{W}} |f_{i(n)}/\int_{\mathcal{W}} f_{i(n)} - f/\int_{\mathcal{W}} f| < \delta$. Then

$$|C_f(T) - C_{f_{i(n)}}(T)| \leq \int_{\mathcal{W}} |\langle e^{-TX_A} u, v\rangle| \cdot \left| \frac{f(A)}{\int f} - \frac{f_{i(n)}}{\int f_{i(n)}} \right| dA \leq \delta \|u\|_{L^\infty} \|v\|_{L^\infty},$$

which implies (5.9) for f by our assumption on f_i and since δ is arbitrarily small. We shall then show (5.9) for functions of the form $\omega(t_1) q(\bar{t}/t_1)$ if $(t_1, \bar{t}) \in \mathbb{R}_+ \times \mathbb{R}^{\kappa-1}$ are coordinates associated to bases of vectors in small cones $\mathcal{C}$ with closure contained in $\mathcal{W} \cup \{0\}$, and $\omega \in C_c^\infty(0,1)$, $q \in C_c^\infty(\mathbb{R}^{\kappa-1})$ such that $\operatorname{supp}(\omega(t_1) q(\bar{t}/t_1)) \subset \mathcal{C}$.

We fix a small open cone $\mathcal{C} \subset \mathcal{W}$ with arbitrarily small conic section with closure contained in $\mathcal{W}$ and choose a basis $(A_j)_{j=1}^\kappa$ of $\mathfrak{a}$ so that $A_j \in \mathcal{C}$. Up to rescaling A_j by some fixed large $T > 0$, we can assume that Lemma 4.14 applies with $T_j = 1/2$ in the construction of $Q(\lambda)$ and $R(\lambda)$. We then identify $\mathfrak{a} \cong \mathbb{R}^\kappa$ by identifying the canonical basis $(e_j)_j$ of $\mathbb{R}^\kappa$ with $(A_j)_j$, and we define a scalar product on $\mathfrak{a}$ by declaring that the A_j are orthonormal. We let $\Sigma = \mathcal{C} \cap \{A_1 + \sum_{j=2}^\kappa t_j A_j \,|\, t_j \in \mathbb{R}\}$ be a hyperplane section of the cone $\mathcal{C}$. Choose $\psi \in C_c^\infty((-1/2, 1/2))$ non-negative even with $\int_{\mathbb{R}} \psi = 1$, and for each $\sigma \in \mathbb{R}^\kappa$, define $\psi_\sigma(t) := \prod_{j=1}^\kappa \psi(t_j - \sigma_j)$. The operators $Q(0)$, $R(0)$ constructed in Lemma 4.14 can be defined, for σ close to e_1, with the cutoff function χ_j such that $-\chi_j'(t_j) = \psi(t_j - \sigma_j)$. We then denote by Q_σ, R_σ the corresponding operators, which in turn are locally uniform in σ. Then μ_v is given by $\mu_v(u) = \lim_{k \to \infty} \langle R_\sigma(0)^k u, v\rangle$ locally uniformly in σ. This means that viewing Σ as an open subset of $e_1 + \mathbb{R}^{\kappa-1}$ containing e_1, taking any $q \in C_c^\infty(\Sigma)$ with $\int_{\mathbb{R}^{\kappa-1}} q(\bar{t})\, d\bar{t} = 1$ and any $\omega \in C_c^\infty((0,1))$ with $\int_0^1 \omega = 1$, we have, for $\sigma(\bar{t}) := (1, \bar{t}) \in \Sigma$,

$$\mu_v(u) = \lim_{N \to \infty} \int_{\mathbb{R}^{\kappa-1}} \frac{1}{N} \sum_{k=1}^N \omega\left(\frac{k}{N}\right) \langle R^k_{\sigma(\bar{t})} u, v\rangle q(\bar{t})\, d\bar{t}. \tag{5.10}$$

Indeed, if $\operatorname{supp} \omega \subset (\varepsilon, 1-\varepsilon)$ for some $\varepsilon > 0$, one can write

$$\begin{aligned}
&\int_{\mathbb{R}^{\kappa-1}} \frac{1}{N} \sum_{k=1}^N \omega\left(\frac{k}{N}\right) \langle R^k_{\sigma(\bar{t})} u, v\rangle q(\bar{t})\, d\bar{t} - \mu_v(u) \\
&\quad = \int_{\mathbb{R}^{\kappa-1}} q(\bar{t}) \Bigg(\frac{1}{N} \sum_{k \in (\varepsilon N, (1-\varepsilon)N)} \omega\left(\frac{k}{N}\right) (\langle R^k_{\sigma(\bar{t})} u, v\rangle - \mu_v(u)) \\
&\qquad\qquad - \mu_v(u) \left(1 - \frac{1}{N} \sum_{k=1}^N \omega\left(\frac{k}{N}\right)\right)\Bigg) d\bar{t}
\end{aligned}$$

and we use $\frac{1}{N}\sum_{k=1}^{N}\omega(\frac{k}{N})\to\int_0^1\omega=1$ as $N\to\infty$, and $\langle R_\sigma^k u,v\rangle\to\mu_v(u)$ uniformly in σ as $k\to\infty$, so that we get the result by dominated convergence.

Recall that R_σ^k is given by the expression

$$R_\sigma^k u=\int_{\mathbb{R}^\kappa}(e^{-\sum_{j=1}^{\kappa}t_jX_j}u)\psi_\sigma^{(k)}(t)\,dt,$$

where $\psi_\sigma^{(k)}$ is the k-th convolution of ψ_σ. To prove (5.9) for a dense subset of $L^1_{\mathrm{comp}}(\mathcal{W})$ it suffices to combine (5.10) and the following lemma (since the space of finite sums of functions of the form $\widetilde{\omega}(t_1)q(\bar{t}/t_1)$ is dense in $L^1_{\mathrm{comp}}(\mathcal{W})$).

Lemma 5.5. *With ω,q as above, and $\widetilde{\omega}(r):=r^{1-\kappa}\omega(r)$, we have, as $N\to\infty$,*

$$\int_{\mathbb{R}^{\kappa-1}}\frac{1}{N}\sum_{k=1}^{N}\omega\Big(\frac{k}{N}\Big)\langle R^k_{\sigma(\bar{t})}u,v\rangle q(\bar{t})\,d\bar{t}\\ -\frac{1}{N^\kappa}\int_0^N\int_{\mathbb{R}^{\kappa-1}}\langle e^{-\sum_j t_jX_j}u,v\rangle\widetilde{\omega}\Big(\frac{t_1}{N}\Big)q\Big(\frac{\bar{t}}{t_1}\Big)\,dt_1\,d\bar{t}\to 0.$$

Proof of Lemma 5.5. Since for $u\in C^\infty(\mathcal{M})$, $\|e^{-\sum_{j=1}^{\kappa}t_jX_j}u\|_{L^\infty}\le\|u\|_{L^\infty}$, it suffices to show that the function

$$\mathbb{R}_+\times\mathbb{R}^{\kappa-1}\ni t=(t_1,\bar{t})\mapsto\\ \int_{\mathbb{R}^{\kappa-1}}\frac{1}{N}\sum_{k=1}^{N}\omega\Big(\frac{k}{N}\Big)\psi^{(k)}_{\sigma(\theta)}(t)q(\theta)d\theta-\frac{1}{N^\kappa}\widetilde{\omega}\Big(\frac{t_1}{N}\Big)q\Big(\frac{\bar{t}}{t_1}\Big)$$

converges as $N\to\infty$ to 0 in $L^1(\mathbb{R}^\kappa)$. Let $\varepsilon>0$ be so small that $\operatorname{supp}\omega\subset(\varepsilon,1-\varepsilon)$. Scaling $t\to tN$, the above convergence statement is equivalent to showing that

$$f_N(t):=N^{\kappa-1}\sum_{k\in\mathbb{Z}\cap(\varepsilon N,(1-\varepsilon)N)}\omega\Big(\frac{k}{N}\Big)\int_{\mathbb{R}^{\kappa-1}}\psi^{(k)}_{\sigma(\theta)}(tN)q(\theta)\,d\theta$$

is such that $\lim_{N\to\infty}\|f_N-h\|_{L^1(\mathbb{R}^\kappa)}=0$ if $h(t):=\frac{1}{T}\widetilde{\omega}(\frac{t_1}{T})q(\frac{T\bar{t}}{t_1})$. First, notice that $\operatorname{supp}(\psi^{(k)}_{\sigma(\theta)}(N\cdot))\subset B(0,2)$ for each $k\le N$, and $\int\psi^{(k)}_{\sigma(\theta)}(t)\,dt=1$. It then suffices to prove that f_N converges in $L^2(\mathbb{R}^\kappa)$ to h. We proceed using the Fourier transform, with $\xi=(\xi_1,\bar{\xi})$,

$$\hat{f}_N(\xi)=\frac{1}{N}\sum_{k=\varepsilon N}^{(1-\varepsilon)N}\omega\Big(\frac{k}{N}\Big)\Big(\hat{\psi}_0\Big(\frac{\xi}{N}\Big)\Big)^k e^{-i\frac{k}{N}\xi_1}\hat{q}\Big(\frac{k}{N}\bar{\xi}\Big).$$

