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Asymptotic stability of planar rarefaction wave to a 2D hyperbolic-elliptic coupling system of the radiating gas on half-space

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Abstract. This paper studies the asymptotic stability of solutions to an initial-boundary value problem for a hyperbolic-elliptic coupling system on the two-dimensional half-space, where the data on the boundary and at the far field are prescribed as u_- and u_+, respectively. We show that the solution to the problem converges to the corresponding planar rarefaction wave for $0 \le u_- < u_+$ as time tends to infinity. To the best of our knowledge, the stability results of planar rarefaction waves on half-space focus primarily on the single viscous conservation law because the rarefaction wave (one-dimensional diffusion wave) of the corresponding one-dimensional problem to scalar viscous conservation law is known. In other words, for a general high-dimensional system of equations, we cannot obtain the stability of planar rarefaction waves on half-space because we cannot construct the rarefaction wave of the corresponding one-dimensional problem. In this paper, we use the structure of the hyperbolic-elliptic coupling system to obtain the monotonic rarefaction wave of the corresponding one-dimensional hyperbolic-elliptic coupling system, and hence give the stability of the planar rarefaction wave on half-space. This can be viewed as the first result for the system of equations on the stability of planar rarefaction waves on half-space.

Keywords: hyperbolic-elliptic coupling system, planar rarefaction wave, L^2-energy method, initial-boundary value problem, asymptotic behavior.

Contents

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1. Introduction

In this paper, we consider the asymptotic stability of solutions to an initial-boundary value problem for a hyperbolic-elliptic coupling system on the two-dimensional half-space:

$$\begin{cases} u_t + f(u)_x + g(u)_y + \operatorname{div} q = 0, & (x,y) \in \mathbb{R}_+ \times \mathbb{R}, \quad t > 0, \\ -\nabla \operatorname{div} q + q + \nabla u = 0, & (x,y) \in \mathbb{R}_+ \times \mathbb{R}, \quad t > 0, \end{cases} \tag{1.1}$$

with initial data

$$u(x,y,0) = u_0(x,y) \tag{1.2}$$

and boundary condition

$$u(0,y,t) = u_-, \quad t > 0, \tag{1.3}$$

$$u(+\infty,y,t) = u_+, \quad q(+\infty,y,t) = 0, \quad t > 0, \tag{1.4}$$

where u and $q = (q_1, q_2)$ are dependent variables with values in $\mathbb{R}$ and $\mathbb{R}^2$, respectively. Both f and g are smooth functions, and $u_\pm$ are constants. We assume that f is strictly convex, i.e., for a certain positive constant α,

$$f''(u) \geq \alpha > 0, \quad u \in \mathbb{R}, \tag{1.5}$$

and that the characteristic speeds $f'(u_\pm)$ satisfy

$$f'(u_-) < f'(u_+). \tag{1.6}$$

Without loss of generality, we also assume

$$f(0) = f'(0) = 0. \tag{1.7}$$

The study of (1.1) is motivated by physical models or the so-called radiative gas model. The model is used to describe the dynamics of a gas in which radiation is present, and it consists of the compressible Euler equations coupling with an elliptic system representing the radiative flux, cf. Vincenti and Kruger [32]. System (1.1), first mentioned by Hamer in [4], simplifies the model for the motion of radiating gases on the two-dimensional half-space. For the deduction of system (1.1), we refer to [1,3,4,32]. In the *in-flow* case of $u_- > 0$, the boundary condition (1.3) is necessary for the single hyperbolic equation $(1.1)_1$. In addition, we also require

$$\operatorname{div} q(0,y,t) = 0 \tag{1.8}$$

for the solvability of the coupling elliptic equation $(1.1)_2$. On the contrary, in the *out-flow* case of $u_- < 0$, the boundary condition (1.3) is sufficient and necessary for the solvability of the hyperbolic-elliptic coupling system (1.1).

Conditions (1.5) and (1.6) give $u_- < u_+$. Referring to [21], we recall that the asymptotic behavior of the solution to the one-dimensional scalar viscous conservation law is classified into the following three cases in accordance with the signs of the characteristic speeds $f'(u_\pm)$:

(a) $f'(u_-) < f'(u_+) \le 0$,

(b) $0 \le f'(u_-) < f'(u_+)$,

(c) $f'(u_-) < 0 < f'(u_+)$.

In case (a), the solution of (1.1) converges to the corresponding stationary solution $(\overline{u}, \overline{q})$ which satisfies

$$\begin{cases} f(\overline{u})_x + \overline{q}_x = 0, & x \in \mathbb{R}_+, \\ -\overline{q}_{xx} + \overline{q} + \overline{u}_x = 0, & x \in \mathbb{R}_+, \\ \overline{u}(0,t) = u_-, & \lim_{x\to+\infty} \overline{u}(x) = u_+. \end{cases}$$

In case (b), the solution behaves as the rarefaction wave which satisfies

$$\begin{cases} r_t + f(r)_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \\ r(x,0) = r_0(x) := \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases} \end{cases}$$

Here $r(x,t) = r(\frac{x}{t})$ is given explicitly by

$$r(x,t) = \begin{cases} u_-, & x \le f'(u_-)t, \\ (f')^{-1}(\frac{x}{t}), & f'(u_-)t \le x \le f'(u_+)t, \\ u_+, & x \ge f'(u_+)t. \end{cases}$$

In case (c), the solution tends to the superposition of stationary solutions and rarefaction waves.

There are many works concerning the asymptotic stability of solutions for different physical systems. Most of them are in the case of one dimension. The pioneer work for stability of nonlinear waves for the Cauchy problem on scalar viscous conservation laws was done by Il'in and Oleinik in [9] in 1960. Then, the rate of convergence toward the rarefaction wave was first obtained by Harabetian in [5] for the viscous Burgers equation and the further work has been investigated by many authors in [7, 23, 24]. For the half-space problem on scalar viscous conservation laws, Liu and Nishihara in [22] considered the asymptotic stability of a viscous shock wave for the case of $u_- > u_+$. For the case where $u_- < u_+$, Liu, Matsumura and Nishihara in [21] first proved the asymptotic stability of rarefaction waves or stationary solutions as well as the superposition of these two kinds of waves. The convergence rate of the rarefaction wave on half-space was found by Nakamura in [25]. By a combination of the weighted L^p energy method and the L^1 estimate, Hashimoto, Ueda and Kawashima in [6] obtained the convergence rate of the superposition of stationary solutions and rarefaction waves. These problems were also considered for the hyperbolic-elliptic coupling system (1.1) on the one-dimensional space. Tanaka in [30] proved that the solution approaches the diffusion wave for the case where $u_- = u_+ = 0$. Kawashima and Nishibata in [14] investigated the existence and asymptotic stability of traveling waves and also obtained the convergence rate for $u_- > u_+$. The remaining case $u_- < u_+$ was studied by Kawashima and Tanaka in [17]. They showed the asymptotic stability of the rarefaction wave and obtained the convergence rate. Recently, the initial boundary value problem of the one-dimensional system (1.1) was also studied thoroughly. Ruan and Zhu in [29] considered the case $0 = u_- < u_+$ for a hyperbolic-elliptic system on the one-dimensional half-space and justified the convergence toward the corresponding rarefaction wave. Moreover, Ji, Zhang and Zhu in [11] further showed the asymptotic stability of the stationary solution, rarefaction wave, and the superposition of these two kinds of waves for the cases $u_- < u_+ \leq 0$, $0 \leq u_- < u_+$ and $u_- < 0 < u_+$, respectively. In addition, the asymptotic stability of rarefaction waves or stationary solutions for the compressible Navier–Stokes equations was fully studied in [13, 16, 18, 27].

In the case of the multi-dimensional space, Xin in [34] first investigated the asymptotic stability of planar rarefaction waves for viscous conservation laws in two dimensions, and then Ito in [10] found the decay rate. Nishikawa in [28] further improved their results without smallness conditions. Kawashima, Nishibata and Nishikawa in [15] first studied the asymptotic stability of planar stationary solutions for viscous conservation laws on the two-dimensional half-space and obtained the convergence rate, and Ueda, Nakamura and Kawashima in [31] improved the decay rate for the degenerate case by using the time weighted L^p energy method. Recently, there are also many papers discussing the asymptotic stability for the hyperbolic-elliptic coupling system in the multi-dimensional whole space. Gao, Ruan and Zhu investigated the asymptotic decay rate for the Cauchy problem of the planar rarefaction wave for a hyperbolic-elliptic system in $\mathbb{R}^n$, $n = 2, 3, 4, 5$, cf. [2, 3]. For the initial-boundary problem for system (1.1)–(1.4), case (a) corresponding to planar stationary solutions was considered by Zhang and Zhu in [35]. However, for case (b) where $0 \leq f'(u_-) < f'(u_+)$, the stability of planar rarefaction waves has been left open. For the initial-boundary value problem of other physically meaningful equations in the multi-dimensional case, there are interesting results about the stability of planar stationary solutions; we refer readers to [12, 26] for the compressible Navier–Stokes equations. However, the corresponding stability results of planar rarefaction waves are quite few. The reason is that it is quite difficult to show the monotonicity of profiles and decay rate of the corresponding one-dimensional equations for the general high-dimensional system of equations. On the other hand, the large-time behavior of solutions to the compressible Navier–Stokes equations is determined by the Riemann problem on the corresponding inviscid Euler system, which contains a planar rarefaction wave in the genuinely nonlinear characteristic fields. This will lead to the emergence of error terms composed of planar rarefaction waves in the perturbation equations, which are only related to x but independent of y such that the integration of these terms over $y \in \mathbb{R}$ is divergent. As far as we know, the stability results of planar rarefaction waves for the multi-dimensional Navier–Stokes equations only consider the case in an infinitely long flat nozzle domain, cf. [19, 20, 33]. Obviously, in this case (y belongs to a bounded domain), the planar rarefaction wave is integrable in y-direction. Therefore, it is meaningful to study the stability of the planar rarefaction wave for an initial-boundary value problem for system (1.1) on the two-dimensional half-space.

Our aim of this paper is to prove the large-time behavior of the solution of (1.1)–(1.8) for case (b), where $0 \le f'(u_-) < f'(u_+)$. Compared with the one-dimensional stability results in [11] and the Cauchy problem in [3], the additional difficulties here lie in the boundary estimates of higher order derivatives, the integration of the planar rarefaction wave over y-direction and the L^1-estimate of the perturbation $V := U - \widetilde{u}$. To overcome the first difficulty, we give the relation between the boundary values in Lemmas 3.6 and 3.31, and utilize the tangential derivative estimates in y-direction and in t-direction to control the boundary terms occurring in the normal direction. In addition, by utilizing the analysis of div-curl decomposition, we convert the boundary term in the form of $p(0, y, t)$ to the form of div $p(0, y, t)$ to complete the H^3-estimates of the perturbation $p := q - Q$. For the second difficulty, this is a hard problem faced by all multi-dimensional equations. To solve the second difficulty, we consider the one-dimensional initial-boundary value problem (2.10)–(2.12) to further approximate the rarefaction wave. Fortunately, the one-dimensional system (2.10)–(2.12) can be rewritten as the following scalar equation form with convolution

$$U_t + f(U)_x + U - KU = 0, \quad x \in \mathbb{R}_+, \quad t > 0, \tag{1.9}$$

where K is the inverse of the elliptic operator $-\partial_x^2 + 1$ in $\mathbb{R}_+$, which is defined in (3.58). It is this scalar equation form that allows us to generalize the monotonicity result in [34] to our problem (2.10)–(2.12). From the maximum principle, we prove that the solution U of system (2.10)–(2.12) satisfies $U_x(x, t) \ge 0$ for any $(x, t) \in \mathbb{R}_+ \times \mathbb{R}_+$ provided that the initial data $U_0(x)$ is a smooth non-decreasing function of x-variable (see details in Lemma 2.4). Moreover, we obtain the decay rate of U and its derivatives, which plays an important role in estimating the perturbation $v := u - U$. Thanks to the structure of (1.9), we can convert the perturbation equations (2.19) into the form of (3.60), which helps us deal with the third difficulty (see in Lemma 3.22).

This paper is organized as follows. In Section 2, we summarize some basic properties of the planar rarefaction wave, which are given in [7, 17]. Then we reformulate the initial-boundary problem (1.1)–(1.3) and present our main theorem. In Section 3, we show the asymptotic behavior for the case of $0 \le f'(u_-) < f'(u_+)$, which corresponds to the planar rarefaction wave. More precisely, we show that if the rarefaction wave strength is suitably weak ($|u_+ - u_-| \ll 1$) and the initial data u_0 in (1.2) is suitably close to the planar rarefaction wave, then the initial-boundary value problem (1.1)–(1.3) has a global-in-time solution which converges to the planar rarefaction wave $r(x, t)$ as time tends to infinity. The specific results can be found in Theorem 2.6.

Notations. Throughout this paper, we denote generic constants by C and c unless they need to be distinguished. We denote $\mathbb{R}_+ \times \mathbb{R}$ by $\mathbb{R}^2_+$. Let $\Omega = \mathbb{R}_+$ or $\mathbb{R}^2_+$. For any nonnegative constant p $(1 \le p \le \infty)$, $L^p(\Omega)$ denotes the usual Lebesgue space over Ω, equipped with the norm $\|\cdot\|_{L^p(\Omega)}$. For any $l \ge 0$, $H^l(\Omega)$ denotes the usual Sobolev space over Ω with the norm $\|\cdot\|_{H^l(\Omega)}$. We use the notation $\nabla^k f$ as in the meaning

$$\nabla^k f = (\partial_x^k f, \partial_x^{(k-1)}\partial_y f, \dots, \partial_x \partial_y^{(k-1)} f, \partial_y^k f),$$

where $f = f(x, y, t)$ and $\nabla^0 f = f$. The notation $\Delta := \partial_x^2 + \partial_y^2$ denotes the Laplacian.

2. Preliminaries and main theorem

2.1. Preliminaries

In order to prove the main Theorem 2.6, we first give some inequalities which will be used later. The following lemma is given in [8].

Lemma 2.1. *Assume that $N \geq 2$ is an integer, $l_1, l_2, \dots, l_N$ are nonnegative integers, $1 \leq p, q, r \leq \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Let $l = l_1 + l_2 + \cdots + l_N$. Then there exists a positive constant $C = C(N, p, q, r, l)$ such that the inequality*

$$\left\| \prod_{j=1}^{N} (\partial_x^{l_j} f_j) \right\|_{L^p(\mathbb{R}_+)} \leq C \|f\|_{L^\infty(\mathbb{R}_+)}^{N-2} \|f\|_{L^q(\mathbb{R}_+)} \|\partial_x^l f\|_{L^r(\mathbb{R}_+)} \tag{2.1}$$

holds for any $f(x) = (f_1(x), f_2(x), \dots, f_N(x))$.

As an application of inequality (2.1), for any $1 \leq p \leq \infty$, [3] points out that

$$\left\| \partial_x^k \left\{ \frac{f'''(w)}{f''(w)} w_x^2 \right\} \right\|_{L^p(\mathbb{R}_+)} \leq C \|\partial_x^{k+2} w\|_{L^p(\mathbb{R}_+)} \tag{2.2}$$

provided $\|w\|_{L^\infty(\mathbb{R}_+)}$ is bounded.

Throughout this paper, we denote the rarefaction wave strength δ by

$$\delta := |u_- - u_+|.$$

As the rarefaction wave $r(x,t)$ is only Lipschitz continuous, we need to find a smooth approximation rarefaction wave through the viscous Burgers equation as in [7,17]. Define $\widetilde{w}(x,t)$ as a solution of the Cauchy problem

$$\begin{cases} \widetilde{w}_t + \widetilde{w}\widetilde{w}_x = \widetilde{w}_{xx}, & x \in \mathbb{R}, \quad t > 0, \\ \widetilde{w}(x,0) = w_0^R(x), & x \in \mathbb{R}. \end{cases} \tag{2.3}$$

For the case of $f'(u_-) > 0$, we define the initial data as

$$w_0^R(x) := \begin{cases} f'(u_-), & x < 0, \\ f'(u_+), & x > 0. \end{cases} \tag{2.4}$$

For the case of $f'(u_-) = 0$, $\widetilde{w}(x,t)$ does not converge to the corresponding rarefaction wave fast enough around the boundary $x = 0$ under the initial condition (2.4). Thus, when $f'(u_-) = 0$, $w_0^R(x)$ is given as

$$w_0^R(x) := \begin{cases} -f'(u_+), & x < 0, \\ f'(u_+), & x > 0, \end{cases}$$

which yields $\widetilde{w}(0,t) = 0$. Using the Hopf–Cole transformation, we can obtain the explicit expression of $\widetilde{w}(x,t)$. From (1.5), we define a smooth approximation $w(x,t)$ of the rarefaction wave as

$$w(x,t) = (f')^{-1}(\widetilde{w}(x,t)). \tag{2.5}$$

Substituting (2.5) into (2.3), we obtain that $w(x,t)$ satisfies the equation

$$\begin{cases} w_t + f(w)_x = w_{xx} + \dfrac{f'''(w)}{f''(w)} w_x^2, & x \in \mathbb{R}, \quad t > 0, \\ w(x,0) = w_0(x) := (f')^{-1}(\widetilde{w}(x,0)), & x \in \mathbb{R}. \end{cases} \tag{2.6}$$

The monotonicity and decay rate of rarefaction wave have been thoroughly studied in [7, 17]. We directly give the properties of the smooth rarefaction wave $w_i(x,t)$, $i = 1, 2$, in Lemma 2.2, where $w_1(x,t)$ corresponds to the case $f'(u_-) = 0$ and $w_2(x,t)$ corresponds to the case $f'(u_-) > 0$.

Lemma 2.2. *For $1 \le p \le \infty$ and $t > 0$, the smooth rarefaction waves $w_i(x,t)$, $i = 1, 2$, satisfy the following properties:*

(i) $0 \le w_2(0,t) - u_- \le C\delta \mathrm{e}^{-c(1+t)}$ *for* $f'(u_-) > 0$ *and* $w_1(0,t) = 0$ *for* $f'(u_-) = 0$;

(ii) $|\partial_x^k \partial_t^l w_2(0,t)| \le C\delta \mathrm{e}^{-c(1+t)}$, $|\partial_x^k \partial_t^l w_1(0,t)| \le C(1+t)^{-\frac{1}{2}(k+l+1)}$, $k + l = 1, 2, 3, 4$, $k, l \in \mathbb{N}$;

(iii) $\|w_i(t) - r(t)\|_{L^p(\mathbb{R})} \le C(1+t)^{-\frac{1}{2}+\frac{1}{2p}}$;

(iv) $\|w_{ix}(t)\|_{L^p(\mathbb{R})} \le C\delta^{\frac{1}{p}}(1+t)^{-1+\frac{1}{p}}$, $\|w_{it}(t)\|_{L^p} \le C\delta^{\frac{1}{p}}(1+t)^{-1+\frac{1}{p}}$;

(v) $\|\partial_x^k \partial_t^l w_i(t)\|_{L^p(\mathbb{R})} \le C\delta(1+t)^{-\frac{1}{2}(k+l-\frac{1}{p})}$, $k + l = 1, 2, 3, 4$, $k, l \in \mathbb{N}$;

(vi) $\|\partial_x^k \partial_t^l w_i(t)\|_{L^p(\mathbb{R})} \le C(1+t)^{-\frac{1}{2}(k+l+1-\frac{1}{p})}$, $k + l = 1, 2, 3, 4$, $k, l \in \mathbb{N}$;

(vii) $w_{ix} > 0$, $x \in \mathbb{R}$.

For the case of $f'(u_-) > 0$, we know that for any $x \in (-\infty, +\infty)$, $w_2(x,t)$ satisfies $u_- < w_2(x,t) < u_+$ due to Lemma 2.2 (vii), which yields $w_2(0,t) \ne u_-$. Therefore, we need to modify $w_2(x,t)$ around the boundary $x = 0$. For simplicity, we still denote w_2 as w in the following. By employing the idea of Nakamura in [25], we define the modified smooth approximation $(\widetilde{u}, \widetilde{q})$ as

$$\begin{cases} \widetilde{u}(x,t) = w(x,t) - \hat{u}(x,t), \\ \widetilde{q}(x,t) = -w_x(x,t) - \hat{q}(x,t), \end{cases} \tag{2.7}$$

where

$$\begin{cases} \hat{u}(x,t) = (w(0,t) - u_-)\mathrm{e}^{-x}, \\ \hat{q}(x,t) = w_{xx}(0,t)\mathrm{e}^{-x}. \end{cases} \tag{2.8}$$

Note that $\hat{u}(0,t) \equiv 0$, if $f'(u_-) = 0$. Substituting (2.7) into (2.6) and capturing $x \in \mathbb{R}_+$, we have

$$\begin{cases} \widetilde{u}_t + f(\widetilde{u})_x = \widetilde{u}_{xx} + \hat{u}_{xx} - \hat{u}_t - (f(\widetilde{u} + \hat{u}) - f(\widetilde{u}))_x + \dfrac{f'''(w)}{f''(w)} w_x^2, \\ \qquad x \in \mathbb{R}_+, \quad t > 0, \\ \widetilde{q} = -\widetilde{u}_x - \hat{u}_x - \hat{q}, \quad x \in \mathbb{R}_+, \quad t > 0, \\ \widetilde{u}(0,t) = u_-, \quad \widetilde{q}_x(0,t) = 0, \quad t > 0, \\ \widetilde{u}(x,0) = \widetilde{u}_0(x) := w_0(x) - \hat{u}(x,0), \quad x \in \mathbb{R}_+. \end{cases} \tag{2.9}$$

From Lemma 2.2, by simple calculations, the following estimates of $\widetilde{u}(x,t)$ can be obtained. For details of the proof, we refer to [7, 17, 25].

Lemma 2.3. *Suppose that $f'(u_-) > 0$. For $1 \le p \le \infty$ and $t > 0$, the smooth rarefaction wave $\widetilde{u}(x,t)$ satisfies the following:*

(i) $|\partial_x^k \partial_t^l \widetilde{u}(0,t)| \le C\delta e^{-c(1+t)}$, $k+l = 1,2,3,4$, $k,l \in \mathbb{N}$;

(ii) $\|\widetilde{u}(t) - r(t)\|_{L^p(\mathbb{R}_+)} \le C(1+t)^{-\frac{1}{2}+\frac{1}{2p}}$;

(iii) $\|\widetilde{u}_x(t)\|_{L^p(\mathbb{R}_+)} \le C\delta^{\frac{1}{p}}(1+t)^{-1+\frac{1}{p}}$, $\|\widetilde{u}_t(t)\|_{L^p} \le C\delta^{\frac{1}{p}}(1+t)^{-1+\frac{1}{p}}$;

(iv) $\|\partial_x^k \partial_t^l \widetilde{u}(t)\|_{L^p(\mathbb{R}_+)} \le C\delta(1+t)^{-\frac{1}{2}(k+l-\frac{1}{p})}$, $k+l = 1,2,3,4$, $k,l \in \mathbb{N}$;

(v) $\|\partial_x^k \partial_t^l \widetilde{u}(t)\|_{L^p(\mathbb{R}_+)} \le C(1+t)^{-\frac{1}{2}(k+l+1-\frac{1}{p})}$, $k+l = 2,3,4$, $k,l \in \mathbb{N}$;

(vi) $\widetilde{u}_x > 0$, $x \in \mathbb{R}$.

Since the terms on the right-hand side of $(2.9)_1$ are not integrable with respect to y, we consider the one-dimensional initial-boundary value problem corresponding to (1.1)–(1.8), which can further approximate $\widetilde{u}$ by the problem

$$\begin{cases} U_t + f(U)_x + Q_x = 0, \\ -Q_{xx} + Q + U_x = 0 \end{cases} \tag{2.10}$$

with initial data

$$U(x,0) = U_0(x) \tag{2.11}$$

and boundary condition

$$U(0,t) = u_-, \quad Q_x(0,t) = 0, \quad U(+\infty,t) = u_+, \quad Q(+\infty,t) = 0. \tag{2.12}$$

Referring to [3], we get the monotonicity of $U(x,t)$ in x-direction by assuming that $\frac{d}{dx}U_0(x) \ge 0$ for $x \in \mathbb{R}_+$.

Lemma 2.4 (Monotonicity of profile). *Assume that $U_0(x)$ is monotonically non-decreasing, i.e.,*

$$\frac{d}{dx}U_0(x) \ge 0, \quad x \in \mathbb{R}_+.$$

Then the solution $(U(x,t), Q(x,t))$ of (2.10)–(2.12) satisfies

$$\frac{\partial}{\partial x}U(x,t) \ge 0, \quad Q(x,t) \le 0, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+.$$

Proof. We take differentiation of $(2.10)_1$ with respect to x and denote $U_x(x,t)$ by $W(x,t)$. Consequently, we get

$$\begin{cases} W_t + f'(U)W_x + f''(U)W^2 + Q_{xx} = 0, \\ \qquad\qquad\qquad\qquad -Q_{xx} + Q + W = 0 \end{cases} \tag{2.13}$$

and

$$W(x,0) = U_x(x,0) = \frac{d}{dx}U_0(x) \ge 0. \tag{2.14}$$

We extend the function $W(x,t)$ such that

$$\widetilde{W}(x,t) = \begin{cases} W(x,t), & x \geq 0, \\ W(-x,t), & x < 0. \end{cases}$$

Then $Q(x,t)$ in (2.13) can be solved as

$$Q(x,t) = \widetilde{Q}(x,t)|_{x\geq 0}$$

and satisfies $Q(x,0) \leq 0$ due to (2.14), where

$$\widetilde{Q}(x,t) = -\frac{1}{2}\int_{\mathbb{R}} \mathrm{e}^{-|x-y|}\widetilde{W}(y,t)\,\mathrm{d}y = -\frac{1}{2}\int_{\mathbb{R}_+} (\mathrm{e}^{-|x-y|} + \mathrm{e}^{-|x+y|})W(y,t)\,\mathrm{d}y.$$

Therefore, we can rewrite (2.13) as

$$\begin{cases} W_t + f'(U)W_x + f''(U)W^2 + Q + W = 0, & (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ Q = -\dfrac{1}{2}\displaystyle\int_{\mathbb{R}_+} (\mathrm{e}^{-|x-y|} + \mathrm{e}^{-|x+y|})W(y,t)\,\mathrm{d}y, & (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+. \end{cases} \tag{2.15}$$

We make the transformation

$$W = \bar{W} - \frac{1}{L}\mathrm{e}^t, \tag{2.16}$$

where L is a positive constant. From $(2.15)_2$, we get

$$\begin{aligned} Q(x,t) =& -\frac{1}{2}\int_{\mathbb{R}_+} (\mathrm{e}^{-|x-y|} + \mathrm{e}^{-|x+y|})\bar{W}(y,t)\,\mathrm{d}y + \frac{1}{L}\mathrm{e}^t \\ :=& \bar{Q} + \frac{1}{L}\mathrm{e}^t. \end{aligned} \tag{2.17}$$

Consequently, we get from (2.14) and (2.15) that

$$\begin{aligned} &\bar{W}_t + f'(U)\bar{W}_x + \Big(f''(U)\bar{W} - \frac{2}{L}\mathrm{e}^t f''(U) + 1\Big)\bar{W} + \bar{Q} \\ &= \frac{1}{L}\mathrm{e}^t\Big(1 - f''(U)\frac{1}{L}\mathrm{e}^t\Big) \end{aligned} \tag{2.18}$$

and

$$\bar{W}(x,0) > 0, \quad \bar{Q}(x,0) < 0.$$

We claim that $\bar{W}(x,t) > 0$ and $\bar{Q}(x,t) < 0$ for any $(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$. In fact, for any $T > 0$, let

$$t_0 = \inf_t\{t \mid \bar{W}(x,t) = 0 \text{ or } \bar{Q}(x,t) = 0\ \forall x \in \mathbb{R}_+,\ t \in (0,T]\}.$$

If t_0 does not exist, the proof is completed. Otherwise, $t_0 \in (0,T]$ and then there exists $x_0 \in \mathbb{R}_+$ such that $\bar{Q}(x_0,t_0) = 0$ and $\bar{W}(x,t_0) \geq 0$ for any $x \in \mathbb{R}_+$, or $\bar{W}(x_0,t_0) = 0$ and $\bar{Q}(x,t_0) \leq 0$ for any $x \in \mathbb{R}_+$. We argue by contradiction for the above two cases.

