J. Eur. Math. Soc. 27, 3369–3406 (2025) DOI 10.4171/JEMS/1435

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(Quasi-)conformal methods in two-dimensional free boundary problems

Received 23 November 2021; revised 27 March 2023

Abstract. In this paper, we relate the theory of quasi-conformal maps to the regularity of the solutions to nonlinear thin-obstacle problems; we prove that the contact set is locally a finite union of intervals and apply this result to the solutions of one-phase Bernoulli free boundary problems with geometric constraint. We also introduce a new conformal hodograph transform, which allows to obtain the precise expansion at branch points of both the solutions to the one-phase problem with geometric constraint and a class of symmetric solutions to the two-phase problem, as well as to construct examples of free boundaries with cusp-like singularities.

Keywords: regularity for free boundary problems, Alt–Caffarelli–Friedman problem, nonlinear thin-obstacle problem, branch points, quasi-conformal maps.

1. Introduction

This note is dedicated to the analysis of the branch singularities arising in two different types of free boundary problems in dimension two: nonlinear thin-obstacle problems and one-phase Bernoulli problems with geometric constraint. In the last part of the paper, we will present some results about branch points of the two-phase problem.

Our main motivation is the description of the structure of branch points arising in free boundary problems of the Bernoulli type. Our main model example is the following one-phase problem with geometric constraint, which for simplicity we state for nonnegative functions u defined on the unit ball B_1 in $\mathbb{R}^d$:

$$\Delta u = 0 \quad \text{in } \Omega_u \subset B_1 \cap \{x_d > 0\}$$

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Mathematics Subject Classification 2020: 35R35 (primary); 49Q10, 47A75 (secondary).

with boundary conditions

$$\begin{aligned} u &= 0 \quad \text{on } B_1 \cap \{x_d = 0\}, \\ |\nabla u| &= 1 \quad \text{on } \partial\Omega_u \cap \{x_d > 0\}, \\ |\nabla u| &\geq 1 \quad \text{on } \partial\Omega_u \cap \{x_d = 0\}, \end{aligned}$$

in which

$$\Omega_u := \{u > 0\}$$

and the geometric constraint is the inclusion $\Omega_u \subset B_1 \cap \{x_d > 0\}$. The (optimal) $C^{1,1/2}$-regularity of the free boundary $\partial\Omega_u \cap B_1$ for this specific problem was proved by Chang-Lara and Savin in [4]. On the other hand, as in the case of other Bernoulli free boundary problems, such as the two-phase problem [9] and the vectorial problem [19], the $C^{1,\alpha}$-regularity of the free boundary $\partial\Omega_u \cap B_1$ by itself does not give any information on the contact set

$$\partial\Omega_u \cap \{x_d = 0\} \cap B_1,$$

nor the structure of its relative boundary in $B_1 \cap \{x_d = 0\}$, which is the set of points at which $\partial\Omega_u$ branches away from $\{x_d = 0\}$. In dimension two, it is natural to expect that this set is discrete and that around each branch point the set $\{u = 0\} \cap \{x_d > 0\}$ forms a cusp. This is precisely the content of one of our main results, Theorem 1.1.

We will study these singularities in two different ways. Firstly, we will prove that branch singularities for minimizers of a general nonlinear thin-obstacle problem are isolated, using the theory of quasi-conformal maps, and then we will deduce the same result for solutions of the problem above via a hodograph transform. Secondly, we will introduce a *conformal hodograph transform* and use it to deduce the result directly. This second method has two advantages: it allows us to give a precise description of the cuspidal behavior of the free boundary at branch singularities and moreover, being reversible, it allows to show that solutions of the two-dimensional one-phase problem with obstacle are in a one-to-one correspondence with solutions to the thin-obstacle problem, thus producing many examples of cuspidal singularities. Finally, we will describe a special symmetric situation in which our techniques are applied to the branch points of solutions to the two-phase problem. Extending our results to the general two-phase situation seems to require an entirely new idea, which is similar in spirit to an analog of Almgren's center manifold for this problem (see [3]).

The quasi-conformal technique is needed to prove Theorem 1.1 and it is the only one available there. On the other hand, the conformal hodograph transform is the only technique which allows to give the precise analytic expansion in Theorems 1.3 (b) and 1.6 (b), and to construct the corresponding examples in Theorems 1.4 and 1.8. Both techniques can be used to prove Theorems 1.3 (a) and 1.6 (a).

We wish to remark that such precise results at branch points, that is, singular points at which the tangent to the free boundary is a plane, usually with multiplicity, are quite rare. To our knowledge, the only such examples are the results of Chang on two-dimensional area minimizing currents [3, 6–8], of Sakai on the two-dimensional obstacle problem [17, 18], and of Lewy on the two-dimensional thin-obstacle problem ([16], and also [15] for a less precise result); like in the present paper, all these results are two-dimensional.

Our approach is similar in spirit to the results of Sakai and Lewy, and makes use of (quasi-)conformal techniques to prove both the local finiteness of the branch set and to give a precise description of the cuspidal behavior at such points. A possible alternative approach, which could also be applicable in higher dimensions, would be to look for a monotone quantity, such as Almgren's frequency function as done for instance in Chang's paper [3]; in fact, for some thin-obstacle problems, as for instance the one involving the classical Laplace operator, the monotonicity of Almgren's frequency function is known (see [2, 15]) and can still be used to get information on the dimension of the branch set (see [12]). However, the operators we study are not regular enough to guarantee the monotonicity of the frequency function, and so we were naturally led to consider (quasi-)conformal techniques. Furthermore, our techniques have the additional benefit of yielding a very precise local description of the free-boundary at branch points (see items (b) of Theorems 1.1, 1.3 and 1.6) in a straightforward way, much simpler than the induction procedure that would be needed using the frequency function as in [3].

Concerning the possible extensions of Theorems 1.1, 1.3 and 1.6 to higher dimensions, we point out that, since the monotonicity of the frequency function does not seem to hold in none of these cases (and it certainly does not hold in the full generality of Theorem 1.1), a dimension reduction argument seems completely out of reach.

1.1. Nonlinear thin-obstacle problem

Let B_1 be the unit ball in $\mathbb{R}^2$ and let

$$B_1^+ := \{(x, y) \in B_1 : y > 0\} \quad \text{and} \quad B_1' = \{(x, y) \in B_1 : y = 0\}.$$

Let $\mathcal{F}: \mathbb{R}^2 \to \mathbb{R}$ be a C^2-regular function, and let $\mathcal{F}_j$, $j = 1, 2$ and $\mathcal{F}_{ij}$, $1 \le i, j \le 2$, be the first and second order partial derivatives of $\mathcal{F}$. Moreover, we identify $\mathbb{R}^2$ with the field of complex numbers $\mathbb{C}$, so we will often think of the functions on $\mathbb{R}^2 = \mathbb{C}$ as functions of two real variables $(x, y) \in \mathbb{R}^2$ and at the same time as a function of one complex variable $z = x + iy \in \mathbb{C}$.

We consider solutions $U \in C^1(B_1^+ \cup B_1')$ of the following nonlinear thin-obstacle problem:

$$\operatorname{div}(\nabla\mathcal{F}(\nabla U)) = 0 \quad \text{in } B_1^+, \tag{1.1}$$

$$U \ge 0 \quad \text{on } B_1', \tag{1.2}$$

$$\nabla\mathcal{F}(\nabla U) \cdot e_2 = 0 \quad \text{on } \{U > 0\} \cap B_1', \tag{1.3}$$

$$\nabla\mathcal{F}(\nabla U) \cdot e_2 \le 0 \quad \text{on } \{U = 0\} \cap B_1', \tag{1.4}$$

where $e_2 = (0, 1)$. Our first main result is the following.

Theorem 1.1 (Nonlinear thin-obstacle). *Suppose that $U \in C^1(B_1^+ \cup B_1')$ is a solution to (1.1)–(1.4) and that $\mathcal{F}: \mathbb{R}^2 \to \mathbb{R}$ is C^2-regular function satisfying*

$$\nabla \mathcal{F}(0) = 0 \quad \textit{and} \quad \nabla^2 \mathcal{F}(0) = \mathrm{Id}. \tag{1.5}$$

Then, the following holds:

(a) *The set of branch points*

$$\mathcal{S}(U) := \{z \in B_1' : U(z) = 0,\ \nabla U(z) = 0\} \tag{1.6}$$

is a discrete (locally finite) subset of B_1'.

(b) *For every point $z_0 \in \mathcal{S}(U)$ (without loss of generality $z_0 = 0$), there are*

- *a radius $r > 0$ and a quasi-conformal homeomorphism $\Psi: B_r \to \Omega$, between B_r and an open set $\Omega \subset B_1$, such that*

$$\Psi \in W^{1,2}_{\mathrm{loc}}(B_r; \mathbb{R}^2), \tag{1.7}$$

$$\mathrm{Im}(\Psi(z)) \equiv 0 \quad \textit{on } \{\mathrm{Im}(z) \equiv 0\}, \tag{1.8}$$

$$|\Psi(z) - z| = o(|z|); \tag{1.9}$$

- *a holomorphic function $\Phi: B_1 \to \mathbb{C}$ of the form*

$$\Phi(z) = az^k + O(z^{k+1}), \quad \textit{where } k \geq 3 \textit{ and } a \in \mathbb{C}, \tag{1.10}$$

such that we can write the solution U as

$$U(z) = \mathrm{Re}(\Phi(\Psi(z))^{1/2}) \quad \textit{for every } z \in B_r(z_0). \tag{1.11}$$

Remark 1.2 (Optimal regularity). We notice that one particular consequence of the previous theorem is the optimal regularity for solutions of the nonlinear thin-obstacle problem (1.1)–(1.4). In fact, if $U \in C^1(B_1^+ \cup B_1')$ is as in Theorem 1.1, then from (1.11), (1.10) and (1.9) it follows that $U \in C^{1,1/2}(B_1^+ \cup B_1')$.

In the case of the classical thin-obstacle problem in which the operator is the Laplacian, that is, $\mathcal{F}(x, y) = x^2 + y^2$, parts (a) and (b) of Theorem 1.1 were obtained by Lewy in [16]; moreover, in this case, claim (a) can also be obtained by means of Almgren's monotonicity formula (see [2,15]); we also notice that for the classical thin-obstacle problem, the map Ψ from Theorem 1.1 is the identity.

However, in order to apply this result to the one-phase problem described in the next subsection, we will be interested in solutions u of the thin-obstacle problem with

$$\mathcal{F}(x, y) := \frac{x^2 + y^2}{1 + y}$$

and for which $\nabla u \in C^{0,1/2}$ and no better. In particular, it is easy to check that U is a solution of an equation of the form

$$\mathrm{div}(A(x)\nabla U) = 0,$$

where $A(x)$ is no better than $C^{0,1/2}$. For these types of equations, the results in [14] cannot be applied (and actually are known to fail), so in order to obtain our result we need to exploit the "quasi-linear" structure of the problem and our approach, based on the use of quasi-conformal maps, seems to be more suitable, although limited to dimension two.

1.2. One-phase problem with geometric constraint

Next, we consider the one-phase problem constrained above a hyperplane, that is, let $u\colon B_1 \cap \{x_d \geq 0\} \to \mathbb{R}$ be a continuous nonnegative function solution of the problem

$$\Delta u = 0 \quad \text{in } \Omega_u := \{u > 0\} \subset B_1, \tag{1.12}$$

$$u = 0 \quad \text{on } B_1 \cap \{x_d = 0\}, \tag{1.13}$$

$$|\nabla u| = 1 \quad \text{on } \partial\Omega_u \cap \{x_d > 0\}, \tag{1.14}$$

$$|\nabla u| \geq 1 \quad \text{on } \partial\Omega_u \cap \{x_d = 0\}. \tag{1.15}$$

In the recent paper by Chang-Lara and Savin [4], it was shown that if u is a viscosity solution of this problem (that is, if the boundary conditions (1.14) and (1.15) are intended in viscosity sense), then in a neighborhood of any contact point $x = (x', 0) \in \partial\Omega_u \cap \{x_d = 0\}$, the boundary $\partial\Omega_u$ is a $C^{1,\alpha}$-regular graph over the hyperplane $\{x_d = 0\}$. More precisely, in a neighborhood of a point $z_0 \in \partial\Omega_u \cap \{x_d = 0\}$, the boundary $\partial\Omega$ is a $C^{1,1/2}$-regular surfaces, that is, there are a radius $\rho > 0$ and a $C^{1,1/2}$-regular function

$$f\colon B'_\rho(z_0) \to [0, +\infty)$$

such that, up to a rotation and translation of the coordinate system, we have

$$\begin{cases} u(x) > 0 & \text{for } x \in (x', x_d) \in B_\rho(z_0) \text{ such that } x_d > f(x'), \\ u(x) = 0 & \text{for } x \in (x', x_d) \in B_\rho(z_0) \text{ such that } x_d \leq f(x'). \end{cases} \tag{1.16}$$

We denote by $\mathscr{C}_1(u)$ the contact set of the free boundary $\partial\Omega_u$ with the hyperplane $\{x_d = 0\}$,

$$\mathscr{C}_1(u) := \{x_d = 0\} \cap \partial\Omega_u,$$

and by $\mathscr{B}_1(u)$ the set of points at which the free boundary separates from $\{x_d = 0\}$,

$$\mathscr{B}_1(u) := \{x \in \mathscr{C}_1(u) : B_r(x) \cap (\partial\Omega_u \setminus \{x_d = 0\}) \neq \emptyset \ \text{ for every } \ r > 0\}.$$