First, for ξ fixed, as $\hat{\psi}_0(\xi)=1+\mathcal{O}(|\xi|^2)$ for small ξ (since $\int\psi=1$ and $\int t\psi(t)\,dt=0$ by assumption on ψ), one has the following pointwise convergence (using Riemann sums):

$$\lim_{N\to\infty}\hat{f}_N(\xi)=\int_{\mathbb{R}}\omega(t_1)\hat{q}(t_1\bar{\xi})e^{-it_1\xi_1}\,dt_1=\hat{h}(\xi).\tag{5.11}$$

To prove L^2 convergence, we use the fact that, since $\psi \geq 0$ is smooth and satisfies $\int \psi = 1$ and $\int t\psi(t)\,dt = 0$, there is $c_0 > 0$ such that

$$|\hat{\psi}_0(\xi)| \leq (1 + c_0|\xi|^2)^{-1}. \tag{5.12}$$

Indeed, it suffices to prove $|\hat{\psi}(s)| \leq (1 + c_0 s^2)^{-1}$ for some $c_0 > 0$; but there is $\varepsilon > 0$ such that this is true for some $c_0 = c_0(\varepsilon)$ when $|s| < \varepsilon$ small (by Taylor expansion at $s = 0$ and since $|\hat{\psi}(s)| < 1$ for $s \neq 0$) and for $|s| > 1/\varepsilon$ large by integration by parts ($\psi \in C_c^\infty$), while for $|s| \in [\varepsilon, 1/\varepsilon]$ there is $c_0'(\varepsilon) > 0$ such that $|\hat{\psi}(s)| \leq (1 + c_0'(\varepsilon)s^2)^{-1}$ because $|\hat{\psi}| \leq 1 - \delta_\varepsilon$ on $[\varepsilon, 1/\varepsilon]$ for some $\delta_\varepsilon > 0$. Thus for $\delta > 0$ arbitrarily small, there is $C > 0$ depending only on $\|\hat{q}\|_{L^\infty}$ and $\|\omega\|_{L^\infty}$ such that

$$\begin{aligned}\int_{|\xi| \geq N^{1/2+\delta}} |\hat{f}_N(\xi)|^2\,d\xi &\leq CN^\kappa \int_{|\xi| \geq N^{-1/2+\delta}} |\hat{\psi}_0(\xi)|^{2\varepsilon N}\,d\xi \\ &\leq CN^\kappa e^{-c_0\varepsilon N^{2\delta}} \int |\hat{\psi}_0(\xi)|\,d\xi \to 0,\end{aligned}$$

where the second inequality holds for large N and the limit is as $N \to \infty$. Next, we will show that for $\delta > 0$ small and $\ell \in \mathbb{N}$, there are N_ℓ, C_ℓ such that for all $N \geq N_\ell$,

$$\forall \xi, |\xi| \in [1, N^{1/2+\delta}], \quad |\hat{f}_N(\xi)| \leq C_\ell(|\xi|^{-\ell} + N^{-\ell(1/2-\delta)}). \tag{5.13}$$

This will prove the convergence of $\hat{f}_N$ to $\hat{h}$ in L^2, since for all $n > 0$, there are T_n and $N_n > 0$ such that for $N \geq N_n$,

$$\int_{|\xi| \geq T_n} |\hat{f}_N(\xi) - \hat{h}(\xi)|^2\,d\xi \leq 1/n$$

and, using dominated convergence and (5.11),

$$\lim_{N \to \infty} \int_{\mathbb{R}^\kappa} |\hat{f}_N(\xi) - \hat{h}(\xi)|^2 \mathbf{1}_{[0,T_n]}(|\xi|)\,d\xi = 0.$$

We next show (5.13). We will use discrete integration by parts to get decay of $\hat{f}_N(\xi)$ in the ξ_1 variable. For $\rho \in C_c^\infty((0,1))$, we define some sequences a_k^m, b_k^m for $k \in \mathbb{Z}$ and $m \in \mathbb{N}$ by induction. First, $b_k^0 := e^{-i\frac{k}{N}\xi_1}$, $a_k^0 = \rho(k/N)$ for $k \in \mathbb{Z}$. Next, for $m \geq 1$ and $k \in \mathbb{Z}$,

$$b_k^m := b_k^{m-1} \frac{e^{-i\xi_1/N}}{1 - e^{-i\xi_1/N}} = e^{-i\xi_1 k/N} \left(\frac{e^{-i\xi_1/N}}{1 - e^{-i\xi_1/N}} \right)^m, \quad a_k^m := a_k^{m-1} - a_{k+1}^{m-1}.$$

Note also that $a_k^m = 0$ for $k < \varepsilon N - m$ and $k > (1-\varepsilon)N$, and that $b_k^m = b_{k-1}^{m+1} - b_k^{m+1}$. Thus,

$$\sum_{k \in \mathbb{Z}} a_k^m b_k^m = \sum_{k \in \mathbb{Z}} a_k^m (b_{k-1}^{m+1} - b_k^{m+1}) = -\sum_{k \in \mathbb{Z}} b_k^{m+1}(a_k^m - a_{k+1}^m) = -\sum_{k \in \mathbb{Z}} b_k^{m+1} a_k^{m+1}.$$

Since ρ is smooth, Taylor expansion gives for each m a constant $C_m > 0$ such that for $N \geq 1$, $|a_k^m| \leq C_m \|\rho\|_{C^m} N^{-m}$. Up to increasing the value of C_m, we also find that $|b_k^m| \leq$

$C_m N^m/|\xi_1|^m$ for $|\xi_1|/N \le \pi/2$. We deduce that for each m, there exists $C_m > 0$ such that for all N large enough,

$$\Big|\sum_{k=1}^{N} \rho(k/N)e^{-i\xi_1 k/N}\Big| \le N \min(C_m\|\rho\|_{C^m}|\xi_1|^{-m}, \|\rho\|_{L^\infty}). \tag{5.14}$$

Now, take $\rho(x) := \hat{q}(x\bar{\xi})\omega(x)e^{xN\log(\hat{\psi}_0(\xi/N))}$, which exists since $|\hat{\psi}_0(\xi/N)-1| \le 1/2$ for N large enough (since $|\xi|/N$ is assumed to be small). Since $q \in C_c^\infty(\mathbb{R}^{\kappa-1})$, for all $\ell \in \mathbb{N}$ there is $C_{\ell,m} > 0$ such that for all $x \in \operatorname{supp}\omega$,

$$|\partial_x^m(\hat{q}(x\bar{\xi})\omega(x))| \le C_{\ell,m}(1+|\bar{\xi}|^2)^{-\ell+m}. \tag{5.15}$$

Since $|N\log(\hat{\psi}_0(\xi/N))| \le C|\xi|^2/N$ for some uniform $C>0$ if $|\xi|/N$ is small (by using (5.12) and $\hat{\psi}_0(\xi) = 1 + \mathcal{O}(|\xi|^2)$), there are C, C' such that for all n and $|\xi|/N$ small enough,

$$|\partial_x^n e^{xN\log(\hat{\psi}_0(\xi/N))}| \le \Big(C\frac{|\xi|^2}{N}\Big)^n e^{Cx|\xi|^2/N} \le C'\Big(C\frac{|\xi|^2}{N}\Big)^n$$

and we finally obtain, by combining this bound with (5.15) (choosing $\ell = 2m-n+j$ for the $m-n$ derivatives), the following bound: for all m, j there are $C_{m,j}, C'_{m,j} > 0$ such that for N large enough,

$$\|\rho\|_{C^m} \le C_{m,j}\sum_{n=0}^{m}(1+|\bar{\xi}|^2)^{-j}\Big(C\frac{|\xi|^2}{N}\Big)^n \le C'_{m,j}(1+|\bar{\xi}|^2)^{-j}\Big(1+\frac{|\xi|^{2m}}{N^m}\Big). \tag{5.16}$$

Combining this bound (by taking $j=m$) with (5.14) implies that for all m, there are $C_m, C'_m > 0$ such that for all N large enough and $|\xi| \in [1, N^{1/2+\delta}]$ and $|\xi_1| \ge 1/2$,

$$|\hat{f}_N(\xi)| \le C_m\frac{1+|\xi|^{2m}N^{-m}}{(1+|\bar{\xi}|^2)^m(1+|\xi_1|)^m} \le C'_m\Big(\frac{1}{|\xi|^m}+\frac{|\xi|^m}{N^m}\Big) \le C'_m\Big(\frac{1}{|\xi|^m}+\frac{1}{N^{m(1/2-\delta)}}\Big),$$

which shows (5.13) whenever $|\xi_1| \ge 1/2$. If $|\xi| \in [1, N^{1/2+\delta}]$ and $|\xi_1| \le 1/2$, one has $|\bar{\xi}| \ge |\xi| - 1/2$ and thus the $\|\rho\|_{L^\infty}$ bound in (5.14) and the bound (5.16) for $m=0$ and j large give

$$|\hat{f}_N(\xi)| \le C_j(1+|\bar{\xi}|^2)^{-j} \le C'_j(1+|\xi|^2)^{-j},$$

which again shows (5.13) for $|\xi_1| \le 1/2$. ∎

As noted in the introduction, we will call the measures μ_v *physical measures*, and μ_1 will be called the *full physical measure*.