Case 1. $\bar{Q}(x_0,t_0)=0$ and $\bar{W}(x,t_0)\geq 0$ for any $x\in\mathbb{R}_+$. From (2.17), we can deduce that $\bar{W}(x,t_0)=0$ for any $x\in\mathbb{R}_+$. It follows that $W(x,t_0)$ and $Q(x,t_0)$ attain their minimum and maximum respectively at the point $x=x_0$. Notice that $\bar{W}_t(x_0,t_0)\leq 0$, $\bar{W}_x(x_0,t_0)=0$. If we choose L sufficiently large satisfying

$$1-f''(U)\frac{1}{L}\mathrm{e}^t\geq 1-f''(U)\frac{1}{L}\mathrm{e}^T>0 \quad \text{on } \mathbb{R}_+\times(0,T],$$

then we get a contradiction at the point (x_0,t_0) from (2.18).

Case 2. $\bar{W}(x_0,t_0)=0$ and $\bar{Q}(x_0,t_0)<0$. In this case, we see that $\bar{W}_t(x_0,t_0)\leq 0$, $\bar{W}_x(x_0,t_0)=0$. Similarly, we can get a contradiction by (2.18) if we choose L sufficiently large.

Therefore, we obtain

$$\bar{W}(x,t)>0,\quad \bar{Q}(x,t)<0,\quad (x,t)\in\mathbb{R}_+\times\mathbb{R}_+.$$

Letting $L\to+\infty$ in (2.16), we have

$$W(x,t)\geq 0,\quad Q(x,t)\leq 0,\quad (x,t)\in\mathbb{R}_+\times\mathbb{R}_+.$$

We complete the proof of Lemma 2.4. ■

Setting

$$\begin{aligned} u(x,y,t)-\tilde{u}(x,t) &= \{U(x,t)-\tilde{u}(x,t)\}+\{u(x,y,t)-U(x,t)\} \\ &:= V(x,t)+v(x,y,t) \end{aligned}$$

and

$$\begin{aligned} q(x,y,t)-\begin{pmatrix}\tilde{q}(x,t)\\0\end{pmatrix} &= \left\{\begin{pmatrix}Q(x,t)\\0\end{pmatrix}-\begin{pmatrix}\tilde{q}(x,t)\\0\end{pmatrix}\right\}+\left\{q(x,y,t)-\begin{pmatrix}Q(x,t)\\0\end{pmatrix}\right\} \\ &:= \begin{pmatrix}P(x,t)\\0\end{pmatrix}+p(x,y,t), \end{aligned}$$

we get two reformulated problems:

$$\begin{cases} V_t+(f(V+\tilde{u})-f(\tilde{u}))_x+P_x=R_1, \\ -P_{xx}+P+V_x=R_2, \\ V(0,t)=0,\quad P_x(0,t)=0, \\ V(x,0)=V_0(x)=U_0(x)-\tilde{u}_0(x) \end{cases} \tag{2.19}$$

and

$$\begin{cases} v_t+(f(v+U)-f(U))_x+g(v+U)_y+\operatorname{div} p=0, \\ -\nabla\operatorname{div} p+p+\nabla v=0, \\ v(0,y,t)=0,\quad \operatorname{div} p(0,y,t)=0, \\ v(x,y,0)=v_0(x,y)=u_0(x,y)-U_0(x), \end{cases} \tag{2.20}$$

where

$$\begin{cases} R_1 := \hat{q}_x + \hat{u}_t + (f(\tilde{u}+\hat{u}) - f(\tilde{u}))_x - \dfrac{f'''(w)}{f''(w)} w_x^2, \\ R_2 := \hat{u}_x + \hat{q} - \tilde{u}_{xxx} - \hat{u}_{xxx} - \hat{q}_{xx}. \end{cases} \tag{2.21}$$

By utilizing Lemmas 2.2 and 2.3, it is not difficult to get the following corollary.

Corollary 2.5. *Suppose that $f'(u_-) > 0$. For $1 \le p \le \infty$ and $t > 0$, $R_1(x,t)$ and $R_2(x,t)$ satisfy*

(i) $\|R_1(t)\|_{L^p(\mathbb{R}_+)} \le C\delta e^{-c(1+t)} + C\delta^{\frac{1}{p}}(1+t)^{-2+\frac{1}{p}}$;

(ii) $\|\partial_x^k \partial_t^l R_1(t)\|_{L^p(\mathbb{R}_+)} \le C\delta e^{-c(1+t)}$
$+ C\min\{\delta(1+t)^{-\frac{k+l+2}{2}+\frac{1}{2p}}, (1+t)^{-\frac{k+l+3}{2}+\frac{1}{2p}}\}$, $k+l = 1,2,3,4$;

(iii) $\|\partial_x^k \partial_t^l R_2(t)\|_{L^p(\mathbb{R}_+)} \le C\delta e^{-c(1+t)}$
$+ C\min\{\delta(1+t)^{-\frac{k+l+3}{2}+\frac{1}{2p}}, (1+t)^{-\frac{k+l+4}{2}+\frac{1}{2p}}\}$, $k+l = 0,1,2,3,4$;

(iv) $|\partial_x^k \partial_t^l R_1(0,t)| \le C\delta e^{-c(1+t)}$, $k,l = 0,1,2,3,4$.

For the case where $f'(u_-) = 0$, $R_1(x,t) = -\frac{f'''(w_1)}{f''(w_1)} w_{1x}^2$ and $R_2(x,t) = -\tilde{u}_{xxx}$, and it holds that

(v) $\|R_1(t)\|_{L^p(\mathbb{R}_+)} \le C\delta^{\frac{1}{p}}(1+t)^{-2+\frac{1}{p}}$;

(vi) $\|\partial_x^k \partial_t^l R_1(t)\|_{L^p(\mathbb{R}_+)} \le C\min\{\delta(1+t)^{-\frac{k+l+2}{2}+\frac{1}{2p}}, (1+t)^{-\frac{k+l+3}{2}+\frac{1}{2p}}\}$, $k+l = 1,2,3,4$;

(vii) $\|\partial_x^k \partial_t^l R_2(t)\|_{L^p(\mathbb{R}_+)} \le C\min\{\delta(1+t)^{-\frac{k+l+3}{2}+\frac{1}{2p}}, (1+t)^{-\frac{k+l+4}{2}+\frac{1}{2p}}\}$, $k+l = 0,1,2,3,4$;

(viii) $|\partial_x^k \partial_t^l R_1(0,t)| \le C\min\{\delta(1+t)^{-\frac{k+l+2}{2}}, (1+t)^{-\frac{k+l+3}{2}}\}$, $k+l = 1,2,3,4$.

Proof. We only give the proof of (i)–(iii) and (viii). The remaining estimates can be obtained by a similar method. From Lemma 2.2 (i), (iv), $(2.8)_1$ and $(2.21)_1$, we have

$$\begin{aligned} \|R_1(t)\|_{L^p(\mathbb{R}_+)} &\le C\delta e^{-c(1+t)} + C\Big(\int_{\mathbb{R}_+} |w_x|^{2p}\,dx\Big)^{\frac{1}{p}} \\ &\le C\delta e^{-c(1+t)} + C\|w_x(t)\|_{L^\infty(\mathbb{R}_+)}\|w_x(t)\|_{L^p(\mathbb{R}_+)} \\ &\le C\delta e^{-c(1+t)} + C(1+t)^{-1}\delta^{\frac{1}{p}}(1+t)^{-1+\frac{1}{p}} \\ &\le C\delta e^{-c(1+t)} + C\delta^{\frac{1}{p}}(1+t)^{-2+\frac{1}{p}}. \end{aligned}$$

Therefore, the desired estimate (i) is obtained.

Next, we try to show the estimate (ii). By utilizing (2.2), we can get

$$\begin{aligned} \|\partial_x^k R_1(t)\|_{L^p(\mathbb{R}_+)} &\le C\delta e^{-c(1+t)} + C\Big\|\partial_x^k\Big\{\frac{f'''(w)}{f''(w)} w_x^2\Big\}\Big\|_{L^p(\mathbb{R}_+)} \\ &\le C\delta e^{-c(1+t)} + C\|\partial_x^{k+2} w(t)\|_{L^p(\mathbb{R}_+)}. \end{aligned}$$

From Lemma 2.2 (v), it follows that

$$\|\partial_x^k R_1(t)\|_{L^p(\mathbb{R}_+)} \le C\delta e^{-c(1+t)} + C\delta(1+t)^{-\frac{1}{2}(k+2-\frac{1}{p})}. \tag{2.22}$$

On the other hand, from Lemma 2.2 (vi), it also holds that

$$\|\partial_x^k R_1(t)\|_{L^p(\mathbb{R}_+)} \leq C\delta \mathrm{e}^{-c(1+t)} + C(1+t)^{-\frac{1}{2}(k+3-\frac{1}{p})}. \tag{2.23}$$

Combining (2.22) and (2.23), we complete the proof of (ii) in Corollary 2.5.

To get (iii), we can get from $(2.21)_2$ that

$$\|\partial_x^k R_2(t)\|_{L^p(\mathbb{R}_+)} \leq C\delta \mathrm{e}^{-c(1+t)} + \|\partial_x^{k+3}\widetilde{u}\|_{L^p(\mathbb{R}_+)}.$$

Estimate (iii) can be obtained by applying Lemma 2.3 (iv) and (v). Finally, we show estimate (viii). In fact, from (vi),

$$|\partial_x^k\partial_t^l R_1(0,t)| \leq \|\partial_x^k\partial_t^l R_1(\cdot,t)\|_{L^\infty(\mathbb{R}_+)} \leq C\min\{\delta(1+t)^{-\frac{k+l+2}{2}}, (1+t)^{-\frac{k+l+3}{2}}\}. \blacksquare$$

We state the main result in this paper as follows.

2.2. *Main theorem*

Theorem 2.6. *Assume that $0 \leq f'(u_-) < f'(u_+)$ holds. Suppose that $u_0(x,y) - r_0 \in L^2(\mathbb{R}_+^2) \cap L^1(\mathbb{R}_+^2)$ and $\nabla u_0 \in H^2(\mathbb{R}_+^2)$. Then there exists a positive constant δ_0 such that if*

$$\|u_0(x,y) - r_0\|_{L^2(\mathbb{R}_+^2)} + \|\nabla u_0\|_{H^2(\mathbb{R}_+^2)} + |u_+ - u_-| \leq \delta_0,$$

then the initial-boundary value problem (1.1)–(1.8) has a unique global solution (u,q) which satisfies

$$\begin{cases} u - r \in C^0([0,\infty); H^3(\mathbb{R}_+^2)), \quad \nabla u - r_x \in L^2(0,\infty; H^2(\mathbb{R}_+^2)), \\ q + r_x \in C^0([0,\infty); H^3(\mathbb{R}_+^2)) \cap L^2(0,\infty; H^3(\mathbb{R}_+^2)), \\ \operatorname{div} q + r_{xx} \in C^0([0,\infty); H^3(\mathbb{R}_+^2)) \cap L^2(0,\infty; H^3(\mathbb{R}_+^2)) \end{cases}$$

and

$$\sup_{(x,y)\in\mathbb{R}_+^2} |\nabla^k(u(x,y,t) - r(x,t))| \to 0 \quad \text{as } t\to\infty,\ k = 0,1,$$

$$\sup_{(x,y)\in\mathbb{R}_+^2} |\nabla^k(q(x,y,t) + r_x(x,t))| \to 0 \quad \text{as } t\to\infty,\ k = 0,1,$$

$$\sup_{(x,y)\in\mathbb{R}_+^2} |\nabla(\operatorname{div} q(x,y,t) + r_{xx}(x,t))| \to 0 \quad \text{as } t\to\infty.$$

3. Asymptotics to planar rarefaction wave

In this section, we focus on case (b) where $0 \leq f'(u_-) < f'(u_+)$. We devote ourselves to showing that the asymptotic behavior of the solution of (1.1)–(1.8) is the corresponding planar rarefaction wave as t tends to infinity. In order to show that Theorem 2.6 is true, we just need to prove the following two results, Theorems 3.1 and 3.2.

Theorem 3.1. *Assume that* (1.5), (1.7) *and* $0 \le f'(u_-) < f'(u_+)$ *hold. Suppose that* $V_0 \in H^4(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$. *Then there exists a small positive constant* δ_1' *such that if* $\|V_0\|_{H^4(\mathbb{R}_+)} + \delta \le \delta_1'$, *then problem* (2.19) *has a unique global solution satisfying*

$$\begin{cases} V \in C^0([0,\infty); H^4(\mathbb{R}_+)) \cap C^0([0,\infty); L^1(\mathbb{R}_+)), \\ P \in C^0([0,\infty); H^5(\mathbb{R}_+)) \cap L^2(0,\infty; H^5(\mathbb{R}_+)), \end{cases}$$

and for a sufficiently large t,

$$\begin{cases} \|V(t)\|_{L^\infty(\mathbb{R}_+)} \le C(1+t)^{-\frac{1}{2}} \log^3(2+t), \\ \|\partial_x^k V(t)\|_{L^\infty(\mathbb{R}_+)} \le C(1+t)^{-\frac{3}{4}} \log^5(2+t), \quad k = 1,2,3, \\ \|\partial_x^k P(t)\|_{L^\infty(\mathbb{R}_+)} \le C(1+t)^{-\frac{3}{4}} \log^5(2+t), \quad k = 0,1,2,3,4. \end{cases}$$

Theorem 3.2. *Assume that* (1.5), (1.7) *and* $0 \le f'(u_-) < f'(u_+)$ *hold. Suppose that* $v_0 \in H^3(\mathbb{R}^2_+) \cap L^1(\mathbb{R}^2_+)$. *Then there exists a small positive constant* δ_2' *such that if* $\|v_0\|_{H^3(\mathbb{R}^2_+)} + \delta \le \delta_2'$, *then problem* (2.20) *has a unique global solution satisfying*

$$\begin{cases} v \in C^0([0,\infty); H^3(\mathbb{R}^2_+)), \quad \nabla v \in L^2(0,\infty; H^2(\mathbb{R}^2_+)), \\ p, \operatorname{div} p \in C^0([0,\infty); H^3(\mathbb{R}^2_+)) \cap L^2(0,\infty; H^3(\mathbb{R}^2_+)) \end{cases}$$

and

$$\begin{cases} \sup\limits_{(x,y)\in\mathbb{R}^2_+} |\nabla^k v(x,y,t)| \to 0 \quad \text{as } t \to \infty,\ k = 0,1, \\ \sup\limits_{(x,y)\in\mathbb{R}^2_+} |\nabla^k p(x,y,t)| \to 0 \quad \text{as } t \to \infty,\ k = 0,1, \\ \sup\limits_{(x,y)\in\mathbb{R}^2_+} |\nabla \operatorname{div} p(x,y,t)| \to 0 \quad \text{as } t \to \infty. \end{cases} \tag{3.1}$$

In the following, we try to prove Theorems 3.1 and 3.2. We note that the main difference between $f'(u_-) > 0$ and $f'(u_-) = 0$ is the boundary values $f'(u_-)\partial_x^k V(0,t)$ and $f'(u_-)\nabla^l v(0,y,t)$, $k = 1,2,3,4$, $l = 1,2,3$. The proof of the case of $f'(u_-) = 0$ is simpler than that of $f'(u_-) > 0$. Thus, we only prove the case of $f'(u_-) > 0$, and the proof of $f'(u_-) = 0$ is omitted.

3.1. Estimates for the perturbation on the one-dimensional half-space

In this section, we consider the initial-boundary value problem on the one-dimensional half-space,

$$\begin{cases} V_t + (f(V+\tilde{u}) - f(\tilde{u}))_x + P_x = R_1, \\ -P_{xx} + P + V_x = R_2 \end{cases} \tag{3.2}$$

with initial data

$$V(x,0) = V_0(x) = U_0(x) - \tilde{u}_0(x) \tag{3.3}$$

and boundary condition

$$V(0,t)=0, \quad P_x(0,t)=0. \tag{3.4}$$

Here, R_1 and R_2 are defined in (2.21). From (3.2), we have

$$(\partial_x^k\partial_t^l V_x)^2 \le 3((\partial_x^k\partial_t^l P_{xx})^2+(\partial_x^k\partial_t^l P)^2+(\partial_x^k\partial_t^l R_2)^2), \tag{3.5}$$

$$(\partial_x^k\partial_t^l P_{xx})^2 \le 3((\partial_x^k\partial_t^l V_x)^2+(\partial_x^k\partial_t^l P)^2+(\partial_x^k\partial_t^l R_2)^2) \tag{3.6}$$

with $k+l=0,1,2,3$, $k,l\in\mathbb{N}$. It will be often used later and plays an important role in a priori estimates.

The solution of the reformulated problem (3.2)–(3.4) is sought in the set of the functional space $X(0,T)$, where for $0\le T\le+\infty$, we define

$$\begin{aligned} X(0,T)=\big\{(V,P)\mid V\in C^0([0,T);H^4(\mathbb{R}_+)),\ V_x\in L^2(0,T;H^3(\mathbb{R}_+)),\\ P\in C^0([0,T);H^5(\mathbb{R}_+))\cap L^2(0,T;H^5(\mathbb{R}_+))\big\}. \end{aligned}$$

Proposition 3.3. *Suppose that the boundary condition and far field states satisfy* $0\le f'(u_-)<f'(u_+)$*, the initial data* $V_0\in H^4(\mathbb{R}_+)$ *and the wavelength* $\delta=|u_--u_+|$ *are sufficiently small. Then there are two positive constants* $\widetilde{\delta}_1$ *and* $C=C(\widetilde{\delta}_1)$ *such that if* $\|V_0\|_{H^4(\mathbb{R}_+)}+\delta\le\widetilde{\delta}_1$*, problem* (3.2)–(3.4) *admits a unique solution* $(V(x,t),P(x,t))\in X(0,+\infty)$ *satisfying*

$$\begin{aligned} &\|V(t)\|^2_{H^4(\mathbb{R}_+)}+\|P(t)\|^2_{H^5(\mathbb{R}_+)}+\int_0^t(\|V_x(\tau)\|^2_{H^3(\mathbb{R}_+)}+\|P(\tau)\|^2_{H^5(\mathbb{R}_+)})\,\mathrm{d}\tau\\ &\quad\le C(\|V_0\|^2_{H^4(\mathbb{R}_+)}+\delta^{\frac12})\quad\forall t\in[0,\infty). \end{aligned}\tag{3.7}$$

Since the proof for the local-in-time existence and uniqueness of the solution to (3.2)–(3.4) is standard, the details will be omitted. By the Sobolev inequality and Lemma 2.3, we get

$$\begin{cases} \sup\limits_{t\ge0}\sum\limits_{k=0}^{3}\|\partial_x^k V(t)\|_{L^\infty(\mathbb{R}_+)}\le C\delta',\\ \sup\limits_{t\ge0}\sum\limits_{k=1}^{3}\|\partial_x^k U(t)\|_{L^\infty(\mathbb{R}_+)}\le C\delta', \end{cases}\tag{3.8}$$

where $\delta'=(\|V_0\|^2_{H^4(\mathbb{R}_+)}+\delta^{\frac12})^{\frac12}$ is sufficiently small, provided that $\widetilde{\delta}_1$ is small enough.

In order to prove Proposition 3.3, it suffices to show the following a priori estimates.

Proposition 3.4 (A priori estimates). *Let* T *be a positive constant. Assume that* $0\le f'(u_-)<f'(u_+)$*. Suppose that problem* (3.2)–(3.4) *has a unique solution* $(V,P)\in X(0,T)$*. Then there exist two positive constants* $\widetilde{\delta}_2(\le\widetilde{\delta}_1)$ *and* $C=C(\widetilde{\delta}_2)$ *such that if* $\|V_0\|_{H^4(\mathbb{R}_+)}+\delta\le\widetilde{\delta}_2$*, then we have the estimate*

$$\begin{aligned} &\|V(t)\|^2_{H^4(\mathbb{R}_+)}+\|P(t)\|^2_{H^5(\mathbb{R}_+)}+\int_0^t(\|V_x(\tau)\|^2_{H^3(\mathbb{R}_+)}+\|P(\tau)\|^2_{H^5(\mathbb{R}_+)})\,\mathrm{d}\tau\\ &\quad\le C(\|V_0\|^2_{H^4(\mathbb{R}_+)}+\delta^{\frac12})\quad\forall t\in[0,T]. \end{aligned}$$

Before proving the a priori estimates, we first give some basic estimates.

Lemma 3.5. *There is a positive constant C such that the following estimates hold:*

(A1) $$\int_{\mathbb{R}_+} (f(V+\widetilde{u}) - f(\widetilde{u}))_x V \,\mathrm{d}x \geq \frac{\alpha}{2} \int_{\mathbb{R}_+} \widetilde{u}_x V^2 \,\mathrm{d}x;$$

(A2) $$\begin{aligned}\int_{\mathbb{R}_+} &(f(V+\widetilde{u}) - f(\widetilde{u}))_{xx} V_x \,\mathrm{d}x \\ &\geq \frac{3\alpha}{2} \int_{\mathbb{R}_+} \widetilde{u}_x V_x^2 \,\mathrm{d}x - \frac{f'(u_-)}{2} V_x^2(0,t) \\ &\quad - C \int_{\mathbb{R}_+} |V_x|(|\widetilde{u}_{xx}||V| + \widetilde{u}_x^2|V| + V_x^2) \,\mathrm{d}x;\end{aligned}$$

(A3) $$\begin{aligned}\int_{\mathbb{R}_+} &(f(V+\widetilde{u}) - f(\widetilde{u}))_{xxx} V_{xx} \,\mathrm{d}x \\ &\geq \frac{5\alpha}{2} \int_{\mathbb{R}_+} \widetilde{u}_x V_{xx}^2 \,\mathrm{d}x - \frac{f'(u_-)}{2} V_{xx}^2(0,t) \\ &\quad - C \int_{\mathbb{R}_+} |V_{xx}| \big\{ (|\widetilde{u}_{xxx}| + |\widetilde{u}_x|^3 + \widetilde{u}_x |\widetilde{u}_{xx}|)|V| \\ &\qquad + (|\widetilde{u}_{xx}| + \widetilde{u}_x^2 + V_x^2)|V_x| + |V_x||V_{xx}| \big\} \,\mathrm{d}x;\end{aligned}$$

(A4) $$\begin{aligned}\int_{\mathbb{R}_+} &(f(V+\widetilde{u}) - f(\widetilde{u}))_{xxxt} V_{xxt} \,\mathrm{d}x \\ &\geq \frac{5\alpha}{2} \int_{\mathbb{R}_+} \widetilde{u}_x V_{xxt}^2 \,\mathrm{d}x - \frac{f'(u_-)}{2} V_{xxt}^2(0,t) - C \int_{\mathbb{R}_+} |V_x||V_{xxt}|^2 \,\mathrm{d}x \\ &\quad - C \int_{\mathbb{R}_+} |V_{xxt}| \big\{ (|V_x|^3 + \widetilde{u}_x^3 + \widetilde{u}_x|\widetilde{u}_{xx}| + |\widetilde{u}_{xx}||V_x| + |\widetilde{u}_{xxx}|)|V_t| \\ &\qquad + (|V_x|^2 + |\widetilde{u}_x|^2 + |\widetilde{u}_{xx}|)|V_{xt}| \big\} \,\mathrm{d}x \\ &\quad - C \int_{\mathbb{R}_+} |V_{xxt}| \big\{ \big(|\widetilde{u}_t||V_x|^2 + |\widetilde{u}_{xt}||V_x| + |\widetilde{u}_{xxt}| + |\widetilde{u}_t||\widetilde{u}_{xx}| \\ &\qquad + \widetilde{u}_x^2|\widetilde{u}_t| + \widetilde{u}_x|\widetilde{u}_{xt}| \big)|V_x| \big\} \,\mathrm{d}x \\ &\quad - C \int_{\mathbb{R}_+} |V_{xxt}| \big\{ \big(|V_t||V_x| + |\widetilde{u}_t||V_x| + |V_t||\widetilde{u}_x| + |\widetilde{u}_t||\widetilde{u}_x| \\ &\qquad + |V_{xt}| + |\widetilde{u}_{xt}| \big)|V_{xx}| + (|V_t| + |\widetilde{u}_t|)|V_{xxx}| \big\} \,\mathrm{d}x \\ &\quad - C \int_{\mathbb{R}_+} |V_{xxt}| \big(|\widetilde{u}_t|\widetilde{u}_x^3 + \widetilde{u}_x^2|\widetilde{u}_{xt}| + |\widetilde{u}_t|\widetilde{u}_x|\widetilde{u}_{xx}| + |\widetilde{u}_{xt}||\widetilde{u}_{xx}| \\ &\qquad + \widetilde{u}_x|\widetilde{u}_{xxt}| + |\widetilde{u}_t||\widetilde{u}_{xxx}| + |\widetilde{u}_{xxxt}| \big)|V| \,\mathrm{d}x;\end{aligned}$$