By $\mathscr{S}_1(u)$ we denote the set of points in $\mathscr{C}_1(u)$ at which u has gradient precisely equal to 1

$$\mathscr{S}_1(u) := \{z \in \mathscr{C}_1(u) : |\nabla u|(z) = 1\}. \tag{1.17}$$

We notice that a priori the set $\mathscr{C}_1(u)$ is no more than a closed subset of $\{x_d = 0\}$. Moreover, if at a point $x = (x', 0)$ we have that $|\nabla u|(x', 0) > 1$, then this point is necessarily in the interior of $\mathscr{C}_1(u)$ in the hyperplane $\{x_d = 0\}$. Thus,

$$\mathscr{S}_1(u) \text{ contains all branch points, } \mathscr{B}_1(u) \subset \mathscr{S}_1(u).$$

Theorem 1.3 (Analyticity at the branch points in the one-phase problem with obstacle). *Let u be a solution of problem* (1.12)–(1.15) *in dimension $d = 2$. Then, the following holds:*

(a) *$\mathcal{S}_1(u)$ is locally finite and $\mathcal{C}_1(u)$ is a locally finite union of disjoint closed intervals of the axis $\{x_2 = 0\}$.*

(b) *For every point $z_0 \in \mathcal{S}_1(u)$, one of the alternatives in Figure 1 holds, that is,*

(b.1) *z_0 is an isolated point of $\mathcal{C}_1(u)$ and, in a neighborhood of z_0, the free boundary $\partial\Omega_u$ is the graph of an analytic function that vanishes only at z_0;*

(b.2) *z_0 lies in the interior of $\mathcal{C}_1(u)$ and there is $r > 0$ such that u is harmonic in $B_r(z_0)$ and $|\nabla u| > 1$ at all points of $\{x_2 = 0\} \cap B_r(z_0)$ except z_0;*

(b.3) *z_0 is an endpoint of a non-trivial interval in the contact set $\mathcal{C}_1(u)$; moreover, there are an interval $\mathcal{I}_\rho = (-\rho, \rho)$ and an analytic function $\phi\colon \mathcal{I}_\rho \to \mathbb{R}$ such that $\phi(0) > 0$ and, up to setting $z_0 = 0$ and rotating the coordinate axis,*

$$f(x) = \begin{cases} 0 & \text{if } x \geq 0, \\ x^{k/2}\,\phi(x) & \text{if } x < 0, \end{cases} \tag{1.18}$$

where f is the function from (1.16).

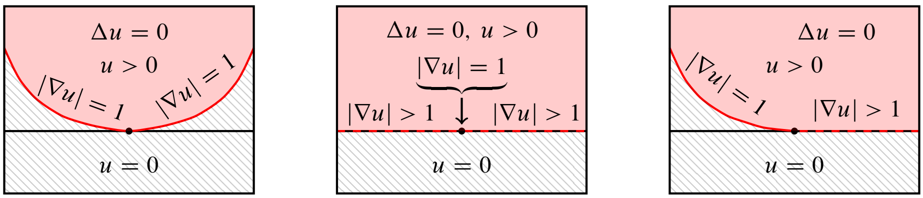

Fig. 1. Branch points for one-phase with obstacle.

As we mentioned above, we will give two proofs of this result. The first will be obtained combining Theorem 1.1 with the standard hodograph transform. The second proof instead, more geometric in spirit, will be achieved via a conformal hodograph transform. This proof has the advantage of being reversible, thus allowing us to construct examples of solutions and free boundaries with any prescribed cuspidal behavior (without invoking any fixed point argument, as usual in the literature).

Theorem 1.4 (Cuspidal points for one-phase problem). *For any positive integer $n \in \mathbb{N}$, there exists a solution of* (1.12)–(1.15) *in dimension $d = 2$ such that* (1.18) *in Theorem* 1.3 *holds with $k = 4n - 1$.*

We point out that Theorems 1.3 and 1.4 can be deduced, respectively, from Theorems 1.6 and 1.8 below by performing an odd reflection.

1.3. Symmetric two-phase problem

Finally, we consider solutions to the two-phase free boundary problem in viscosity sense, that is, we let $u\colon B_1 \to \mathbb{R}$ be a continuous function and denote by u_+ and u_- the functions

$$u_+ = \max\{u, 0\} \quad \text{and} \quad u_- := \min\{u, 0\},$$

and by Ω_u^+ and Ω_u^- the sets

$$\Omega_u^\pm := \{\pm u > 0\}.$$

Notice that with this notation, u_- is negative. Then u is a viscosity solution of the problem

$$\Delta u = 0 \quad \text{in } \Omega_u^+ \cup \Omega_u^-, \tag{1.19}$$

$$|\nabla u_+| = 1 \quad \text{on } \partial\Omega_u^+ \setminus \partial\Omega_u^- \cap B_1, \tag{1.20}$$

$$|\nabla u_-| = 1 \quad \text{on } \partial\Omega_u^- \setminus \partial\Omega_u^+ \cap B_1, \tag{1.21}$$

$$|\nabla u_+| = |\nabla u_-| \geq 1 \quad \text{on } \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_1. \tag{1.22}$$

In [9], we proved that if u is a viscosity solution of this problem in any dimension $d \geq 2$, then in a neighborhood of any two-phase point

$$x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_1,$$

both free boundaries $\partial\Omega_u^+ \cap B_1$ and $\partial\Omega_u^- \cap B_1$ are $C^{1,\alpha}$-regular. Thus, by the classical elliptic regularity theory, also the functions $u_\pm$ are $C^{1,\alpha}$-regular on $\bar{\Omega}_u^+ \cap B_1$ and $\bar{\Omega}_u^- \cap B_1$, respectively, and equations (1.19)–(1.22) hold in the classical sense.

We will denote by $\mathcal{C}_2(u_+, u_-)$ the two-phase free boundary, which is the contact set between the free boundaries $\partial\Omega_u^+$ and $\partial\Omega_u^-$, and by $\mathcal{O}_\pm$ the remaining one-phase parts:

$$\mathcal{C}_2(u_+, u_-) := \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_1 \quad \text{and} \quad \mathcal{O}_\pm := (\partial\Omega_u^\pm \cap B_1) \setminus \mathcal{C}_2(u_+, u_-).$$

We notice that the set $\mathcal{C}_2(u_+, u_-)$ is closed, while $\mathcal{O}_+$ and $\mathcal{O}_-$ are relatively open subsets of $\partial\Omega_u^\pm \cap B_1$, respectively. We define the set of branch points $\mathcal{B}_2(u_+, u_-)$ as the set of points at which the two free boundaries $\partial\Omega_u^\pm$ are separated, that is,

$$\mathcal{B}_2(u_+, u_-) = \{x \in \mathcal{C}_2(u_+, u_-) : B_r(x) \cap \mathcal{O}_\pm \neq \emptyset \text{ for every } r > 0\}. \tag{1.23}$$

By C^1-regularity of $u_\pm$, if $x \in (\partial\Omega_u^+ \cup \partial\Omega_u^-) \cap B_1$ is such that

$$|\nabla u_+|(x) > 1 \quad \text{or} \quad |\nabla u_-|(x) > 1,$$

then it is necessarily a two-phase non-branch point $x \in \mathcal{C}_2(u_+, u_-) \setminus \mathcal{B}_2(u_+, u_-)$. In particular, this implies that the set

$$\mathcal{S}_2(u_+, u_-) := \{x \in \mathcal{C}_2(u_+, u_-) : |\nabla u_+|(x) = |\nabla u_-|(x) = 1\}, \tag{1.24}$$

contains the set of branch points $\mathcal{B}_2(u_+, u_-)$.

In dimension $d = 2$, $\partial\Omega_u^{\pm}$ are locally parametrized by two $C^{1,\alpha}$ curves. Precisely, suppose that $z_0 = (x_0, y_0) \in \mathcal{C}_2(u_+, u_-)$, without loss of generality we may assume that $z_0 = (0, 0)$, and that there are an interval $\mathcal{I}_\rho := (-\rho, \rho)$ and two $C^{1,\alpha}$-regular functions

$$f_\pm \colon \mathcal{I}_\rho \to \mathbb{R},$$

such that

$$f_+ \geq f_- \text{ on } \mathcal{I}_\rho \quad \text{and} \quad f_+(0) = f_-(0) = \partial_x f_+(0) = \partial_x f_-(0) = 0,$$

and, up to rotations and translations,

$$\begin{cases} u(x,y) > 0 & \text{for } (x,y) \in \mathcal{I}_\rho \times \mathcal{I}_\rho \text{ such that } y > f_+(x), \\ u(x,y) = 0 & \text{for } (x,y) \in \mathcal{I}_\rho \times \mathcal{I}_\rho \text{ such that } f_-(x) \leq y \leq f_+(x), \\ u(x,y) < 0 & \text{for } (x,y) \in \mathcal{I}_\rho \times \mathcal{I}_\rho \text{ such that } y < f_-(x). \end{cases} \tag{1.25}$$

Thus, in the square $\mathcal{I}_\rho \times \mathcal{I}_\rho$, the one-phase parts $\mathcal{O}_+$ and $\mathcal{O}_-$ of the free boundary are the union of $C^{1,\alpha}$-regular (actually analytic) graphs over a countable family of disjoint open intervals

$$\mathcal{O}_\pm := \bigcup_{i \in \mathbb{N}} \Gamma_\pm^i,$$

where for every $i \in \mathbb{N}$, there is an open interval $\mathcal{I}_i \subset \mathcal{I}_\rho$ such that

$$\Gamma_\pm^i = \{(x, f_\pm(x)) : x \in \mathcal{I}_i\}. \tag{1.26}$$

Definition 1.5 (Symmetric solutions of the two-phase problem). In dimension $d = 2$, we will say that a continuous function $u \colon B_1 \to \mathbb{R}$ is a *symmetric solution to the two-phase problem* if u satisfies (1.19)–(1.22) and moreover

$$\mathcal{H}^1(\Gamma_+^i) = \mathcal{H}^1(\Gamma_-^i) \quad \text{for every } i \in \mathbb{N} \text{ such that } \overline{\mathcal{I}_i} \subset \mathcal{I}_\rho.$$

The main result of this section is the following.

Theorem 1.6 (Cuspidal points for the symmetric solutions of the two-phase problem). *Let $u \colon B_1 \to \mathbb{R}$ be a viscosity solution of the two-phase problem* (1.19)–(1.22). *Then the following holds:*

(a) *If u is symmetric in the sense of Definition* 1.5, *then the singular set $\mathcal{S}_2(u_+, u_-)$ defined in* (1.24) *is locally finite, so in particular the two-phase free boundary*

$$\mathcal{C}_2(u_+, u_-) = (\partial\Omega_u^+ \cup \partial\Omega_u^-) \cap B_1$$

is a locally finite union of disjoint $C^{1,\alpha}$-arcs;

(b) *If $z_0 \in \mathcal{S}_2(u_+, u_-)$ is an isolated point of $\mathcal{S}_2(u_+, u_-)$, then one of the alternatives in Figure* 2 *holds, that is,*

(b.1) *z_0 is an isolated point of $\mathcal{C}_2(u_+, u_-)$ and, in a neighborhood of z_0, the free boundaries $\partial\Omega_u^+$ and $\partial\Omega_u^-$ are analytic graphs meeting only in z_0;*

(b.2) *z_0 lies in the interior of $\mathcal{C}_2(u_+, u_-)$ and moreover there is $r > 0$ such that $\Delta u = 0$ in $B_r(z_0)$ and $|\nabla u| > 1$ at all points of $\{u = 0\} \cap B_r(z_0)$ except z_0;*

(b.3) *z_0 is an endpoint of a non-trivial arc in $\mathcal{C}_2(u_+, u_-)$, and there are an interval $\mathcal{I}_\rho = (-\rho, \rho)$, a constant $k \in \mathbb{N}$, $k \geq 3$, and an analytic function $\phi\colon \mathcal{I}_\rho \to \mathbb{R}$ such that $\phi(0) \neq 0$ and, up to setting $z_0 = 0$ and changing the coordinates,*

$$f_+(x) - f_-(x) = \begin{cases} x^{k/2}\,\phi(|x|^{1/2}) & \text{if } x \leq 0, \\ 0 & \text{if } x \geq 0. \end{cases} \tag{1.27}$$

Precisely, there are analytic functions Φ, $\beta_\pm$ and Θ such that for every $x \leq 0$

$$f_\pm(x) = \Phi(x + |x|^{5/2}\beta_\pm(|x|^{1/2})) \pm \Psi(x + |x|^{5/2}\beta_\pm(|x|^{1/2})), \tag{1.28}$$

where Ψ is of the form $\Psi(x) = |x|^{3/2}\Theta(x)$.

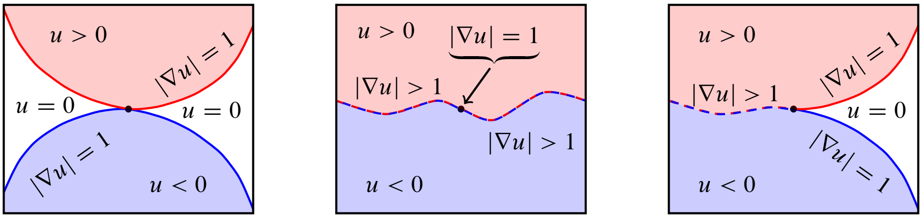

Fig. 2. Branch points for two-phase.

Notice that (a) of the previous theorem requires that the function u is symmetric in the generalized sense of Definition 1.5, while (b.3) is always true at isolated branch points. The question of whether the statement of Theorem 1.6 (a) is true without the generalized symmetry assumption is extremely interesting and would probably require the introduction of new techniques.