5.2. *Imaginary Ruelle–Taylor resonances for volume-preserving actions*

In this section, we are going to study the dimensions of the Ruelle–Taylor resonance at $\lambda = 0$ in the case where there is a smooth measure preserved by the action. First, we want to prove the following.

Proposition 5.6. *Assume that there is a smooth invariant measure μ for the action, i.e. $\mathcal{L}_{X_A}\mu = 0$ for each $A \in \mathcal{W}$. Then, for each $\lambda \in i\mathfrak{a}^*$ imaginary, there is an injective map*

$$\ker_{C^{-\infty}_{E_u^*}\Lambda^j} d_{X+\lambda}/\mathrm{ran}_{C^{-\infty}_{E_u^*}\Lambda^j} d_{X+\lambda} \to \ker_{C^\infty\Lambda^j} d_{X+\lambda}/\mathrm{ran}_{C^\infty\Lambda^j} d_{X+\lambda}. \tag{5.17}$$

Proof. We shall use an argument inspired by [20, Section 6] in rank 1. As in [20, Section 6], it will be technically convenient to use a semiclassical Weyl quantization Op_h and semiclassical wavefront set, the semiclassical parameter being denoted $h > 0$. The interested reader can consult the book [60] for the details on semiclassical calculus, and [18, Appendix E] that sumarizes all the necessary results used here. For a shorter summary see also [16, appendix]. We will use the classes of semiclassical operators $\Psi_h^k(\mathcal{M})$ (see [60, Section 14.2]), the semiclassical wavefront set $\mathrm{WF}_h(u)$ (resp. $\mathrm{WF}_h(A)$) of a distribution u (resp. of an operator $A \in \Psi_h^k(\mathcal{M})$) depending on a small parameter $h > 0$ (see [60, Section 8.4.2] or [18, Sections E.2.1, E.2.3]). The semiclassical wavefront set is a subset of the fiber radial compactification $\overline{T}^*\mathcal{M}$ of the cotangent bundle $T^*\mathcal{M}$; see [18, Section E.1.3].

Fix a basis $A_1, \dots, A_\kappa \in \mathcal{W}$ close to A_1 and write $\lambda_j := \lambda(A_j)$ and $X_{A_j}(\lambda) := X_{A_j} + \lambda_j$. Let $T_j > 0$ for $j = 1, \dots, \kappa$, let $\varepsilon > 0$ be small and consider $\chi_j \in C_c^\infty([0,\infty); [0,1])$ non-increasing with $\chi_j = 1$ in $[0, T_j]$ and $\mathrm{supp}\,\chi_j \subset [0, T_j + \varepsilon]$. We use the parametrix $Q(\lambda) = \delta_{\mathbf{Q}(\lambda)}$ of the proof of Lemma 4.14 and get (4.15) with those χ_j. As in the proof of Lemma 4.14, $F(\lambda) - \mathrm{Id} = R(\lambda) + K(\lambda)$ with $K(\lambda)$ compact on $\mathcal{H}_{NG}$ and $\|R(\lambda)\| < 1/2$, and by the Remark following the proof of Lemma 4.14, we can choose $T_1 > 0$ large and $T_j > 0$ small for $j = 2, \dots, \kappa$ so that this still holds. Using Lemma 4.13, we deduce $\mathrm{ran}\,\Pi_0(\lambda) \subset C^{-\infty}_{E_u^*}(\mathcal{M}; \Lambda\mathfrak{a}^*)$ if $\Pi_0(\lambda)$ is the spectral projector of $F(\lambda)$ at $z = 0$. We will show that the range of the spectral projector $\Pi_0(\lambda)$ at $z = 0$ of $F(\lambda)$ actually satisfies

$$\mathrm{ran}\,\Pi_0(\lambda) \subset C^\infty(\mathcal{M}; \Lambda\mathfrak{a}^*). \tag{5.18}$$

Since $F(\lambda)$ is a scalar operator, we can work on scalar-valued distributions, and we shall then identify $F(\lambda)$ with an operator $\mathcal{H}_{NG} \to \mathcal{H}_{NG}$ for some $N > 0$ large enough, and fixed.

Using Lemma 5.1, $z = 1$ is at most a pole of order 1 of $(\mathrm{Id} - F(\lambda) - z)^{-1}$, so that each $u \in \mathrm{ran}\,\Pi_0(\lambda)$ satisfies $F(\lambda)u = 0$. Then let $u \in \mathcal{H}_{NG}$ be such that $F(\lambda)u = 0$.

Recall from [16, (2.6)] that $\mathrm{WF}(u) = \mathrm{WF}_h(u) \cap T^*\mathcal{M} \setminus \{0\}$. We are now going to show that $\mathrm{WF}_h(u) \cap \{(x,\xi) \in E_u^* \mid |\xi| \in [c_1, c_2]\} = \emptyset$ for some $0 < c_1 < c_2$ by using the equation $F(\lambda)u = 0$, the propagation of semiclassical wavefront sets (Egorov theorem [60, Theorem 11.12]) and the explicit expression of $F(\lambda)$ in terms of the propagators $e^{-tX_{A_j}(\lambda)}$.

For $T_1 > 0$ large enough but fixed and $T_2, \dots, T_\kappa$ small enough, one can find a closed neighborhood W_u of $E_u^* \cap \partial\overline{T}^*\mathcal{M}$ in the fiber radial compactification of $T^*\mathcal{M}$, which is conic for $|\xi|$ large, $0 < c_1 < c_2$ such that for all $t_1 \in [T_1/2, T_1 + \varepsilon]$ and all $t_j \in [0, T_j + \varepsilon]$ for $j \geq 2$ we have

$$W_u \subset e^{-\sum_{j=1}^\kappa t_j X_{A_j}^H}(W_u) \quad \text{and} \quad \{(x,\xi) \in E_u^* \mid |\xi| \in [c_1, c_2]\} \subset e^{-\sum_j t_j X_{A_j}^H}(W_u) \setminus W_u.$$

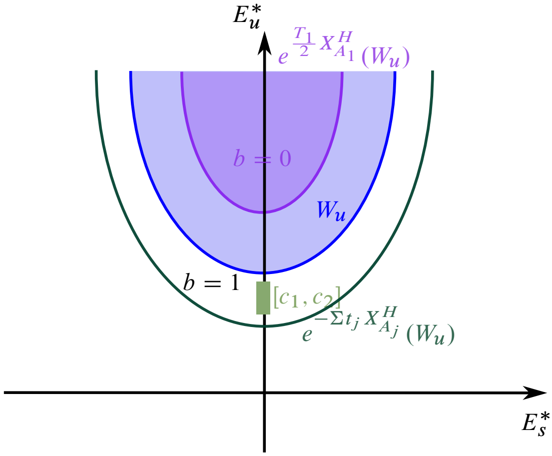

Fig. 2. Schematic sketch of the phase space regions appearing in the proof of Proposition 5.6.