(A5) $$\begin{aligned}\int_{\mathbb{R}_+} &(f(V+\widetilde{u}) - f(\widetilde{u}))_{xxxtt} V_{xxtt} \,\mathrm{d}x \\ &\geq \frac{5\alpha}{2} \int_{\mathbb{R}_+} \widetilde{u}_x V_{xxtt}^2 \,\mathrm{d}x - \frac{f'(u_-)}{2} V_{xxtt}^2(0,t) \\ &\quad - C \int_{\mathbb{R}_+} |V_{xxtt}| \{ |V_x||V_{xxtt}| + (|V_t| + |\widetilde{u}_t|)|V_{xxxt}| \} \,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&- C\int_{\mathbb{R}_+} |V_{xxtt}|\big\{(\widetilde{u}_t^2 + |\widetilde{u}_{tt}|)(V_x^2 + \widetilde{u}_x^2)\\
&\qquad + (|\widetilde{u}_t\widetilde{u}_{xt}| + |\widetilde{u}_{xtt}|)(|V_x| + \widetilde{u}_x) + V_{xt}^2 + |\widetilde{u}_{xx}|V_t^2\\
&\qquad + |\widetilde{u}_{xxt}|(|V_t| + |\widetilde{u}_t|)\big\}|V_x|\,\mathrm{d}x\\
&- C\int_{\mathbb{R}_+} |V_{xxtt}|\big\{(|\widetilde{u}_{xxtt}| + |\widetilde{u}_{xx}|\widetilde{u}_t^2 + \widetilde{u}_{xt}^2 + |\widetilde{u}_{tt}\widetilde{u}_{xx}|)|V_x|\\
&\qquad + (|\widetilde{u}_{xx}\widetilde{u}_{xt}| + \widetilde{u}_x|\widetilde{u}_{xxt}| + |\widetilde{u}_{xxxt}|)|V_t|\big\}\,\mathrm{d}x\\
&- C\int_{\mathbb{R}_+} |V_{xxtt}|\big\{(|V_x|^3 + \widetilde{u}_x^3 + \widetilde{u}_x|\widetilde{u}_{xx}| + |\widetilde{u}_{xxx}|)(|V_t| + |\widetilde{u}_t|)\\
&\qquad + |\widetilde{u}_{xt}|(V_x^2 + \widetilde{u}_x^2)\big\}|V_t|\,\mathrm{d}x\\
&- C\int_{\mathbb{R}_+} |V_{xxtt}|\big\{(|V_x| + \widetilde{u}_x)(V_t^2 + \widetilde{u}_t^2) + (|V_x| + \widetilde{u}_x)(|V_{tt}| + |\widetilde{u}_{tt}|)\\
&\qquad + (|V_t| + |\widetilde{u}_t|)(|V_{xt}| + |\widetilde{u}_{xt}|) + |\widetilde{u}_{xtt}|\big\}|V_{xx}|\,\mathrm{d}x\\
&- C\int_{\mathbb{R}_+} |V_{xxtt}|\big\{(|V_t| + |\widetilde{u}_t|)(V_x^2 + \widetilde{u}_x^2) + |\widetilde{u}_{xx}|(|V_t| + |\widetilde{u}_t|)\\
&\qquad + \widetilde{u}_x(|V_{xt}| + |\widetilde{u}_{xt}|) + |\widetilde{u}_{xxt}|\big\}|V_{xt}|\,\mathrm{d}x\\
&- C\int_{\mathbb{R}_+} |V_{xxtt}|\big\{(|V_x|^3 + \widetilde{u}_x^3 + |\widetilde{u}_{xx}V_x| + |\widetilde{u}_{xx}|\widetilde{u}_x)|V_{tt}|\\
&\qquad + (V_x^2 + \widetilde{u}_x^2 + |V_{xx}| + |\widetilde{u}_{xx}|)|V_{xtt}|\big\}\,\mathrm{d}x\\
&- C\int_{\mathbb{R}_+} |V_{xxtt}|\big\{(V_t^2 + \widetilde{u}_t^2 + |V_{tt}| + |\widetilde{u}_{tt}|)|V_{xxx}|\\
&\qquad + \big(|V_xV_t| + \widetilde{u}_x|V_t| + |\widetilde{u}_tV_x| + \widetilde{u}_x|\widetilde{u}_t| + |V_{xt}|\\
&\qquad\quad + |\widetilde{u}_{xt}|\big)|V_{xxt}|\big\}\,\mathrm{d}x\\
&- C\int_{\mathbb{R}_+} |V_{xxtt}|\big\{\widetilde{u}_x|\widetilde{u}_{xxtt}| + \widetilde{u}_x^3|\widetilde{u}_{tt}| + \widetilde{u}_x^2|\widetilde{u}_{xt}| + \widetilde{u}_x\widetilde{u}_{xt}^2 + \widetilde{u}_t^2\widetilde{u}_x|\widetilde{u}_{xx}|\\
&\qquad + \widetilde{u}_x^2|\widetilde{u}_{xtt}| + |\widetilde{u}_{tt}||\widetilde{u}_{xxx}| + \widetilde{u}_x|\widetilde{u}_{tt}||\widetilde{u}_{xx}|\big\}|V|\,\mathrm{d}x\\
&- C\int_{\mathbb{R}_+} |V_{xxtt}|\big\{|\widetilde{u}_t\widetilde{u}_{xt}\widetilde{u}_{xx}| + \widetilde{u}_x|\widetilde{u}_t\widetilde{u}_{xxt}| + |\widetilde{u}_{xx}\widetilde{u}_{xtt}| + |\widetilde{u}_{xxt}\widetilde{u}_{xt}|\\
&\qquad + \widetilde{u}_t^2|\widetilde{u}_{xxx}| + |\widetilde{u}_t\widetilde{u}_{xxxt}| + |\widetilde{u}_{xxxtt}|\big\}|V|\,\mathrm{d}x.
\end{aligned}
$$

Proof. By direct calculations, we can obtain

$$
\begin{aligned}
(f(V+\widetilde{u}) - f(\widetilde{u}))_x V &= (f(V+\widetilde{u}) - f(\widetilde{u}) - f'(\widetilde{u})V)\widetilde{u}_x\\
&\quad + \left\{(f(V+\widetilde{u}) - f(\widetilde{u}))V - \int_{\widetilde{u}}^{V+\widetilde{u}} f(s)\,\mathrm{d}s + f(\widetilde{u})V\right\}_x
\end{aligned}
$$

and

$$
\begin{aligned}
(f(V+\widetilde{u}) - f(\widetilde{u}))_{xx}V_x &= \frac{1}{2}f''(V+\widetilde{u})V_x^3 + \frac{3}{2}f''(v+\widetilde{u})\widetilde{u}_xV_x^2\\
&\quad + \frac{1}{2}\{f'(V+\widetilde{u})V_x^2\}_x + (f''(V+\widetilde{u}) - f''(\widetilde{u}))\widetilde{u}_x^2V_x\\
&\quad + (f'(V+\widetilde{u}) - f'(\widetilde{u}))\widetilde{u}_{xx}V_x.
\end{aligned}
$$

Integrating the above two equations over $\mathbb{R}_+$, from $V(0,t)=0$, we have

$$\int_{\mathbb{R}_+}(f(V+\tilde{u})-f(\tilde{u}))_x V\,\mathrm{d}x \geq \frac{\alpha}{2}\int_{\mathbb{R}_+}\tilde{u}_x V^2\,\mathrm{d}x$$

and

$$\int_{\mathbb{R}_+}(f(V+\tilde{u})-f(\tilde{u}))_{xx} V_x\,\mathrm{d}x \geq \frac{3\alpha}{2}\int_{\mathbb{R}_+}\tilde{u}_x V_x^2\,\mathrm{d}x - \frac{f'(u_-)}{2}V_x^2(0,t) - C\int_{\mathbb{R}_+}|V_x|(|\tilde{u}_{xx}||V|+\tilde{u}_x^2|V|+V_x^2)\,\mathrm{d}x,$$

which yields the desired estimates (A1) and (A2). The inequalities (A3)–(A5) can be obtained by using the similar approach. ■

Lemma 3.6. *Suppose that $f(u_-)>0$. The solution $V(x,t)$ satisfies the following boundary estimates:*

(B1) $V_x^2(0,t)\leq C\delta \mathrm{e}^{-c(1+t)}$;
(B2) $V_{xt}^2(0,t)\leq C\delta \mathrm{e}^{-c(1+t)}$;
(B3) $V_{xx}^2(0,t)\leq C\delta \mathrm{e}^{-c(1+t)}+CP_{xx}^2(0,t)$;
(B4) $V_{xtt}^2(0,t)\leq C\delta \mathrm{e}^{-c(1+t)}$;
(B5) $V_{xxt}^2(0,t)\leq C\delta \mathrm{e}^{-c(1+t)}+CP_{xxt}^2(0,t)$;
(B6) $V_{xttt}^2(0,t)\leq C\delta \mathrm{e}^{-c(1+t)}$;
(B7) $V_{xxtt}^2(0,t)\leq C\delta \mathrm{e}^{-c(1+t)}+CP_{xxtt}^2(0,t)$.

Proof. From $(3.2)_1$ and (3.4), it holds that

$$f'(u_-)V_x(0,t)=R_1(0,t)-P_x(0,t)=R_1(0,t). \tag{3.9}$$

By utilizing Corollary 2.5 (iv), we have

$$V_x^2(0,t)\leq C\delta \mathrm{e}^{-c(1+t)}.$$

Differentiating (3.9) with respect to t, we get

$$f'(u_-)V_{xt}(0,t)=R_{1t}(0,t).$$

It follows that

$$V_{xt}^2(0,t)\leq CR_{1t}^2(0,t)\leq C\delta \mathrm{e}^{-c(1+t)}.$$

Similarly, we can get the remaining boundary estimates (B3)–(B7) by utilizing $(3.2)_1$. ■

Remark 3.7. For the case of $f'(u_-)=0$, the boundary terms (B1)–(B7) in Lemma 3.6 will disappear because the coefficients of all boundary terms $(\partial_x^k\partial_t^l V(0,y,t))^2$ are $\frac{f'(u_-)}{2}$ which is given in Lemma 3.5 (A2)–(A5). Similarly, the boundary terms (D3)–(D5) in Lemma 3.31 will also disappear. In other words, in the case of $f'(u_-)=0$, the result of the initial boundary value problem (1.1)–(1.8) is the same as that of the Cauchy problem. For details, we refer to [3].

3.1.1. A priori estimates. In this section, we will prove Proposition 3.4 under the a priori assumption

$$\|V_x(t)\|_{H^4(\mathbb{R}_+)} \le \varepsilon_0,$$

where $0 < \varepsilon_0 \ll 1$. By the Sobolev inequality, there exists a positive constant C such that

$$\|\partial_x^k V(t)\|_{L^\infty(\mathbb{R}_+)} \le C\varepsilon_0, \quad k = 0, 1, 2, 3.$$

For simplicity, we divide the proof of the a priori estimates into several lemmas.

Lemma 3.8. *Under the same assumption as Proposition 3.4, there exists a positive constant C such that*

$$\|V(t)\|^2_{L^2(\mathbb{R}_+)} + \int_0^t (\|\sqrt{\tilde{u}_x}V(\tau)\|^2_{L^2(\mathbb{R}_+)} + \|P_x(\tau)\|^2_{L^2(\mathbb{R}_+)} + \|P(\tau)\|^2_{L^2(\mathbb{R}_+)})\,\mathrm{d}\tau \\ \le C(\|V_0\|^2_{H^4(\mathbb{R}_+)} + \delta^{\frac{1}{2}}). \tag{3.10}$$

Proof. We can get from $(3.2)_1 \times V + (3.2)_2 \times P$ that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}V^2 + (f(V+\tilde{u}) - f(\tilde{u}))_x V + P_x^2 + P^2 + \{PV - P_x P\}_x \\ = R_1 V + R_2 P. \tag{3.11}$$

For $V(0,t) = 0$ and $P_x(0,t) = 0$, the terms in $\{\cdots\}_x$ disappear after integration in $x \in \mathbb{R}_+$. Thus, integrating (3.11) over $\mathbb{R}_+$, by Lemma 3.5 (A1), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|V(t)\|^2_{L^2(\mathbb{R}_+)} + \frac{\alpha}{2}\|\sqrt{\tilde{u}_x}V(t)\|^2_{L^2(\mathbb{R}_+)} + \|P_x(t)\|^2_{L^2(\mathbb{R}_+)} + \|P(t)\|^2_{L^2(\mathbb{R}_+)} \\ \le C\int_{\mathbb{R}_+} (|R_1 V| + |R_2 P|)\,\mathrm{d}x. \tag{3.12}$$

We treat the terms on the right-hand side of (3.12) as follows. From Corollary 2.5 (i) and (iii),

$$\begin{aligned} C\int_{\mathbb{R}_+} |R_1 V|\,\mathrm{d}x &\le C\|R_1(t)\|_{L^2(\mathbb{R}_+)}\|V(t)\|_{L^2(\mathbb{R}_+)} \\ &\le C(\delta^{\frac{1}{2}}(1+t)^{-\frac{3}{2}} + \delta \mathrm{e}^{-c(1+t)})(1 + \|V(t)\|^2_{L^2(\mathbb{R}_+)}), \\ C\int_{\mathbb{R}_+} |R_2 P|\,\mathrm{d}x &\le \frac{1}{4}\|P(t)\|^2_{L^2(\mathbb{R}_+)} + C\|R_2(t)\|^2_{L^2(\mathbb{R}_+)} \\ &\le \frac{1}{4}\|P(t)\|^2_{L^2(\mathbb{R}_+)} + C\delta^2(1+t)^{-\frac{5}{2}} + C\delta \mathrm{e}^{-c(1+t)}. \end{aligned} \tag{3.13}$$

Substituting (3.13) into (3.12), we can deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|V(t)\|^2_{L^2(\mathbb{R}_+)} + \|\sqrt{\tilde{u}_x}V(t)\|^2_{L^2(\mathbb{R}_+)} + \|P_x(t)\|^2_{L^2(\mathbb{R}_+)} + \|P(t)\|^2_{L^2(\mathbb{R}_+)} \\ \le C\delta^{\frac{1}{2}}((1+t)^{-\frac{3}{2}} + \mathrm{e}^{-c(1+t)})(1 + \|V(t)\|^2_{L^2(\mathbb{R}_+)}). \tag{3.14}$$

Integrating (3.14) over $[0,t]$, for some small δ, we obtain the desired estimate (3.10). ∎

Lemma 3.9. *Under the same assumptions as Proposition 3.4, there exists a positive constant C such that*

$$\|V_x(t)\|^2_{L^2(\mathbb{R}_+)} + \int_0^t (\|\sqrt{\tilde{u}_x} V_x(\tau)\|^2_{L^2(\mathbb{R}_+)} + \|P_{xx}(\tau)\|^2_{L^2(\mathbb{R}_+)} + \|P_x(\tau)\|^2_{L^2(\mathbb{R}_+)})\,\mathrm{d}\tau \le C(\|V_0\|^2_{H^4(\mathbb{R}_+)} + \delta^{\frac{1}{2}}). \tag{3.15}$$

Proof. We can obtain from $\partial_x(3.2)_1 \times V_x - (3.2)_2 \times P_{xx}$ that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} V_x^2 + (f(V+\tilde{u}) - f(\tilde{u}))_{xx} V_x + P_{xx}^2 + P_x^2 - \{P_x P\}_x = R_{1x} V_x - R_2 P_{xx}. \tag{3.16}$$

From Lemma 3.5 (A2), integrating (3.16) over $\mathbb{R}_+$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|V_x(t)\|^2_{L^2(\mathbb{R}_+)} + \frac{3\alpha}{2}\|\sqrt{\tilde{u}_x} V_x(t)\|^2_{L^2(\mathbb{R}_+)} + \|P_{xx}(t)\|^2_{L^2(\mathbb{R}_+)} + \|P_x(t)\|^2_{L^2(\mathbb{R}_+)} \le \frac{f'(u_-)}{2} V_x^2(0,t) + C\int_{\mathbb{R}_+} |V_x|(|\tilde{u}_{xx}||V| + \tilde{u}_x^2|V| + V_x^2)\,\mathrm{d}x + \int_{\mathbb{R}_+} (|R_{1x} V_x| + |R_2 P_{xx}|)\,\mathrm{d}x. \tag{3.17}$$

Now we estimate the terms on the right-hand side of (3.17) one by one. By applying (2.1) and Lemma 2.3 (iv) with $k = 2$, the second term can be estimated as

$$C\int_{\mathbb{R}_+} |V_x|(|\tilde{u}_{xx}||V| + \tilde{u}_x^2|V| + V_x^2)\,\mathrm{d}x \le (\|V_x(t)\|_{L^\infty(\mathbb{R}_+)} + \mu)\|V_x(t)\|^2_{L^2(\mathbb{R}_+)} + C\mu^{-1}\|\tilde{u}_{xx}(t)\|^2_{L^\infty(\mathbb{R}_+)}\|V(t)\|^2_{L^2(\mathbb{R}_+)} \le (\varepsilon_0 + \mu)\|V_x(t)\|^2_{L^2(\mathbb{R}_+)} + C\mu^{-1}\delta^2(1+t)^{-2}. \tag{3.18}$$

The last term of (3.17) is bounded by

$$\int_{\mathbb{R}_+} (|R_{1x} V_x| + |R_2 P_{xx}|)\,\mathrm{d}x \le \mu(\|V_x(t)\|^2_{L^2(\mathbb{R}_+)} + \|P_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}) + C\mu^{-1}(\|R_{1x}(t)\|^2_{L^2(\mathbb{R}_+)} + \|R_2(t)\|^2_{L^2(\mathbb{R}_+)}) \le \mu(\|V_x(t)\|^2_{L^2(\mathbb{R}_+)} + \|P_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}) + C\mu^{-1}\delta^2(1+t)^{-\frac{5}{2}}. \tag{3.19}$$

Substituting (3.18)–(3.19) into (3.17), by employing Lemma 3.6 (B1), for some small but fixed μ, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|V_x(t)\|^2_{L^2(\mathbb{R}_+)} + \|\sqrt{\tilde{u}_x} V_x(t)\|^2_{L^2(\mathbb{R}_+)} + \|P_{xx}(t)\|^2_{L^2(\mathbb{R}_+)} + \|P_x(t)\|^2_{L^2(\mathbb{R}_+)} \le C(\varepsilon_0 + \mu)\|V_x(t)\|^2_{L^2(\mathbb{R}_+)} + C\delta^2(1+t)^{-2} + C\delta \mathrm{e}^{-c(1+t)}. \tag{3.20}$$

The inequality (3.5) with $k = l = 0$ gives

$$\|V_x(t)\|^2_{L^2(\mathbb{R}_+)} \le 3(\|P_{xx}(t)\|^2_{L^2(\mathbb{R}_+)} + \|P(t)\|^2_{L^2(\mathbb{R}_+)} + \|R_2(t)\|^2_{L^2(\mathbb{R}_+)}). \tag{3.21}$$

It follows from (3.20) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|V_x(t)\|^2_{L^2(\mathbb{R}_+)} + \|\sqrt{\tilde{u}_x}V_x(t)\|^2_{L^2(\mathbb{R}_+)} + \|P_{xx}(t)\|^2_{L^2(\mathbb{R}_+)} + \|P_x(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$\leq C\|P(t)\|^2_{L^2(\mathbb{R}_+)} + C\delta^2(1+t)^{-2} + C\delta \mathrm{e}^{-c(1+t)}. \tag{3.22}$$

Integrating (3.22) over $[0, t]$ and combining (3.10), we get the desired estimate (3.15). ■

Combining (3.10), (3.15) and (3.21), we can easily obtain the following corollary.

Corollary 3.10. *Under the same assumptions as Proposition 3.4, there exists a positive constant C such that*

$$\int_0^t \|V_x(t)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau \leq C(\|V_0\|^2_{H^4(\mathbb{R}_+)} + \delta^{\frac{1}{2}}). \tag{3.23}$$

Lemma 3.11. *Under the same assumptions as Proposition 3.4, there exists a positive constant C such that*

$$\|P(t)\|^2_{H^2(\mathbb{R}_+)} \leq C(\|V_0\|^2_{H^4(\mathbb{R}_+)} + \delta^{\frac{1}{2}}).$$

Proof. Rewriting $(3.2)_2$ in the form

$$P_{xx} - P = V_x - R_1$$

and squaring this equation, we get

$$P^2_{xx} + 2P^2_x + P^2 - 2\{P_x P\}_x = V^2_x + R^2_1 - 2V_x R_1. \tag{3.24}$$

Integrating (3.24) over $\mathbb{R}_+$, from $P_x(0, t) = 0$, combining (3.23) and Corollary 2.5 (i), we obtain

$$\|P(t)\|^2_{H^2(\mathbb{R}_+)} \leq 2(\|V_x(t)\|^2_{L^2(\mathbb{R}_+)} + \|R_1(t)\|^2_{L^2(\mathbb{R}_+)})$$
$$\leq C(\|V_0\|^2_{H^4(\mathbb{R}_+)} + \delta^{\frac{1}{2}}), \tag{3.25}$$

which completes the proof of Lemma 3.11. ■

Lemma 3.12. *Under the same assumptions as Proposition 3.4, there exists a positive constant C such that*

$$\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)} + \|P_{xxx}(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$+ \int_0^t (\|\sqrt{\tilde{u}_x}V_{xx}(\tau)\|^2_{L^2(\mathbb{R}_+)} + \|P_{xx}(\tau)\|^2_{H^1(\mathbb{R}_+)} + \|V_{xx}(\tau)\|^2_{L^2(\mathbb{R}_+)})\,\mathrm{d}\tau$$
$$\leq C(\|V_0\|^2_{H^4(\mathbb{R}_+)} + \delta^{\frac{1}{2}}). \tag{3.26}$$

Proof. We can get from $\partial^2_x(3.2)_1 \times V_{xx} - \partial_x(3.2)_2 \times P_{xxx}$ that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}V^2_{xx} + (f(V+\tilde{u}) - f(\tilde{u}))_{xxx}V_{xx} + P^2_{xxx} + P^2_{xx} - \{P_{xx}P_x\}_x$$
$$= R_{1xx}V_{xx} - R_{2x}P_{xxx}. \tag{3.27}$$

Integrating (3.27) over $\mathbb{R}_+$ and further employing Lemma 3.5 (A3), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}+\frac{5\alpha}{2}\|\sqrt{\tilde{u}_x}V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}+\|P_{xxx}(t)\|^2_{L^2(\mathbb{R}_+)}+\|P_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}\\
&\quad\le \frac{f'(u_-)}{2}V^2_{xx}(0,t)+C\int_{\mathbb{R}_+}|V_{xx}|\big\{(|\tilde{u}_{xxx}|+|\tilde{u}_x|^3+\tilde{u}_x|\tilde{u}_{xx}|)|V|\\
&\qquad\qquad +(|\tilde{u}_{xx}|+\tilde{u}_x^2+V_x^2)|V_x|+|V_x||V_{xx}|\big\}\,\mathrm{d}x\\
&\qquad+\int_{\mathbb{R}_+}(|R_{1xx}V_{xx}|+|R_{2x}P_{xxx}|)\,\mathrm{d}x. \qquad (3.28)
\end{aligned}$$

The terms on the right-hand side of (3.28) can be estimated as follows. By using Lemma 3.6 (B3), the first term can be estimated as

$$\frac{f'(u_-)}{2}V^2_{xx}(0,t)\le C\delta \mathrm{e}^{-c(1+t)}+\frac{1}{4}\|P_{xxx}(t)\|^2_{L^2(\mathbb{R}_+)}+C\|P_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}. \qquad (3.29)$$

Using the Cauchy inequality and (2.1), we have

$$\begin{aligned}
&C\int_{\mathbb{R}_+}|V_{xx}|\{(|\tilde{u}_{xxx}|+|\tilde{u}_x|^3+\tilde{u}_x|\tilde{u}_{xx}|)|V|+(|\tilde{u}_{xx}|+\tilde{u}_x^2+V_x^2)|V_x|+|V_x||V_{xx}|\}\,\mathrm{d}x\\
&\quad\le(\|V_x\|_{L^\infty(\mathbb{R}_+)}+\mu)\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}+C\mu^{-1}\|V_x(t)\|^2_{L^2(\mathbb{R}_+)}\\
&\qquad+C\mu^{-1}\|\tilde{u}_{xxx}(t)\|^2_{L^\infty(\mathbb{R}_+)}\|V(t)\|^2_{L^2(\mathbb{R}_+)}\\
&\quad\le C(\varepsilon_0+\mu)\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}+C\mu^{-1}\|V_x(t)\|^2_{L^2(\mathbb{R}_+)}+C\delta^2(1+t)^{-3}, \qquad (3.30)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}_+}|R_{1xx}V_{xx}|+|R_{2x}P_{xxx}|\,\mathrm{d}x &\le \mu(\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}+\|P_{xxx}(t)\|^2_{L^2(\mathbb{R}_+)})\\
&\quad+C\mu^{-1}(\|R_{1xx}(t)\|^2_{L^2(\mathbb{R}_+)}+\|R_{2x}(t)\|^2_{L^2(\mathbb{R}_+)})\\
&\le \mu(\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}+\|P_{xxx}(t)\|^2_{L^2(\mathbb{R}_+)})\\
&\quad+C\mu^{-1}\delta^2(1+t)^{-\frac{7}{2}}+C\mu^{-1}\delta \mathrm{e}^{-c(1+t)}. \qquad (3.31)
\end{aligned}$$

Substituting (3.29)–(3.31) into (3.28) and choosing small but fixed μ, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}+\|\sqrt{\tilde{u}_x}V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}+\|P_{xxx}(t)\|^2_{L^2(\mathbb{R}_+)}+\|P_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}\\
&\quad\le(\varepsilon_0+\mu)\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}+C(\|V_x(t)\|^2_{L^2(\mathbb{R}_+)}+\|P_{xx}(t)\|^2_{L^2(\mathbb{R}_+)})\\
&\qquad+C\delta^2(1+t)^{-3}+C\delta \mathrm{e}^{-c(1+t)}. \qquad (3.32)
\end{aligned}$$

From (3.5) with $k=1$ and $l=0$, it follows that

$$\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}\le 3(\|P_{xxx}(t)\|^2_{L^2(\mathbb{R}_+)}+\|P_x(t)\|^2_{L^2(\mathbb{R}_+)}+\|R_{2x}(t)\|^2_{L^2(\mathbb{R}_+)}). \qquad (3.33)$$

We substitute (3.33) into (3.32) and then integrate the resulting inequality over $[0,t]$. Consequently, combining (3.15) and (3.23), for some small δ, μ and ε_0, we obtain

$$\begin{aligned}
&\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}+\int_0^t(\|\sqrt{\tilde{u}_x}V_{xx}(\tau)\|^2_{L^2(\mathbb{R}_+)}+\|P_{xxx}(\tau)\|^2_{L^2(\mathbb{R}_+)}+\|P_{xx}(\tau)\|^2_{L^2(\mathbb{R}_+)})\,\mathrm{d}\tau\\
&\quad\le C(\|V_0\|^2_{H^4(\mathbb{R}_+)}+\delta^{\frac{1}{2}}).
\end{aligned}$$

Using relation (3.33) again, we can deduce that

$$\int_0^t \|V_{xx}(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau \le C(\|V_0\|^2_{H^4(\mathbb{R}_+)} + \delta^{\frac{1}{2}}).$$

On the other hand, (3.6) with $k = 1$ and $l = 0$ gives

$$\|P_{xxx}\|^2_{L^2(\mathbb{R}_+)} \le C(\|P_{xx}\|^2_{L^2(\mathbb{R}_+)} + \|V_{xx}\|^2_{L^2(\mathbb{R}_+)} + \|R_{2x}\|^2_{L^2(\mathbb{R}_+)}),$$

which completes the proof of (3.26). ∎

Lemma 3.13. *Under the same assumptions as Proposition 3.4, there exists a positive constant C such that*

$$\begin{aligned}\|V_t(t)\|^2_{H^1(\mathbb{R}_+)} + \int_0^t (\|V_t(\tau)\|^2_{H^1(\mathbb{R}_+)} + \|P_t(\tau)\|^2_{H^2(\mathbb{R}_+)})\,\mathrm{d}\tau \\ \le C(\|V_0\|^2_{H^4(\mathbb{R}_+)} + \delta^{\frac{1}{2}}).\end{aligned} \tag{3.34}$$