We also have the following result, which simply follows from the fact that if z_0 is an isolated point of $\mathcal{B}_2(u_+, u_-)$, then it is also an isolated point of $\mathcal{S}_2(u_+, u_-)$ for which Theorem 1.6 (b.2) does not hold.

Corollary 1.7 (Isolated cuspidal points of two-phase problem). *Let u be a solution of the two-phase problem as in Definition 1.5. If $z_0 \in \mathcal{B}_2(u_+, u_-)$ is an isolated point of the set $\mathcal{B}_2(u_+, u_-)$ defined in (1.23), then at least one of points (b.1) and (b.3) is true at z_0.*

We will prove Theorem 1.6 in Section 5, where we will also discuss the obstructions in applying the conformal hodograph transform to the study of the branch points of the two-phase problem in the absence of symmetries or in the presence of weights $\lambda_\pm$ on the volume of the positivity and the negativity sets.

Finally, as in Theorem 1.4, by reversing the argument from the proof of Theorem 1.6, we can construct two-phase cusps with prescribed behavior.

Theorem 1.8 (Cuspidal points for two-phase problem). *For any positive integer $n \in \mathbb{N}$, there exists a solution of* (1.19)–(1.22) *in dimension $d = 2$ such that* (1.27) *holds with $k = 4n - 1$ and* (1.28) *holds with $\Phi(x) = x^m + o(x)$, where $m \geq 2$.*

The particular case $\Phi \equiv 0$ is an immediate consequence from Theorem 1.4 as a solution of the one-phase problem, together with its reflection, gives a solution of the two-phase one. However, the same method provides also non-symmetric examples in which the asymmetry is given by the function Φ.

We notice that the examples constructed in Theorem 1.8 are minimizers of the two-phase functional. Indeed, any flat monotone solution to (1.19)–(1.22) is unique and so it minimizes the two-phase functional; we prove this in Appendix A as a direct consequence of the maximum principle. We refer to the recent work [5] for some interesting examples of almost-minimizing free boundaries.

2. Nonlinear thin-obstacle problem

In this section, we prove Theorem 1.1 using the theory of quasi-conformal map.

2.1. *Notation and known results*

Let $U \in C^1(B_1^+ \cup B_1')$ be a solution of the thin-obstacle problem (1.1)–(1.4), where the function $\mathcal{F}: \mathbb{R}^2 \to \mathbb{R}$ is C^2-regular.

2.1.1. Variational inequality formulation. System (1.1)–(1.4) can be equivalently written in the form of a variational inequality. Precisely, the following are equivalent:

(1) $U \in C^1(B_1^+ \cup B_1')$ and satisfies (1.1)–(1.4);

(2) $U \in H^1_{\mathrm{loc}}(B_1^+ \cup B_1')$ (that is, $u \in H^1(B_r^+)$ for every $r < 1$) and

$$\int_{B_1^+} \nabla\mathcal{F}(\nabla U) \cdot \nabla(U - v)\, dx \leq 0 \quad \text{for every } v \in \mathcal{K}_U, \tag{2.1}$$

where $\mathcal{K}_U$ is the convex set

$$\mathcal{K}_U := \big\{v \in H^1_{\mathrm{loc}}(B_1^+ \cup B_1') : v \geq 0 \text{ on } B_1',\ v = U \text{ in a neighborhood of } \partial B_1 \cap \{x_d > 0\}\big\}.$$

Indeed, the implication (1) $\Rightarrow$ (2) follows simply by an integration by parts, while (2) $\Rightarrow$ (1) was proved by Frehse [13]. In particular, if $U \in H^1(B_1^+)$ minimizes the integral functional

$$\mathcal{J}(v) := \int_{B_1^+} \mathcal{F}(\nabla v)\, dx \tag{2.2}$$

among all functions in $\mathcal{K}_U$, then U satisfies the variational inequality (2.1).

2.1.2. Higher regularity of the solutions. It was proved by Frehse in [13, Lemma 2.2] that if $U \in H^1(B_1^+)$ is a solution of the variational inequality (2.1), then U is in $H^2(B_r^+)$ for every $r < 1$. Moreover, in [10, Theorem 4.1] it was shown that the solution U is actually in $C^{1,\alpha}(B_1^+ \cup B_1')$ for some $\alpha > 0$.

2.2. *Local finiteness of the set of branch points*

In this subsection, we prove Theorem 1.1 (a). We introduce a special function Q that we prove to be quasi-regular in the half-ball, then we obtain Theorem 1.1 (a) by applying Stoïlow's factorization theorem for quasi-conformal and quasi-regular maps (see [1, Chapter 5]).

Given a solution $U\colon B_1 \cap \{y \geq 0\} \to \mathbb{R}$ of (1.1)–(1.4), we consider the function

$$Q\colon B_1^+ \cap \{y \geq 0\} \to \mathbb{C}, \quad Q(x+iy) = \partial_x U - i\mathcal{F}_2(\nabla U(x,y)). \tag{2.3}$$

We gather the fundamental properties of this function in the next lemma.

Lemma 2.1. *The function Q defined in (2.3) satisfies the following properties:*

(1) $Q^2 \in W^{1,2}(B_r^+;\mathbb{C})$ *for every $r < 1$;*

(2) *there is $r_0 > 0$ such that, for every $r < r_0$, Q satisfies the Beltrami equation*

$$\partial_{\bar z} Q = \mu(\nabla U, \nabla^2 U)\partial_z Q \quad \textit{in } B_r^+,$$

and if for some $\delta \in (0,1]$

$$\|\mathrm{Id} - \nabla^2\mathcal{F}(\nabla U(z))\|_2 \leq \delta \quad \textit{for every } z = (x,y) \in B_r^+,$$

then

$$|\mu(\nabla U(z), \nabla^2 U(z))| \leq \frac{\delta}{\sqrt{4 - 4\delta - \delta^2}} \quad \textit{for every } z = (x,y) \in B_r^+,$$

where for any real matrix $A = (a_{ij})_{ij}$, $\|A\|_2 := (\sum_{i,j} a_{ij}^2)^{1/2}$.

In particular, properties (1) and (2) imply that Q is a quasi-conformal map.

Proof. We first prove (1). By [13], we know that $U \in H^2(B_r^+)$ and that $|\nabla U| \in L^\infty(B_r^+)$. Thus, (1) follows directly by the definition of Q. Let us now prove (2).

For simplicity, we set

$$A := \partial_x U \quad \text{and} \quad B := \mathcal{F}_2(\nabla U).$$

Thus, $Q = A - iB$ and

$$\begin{cases} \partial_{\bar z} Q = \dfrac{1}{2}(\partial_x + i\partial_y)(A - iB) = \dfrac{1}{2}(\partial_x A + \partial_y B) + \dfrac{i}{2}(\partial_y A - \partial_x B), \\ \partial_z Q = \dfrac{1}{2}(\partial_x - i\partial_y)(A - iB) = \dfrac{1}{2}(\partial_x A - \partial_y B) - \dfrac{i}{2}(\partial_y A + \partial_x B), \end{cases}$$

which implies

$$\begin{cases} 4|\partial_{\bar z} Q|^2 = (\partial_x A + \partial_y B)^2 + (\partial_y A - \partial_x B)^2, \\ 4|\partial_z Q|^2 = (\partial_x A - \partial_y B)^2 + (\partial_y A + \partial_x B)^2. \end{cases} \tag{2.4}$$

We first compute

$$\begin{cases} \partial_x A = \partial_{xx} U, \\ \partial_y A = \partial_{xy} U, \\ \partial_x B = \mathcal{F}_{12}(\nabla U)\partial_{xx} U + \mathcal{F}_{22}(\nabla U)\partial_{xy} U, \\ \partial_y B = \mathcal{F}_{12}(\nabla U)\partial_{xy} U + \mathcal{F}_{22}(\nabla U)\partial_{yy} U, \end{cases}$$

and, using the equation for U, we obtain

$$\begin{cases} \partial_x A + \partial_y B = (1 - \mathcal{F}_{11}(\nabla U))\partial_{xx} U - \mathcal{F}_{12}(\nabla U)\partial_{xy} U, \\ \partial_y A - \partial_x B = -\mathcal{F}_{12}(\nabla U)\partial_{xx} U + (1 - \mathcal{F}_{22}(\nabla U))\partial_{xy} U. \end{cases} \tag{2.5}$$

For simplicity, we use the following notation:

$$m_{ij} := \delta_{ij} - \mathcal{F}_{ij}(\nabla U) \quad \text{for every } 1 \le i, j \le 2,$$

and

$$\mathcal{M} := \mathrm{Id} - \nabla^2 \mathcal{F}(\nabla U) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}.$$

We also set

$$\|\mathcal{M}\|_2^2 := m_{11}^2 + 2m_{12}^2 + m_{22}^2.$$

Then, by (2.5) and the Cauchy–Schwarz inequality, we immediately obtain

$$(\partial_x A + \partial_y B)^2 + (\partial_y A - \partial_x B)^2 \le \|\mathcal{M}\|_2^2 |\nabla A|^2. \tag{2.6}$$

In order to estimate $|\partial_z Q|^2$ in (2.4), we write

$$\begin{aligned} &(\partial_x A - \partial_y B)^2 + (\partial_y A + \partial_x B)^2 \\ &\quad = (2\partial_x A - (\partial_x A + \partial_y B))^2 + (2\partial_y A - (\partial_y A - \partial_x B))^2 \\ &\quad = 4|\nabla A|^2 - 4\nabla A \cdot \mathcal{M}(\nabla A) + (\partial_x A + \partial_y B)^2 + (\partial_y A - \partial_x B)^2 \\ &\quad =: 4|\nabla A|^2 + \mathcal{R}, \end{aligned}$$

where by (2.5) and (2.6) we have the estimate

$$|\mathcal{R}| \le (4\|\mathcal{M}\|_2 + \|\mathcal{M}\|_2^2)|\nabla A|^2.$$

Now, if at some point $\nabla A = 0$, then

$$\partial_z Q = \partial_{\bar z} Q = 0.$$

Thus, we can define μ as follows:

$$\mu = 0 \quad \text{if } \nabla A = 0, \qquad \mu = \frac{\partial_{\bar z} Q}{\partial_z Q} \quad \text{if } \nabla A \neq 0.$$

Since A, $\partial_{\bar{z}} Q$ and $\partial_z Q$ are all functions of ∇U and $\nabla^2 U$, μ also can be written in terms of the same variables, that is, $\mu = \mu(\nabla U, \nabla^2 U)$. We notice that with this definition, μ remains bounded. Indeed,

$$|\mu|^2 = \left|\frac{\partial_{\bar{z}} Q}{\partial_z Q}\right|^2 \leq \frac{\|\mathcal{M}\|_2^2}{4 - 4\|\mathcal{M}\|_2 + \|\mathcal{M}\|_2^2},$$

so that for r sufficiently small, the conclusion follows. ∎

Proof of Theorem 1.1 (a). Let Q be the function defined in (2.3) and let

$$S(z) := \begin{cases} Q(z)^2 & \text{if } \operatorname{Im}(z) \geq 0, \\ \overline{S}(\bar{z}) & \text{if } \operatorname{Im}(z) \leq 0. \end{cases}$$

We notice that

$$\operatorname{Im}(Q^2(z)) = \partial_x U \cdot \mathcal{F}_2(\nabla U) = 0 \quad \text{on } \{\operatorname{Im}(z) = 0\},$$

so that the function S is in $W^{1,2}(B_r)$ and satisfies the Beltrami equation

$$\partial_{\bar{z}} S = \psi(z)\, \partial_z S \quad \text{in } B_r^+,$$

where

$$\psi(z) = \psi(x + iy) := \begin{cases} \mu(\nabla U(x, y), \nabla^2 U(x, y)) & \text{if } \operatorname{Im}(z) \geq 0, \\ \overline{\psi}(\bar{z}) & \text{if } \operatorname{Im}(z) \leq 0. \end{cases}$$

Thus, by [1, Theorem 5.5.2], we conclude that the zeros of the function S are isolated, which is the claim. ∎

2.3. *Local behavior of the solutions at branch points*

In this subsection, we prove Theorem 1.1 (b). Given a branch point $z_0 \in \mathcal{S}$, we construct a quasi-regular mapping whose real part is precisely the solution U. Assume that $z_0 = 0$. We consider the case where there exists a radius $r > 0$ such that

$$\{U = 0\} \cap B_r' = \{x \leq 0\} \cap B_r' \quad \text{and} \quad \{U > 0\} \cap B_r' = \{x > 0\} \cap B_r', \tag{2.7}$$

which is the case of a branch point, the other two cases in Figure 1 being analogous.

We notice that the differential form

$$\alpha = -\mathcal{F}_2(\nabla U)\, dx + \mathcal{F}_1(\nabla U)\, dy$$

is closed in B_r^+, and so the potential

$$V\colon B_r^+ \cup B_r' \to \mathbb{R}, \quad V(x, y) := \int_0^1 (-\mathcal{F}_2(\nabla U(tx, ty))x + \mathcal{F}_1(\nabla U(tx, ty))y)\, dt$$

is Lipschitz continuous in $B_r^+ \cup B_r'$, C^2 in B_r^+ and satisfies

$$\begin{cases} \partial_x V = -\mathcal{F}_2(\nabla U) & \text{in } B_r^+, \\ \partial_y V = \mathcal{F}_1(\nabla U) & \text{in } B_r^+, \\ UV = 0 & \text{on } B_r', \end{cases}$$

where the last equality follows from (2.7), (1.3) and the very definition of V. We next define the complex function

$$P\colon B_r^+ \cap \{y \geq 0\} \to \mathbb{C}, \quad P(x+iy) = U(x,y) + iV(x,y). \tag{2.8}$$

Remark 2.2. Notice that, by the definition of V, we have $\partial_x P = Q$ in B_r^+.