First we choose $b_0 \in S^0(\mathcal{M})$ with the following properties: first,

$$b_0 \geq 0, \quad b_0(x,\xi) = 1 \text{ in } T^*\mathcal{M} \setminus W_u, \quad b_0(x,\xi) = 0 \text{ in } e^{\frac{T_1}{2}X^H_{A_1}}(W_u);$$

then, for each $t = (t_1, \ldots, t_\kappa)$ with $t_j \in [T_j, T_j + \varepsilon]$ the symbol

$$0 \leq b_0(x,\xi) - b_0(e^{\sum_{j=1}^{\kappa} t_j X^H_{A_j}}(x,\xi))$$

is equal to 1 on $\{(x,\xi) \in E_u^* \mid |\xi| \in [c_1, c_2]\}$; and third, by a partition of unity we can ensure that there is $c_0 \in S^0(\mathcal{M})$ such that $b_0^2 + c_0^2 = 1$. If $B_0 = \mathrm{Op}_h(b_0)$ and $C_0 = \mathrm{Op}_h(c_0)$ are the corresponding operators then $B_0B_0^* + C_0C_0^* = \mathrm{Id} + hR$ with $R \in \Psi_h^{-1}(\mathcal{M})$. As in the construction of a parametrix, one can modify the symbols by lower order terms and get symbols b, c such that for $B = \mathrm{Op}_h(b)$ and $C = \mathrm{Op}_h(c)$ we have $\mathrm{Id} - B^*B - C^*C \in h^\infty\Psi_h^0(\mathcal{M})$. Indeed, for $S \in \Psi_h^{-1}(\mathcal{M})$ with real principal symbol (using $\mathrm{Op}_h(q) - \mathrm{Op}_h(q)^* \in h\Psi_h^0(\mathcal{M})$ if $q \in S^0(\mathcal{M})$ is real-valued), we have

$$(B_0 + hS)(B_0 + hS)^* + (C_0 + hS)(C_0 + hS)^* - \mathrm{Id} - h(R + 2S(B_0 + C_0)) \in h^2\Psi_h^{-2}(\mathcal{M})$$

and, as $B_0 + C_0$ is semiclassically elliptic, we can invert it microlocally and find $s \in S^0(\mathcal{M})$ such that $S = \mathrm{Op}_h(s)$ satisfies $R + 2S(B_0 + C_0) \in h\Psi^0(\mathcal{M})$ and we gain a power of h if we correct B_0, C_0 by hS. This argument can then be iterated. Note furthermore that the regions where $b_0 = 0$ respectively $b_0 = 1$ are still valid for b.

Note that the escape function G can be chosen so that the order function m is non-negative in the region $T^*\mathcal{M} \setminus W_u$ for $|\xi|$ large enough. Since $u \in \mathcal{H}_{NG}$, we thus have $Bu \in L^2$. Let $\tilde{\chi} \in C^\infty(\mathbb{R}^\kappa)$ be given by $\tilde{\chi}(t) = (-1)^\kappa \prod_{j=1}^{\kappa} \chi_j'(t_j) \geq 0$ for $t \in \mathbb{R}^\kappa$. Recalling that $F(\lambda)$ is the operator introduced in (4.15), we can write, using the semiclassical

Egorov lemma in its simple form of a coordinate change [18, Proposition E19],

$$Bu = B(\mathrm{Id} - F(\lambda))u = \int_{(\mathbb{R}+)^\kappa} Be^{-\sum_{j=1}^{\kappa} t_j X_{A_j}(\lambda)} u\tilde{\chi}(t)\, dt_1 \dots dt_\kappa$$
$$= \int_{(\mathbb{R}+)^\kappa} e^{-\sum_{j=1}^{\kappa} t_j X_{A_j}(\lambda)} B_t u\tilde{\chi}(t)\, dt_1 \dots dt_\kappa$$

with $B_t - \mathrm{Op}_h(b \circ e^{\sum_j t_j X^H_{A_j}}) \in h\Psi_h^{-1}(\mathcal{M})$ and $\mathrm{WF}_h(B_t) \subset e^{-\sum_j t_j X^H_{A_j}}(\mathrm{WF}_h(B))$. This gives, using $\|e^{-t_j X_j(\lambda)}\|_{\mathcal{L}(L^2)} = 1$ because μ is invariant, and using Cauchy–Schwarz,

$$\|Bu\|^2_{L^2} = \|B(\mathrm{Id} - F(\lambda))u\|^2_{L^2} \le \int_{(\mathbb{R}+)^\kappa} \|B_t u\|^2_{L^2} \tilde{\chi}(t)\, dt \cdot \int_{(\mathbb{R}+)^\kappa} \tilde{\chi}(t)\, dt.$$

We can then write, since $\int_{(\mathbb{R}+)^\kappa} \tilde{\chi}(t)\, dt = 1$,

$$\int_{(\mathbb{R}+)^\kappa} (\|Bu\|^2_{L^2} - \|B_t u\|^2_{L^2})\tilde{\chi}(t)\, dt \le 0. \tag{5.19}$$

Next, recall that $\mathrm{supp}\, \tilde{\chi}(t) \subset \prod_{j\ge1}[T_j, T_j + \varepsilon]$. We claim that for $t \in \mathrm{supp}\, \tilde{\chi}$ there is $e_t \in S^0(\mathcal{M}; [0, 1])$ such that $B^*B - (B_t^* B_t + E_t^* E_t) \in h^\infty \Psi_h^0(\mathcal{M})$ for $E_t := \mathrm{Op}_h(e_t)$ and $e_t(x, \xi) = 1 + \mathcal{O}(h)$ in $\{(x, \xi) \in E_u^* \mid |\xi| \in [c_1, c_2]\}$. Indeed, E_t is microlocally equal to $C_t := e^{\sum_{j=1}^{\kappa} t_j X_{A_j}(\lambda)} C e^{-\sum_{j=1}^{\kappa} t_j X_{A_j}(\lambda)}$ on $\mathrm{WF}_h(B_t)$ and to B on the complement of $\mathrm{WF}_h(B_t)$. This implies, thanks to (5.19),

$$\int_{(\mathbb{R}+)^\kappa} \|E_t u\|^2_{L^2} \tilde{\chi}(t)\, dt = \mathcal{O}(h^\infty).$$

There are $f, g_t \in S^0(\mathcal{M}; [0, 1])$, with $f = 1 + \mathcal{O}(h)$ on $\{(x, \xi) \in E_u^* \mid |\xi| \in [c_1, c_2]\}$ and f independent of t, such that for $t \in \mathrm{supp}\, \tilde{\chi}$, $E_t^* E_t - (F^* F + G_t^* G_t) \in h^\infty \Psi_h^0(\mathcal{M})$, where $F = \mathrm{Op}_h(f)$ and $G_t = \mathrm{Op}_h(g_t)$. We thus obtain

$$\|Fu\|^2_{L^2} \le \int_{(\mathbb{R}+)^\kappa} \|E_t u\|^2_{L^2} \tilde{\chi}(t)\, dt + \mathcal{O}(h^\infty) = \mathcal{O}(h^\infty),$$

which implies that $\mathrm{WF}_h(u) \cap \{(x, \xi) \in E_u^* \mid |\xi| \in [c_1, c_2]\} = \emptyset$. We then conclude that $\mathrm{WF}(u) \cap E_u^* = \emptyset$, which also shows that $u \in C^\infty$ and (5.18) holds.

Then we define the following map:

$$\begin{aligned} \mathcal{I} : \ker_{\mathrm{ran}\, \Pi_0(\lambda)} d_{X(\lambda)} / \mathrm{ran}_{\mathrm{ran}\, \Pi_0(\lambda)} d_{X(\lambda)} &\to \ker_{C^\infty \Lambda} d_{X(\lambda)} / \mathrm{ran}_{C^\infty \Lambda} d_{X(\lambda)}, \\ u + \mathrm{ran}_{\mathrm{ran}\, \Pi_0(\lambda)} d_{X(\lambda)} &\mapsto u + \mathrm{ran}_{C^\infty \Lambda} d_{X(\lambda)}, \end{aligned} \tag{5.20}$$

which is well-defined since $\mathrm{ran}\, \Pi_0(\lambda) \subset C^\infty \Lambda$. We claim that this map is injective: Let $u = d_{X(\lambda)} v \in \mathrm{ran}\, \Pi_0(\lambda)$ with $v \in C^\infty \Lambda^j$. We need to show that $u = d_{X(\lambda)} w$ for some $w \in \mathrm{ran}\, \Pi_0(\lambda)$. But it suffices to use $[d_{X(\lambda)}, \Pi_0(\lambda)] = 0$ to see that $u = \Pi_0(\lambda)u = d_{X(\lambda)} \Pi_0(\lambda) v$. This proves the claim and concludes the proof of the lemma by using also the isomorphism (4.14). ∎

Lemma 5.7. *Assume that there is a smooth invariant measure μ for the action, i.e. $\mathcal{L}_{X_A}\mu = 0$ for each $A \in \mathfrak{a}$, and* $\operatorname{supp}\mu = \mathcal{M}$. *Then the periodic tori are dense in $\mathcal{M}$.*

Proof. Since $\mathcal{M}$ is compact, the measure is finite, so we can apply Poincaré's recurrence theorem: almost every point x of $\mathcal{M}$ is recurrent, i.e. its orbit comes back infinitely close to x infinitely many times (and this for each direction of the action). Katok–Spatzier [39, Theorem 2.4] proved a closing lemma for Anosov actions: there are $C, \delta > 0$ such that whenever there are $x \in \mathcal{M}$ and $t \in \mathcal{W}$ with $d(\tau(t)x, x) < \delta$ and $\|t\| > C$, then there is a periodic torus for the action at distance at most $\frac{1}{\delta} d(\tau(t)x, x)$ from x. ∎

Proposition 5.8. *Assume that there is a smooth invariant measure μ for the action, with* $\operatorname{supp}\mu = \mathcal{M}$. *Then*

$$\dim\big(\ker_{C^{-\infty}_{E_u^*}\Lambda^j}\, d_X / \operatorname{ran}_{C^{-\infty}_{E_u^*}\Lambda^j}\, d_X\big) = \dim \Lambda^j \mathfrak{a}^* = \binom{\kappa}{j}$$

and the cohomology space is generated by the constant forms $e'_{i_1} \wedge \cdots \wedge e'_{i_\kappa}$ if $(e'_j)_j$ is a basis of $\mathfrak{a}^*$.