Proof. Equation $(3.2)_1$ gives

$$\begin{aligned}&\int_{\mathbb{R}_+} V_t^2\,\mathrm{d}x \le C\int_{\mathbb{R}_+} (V_x^2 + \tilde{u}_x^2 V^2 + P_x^2 + R_1^2)\,\mathrm{d}x,\\ &\int_{\mathbb{R}_+} V_{xt}^2\,\mathrm{d}x \le C\int_{\mathbb{R}_+} (V_{xx}^2 + V_x^4 + \tilde{u}_x^2 V_x^2 + \tilde{u}_{xx}^2 V^2 + \tilde{u}_x^4 V^2 + P_{xx}^2 + R_{1x}^2)\,\mathrm{d}x.\end{aligned} \tag{3.35}$$

It follows that

$$\|V_t(t)\|^2_{L^2(\mathbb{R}_+)} \le \|V(t)\|^2_{H^1(\mathbb{R}_+)} + \|P_x(t)\|^2_{L^2(\mathbb{R}_+)} + C\delta \le C(\|V_0\|^2_{H^4(\mathbb{R}_+)} + \delta^{\frac{1}{2}}),$$
$$\|V_{xt}(t)\|^2_{L^2(\mathbb{R}_+)} \le \|V(t)\|^2_{H^2(\mathbb{R}_+)} + \|P_{xx}(t)\|^2_{L^2(\mathbb{R}_+)} + C\delta \le C(\|V_0\|^2_{H^4(\mathbb{R}_+)} + \delta^{\frac{1}{2}})$$

and

$$\begin{aligned}&\int_0^t \|V_t(\tau)\|^2_{H^1(\mathbb{R}_+)}\,\mathrm{d}\tau\\ &\quad\le C\int_0^t (\|V_x(\tau)\|^2_{H^1(\mathbb{R}_+)} + \|\sqrt{\tilde{u}_x}V(\tau)\|^2_{L^2(\mathbb{R}_+)} + \|P_x(\tau)\|^2_{H^1(\mathbb{R}_+)})\,\mathrm{d}\tau + C\delta\\ &\quad\le C(\|V_0\|^2_{H^4(\mathbb{R}_+)} + \delta^{\frac{1}{2}}).\end{aligned}$$

From $(3.2)_2$ and $P_{xt}(0,t) = 0$, it is easy to obtain

$$\begin{aligned}\|P_t(t)\|^2_{H^2(\mathbb{R}_+)} &\le C\|V_{xt}(t)\|^2_{L^2(\mathbb{R}_+)} + C\|R_{2t}(t)\|^2_{L^2(\mathbb{R}_+)}\\ &\le C\|V_{xt}(t)\|^2_{L^2(\mathbb{R}_+)} + C\delta(1+t)^{-\frac{7}{2}}.\end{aligned} \tag{3.36}$$

Integrating (3.36) over $[0,t]$, we can obtain (3.34). Therefore, we complete the proof of Lemma 3.13. ∎

Lemma 3.14. *Under the same assumptions as Proposition 3.4, there exists a positive constant C such that*

$$\|V_{xxt}(t)\|^2_{L^2(\mathbb{R}_+)} + \int_0^t (\|P_{xxxt}(\tau)\|^2_{L^2(\mathbb{R}_+)} + \|P_{xxt}(\tau)\|^2_{L^2(\mathbb{R}_+)} + \|V_{xxt}(\tau)\|^2_{L^2(\mathbb{R}_+)})\,\mathrm{d}\tau$$
$$\leq C(\|V_0\|^2_{H^4(\mathbb{R}_+)} + \delta^{\frac{1}{2}}) + C(\|V_0\|^2_{H^4(\mathbb{R}_+)} + \delta^{\frac{1}{2}}) \int_0^t \|V_{xxx}(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau. \quad (3.37)$$

Proof. We can get from $\partial_{xxt}(3.2)_1 \times V_{xxt} - \partial_{xt}(3.2)_2 \times P_{xxxt}$ that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} V^2_{xxt} + (f(V+\tilde{u}) - f(\tilde{u}))_{xxxt} V_{xxt} - \{P_{xxt}P_{xt}\}_x + P^2_{xxxt} + P^2_{xxt}$$
$$= R_{1xxt}V_{xxt} - R_{2xt}P_{xxxt}. \quad (3.38)$$

Integrating (3.38) over $\mathbb{R}_+$, using Lemma 3.5 (A4), (2.1) and the Cauchy inequality, we can get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|V_{xxt}(t)\|^2_{L^2(\mathbb{R}_+)} + \frac{5\alpha}{2}\|\sqrt{\tilde{u}_x}V_{xxt}(t)\|^2_{L^2(\mathbb{R}_+)} + \|P_{xxxt}(t)\|^2_{L^2(\mathbb{R}_+)} + \|P_{xxt}(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$\leq C V^2_{xxt}(0,t) + (\|V_x(t)\|_{L^\infty(\mathbb{R}_+)} + \mu)\|V_{xxt}(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$+ C\mu^{-1}(\|V_t\|^2_{L^\infty(\mathbb{R}_+)} + \|\tilde{u}_t\|^2_{L^\infty(\mathbb{R}_+)})\|V_{xxx}(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$+ C\mu^{-1}\big(\|V_x\|^6_{L^\infty(\mathbb{R}_+)} + \|\tilde{u}_{xx}(t)\|^2_{L^\infty(\mathbb{R}_+)}\|V_x\|^2_{L^\infty(\mathbb{R}_+)}$$
$$+ \|\tilde{u}_{xxx}\|^2_{L^\infty(\mathbb{R}_+)}\big)\|V_t(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$+ C\mu^{-1}\big(\|V_x\|^4_{L^\infty(\mathbb{R}_+)}\|\tilde{u}_t\|^2_{L^\infty(\mathbb{R}_+)} + \|\tilde{u}_{xt}\|^2_{L^\infty(\mathbb{R}_+)}\|V_x(t)\|^2_{L^\infty(\mathbb{R}_+)}$$
$$+ \|\tilde{u}_{xxt}\|^2_{L^\infty(\mathbb{R}_+)}\big)\|V_x(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$+ C\mu^{-1}(\|V_x\|^4_{L^\infty(\mathbb{R}_+)} + \|\tilde{u}_{xx}\|^2_{L^\infty(\mathbb{R}_+)})\|V_{xt}(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$+ \|\tilde{u}_{xxxxt}\|^2_{L^\infty(\mathbb{R}_+)}\|V(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$+ C\mu^{-1}\big(\|V_{xt}\|^2_{L^\infty(\mathbb{R}_+)} + \|\tilde{u}_{xt}\|^2_{L^\infty(\mathbb{R}_+)} + \|V_tV_x\|_{L^\infty(\mathbb{R}_+)} + \|\tilde{u}_tV_x\|_{L^\infty(\mathbb{R}_+)}$$
$$+ \|V_t\tilde{u}_x\|_{L^\infty(\mathbb{R}_+)}\big)\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)} + \mu\|P_{xxxt}(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$+ C\mu^{-1}(\|R_{1xxt}(t)\|^2_{L^2(\mathbb{R}_+)} + \|R_{2xt}(t)\|^2_{L^2(\mathbb{R}_+)}). \quad (3.39)$$

By Lemma 3.6 (B5), the first term on the right-hand side of (3.39) is bounded by

$$V^2_{xxt}(0,t) \leq C\delta \mathrm{e}^{-c(1+t)} + \frac{1}{4}\|P_{xxxt}(t)\|^2_{L^2(\mathbb{R}_+)} + C\|P_{xxt}(t)\|^2_{L^2(\mathbb{R}_+)}. \quad (3.40)$$

From (3.5) with $k=1$, $l=1$, the second term on the right-hand side is bounded by

$$\|V_{xxt}\|^2_{L^2(\mathbb{R}_+)} \leq 3(\|P_{xxxt}\|^2_{L^2(\mathbb{R}_+)} + \|P_{xt}\|^2_{L^2(\mathbb{R}_+)} + \|R_{2xt}\|^2_{L^2(\mathbb{R}_+)}). \quad (3.41)$$

Substituting (3.40)–(3.41) into (3.39), and then integrating the resulting inequality over $[0,t]$, choosing small but fixed μ, we get

$$\begin{aligned}
&\|V_{xxt}(t)\|^2_{L^2(\mathbb{R}_+)}\\
&\quad+\int_0^t(\|\sqrt{\tilde{u}_x}V_{xxt}(\tau)\|^2_{L^2(\mathbb{R}_+)}+\|P_{xxxt}(\tau)\|^2_{L^2(\mathbb{R}_+)}+\|P_{xxt}(\tau)\|^2_{L^2(\mathbb{R}_+)})\,\mathrm{d}\tau\\
&\quad\le C\|V_0\|^2_{H^4(\mathbb{R}_+)}+C\int_0^t(\|V_t(\tau)\|^2_{H^1(\mathbb{R}_+)}+\|V_x(\tau)\|^2_{H^1(\mathbb{R}_+)}+\|P_{xt}(\tau)\|^2_{H^1(\mathbb{R}_+)})\,\mathrm{d}\tau\\
&\qquad+C(\|V_0\|^2_{H^4(\mathbb{R}_+)}+\delta^{\frac{1}{2}})\int_0^t\|V_{xxx}(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau+C\delta.
\end{aligned}$$

By using (3.41) again, and combining (3.23), (3.26) and (3.34), the desired estimate (3.37) can be proved. ■

Lemma 3.15. *Under the same assumptions as Proposition 3.4, there exists a positive constant C such that*

$$\|V_{xxx}(t)\|^2_{L^2(\mathbb{R}_+)}+\int_0^t\|V_{xxx}(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau\le C(\|V_0\|^2_{H^4(\mathbb{R}_+)}+\delta^{\frac{1}{2}}).\tag{3.42}$$

Proof. By $f'(u_+)>0$ and using Taylor's expansion, we can get from $\partial_x^2(3.2)_1$ that

$$\begin{aligned}
f'(u_+)V_{xxx}=&-f''(\xi)(V+\tilde{u}-u_+)V_{xxx}-f'''(V+\tilde{u})(V_x+\tilde{u}_x)^3\\
&-3f''(V+\tilde{u})(V_x+\tilde{u}_x)(V_{xx}+\tilde{u}_{xx})-f'(V+\tilde{u})\tilde{u}_{xxx}\\
&+f'''(\tilde{u})\tilde{u}_x^3+3f''(\tilde{u})\tilde{u}_x\tilde{u}_{xx}+f'(\tilde{u})\tilde{u}_{xxx}\\
&+R_{1xx}-P_{xxx}-V_{xxt},
\end{aligned}\tag{3.43}$$

where ξ is between $V+\tilde{u}$ and u_+.

Squaring (3.43) and then integrating the resulting equation over $\mathbb{R}_+$, we have

$$\begin{aligned}
&\int_{\mathbb{R}_+}(f'(u_+)V_{xxx})^2\,\mathrm{d}x\\
&\quad\le C\int_{\mathbb{R}_+}(V+\tilde{u}-u_+)^2V_{xxx}^2\,\mathrm{d}x+C\int_{\mathbb{R}_+}(\tilde{u}_x^6V^2+\tilde{u}_x^2\tilde{u}_{xx}^2V^2+\tilde{u}_{xxx}^2V^2)\,\mathrm{d}x\\
&\qquad+C\int_{\mathbb{R}_+}\big(V_x^6+\tilde{u}_x^2V_x^4+\tilde{u}_x^4V_x^2+V_x^2V_{xx}^2+\tilde{u}_{xx}^2V_x^2+\tilde{u}_x^2V_{xx}^2+R_{1xx}^2\\
&\qquad\qquad+P_{xxx}^2+V_{xxt}^2\big)\,\mathrm{d}x\\
&\quad\le C(\varepsilon_0+\delta)\int_{\mathbb{R}_+}V_{xxx}^2\,\mathrm{d}x\\
&\qquad+C(\|\tilde{u}_x\|^6_{L^\infty(\mathbb{R}_+)}+\|\tilde{u}_x\tilde{u}_{xx}\|^2_{L^\infty(\mathbb{R}_+)}+\|\tilde{u}_{xxx}\|^2_{L^\infty(\mathbb{R}_+)})\|V(t)\|^2_{L^2(\mathbb{R}_+)}\\
&\qquad+C(\|V_x\|^4_{L^\infty(\mathbb{R}_+)}+\|\tilde{u}_x\|^4_{L^\infty(\mathbb{R}_+)}+\|\tilde{u}_{xx}\|^2_{L^\infty(\mathbb{R}_+)})\|V_x(t)\|^2_{L^2(\mathbb{R}_+)}\\
&\qquad+C\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}+C\|R_{1xx}(t)\|^2_{L^2(\mathbb{R}_+)}+C\|P_{xxx}(t)\|^2_{L^2(\mathbb{R}_+)}\\
&\qquad+C\|V_{xxt}(t)\|^2_{L^2(\mathbb{R}_+)}.
\end{aligned}\tag{3.44}$$

Combining (3.26) and (3.37), we can obtain the desired inequality (3.42) for some small δ and ε_0. ■

Equation (3.6) gives

$$\|P_{xxxx}\|^2_{L^2(\mathbb{R}_+)} \le C(\|P_{xx}\|^2_{L^2(\mathbb{R}_+)} + \|V_{xxx}\|^2_{L^2(\mathbb{R}_+)} + \|R_{2xx}\|^2_{L^2(\mathbb{R}_+)}).$$

Consequently, we have the following corollary.

Corollary 3.16. *Under the same assumptions as Proposition 3.4, there exists a positive constant C such that*

$$\|P_{xxxx}(t)\|^2_{L^2(\mathbb{R}_+)} + \int_0^t \|P_{xxxx}(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau \le C(\|V_0\|^2_{H^4(\mathbb{R}_+)} + \delta^{\frac{1}{2}}).$$

Lemma 3.17. *Under the same assumptions as Proposition 3.4, there exists a positive constant C such that*

$$\int_0^t (\|V_{tt}(\tau)\|^2_{H^1(\mathbb{R}_+)} + \|P_{tt}(\tau)\|^2_{H^2(\mathbb{R}_+)})\,\mathrm{d}\tau \le C(\|V_0\|^2_{H^4(\mathbb{R}_+)} + \delta^{\frac{1}{2}}). \tag{3.45}$$

Proof. We can get from ∂_t(3.2) and $\partial_t\partial_x$(3.2) that

$$\begin{aligned}\int_{\mathbb{R}_+} V_{tt}^2\,\mathrm{d}x &\le C\int_{\mathbb{R}} (V_x^2V_t^2 + \tilde{u}_t^2V_x^2 + \tilde{u}_x^2V_t^2 + \tilde{u}_x^2\tilde{u}_t^2V^2 + V_{xt}^2 + \tilde{u}_{xt}^2V^2 + P_{xt}^2 + R_{1t}^2)\,\mathrm{d}x \\ &\le C(\|V_t\|^2_{L^\infty(\mathbb{R}_+)} + \|\tilde{u}_t\|^2_{L^\infty(\mathbb{R}_+)})\|V_x\|^2_{L^2(\mathbb{R}_+)} + C\|\tilde{u}_x\|^2_{L^\infty}\|V_t\|^2_{L^2(\mathbb{R}_+)} \\ &\quad + C\|V_{xt}\|^2_{L^2(\mathbb{R}_+)} + C(\|\tilde{u}_x\tilde{u}_t\|^2_{L^\infty(\mathbb{R}_+)} + \|\tilde{u}_{xt}\|^2_{L^\infty(\mathbb{R}_+)})\|V\|^2_{L^2(\mathbb{R}_+)} \\ &\quad + C\|P_{xt}\|^2_{L^2(\mathbb{R}_+)} + C\|R_{1t}\|^2_{L^2(\mathbb{R}_+)}\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}_+} V_{xtt}^2\,\mathrm{d}x \le C\big(&\|V_x\|^2_{L^\infty(\mathbb{R}_+)} + \|\tilde{u}_xV_x\|^2_{L^\infty(\mathbb{R}_+)} + \|\tilde{u}_x\|^4_{L^\infty(\mathbb{R}_+)} \\ &+ \|\tilde{u}_{xx}\|^2_{L^\infty(\mathbb{R}_+)}\big)\|V_t(t)\|^2_{L^2(\mathbb{R}_+)} \\ &+ C(\|\tilde{u}_tV_x\|^2_{L^\infty(\mathbb{R}_+)} + \|\tilde{u}_x\tilde{u}_t\|^2_{L^\infty(\mathbb{R}_+)})\|V_x(t)\|^2_{L^2(\mathbb{R}_+)} \\ &+ C\|V_{xt}(t)\|^2_{H^1(\mathbb{R}_+)} + C\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)} \\ &+ C\big(\|\tilde{u}_t\tilde{u}_{xx}\|^2_{L^\infty(\mathbb{R}_+)} + \|\tilde{u}_{xxt}\|^2_{L^\infty(\mathbb{R}_+)} + \|\tilde{u}_x\tilde{u}_{xt}\|^2_{L^\infty(\mathbb{R}_+)} \\ &\quad + \|\tilde{u}_t\tilde{u}_x^2\|^2_{L^\infty(\mathbb{R}_+)}\big)\|V(t)\|^2_{L^2(\mathbb{R}_+)} \\ &+ C\|P_{xxt}(t)\|^2_{L^2(\mathbb{R}_+)} + C\|R_{1xt}(t)\|^2_{L^2(\mathbb{R}_+)}.\end{aligned}$$

On the other hand, we can get from $(3.2)_2$ and $P_{xtt}(0,t) = 0$ that

$$\|P_{tt}(t)\|^2_{H^2(\mathbb{R}_+)} \le C(\|V_{xtt}(t)\|^2_{L^2(\mathbb{R}_+)} + \|V_{tt}(t)\|^2_{L^2(\mathbb{R}_+)}).$$

Integrating the above three inequalities over $[0,t]$ and using Corollary 2.5, the desired estimate (3.45) can be obtained. ■

Lemma 3.18. *Under the same assumptions as Proposition 3.4, there exists a positive constant C such that*

$$\|V_{xxtt}(t)\|^2_{L^2(\mathbb{R}_+)}+\int_0^t(\|V_{xxtt}(\tau)\|^2_{L^2(\mathbb{R}_+)}+\|P_{xxxtt}(\tau)\|^2_{L^2(\mathbb{R}_+)}+\|P_{xxtt}(\tau)\|^2_{L^2(\mathbb{R}_+)})\,\mathrm{d}\tau$$
$$\leq C(\|V_0\|^2_{H^4(\mathbb{R}_+)}+\delta^{\frac{1}{2}})+C(\|V_0\|^2_{H^4(\mathbb{R}_+)}+\delta^2)\int_0^t\|V_{xxxt}(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau. \quad (3.46)$$

Proof. We can get from $\partial_x^2\partial_t^2(3.2)_1\times V_{xxtt}-\partial_x\partial_t^2(3.2)_2\times P_{xxxtt}$ that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}V^2_{xxtt}+(f(V+\tilde{u})-f(\tilde{u}))_{xxxtt}V_{xxtt}+P^2_{xxxtt}+P^2_{xxtt}-\{P_{xxtt}P_{xtt}\}_x$$
$$=R_{1xxtt}V_{xxtt}-P_{xxxtt}R_{2xtt}. \quad (3.47)$$

Integrating (3.47) over $\mathbb{R}_+$, using Lemma 3.5 (A5) and Lemma 3.6 (B7), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|V_{xxtt}(t)\|^2_{L^2(\mathbb{R}_+)}+\frac{5\alpha}{2}\|\sqrt{\tilde{u}_x}V_{xxtt}(t)\|^2_{L^2(\mathbb{R}_+)}+\|P_{xxxtt}(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$+\|P_{xxtt}(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$\leq CV^2_{xxtt}(0,t)+(\|V_x\|_{L^\infty(\mathbb{R}_+)}+\mu)\|V_{xxtt}(t)\|^2_{L^2(\mathbb{R}_+)}+\mu\|P_{xxxtt}(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$+C\mu^{-1}\|\tilde{u}_{xxxtt}\|^2_{L^\infty(\mathbb{R}_+)}\|V(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$+C\mu^{-1}(\|V_x(t)\|^2_{H^2(\mathbb{R}_+)}+\|V_{tt}\|^2_{H^1(\mathbb{R}_+)}+\|V_t(t)\|^2_{H^2(\mathbb{R}_+)})$$
$$+C\mu^{-1}(\|V_t\|^2_{L^\infty(\mathbb{R}_+)}+\|\tilde{u}_t\|^2_{L^\infty(\mathbb{R}_+)})\|V_{xxxt}\|^2_{L^2(\mathbb{R}_+)}$$
$$+C\mu^{-1}(\|R_{1xxtt}(t)\|^2_{L^2(\mathbb{R}_+)}+\|R_{2xtt}(t)\|^2_{L^2(\mathbb{R}_+)})$$
$$\leq C(\varepsilon_0+\mu)\|V_{xxtt}(t)\|^2_{L^2(\mathbb{R}_+)}+2\mu\|P_{xxxtt}(t)\|^2_{L^2(\mathbb{R}_+)}+C\delta^2(1+t)^{-5}$$
$$+C\delta\mathrm{e}^{-c(1+t)}+C\|V_x(t)\|^2_{H^2(\mathbb{R}_+)}+C\|V_{tt}(t)\|^2_{H^1(\mathbb{R}_+)}+C\|V_t(t)\|^2_{H^2(\mathbb{R}_+)}$$
$$+C(\|V_t\|^2_{L^\infty(\mathbb{R}_+)}+\|\tilde{u}_t\|^2_{L^\infty(\mathbb{R}_+)})\|V_{xxxt}(t)\|^2_{L^2(\mathbb{R}_+)}+C\|P_{xxtt}(t)\|^2_{L^2(\mathbb{R}_+)}.$$

From (3.5) with $k=1$ and $l=2$, the first term on the right-hand side can be treated as

$$\|V_{xxtt}(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$\leq 3(\|P_{xxxtt}(t)\|^2_{L^2(\mathbb{R}_+)}+\|P_{xtt}(t)\|^2_{L^2(\mathbb{R}_+)}+\|R_{2xtt}\|^2_{L^2(\mathbb{R}_+)}). \quad (3.48)$$

It then follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|V_{xxtt}(t)\|^2_{L^2(\mathbb{R}_+)}+\|\sqrt{\tilde{u}_x}V_{xxtt}(t)\|^2_{L^2(\mathbb{R}_+)}+\|P_{xxxtt}(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$+\|P_{xxtt}(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$\leq C(\|P_{xtt}(t)\|^2_{H^1(\mathbb{R}_+)}+\|V_x(t)\|^2_{H^2(\mathbb{R}_+)}+\|V_{tt}(t)\|^2_{H^1(\mathbb{R}_+)}+\|V_t(t)\|^2_{H^2(\mathbb{R}_+)})$$
$$+C(\|V_t\|^2_{L^\infty(\mathbb{R}_+)}+\|\tilde{u}_t\|^2_{L^\infty(\mathbb{R}_+)})\|V_{xxxt}\|^2_{L^2(\mathbb{R}_+)}+C\delta^2(1+t)^{-5}$$
$$+C\delta\mathrm{e}^{-c(1+t)}. \quad (3.49)$$

Integrating (3.49) over $[0,t]$ and using (3.48) again, estimate (3.46) can be proved. ∎

Lemma 3.19. *Under the same assumptions as Proposition 3.4, there exists a positive constant C such that*

$$\|V_{xxxt}(t)\|^2_{L^2(\mathbb{R}_+)} + \int_0^t \|V_{xxxt}(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau \le C(\|V_0\|^2_{H^4(\mathbb{R}_+)} + \delta^{\frac{1}{2}}). \tag{3.50}$$

Proof. By applying Taylor's expansion, we can get from $\partial_x^2\partial_t(3.2)_1$ that

$$\begin{aligned} f'(u_+)V_{xxxt} =& - f''(\xi)(V+\tilde{u}-u_+)V_{xxxt} - f^{(4)}(V+\tilde{u})(V_t+\tilde{u}_t)(V_x+\tilde{u}_x)^3 \\ & - 3f^{(3)}(V+\tilde{u})(V_x+\tilde{u}_x)^2(V_{xt}+\tilde{u}_{xt}) \\ & - 3f^{(3)}(V+\tilde{u})(V_t+\tilde{u}_t)(V_x+\tilde{u}_x)(V_{xx}+\tilde{u}_{xx}) \\ & - 3f''(V+\tilde{u})(V_{xt}+\tilde{u}_{xt})(V_{xx}+\tilde{u}_{xx}) \\ & - 3f''(V+\tilde{u})(V_x+\tilde{u}_x)(V_{xxt}+\tilde{u}_{xxt}) \\ & - f''(V+\tilde{u})(V_t+\tilde{u}_t)(V_{xxx}+\tilde{u}_{xxx}) - f'(V+\tilde{u})\tilde{u}_{xxxt} \\ & - f^{(4)}(\tilde{u})\tilde{u}_t\tilde{u}_x^3 - 3f^{(3)}(\tilde{u})\tilde{u}_x^2\tilde{u}_{xt} - 3f^{(3)}\tilde{u}_t\tilde{u}_x\tilde{u}_{xx} - 3f''(\tilde{u})\tilde{u}_{xt}\tilde{u}_{xx} \\ & - 3f''(\tilde{u})\tilde{u}_x\tilde{u}_{xxt} - f''(\tilde{u})\tilde{u}_t\tilde{u}_{xxx} - f'(\tilde{u})\tilde{u}_{xxxt} \\ & - V_{xxtt} - P_{xxxt} + R_{1xxt}, \end{aligned} \tag{3.51}$$

where ξ is between $V+\tilde{u}$ and u_+.