We now prove the following lemma.

Lemma 2.3. *The function P defined in* (2.8) *satisfies the following properties:*

(1) $P^2 \in W^{1,\infty}_{\mathrm{loc}}(B_1^+ \cup B_1')$*;*

(2) *P is a solution of the Beltrami equation*

$$\partial_{\bar z} P = \eta(\nabla U)\, \partial_z P \quad \textit{in } B_r^+, \tag{2.9}$$

where $\eta(\nabla U) = o(|\nabla U|)$.

Proof. The first claim follows from the Lipschitz continuity of U and V. In order to prove the second claim, we compute

$$\begin{cases} 2\partial_{\bar z} P = (\partial_x + i\partial_y)(U + iV) = (\partial_x U - \mathcal{F}_1(\nabla U)) + i(\partial_y U - \mathcal{F}_2(\nabla U)), \\ 2\partial_z P = (\partial_x - i\partial_y)(U + iV) = (\partial_x U + \mathcal{F}_1(\nabla U)) - i(\partial_y U + \mathcal{F}_2(\nabla U)). \end{cases}$$

Now, by the differentiability of $\mathcal{F}_1$ and $\mathcal{F}_2$ at zero and (1.5), we can write

$$\mathcal{F}_1(X) - X_1 = \varepsilon_1(X)|X| \quad \text{and} \quad \mathcal{F}_2(X) - X_2 = \varepsilon_2(X)|X|$$

for every

$$X = (X_1, X_2) \in \mathbb{R}^2,$$

where the functions ε_1 and ε_2 are such that

$$\lim_{|X|\to 0} \varepsilon_1(X) = \lim_{|X|\to 0} \varepsilon_2(X) = 0,$$

from which the first part of the claim follows. ∎

Proof of Theorem 1.1 (b). Let P be the function defined in (2.8) and let

$$T(z) := \begin{cases} P(z)^2 & \text{if } \mathrm{Im}(z) \geq 0, \\ \overline{T}(\bar z) & \text{if } \mathrm{Im}(z) \leq 0. \end{cases}$$

Then

$$\mathrm{Im}(P^2(z)) = U(z)V(z) = 0 \quad \text{on } \{\mathrm{Im}(z) = 0\},$$

so T is Lipschitz continuous on B_r and satisfies the Beltrami equation

$$\partial_{\bar{z}} T = \phi(z)\, \partial_z T \quad \text{in } B_r, \tag{2.10}$$

where ϕ is the extension over the whole B_r of the Beltrami coefficient $\eta(\nabla U)$ from (2.9),

$$\phi(z) = \phi(x+iy) := \begin{cases} \eta(\nabla U(x,y)) & \text{if } \mathrm{Im}(z) \geq 0, \\ \bar{\phi}(\bar{z}) & \text{if } \mathrm{Im}(z) \leq 0. \end{cases}$$

According to [1, Theorem 5.5.1 and Corollary 5.5.3], there exist a homeomorphism $\Psi \in W^{1,2}(B_r; B_1)$, which is a solution of (2.10) and such that $\Psi(0) = 0$ and $\Psi(\rho) = \rho$, for some $\rho < r$, and a holomorphic function $\Phi\colon \Omega \to \mathbb{C}$ such that

$$T(z) = \Phi(\Psi(z)) \quad \forall z \in B_r. \tag{2.11}$$

Next we prove (1.8). Observe that if Ψ is a solution to (2.10), then also $\bar{\Psi}(\bar{z})$ is a solution to (2.10), and moreover $\bar{\Psi}(0) = \Psi(0) = 0$ and $\bar{\Psi}(\rho) = \Psi(\rho) = 1$. It follows, by the uniqueness of normalized solutions, that $\bar{\Psi}(z) = \Psi(z)$, which implies (1.8).

Finally, we come to equation (1.9). Suppose by contradiction that (1.9) is false. Then, there is a sequence of radii $\rho_k \to 0$ such that the sequence of homeomorphisms $\Psi_k \in W^{1,2}(B_r, B_1)$, solutions of

$$\partial_{\bar{z}} \Psi_k = \phi(z)\, \partial_z \Psi \text{ in } B_r, \quad \Psi_k(0) = 0, \quad \Psi_k(\rho_k) = \rho_k,$$

does not converge uniformly to the function z. Now consider the sequence of functions $\widetilde{\Psi}_k(z) := \rho_k^{-1} \Psi_k(\rho_k z)$, then they are solutions of

$$\partial_{\bar{z}} \widetilde{\Psi}_k = \phi(\rho_k z)\, \partial_z \widetilde{\Psi} \text{ in } B_{r/\rho_k}, \quad \widetilde{\Psi}_k(0) = 0, \quad \widetilde{\Psi}_k(1) = 1.$$

Reasoning as in the proof of Lemma 2.3 and using the fact that $\nabla U(\rho_k z) \to 0$ as $k \to \infty$, since $U \in C^1$ and $\nabla U(0) = 0$, we have

$$\lim_{k \to 0} \phi(\rho_k z) = 0 \quad \text{a.e. } z \in B_{r/\rho_k}.$$

Using [1, Lemma 5.3.5], we have that $\widetilde{\Phi}_k$ converges locally uniformly to a homeomorphism $\widetilde{\Psi}\colon \mathbb{C} \to \mathbb{C}$, which is a solution of

$$\partial_{\bar{z}} \widetilde{\Psi} = 0 \text{ in } \mathbb{C}, \quad \widetilde{\Psi}(0) = 0, \quad \widetilde{\Psi}(1) = 1.$$

But this implies that $\widetilde{\Psi}(z) = z$, which is a contradiction for k sufficiently large.

In particular, notice that if $\Phi(z) = z^k + O(z^{k+1})$, then the C^1-regularity of solutions to the nonlinear thin-obstacle problem (see, for instance, [11]) implies that $k \geq 3$. ∎

3. Theorem 1.3: Proof via quasi-conformal maps

In this section, we will prove Theorem 1.3 as a consequence of Theorem 1.1 combined with an application of the hodograph transform.

3.1. *The hodograph transform*

In this section, we write the hodograph transformation of a solution u of (1.12)–(1.15). We do this in every dimension $d \geq 2$.

3.1.1. Notation. We adopt the following notation. We write every point $x \in \mathbb{R}^d$ in coordinates as $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$. For every $\rho > 0$, we denote by B_ρ and B'_ρ the balls centered in zero of radius ρ in $\mathbb{R}^d$ and $\mathbb{R}^{d-1}$, respectively. We will identify $\mathbb{R}^{d-1}$ with the hyperplane $\mathbb{R}^{d-1} \times \{0\} \subset \mathbb{R}^d$, thus

$$B'_\rho = B_\rho \cap \{x_d = 0\} \quad \text{and} \quad B^+_\rho = B_\rho \cap \{x_d > 0\}.$$

We denote by $\nabla_{x'}$ the gradient with respect to $x' = (x_1, \dots, x_{d-1})$, the first $d-1$ coordinates. Thus, for every function $u: \mathbb{R}^d \to \mathbb{R}$, we can write the full gradient ∇u as

$$\nabla u = (\nabla_{x'} u, \partial_d u) \quad \text{and} \quad |\nabla u|^2 = |\nabla_{x'} u|^2 + |\partial_d u|^2.$$

Let us assume that $0 \in \mathcal{S}_1(u)$, that is, 0 is a branch point, and let $f \in C^{1,\alpha}$ be the function that locally describes the free boundary $\partial\Omega_u$ as in (1.16), so that

$$f(0) = 0 \quad \text{and} \quad \nabla_{x'} f(0) = 0.$$

Now since $u(x', f(x'))$ vanishes for every $x' \in B'_\rho$, we have that $\nabla_{x'} u(0) = 0$. Thus

$$\nabla u(0) = \partial_d u(0)\, e_d \quad \text{and} \quad \partial_d u(0) \geq 1.$$

3.1.2. The hodograph transform. Let $0 \in \partial\Omega_u \cap \{x_d = 0\}$ and $f: B'_\rho \to [0, +\infty)$ be as above. We consider the change of coordinates

$$y' = x', \quad y_d = u(x', x_d).$$

Since $u \in C^{1,\alpha}(\overline{\Omega}_u \cap B_1)$ and since $\partial_d u(0) \geq 1 > 0$, we have that the function

$$T\colon B_\rho \cap \overline{\Omega}_u \to \mathbb{R}^d \cap \{y_d \geq 0\}, \quad T(x', x_d) = (y', y_d),$$

is invertible for ρ small enough. In particular, the set $T(B_\rho \cap \overline{\Omega}_u)$ is an open neighborhood of 0 in the upper half-space $\mathbb{R}_d \cap \{y_d \geq 0\}$. Let

$$S\colon T(B_\rho \cap \overline{\Omega}_u) \to B_\rho \cap \overline{\Omega}_u, \quad S(y', y_d) = (x', x_d),$$

be the inverse of T. Since the map T does not change the first $d-1$ coordinates, there is a $C^{1,\alpha}$-regular function v, defined on the set $T(B_\rho \cap \overline{\Omega}_u)$, such that

$$S(y', y_d) = (y', v(y', y_d)).$$

We will write this in coordinates as

$$x' = y', \quad x_d = v(y', y_d).$$

Remark 3.1. The function v contains all the information of the free boundary $\partial\Omega_u$. Precisely, for every x' in a neighborhood of $0 \in \mathbb{R}^{d-1}$, we have

$$v(x', 0) = f(x'). \tag{3.1}$$

Indeed, it is immediate to check that for any point (x', x_d) in a neighborhood of zero,

$$x_d = f(x') \;\Leftrightarrow\; (x', x_d) \in \partial\Omega_u \;\Leftrightarrow\; x_d = v(x', u(x', x_d)) = v(x', 0).$$

As a consequence of (3.1), we get that

$$v(x', 0) \geq 0 \quad \text{for every } x' \text{ in a neighborhood of zero in } \mathbb{R}^{d-1}.$$

Lemma 3.2 (Hodograph transform). *Let u, T, B_ρ and v be as above. Then, there is $r > 0$ such that*

$$B_r \cap \{x_d \geq 0\} \subset T(B_\rho \cap \overline{\Omega}_u)$$

and such that the function

$$v\colon B_r \cap \{x_d \geq 0\} \to \mathbb{R}$$

exists, is $C^{1,\alpha}$ in $B_r \cap \{x_d \geq 0\}$ and C^∞ in $B_r \cap \{x_d > 0\}$. Moreover, the function

$$w\colon B_r \cap \{x_d \geq 0\} \to \mathbb{R}, \quad w(x', x_d) = v(x', x_d) - x_d,$$

solves the nonlinear thin-obstacle problem

$$\operatorname{div}(\nabla\mathcal{F}(\nabla w)) = 0 \quad \text{in } B_r^+, \tag{3.2}$$

$$w \geq 0 \quad \text{on } B_r', \tag{3.3}$$

$$\mathcal{F}_d(\nabla w) = 0 \quad \text{on } \{w > 0\} \cap B_r', \tag{3.4}$$

$$\mathcal{F}_d(\nabla w) \leq 0 \quad \text{on } \{w = 0\} \cap B_r' \tag{3.5}$$

for the nonlinearity

$$\mathcal{F}(x', x_d) := \frac{|x'|^2 + x_d^2}{1 + x_d}.$$

Remark 3.3. We notice that (3.1) implies that the contact sets of the solution of the one-phase problem u and the solution of the nonlinear thin-obstacle problem w are mapped one into the other,

$$\mathcal{C}_1(u) = \partial\Omega_u \cap B_r' = S(\{w = 0\} \cap B_r')$$

as well as the singular sets defined in (1.6) and (1.17)

$$\mathcal{S}_1(u) = B_r' \cap \{u = 0\} \cap \{|\nabla u| = 1\} = S(B_r' \cap \{w = 0\} \cap \{|\nabla w| = 0\}).$$

Proof of Lemma 3.2. We first notice that

$$w(x', 0) = v(x', 0) = f(x') \quad \text{for every } x' \in B_r'.$$

This proves (3.3) and the first part of (3.5). Next, we notice that since

$$v(x',u(x',x_d)) = x_d \quad \text{for every } (x',x_d) \in B_\rho \cap \overline{\Omega}_u,$$

we have that

$$\partial_i v_+(x',u_+(x',x_d)) + \partial_d v_+(x',u(x',x_d))\partial_i u_+(x',x_d) = 0 \tag{3.6}$$

for $i = 1,\dots,d-1$, and

$$\partial_d v(x',u(x',x_d))\partial_d u(x',x_d) \equiv 1. \tag{3.7}$$

Thus, we can compute

$$(1+\partial_d w(x',0))\partial_d u(x',f(x')) \equiv 1, \tag{3.8}$$

and since $\partial_d u(x',0) \geq 1$, we also obtain the second part of (3.5).