Proof. In the proof of Proposition 5.6 with $\lambda = 0$, we have defined an operator $F(0)$ that is Fredholm on $\mathcal{H}_{NG}$ and $\Pi_0(0)$ is its spectral projector at $z = 0$, with $\operatorname{ran}\Pi_0(0) \subset C^\infty(\mathcal{M})$. Recall also that $F(0)$ is scalar and can thus be considered as an operator on functions. Let us show that $\operatorname{ran}\Pi_0(0) = \mathbb{R}$ consists only of constants under our assumptions. Pick $u \in C^\infty(\mathcal{M})$ such that $F(0)u = 0$. Let $x \in \mathcal{M}$ belong to a closed orbit in the Weyl chamber, i.e. $\varphi_{t_0}^{X_A}(x) = x$ for some $A \in \mathcal{W}$ and $t_0 > 0$. Then it is a classical result that the orbit $T_x := \{\varphi_s^{X_{\tilde{A}}}(x) \mid s \in \mathbb{R}, \tilde{A} \in \mathfrak{a}\}$ is a closed κ-dimensional torus (a proof compatible with the present notation can be found e.g. in [29, Lemma 3.1]). It is isomorphic to $\mathbb{R}^\kappa/\mathbb{Z}^\kappa$ by the map

$$\psi_x : \mathbb{R}^\kappa \ni t \mapsto \tau\Big(\sum_{j=1}^{\kappa} t_j A'_j\Big)(x)$$

for some basis $A'_i \in \mathfrak{a}$. Note that $\psi_x^*(e^{\sum_\ell s_\ell X_{A'_\ell}} u)(t) = \psi_x^* u(t+s)$. Let us restrict the identity $F(0)u = 0$ or equivalently $R(0)u = u$ to T_x. We can decompose $v := \psi_x^* u$ into a Fourier series:

$$v(t) = \sum_{k \in \mathbb{Z}^\kappa} e^{2i\pi k.t} v_k, \quad t \in \mathbb{R}^\kappa.$$

Recall that the basis for which $R(0)$ was constructed in Proposition 5.6 was denoted by $A_1, \dots, A_\kappa \in \mathfrak{a}$. We can express this basis in terms of the basis A'_j of the periodic torus via some base change matrix: $A_j = \sum_i M_{ij} A'_i$ (using $\sum_{\ell=1}^{\kappa} s_\ell A_\ell = \sum_{\ell,i} s_\ell M_{i\ell} A'_i$). The identity $Ru(x) = u(x)$ implies

$$\begin{aligned}\sum_{k \in \mathbb{Z}^\kappa} e^{2i\pi k.t} v_k (\psi_x^* R(0)u)(t) &= \int_{(\mathbb{R}+)^\kappa} (\psi_x^* u)(t - Ms)\tilde{\chi}(s)\, ds \\ &= \sum_{k \in \mathbb{Z}^\kappa} v_k e^{2i\pi k.t} \int_{(\mathbb{R}+)^\kappa} e^{-2i\pi k.Ms} \tilde{\chi}(s)\, ds\end{aligned}$$

with $M = (M_{ij})_{ij}$ real-valued and $\widetilde{\chi}$ defined in the proof of Proposition 5.6. This shows that for each $k \in \mathbb{Z}^\kappa$,

$$v_k = 0 \text{ or } \int_{(\mathbb{R}^+)^\kappa} (e^{-2i\pi k.Ms} - 1)\widetilde{\chi}(s)\, ds = 0.$$

Since $\widetilde{\chi} \geq 0$ and $\widetilde{\chi}(s) > 0$ in some open set, and M is invertible, we see that either $v_k = 0$ or $k = 0$, i.e. $v(t) = v(0)$ is constant. Therefore u is constant on each periodic torus. Since u is smooth and the periodic tori are dense, this implies that $d_X u = 0$ and $u(\varphi_t^{X_A}(x)) = u(x)$ for each $x \in \mathcal{M}$, $t \in \mathbb{R}$ and $A \in \mathfrak{a}$. Taking $A \in \mathcal{W}$, there is $\nu > 0$ such that for each $t > 0$ large enough, $|d\varphi_t^{X_A} v| \leq e^{-\nu t}|v|$ for each $v \in E_s$. Thus

$$|du_x(v)| = |du_{\varphi_t^{X_A}(x)} d\varphi_t^{X_A}(x)v| \leq \|du\| e^{-\nu t}|v|.$$

Letting $t \to \infty$, we conclude that $du|_{E_s} = 0$. The same argument with $t < 0$ shows that $du|_{E_u} = 0$ and therefore $du = 0$. Since $F(0)1 = 0$, this shows that, when viewed as an operator on $\Lambda\mathfrak{a}^*$, ran $\Pi_0(0)$ is exactly the space of constant forms. We can then use the isomorphism (4.14) to conclude the proof since it is seen directly that the constant forms $e'_{i_1} \wedge \cdots \wedge e'_{i_j}$ form a basis of $\ker d_X / \operatorname{ran} d_X$ on ran $\Pi_0(0)$ (as $d_X|_{\operatorname{ran}\Pi_0(0)} = 0$). ■

Note that in [39] Katok–Spatzier study the first cohomology group of dynamical systems and show that any smooth cocycle is smoothly conjugate to a constant function [39, Theorem 2.9 (a)]. In our language of Taylor complexes, this result implies that for standard Anosov actions $\dim(\ker_{C^\infty\Lambda^1} d_X / \operatorname{ran}_{C^\infty\Lambda^1} d_X) = \kappa$ and is spanned by the constant forms. Combining this fact with Proposition 5.8, we obtain the following result.

Corollary 5.9. *If the Anosov $\mathbb{R}^\kappa$-action is standard in the sense of* [39]*, then the map* (5.17) *is an isomorphism for $j = 1$.*

5.3. Ruelle–Taylor resonances and mixing properties

In this section we do not assume anymore that a volume measure is preserved, and want to establish the following relation of Ruelle–Taylor resonances and mixing properties.

Proposition 5.10. *Let X be an Anosov action on $\mathcal{M}$. Then the following are equivalent:*

(1) *There is a direction $A_0 \in \mathfrak{a}$ such that $\varphi_t^{X_{A_0}}$ is weakly mixing with respect to the full physical measure μ_1.*
(2) *0 is the only Ruelle–Taylor resonance on $i\mathfrak{a}^*$ and there is a unique normalized physical measure μ_1.*
(3) *For each $A \in \mathcal{W}$, $\varphi_t^{X_A}$ is strongly mixing with respect to the full physical measure μ_1.*

Proof. Obviously (3)⇒(1). So let us prove (1)⇒(2): Assume that there is either a non-zero Ruelle–Taylor resonance $i\lambda \in i\mathfrak{a}^*$ or a non-unique normalized physical measure. Then by Proposition 5.4 (5) there is a non-constant bounded density $f \in L^\infty(\mathcal{M}, \mu_1)$ with $X_A f = i\lambda(A) f$ for all $A \in \mathfrak{a}$ (setting $\lambda = 0$ if the density comes from the non-uniqueness of the physical measure). As f is non-constant there exists $g \in L^\infty(M, \mu_1)$ with $\int g\, d\mu_1 = 0$ but $\int gf\, d\mu_1 \neq 0$. With these two functions, we have the correlation function

$$C_{f,g}(t; A_0) := \int_{\mathcal{M}} g(\varphi_{-t}^{X_{A_0}})^* f\, d\mu_1 - \int_{\mathcal{M}} g\, d\mu_1 \cdot \int_{\mathcal{M}} f\, d\mu_1 = e^{-i\lambda(A_0)t} \int_{\mathcal{M}} gf\, d\mu_1,$$

so $\varphi_t^{X_{A_0}}$ is not weakly mixing.