Square (3.51) and then integrate the resulting equation over $\mathbb{R}_+$. Consequently, we choose small ε_0 and δ such that the first term

$$\int_{\mathbb{R}_+} (V+\tilde{u}-u_+)^2 V_{xxxt}^2\,\mathrm{d}x \le C(\varepsilon_0+\delta)^2 \int_{\mathbb{R}_+} V_{xxxt}^2\,\mathrm{d}x$$

on the right-hand side of (3.51) can be absorbed into the left-hand side of (3.51). Then, we get

$$\begin{aligned} \int_{\mathbb{R}_+} V_{xxxt}^2\,\mathrm{d}x \le\,& C\big(\|V_x\|^4_{L^\infty(\mathbb{R}_+)} + \|\tilde{u}_x\|^4_{L^\infty(\mathbb{R}_+)} + \|\tilde{u}_x\tilde{u}_{xx}\|^2_{L^\infty(\mathbb{R}_+)} \\ & + \|\tilde{u}_{xxx}\|^2_{L^\infty(\mathbb{R}_+)}\big)\|V_t(t)\|^2_{L^2(\mathbb{R}_+)} \\ & + C\big(\|\tilde{u}_tV_x^2\|^2_{L^\infty(\mathbb{R}_+)} + \|\tilde{u}_{xt}V_x\|^2_{L^\infty(\mathbb{R}_+)} + \|\tilde{u}_{xx}V_t\|^2_{L^\infty(\mathbb{R}_+)} \\ & + \|\tilde{u}_{xxt}\|^2_{L^\infty(\mathbb{R}_+)}\big)\|V_x(t)\|^2_{L^2(\mathbb{R}_+)} \\ & + C\big(\|\tilde{u}_t\tilde{u}_x^3\|^2_{L^\infty(\mathbb{R}_+)} + \|\tilde{u}_{xt}\tilde{u}_x^2\|^2_{L^\infty(\mathbb{R}_+)} + \|\tilde{u}_t\tilde{u}_x\tilde{u}_{xx}\|^2_{L^\infty(\mathbb{R}_+)} \\ & + \|\tilde{u}_{xxxt}\|^2_{L^\infty(\mathbb{R}_+)}\big)\|V(t)\|^2_{L^2(\mathbb{R}_+)} \\ & + C\big(\|\tilde{u}_{xx}\tilde{u}_{xt}\|^2_{L^\infty(\mathbb{R}_+)} + \|\tilde{u}_{xxt}\tilde{u}_x\|^2_{L^\infty(\mathbb{R}_+)} \\ & + \|\tilde{u}_{xxx}\tilde{u}_t\|^2_{L^\infty(\mathbb{R}_+)}\big)\|V(t)\|^2_{L^2(\mathbb{R}_+)} + C\|V_{xt}(t)\|^2_{H^1(\mathbb{R}_+)} \\ & + C\big(\|V_{xx}(t)\|^2_{H^1(\mathbb{R}_+)} + \|V_{xxtt}(t)\|^2_{L^2(\mathbb{R}_+)} + \|P_{xxtt}(t)\|^2_{L^2(\mathbb{R}_+)} \\ & + \|R_{1xxt}(t)\|^2_{L^2(\mathbb{R}_+)}\big). \end{aligned}$$

Combining Lemmas 3.8–3.18, we can easily obtain the desired estimate (3.50). ∎

Lemma 3.20. *Under the same assumptions as Proposition 3.4, there exists a positive constant C such that*

$$\|\partial_x^4 V(t)\|_{L^2(\mathbb{R}_+)}^2 + \|\partial_x^5 P(t)\|_{L^2(\mathbb{R}_+)}^2 + \int_0^t (\|\partial_x^4 V(\tau)\|_{L^2(\mathbb{R}_+)}^2 + \|\partial_x^4 P(\tau)\|_{H^1(\mathbb{R}_+)}^2)\,d\tau$$
$$\leq C(\|V_0\|_{H^4(\mathbb{R}_+)}^2 + \delta^{\frac{1}{2}}). \tag{3.52}$$

Proof. Similar to (3.51), we can get from $\partial_x^3(3.2)_1$ that

$$\begin{aligned} f'(u_+)\partial_x^4 V = & -f''(\xi)(V+\tilde{u}-u_+)\partial_x^4 V - f'(V+\tilde{u})\partial_x^4\tilde{u} - f^{(4)}(V+\tilde{u})(V_x+\tilde{u}_x)^4 \\ & - 6f^{(3)}(V+\tilde{u})(V_x+\tilde{u}_x)^2(V_{xx}+\tilde{u}_{xx}) - 3f''(V+\tilde{u})(V_{xx}+\tilde{u}_{xx})^2 \\ & - 4f''(V+\tilde{u})(V_x+\tilde{u}_x)(V_{xxx}+\tilde{u}_{xxx}) - f^{(4)}(\tilde{u})\tilde{u}_x^4 - 6f^{(3)}(\tilde{u})\tilde{u}_x^2\tilde{u}_{xx} \\ & - 3f''(\tilde{u})\tilde{u}_{xx}^2 - 4f''(\tilde{u})\tilde{u}_x\tilde{u}_{xxx} - f'(\tilde{u})\tilde{u}_{xxxx} - V_{xxxt} \\ & - \partial_x^4 P + R_{1xxx}, \end{aligned} \tag{3.53}$$

where ξ is between $V+\tilde{u}$ and u_+.

Squaring (3.53) and then integrating the resulting equation over $\mathbb{R}_+$, we obtain

$$\begin{aligned} \int_{\mathbb{R}_+} (\partial_x^4 V)^2\,dx \leq\ & C\big(\|V_x\|_{L^\infty(\mathbb{R}_+)}^6 + \|\tilde{u}_x\|_{L^\infty(\mathbb{R}_+)}^6 + \|\tilde{u}_{xxx}\|_{L^\infty(\mathbb{R}_+)}^2 \\ & + \|\tilde{u}_x\tilde{u}_{xx}\|_{L^\infty(\mathbb{R}_+)}^2 + \|\tilde{u}_{xx}V_x\|_{L^\infty(\mathbb{R}_+)}^2\big)\|V_x(t)\|_{L^2(\mathbb{R}_+)}^2 \\ & + C\big(\|\partial_x^4\tilde{u}\|_{L^\infty(\mathbb{R}_+)}^2 + \|\tilde{u}_x\|_{L^\infty(\mathbb{R}_+)}^8 + \|\tilde{u}_x\tilde{u}_{xxx}\|_{L^\infty(\mathbb{R}_+)}^2 \\ & + \|\tilde{u}_{xx}\|_{L^\infty(\mathbb{R}_+)}^4 + \|\tilde{u}_x^2\tilde{u}_{xx}\|_{L^\infty(\mathbb{R}_+)}^2\big)\|V(t)\|_{L^2(\mathbb{R}_+)}^2 \\ & + C(\|V_{xx}(t)\|_{H^1(\mathbb{R}_+)}^2 + \|\partial_x^4 P(t)\|_{L^2(\mathbb{R}_+)}^2 + \|R_{1xxx}(t)\|_{L^2(\mathbb{R}_+)}^2). \end{aligned}$$

On the other hand, utilizing (3.6) with $k=3$ and $l=0$, the fifth derivative of P can be estimated as

$$\|\partial_x^5 P(t)\|_{L^2(\mathbb{R}_+)}^2 \leq C(\|P_{xxx}(t)\|_{L^2(\mathbb{R}_+)}^2 + \|\partial_x^4 V(t)\|_{L^2(\mathbb{R}_+)}^2 + \|R_{2xxx}(t)\|_{L^2(\mathbb{R}_+)}^2).$$

By combining Lemmas 3.9, 3.12, 3.15, 3.19 and Corollaries 3.10, 3.16, the desired estimate (3.52) can be obtained. ∎

Combining Lemmas 3.8–3.20, we finish the proof of the a priori estimates (Proposition 3.4). Next, we devote ourselves to obtaining the decay rate of V by employing the L^1-estimate.

3.1.2. Decay estimates. To give the decay estimates for the perturbation V, we further assume that $V_0 \in L^1(\mathbb{R}_+)$. We define $\phi_\mu(x)$ and $\Phi_\mu(x)$ as follows:

$$\begin{aligned} \phi_\mu(x) &:= (\rho_\mu * \mathrm{sgn})(x) = \int_{-\infty}^{+\infty} \rho_\mu(x-y)\,\mathrm{sgn}(y)\,dy, \\ \Phi_\mu(x) &:= \int_0^x \phi_\mu(y)\,dy, \end{aligned} \tag{3.54}$$

where "sgn" is a usual signature function defined as

$$\operatorname{sgn}(y) := \begin{cases} -1, & y < 0, \\ 0, & y = 0, \\ 1, & y > 0. \end{cases}$$

The symbol ρ_μ denotes the Friedrichs mollifier defined as

$$\rho_\mu(x) := \frac{1}{\mu}\rho\Big(\frac{x}{\mu}\Big),$$

where ρ is a smooth function which has a compact support and satisfies

$$\int_{-\infty}^{+\infty} \rho(x)\,\mathrm{d}x = 1.$$

We here recall the following properties of $\phi_\mu(x)$ and $\Phi_\mu(x)$. The details can be found in [6, 10].

Lemma 3.21. *Suppose that $\phi_\mu(x)$ and $\Phi_\mu(x)$ are defined in* (3.54). *Then $\phi_\mu(x)$ and $\Phi_\mu(x)$ satisfy*

(1) $\lim_{\mu\to 0} \phi_\mu(x) = \operatorname{sgn}(x)$, $x \in \mathbb{R}$,

(2) $\lim_{\mu\to 0} \Phi_\mu(x) = |x|$, $x \in \mathbb{R}$,

(3) $\phi_\mu(0) = 0$,

(4) $\frac{\mathrm{d}}{\mathrm{d}x}\phi_\mu(x) = 2\rho_\mu(x) \geq 0$, $x \in \mathbb{R}$.

By utilizing Lemma 3.21, we can obtain the following L^1-estimate.

Lemma 3.22 (L^1-estimate). *Suppose that $V_0 \in L^1(\mathbb{R}_+) \cap H^3(\mathbb{R}_+)$, then the solution V of problem* (3.2)–(3.4) *satisfies*

$$\|V(t)\|_{L^1(\mathbb{R}_+)} \leq C(\|V_0\|_{L^1(\mathbb{R}_+)} + \delta \log(2+t)). \tag{3.55}$$

Proof. We denote $F(x,t)$ by

$$F(x,t) = -V_x(x,t) + R_2(x,t)$$

and then extend the function $F(x,t)$ such that

$$\widetilde{F}(x,t) := \begin{cases} F(x,t), & x \geq 0, \\ F(-x,t), & x < 0. \end{cases}$$

Then P in $(3.2)_2$ can be solved as

$$\begin{aligned} P(x,t) &= \frac{1}{2}\int_{\mathbb{R}} \mathrm{e}^{-|x-y|} F(y,t)\,\mathrm{d}y \\ &= \frac{1}{2}\int_{\mathbb{R}_+} (\mathrm{e}^{-|x-y|} + \mathrm{e}^{-|x+y|})(-V_x(y,t) + R_2(y,t))\,\mathrm{d}y, \quad x \in \mathbb{R}_+. \end{aligned} \tag{3.56}$$

Differentiating (3.56) with respect to x, we get

$$\begin{aligned}P_x(x,t) &= \frac{1}{2}\int_{\mathbb{R}_+}\Big(\frac{\partial \mathrm{e}^{-|x-y|}}{\partial x}+\frac{\partial \mathrm{e}^{-|x+y|}}{\partial x}\Big)(-V_x(y,t)+R_2(y,t))\,\mathrm{d}y\\ &= \frac{1}{2}\int_{\mathbb{R}_+}\Big(\frac{-\partial \mathrm{e}^{-|x-y|}}{\partial y}+\frac{\partial \mathrm{e}^{-|x+y|}}{\partial y}\Big)(-V_x(y,t)+R_2(y,t))\,\mathrm{d}y\\ &= \frac{1}{2}\int_{\mathbb{R}_+}(\mathrm{e}^{-|x-y|}-\mathrm{e}^{-|x+y|})(-V_{xx}(y,t)+R_{2x}(y,t))\,\mathrm{d}y\\ &= V-\frac{1}{2}\int_{\mathbb{R}_+}(\mathrm{e}^{-|x-y|}-\mathrm{e}^{-|x+y|})(V(y,t)-R_{2x}(y,t))\,\mathrm{d}y.\end{aligned} \tag{3.57}$$

It is easy to verify that $P_x(0,t)=0$. In deriving the last equality of (3.57), we have used the fact that

$$\begin{cases} V(x,t)=\int_{\mathbb{R}_+}(\mathrm{e}^{-|x-y|}-\mathrm{e}^{-|x+y|})(-V_{xx}(y,t)+V(y,t))\,\mathrm{d}y, & x\in\mathbb{R}_+,\\ V(0,t)=0. \end{cases}$$

We define the operator K as

$$K(f)(x)=\frac{1}{2}\int_{\mathbb{R}_+}(\mathrm{e}^{-|x-y|}-\mathrm{e}^{-|x+y|})f(y)\,\mathrm{d}y. \tag{3.58}$$

Then equation (3.57) can be rewritten as

$$P_x=V-KV+KR_{2x}. \tag{3.59}$$

Substituting (3.59) into $(3.2)_1$, we obtain

$$V_t+(f(V+\tilde{u})-f(\tilde{u}))_x+V-KV=R_1-KR_{2x}. \tag{3.60}$$

Multiplying (3.60) by $\phi_\mu(V)$ and then integrating the resulting equation over $\mathbb{R}_+\times[0,t]$, we have

$$\begin{aligned}&\int_{\mathbb{R}_+}\Phi_\mu(V)\,\mathrm{d}x+\int_0^t\int_{\mathbb{R}_+}\phi_\mu(V)(f(V+\tilde{u})-f(\tilde{u}))_x\,\mathrm{d}x\mathrm{d}\tau\\ &\quad+\int_0^t\int_{\mathbb{R}_+}\phi_\mu(V)(V-KV)\,\mathrm{d}x\mathrm{d}\tau\\ &\quad=\int_{\mathbb{R}_+}\Phi_\mu(V_0)\,\mathrm{d}x+\int_0^t\int_{\mathbb{R}_+}\phi_\mu(V)(R_1-KR_{2x})\,\mathrm{d}x\mathrm{d}\tau.\end{aligned}$$

Letting $\mu\to 0$, we can obtain that

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}_+}\phi_\mu(V)(f(V+\tilde{u})-f(\tilde{u}))_x\,\mathrm{d}x\mathrm{d}\tau\\ &\quad=\int_0^t\int_{\mathbb{R}_+}\int_0^V 2\rho_\mu(\eta)(f'(\eta+\tilde{u})-f'(\tilde{u}))\tilde{u}_x\,\mathrm{d}\eta\mathrm{d}x\mathrm{d}\tau\ge 0.\end{aligned}$$

By using Young's inequality

$$\|f * g\|_{L^r(\mathbb{R}_+)} \leq \|f\|_{L^p(\mathbb{R}_+)} \|g\|_{L^q(\mathbb{R}_+)},$$

where $1 \leq r, p, q \leq \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ and "$*$" denotes the convolution with respect to the space variable x, we can deduce that

$$\|Kf\|_{L^1(\mathbb{R}_+)} \leq \|f\|_{L^1(\mathbb{R}_+)}.$$

It follows that

$$\int_{\mathbb{R}_+} \phi_\mu(V)(V - KV)\,\mathrm{d}x \geq 0$$

and

$$\begin{aligned}\int_{\mathbb{R}_+} \phi_\mu(V)(R_1 - KR_{2x})\,\mathrm{d}x &\leq \|R_1\|_{L^1(\mathbb{R}_+)} + \|R_{2x}\|_{L^1(\mathbb{R}_+)} \\ &\leq C\delta(1+t)^{-1} + C\delta \mathrm{e}^{-c(1+t)}\end{aligned}$$

for $\mu \to 0$. Thus, letting $\mu \to 0$, we can obtain the desired estimate (3.55). ∎

Proposition 3.23 (Decay estimates of V). *Suppose that $f'(u_-) > 0$ and $(V(x,t), P(x,t))$ is a solution of problem (3.2)–(3.4). Then for $\varepsilon \in (0, \frac{1}{2})$ and sufficiently large t, the solution $(V(x,t), P(x,t))$ satisfies*

$$\begin{aligned}&(1+t)^{\frac{1}{2}+\varepsilon} \int_{\mathbb{R}_+} (|V|^2 + |V_x|^2)\,\mathrm{d}x + \int_0^t (1+\tau)^{\frac{1}{2}+\varepsilon} \int_{\mathbb{R}_+} \tilde{u}_x (|V|^2 + |V_x|^2)\,\mathrm{d}x\mathrm{d}\tau \\ &\quad + \int_0^t (1+\tau)^{\frac{1}{2}+\varepsilon} \int_{\mathbb{R}_+} |V_x|^2\,\mathrm{d}x\mathrm{d}\tau \leq C(1+t)^{\varepsilon} \log^2(2+t), \end{aligned} \tag{3.61}$$

$$\begin{aligned}&(1+t)^{\frac{3}{2}+\varepsilon} \int_{\mathbb{R}_+} (|V_x|^2 + |V_{xx}|^2)\,\mathrm{d}x + \int_0^t (1+\tau)^{\frac{1}{2}+\varepsilon} \int_{\mathbb{R}_+} \tilde{u}_x (|V_x|^2 + |V_{xx}|^2)\,\mathrm{d}x\mathrm{d}\tau \\ &\quad + \int_0^t (1+\tau)^{\frac{1}{2}+\varepsilon} \int_{\mathbb{R}_+} |V_{xx}|^2\,\mathrm{d}x\mathrm{d}\tau \leq C(1+t)^{\varepsilon} \log^{10}(2+t), \end{aligned} \tag{3.62}$$

$$\begin{aligned}&(1+t)^{\frac{3}{2}+\varepsilon} \int_{\mathbb{R}_+} \Big(|V_{xxx}|^2 + \sum_{j=0}^{2} |\partial_x^j V_t|^2\Big)\,\mathrm{d}x + \int_0^t (1+\tau)^{\frac{3}{2}+\varepsilon} \int_{\mathbb{R}_+} |V_{xxx}|^2\,\mathrm{d}x\mathrm{d}\tau \\ &\quad \leq C(1+t)^{\varepsilon} \log^{10}(2+t), \end{aligned} \tag{3.63}$$

$$\begin{aligned}&(1+t)^{\frac{3}{2}+\varepsilon} \int_{\mathbb{R}_+} \Big(|\partial_x^4 V|^2 + \sum_{j=0}^{2} |\partial_x^j V_{tt}|^2 + |V_{xxxt}|^2\Big)\,\mathrm{d}x + \int_0^t (1+\tau)^{\frac{3}{2}+\varepsilon} \int_{\mathbb{R}_+} |\partial_x^4 V|^2\,\mathrm{d}x\mathrm{d}\tau \\ &\quad \leq C(1+t)^{\varepsilon} \log^{10}(2+t) \end{aligned} \tag{3.64}$$

and

$$(1+t)^{\frac{3}{2}+\varepsilon} \int_{\mathbb{R}_+} |\partial_x^j P|^2\,\mathrm{d}x \leq C(1+t)^{\varepsilon} \log^{10}(2+t), \quad j = 0, 1, 2, 3, 4, 5. \tag{3.65}$$

Proof. Adding (3.14) and (3.20), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|V(t)\|^2_{L^2(\mathbb{R}_+)} + \|V_x(t)\|^2_{L^2(\mathbb{R}_+)}) + \|\sqrt{\tilde{u}_x}V(t)\|^2_{L^2(\mathbb{R}_+)} + \|\sqrt{\tilde{u}_x}V_x(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$+ \|P_{xx}(t)\|^2_{L^2(\mathbb{R}_+)} + 2\|P_x(t)\|^2_{L^2(\mathbb{R}_+)} + \|P(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$\le C(\varepsilon_0+\mu)\|V_x(t)\|^2_{L^2(\mathbb{R}_+)} + C(1+t)^{-\frac{3}{2}} + C\mathrm{e}^{-c(1+t)}. \tag{3.66}$$

By $(3.2)_2$, we get

$$\int_{\mathbb{R}_+}(P_{xx}^2 + 2P_x + P^2)\,\mathrm{d}x = \int_{\mathbb{R}_+}(V_x^2 + 2R_2V_x + R_2^2)\,\mathrm{d}x,$$

which implies that

$$\int_{\mathbb{R}_+}V_x^2\,\mathrm{d}x \le \int_{\mathbb{R}_+}(P_{xx}^2 + 2P_x + P^2)\,\mathrm{d}x + C\int_{\mathbb{R}_+}R_2^2\,\mathrm{d}x$$
$$\le \int_{\mathbb{R}_+}(P_{xx}^2 + 2P_x + P^2)\,\mathrm{d}x + C(1+t)^{-\frac{7}{2}} + C\mathrm{e}^{-c(1+t)}. \tag{3.67}$$

Therefore, adding (3.66) and (3.67), and then multiplying the resulting inequality by $(1+t)^{\frac{1}{2}+\varepsilon}$ and integrating it over $[0,t]$, we have

$$(1+t)^{\frac{1}{2}+\varepsilon}(\|V(t)\|^2_{L^2(\mathbb{R}_+)} + \|V_x(t)\|^2_{L^2(\mathbb{R}_+)}) + \int_0^t(1+\tau)^{\frac{1}{2}+\varepsilon}\|\sqrt{\tilde{u}_x}V(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau$$
$$+ \int_0^t(1+\tau)^{\frac{1}{2}+\varepsilon}\|\sqrt{\tilde{u}_x}V_x(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau + \int_0^t(1+\tau)^{\frac{1}{2}+\varepsilon}\|V_x(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau$$
$$\le C\|V_0\|^2_{H^4(\mathbb{R}_+)}(1+t)^{\varepsilon} + \left(\frac{1}{2}+\varepsilon\right)\int_0^t(1+\tau)^{-\frac{1}{2}+\varepsilon}\|V(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau$$
$$+ \left(\frac{1}{2}+\varepsilon\right)\int_0^t(1+\tau)^{-\frac{1}{2}+\varepsilon}\|V_x(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau. \tag{3.68}$$

By employing the Gagliardo–Nirenberg inequality

$$\|f\|^2_{L^2(\mathbb{R}_+)} \le C\|f\|^{\frac{4}{3}}_{L^1(\mathbb{R}_+)}\|f_x\|^{\frac{2}{3}}_{L^2(\mathbb{R}_+)},$$

the second term on the right-hand side of (3.68) can be estimated as

$$\left(\frac{1}{2}+\varepsilon\right)\int_0^t(1+\tau)^{-\frac{1}{2}+\varepsilon}\|V(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau$$
$$\le C\int_0^t(1+\tau)^{-1+\varepsilon}\|V(\tau)\|^2_{L^1(\mathbb{R}_+)}\,\mathrm{d}\tau + \frac{1}{4}\int_0^t(1+\tau)^{\frac{1}{2}+\varepsilon}\|V_x(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau$$
$$\le C(1+t)^{\varepsilon}\log^2(2+t) + \frac{1}{4}\int_0^t(1+\tau)^{\frac{1}{2}+\varepsilon}\|V_x(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau. \tag{3.69}$$

We treat the last term on the right-hand side of (3.68). For any $\varepsilon \in (0, \frac{1}{2})$, it holds that

$$\left(\frac{1}{2}+\varepsilon\right)\int_0^t (1+\tau)^{-\frac{1}{2}+\varepsilon}\|V_x(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau \le C\int_0^t \|V_x(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau$$
$$\le C(\|V_0\|^2_{H^4(\mathbb{R}_+)}+\delta^{\frac{1}{2}}). \tag{3.70}$$

Substituting (3.69)–(3.70) into (3.68), we obtain (3.61).

Next, we prove (3.62). From (3.17) and Lemma 3.6 (B1), we can get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|V_x(t)\|^2_{L^2(\mathbb{R}_+)} + \|\sqrt{\tilde{u}_x}V_x(t)\|^2_{L^2(\mathbb{R}_+)} + \|P_{xx}(t)\|^2_{L^2(\mathbb{R}_+)} + \|P_x(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$\le C\mathrm{e}^{-c(1+t)} + (1+t)^{-1}\int_{\mathbb{R}_+} V_x^2\,\mathrm{d}x + C(1+t)\int_{\mathbb{R}} R_{1x}^2\,\mathrm{d}x + \mu\int_{\mathbb{R}_+} P_{xx}^2\,\mathrm{d}x$$
$$+ C\mu^{-1}\int_{\mathbb{R}_+} R_2^2\,\mathrm{d}x + C\int_{\mathbb{R}_+} |V_x|(|\tilde{u}_{xx}||V| + \tilde{u}_x^2|V| + V_x^2)\,\mathrm{d}x. \tag{3.71}$$

The last term can be estimated as follows. From (2.1) and (3.61),

$$\int_{\mathbb{R}_+} |V_x|(|\tilde{u}_{xx}| + \tilde{u}_x^2)|V|\,\mathrm{d}x$$
$$\le \|\tilde{u}_{xx}(t)\|_{L^\infty(\mathbb{R}_+)}\|V_x(t)\|_{L^2(\mathbb{R}_+)}\|V(t)\|_{L^2(\mathbb{R}_+)}$$
$$\le \|\tilde{u}_{xx}\|^{\frac{2}{3}}_{L^\infty}\|V_x(t)\|^2_{L^2(\mathbb{R}_+)} + \|\tilde{u}_{xx}\|^{\frac{4}{3}}_{L^\infty}\|V(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$\le (1+t)^{-1}\|V_x(t)\|^2_{L^2(\mathbb{R}_+)} + C(1+t)^{-2}\|V(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$\le (1+t)^{-1}\|V_x(t)\|^2_{L^2(\mathbb{R}_+)} + C(1+t)^{-\frac{5}{2}}\log^2(2+t). \tag{3.72}$$

Using the Gagliardo–Nirenberg inequality

$$\|f_x\|^3_{L^3(\mathbb{R}_+)} \le C\|f\|^{\frac{5}{4}}_{L^2(\mathbb{R}_+)}\|f_{xx}\|^{\frac{7}{4}}_{L^2(\mathbb{R}_+)},$$

we have

$$\int_{\mathbb{R}_+} |V_x|^3\,\mathrm{d}x \le \mu\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)} + C\mu^{-1}\|V(t)\|^{10}_{L^2(\mathbb{R}_+)}$$
$$\le \mu\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)} + C\mu^{-1}(1+t)^{-\frac{5}{2}}\log^{10}(2+t). \tag{3.73}$$

Substituting (3.72)–(3.73) into (3.71), for some small but fixed μ, we can deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|V_x(t)\|^2_{L^2(\mathbb{R}_+)} + \|\sqrt{\tilde{u}_x}V_x(t)\|^2_{L^2(\mathbb{R}_+)} + \|P_{xx}(t)\|^2_{L^2(\mathbb{R}_+)} + \|P_x(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$\le \mu\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)} + C(1+t)^{-1}\|V_x(t)\|^2_{L^2(\mathbb{R}_+)}$$
$$+ C(1+t)^{-\frac{5}{2}}\log^{10}(2+t) + C\mathrm{e}^{-c(1+t)}. \tag{3.74}$$

By (3.29), we can get from (3.28) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}+\|\sqrt{\tilde{u}_x}V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}+\|P_{xxx}(t)\|^2_{L^2(\mathbb{R}_+)}+\|P_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}\\
&\quad\le C(\varepsilon_0+\mu)\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}+CV_{xx}^2(0,t)+\mu\|P_{xxx}(t)\|^2_{L^2(\mathbb{R}_+)}\\
&\qquad+C\mu^{-1}(\|R_{1xx}\|^2_{L^2(\mathbb{R}_+)}+\|R_{2x}\|^2_{L^2(\mathbb{R}_+)})\\
&\qquad+C\mu^{-1}\big(\|\tilde{u}_{xxx}\|^2_{L^\infty(\mathbb{R}_+)}\|V(t)\|^2_{L^2(\mathbb{R}_+)}+\|\tilde{u}_{xx}\|^2_{L^\infty(\mathbb{R}_+)}\|V_x(t)\|^2_{L^2(\mathbb{R}_+)}\\
&\qquad\qquad+\|V_x(t)\|^6_{L^6(\mathbb{R}_+)}\big)\\
&\quad\le C(\varepsilon_0+\mu)\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}+\mu\|P_{xxx}(t)\|^2_{L^2(\mathbb{R}_+)}+C\mathrm{e}^{-c(1+t)}\\
&\qquad+C\|P_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}+C(1+t)^{-4}\|V(t)\|^2_{L^2(\mathbb{R}_+)}\\
&\qquad+C(1+t)^{-3}\|V_x(t)\|^2_{L^2(\mathbb{R}_+)}+C(1+t)^{-\frac{9}{2}}.
\end{aligned}\tag{3.75}
$$