Next, in order to prove that the boundary condition (3.4) holds, we notice that it is equivalent to

$$(\partial_d v(x',0))^2 = 1 + |\nabla_{x'} f(x')|^2 \quad \text{for } x' \in B_r' \cap \{f>0\},$$

and, in view of (3.8), it is also equivalent to

$$(\partial_d u(x',f(x')))^2(1+|\nabla_{x'} f(x')|^2) = 1 \quad \text{for } x' \in B_r' \cap \{f>0\},$$

which is a consequence of the identity

$$\partial_i u(x',f(x')) + \partial_d u(x',f(x'))\partial_i f(x') \equiv 0 \quad \text{for every } i = 1,\dots,d-1,$$

and the boundary condition

$$(-\nabla_{x'} f(x'),1)\cdot\nabla u(x',f(x')) = -(|\nabla_{x'} f(x')|^2+1)^{1/2} \quad \text{on } \{f>0\}.$$

In order to prove (3.2), we notice that in Ω_u, u is a local minimizer of the Dirichlet integral

$$J(u) = \int |\nabla u|^2\,dx,$$

which can be expressed in terms of w by applying (3.6) and (3.7),

$$|\nabla u|^2(x',x_d) = \frac{|\nabla_{x'} v|^2(x',u(x',x_d))+1}{|\partial_d v|^2(x',u(x',x_d))} \quad \text{and} \quad \det(\nabla T)(x',x_d) = \partial_d u(x',x_d).$$

Now, by the change of coordinates $y' = x'$, $y_d = u(x',x_d)$, we get

$$\begin{aligned}\int_{B_\rho\cap\overline{\Omega}_u} |\nabla u|^2\,dx &= \int \frac{|\nabla_{y'} v|^2(y',y_d)+1}{|\partial_d v|^2(y',y_d)}\,\frac{1}{|\partial_d u(x',x_d)|}\,dy \\ &= \int \frac{|\nabla_{y'} v|^2(y',y_d)+1}{\partial_d v(y',y_d)}\,dy,\end{aligned}$$

where all the integrals in dy are over $T(B_\rho \cap \bar{\Omega}_u)$. Now, by the definition of w, we get

$$\int_{B_\rho \cap \Omega_u} |\nabla u|^2\, dx = \int_{T(B_\rho \cap \bar{\Omega}_u)} \Big(\frac{|\nabla w|^2(y', y_d)}{1 + \partial_d w(y', y_d)} + 2 \Big)\, dy.$$

Thus, w minimizes the functional

$$J(w) = \int \frac{|\nabla w|^2(y', y_d)}{1 + \partial_d w(y', y_d)}\, dy$$

in the open set $T(B_\rho \cap \Omega_u)$ with respect to perturbations of the form $w + \varepsilon\varphi$ for small ε and smooth φ. This concludes the proof of Lemma 3.2. ∎

Proof of Theorem 1.3. Now Theorem 1.3 follows by combining Lemma 3.2 with Theorem 1.1. ∎

4. Theorems 1.3 and 1.4: Proof via conformal hodograph transform

In this section, we prove Theorem 1.3 by introducing a new, conformal version, of the hodograph transform, which not only provides another proof of the fact that the one-phase branch points are isolated, but also provides the full expansion of the solution, and a way to construct examples of solutions with prescribed vanishing order (see Theorem 1.4).

4.1. The harmonic conjugate

Let u be a solution of the one-phase problem (1.12)–(1.15), let $\mathcal{S}_1(u)$ be the singular set defined in (1.17) and let $0 \in \mathcal{S}_1(u)$. Let $\mathcal{I}_\rho = (-\rho, \rho)$ and let $f : \mathcal{I}_\rho \to \mathbb{R}$ be the $C^{1,\alpha}$ function from (1.16) that describes locally the free boundary $\partial\Omega_u \cap B_\rho$; we recall that f is nonnegative and $f(0) = f'(0) = 0$. Now, since the function

$$\mathcal{I}_\rho \ni x \mapsto u(x, f(x))$$

vanishes for every $x \in \mathcal{I}_\rho$, we have that $\partial_x u(0,0) = 0$. Thus

$$\nabla u(0,0) = \partial_y u(0,0)\, e_2 \quad \text{and} \quad \partial_y u(0,0) \geq 1,$$

where $e_2 = (0,1)$. We next define the open set

$$\Omega_\rho = \{(x,y) \in \mathcal{I}_\rho \times \mathcal{I}_\rho : f(x) > y\},$$

and the boundary

$$\Gamma_\rho := \{(x,y) \in \mathcal{I}_\rho \times \mathcal{I}_\rho : f(x) = y\}.$$

Since Ω_ρ is simply connected and u is harmonic in Ω_ρ, there is a function

$$U : \Omega_\rho \cup \Gamma_\rho \to \mathbb{R}$$

which solves the problem

$$U(0,0) = 0, \quad \partial_x U = \partial_y u \quad \text{and} \quad \partial_y U = -\partial_x u \quad \text{in } \Omega_\rho.$$

We recall that for any $(x, y) \in \Omega_\rho \cup \Gamma_\rho$, $U(x, y)$ is the line integral $\int_\sigma \alpha$ of the 1-form

$$\alpha := \partial_y u(x, y)\, dx - \partial_x u(x, y)\, dy$$

over any curve

$$\sigma\colon [0, 1] \to \Omega_\rho \cup \Gamma_\rho$$

connecting the origin $(0, 0)$ to (x, y). In particular, U is as regular as u,

$$U \in C^{1,\alpha}(\Omega_\rho \cup \Gamma_\rho).$$

If we choose σ to be the curve parametrizing the free boundary Γ_ρ,

$$\sigma\colon [0, x] \to \mathbb{R}^2, \quad \sigma(t) = (t, f(t)),$$

then, by integrating α over σ and using that

$$\partial_x u(t, f(t)) + f'(t)\partial_y u(t, f(t)) = 0 \quad \text{for every } t \in \mathcal{I}_\rho,$$

we obtain the formula

$$\begin{aligned} U(x, f(x)) &:= \int_0^x (\partial_y u(t, f(t)) - \partial_x u(t, f(t)) f'(t))\, dt \\ &= \int_0^x |\nabla u|(t, f(t)) \sqrt{1 + f'(t)^2}\, dt = \int_\sigma |\nabla u|. \end{aligned}$$

In what follows, we will use the notation

$$\eta(x) := U(x, f(x)) = \int_\sigma |\nabla u|.$$

4.2. The conformal hodograph transform

With the notation from Section 4.1, we consider the change of coordinates

$$x' = U(x, y), \quad y' = u(x, y),$$

given by the $C^{1,\alpha}$-regular map

$$T\colon \Omega_\rho \cup \Gamma_\rho \to \mathbb{R}^2 \cap \{y' \geq 0\}, \quad T(x, y) = (x', y').$$

Now, by the definition of U and the fact that $\partial_y u(0, 0) \geq 1$, we have that the map T is invertible for ρ small enough. In particular, the set $T(\Omega_\rho \cup \Gamma_\rho)$ is an open neighborhood of $(0, 0)$ in the upper half-plane $\mathbb{R}^2 \cap \{y' \geq 0\}$. Let

$$S\colon T(\Omega_\rho \cup \Gamma_\rho) \to \Omega_\rho \cup \Gamma_\rho, \quad S(x', y') = (x, y),$$

be the inverse of T. We can write S as

$$S(x', y') = (V(x', y'), v(x', y')),$$

which in coordinates reads as

$$x = V(x', y'), \quad y = v(x', y').$$

As in the case of the classical hodograph transform, the function v contains all the information of the free boundary Γ_ρ. Precisely, for every $x \in \mathcal{I}_\rho$, we have

$$y = f(x) \iff (x, y) \in \Gamma_\rho \iff y = v(U(x, y), u(x, y)) = v(x', 0).$$

As a consequence, we obtain the equation

$$f(x) = v(\eta(x), 0) \quad \text{for every } x \in \mathcal{I}_\rho.$$

In particular, for $x' \in \mathbb{R}$ in a neighborhood of zero, $v(x', 0) \geq 0$ and

$$v(x', 0) > 0 \iff f(\eta^{-1}(x')) > 0.$$

Remark 4.1. We notice that, in terms of the contact sets

$$\mathcal{C}_1(u) = \{y = 0\} \cap \partial\Omega_u \quad \text{and} \quad \mathcal{C}(v) = \{y' = 0\} \cap \{v(x', 0) = 0\},$$

the map η is locally a C^1-diffeomorphism, which is sending $\mathcal{C}_1(u)$ into $\mathcal{C}(v)$.

Lemma 4.2 (Equations for v). *Let $T = (U, u)$ and $S = (V, v)$ be as above. Then, there is $r > 0$ such that*

$$B_r \cap \{y' \geq 0\} \subset T(\Omega_\rho \cup \Gamma_\rho)$$

and such that the function

$$v\colon B_r \cap \{y' \geq 0\} \to \mathbb{R}$$

is $C^{1,\alpha}$-regular in $B_r \cap \{y' \geq 0\}$ and C^∞-regular in $B_r \cap \{y' > 0\}$.

Moreover, if we denote by $\mathcal{C}_v$ the contact set

$$\mathcal{C}_v := \{(x', 0) : x' = \eta(x),\ x \in \mathcal{I}_\rho,\ f(x) = 0\}, \tag{4.1}$$

then v solves the problem

$$\begin{aligned}
\Delta v = 0 &\quad \text{in } B_r \cap \{y' > 0\}, &(4.2)\\
v \geq 0 &\quad \text{on } B_r \cap \{y' = 0\}, &(4.3)\\
|\nabla v| = 1 &\quad \text{on } B_r \cap \{y' = 0\} \setminus \mathcal{C}_v, &(4.4)\\
v = 0,\ |\nabla v| \leq 1 &\quad \text{on } B_r \cap \{y' = 0\} \cap \mathcal{C}_v. &(4.5)
\end{aligned}$$

Moreover, for every $x \in \Gamma_\rho$, we have the identities

$$f'(x) = \frac{\partial_{x'} v(\eta(x), 0)}{\partial_{y'} v(\eta(x), 0)} \quad \text{and} \quad \eta'(x) = \frac{1}{\partial_{y'} v(\eta(x), 0)}. \tag{4.6}$$

Proof. We start by proving that v satisfies equations (4.2)–(4.5). First notice that v is harmonic since it is the second component of a conformal map. Moreover, since

$$v(U(x,y),u(x,y)) = y \quad \text{for every } (x,y) \in \Omega_\rho,$$

taking the derivatives with respect to x and y, we obtain that

$$\partial_{x'} v(U(x,y),u(x,y))\partial_x U(x,y) + \partial_{y'} v(U(x,y),u(x,y))\partial_x u(x,y) = 0,$$
$$\partial_{x'} v(U(x,y),u(x,y))\partial_y U(x,y) + \partial_{y'} v(U(x,y),u(x,y))\partial_y u(x,y) = 1.$$

By exploiting that $\partial_x U = \partial_y u$ and $\partial_y U = -\partial_x u$, we get

$$\partial_{x'} v(x',y')\partial_y u(x,y) + \partial_{y'} v(x',y')\partial_x u(x,y) = 0, \tag{4.7}$$
$$-\partial_{x'} v(x',y')\partial_x u(x,y) + \partial_{y'} v(x',y')\partial_y u(x,y) = 1. \tag{4.8}$$

Solving the system (4.7)–(4.8) leads to

$$\partial_{y'} v(x',y') = \frac{\partial_y u(x,y)}{|\nabla u|^2(x,y)} \quad \text{and} \quad \partial_{x'} v(x',y') = -\frac{\partial_x u(x,y)}{|\nabla u|^2(x,y)}. \tag{4.9}$$

Thus, we obtain

$$|\nabla u|(x,y)\,|\nabla v|(x',y') = 1, \tag{4.10}$$

which gives both (4.4) and (4.5). We next prove (4.6). Using that $u(x,f(x)) \equiv 0$, we get

$$f'(x) = -\frac{\partial_x u(x,f(x))}{\partial_y u(x,f(x))},$$

which together with (4.9) gives the first part of (4.6). For the second part, we notice that the identity $v(\eta(x),0) = f(x)$ gives that

$$f'(x) = \eta'(x)\partial_{x'} v(\eta(x),0),$$

which, combined with the first identity in (4.6), concludes the proof. ■

4.3. *Proof of Theorem* 1.3

Let v be as in the previous section and let

$$Q := \partial_{z'} v = \partial_{x'} v - i\partial_{y'} v,$$

where $z' = x' + iy'$. Since v satisfies (4.2)–(4.5), we get that

$$\begin{cases} \partial_{\bar z'} Q = 0 & \text{in } B_r \cap \{y' > 0\}, \\ |Q| = 1 & \text{on } B_r \cap \{y' = 0\} \setminus \mathcal{C}_v, \\ \operatorname{Re} Q = 0 & \text{on } B_r \cap \{y' = 0\} \cap \mathcal{C}_v, \end{cases}$$

where the set $\mathcal{C}_v$ was defined in (4.1).

Consider now the function

$$P = -i\,\frac{Q+i}{Q-i} = -i\,\frac{(Q+i)(\bar{Q}+i)}{|Q-i|^2} = \frac{2\,\mathrm{Re}\,Q}{|Q-i|^2} - i\,\frac{|Q|^2-1}{|Q-i|^2}.$$

Then, we have that $P(0)=0$ and

$$\begin{cases} \partial_{\bar{z}'} P = 0 & \text{in } B_r \cap \{y'>0\}, \\ \mathrm{Re}\,P = 0 & \text{on } B_r \cap \{y'=0\} \cap \mathcal{C}_v, \\ \mathrm{Im}\,P = 0 & \text{on } B_r \cap \{y'=0\} \setminus \mathcal{C}_v, \end{cases}$$

which implies that $P^2(0)=0$ and

$$\begin{cases} \partial_{\bar{z}'}(P^2) = 0 & \text{in } B_r \cap \{y'>0\}, \\ \mathrm{Im}(P^2) = 0 & \text{on } B_r \cap \{y'=0\}. \end{cases}$$

As a consequence, the zero set

$$\mathcal{Z}(P) = \{z' \in B_r : P(z')=0,\ \mathrm{Im}\,z'=0\}$$

is discrete or coincides with $B_r \cap \{y'=0\}$. Now, Theorem 1.3 (a) follows since

$$P(z')=0 \ \Leftrightarrow\ \begin{cases} \partial_x u(x,y) = 0, \\ \partial_y u(x,y) = 1, \end{cases}$$

that is, every branch point $(x,y) \in \mathcal{S}_1(u)$ corresponds to a zero z' of P.