We will now prove (2)⇒(3) using the regularity of a joint spectral measure: Let us first introduce these measures. We consider the space $L^2(M, \mu_1)$. Since the measure μ_1 is flow-invariant, the flow acts as unitary operators on $L^2(M, \mu_1)$. In particular, for each $A \in \mathfrak{a}$, X_A is skew-adjoint when acting on $L^2(M, \mu_1)$ with domain

$$\begin{aligned}\mathcal{D}(X_A) &= \left\{ u \in L^2(M, \mu_1) \;\middle|\; \lim_t \frac{1}{t}(e^{tX_A}u - u) \text{ exists} \right\} \\ &= \{u \in L^2(M, \mu_1) \mid X_A u \in L^2(M, \mu_1)\}.\end{aligned}$$

Additionally, since the flows commute, the X_A are *strongly commuting*, so that we can apply the joint spectral theorem [51, Theorem 5.21]. There exists a Borel, $L^2(M, \mu_1)$-projector-valued measure ν on $\mathfrak{a}^*$ such that for $u \in L^2(M, \mu_1)$,

$$u = \int_{\mathfrak{a}^*} d\nu(\vartheta)u, \quad X_A u = \int_{\mathfrak{a}^*} i\vartheta(A)\, d\nu(\vartheta)u \quad \text{for all } A \in \mathfrak{a}.$$

We will prove the following regularity result for these measures.

Lemma 5.11. *Let X be an Anosov action. Assume that there is no non-zero purely imaginary Ruelle–Taylor resonance and there is a unique normalized physical measure. For any $f, g \in C^\infty(\mathcal{M})$ with $\int_{\mathcal{M}} f\, d\mu_1 = \int_{\mathcal{M}} g\, d\mu_1 = 0$, consider $\nu_{f,g}(\theta) := \langle \nu(\theta) f, g\rangle_{L^2(\mathcal{M},\mu_1)}$, which is a finite complex-valued measure on $\mathfrak{a}^*$. Then the analytic wavefront set* [6] [7] $\mathrm{WF}_a(\nu_{f,g}) \subset \mathfrak{a}^* \times \mathfrak{a}$ *fulfills*

$$\mathrm{WF}_a(\nu_{f,g}) \cap (\mathfrak{a}^* \times \mathcal{W}) = \emptyset.$$

Before proving this lemma let us show that it implies (3). Take $A_0 \in \mathcal{W}$, and f, g as in the lemma. Then the spectral theorem yields

$$C_{f,g}(t; A_0) = \int_{\mathcal{M}} g(\varphi_{-t}^{X_{A_0}})^* f\, d\mu_1 = \int_{\mathfrak{a}^*} e^{-i\vartheta(A_0)t}\, d\nu_{f,\overline{g}}(\vartheta).$$

Given any $\varepsilon > 0$, using the fact that $\nu_{f,\overline{g}}$ is finite, there is a cutoff function $\chi_K \in C_c^\infty(\mathfrak{a}^*; [0, 1])$ equal to 1 on a sufficiently large compact set $K \subset \mathfrak{a}^*$ such that

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[6] See Folland [24, Section 3.3] for the definition and basic properties of the analytic wavefront set. In our proof, we need to use a non-quadratic phase, and only the quadratic case is treated by Folland; however, this is just a slight technical hurdle, as mentioned by Folland at the start of p. 160. For completeness, however, we will refer to [52] (in French).

[7] The usual C^∞ wavefront set is contained in the analytic wavefront set, i.e. $\mathrm{WF} \subset \mathrm{WF}_a$; see [24, Theorem 3.22].

$|\int_{\mathfrak{a}^*} e^{-i\vartheta(A_0)t}(1-\chi_K)\,d\nu_{f,\overline{g}}(\vartheta)| \le \varepsilon/2$ uniformly in t. Furthermore, by the fact that the wavefront set is empty in the direction of the Weyl chamber $\mathcal{W}$ we deduce that there is T such that $|\int_{\mathfrak{a}^*} e^{-i\vartheta(A_0)t}\chi_K\,d\nu_{f,\overline{g}}(\vartheta)| \le \varepsilon/2$ for any $t > T$, thus $\lim_{t\to\infty} C_{f,g}(t, A_0) = 0$. The passage to arbitrary $L^2(\mathcal{M}, \mu_1)$ functions follows by the density of the smooth functions. ∎

Proof of Lemma 5.11. Let us pick any $A_0 \in \mathcal{W}$ and a basis $A_1, \ldots, A_\kappa \in \mathcal{W}$ such that these elements span an open cone around A_0. With this basis we identify the joint spectral measure with a measure on $\mathbb{R}^\kappa$. Recall the definition of $R_\sigma(i\lambda)$ from the proof of Proposition 5.4 which was based on the choice of an even, positive $\psi \in C^\infty((-1/2, 1/2))$ with $\int \psi = 1$ and some $\sigma \in \mathbb{R}^\kappa_+$. Using the spectral theorem we calculate, for any $f, g \in L^2(\mathcal{M}, \mu_1)$,

$$\langle R_\sigma(i\lambda)^k f, g\rangle_{L^2(\mathcal{M},\mu_1)} = \int_{\mathbb{R}^\kappa} \hat{\Psi}(\vartheta+\lambda)^k e^{-ik\sigma(\vartheta+\lambda)}\,d\nu_{f,g}(\vartheta), \tag{5.21}$$

where $\Psi(t) := \prod_{j=1}^\kappa \psi(t_j)$. Now let us define the closed subspaces

$$\mathcal{H}_{NG,0} := \left\{u \in \mathcal{H}_{NG} \,\middle|\, \int u\,d\mu_1 = 0\right\} \subset \mathcal{H}_{NG}.$$

Note that these are well-defined for sufficiently large N because $\mu_1 \in \mathcal{H}_{-NG}$. Furthermore, from the invariance of μ_1 under the Anosov actions the spaces $\mathcal{H}_{NG,0}$ are preserved by $R_\sigma(i\lambda)$. Now the assumption that there is no imaginary Ruelle–Taylor resonance except zero and that there is a unique normalized physical measure imply (together with the findings of Section 5.1) that $R_\sigma(i\lambda)$ has spectral radius < 1 on $\mathcal{H}_{NG,0}$ for any $\lambda \in \mathbb{R}^\kappa$ and $\sigma \in \mathbb{R}^\kappa_+$ sufficiently large. Thus there are $C_{\sigma,\lambda}, \varepsilon_{\sigma,\lambda} > 0$, locally uniform in σ, λ, such that $\|R_\sigma(i\lambda)^k\|_{\mathcal{H}_{NG,0}} \le C_{\sigma,\lambda} e^{-\varepsilon_{\sigma,\lambda}k}$. Now let f, g be as in the assumption of our lemma. Then we can estimate

$$\langle R_\sigma(i\lambda)^k f, g\rangle_{L^2(\mathcal{M},\mu_1)} \le \|R_\sigma(i\lambda)^k f\|_{\mathcal{H}_{NG,0}} \|g\mu_1\|_{\mathcal{H}_{-NG}} \le C_{f,g,\sigma,\lambda} e^{-\varepsilon_{\sigma,\lambda}k}.$$

Let us come back to the expression (5.21) involving the spectral measures. By the properties of ψ we deduce that near zero, $\hat{\Psi}(\xi) = \exp(-S(\xi))$ with some analytic function $S(\xi) = a|\xi|^2 + \mathcal{O}(|\xi|^4)$. Furthermore, for any $\delta > 0$, there is $\varepsilon_2 > 0$ such that $\hat{\Psi}(\xi) < e^{-\varepsilon_2}$ for $|\xi| > \delta$. Choosing a cutoff function $\chi \in C_c^\infty((-3\delta, 3\delta)^\kappa)$ with $\chi(\xi) = 1$ for $|\xi| < 2\delta$, by the boundedness of $\nu_{f,g}$ we get, for an arbitrary fixed $\lambda_0 \in \mathbb{R}^\kappa$,

$$\left|\langle R_\sigma(i\lambda)^k f, g\rangle_{L^2(\mathcal{M},\mu_1)} - \int_{\mathbb{R}^\kappa} \hat{\Psi}(\vartheta+\lambda)^k e^{-ik\sigma(\vartheta+\lambda)}\chi(\vartheta+\lambda_0)\,d\nu_{f,g}(\vartheta)\right| \le Ce^{-\varepsilon_2 k}$$

uniformly for $\sigma \in \mathbb{R}^\kappa_+$, $|\lambda - \lambda_0| < \delta$. Putting everything together we get

$$\left|\int_{\mathbb{R}^\kappa} e^{-kS(\vartheta+\lambda)-ik\sigma(\vartheta+\lambda)}\chi(\vartheta+\lambda_0)\,d\nu_{f,g}(\vartheta)\right| \le \tilde{C}e^{-\tilde{\varepsilon}k}$$

with $\tilde{C}, \tilde{\varepsilon} > 0$ locally uniform in $|\lambda - \lambda_0| < \delta$ and $\sigma \in \mathbb{R}^\kappa_+$. In the expression on the left hand side, we recognize a semiclassical Fourier–Bros–Iagolnitzer (FBI) transform of the distribution $d\nu_{f,g}$ at parameters (λ, σ), with $h = 1/k$. That it decays exponentially as $h \to 0$, uniformly in λ close to λ_0 and uniformly in $\sigma \in \mathbb{R}_+^\kappa$, is the definition of

$$\mathrm{WF}_a(d\nu_{f,g}) \cap \{\lambda_0\} \times \mathbb{R}_+^\kappa = \emptyset.$$

Here WF_a is the *analytic* wavefront set of [52, Définition 6.1, Proposition 6.2].