In deriving of the last inequality of (3.75), we have used the fact that

$$
\|V_x(t)\|^6_{L^6(\mathbb{R}_+)}\le C\|V(t)\|^4_{L^2(\mathbb{R}_+)}\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}\le C\varepsilon_0^4\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}.
$$

We multiply (3.74) by a sufficiently large number λ such that the fourth term on the right-hand side of (3.75) can be absorbed into the third on the left-hand side of (3.74), and then add the resulting inequality to (3.75). Consequently, choosing small ε_0 and μ and using (3.33), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|V_x(t)\|^2_{L^2(\mathbb{R}_+)}+\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)})+\|\sqrt{\tilde{u}_x}V_x(t)\|^2_{L^2(\mathbb{R}_+)}\\
&\quad+\|\sqrt{\tilde{u}_x}V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}+\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}+\|P_x(t)\|^2_{H^2(\mathbb{R}_+)}\\
&\quad\le C(1+t)^{-1}\|V_x(t)\|^2_{L^2(\mathbb{R}_+)}+C(1+t)^{-4}\|V(t)\|^2_{L^2(\mathbb{R}_+)}\\
&\qquad+C(1+t)^{-\frac{5}{2}}\log^{10}(2+t)+C\mathrm{e}^{-c(1+t)}.
\end{aligned}\tag{3.76}
$$

Multiplying (3.76) by $(1+t)^{\frac{3}{2}+\varepsilon}$ and then integrating it over $[0,t]$, by (3.7), we can deduce that

$$
\begin{aligned}
&(1+t)^{\frac{3}{2}+\varepsilon}\|V_x(t)\|^2_{H^1(\mathbb{R}_+)}\\
&\quad+\int_0^t(1+\tau)^{\frac{3}{2}+\varepsilon}(\|\sqrt{\tilde{u}_x}V_x(\tau)\|^2_{L^2(\mathbb{R}_+)}+\|\sqrt{\tilde{u}_x}V_{xx}(\tau)\|^2_{L^2(\mathbb{R}_+)})\,\mathrm{d}\tau\\
&\quad+\int_0^t(1+\tau)^{\frac{3}{2}+\varepsilon}(\|V_{xx}(\tau)\|^2_{L^2(\mathbb{R}_+)}+\|P_x(\tau)\|^2_{H^2(\mathbb{R}_+)})\,\mathrm{d}\tau\\
&\quad\le C(1+t)^{\varepsilon}\log^{10}(2+t)+\Big(\frac{3}{2}+\varepsilon\Big)\int_0^t(1+\tau)^{\frac{1}{2}+\varepsilon}\|V_x(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau\\
&\qquad+\Big(\frac{3}{2}+\varepsilon\Big)\int_0^t(1+\tau)^{\frac{1}{2}+\varepsilon}\|V_{xx}(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau.
\end{aligned}
$$

For sufficiently large T such that $(\frac{3}{2}+\varepsilon)\frac{1}{1+T} \leq \frac{1}{2}$, the last term can be estimated as

$$\begin{aligned}
&\left(\frac{3}{2}+\varepsilon\right)\int_0^t (1+\tau)^{\frac{1}{2}+\varepsilon}\|V_{xx}(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau \\
&\quad = \left(\frac{3}{2}+\varepsilon\right)\int_0^T (1+\tau)^{\frac{1}{2}+\varepsilon}\|V_{xx}(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau \\
&\qquad + \left(\frac{3}{2}+\varepsilon\right)\frac{1}{1+T}\int_T^t (1+\tau)^{\frac{3}{2}+\varepsilon}\|V_{xx}(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau \\
&\quad \leq C\int_0^t \|V_{xx}(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau + \frac{1}{2}\int_0^t (1+\tau)^{\frac{3}{2}+\varepsilon}\|V_{xx}(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau. \qquad (3.77)
\end{aligned}$$

Combining (3.26) and (3.61), we have

$$\begin{aligned}
&(1+t)^{\frac{3}{2}+\varepsilon}\|V_x(t)\|^2_{H^1(\mathbb{R}_+)} + \int_0^t (1+\tau)^{\frac{3}{2}+\varepsilon}\big(\|\sqrt{\tilde{u}_x}V_x(\tau)\|^2_{L^2(\mathbb{R}_+)} \\
&\quad + \|\sqrt{\tilde{u}_x}V_{xx}(\tau)\|^2_{L^2(\mathbb{R}_+)}\big)\,\mathrm{d}\tau + \int_0^t (1+\tau)^{\frac{3}{2}+\varepsilon}\big(\|V_{xx}(\tau)\|^2_{L^2(\mathbb{R}_+)} \\
&\quad + \|P_x(\tau)\|^2_{H^2(\mathbb{R}_+)}\big)\,\mathrm{d}\tau \leq C(1+t)^{\varepsilon}\log^{10}(2+t), \qquad (3.78)
\end{aligned}$$

which completes the proof of (3.62). Moreover, it follows that

$$\|V_x(t)\|^2_{L^\infty(\mathbb{R}_+)} \leq C(1+t)^{-\frac{3}{2}}\log^{10}(2+t). \qquad (3.79)$$

On the other hand, using (3.25) and (3.6) with $k=1$, we also have

$$(1+t)^{\frac{3}{2}+\varepsilon}\|P(t)\|^2_{H^3(\mathbb{R}_+)} \leq C(1+t)^{\varepsilon}\log^{10}(2+t). \qquad (3.80)$$

We now show (3.62). We can get from (3.35) that

$$\begin{aligned}
&(1+t)^{\frac{3}{2}+\varepsilon}\|V_t(t)\|^2_{L^2(\mathbb{R}_+)} + \int_0^t (1+\tau)^{\frac{1}{2}+\varepsilon}\|V_t(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau \\
&\quad \leq C(1+t)^{\varepsilon}\log^{10}(2+t), \\
&(1+t)^{\frac{3}{2}+\varepsilon}\|V_{xt}(t)\|^2_{L^2(\mathbb{R}_+)} + \int_0^t (1+\tau)^{\frac{3}{2}+\varepsilon}\|V_{xt}(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau \\
&\quad \leq C(1+t)^{\varepsilon}\log^{10}(2+t).
\end{aligned}$$

It follows from (3.36) that

$$\int_0^t (1+\tau)^{\frac{3}{2}+\varepsilon}\|P_t(\tau)\|^2_{H^2(\mathbb{R}_+)}\,\mathrm{d}\tau \leq C(1+t)^{\varepsilon}\log^{10}(2+t). \qquad (3.81)$$

By utilizing (3.40)–(3.41) and (3.79), we can get from (3.39) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|V_{xxt}(t)\|^2_{L^2(\mathbb{R}_+)} + \|\sqrt{\tilde{u}_x}V_{xxt}(t)\|^2_{L^2(\mathbb{R}_+)} + \|P_{xxxt}(t)\|^2_{L^2(\mathbb{R}_+)} \\
&\quad + \|P_{xxt}(t)\|^2_{L^2(\mathbb{R}_+)} + \|V_{xxt}(t)\|^2_{L^2(\mathbb{R}_+)}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\mathrm{e}^{-c(1+t)} + C(\varepsilon_0+\mu)\|V_{xxt}(t)\|^2_{L^2(\mathbb{R}_+)} + C(1+t)^{-4}\|V_x(t)\|^2_{L^2(\mathbb{R}_+)}\\
&\quad + C(1+t)^{-4}\|V_t(t)\|^2_{L^2(\mathbb{R}_+)} + C\|V_{xt}(t)\|^2_{L^2(\mathbb{R}_+)} + C\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)}\\
&\quad + C\|P_{xt}(t)\|^2_{H^1(\mathbb{R}_+)} + C(\varepsilon_0+\delta)\|V_{xxx}(t)\|^2_{L^2(\mathbb{R}_+)}\\
&\quad + C(1+t)^{-6}\|V(t)\|^2_{L^2(\mathbb{R}_+)} + C(1+t)^{-\frac{11}{2}}.
\end{aligned}
\tag{3.82}
$$

Multiplying (3.82) by $(1+t)^{\frac{3}{2}+\varepsilon}$ and then integrating (3.82) over $[0,t]$, for some small ε_0 and μ, we obtain

$$
\begin{aligned}
&(1+t)^{\frac{3}{2}+\varepsilon}\|V_{xxt}(t)\|^2_{L^2(\mathbb{R}_+)} + \int_0^t (1+\tau)^{\frac{3}{2}+\varepsilon}\big(\|\sqrt{\tilde{u}_x}V_{xxt}(\tau)\|^2_{L^2(\mathbb{R}_+)}\\
&\quad + \|P_{xxt}(\tau)\|^2_{H^1(\mathbb{R}_+)} + \|V_{xxt}(\tau)\|^2_{L^2(\mathbb{R}_+)}\big)\,\mathrm{d}\tau\\
&\quad \leq C(1+t)^{\varepsilon}\log^{10}(2+t) + \left(\frac{3}{2}+\varepsilon\right)\int_0^t (1+\tau)^{\frac{1}{2}+\varepsilon}\|V_{xxt}(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau\\
&\qquad + C(\varepsilon_0+\delta)\int_0^t (1+\tau)^{\frac{3}{2}+\varepsilon}\|V_{xxx}(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau.
\end{aligned}
\tag{3.83}
$$

The second term on the right-hand side of (3.83) can be estimated by using the similar method in (3.77). To deal with the last term of (3.83), we can get from (3.44) that

$$
\begin{aligned}
\|V_{xxx}(t)\|^2_{L^2(\mathbb{R}_+)} &\leq C(1+t)^{-6}\|V(t)\|^2_{L^2(\mathbb{R}_+)} + C(1+t)^{-3}\log^{20}(2+t)\|V_x(t)\|^2_{L^2(\mathbb{R}_+)}\\
&\quad + C\|V_{xx}(t)\|^2_{L^2(\mathbb{R}_+)} + C\|P_{xxx}(t)\|^2_{L^2(\mathbb{R}_+)}\\
&\quad + C\|V_{xxt}\|^2_{H^1(\mathbb{R}_+)} + C(1+t)^{-\frac{9}{2}}.
\end{aligned}
\tag{3.84}
$$

By combining (3.62), (3.80) and (3.81), it follows from (3.84) that

$$
\begin{aligned}
&(1+t)^{\frac{3}{2}+\varepsilon}\|V_{xxx}(t)\|^2_{L^2(\mathbb{R}_+)} + \int_0^t (1+\tau)^{\frac{3}{2}+\varepsilon}\|V_{xxx}(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau\\
&\quad \leq C(1+t)^{\varepsilon}\log^{10}(2+t).
\end{aligned}
$$

Therefore, we complete the proof of (3.63). By using the similar method, we can prove inequality (3.64). By utilizing (3.6) with $k=2,3$ and combining (3.61)–(3.64) and (3.78), we can prove (3.65). Thus, we complete the proof of Proposition 3.23. ■

Remark 3.24. We cannot multiply inequality (3.75) by $(1+t)^{\frac{5}{2}+\varepsilon}$ as in [3] because the decay rate of the boundary term $f'(u_-)V_{xx}^2(0,t)$ is $O(t^{-\frac{3}{2}})$ if $f'(u_-)$ is not large enough by using $f'(u_-)V_{xx}^2(0,t) \leq C\frac{1}{f'(u_-)}P_{xx}^2(0,t) + C\mathrm{e}^{-c(1+t)} \leq C\|P_{xx}\|^2_{H^1(\mathbb{R}_+)} + C\mathrm{e}^{-c(1+t)}$. Although our decay rate is not as good as that of the Cauchy problem in [3], we focus on the stability of the initial-boundary value problem of (1.1) rather than calculating the optimal decay rate.

Remark 3.25. For the case of $f'(u_-)=0$, we can get the better decay estimate like in [3] due to the boundary term $f'(u_-)\partial_x^k V(0,t)=0$ $(k=1,2,3,4)$. For the details, one can refer to [3].

By the Sobolev inequality and Proposition 3.23, we obtain the following corollary.

Corollary 3.26. *The solution* (V, P) *of* (3.2)–(3.4) *satisfies*

$$\begin{cases} \|V(t)\|_{L^\infty(\mathbb{R}_+)} \le C(1+t)^{-\frac{1}{2}} \log^3(2+t), \\ \|\partial_x^k V(t)\|_{L^\infty(\mathbb{R}_+)} \le C(1+t)^{-\frac{3}{4}} \log^5(2+t), \quad k = 1, 2, 3, \\ \|\partial_x^k V_t(t)\|_{L^\infty(\mathbb{R}_+)} \le C(1+t)^{-\frac{3}{4}} \log^5(2+t), \quad k = 0, 1, 2, \\ \|\partial_x^k P(t)\|_{L^\infty(\mathbb{R}_+)} \le C(1+t)^{-\frac{3}{4}} \log^5(2+t), \quad k = 0, 1, 2, 3, 4. \end{cases}$$

Combining the results of Proposition 3.3 and Corollary 3.26, we can complete the proof of Theorem 3.1.

Furthermore, by Corollaries 3.26, 3.23 and Lemma 2.3, we get the following corollary.

Corollary 3.27. *The solution* (U, Q) *of* (2.10)–(2.12) *satisfies*

$$\begin{cases} \|\partial_x^k U(t)\|_{L^2(\mathbb{R}_+)} \le C \min\{\delta', (1+t)^{-\frac{3}{4}} \log^5(2+t)\}, \quad k = 2, 3, 4, \\ \|\partial_x^k U_t(t)\|_{L^2(\mathbb{R}_+)} \le C \min\{\delta', (1+t)^{-\frac{3}{4}} \log^5(2+t)\}, \quad k = 1, 2, 3, \\ \|\partial_x^k U(t)\|_{L^\infty(\mathbb{R}_+)} \le C \min\{\delta', (1+t)^{-\frac{3}{4}} \log^5(2+t)\}, \quad k = 2, 3, \\ \|\partial_x^k U_t(t)\|_{L^\infty(\mathbb{R}_+)} \le C \min\{\delta', (1+t)^{-\frac{3}{4}} \log^5(2+t)\}, \quad k = 1, 2, \end{cases} \tag{3.85}$$

where δ' *is defined in* (3.8).

3.2. *Estimates for the perturbation on the two-dimensional half-space*

In this section, we consider the initial-boundary value problem on the two-dimensional half-space:

$$\begin{cases} v_t + (f(v+U) - f(U))_x + g(v+U)_y + \operatorname{div} p = 0, \\ -\nabla \operatorname{div} p + p + \nabla v = 0 \end{cases} \tag{3.86}$$

with initial data

$$v(x, y, 0) = v_0(x, y) = u_0(x, y) - U_0(x)$$

and boundary condition

$$v(0, y, t) = 0, \quad \operatorname{div} p(0, y, t) = 0. \tag{3.87}$$

By calculating the curl of $(3.86)_2$, we can deduce that

$$p_{1y} = p_{2x}, \tag{3.88}$$

which will be used in estimating the perturbation p. Note that equation $(3.86)_2$ is equivalent to

$$\begin{cases} -\operatorname{div} p_x + p_1 + v_x = 0, \\ -\operatorname{div} p_y + p_2 + v_y = 0. \end{cases} \tag{3.89}$$

The global existence follows from the combination of the local estimate in Proposition 3.28 and the a priori estimates in Proposition 3.29. In this section, we will devote ourselves to establishing a priori estimates under the a priori assumption

$$\sup_{t\geq 0} \|v(t)\|^2_{H^3(\mathbb{R}^2_+)} \leq \varepsilon_1^2,$$

where $0 < \varepsilon_1 \ll 1$. Then by the Sobolev inequality

$$\|f\|^2_{L^\infty(\mathbb{R}^2_+)} \leq C\|f\|^2_{H^2(\mathbb{R}^2_+)},$$

we obtain

$$\|\nabla v(t)\|_{L^\infty(\mathbb{R}^2_+)} \leq C\varepsilon_1.$$

For any $0 < T \leq \infty$, we seek for the solution of the initial boundary value problem (3.86)–(3.87) in the set of functions $\widetilde{X}(0,T)$ defined by

$$\begin{aligned}\widetilde{X}(0,T) = \big\{(v,p) \mid\ & v \in C^0([0,T);H^3),\ \nabla v \in L^2(0,T;H^2)\\ & p \in C^0([0,T);H^3)\cap L^2(0,T;H^3),\\ & \operatorname{div} p \in C^0([0,T);H^3)\cap L^2(0,T;H^3)\big\}.\end{aligned}$$

In order to state the results on the a priori estimates, we define M_0 by

$$M_0^2 := \|v_0\|^2_{H^3(\mathbb{R}^2_+)}.$$

Proposition 3.28 (Local existence). *Suppose the boundary condition satisfies*

$$0 \leq f'(u_-) < f'(u_+)$$

and the initial data satisfies $v_0 \in H^3(\mathbb{R}^2_+)$. Also suppose that the initial data $\|v_0\|_{H^3(\mathbb{R}^2_+)}$ and δ are both small enough. Then there are two positive constants C and T_0 such that problem (3.86)–(3.87) has a unique solution $(v,p) \in \widetilde{X}(0,T_0)$, which satisfies

$$\begin{aligned}&\|v(t)\|^2_{H^3(\mathbb{R}^2_+)} + \|p(t)\|^2_{H^3(\mathbb{R}^2_+)} + \|\operatorname{div} p(t)\|^2_{H^3(\mathbb{R}^2_+)}\\ &\quad + \int_0^t \big(\|\nabla v(\tau)\|^2_{H^2(\mathbb{R}^2_+)} + \|p(\tau)\|^2_{H^3(\mathbb{R}^2_+)} + \|\operatorname{div} p(\tau)\|^2_{H^3(\mathbb{R}^2_+)}\big)\,\mathrm{d}\tau\\ &\quad \leq C\|v_0\|^2_{H^3(\mathbb{R}^2_+)} \quad \forall t \in [0,T_0].\end{aligned}$$

Proposition 3.29 (A priori estimates). *Let T be a positive constant. Suppose that $0 \leq f'(u_-) < f'(u_+)$ and problem (3.86)–(3.87) has a unique solution $(v,p) \in \widetilde{X}(0,T)$. Then there exist positive constants C and δ_3' such that if $\|v_0\|_{H^3(\mathbb{R}^2_+)} + \delta \leq \delta_3'$ $(0 < \delta_3' \ll 1)$, then we get the estimate*

$$\begin{aligned}&\|v(t)\|^2_{H^3(\mathbb{R}^2_+)} + \|p(t)\|^2_{H^3(\mathbb{R}^2_+)} + \|\operatorname{div} p(t)\|^2_{H^3(\mathbb{R}^2_+)}\\ &\quad + \int_0^t \big(\|\nabla v(\tau)\|^2_{H^2(\mathbb{R}^2_+)} + \|p(\tau)\|^2_{H^3(\mathbb{R}^2_+)} + \|\operatorname{div} p(\tau)\|^2_{H^3(\mathbb{R}^2_+)}\big)\,\mathrm{d}\tau\\ &\quad \leq C\|v_0\|^2_{H^3(\mathbb{R}^2_+)} \quad \forall t \in [0,T].\end{aligned}$$

Lemma 3.30. *Under the same assumptions of Proposition 3.29, there exists a positive constant C such that the following estimates hold:*

(C1) $$\iint_{\mathbb{R}^2_+} (f(v+U)-f(U))_x v \,\mathrm{d}x\mathrm{d}y \geq \frac{\alpha}{2}\iint_{\mathbb{R}^2_+} U_x v^2 \,\mathrm{d}x\mathrm{d}y;$$

(C2) $$\begin{aligned}&\iint_{\mathbb{R}^2_+} \nabla(f(v+U)-f(U))_x\cdot\nabla v \,\mathrm{d}x\\ &\geq \frac{\alpha}{2}\iint_{\mathbb{R}^2_+} U_x|\nabla v|^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}y+\alpha\iint_{\mathbb{R}^2_+} U_x|v_x|^2\,\mathrm{d}x\mathrm{d}y\\ &\quad-\frac{f'(u_-)}{2}\int_{\mathbb{R}}|\nabla v(0,y,t)|^2\,\mathrm{d}y\\ &\quad-C\iint_{\mathbb{R}^2_+}|\nabla v|(|\nabla v|^2+U_x^2|v|+|U_{xx}||v|)\,\mathrm{d}x\mathrm{d}y;\end{aligned}$$

(C3) $$\begin{aligned}&\iint_{\mathbb{R}^2_+} \Delta(f(v+U)-f(U))_x\Delta v\,\mathrm{d}x\mathrm{d}y\\ &\geq \frac{\alpha}{2}\iint_{\mathbb{R}^2_+} U_x|\Delta v|^2\,\mathrm{d}x\mathrm{d}y-\frac{f'(u_-)}{2}\int_{\mathbb{R}}(\Delta v(0,y,t))^2\,\mathrm{d}y\\ &\quad-C\iint_{\mathbb{R}^2_+}\{|\nabla v||\nabla^2 v|^2+(|\nabla v|+|U_{xx}|+U_x)|\nabla v|\Delta v\}\,\mathrm{d}x\mathrm{d}y\\ &\quad-C\iint_{\mathbb{R}^2_+}(|U_{xxx}|+U_x|U_{xx}|+U_x^3)|v|(\Delta v)\,\mathrm{d}x\mathrm{d}y;\end{aligned}$$

(C4) $$\begin{aligned}&\iint_{\mathbb{R}^2_+}(f(v+U)-f(U))_{xyy}v_{yy}\,\mathrm{d}x\mathrm{d}y\\ &\geq\frac{\alpha}{2}\iint_{\mathbb{R}^2_+}U_x v_{yy}^2\,\mathrm{d}x\mathrm{d}y-C\iint_{\mathbb{R}^2_+}\{|\nabla v\|\nabla^2 v|^2+(|\nabla v|+U_x)|\nabla v||\nabla^2 v|\}\,\mathrm{d}x\mathrm{d}y;\end{aligned}$$

(C5) $$\begin{aligned}&\iint_{\mathbb{R}^2_+}\nabla(f(v+U)-f(U))_{xyy}\cdot\nabla v_{yy}\,\mathrm{d}x\mathrm{d}y\\ &\geq\frac{\alpha}{2}\iint_{\mathbb{R}^2_+}U_x|\nabla v_{yy}|^2\,\mathrm{d}x\mathrm{d}y+\alpha\iint_{\mathbb{R}^2_+}U_x|v_{xyy}|^2\,\mathrm{d}x\mathrm{d}y\\ &\quad-\frac{f'(u_-)}{2}\int_{\mathbb{R}}|\nabla v_{yy}(0,y,t)|^2\,\mathrm{d}y\\ &\quad-C\iint_{\mathbb{R}^2_+}\{(|\nabla v|+U_x)|\nabla v||\nabla v_{yy}|+(|\nabla v|+U_x+|U_{xx}|)|\nabla^2 v||\nabla v_{yy}|\}\,\mathrm{d}x\mathrm{d}y\\ &\quad-C\iint_{\mathbb{R}^2_+}\{|\nabla^2 v|^2|\nabla v_{yy}|+(|\nabla v|+U_x)|\nabla^3 v|^2\}\,\mathrm{d}x\mathrm{d}y;\end{aligned}$$

(C6) $\iint_{\mathbb{R}^2_+} \nabla\Delta(f(v+U)-f(U))_x \cdot \nabla\Delta v \, dxdy$

$$\geq \frac{\alpha}{2}\iint_{\mathbb{R}^2_+} U_x|\nabla\Delta v|^2 \, dxdy + \alpha\iint_{\mathbb{R}^2_+} U_x|\Delta v_x|^2 \, dxdy$$
$$- \frac{f'(u_-)}{2}\int_{\mathbb{R}} |\nabla\Delta v(0,y,t)|^2 \, dy - C\left(\|\nabla v\|_{L^\infty(\mathbb{R}_+)} + \sum_{k=1}^{3}\|\partial_x^k U\|_{L^\infty(\mathbb{R}^2_+)}\right)$$
$$\times \iint_{\mathbb{R}^2_+} (|\nabla v|^2 + |\nabla^2 v|^2 + |\nabla^2 v|^4 + |\nabla^3 v|^2) \, dxdy$$
$$- C\iint_{\mathbb{R}^2_+} (|U_{xxxx}| + |U_x|^4 + |U_{xx}|^2 + |U_x||U_{xxx}| + |U_x|^2|U_{xx}|)$$
$$\times |v||\Delta v_x| \, dxdy;$$

(C7) $\iint_{\mathbb{R}^2_+} (g(v+U)_y)v \, dxdy = 0;$

(C8) $\iint_{\mathbb{R}^2_+} \nabla(g(v+U)_y)\cdot\nabla v \, dxdy$

$$\leq C(\|\nabla v\|_{L^\infty(\mathbb{R}^2_+)} + \|U_x\|_{L^\infty(\mathbb{R}^2_+)})\iint_{\mathbb{R}^2_+} |\nabla v|^2 \, dxdy;$$

(C9) $\iint_{\mathbb{R}^2_+} \Delta(g(v+U)_y)\Delta v \, dxdy$

$$\leq C(\|\nabla v\|_{L^\infty(\mathbb{R}^2_+)} + \sum_{k=1}^{2}\|\partial_x^k U\|_{L^\infty(\mathbb{R}^2_+)})\iint_{\mathbb{R}^2_+} (|\nabla v|^2 + |\nabla^2 v|^2) \, dxdy;$$

(C10) $\iint_{\mathbb{R}^2_+} (g(v+U)_{yyy}v_{yy} \, dxdy$

$$\leq C(\|\nabla v\|_{L^\infty(\mathbb{R}^2_+)} + \|U_x\|_{L^\infty(\mathbb{R}^2_+)})\iint_{\mathbb{R}^2_+} (|\nabla v|^2 + |\nabla^2 v|^2) \, dxdy;$$

(C11) $\iint_{\mathbb{R}^2_+} \nabla(g(v+U)_{yyy})\cdot\nabla v_{yy} \, dxdy$

$$\leq C\iint_{\mathbb{R}^2_+} \{(|\nabla v| + |U_x|)|\nabla v| + (|\nabla v| + |U_x|)|\nabla^2 v| + |\nabla^2 v|^2 + |\nabla v||\nabla v_{yy}|\}$$
$$\times |\nabla v_{yy}| \, dxdy;$$

(C12) $\iint_{\mathbb{R}^2_+} \nabla\Delta(g(v+U)_y)\nabla\Delta v \, dxdy$

$$\leq C(\|\nabla v\|_{L^\infty(\mathbb{R}^2_+)} + \|U_x\|_{L^\infty(\mathbb{R}_+)})\iint_{\mathbb{R}^2_+} (|\nabla v|^2 + |\nabla^2 v|^2 + |\nabla^3 v|^2) \, dxdy$$
$$+ C\iint_{\mathbb{R}^2_+} |\nabla^2 v|^4 \, dxdy.$$