We next prove Theorem 1.3 (b). Let $z_0 = 0$ be an isolated point of $\mathcal{S}_1(u)$ and $z_0' = 0$ be the corresponding point in $\mathcal{Z}(P)$. Since zero is an isolated point of $\mathcal{Z}(P)$ and since

$$\mathrm{Re}\,P(z') \cdot \mathrm{Im}\,P(z') = 0 \quad \text{on } \{\mathrm{Im}\,z'=0\},$$

we have the following three possibilities in a neighborhood of zero:

(1) $\mathrm{Re}\,P(z') \equiv 0$ on $\{y'=0\}$, and $\mathrm{Im}\,P(z') \neq 0$ on $\{y'=0\} \setminus \{x'=0\}$;

(2) $\mathrm{Im}\,P(z') \equiv 0$ on $\{y'=0\}$, and $\mathrm{Re}\,P(z') \neq 0$ on $\{y'=0\} \setminus \{x'=0\}$;

(3) up to changing the direction of the real axis $\{y'=0\}$, we have

$$\begin{cases} \mathrm{Re}\,P(z') \equiv 0 \quad \text{and} \quad \mathrm{Im}\,P(z') \neq 0 \quad \text{on } \{y'=0\} \cap \{x'>0\}, \\ \mathrm{Re}\,P(z') \neq 0 \quad \text{and} \quad \mathrm{Im}\,P(z') \equiv 0 \quad \text{on } \{y'=0\} \cap \{x'<0\}. \end{cases}$$

We will show that each of these cases corresponds to one of the points (b.1), (b.2) and (b.3) of Theorem 1.3. We first suppose that (3) holds. Then P solves the problem

$$\begin{cases} \partial_{\bar{z}'} P = 0 & \text{in } B_r \cap \{y'>0\}, \\ \mathrm{Re}\,P = 0 & \text{on } B_r' \cap \{x' \geq 0\}, \\ \mathrm{Im}\,P = 0 & \text{on } B_r' \cap \{x'<0\}. \end{cases}$$

We next notice that

$$\partial_{x'} v - i\partial_{y'} v = Q = \frac{1+iP}{P+i} = \frac{2\mathrm{Re}(P)}{|P+i|^2} - i\frac{1-|P|^2}{|P+i|^2},$$

so that

$$\partial_{x'} v = \frac{2\,\mathrm{Re}(P)}{|P+i|^2} \quad \text{and} \quad \partial_{y'} v = \frac{1-|P|^2}{|P+i|^2}.$$

In particular, since the function η is increasing and $\eta(0) = 0$, we get

$$\partial_{x'} v(\eta(x), 0) \equiv 0 \quad \text{for } x \geq 0.$$

Integrating this identity and taking into account that $v(\eta(0), 0) = v(0, 0) = 0$, we obtain

$$f(x) = v(\eta(x), 0) = \int_0^x \partial_{x'} v(\eta(t), 0)\eta'(t)\, dt = 0 \quad \text{for } x \geq 0.$$

Conversely, assume that $x < 0$ and let $x' = \eta(x) < 0$. Then, $\mathrm{Im}(P(x')) = 0$ and

$$\partial_{x'} v(x', 0) = \frac{2P(x')}{1+P^2(x')} \quad \text{and} \quad \partial_{y'} v(x', 0) = \frac{1-P^2(x')}{1+P^2(x')} \quad \text{for } z' = x' < 0.$$

In particular, from (4.6) it follows that

$$\begin{cases} \eta'(x) = \dfrac{1+P^2(\eta(x))}{1-P^2(\eta(x))} & \text{if } x < 0, \\ \eta(0) = 0, \end{cases}$$

which implies, by Cauchy–Kovalevskaya theorem, that $\eta\colon (-\rho, 0] \to \mathbb{R}$ is an analytic function with $\eta'(0) = 1$ since $P(0) = 0$. Since for $x < 0$, we have

$$\eta'(x) = \sqrt{1+f'(x)^2} \Rightarrow f'(x) = \sqrt{\eta'(x)^2 - 1}, \tag{4.11}$$

we get that $f'\colon (-\rho, 0] \to \mathbb{R}$ is of the form

$$f'(x) = x^{k/2}\psi(x)$$

for some $k \geq 1$ and some analytic function $\psi\colon (-\rho, 0] \to \mathbb{R}$ with $\psi(0) > 0$. It follows that there is an analytic function ϕ such that $\phi(0) > 0$ and

$$f(x) = 0 \ \text{ if } x \geq 0 \quad \text{and} \quad f(x) = x^{(k+2)/2}\phi(x) \ \text{ if } x < 0.$$

Suppose now that (2) holds. Then $\mathrm{Im}\, P \equiv 0$ on the real axis $\{y' = 0\}$, and so P (not only P^2) is a holomorphic function. As a consequence, Q is also holomorphic. Thus, $\partial_{y'} v(x', 0)$ is analytic. Since $\eta\colon (-\rho, \rho) \to \mathbb{R}$ solves the equation

$$\eta'(x) = \frac{1}{\partial_{y'} v(\eta(x), 0)}, \quad \eta(0) = 0,$$

we get that η is analytic and, by (4.11), so is f. This gives (b.2).

Finally, we suppose that (1) holds. Since $\mathrm{Im}\, P \neq 0$ on $\{y' = 0\} \setminus \{0\}$, we get that the contact set $\mathcal{C}_v$ contains a neighborhood of zero. As a consequence, also the contact set $\mathcal{C}_1(u)$ contains a neighborhood of zero (see Remark 4.1), from which we obtain (b.1).

4.4. *Proof of Theorem* 1.4

Finally, we come to the proof of Theorem 1.4, which is obtained by reversing the construction from the previous subsection.

Proof of Theorem 1.4. For any k of the form $k = 2n - 3/2$ with $n \in \mathbb{N}_{\geq 1}$, we define

$$P(z) = (iz)^k = \rho^k(-\sin(k\theta) + i\cos(k\theta)).$$

In particular, setting $\mathcal{C}_P := \{(x,0) \in \mathbb{R}^2 : x \geq 0\}$, we have

$$\begin{cases} \partial_{\bar z} P = 0 & \text{in } \{y > 0\}, \\ \mathrm{Re}\, P = 0,\ \mathrm{Im}\, P > 0 & \text{on } \{x > 0\}, \\ \mathrm{Re}\, P < 0,\ \mathrm{Im}\, P = 0 & \text{on } \{x < 0\}. \end{cases}$$

Then we consider a radius $r \in (0,1)$ and the function $Q\colon B_r \cap \{y \geq 0\} \to \mathbb{C}$,

$$Q = \frac{1 + iP}{P + i} = \frac{2\,\mathrm{Re}(P)}{|P+i|^2} - i\,\frac{1 - |P|^2}{|P+i|^2}.$$

Notice that Q is still conformal in $B_r \cap \{y > 0\}$ and that we have

$$\begin{cases} \partial_{\bar z} Q = 0 & \text{in } \{y > 0\}, \\ \mathrm{Re}\, Q = 0,\ \mathrm{Im}\, Q \in (-1,0),\ |Q| < 1 & \text{on } \{x > 0\}, \\ \mathrm{Re}\, Q < 0,\ \mathrm{Im}\, Q \in (-1,0),\ |Q| = 1 & \text{on } \{x < 0\}. \end{cases}$$

Since $B_r \cap \{y > 0\}$ is simply connected, there is a function $v\colon B_r \cap \{y \geq 0\} \to \mathbb{R}$ such that

$$\partial_z v = \partial_x v - i\partial_y v = Q \quad \text{in } B_r \cap \{y > 0\}.$$

Precisely, for every $z = x + iy$ in $B_r \cap \{y \geq 0\}$, v is given by the formula

$$v(z) = v(x,y) = \int_0^1 (x\,\mathrm{Re}\, Q(tz) - y\,\mathrm{Im}\, Q(tz))\,dt.$$

Thus, v is a solution to the problem

$$\begin{cases} \Delta v = 0 & \text{in } B_r \cap \{y > 0\}, \\ v = 0,\ |\nabla v| < 1 & \text{on } B_r \cap \{x > 0\}, \\ v > 0,\ |\nabla v| = 1 & \text{on } B_r \cap \{x < 0\}. \end{cases}$$

Moreover, we notice that

$$v(0,0) = 0 \quad \text{and} \quad \partial_y v(0,0) = 1.$$

Thus, by choosing $r > 0$ small enough, we may suppose that $v > 0$ in $B_r \cap \{y > 0\}$. We next consider the harmonic conjugate $V\colon B_r \cap \{y > 0\} \to \mathbb{R}$ of v and the inverse hodograph transform

$$S\colon B_r \cap \{y \geq 0\} \to \mathbb{R}^2, \quad S(x,y) := (V(x,y), v(x,y)).$$

Tracing backwards the argument from Section 4.2, we have that when r is small enough, S is a diffeomorphism; we can then consider its inverse

$$T\colon S(B_r \cap \{y \geq 0\}) \to B_r \cap \{y \geq 0\}, \quad T(x', y') = (U(x', y'), u(x', y')),$$

where we notice that the positivity set $\Omega_u = \{u > 0\}$ of the second component u of T is precisely $S(B_r \cap \{y > 0\})$ and that, since $v \geq 0$, $\Omega_u = S(B_r \cap \{y > 0\})$ is contained in the upper half-plane $\{y' > 0\}$. Now, reasoning as in Lemma 4.2 (see (4.10)), we get that

$$|\nabla u(x', y')|\,|\nabla v(x, y)| = 1,$$

and that, in a small ball B_ρ, u is a solution to the problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega_u \cap B_\rho, \\ u &= 0 \quad \text{on } B_\rho \cap \{y' = 0\}, \\ |\nabla u| &= 1 \quad \text{on } \partial\Omega_u \cap \{y' > 0\}, \\ |\nabla u| &\geq 1 \quad \text{on } \partial\Omega_u \cap \{y' = 0\}, \end{aligned}$$

where $\partial\Omega_u \cap \{y' = 0\} = \{x' \geq 0\} \cap \{y' = 0\}$ and $|\nabla u| \geq 1$ on $\{x' \geq 0\} \cap \{y' = 0\}$. We now define the function f describing the boundary $\partial\Omega_u$ (see (1.16)) and the function $\eta(x) = U(x, f(x))$ to be as in the proof of Theorem 1.3. Then, η is a solution to

$$\begin{cases} \eta'(x) = \dfrac{1 + P^2(\eta(x))}{1 - P^2(\eta(x))} & \text{if } x < 0, \\ \eta(0) = 0 \end{cases}$$

and so, it is analytic since $P^2(z) = iz^{4n-3}$ with $n \in \mathbb{N}$. Finally, since $\eta(x) = x + o(x)$, we can write the function η as

$$|\eta(x)|^{1/2} = |x|^{1/2}\psi(x) \quad \text{for } x \leq 0,$$

where ψ is analytic and $\psi(0) = 1$. Thus, we get the precise form of f by the formula

$$f(x) = v(\eta(x), 0) = \begin{cases} \displaystyle\int_0^x \frac{-|\eta(t)|^{2n-1/2}}{|\eta(t)|^{4n-3} + 1}\,dt & \text{if } x < 0, \\ 0 & \text{if } x \geq 0, \end{cases}$$

and we notice that $f(x) = |x|^{2n-1/2}(1 + o(1))$ for $x < 0$. This concludes the proof. ∎

5. The symmetric two-phase problem and some remarks

Let $0 = z_0 \in \mathcal{S}$ and let $f_\pm$ be as in (1.25). We define

$$\begin{aligned} \Omega_\rho^\pm &= \{(x, y) \in \mathcal{I}_\rho \times \mathcal{I}_\rho : f_\pm(x) > y\}, \\ \Gamma_\rho^\pm &:= \{(x, y) \in \mathcal{I}_\rho \times \mathcal{I}_\rho : f_\pm(x) = y\}. \end{aligned}$$

In what follows, we perform the hodograph transform of u_+ in Ω_ρ^+ and of u_- in Ω_ρ^-. In order to simplify the notation, we set

$$i := + \ \text{ or } \ -\,.$$

Let $\eta_\pm$, $T_\pm = (U_\pm, u_\pm)$, $S_\pm = (V_\pm, v_\pm)$ be the functions constructed in Sections 4.1 and 4.2 separately for u_+ and u_-. Recall that the functions v_i, $i = \pm$, contain all the information of the free boundaries Γ_ρ^i. Precisely, for every $x \in \mathcal{J}_\rho$, we have

$$y = f_i(x) \ \Leftrightarrow\ (x,y) \in \Gamma_\rho^i \ \Leftrightarrow\ y = v_i(U_i(x,y), u_i(x,y)) = v_i(x',0).$$

As a consequence, we get the equation

$$f_i(x) = v_i(\eta_i(x), 0) \quad \text{for every } x \in \mathcal{J}_\rho.$$

In particular, we have

$$v_+(\eta_+(x),0) \ge v_-(\eta_-(x),0) \quad \text{for every } x \in \mathcal{J}_\rho. \tag{5.1}$$