Recall furthermore that we identified $\mathfrak{a}^* \cong \mathbb{R}^\kappa$ by the above choice of the basis A_j, thus our result implies that there is no analytic wavefront set in $\mathfrak{a}^* \times \{\sum c_j A_j \mid c_j \in \mathbb{R}_+\}$, but as the A_j span an arbitrary subcone of $\mathcal{W}$ we also get the absence of analytic wavefront set in $\mathfrak{a}^* \times \mathcal{W}$ and we have completed the proof of Lemma 5.11. ∎

Appendix A. Tools from microlocal analysis

We recall here some essentials of microlocal analysis. In this paper, we are working with pseudodifferential operators acting on $C^\infty(\mathcal{M}; E) \otimes \Lambda\mathfrak{a}_\mathbb{C}^* \cong C^\infty(\mathcal{M}; E \otimes \Lambda\mathfrak{a}_\mathbb{C}^*)$. Note that by fixing an arbitrary scalar product on $\mathfrak{a}^*$ the bundle $E \otimes \Lambda := E \otimes \Lambda\mathfrak{a}_\mathbb{C}^* \to \mathcal{M}$ is again a Riemannian bundle. We will therefore introduce notations for pseudodifferential operators on general Riemannian bundles $E \to \mathcal{M}$ over a compact Riemannian manifold $\mathcal{M}$. Only when we want to exploit some specific structures of $E \otimes \Lambda$, will we refer to this particular bundle.

For more details we refer to standard references such as [27]. For the details concerning anisotropic calculus we refer to [20].

Definition A.1. Let $k \in \mathbb{R}$ and $1/2 < \rho \leq 1$. Then the *standard symbol space* $S_\rho^k(\mathcal{M}; E)$ is the space of functions $a \in C^\infty(T^*\mathcal{M}; \mathrm{End}(E))$ for which in any local chart $U \subset \mathbb{R}^n$ of $\mathcal{M}$ and any local trivialization of the bundle, for any compact set $K \subset U$ and any $\alpha, \beta \in \mathbb{N}^n$,

$$\sup_{(x,\xi)\in T^*U,\, x\in K} \|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\| \langle\xi\rangle^{-(k-\rho|\beta|+(1-\rho)|\alpha|)} < \infty.$$

Given a zeroth order symbol $m(x,\xi) \in S_1^0(\mathcal{M})$, the *anisotropic symbol space* $S_\rho^{m(x,\xi)}(\mathcal{M}; E)$ is the space of functions $a \in C^\infty(T^*\mathcal{M}; \mathrm{End}(E))$ for which in any local chart $U \subset \mathbb{R}^n$, for any compact set $K \subset U$ and any $\alpha, \beta \in \mathbb{N}^n$,

$$\sup_{(x,\xi)\in T^*U,\, x\in K} \|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\| \langle\xi\rangle^{-(m(x,\xi)-\rho|\beta|+(1-\rho)|\alpha|)} < \infty.$$

We furthermore set[8]

$$S^{-\infty}(\mathcal{M}; E) := \bigcap_{k>0} S_\rho^{-k}(\mathcal{M}; E), \qquad S^{\infty}(\mathcal{M}; E) := \bigcup_{k>0} S_\rho^{k}(\mathcal{M}; E),$$
$$S_\rho^{m+}(\mathcal{M}; E) := \bigcap_{\varepsilon>0} S_\rho^{m+\varepsilon}(\mathcal{M}; E), \quad S_{\rho-}^{m}(\mathcal{M}; E) := \bigcup_{\varepsilon>0} S_{\rho-\varepsilon}^{m}(\mathcal{M}; E).$$

[8] Note that $\bigcap_{k>0} S_\rho^{-k}(\mathcal{M}; E)$ is independent of $1/2 < \rho \leq 1$ and we therefore drop the ρ index in the notation of $S^{-\infty}(\mathcal{M}; E)$.

Note that by setting $m(x,\xi) = k \in \mathbb{R}$ the standard symbols are a special case of anisotropic symbols. We will therefore mostly introduce the notation in the anisotropic setting as it contains the standard symbols as a special case. Furthermore, note that $x \mapsto \mathrm{Id}_{E_x}$ is a global smooth section of $\mathrm{End}(E) \to \mathcal{M}$ and multiplication with this section yields a canonical embedding $S^\infty_\rho(\mathcal{M}) \hookrightarrow S^\infty_\rho(\mathcal{M}; E)$. We will refer to symbols in the image of this embedding as *scalar symbols*.

After fixing a finite atlas and a suitable partition of unity on $\mathcal{M}$ one can define a quantization (see e.g. [18, E.1.7]) that associates to any $a \in S^\infty_\rho(\mathcal{M}; E)$ a continuous operator $\mathrm{Op}(a) : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E)$ which extends to a continuous operator $\mathrm{Op}(a) : C^{-\infty}(\mathcal{M}; E) \to C^{-\infty}(\mathcal{M}; E)$. We denote by $\Psi^{-\infty}(\mathcal{M}; E)$ the space of smoothing operators $A : C^{-\infty}(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E)$. The quantization has the property that $\mathrm{Op}(S^{-\infty}(\mathcal{M}; E)) \subset \Psi^{-\infty}(\mathcal{M}; E)$. We say that $A \in \Psi^m_\rho(\mathcal{M}; E)$ iff there is $a \in S^m_\rho(\mathcal{M}; E)$ such that $A - \mathrm{Op}(a) \in \Psi^{-\infty}(\mathcal{M}; E)$. When $\rho = 1$, we will drop the ρ index and write $S^m(\mathcal{M}; E)$ and $\Psi^m(\mathcal{M}; E)$ instead of $S^m_1(\mathcal{M}; E)$ and $\Psi^m_1(\mathcal{M}; E)$.

With any $A \in \Psi^m_\rho(\mathcal{M}; E)$ one can associate its principal symbol

$$\sigma^m_p(A) \in S^m_\rho(\mathcal{M}; E)/S^{m-2\rho+1}_\rho(\mathcal{M}; E).$$

The principal symbol is an inverse to Op in the sense that

$$\sigma^m_p \circ \mathrm{Op} : S^m_\rho \to S^m_\rho / S^{m-2\rho+1}_\rho \quad \text{and} \quad \mathrm{Op} \circ \sigma^m_p : \Psi^m_\rho \to \Psi^m_\rho / \Psi^{m-2\rho+1}_\rho$$

are simply the projections on the respective quotients.

Example A.2. Any k-th order differential operator P with smooth coefficients on the bundle $E \to \mathcal{M}$ is in $\Psi^k_1(\mathcal{M}; E)$ and a representative of its principal symbol $\sigma^k_p(P)$ can be calculated by

$$[\sigma^k_p(P)(x,\xi)]u(x) = \lim_{t\to\infty} t^{-k}[e^{-it\phi} P(e^{it\phi}u)](x),$$

where $u \in C^\infty(\mathcal{M}; E)$ and $\phi \in C^\infty(\mathcal{M})$ is a phase function with $d\phi(x) = \xi$ (see e.g. [36, (6.4.6′)]). As a direct consequence we get:

(1) For any vector field $X \in C^\infty(\mathcal{M}; T^*\mathcal{M}) \subset \Psi^1_1(\mathcal{M})$ we have $\sigma^1_p(X)(x,\xi) = i\xi(X(x))$.

(2) If $\mathbf{X} : \mathfrak{a} \to \mathrm{Diff}^1(\mathcal{M}; E) \subset \Psi^1_1(\mathcal{M}; E)$ is an admissible lift of an Anosov action, then for all $A \in \mathfrak{a}$ the principal symbol $\sigma^1_p(\mathbf{X}_A)(x,\xi) = i\xi(X_A(x))\,\mathrm{Id}_{E_x}$ is scalar.