Proof. We only give the proof of $(C3)$ and $(C6)$, and the rest can be calculated in the similar way. Notice that $\Delta(fg) = (\Delta f)g + f(\Delta g) + 2\nabla f \cdot \nabla g$. By direct calculation, we have

$$
\begin{aligned}
&\Delta(f(v+U)-f(U))_x\Delta v\\
&\quad= \Delta f'(v+U)(v_x+U_x)\Delta v + f'(v+U)(\Delta v_x + U_{xxx})\Delta v\\
&\qquad + 2f''(v+U)(\nabla v\cdot\nabla v_x + v_xU_{xx} + U_xv_{xx} + U_xU_{xx})\Delta v\\
&\qquad - \Delta f'(U)U_x\Delta v - f'(U)U_{xxx}\Delta v - 2f''(U)U_xU_{xx}\Delta v\\
&\quad= f'''(v+U)(v_x^3 + U_xv_x^2 + U_x^2v_x)\Delta v + f''(v+U)(v_x+U_x)(\Delta v)^2\\
&\qquad + f''(v+u)U_{xx}v_x\Delta v + f'''(v+U)(v_x+U_x)v_y^2(\Delta v) + f(v+U)\Delta v_x\Delta v\\
&\qquad + 2f''(v+U)(\nabla v\cdot\nabla v_x + v_xU_{xx} + U_xv_{xx})\Delta v\\
&\qquad + (f'''(v+U)-f'''(U))U_x^3\Delta v + (f'(v+U)-f'(U))U_{xxx}\Delta v\\
&\qquad + 3(f''(v+U)-f''(U))U_xU_{xx}\Delta v,
\end{aligned}
\tag{3.90}
$$

where we have used the fact that

$$
\Delta f'(v+U) = f'''(v+U)(v_x+U_x)^2 + f''(v+U)(\Delta v + U_{xx}) + f'''(v+U)v_y^2.
$$

Integrating (3.90) over $\mathbb{R}^2_+$, we can obtain $(C3)$. Based on the equations, we have

$$
\begin{aligned}
&\nabla\Delta(f(v+U)-f(U))_x\cdot\nabla\Delta v\\
&\quad= \nabla\Delta f'(v+U)\cdot\nabla\Delta v(v_x+U_x) + \Delta f'(v+U)(\nabla v_x\cdot\nabla\Delta v + U_{xx}\Delta v)\\
&\qquad + f''(v+U)(\nabla v\cdot\nabla\Delta v + U_x\Delta v_x)(\Delta v_x + U_{xxx})\\
&\qquad + f'(v+U)(\nabla\Delta v_x\cdot\nabla\Delta v + U_{xxxx}\Delta v_x)\\
&\qquad + 2f'''(v+U)(\nabla v\cdot\nabla\Delta v + U_x\Delta v_x)(\nabla v\cdot\nabla v_x + v_xU_{xx} + U_xv_{xx} + U_xU_{xx})\\
&\qquad + 2f''(v+U)\big(\nabla v_x\cdot\nabla\Delta v v_{xx} + v_x\nabla v_{xx}\cdot\nabla\Delta v + \nabla v_y\cdot\nabla\Delta v v_{xy}\\
&\qquad\qquad + v_y\nabla v_{xy}\cdot\nabla\Delta v + \nabla v_x\cdot\nabla\Delta v U_{xx}\big) + 2f''(v+U)\\
&\qquad \times(v_xU_{xxx}\Delta v_x + v_{xx}U_{xx}\Delta v_x + U_x\nabla v_{xx}\cdot\nabla\Delta v + U_{xx}^2\Delta v_x + U_xU_{xxx}\Delta v_x)\\
&\qquad - \big(f^{(4)}(U)U_x^4 + 6f'''(U)U_x^2U_{xx} + 3f'(U)U_{xx}^2 + 4f'(U)U_xU_{xxx}\\
&\qquad\quad + f'(U)U_{xxxx}\big)\Delta v_x.
\end{aligned}
\tag{3.91}
$$

The first term on the right-hand side is equal to

$$
\begin{aligned}
&\nabla\Delta f'(v+U)\cdot\nabla\Delta v(v_x+U_x)\\
&\quad= f^{(4)}(v+U)(\nabla v\cdot\nabla\Delta v + U_x\Delta v_x)(v_x+U_x)^3\\
&\qquad + 2f'''(v+U)(v_x+U_x)^2(\nabla v\cdot\nabla\Delta v + U_x\Delta v_x)\\
&\qquad + f'''(v+U)(\nabla v\cdot\nabla\Delta v + U_x\Delta v_x)(\Delta v + U_{xx})(v_x+U_x)\\
&\qquad + f''(v+U)(|\nabla\Delta v|^2 + U_{xxx}\Delta v_x)(v_x+U_x)\\
&\qquad + f^{(4)}(v+U)(\nabla v\cdot\nabla\Delta v + U_x\Delta v_x)v_y^2(v_x+U_x)\\
&\qquad + 2f'''(v+U)v_y\nabla v_y\cdot\nabla\Delta v(v_x+U_x).
\end{aligned}
\tag{3.92}
$$

Substituting (3.92) into (3.91) and then integrating it over $\mathbb{R}^2_+$, by using the Cauchy inequality, estimate (C6) can be obtained. ■

Lemma 3.31. *The solution $v(x,y,t)$ of (3.86)–(3.87) satisfies the following boundary estimates:*

(D1) $\partial_y^k \partial_t^l v(0,t) = 0,\ k,l = 0,1,2,\dots;$

(D2) $v_x(0,y,t) = 0;$

(D3) $v_{xx}^2(0,y,t) \le C(\operatorname{div} p_x(0,y,t))^2;$

(D4) $v_{xxy}^2(0,y,t) \le C(\operatorname{div} p_{xy}(0,y,t))^2;$

(D5) $v_{xxx}^2(0,y,t) \le C(\operatorname{div} p_x(0,t))^2 + C(\operatorname{div} p_{xt}(0,y,t))^2 + C(\operatorname{div} p_{xy}(0,y,t))^2 + C(\operatorname{div} p_{xx}(0,y,t))^2.$

3.2.1. A priori estimates.

Lemma 3.32. *Under the same assumptions of Proposition 3.29, there exists a positive constant C such that for $0 \le t \le T$,*

$$\begin{aligned}&\|v(t)\|^2_{H^1(\mathbb{R}^2_+)} + \int_0^t (\|\sqrt{U_x} v(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\sqrt{U_x}\nabla v(t)\|^2_{L^2(\mathbb{R}^2_+)})\,\mathrm{d}\tau\\&+\int_0^t (\|\operatorname{div} p(t)\|^2_{H^1(\mathbb{R}^2_+)} + \|p(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\nabla v(t)\|^2_{L^2(\mathbb{R}^2_+)})\,\mathrm{d}\tau\\&\le CM_0^2.\end{aligned}$$

Proof. We can get from $(3.86)_1 \times v + (3.86)_2 \cdot p$ that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}v^2 + (f(v+U) - f(U))_x v + (g(v+U)_y)v + (\operatorname{div} p)^2 + |p|^2\\&\quad + \operatorname{div}\{pv - \operatorname{div} pp\} = 0.\end{aligned} \tag{3.93}$$

According to Lemma 3.30 (C1), (C7) and boundary condition (3.87), integrating (3.93) over $\mathbb{R}^2_+ \times [0,t]$, we have

$$\begin{aligned}&\|v(t)\|^2_{L^2(\mathbb{R}^2_+)} + \int_0^t \|\sqrt{U_x} v(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau\\&\quad+\int_0^t (\|\operatorname{div} p(\tau)\|^2_{L^2(\mathbb{R}^2_+)} + \|p(\tau)\|^2_{L^2(\mathbb{R}^2_+)})\,\mathrm{d}\tau\\&\quad\le CM_0^2.\end{aligned} \tag{3.94}$$

On the other hand, we can get from $\nabla(3.86)_1 \cdot \nabla v - (3.86)_2 \cdot \nabla \operatorname{div} p$ that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla v|^2 + \nabla(f(v+U) - f(U))_x \cdot \nabla v + \nabla(g(v+U)_y) \cdot \nabla v + |\nabla \operatorname{div} p|^2\\&\quad + (\operatorname{div} p)^2 - \operatorname{divp}\{\operatorname{div} pp\} = 0.\end{aligned} \tag{3.95}$$

Integrating (3.95) over $\mathbb{R}^2_+$, using Lemma 3.30 (C2), (C8) and Lemma 3.31 (D1)–(D2), combining (3.94) and (3.85), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\sqrt{U_x}\nabla v(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\nabla \operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)} \\
&\quad \le C\|\nabla v\|_{L^\infty(\mathbb{R}^2_+)}\|\nabla v(t)\|^2_{L^2(\mathbb{R}^2_+)} + C\|v(t)\|^2_{L^2(\mathbb{R}^2_+)}\|U_{xx}\|^2_{L^\infty(\mathbb{R}_+)} \\
&\qquad + C\|\sqrt{U_x}v(t)\|^2_{L^2(\mathbb{R}^2_+)} \\
&\quad \le C\varepsilon_1\|\nabla v(t)\|^2_{L^2(\mathbb{R}^2_+)} + CM_0^2(1+t)^{-\frac{3}{2}}\log^{10}(2+t) \\
&\qquad + C\|\sqrt{U_x}v(t)\|^2_{L^2(\mathbb{R}^2_+)}.
\end{aligned} \tag{3.96}$$

From $(3.86)_2$, the first term on the right-hand side can be estimated as

$$\|\nabla v\|^2_{L^2(\mathbb{R}^2_+)} \le 2(\|\nabla \operatorname{div} p\|^2_{L^2(\mathbb{R}^2_+)} + \|p\|^2_{L^2(\mathbb{R}^2_+)}). \tag{3.97}$$

It then follows from (3.96) after integration over $[0,t]$ that

$$\begin{aligned}
&\|\nabla v(t)\|^2_{L^2(\mathbb{R}^2_+)} + \int_0^t \|\sqrt{U_x}\nabla v(\tau)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\tau \\
&\quad + \int_0^t (\|\nabla \operatorname{div} p(\tau)\|^2_{L^2(\mathbb{R}^2_+)} + \|\operatorname{div} p(\tau)\|^2_{L^2(\mathbb{R}^2_+)})\,\mathrm{d}\tau \le CM_0^2
\end{aligned}$$

for some small ε_1. Using (3.97) again, we get

$$\int_0^t \|\nabla v(\tau)\|^2_{L^2(\mathbb{R}^2_+)}\,\mathrm{d}\tau \le CM_0^2,$$

which completes the proof of Lemma 3.32. ∎

Lemma 3.33. *Under the same assumptions of Proposition 3.29, there exists a positive constant C such that for $0 \le t \le T$,*

$$\begin{aligned}
&\|\nabla^2 v(t)\|^2_{L^2(\mathbb{R}^2_+)} + \int_0^t \Big(\|\sqrt{U_x}\Delta v(\tau)\|^2_{L^2(\mathbb{R}^2_+)} + \|\nabla \operatorname{div} p(\tau)\|^2_{H^1(\mathbb{R}^2_+)} \\
&\quad + \|\nabla^2 v(\tau)\|^2_{L^2(\mathbb{R}^2_+)}\Big)\,\mathrm{d}\tau \le CM_0^2.
\end{aligned}$$

Proof. We can get from $\Delta(3.86)_1 \times \Delta v - \operatorname{div}(3.86)_2 \times \Delta \operatorname{div} p$ that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\Delta v)^2 + \Delta((f(v+U) - f(U))_x)\Delta v + \Delta(g(v+U)_y)\Delta v + (\Delta \operatorname{div} p)^2 \\
&\quad + |\nabla \operatorname{div} p|^2 - \operatorname{div}\{\operatorname{div} p \nabla \operatorname{div} p\} = 0.
\end{aligned} \tag{3.98}$$

Notice that

$$\begin{aligned}
(\Delta \operatorname{div} p)^2 &= (\operatorname{div} p_{xx})^2 + (\operatorname{div} p_{yy})^2 + 2(\operatorname{div} p_{xy})^2 + 2\{\operatorname{div} p_{xx} \operatorname{div} p_y\}_y \\
&\quad - 2\{\operatorname{div} p_{xy} \operatorname{div} p_y\}_x, \\
(\Delta v)^2 &= (v_{xx})^2 + (v_{yy})^2 + 2(v_{xy})^2 + 2\{v_{xx}v_y\}_y - 2\{v_{xy}v_y\}_x,
\end{aligned}$$

which implies that

$$\iint_{\mathbb{R}^2_+} (\Delta \operatorname{div} p)^2 \,\mathrm{d}x\mathrm{d}y = \iint_{\mathbb{R}^2_+} |\nabla^2 \operatorname{div} p|^2 \,\mathrm{d}x\mathrm{d}y, \tag{3.99}$$

$$\iint_{\mathbb{R}^2_+} (\Delta v)^2 \,\mathrm{d}x\mathrm{d}y = \iint_{\mathbb{R}^2_+} |\nabla^2 v|^2 \,\mathrm{d}x\mathrm{d}y. \tag{3.100}$$

Integrating (3.98) over $\mathbb{R}^2_+$, by utilizing Lemma 3.30 (C3), (C9), (3.99) and the Cauchy inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2 v(t)\|^2_{L^2(\mathbb{R}^2_+)} + \frac{\alpha}{2}\|\sqrt{U_x}\Delta v(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\nabla^2 \operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)} \\
&\quad + \|\nabla \operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)} \\
&\quad \le C\int_{\mathbb{R}} v^2_{xx}(0,y,t)\,\mathrm{d}y + C(\|\nabla v\|_{L^\infty(\mathbb{R}^2_+)} + \mu)\|\nabla^2 v(t)\|^2_{L^2(\mathbb{R}^2_+)} \\
&\qquad + C\mu^{-1}\|\nabla v(t)\|^2_{L^2(\mathbb{R}^2_+)} + C\mu^{-1}\|v(t)\|^2_{L^2(\mathbb{R}^2_+)}\|U_{xxx}(t)\|^2_{L^\infty(\mathbb{R}_+)} \\
&\qquad + C\mu^{-1}\|\sqrt{U_x}v(t)\|^2_{L^2(\mathbb{R}_+)}.
\end{aligned} \tag{3.101}$$

By Lemma 3.31 (D3), for the first term on the right-hand side of (3.101), we have

$$\int_{\mathbb{R}} v^2_{xx}(0,y,t)\,\mathrm{d}y \le \frac{1}{4}\|\operatorname{div} p_{xx}(t)\|^2_{L^2(\mathbb{R}^2_+)} + C\|\operatorname{div} p_x(t)\|^2_{L^2(\mathbb{R}^2_+)}. \tag{3.102}$$

From (3.100) and $(3.86)_2$, we can treat the second term as

$$\begin{aligned}
\|\nabla^2 v(t)\|^2_{L^2(\mathbb{R}^2_+)} &= \|\Delta v(t)\|^2_{L^2(\mathbb{R}^2_+)} \\
&\le 2(\|\Delta \operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)}).
\end{aligned} \tag{3.103}$$

The former term of (3.103) can be absorbed into the third term on the left-hand side of (3.101) if ε_1 and μ are small enough. Substituting (3.102) and (3.103) into (3.101), we can deduce that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2 v(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\sqrt{U_x}\Delta v(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\nabla^2 \operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)} \\
&\quad + \|\nabla \operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)} \\
&\quad \le C\|\operatorname{div} p(t)\|^2_{H^1(\mathbb{R}^2_+)} + C\|\nabla v(t)\|^2_{L^2(\mathbb{R}^2_+)} + C\|\sqrt{U_x}v(t)\|^2_{L^\infty(\mathbb{R}_+)} \\
&\qquad + CM_0^2(1+t)^{-\frac{3}{2}}\log^{10}(2+t).
\end{aligned}$$

Integrating the above inequality over $[0,t]$, we get

$$\|\nabla^2 v(t)\|^2_{L^2(\mathbb{R}^2_+)} + \int_0^t (\|\sqrt{U_x}\Delta v(\tau)\|^2_{L^2(\mathbb{R}^2_+)} + \|\nabla \operatorname{div} p(\tau)\|^2_{H^1(\mathbb{R}^2_+)})\,\mathrm{d}\tau \le CM_0^2.$$

Using (3.103) again, we complete the proof of Lemma 3.33. ■

Lemma 3.34. *Under the same assumptions of Proposition 3.29, there exists a positive constant C such that for $0 \le t \le T$,*

$$\|v_{yy}(t)\|^2_{H^1(\mathbb{R}^2_+)} + \int_0^t (\|\sqrt{U_x} v_{yy}(\tau)\|^2_{L^2(\mathbb{R}^2_+)} + \frac{\alpha}{2}\|\sqrt{U_x}\nabla v_{yy}(\tau)\|^2_{L^2(\mathbb{R}^2_+)})\,\mathrm{d}\tau + \int_0^t (\|\nabla v_{yy}(\tau)\|^2_{L^2(\mathbb{R}^2_+)} + \|\mathrm{div}\, p_{yy}(\tau)\|^2_{H^1(\mathbb{R}^2_+)} + \|p_{yy}(\tau)\|^2_{L^2(\mathbb{R}^2_+)})\,\mathrm{d}\tau \le CM_0^2 + C(M_0+\delta')\int_0^t \|\nabla^3 v(\tau)\|^2_{L^2(\mathbb{R}^2_+)}\,\mathrm{d}\tau. \tag{3.104}$$

Proof. We can get from the expression $\partial_y^2(3.86)_1 \times v_{yy} + (3.86)_2 \cdot p_{yy} + \nabla\partial_y^2(3.86)_1 \cdot \nabla v_{yy} - \partial_y^2(3.86)_2 \cdot \nabla\,\mathrm{div}\, p_{yy}$ that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(v_{yy}^2 + |\nabla v_{yy}|^2) + (f(v+U)-f(U))_{xyy}v_{yy} + \nabla(f(v+U)-f(U))_{xyy}\cdot\nabla v_{yy} + g(v+U)_{yyy}v_{yy} + \nabla(g(v+U)_{yyy})\cdot\nabla v_{yy} + |\nabla\,\mathrm{div}\, p_{yy}|^2 - 2\nabla\,\mathrm{div}\, p_{yy}\cdot p_{yy} + |p_{yy}|^2 + \mathrm{div}\{v_{yy}p_{yy}\} = 0. \tag{3.105}$$

Integrating (3.105) over $\mathbb{R}^2_+$, by utilizing Lemma 3.30 (C4)–(C5) and (C10)–(C11), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_{yy}(t)\|^2_{H^1(\mathbb{R}^2_+)} + \frac{\alpha}{2}\|\sqrt{U_x}v_{yy}(t)\|^2_{L^2(\mathbb{R}^2_+)} + \frac{\alpha}{2}\|\sqrt{U_x}\nabla v_{yy}(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\nabla v_{yy}(t)\|^2_{L^2(\mathbb{R}^2_+)} \le C(\|\nabla v\|_{L^\infty(\mathbb{R}^2_+)} + \|U_x\|_{L^\infty(\mathbb{R}_+)})\|\nabla^3 v(t)\|^2_{L^2(\mathbb{R}^2_+)} + C\|\nabla v(t)\|^2_{H^1(\mathbb{R}^2_+)} + C\iint_{\mathbb{R}^2_+}|\nabla^2 v|^4\,\mathrm{d}x\mathrm{d}y. \tag{3.106}$$

Here we have used the fact from $(3.86)_2$ that

$$|\nabla v_{yy}|^2 = |\nabla\,\mathrm{div}\, p_{yy}|^2 - 2\nabla\,\mathrm{div}\, p_{yy}\cdot p_{yy} + |p_{yy}|^2.$$

The last term of (3.106) can be estimated as

$$\begin{aligned} C\iint_{\mathbb{R}^2_+}|\nabla^2 v|^4\,\mathrm{d}x\mathrm{d}y &\le C\int_{\mathbb{R}_+}\sup_{y\in\mathbb{R}}|\nabla^2 v|^2\,\mathrm{d}x\int_{\mathbb{R}}\sup_{x\in\mathbb{R}_+}|\nabla^2 v|^2\,\mathrm{d}y \\ &\le C\int_{\mathbb{R}_+}\|\nabla^2 v(x,\cdot,t)\|_{L^2(\mathbb{R})}\|\nabla^2 v_y(x,\cdot,t)\|_{L^2(\mathbb{R})}\,\mathrm{d}x \\ &\quad\times\int_{\mathbb{R}}\|\nabla^2 v(\cdot,y,t)\|_{L^2(\mathbb{R}_+)}\|\nabla^2 v_x(\cdot,y,t)\|_{L^2(\mathbb{R}_+)}\,\mathrm{d}y \\ &\le C\|\nabla^2 v(t)\|^2_{L^2(\mathbb{R}^2_+)}\|\nabla^3 v(t)\|^2_{L^2(\mathbb{R}^2_+)} \\ &\le CM_0^2\|\nabla^3 v(t)\|^2_{L^2(\mathbb{R}^2_+)}. \end{aligned} \tag{3.107}$$

Substituting (3.107) into (3.106), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|v_{yy}(t)\|^2_{H^1(\mathbb{R}^2_+)} + \|\sqrt{U_x}v_{yy}(t)\|^2_{L^2(\mathbb{R}^2_+)} + \frac{\alpha}{2}\|\sqrt{U_x}\nabla v_{yy}(t)\|^2_{L^2(\mathbb{R}^2_+)}\\
&\quad + \|\nabla v_{yy}(t)\|^2_{L^2(\mathbb{R}^2_+)}\\
&\quad \le C\|\nabla v(t)\|^2_{H^1(\mathbb{R}^2_+)} + C(M_0+\delta')\|\nabla^3 v(t)\|^2_{L^2(\mathbb{R}^2_+)}.
\end{aligned} \tag{3.108}$$

Integrating (3.108) over $[0,t]$, we can get

$$\begin{aligned}
&\|v_{yy}(t)\|^2_{H^1(\mathbb{R}^2_+)} + \int_0^t (\|\sqrt{U_x}v_{yy}(\tau)\|^2_{L^2(\mathbb{R}^2_+)} + \|\sqrt{U_x}\nabla v_{yy}(\tau)\|^2_{L^2(\mathbb{R}^2_+)})\,\mathrm{d}\tau\\
&\quad + \int_0^t \|\nabla v_{yy}(\tau)\|^2_{L^2(\mathbb{R}^2_+)}\,\mathrm{d}\tau\\
&\quad \le CM_0^2 + C(M_0+\delta')\int_0^t \|\nabla^3 v(\tau)\|^2_{L^2(\mathbb{R}^2_+)}\,\mathrm{d}\tau.
\end{aligned}$$

Furthermore, equation $(3.86)_2$ gives

$$|\nabla\,\mathrm{div}\,p_{yy}|^2 + |p_{yy}|^2 + 2(\mathrm{div}\,p_{yy})^2 - 2\,\mathrm{div}\{\mathrm{div}\,p_{yy}\,p_{yy}\} = |\nabla v_{yy}|^2,$$

which implies

$$\|\mathrm{div}\,p_{yy}(t)\|^2_{H^1(\mathbb{R}^2_+)} + \|p_{yy}(t)\|^2_{L^2(\mathbb{R}^2_+)} \le \|\nabla v_{yy}(t)\|^2_{L^2(\mathbb{R}^2_+)},$$

due to $\mathrm{div}\,p_{yy}(0,y,t)=0$.