Lemma 5.1. *There is $r > 0$ such that*

$$B_r \cap \{y' \ge 0\} \subset T_+(\Omega_\rho^+ \cup \Gamma_\rho^+) \quad \textit{and} \quad B_r \cap \{y' \le 0\} \subset T_-(\Omega_\rho^- \cup \Gamma_\rho^-).$$

The functions

$$v_\pm \colon B_r \cap \{\pm y' \ge 0\} \to \mathbb{R}$$

are both $C^{1,\alpha}$-regular respectively in the half-disks $B_r \cap \{\pm y' \ge 0\}$ up to the hyperplane $\{y' = 0\}$, and are C^∞-regular respectively in $B_r \cap \{\pm y' > 0\}$. Furthermore, they solve the following thin two-membrane problem:

$$\begin{aligned}
\Delta v_+ &= 0 && \textit{in } B_r \cap \{y' > 0\},\\
\Delta v_- &= 0 && \textit{in } B_r \cap \{y' < 0\},\\
v_+(\eta_+(x),0) &\ge v_-(\eta_-(x),0) && \textit{for } x \in \mathcal{J}_\rho,\\
|\nabla v_\pm|(\eta_\pm(x),0) &= 1 && \textit{if } v_+(\eta_+(x),0) > v_-(\eta_-(x),0),\\
\eta_+'(x)\partial_{y'} v_+(\eta_+(x),0) = \eta_-'(x)\partial_{y'} v_-(\eta_-(x),0) &\le 1 && \textit{if } v_+(\eta_+(x),0) = v_-(\eta_-(x),0).
\end{aligned}$$

Moreover, for every $x \in \Gamma_\rho$ we have the identities

$$f_\pm'(x) = \pm\frac{\partial_{x'} v_\pm(\eta_\pm(x),0)}{\partial_{y'} v_\pm(\eta_\pm(x),0)} \quad \textit{and} \quad \eta_\pm'(x) = \frac{1}{\partial_{y'} v_\pm(\eta_\pm(x),0)}. \tag{5.2}$$

Proof. We reason precisely as in Lemma 4.2. Since

$$v_i(U_i(x,y), u_i(x,y)) = y \quad \text{for every } (x,y) \in \Omega_\rho^i,$$

taking the derivatives with respect to x and y, we obtain that

$$\begin{cases}
\partial_{x'} v_i(U_i(x,y), u_i(x,y))\partial_x U_i(x,y) + \partial_{y'} v_i(U_i(x,y), u_i(x,y))\partial_x u_i(x,y) = 0,\\
\partial_{x'} v_i(U_i(x,y), u_i(x,y))\partial_y U_i(x,y) + \partial_{y'} v_i(U_i(x,y), u_i(x,y))\partial_y u_i(x,y) = 1.
\end{cases}$$

Since $\partial_x U_i = \partial_y u_i$ and $\partial_y U_i = -\partial_x u_i$, we get

$$\begin{cases} -\partial_{x'} v_i(x', y')\partial_y u_i(x, y) + \partial_{y'} v_i(x', y')\partial_x u_i(x, y) = 0, \\ \partial_{x'} v_i(x', y')\partial_x u_i(x, y) + \partial_{y'} v_i(x', y')\partial_y u_i(x, y) = 1. \end{cases}$$

When $y' = 0$, we can write

$$x' = \eta_i(x) \quad \text{and} \quad y = f_i(x).$$

Thus, we have

$$\begin{cases} -\partial_{x'} v_i(\eta_i(x), 0)\partial_y u_i(x, f_i(x)) + \partial_{y'} v_i(\eta_i(x), 0)\partial_x u_i(x, f_i(x)) = 0, \\ \partial_{x'} v_i(\eta_i(x), 0)\partial_x u_i(x, f_i(x)) + \partial_{y'} v_i(\eta_i(x), 0)\partial_y u_i(x, f_i(x)) = 1, \end{cases}$$

which we will simply write as

$$\begin{cases} -\partial_{x'} v_i \partial_y u_i + \partial_{y'} v_i \partial_x u_i = 0, \\ \partial_{x'} v_i \partial_x u_i + \partial_{y'} v_i \partial_y u_i = 1, \end{cases} \tag{5.3}$$

and we remember that all the derivatives of v are computed at $(\eta_i(x), 0)$, while all the derivatives of u are calculated at $(x, f_i(x))$. We next consider two cases.

Case 1: $v_+(\eta_+(x), 0) = v_-(\eta_-(x), 0)$. We set

$$f(x) := f_+(x) = f_-(x) \quad \text{and} \quad f'(x) := f'_+(x) = f'_-(x),$$

and we notice that we have the system

$$\partial_x u_+ + f'(x)\partial_y u_+ = 0 = \partial_x u_- + f'(x)\partial_y u_-, \tag{5.4}$$

$$-f'(x)\partial_x u_+ + \partial_y u_+ = -f'(x)\partial_x u_- + \partial_y u_-, \tag{5.5}$$

$$-f'(x)\partial_x u_\pm + \partial_y u_\pm \geq (1 + (f'(x))^2)^{1/2}, \tag{5.6}$$

where again all the partial derivatives of u_+ and u_- are computed at $(x, f(x))$.

Now, using (5.4) in (5.5) and (5.6), we get

$$\partial_y u_+ = \partial_y u_-, \tag{5.7}$$

$$\sqrt{1 + (f'(x))^2}\partial_y u_\pm \geq 1. \tag{5.8}$$

On the other hand, using (5.4) in the system (5.3), we obtain

$$\begin{cases} (\partial_{x'} v_i + \partial_{y'} v_i\, f'(x))\partial_y u_i = 0, \\ (-f'(x)\partial_{x'} v_i + \partial_{y'} v_i)\partial_y u_i = 1, \end{cases} \tag{5.9}$$

so we get

$$(1 + f'(x)^2)\partial_{y'} v_\pm \partial_y u_\pm = 1,$$

which gives that

$$\partial_{y'} v_+ = \partial_{y'} v_-, \quad \partial_{x'} v_+ = \partial_{x'} v_- \quad \text{and} \quad \sqrt{1 + (f'(x))^2}\partial_{y'} v_\pm \leq 1,$$

all the derivatives of $v_\pm$ being calculated in $(\eta_\pm(x), 0)$.

Case 2: $v_+(\eta_+(x),0) > v_-(\eta_-(x),0)$. In this case, the two free boundaries are separated, that is, $f_+ > f_-$ in a neighborhood of x. Then, for each $i = \pm$, we can proceed as in the proof of (4.4) in Lemma 4.2.

Finally, we notice that (5.2) follows by taking the reflection

$$\overline{u}(x,y) := -u_-(x,-y)$$

and applying the identities from (4.6) to u_+ and $\overline{u}$. ■

When u is a symmetric solution to the two-phase problem, we have the following assertion.

Corollary 5.2. *Let u be a symmetric solution to the two-phase problem, then, up to taking a smaller radius $r > 0$, the functions $v_\pm$ constructed in Lemma 5.1 satisfy*

$$\begin{aligned} \Delta v_+ &= 0 && \text{in } B_r \cap \{y' > 0\}, \\ \Delta v_- &= 0 && \text{in } B_r \cap \{y' < 0\}, \\ |\nabla v_\pm|(x',0) &= 1 && \text{when } x' \in B'_r \setminus \mathcal{C}_v, \\ |\nabla v_+|(x',0) = |\nabla v_-|(x',0) &\le 1 && \text{when } B'_r \cap \mathcal{C}_v, \end{aligned}$$

where we denote by $\mathcal{C}_v$ the contact set

$$\mathcal{C}_v := \{(x',0) : x' = \eta(x),\ x \in \mathcal{I}_\rho,\ f_+(x) = f_-(x)\}. \tag{5.10}$$

Proof. By definition,

$$\eta_\pm(x) = \int_0^x |\nabla u_\pm|(t, f_\pm(t)) \sqrt{1 + |f'_\pm(t)|^2}\, dt.$$

Let $\mathcal{I}_i$ be the intervals defined in (1.26), then notice that

- if $t \in \mathcal{I}_i$, then $|\nabla u_\pm|(t, f_\pm(t)) = 1$;
- if $t \in (-\rho,\rho) \setminus (\bigcup_i \mathcal{I}_i)$, then

$$f_+(t) = f_-(t) \quad \text{and} \quad |\nabla u_+|(t, f(t)) = |\nabla u_-|(t, f(t)).$$

In particular, the first item implies that

$$\eta_+(\mathcal{I}_i) = \eta_-(\mathcal{I}_i) \quad \forall i,$$

which combined with the second item implies that

$$\eta_+(\{x \in (-\rho,\rho) : f_+(x) > f_-(x)\}) = \eta_-(\{x \in (-\rho,\rho) : f_+(x) > f_-(x)\}).$$

Then the conclusion follows from the previous lemma. ■

Remark 5.3. Notice that, in the above proof, we are not claiming that $\eta_+ \equiv \eta_-$, but only that branch points are sent in branch points.

5.1. Proof of Theorem 1.6 (a)

Let $v_\pm$ be the functions from Corollary 5.2 and let

$$Q_\pm := \partial_{x'} v_\pm - i\partial_{y'} v_\pm. \tag{5.11}$$

As in the proof of Theorem 1.3, we have that Q is a solution to

$$\begin{cases} \partial_{\bar z} Q_\pm = 0 & \text{in } B_r \cap \{\pm y' > 0\}, \\ |Q_\pm| = 1 & \text{on } B_r \cap \{y' = 0\} \setminus \mathcal{C}_v, \\ Q_+ = Q_- & \text{on } B_r \cap \{y' = 0\} \cap \mathcal{C}_v. \end{cases} \tag{5.12}$$

We then define

$$P_\pm = -i\,\frac{Q_\pm + i}{Q_\pm - i} = -i\,\frac{(Q_\pm + i)(\bar Q_\pm + i)}{|Q_\pm - i|^2} = \frac{2\,\mathrm{Re}\, Q_\pm}{|Q_\pm - i|^2} - i\,\frac{|Q_\pm|^2 - 1}{|Q_\pm + i|^2}, \tag{5.13}$$

and we notice that

$$\begin{cases} \partial_{\bar z} P_\pm = 0 & \text{in } B_r \cap \{\pm y' > 0\}, \\ P_+ = P_- & \text{on } B_r \cap \{y' = 0\} \cap \mathcal{C}_v, \\ \mathrm{Im}\, P_\pm = 0 & \text{on } B_r \cap \{y' = 0\} \setminus \mathcal{C}_v. \end{cases}$$

We now consider the reflection

$$P' \colon B_r \cap \{y' \geq 0\} \to \mathbb{C}, \quad P'(z) := \bar P_-(\bar z),$$

so that the functions P_+ and P' are both defined on the same domain, and we can take

$$M(z) := \frac{P_+(z) + P'(z)}{2} \quad \text{and} \quad D(z) := \frac{P_+(z) - P'(z)}{2}, \tag{5.14}$$

which satisfy the equations

$$\begin{cases} \partial_{\bar z} M = 0 & \text{in } B_r \cap \{y' > 0\}, \\ \mathrm{Im}\, M = 0 & \text{on } B_r \cap \{y' = 0\} \end{cases} \tag{5.15}$$

and

$$\begin{cases} \partial_{\bar z} D = 0 & \text{in } B_r \cap \{y' > 0\}, \\ \mathrm{Re} D = 0 & \text{on } B_r \cap \{y' = 0\} \cap \mathcal{C}_v, \\ \mathrm{Im}\, D = 0 & \text{on } B_r \cap \{y' = 0\} \setminus \mathcal{C}_v. \end{cases}$$

Reasoning as in the proof of Theorem 1.3, we get that $\mathrm{Im}(D^2) = 2\,\mathrm{Re}\, D\,\mathrm{Im}\, D = 0$ on $\{y' = 0\}$ so that D^2 can be extended to a conformal map on the whole of B_r, so the set

$$\{D = 0\} \cap B_r \cap \{y' = 0\}$$

is either discrete or coincides with $B_r \cap \{y' = 0\}$. This proves Theorem 1.6 (a) since at every z' on the real line $\{y' = 0\}$ we have

$$D(z') = 0 \;\Leftrightarrow\; \begin{cases} P^+ = P^-, \\ \mathrm{Im} P_\pm = 0 \end{cases} \;\Leftrightarrow\; \begin{cases} Q^+ = Q^-, \\ |Q_\pm| = 1 \end{cases} \;\Leftrightarrow\; \begin{cases} \nabla u_+ = \nabla u_-, \\ |\nabla u_\pm| = 1, \end{cases}$$

that is, every branch point of u corresponds to a zero of D.

5.2. Proof of Theorem 1.6 (b) and Corollary 5.2

Remark 5.4. We notice that in this part of Theorem 1.6, we do not assume any symmetry of the solutions, but only that the branch points are isolated.

Let $z_0 \in \mathcal{S}_2(u_+, u_-)$ be an isolated point of $\mathcal{S}_2(u_+, u_-)$. If z_0 is in the interior of the contact set $\mathcal{C}_2(u_+, u_-)$, then (b.2) is immediate as the function $u = u_+ - u_-$ is harmonic in a neighborhood of z_0. Suppose then that z_0 is a branch point: $z_0 \in \mathcal{B}_2(u_+, u_-)$; moreover, since $\mathcal{B}_2 \subset \mathcal{S}_2$, we have that z_0 is isolated in the set of branch points $\mathcal{B}_2(u_+, u_-)$. This means that in order to complete the proof of Theorem 1.6 (b) we only need to prove Corollary 5.2. We set $z_0 = 0$ and consider the following two cases.