(3) To express the principal symbol of the exterior derivative $d_{\mathbf{X}} \in \Psi^1_1(\mathcal{M}; E \otimes \Lambda\mathfrak{a}^*_{\mathbb{C}})$ of X, consider the smooth map $T^*\mathcal{M} \ni (x,\xi) \mapsto \xi(X_\bullet(x)) \in \Lambda^1\mathfrak{a}^*$. With its help we calculate, for $v \in E_x$ and $\omega \in \Lambda\mathfrak{a}^*$,

$$\sigma^1_p(d_{\mathbf{X}})(x,\xi)(v \otimes \omega) = i v \otimes \big(\xi(X_\bullet(x)) \wedge \omega\big).$$

(Thus $\sigma^1_p(d_{\mathbf{X}})$ is scalar on the E-component but not on the $\Lambda\mathfrak{a}^*$-component as it increases the order of differential forms.)

Proposition A.3. *Let $A \in \Psi^{m_1(x,\xi)}_\rho(\mathcal{M}; E)$ and $B \in \Psi^{m_2(x,\xi)}_\rho(\mathcal{M}; E)$. Then*

$$AB \in \Psi^{m_1+m_2}_\rho(\mathcal{M}; E) \text{ and } \sigma^{m_1+m_2}_p(AB) = \sigma^{m_1}_p(A)\sigma^{m_2}_p(B) \bmod S^{m_1+m_2-2\rho+1}_\rho(\mathcal{M}; E).$$

Definition A.4. Given $a \in S_\rho^{m(x,\xi)}(\mathcal{M}; E)$, we define its *elliptic set* to be the open cone $\mathrm{ell}^{m(x,\xi)}(a) \subset T^*\mathcal{M} \setminus \{0\}$ which consists of all $(x_0, \xi_0) \in T^*\mathcal{M} \setminus \{0\}$ for which there is a $C > 0$ and a function $\chi \in C^\infty(T^*\mathcal{M})$, positively homogeneous of degree zero for $|\xi| \geq C$, and with $\chi(x_0, C\xi_0/|\xi_0|) > 0$, such that $a(x,\xi) \in \mathrm{End}(E_x)$ is invertible for all $(x,\xi) \in \mathrm{supp}\,\chi$ and $\chi a^{-1} \in S_\rho^{-m(x,\xi)}(\mathcal{M}; E)$. We call $a \in S_\rho^{m(x,\xi)}(\mathcal{M}, E)$ *elliptic* if $\mathrm{ell}^{m(x,\xi)}(a) = T^*\mathcal{M} \setminus \{0\}$.

As a direct consequence of the chain rule for derivatives and the symbol estimates we get the following.

Lemma A.5. *If $a \in S_\rho^{m(x,\xi)}(\mathcal{M}; E)$ is a scalar symbol, then $(x_0, \xi_0) \in \mathrm{ell}^{m(x,\xi)}(a)$ if there exists an open cone $\Gamma \subset T^*\mathcal{M}$ containing (x_0, ξ_0) and $C > 0$ such that*

$$|a(x,\xi)| \geq \frac{1}{C}\langle\xi\rangle^{m(x,\xi)} \quad \textit{for all } (x,\xi) \in \Gamma \cap \{|\xi| > C\}.$$

One checks that for $a \in S_\rho^{m(x,\xi)}(\mathcal{M}; E)$ and $r \in S_\rho^{m(x,\xi)-\varepsilon}(\mathcal{M}; E)$ one has $\mathrm{ell}^{m(x,\xi)}(a) = \mathrm{ell}^{m(x,\xi)}(a + r)$, which allows us to define the elliptic set of an operator $A \in \Psi^{m(x,\xi)}(\mathcal{M}; E)$ via its principal symbol: $\mathrm{ell}^{m(x,\xi)}(A) := \mathrm{ell}^{m(x,\xi)}(\sigma_p^{m(x,\xi)}(A))$.

Definition A.6. Given $A = \mathrm{Op}(a) \bmod \Psi^{-\infty}(\mathcal{M}; E)$, we define its *wavefront set* to be the closed cone $\mathrm{WF}(A) \subset T^*\mathcal{M} \setminus \{0\}$ which is the complement of all $(x_0, \xi_0) \in T^*\mathcal{M} \setminus \{0\}$ for which there is an open cone $\Gamma \subset T^*\mathcal{M}$ around (x_0, ξ_0) such that for all $N > 0$ and $\alpha, \beta \in \mathbb{N}^n$ there is $C_{N,\alpha,\beta}$ such that

$$\|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\| \leq C_{N,\alpha,\beta}\langle\xi\rangle^{-N} \quad \text{for all } (x,\xi) \in \Gamma.$$

The wavefront set has the following property for the product of two pseudodifferential operators $A, B \in \Psi_\rho^\infty(\mathcal{M}; E)$:

$$\mathrm{WF}(AB) \subset \mathrm{WF}(A) \cap \mathrm{WF}(B).$$

We crucially use the following constructions of microlocal parametrices.

Lemma A.7. *If $A \in \Psi_\rho^{m_1(x,\xi)}(\mathcal{M}; E)$, $B \in \Psi_\rho^{m_2(x,\xi)}(\mathcal{M}; E)$ and $\mathrm{WF}(B) \subset \mathrm{ell}^{m_1(x,\xi)}(A)$, then there is $Q \in \Psi_\rho^{m_2(x,\xi)-m_1(x,\xi)}$ with $\mathrm{WF}(Q) \subset \mathrm{WF}(B)$ such that*

$$AQ - B \in \Psi^{-\infty}(\mathcal{M}; E).$$

If furthermore A and B are holomorphic families of operators, then Q can be chosen to be holomorphic as well.

As a consequence of Lemma A.7, if $A \in \Psi_\rho^{m_1}(\mathcal{M}; E)$ and $B \in \Psi_\rho^{m_2}(\mathcal{M}; E)$, then

$$\mathrm{ell}^{m_1}(A) \cap \mathrm{WF}(B) \subset \mathrm{WF}(AB). \tag{A.1}$$

We also have the following particular case of Egorov's lemma.

Lemma A.8. *Let $F \in \mathrm{Diffeo}(\mathcal{M})$ be a smooth diffeomorphism and let $\widetilde{F} \in \mathrm{Diffeo}(E)$ be a lift of F, i.e. $\widetilde{F}$ acts linearly in the fibers and $\pi \circ \widetilde{F} = F \circ \pi$ for the fiber projection $\pi : E \to \mathcal{M}$. Define the* transfer operator

$$L_F : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E), \quad (L_F u)(x) := \widetilde{F}^{-1}(F(x), u(F(x))).$$

Then for each $A \in \Psi_\rho^m(\mathcal{M}; E)$, we have $L_F A L_F^{-1} \in \Psi_\rho^{m\circ\Phi}(\mathcal{M}; E)$ with $\Phi(x, \xi) := (F(x), (dF^{-1})^T \xi)$ and

$$\sigma_p^{m\circ\Phi}(L_F A L_F^{-1})(x, \xi) = \widetilde{F}^{-1}(F(x), \cdot) \circ \sigma_p^m(A)(\Phi(x, \xi)) \circ \widetilde{F}(x, \cdot).$$

Proposition A.9 (L^2-boundedness). *Let $A \in \Psi_\rho^0(\mathcal{M}; E)$. Then A can be extended from an operator on $C^\infty(\mathcal{M}; E)$ to a bounded operator on $L^2(\mathcal{M}; E)$. Furthermore, for any*

$$C > \limsup_{|\xi|\to\infty} \|\sigma_p^0(A)(x, \xi)\|,$$

there exists a decomposition $A = K + R$, where $K \in \Psi^{-\infty}(\mathcal{M}; E)$ is a smoothing and hence L^2-compact operator and $\|R\|_{L^2\to L^2} \leq C$. If A_t is a smooth family in $\Psi_\rho^0(\mathcal{M}; E)$ for $t \in [t_1, t_2]$, the decomposition $A_t = R_t + K_t$ can be chosen so that $t \mapsto R_t$ and $t \mapsto K_t$ are continuous in t.

Proof. See [20, Lemma 14]. The continuity in t is straightforward from the proof. ∎

We conclude this appendix by mentioning that one can use a small semiclassical parameter $h > 0$ in the quantization, in which case we shall write Op_h, by using the expression in a local chart

$$\mathrm{Op}_h(a) f(x) = \frac{1}{(2\pi h)^n} \int e^{\frac{i(x-x')\xi}{h}} a(x, \xi) f(x')\, d\xi\, dx'$$

if a is supported in a chart. We do not use this semiclassical quantization except in Section 5.2 and we refer to [18, Appendix E] for the results on semiclassical pseudo-differential operators that we use. One of their advantages is that one can get the estimate $\|\mathrm{Op}_h(a)\|_{L^2\to L^2} \leq \sup_{x,\xi} |a(x, \xi)| + \mathcal{O}(h)$ for small $h > 0$ and if $a \in S^0(\mathcal{M}; E)$.

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