Integrating the above inequality over $[0,t]$, the desired estimates (3.104) can be obtained, which completes the proof of Lemma 3.34. ■

Lemma 3.35. *Under the same assumptions of Proposition 3.29, there exists a positive constant C such that for $0\le t\le T$,*

$$\begin{aligned}
&\|v_t(t)\|^2_{H^1(\mathbb{R}^2_+)} + \|\mathrm{div}\,p(t)\|^2_{H^1(\mathbb{R}^2_+)} + \|p(t)\|^2_{L^2(\mathbb{R}^2_+)}\\
&\quad + \int_0^t (\|v_t(\tau)\|^2_{H^1(\mathbb{R}^2_+)} + \|\mathrm{div}\,p_t(\tau)\|^2_{H^1(\mathbb{R}^2_+)} + \|p_t(\tau)\|^2_{L^2(\mathbb{R}^2_+)})\,\mathrm{d}\tau\\
&\quad \le CM_0^2.
\end{aligned} \tag{3.109}$$

Proof. We can get from $(3.86)_1$ and $\nabla(3.86)_1$ that

$$\begin{aligned}
&\iint_{\mathbb{R}^2_+} v_t^2\,\mathrm{d}x\mathrm{d}y \le C\iint_{\mathbb{R}^2_+} (v_x^2 + U_x^2v^2 + v_y^2 + (\mathrm{div}\,p)^2)\,\mathrm{d}x\mathrm{d}y,\\
&\iint_{\mathbb{R}^2_+} |\nabla v_t|^2\,\mathrm{d}x\mathrm{d}y \le C\iint_{\mathbb{R}^2_+} (|\nabla v|^4 + U_x^2|\nabla v|^2 + U_x^4v^2 + |\nabla^2 v|^2\\
&\qquad + U_{xx}^2v^2 + |\nabla\,\mathrm{div}\,p|^2)\,\mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{3.110}$$

Integrating (3.110) over $[0, t]$, by Lemmas 3.32 and 3.33, we have

$$\int_0^t \|v_t(\tau)\|^2_{L^2(\mathbb{R}^2_+)} \,d\tau \le C \int_0^t \left(\|\nabla v(\tau)\|^2_{L^2(\mathbb{R}^2_+)} + \|\sqrt{U_x} v(\tau)\|^2_{L^2(\mathbb{R}^2_+)} + \|\operatorname{div} p(\tau)\|^2_{L^2(\mathbb{R}^2_+)}\right) d\tau \le CM_0^2,$$

$$\int_0^t \|\nabla v_t(\tau)\|^2_{L^2(\mathbb{R}^2_+)} \,d\tau \le C \int_0^t \left(\|\nabla v(\tau)\|^2_{H^1(\mathbb{R}^2_+)} + \|\sqrt{U_x} v(\tau)\|^2_{L^2(\mathbb{R}^2_+)} + \|\nabla \operatorname{div} p(\tau)\|^2_{L^2(\mathbb{R}^2_+)} + \|U_{xx}(\tau)\|^2_{L^\infty(\mathbb{R}_+)}\right) d\tau \le CM_0^2.$$

On the other hand, rewriting $(3.86)_2$ in the form $\nabla \operatorname{div} p - p = \nabla v$ and squaring the resulting equation, we can get

$$|\nabla \operatorname{div} p|^2 + 2(\operatorname{div} p)^2 + |p|^2 - 2 \operatorname{div}\{\operatorname{div} p p\} = |\nabla v|^2.$$

From $\operatorname{div} p(0, y, t) = 0$, we further get

$$\|\operatorname{div} p(t)\|^2_{H^1(\mathbb{R}^2_+)} + \|p(t)\|^2_{L^2(\mathbb{R}^2_+)} \le \|\nabla v(t)\|^2_{H^1(\mathbb{R}^2_+)} \le CM_0^2, \tag{3.111}$$

it follows from (3.110) that

$$\|v_t(t)\|^2_{H^1(\mathbb{R}^2_+)} \le C\|v(t)\|^2_{H^2(\mathbb{R}^2_+)} + C\|\operatorname{div} p(t)\|^2_{H^1(\mathbb{R}^2_+)} \le CM_0^2.$$

Similar to (3.111), from $\operatorname{div} p_t(0, y, t) = 0$, we also get

$$\|\operatorname{div} p_t(t)\|^2_{H^1(\mathbb{R}^2_+)} + \|p_t(t)\|^2_{L^2(\mathbb{R}^2_+)} \le \|\nabla v_t(t)\|^2_{H^1(\mathbb{R}^2_+)}. \tag{3.112}$$

Integrating (3.112) over $[0, t]$, the desired estimate (3.109) can be obtained. Hence, we complete the proof of Lemma 3.35. ■

Lemma 3.36. *Under the same assumptions of Proposition 3.29, there exists a positive constant C such that for $0 \le t \le T$,*

$$\|\nabla^3 v(t)\|^2_{L^2(\mathbb{R}^2_+)} + \int_0^t \|\sqrt{U_x} \nabla \Delta v(t)\|^2_{L^2(\mathbb{R}^2_+)} \,d\tau + \int_0^t (\|\nabla^3 \operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\nabla^3 v(t)\|^2_{L^2(\mathbb{R}^2_+)}) \,d\tau \le CM_0^2. \tag{3.113}$$

Proof. We can get from $\nabla\Delta(3.86)_1 \cdot \nabla\Delta v - \nabla \operatorname{div}(3.86) \cdot \nabla\Delta \operatorname{div} p$ that

$$\frac{1}{2}\frac{d}{dt}|\nabla\Delta v|^2 + \nabla\Delta(f(v+U) - f(U))_x \cdot \nabla\Delta v + \nabla\Delta(g(v+U)_y) \cdot \nabla\Delta v + |\nabla\Delta \operatorname{div} p|^2 + (\Delta \operatorname{div} p)^2 - \operatorname{div}\{\Delta \operatorname{div} p \nabla \operatorname{div} p\} = 0. \tag{3.114}$$

Integrating (3.114) over $\mathbb{R}^2_+$, from Lemma 3.30 (C6) and (C12), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta v(t)\|^2_{L^2(\mathbb{R}^2_+)} + \frac{\alpha}{2}\|\sqrt{U_x}\nabla\Delta v(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\nabla\Delta \operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)} \\
&\quad + \|\Delta \operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)} \\
&\quad \le C\int_{\mathbb{R}} |\nabla\Delta v(0,y,t)|^2\,\mathrm{d}y + C\|\nabla v(t)\|^2_{H^1(\mathbb{R}^2_+)} \\
&\qquad + C(\|\nabla v\|_{L^\infty(\mathbb{R}^2_+)} + \delta')\|\nabla^3 v(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|U_x v(t)\|^2_{L^2(\mathbb{R}^2_+)} \\
&\qquad + \|U_{xx}\|^4_{L^\infty(\mathbb{R}_+)}\|v(t)\|^2_{L^2(\mathbb{R}^2_+)} + C\iint_{\mathbb{R}^2_+} |\nabla^2 v|^4\,\mathrm{d}x\mathrm{d}y \\
&\qquad + C\int_{\mathbb{R}_+} |U_{xxxx}|^2\,\mathrm{d}x \int_{\mathbb{R}} \sup_{x\in\mathbb{R}_+} |v|^2\,\mathrm{d}y. \qquad (3.115)
\end{aligned}$$

The terms on the right-hand side of (3.115) can be estimated as follows. From Lemma 3.31,

$$\begin{aligned}
\int_{\mathbb{R}} |\nabla\Delta v(0,y,t)|^2\,\mathrm{d}y &= \int_{\mathbb{R}} (v^2_{xxx}(0,y,t) + v^2_{xxy}(0,y,t))\,\mathrm{d}y \\
&\le C\int_{\mathbb{R}} (\operatorname{div} p_x(0,t))^2\,\mathrm{d}y + C\int_{\mathbb{R}} (\operatorname{div} p_{xt}(0,y,t))^2\,\mathrm{d}y \\
&\quad + C\int_{\mathbb{R}} (\operatorname{div} p_{xy}(0,y,t))^2\,\mathrm{d}y + C\int_{\mathbb{R}} (\operatorname{div} p_{xx}(0,y,t))^2\,\mathrm{d}y \\
&\le \frac{1}{8}\|\nabla^3 \operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)} + C\|\nabla \operatorname{div} p(t)\|^2_{H^1(\mathbb{R}^2_+)} \\
&\quad + C\int_{\mathbb{R}} (\operatorname{div} p_{xt}(0,y,t))^2\,\mathrm{d}y \\
&\le \frac{1}{4}(\|\nabla\Delta \operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\nabla \operatorname{div} p_{yy}(t)\|^2_{L^2(\mathbb{R}^2_+)}) \\
&\quad + C\|\nabla \operatorname{div} p(t)\|^2_{H^1(\mathbb{R}^2_+)} \\
&\quad + C\int_{\mathbb{R}} (\operatorname{div} p_{xt}(0,y,t))^2\,\mathrm{d}y. \qquad (3.116)
\end{aligned}$$

To treat the last term of (3.116), we obtain from $\partial_t(3.89)_1$ that

$$\begin{aligned}
\int_{\mathbb{R}} (\operatorname{div} p_{xt}(0,y,t))^2\,\mathrm{d}y &= \int_{\mathbb{R}} |p_{1t}(0,y,t)|^2\,\mathrm{d}y \\
&\le \int_{\mathbb{R}} \|p_{1t}(t)\|^2_{L^\infty(\mathbb{R}_+)}\,\mathrm{d}y \\
&\le C(\|p_{1xt}(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|p_{1t}(t)\|^2_{L^2(\mathbb{R}^2_+)}).
\end{aligned}$$

Differentiating $(3.89)_1$ with respect to t and then squaring this equation, we get

$$(\operatorname{div} p_{xt} - p_{1t})^2 = v^2_{xt},$$

which implies that

$$(\mathrm{div}\, p_{xt})^2 + |p_{1t}|^2 - 2\{\mathrm{div}\, p_t p_{1t}\}_x + 2p_{1xt}^2 = v_{xt}^2 - 2p_{1xt}p_{2yt}. \tag{3.117}$$

Integrating (3.117) over $\mathbb{R}_+^2$, from $\mathrm{div}\, p_t(0, y, t) = 0$, we have

$$\begin{aligned}
&\iint_{\mathbb{R}_+^2} (\mathrm{div}\, p_{xt})^2 \,\mathrm{d}x\mathrm{d}y + 2\iint_{\mathbb{R}_+^2} (p_{1xt})^2 \,\mathrm{d}x\mathrm{d}y + \iint_{\mathbb{R}_+^2} (p_{1t})^2 \,\mathrm{d}x\mathrm{d}y \\
&\quad \le \iint_{\mathbb{R}_+^2} v_{xt}^2 \,\mathrm{d}x\mathrm{d}y + \iint_{\mathbb{R}_+^2} (p_{1xt} + p_{2yt})^2 \,\mathrm{d}x\mathrm{d}y \\
&\quad \le \iint_{\mathbb{R}_+^2} v_{xt}^2 \,\mathrm{d}x\mathrm{d}y + \iint_{\mathbb{R}_+^2} (\mathrm{div}\, p_t)^2 \,\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Therefore, it follows from (3.116) that

$$\begin{aligned}
\int_{\mathbb{R}} |\nabla\Delta v(0, y, t)|^2 \,\mathrm{d}y &\le \frac{1}{4}(\|\nabla\Delta\, \mathrm{div}\, p(t)\|_{L^2(\mathbb{R}_+^2)}^2 + \|\nabla\, \mathrm{div}\, p_{yy}(t)\|_{L^2(\mathbb{R}_+^2)}^2) \\
&\quad + C\big(\|\nabla\, \mathrm{div}\, p(t)\|_{H^1(\mathbb{R}_+^2)}^2 + \|\nabla v_t(t)\|_{L^2(\mathbb{R}_+^2)}^2 \\
&\quad + \|\mathrm{div}\, p_t(t)\|_{L^2(\mathbb{R}_+^2)}^2\big).
\end{aligned} \tag{3.118}$$

Similar to (3.107), we can deduce that

$$\iint_{\mathbb{R}_+^2} |\nabla^2 v|^4 \,\mathrm{d}x\mathrm{d}y \le CM_0^2 \|\nabla^3 v(t)\|_{L^2(\mathbb{R}_+^2)}^2. \tag{3.119}$$

From (3.85),

$$\begin{aligned}
\int_{\mathbb{R}_+} |U_{xxxx}|^2 \,\mathrm{d}x \int_{\mathbb{R}} \sup_{x\in\mathbb{R}_+} |v|^2 \,\mathrm{d}y &\le \int_{\mathbb{R}_+} |U_{xxxx}|^2 \,\mathrm{d}x \iint_{\mathbb{R}_+^2} (v^2 + v_x^2) \,\mathrm{d}x\mathrm{d}y \\
&\le CM_0^2 \|U_{xxxx}(t)\|_{L^2(\mathbb{R}_+)}^2 \\
&\le CM_0^2 (1+t)^{-\frac{3}{2}} \log^{10}(2+t).
\end{aligned} \tag{3.120}$$

Substituting (3.118)–(3.120) into (3.115), for sufficiently small δ' we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta v(t)\|_{L^2(\mathbb{R}_+^2)}^2 + \|\sqrt{U_x}\nabla\Delta v(t)\|_{L^2(\mathbb{R}_+^2)}^2 + \|\nabla\Delta\, \mathrm{div}\, p(t)\|_{L^2(\mathbb{R}_+^2)}^2 \\
&\quad + \|\Delta\, \mathrm{div}\, p(t)\|_{L^2(\mathbb{R}_+^2)}^2 \\
&\quad \le C\|\nabla v(t)\|_{H^1(\mathbb{R}_+^2)}^2 + C(M_0 + \delta')\|\nabla^3 v(t)\|_{L^2(\mathbb{R}_+^2)}^2 \\
&\qquad + CM_0^2(1+t)^{-\frac{3}{2}} \log^{10}(2+t) + C\|\nabla v_t(t)\|_{L^2(\mathbb{R}_+^2)}^2 \\
&\qquad + \frac{1}{4}(\|\nabla\Delta\, \mathrm{div}\, p(t)\|_{L^2(\mathbb{R}_+^2)}^2 + \|\nabla\, \mathrm{div}\, p_{yy}(t)\|_{L^2(\mathbb{R}_+^2)}^2) \\
&\qquad + C\|\nabla\, \mathrm{div}\, p(t)\|_{H^1(\mathbb{R}_+^2)}^2 + C\|\mathrm{div}\, p_t(t)\|_{L^2(\mathbb{R}_+^2)}^2 + \|\sqrt{U_x} v(t)\|_{L^2(\mathbb{R}_+^2)}^2.
\end{aligned} \tag{3.121}$$

Integrating (3.121) over $[0, t]$, substituting (3.109) and (3.104) into the resulting inequality, we can get

$$\begin{aligned}
&\|\nabla\Delta v(t)\|^2_{L^2(\mathbb{R}^2_+)} \\
&\quad + \int_0^t (\|\sqrt{U_x}\nabla\Delta v(\tau)\|^2_{L^2(\mathbb{R}^2_+)} + \|\nabla\Delta \operatorname{div} p(\tau)\|^2_{L^2(\mathbb{R}^2_+)} + \|\Delta \operatorname{div} p(\tau)\|^2_{L^2(\mathbb{R}^2_+)})\, d\tau \\
&\quad \le CM_0^2 + C(M_0+\delta')\int_0^t \|\nabla^3 v(\tau)\|^2_{L^2(\mathbb{R}^2_+)}\, d\tau. \qquad (3.122)
\end{aligned}$$

From $(3.86)_2$, combining (3.104), the last term can be estimated as

$$\begin{aligned}
\int_0^t \|\nabla^3 v(\tau)\|^2_{L^2(\mathbb{R}^2_+)}\, d\tau &\le C\int_0^t \|\nabla\Delta v(\tau)\|^2_{L^2(\mathbb{R}^2_+)}\, d\tau + C\int_0^t \|\nabla v_{yy}(\tau)\|^2_{L^2(\mathbb{R}^2_+)}\, d\tau \\
&\le C\int_0^t (\|\nabla\Delta \operatorname{div} p(\tau)\|^2_{L^2(\mathbb{R}^2_+)} + \|\nabla \operatorname{div} p(\tau)\|^2_{L^2(\mathbb{R}^2_+)})\, d\tau \\
&\quad + C\int_0^t (\|\nabla \operatorname{div} p_{yy}(\tau)\|^2_{L^2(\mathbb{R}^2_+)} + \|p_{yy}(\tau)\|^2_{L^2(\mathbb{R}^2_+)})\, d\tau \\
&\le CM_0^2 + C(M_0+\delta')\int_0^t \|\nabla^3 v(\tau)\|^2_{L^2(\mathbb{R}^2_+)}\, d\tau,
\end{aligned}$$

which implies that

$$\int_0^t \|\nabla^3 v(\tau)\|^2_{L^2(\mathbb{R}^2_+)}\, d\tau \le CM_0^2 \qquad (3.123)$$

for some small M_0 and δ'. Combining (3.122), (3.104) and (3.123), the desired estimate (3.113) can be directly obtained. ∎

Lemma 3.37. *Under the same assumptions of Proposition 3.29, there exists a positive constant C such that for $0 \le t \le T$,*

$$\begin{aligned}
&\|\nabla^2 \operatorname{div} p(t)\|^2_{H^1(\mathbb{R}^2_+)} + \|\nabla p(t)\|^2_{H^2(\mathbb{R}^2_+)} \\
&\quad + \int_0^t (\|\nabla^2 \operatorname{div} p(\tau)\|^2_{H^1(\mathbb{R}^2_+)} + \|\nabla p(\tau)\|^2_{H^2(\mathbb{R}^2_+)})\, d\tau \le CM_0^2. \qquad (3.124)
\end{aligned}$$

Proof. Making use of $(3.86)_2$, we can deduce that

$$\begin{aligned}
&\|\nabla^2 \operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)} = \|\Delta \operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)} \le C(\|\operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\Delta v(t)\|^2_{L^2(\mathbb{R}^2_+)}), \\
&\|\nabla\Delta \operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)} \le C(\|\nabla \operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\nabla\Delta v(t)\|^2_{L^2(\mathbb{R}^2_+)}), \\
&\|\nabla^3 \operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)} \le \|\nabla \operatorname{div} p_{yy}(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\nabla\Delta \operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)}
\end{aligned}$$

and

$$\|\nabla \operatorname{div} p_{yy}(t)\|^2_{L^2(\mathbb{R}^2_+)} + 2\|\operatorname{div} p_{yy}(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|p_{yy}(t)\|^2_{L^2(\mathbb{R}^2_+)} \le \|\nabla v_{yy}(t)\|^2_{L^2(\mathbb{R}^2_+)}.$$

Therefore, it holds that

$$\|\nabla^2 \operatorname{div} p(t)\|^2_{H^1(\mathbb{R}^2_+)} + \int_0^t \|\nabla^2 \operatorname{div} p(\tau)\|^2_{H^1(\mathbb{R}^2_+)} \mathrm{d}\tau \leq CM_0^2. \tag{3.125}$$

By utilizing (3.86)$_2$ again, we have

$$\|\nabla p(t)\|^2_{L^2(\mathbb{R}^2_+)} \leq C(\|\nabla^2 \operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\nabla^2 v(t)\|^2_{L^2(\mathbb{R}^2_+)}),$$
$$\|\nabla^2 p(t)\|^2_{L^2(\mathbb{R}^2_+)} \leq C(\|\nabla^3 \operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\nabla^3 v(t)\|^2_{L^2(\mathbb{R}^2_+)}).$$

In order to show the L^2-estimate of $\nabla^3 p$, we know the definition of $\nabla^3 p$ is that

$$\nabla^3 p = \begin{pmatrix} p_{1xxx} & p_{1xxy} & p_{1xyy} & p_{1yyy} \\ p_{2xxx} & p_{2xxy} & p_{2xyy} & p_{2yyy} \end{pmatrix}.$$

From (3.88), we note that

$$\operatorname{div} p_{xx} - p_{1xxx} = p_{2xxy} = p_{1xyy} = \operatorname{div} p_{yy} - p_{2yyy}, \tag{3.126}$$
$$p_{2xxx} = p_{1xxy} = \operatorname{div} p_{xy} - p_{2xyy} = \operatorname{div} p_{xy} - p_{1yyy}. \tag{3.127}$$

Therefore, we just need to show the L^2-estimate of p_{2xyy} and p_{2yyy}. Once we get these two estimates, combined with the obtained estimates of $\nabla^2 \operatorname{div} p$, we can deduce the estimate of $\nabla^3 p$ immediately by using (3.126)–(3.127). Now, we estimate p_{2xyy} and p_{2yyy}.

We can get from $\partial_x \partial_y$(3.89)$_2$ that

$$(\operatorname{div} p_{xyy})^2 + p^2_{2xy} + 2p^2_{2xyy} - 2\{\operatorname{div} p_{xy} p_{2xy}\}_y = v^2_{xyy} - 2p_{1xxy} p_{2xyy}. \tag{3.128}$$

Integrating (3.128) over $\mathbb{R}^2_+$, we have

$$\begin{aligned} &\iint_{\mathbb{R}^2_+} (\operatorname{div} p_{xyy})^2 \,\mathrm{d}x\mathrm{d}y + \iint_{\mathbb{R}^2_+} p^2_{2xy} \,\mathrm{d}x\mathrm{d}y + 2\iint_{\mathbb{R}^2_+} p^2_{2xyy} \,\mathrm{d}x\mathrm{d}y \\ &\quad \leq \iint_{\mathbb{R}^2_+} v^2_{xyy} \,\mathrm{d}x\mathrm{d}y + \iint_{\mathbb{R}^2_+} (p_{1xxy} + p_{2xyy})^2 \,\mathrm{d}x\mathrm{d}y \\ &\quad \leq \iint_{\mathbb{R}^2_+} v^2_{xyy} \,\mathrm{d}x\mathrm{d}y + \iint_{\mathbb{R}^2_+} (\operatorname{div} p_{xy})^2 \,\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Similarly, we can get from ∂_y^2(3.89)$_2$ that

$$(\operatorname{div} p_{yyy})^2 + p^2_{2yy} + 2p^2_{2yyy} - 2\{\operatorname{div} p_{yy} p_{2yy}\}_y = v^2_{yyy} - 2p_{1xyy} p_{2yyy}.$$

It follows that

$$\begin{aligned} &\iint_{\mathbb{R}^2_+} (\operatorname{div} p_{yyy})^2 \,\mathrm{d}x\mathrm{d}y + \iint_{\mathbb{R}^2_+} p^2_{2yy} \,\mathrm{d}x\mathrm{d}y + 2\iint_{\mathbb{R}^2_+} p^2_{2yyy} \,\mathrm{d}x\mathrm{d}y \\ &\quad \leq \iint_{\mathbb{R}^2_+} v^2_{yyy} \,\mathrm{d}x\mathrm{d}y + \iint_{\mathbb{R}^2_+} (\operatorname{div} p_{yy})^2 \,\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Combining (3.113) and (3.125), we can obtain the desired estimate (3.124). ■

3.2.2. Large-time behavior. The combination of the existence and uniqueness of the local solution and the a priori estimates can extend the local solution for problem (3.86)–(3.87) globally, that is,

$$\begin{cases} v \in C^0([0,\infty); H^3(\mathbb{R}^2_+)) \cap C^1([0,\infty); H^2(\mathbb{R}^2_+)),\ \nabla v \in L^2(0,\infty; H^2(\mathbb{R}^2_+)),\\ p \in C^0([0,\infty); H^3(\mathbb{R}^2_+)) \cap L^2(0,\infty; H^3(\mathbb{R}^2_+)),\\ \operatorname{div} p \in C^0([0,\infty); H^3(\mathbb{R}^2_+)) \cap L^2(0,\infty; H^3(\mathbb{R}^2_+)),\\ v_t \in L^2(0,\infty; H^1(\mathbb{R}^2_+)),\ p_t \in L^2(0,\infty; L^2(\mathbb{R}^2_+)),\ \operatorname{div} p_t \in L^2(0,\infty; H^1(\mathbb{R}^2_+)). \end{cases}$$

It follows that for $t \geq 0$,

$$\begin{aligned} &\|v(t)\|^2_{H^3(\mathbb{R}^2_+)} + \|v_t(t)\|^2_{H^1(\mathbb{R}^2_+)} + \|p(t)\|^2_{H^3(\mathbb{R}^2_+)} + \|\operatorname{div} p(t)\|^2_{H^3(\mathbb{R}^2_+)}\\ &\quad + \int_0^\infty (\|\nabla v(t)\|^2_{H^2(\mathbb{R}^2_+)} + \|v_t(t)\|^2_{H^1(\mathbb{R}^2_+)})\, d\tau\\ &\quad + \int_0^\infty \big(\|p(t)\|^2_{H^3(\mathbb{R}^2_+)} + \|\operatorname{div} p(t)\|^2_{H^3(\mathbb{R}^2_+)} + \|p_t(t)\|^2_{L^2(\mathbb{R}^2_+)}\\ &\qquad + \|\operatorname{div} p_t(t)\|^2_{H^1(\mathbb{R}^2_+)}\big)\, d\tau < \infty, \end{aligned} \tag{3.129}$$

In order to show the large-time behavior (3.1) in Theorem 3.2 from (3.129), we just need to prove that

$$\begin{aligned} &\int_0^\infty \Big|\frac{d}{dt}(\|\nabla v(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\nabla^2 v(t)\|^2_{L^2(\mathbb{R}^2_+)})\Big|\, dt < \infty,\\ &\int_0^\infty \Big|\frac{d}{dt}(\|\nabla \operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\operatorname{div} p(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|p(t)\|^2_{L^2(\mathbb{R}^2_+)})\Big|\, dt < \infty. \end{aligned} \tag{3.130}$$

Once we get the above two inequalities, combining (3.129), we have

$$\begin{aligned} &\|\nabla v(t)\|_{L^2(\mathbb{R}^2_+)},\quad \|\nabla^2 v(t)\|_{L^2(\mathbb{R}^2_+)},\quad \|\nabla \operatorname{div} p(t)\|_{L^2(\mathbb{R}^2_+)},\\ &\|\operatorname{div} p(t)\|_{L^2(\mathbb{R}^2_+)},\quad \|p(t)\|_{L^2(\mathbb{R}^2_+)} \to 0 \quad \text{as } t \to \infty. \end{aligned}$$

By utilizing the Gagliardo–Nirenberg inequality

$$\begin{aligned} \|f\|_{L^\infty(\mathbb{R}^2_+)} &\leq C\|f\|^{\frac12}_{L^2(\mathbb{R}^2_+)}\|\nabla^2 f\|^{\frac12}_{L^2(\mathbb{R}^2_+)},\\ \|\nabla f\|_{L^\infty(\mathbb{R}^2_+)} &\leq C\|f\|^{\frac13}_{L^2(\mathbb{R}^2_+)}\|\nabla^3 f\|^{\frac23}_{L^2(\mathbb{R}^2_+)}, \end{aligned}$$

we can deduce that as $t \to \infty$,

$$\begin{aligned} \|\nabla^k v(t)\|_{L^\infty(\mathbb{R}^2_+)} &\leq C\|\nabla^k v(t)\|^{\frac12}_{L^2(\mathbb{R}^2_+)}\|\nabla^{k+2} v(t)\|^{\frac12}_{L^2(\mathbb{R}^2_+)} \to 0, \quad k = 0, 1,\\ \|p(t)\|_{L^\infty(\mathbb{R}^2_+)} &\leq C\|p(t)\|^{\frac12}_{L^2(\mathbb{R}^2_+)}\|\nabla^2 p(t)\|^{\frac12}_{L^2(\mathbb{R}^2_+)} \to 0,\\ \|\nabla p(t)\|_{L^\infty(\mathbb{R}^2_+)} &\leq C\|p(t)\|^{\frac13}_{L^2}\|\nabla^3 p(t)\|^{\frac23}_{L^2(\mathbb{R}^2_+)} \to 0,\\ \|\nabla \operatorname{div} p(t)\|_{L^\infty(\mathbb{R}^2_+)} &\leq C\|\nabla \operatorname{div} p(t)\|^{\frac12}_{L^2(\mathbb{R}^2_+)}\|\nabla^3 \operatorname{div} p(t)\|^{\frac12}_{L^2(\mathbb{R}^2_+)} \to 0. \end{aligned}$$

In order to prove (3.130), the key is to estimate $\nabla^2 v_t$. We can get from $\Delta(3.86)_1$ that

$$\|\Delta v_t(t)\|^2_{L^2(\mathbb{R}^2_+)} \leq C\big(\|\nabla v(t)\|^2_{H^2(\mathbb{R}^2_+)} + \|\sqrt{U_x}v(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|U_{xxx}\|^2_{L^\infty(\mathbb{R}_+)}\|v(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\Delta\,\mathrm{div}\,p(t)\|^2_{L^2(\mathbb{R}^2_+)}\big).$$

Similar to (3.100), we can deduce that

$$\|\Delta v_t(t)\|^2_{L^2(\mathbb{R}^2_+)} = \|\nabla^2 v_t(t)\|^2_{L^2(\mathbb{R}^2_+)}.$$

It follows that

$$\int_0^\infty \|\nabla^2 v_t(t)\|^2_{L^2(\mathbb{R}^2_+)}\,\mathrm{d}t < \infty.$$

Then, we can obtain

$$\begin{aligned}
&\int_0^\infty \Big|\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla v(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\nabla^2 v(t)\|^2_{L^2(\mathbb{R}^2_+)})\Big|\,\mathrm{d}t \\
&\quad\leq C\int_0^\infty \big(\|\nabla v(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\nabla v_t(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\nabla^2 v(t)\|^2_{L^2(\mathbb{R}^2_+)} \\
&\qquad + \|\nabla^2 v_t(t)\|^2_{L^2(\mathbb{R}^2_+)}\big)\,\mathrm{d}t < \infty
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty \Big|\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla\,\mathrm{div}\,p(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\mathrm{div}\,p(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|p(t)\|^2_{L^2(\mathbb{R}^2_+)})\Big|\mathrm{d}t \\
&\quad\leq C\int_0^\infty (\|\nabla\,\mathrm{div}\,p(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\mathrm{div}\,p(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|p(t)\|^2_{L^2(\mathbb{R}^2_+)})\,\mathrm{d}t \\
&\qquad + C\int_0^\infty (\|\nabla\,\mathrm{div}\,p_t(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|\mathrm{div}\,p_t(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|p_t(t)\|^2_{L^2(\mathbb{R}^2_+)})\,\mathrm{d}t < \infty.
\end{aligned}$$

Therefore, the proof of Theorem 3.2 is completed. Combined with the proved Theorem 3.1, we prove the main Theorem 2.6.

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