Case 1: 0 is isolated also as point of the contact set $\mathcal{C}_2(u_+, u_-)$, that is,

$$B_r \cap \mathcal{C}_2(u_+, u_-) = \{0\}$$

for some radius $r > 0$. In this case, on the free boundaries $\partial\Omega_u^\pm$ we have that $|\nabla u_\pm| = 1$ and so, Corollary 5.2 (b.1) follows as in the proof of Theorem 1.3 (b.1).

Case 2: 0 is not isolated in the set $\mathcal{C}_2(u_+, u_-)$. Then, since there are no other branch points in a neighborhood of 0, we can assume that:

$$f_+(x) = f_-(x) \quad \text{when } x \geq 0 \quad \text{and} \quad f_+(x) > f_-(x) \quad \text{when } x < 0.$$

As above, we define $\eta_\pm$ as

$$\eta_\pm(x) = \int_0^x |\nabla u_\pm|(t, f_\pm(t))| \sqrt{1 + (f_\pm'(t))^2}\, dt, \tag{5.16}$$

while $v_\pm$ are the hodograph transforms of $u_\pm$, for which we recall the identities

$$f_\pm(x) = v_\pm(\eta_\pm(x), 0) \quad \text{and} \quad |\nabla v_\pm|(\eta_\pm(x), 0) = \frac{1}{|\nabla u|(x, f_\pm(x))}$$

for every x in a neighborhood of zero. Then, since $\eta_+(x) = \eta_-(x)$ for $x \geq 0$, we get that:

$$\begin{cases} \quad v_+(x', 0) = v_-(x', 0),\ \nabla v_+(x', 0) = \nabla v_-(x', 0) & \text{when } x' \geq 0, \\ |\nabla v_+|(x', 0) = |\nabla v_-|(x', 0) & \text{when } x' < 0. \end{cases}$$

Remark 5.5. Notice that when $x < 0$, we cannot say if $\eta_+(x) = \eta_-(x)$. In particular, we cannot say if $v_+(x', 0) \geq v_-(x', 0)$ when $x' < 0$ and so, we do not know if $\{x' \geq 0\}$ is the contact set $\{x' : v_+(x', 0) = v_-(x', 0)\}$.

We next consider the functions $Q_\pm$ and $P_\pm$ given by (5.11) and (5.13), and the functions D and M defined in (5.14). Then, in a neighborhood $(-r, r) \times [0, r)$ of zero, the difference D satisfies

$$\begin{cases} \partial_{\overline{z}} D = 0 & \text{in } (-r, r) \times (0, r), \\ \operatorname{Re} D = 0 & \text{on } (0, r) \times \{0\}, \\ \operatorname{Im} D = 0 & \text{on } (-r, 0) \times \{0\}. \end{cases} \tag{5.17}$$

Recall that by the definitions of M, D and P', we have

$$P_+(z) = M(z) + D(z) \quad \text{and} \quad P_-(z) = \overline{M(\overline{z})} - \overline{D(\overline{z})},$$

and moreover

$$\partial_{x'} v_\pm = \mathrm{Re}(Q_\pm) = \frac{2\ \mathrm{Re}(P_\pm)}{|P_\pm + i|^2} \quad \text{and} \quad \partial_{y'} v_\pm = -\,\mathrm{Im}(Q_\pm) = \frac{1 - |P_\pm|^2}{|P_\pm + i|^2}.$$

We set $g_\pm(x') := \eta_\pm^{-1}(x')$ and $\widetilde{f}_\pm(x') := f_\pm(g_\pm(x'))$. Since

$$f_\pm(x) = v_\pm(\eta_\pm(x), 0) \quad \text{and} \quad \eta'_\pm(x) = \frac{1}{\partial_{y'} v_\pm(\eta_\pm(x), 0)},$$

we get that

$$\widetilde{f}_\pm(x') = v_\pm(x', 0) \quad \text{and} \quad g'_\pm(x') = \partial_{y'} v_\pm(x', 0).$$

In particular,

$$\widetilde{f}_\pm(x') = \int_0^{x'} \partial_{x'} v_\pm(t, 0)\, dt = \int_0^{x'} \frac{2\,\mathrm{Re}(P_\pm(t))}{|P_\pm(t) + i|^2}\, dt$$

and

$$g_\pm(x') = \int_0^{x'} \partial_{y'} v_\pm(t, 0)\, dt = \int_0^{x'} \frac{1 - |P_\pm(t)|^2}{|P_\pm(t) + i|^2}\, dt.$$

Now, by (5.17) and (5.15), we have that

$$M = \mathrm{Re}\, M \quad \text{and} \quad D = i\, \mathrm{Im}\, D \quad \text{on } [0, r) \times \{0\},$$

which gives that on $[0, r) \times \{0\}$, $P_+ = P_-$, precisely,

$$\mathrm{Re}(P_+) = \mathrm{Re}(P_-) = M \quad \text{and} \quad \mathrm{Im}(P_+) = \mathrm{Im}(P_-) = \mathrm{Im}\, D = -iD.$$

This implies that

$$\widetilde{f}_\pm(x') = \int_0^{x'} \frac{2M(t)}{M^2(t) + (1 + \mathrm{Im}\, D(t))^2}\, dt,$$

so that $\widetilde{f}_+ \equiv \widetilde{f}_-$ on $\{x' \geq 0\}$. Similarly,

$$g_\pm(x') = \int_0^{x'} \frac{1 - M^2(t) - (\mathrm{Im}\, D(t))^2}{M^2(t) + (1 + \mathrm{Im}\, D(t))^2}\, dt,$$

which again implies that $g_+ \equiv g_-$. Combining these two identities, we get that

$$f_+ \equiv f_- \quad \text{on } \{x' \geq 0\}.$$

Using again (5.17) and (5.15), this time for $x' \leq 0$, we get that

$$M = \mathrm{Re}\, M \quad \text{and} \quad D = \mathrm{Re}\, D \quad \text{on } (-r, 0) \times \{0\},$$

which implies that $P_\pm$ are both real and

$$P_+ = M + D \quad \text{and} \quad P_- = M - D \quad \text{on } (-r, 0) \times \{0\}.$$

As above, we compute

$$\widetilde{f}_\pm(x') = 2\int_0^{x'} \frac{M(t) \pm D(t)}{1 + (M(t) \pm D(t))^2}\, dt$$

and

$$g_\pm(x') = \int_0^{x'} \frac{1 - (M(t) \pm D(t))^2}{1 + (M(t) \pm D(t))^2}\, dt.$$

We now define

$$\Psi(x') := \frac{\widetilde{f}_+(x') - \widetilde{f}_-(x')}{2} = 2\int_0^{x'} D(t)\frac{1 + D^2 - M^2}{(1 + M^2 + D^2)^2 - 4D^2M^2}\, dt$$

and

$$\Phi(x') := \frac{\widetilde{f}_+(x') + \widetilde{f}_-(x')}{2} = 2\int_0^{x'} M(t)\frac{1 + M^2 - D^2}{(1 + M^2 + D^2)^2 - 4D^2M^2}\, dt,$$

and we notice that

- Φ is an analytic function of the form $\Phi(x') = O(x'^2)$;
- Ψ is of the form $\Psi(x') = (x')^{3/2}\Theta(x')$, where Θ is an analytic function.

Also, let

$$\psi := \frac{g_+(x') - g_-(x')}{2} = \int_0^{x'} \frac{-4D(t)M(t)}{(M^2 + D^2 + 1)^2 - 4M^2D^2}\, dt$$

and

$$\phi := \frac{g_+(x') + g_-(x')}{2} = \int_0^{x'} \frac{1 - (M^2 - D^2)^2}{(M^2 + D^2 + 1)^2 - 4M^2D^2}\, dt,$$

where, as above,

- ϕ is an analytic function of the form $\phi(x') = x' + o(x')$;
- ψ is of the form $\psi(x') = (x')^{5/2}\theta(x')$, where θ is an analytic function.

Therefore, we have

$$\begin{cases} f_+(\phi(x') + \psi(x')) - f_-(\phi(x') - \psi(x')) = 2\Psi(x'), \\ f_+(\phi(x') + \psi(x')) + f_-(\phi(x') - \psi(x')) = 2\Phi(x') \end{cases}$$

and

$$f_+(\phi(x') + \psi(x')) = \Phi(x') + \Psi(x'),$$
$$f_-(\phi(x') - \psi(x')) = \Phi(x') - \Psi(x').$$

Since $\eta_\pm$ is the inverse of $\phi \pm \psi$, we get that $\eta_\pm$ are of the form

$$\eta_\pm(x) = x + x^{5/2}\beta_\pm(x^{1/2}),$$

where $\beta_\pm$ are analytic functions. Thus,

$$f_\pm(x) = \Phi(x + x^{5/2}\beta_\pm(x^{1/2})) \pm \Psi(x + x^{5/2}\beta_\pm(x^{1/2})),$$

which concludes the proof of Corollary 5.2 and Theorem 1.6 (b.3).

5.3. Remarks on the non-symmetric case

For non-symmetric solutions, or more generally when different weights are put on the gradients of $u_\pm$ (as in the more general Alt–Caffarelli–Friedman energy, see, for instance, [9]), we cannot guarantee the validity of Corollary 5.2, and so branch points of the original problem might not be sent into branch points of the thin two-membrane problem. In fact, suppose that $(x_0, f_\pm(x_0))$ and $(x_1, f_\pm(x_1))$ are two consecutive points in $\mathcal{B}_2(u_+, u_-)$ such that $x_0 < x_1$ and

$$\begin{cases} f_+(x) = f_-(x) & \text{when } x \le x_0, \\ f_+(x) > f_-(x) & \text{when } x_0 < x < x_1, \\ f_+(x) = f_-(x) & \text{when } x \ge x_1. \end{cases}$$

Suppose that $x_0 = 0$ and define $\eta_\pm$ as in (5.16). Now, we might have that

$$\eta_+(x_1) = \int_0^{x_1} \sqrt{1 + (f_+'(t))^2}\, dt > \int_0^{x_1} \sqrt{1 + (f_-'(t))^2}\, dt = \eta_-(x_1). \tag{5.18}$$

But then, for a generic point x' between $\eta_-(x_1)$ and $\eta_+(x_1)$, we get that

$$|\nabla v_+|(x', 0) = 1,$$

while $|\nabla v_-|(x', 0) < 1$, so that equations (5.12) for $Q_\pm$ are not satisfied.

We notice that the symmetry assumption in point (a) of Theorem 1.6 is precisely what prevents (5.18) from happening. In particular, this assumption is fulfilled when

$$f_+(x) + f_-(x) \equiv 0 \quad \text{on } B_1'. \tag{5.19}$$

We also notice that (5.19) is equivalent to assuming that $\eta_+ \equiv \eta_-$.

Lemma 5.6. *Suppose that $\eta_+ \equiv \eta_-$ on $(-1, 1)$, then $u_\pm \colon B_1^\pm \cup B_1' \to \mathbb{R}$ and moreover*

$$u_-(x, y) = -u_+(x, -y) \quad \textit{and} \quad f_+(x) + f_-(x) = 0 \quad \textit{for every } x \in (-1, 1).$$

Proof. Since $\eta_+' \equiv \eta_-'$, (5.2) implies that $\partial_{y'} v_+(\eta_+(x), 0) = \partial_{y'} v_-(\eta_-(x), 0)$. In particular,

- if $f_+(x) > f_-(x)$, then $|\nabla v_\pm(\eta(x), 0)| = 1$ and so $\partial_x v_+(\eta_+(x), 0) = \partial_x v_-(\eta_-(x), 0)$;
- if $f_+(x) = f_-(x)$, then $\partial_x v_+(\eta_+(x), 0) = \partial_x v_-(\eta_-(x), 0)$.

In conclusion, we have that

$$\nabla v_+(\eta_+(x), 0) = \nabla v_-(\eta_-(x), 0),$$

which, using again (5.2), implies that $f_+'(x) \equiv -f_-'(x)$. Since $f_\pm(0) = 0$, integrating we get

$$f_+(x) + f_-(x) = \int_0^x (f_+'(t) + f_-'(t))\, dt = 0.$$

Finally, $u_-(x, y) + u_+(x, -y)$ is a harmonic function in Ω_u^- which vanishes together with its gradient on $\partial\Omega_u^-$. This implies that

$$u_-(x, y) + u_+(x, -y) = 0$$

for every $(x, y) \in \Omega_u^-$. ∎

Appendix A. The flat monotone solutions are minimizers

In this section, we show that the solutions constructed in Theorems 1.4 and 1.8 are minimizers. We prove this fact for monotone solutions to the two-phase problem, the one-phase case being analogous.

Theorem A.1. *There is a constant $\varepsilon > 0$ depending only on the dimension d such that the following holds. Let B_r' be the ball of radius r in $\mathbb{R}^{d-1}$; let $\eta_\pm\colon B_2' \to \mathbb{R}$ be two C^1-regular functions with $\eta_\pm(0) = |\nabla_{x'}\eta_\pm(0)| = 0$ and*

$$|\eta_\pm| + |\nabla\eta_\pm| \le \varepsilon \quad \text{on } B_2'.$$

Let

$$\Gamma_\pm := \{(x', \eta_\pm(x')) : x' \in B_2'\}$$

and

$$\Omega_\pm := \{(x', x_d) \in B_2' \times (-2, 2) : \pm x_d > \eta_\pm(x')\}.$$

Let $u_\pm\colon \Omega_\pm \cup \Gamma_\pm \to \mathbb{R}$ be two C^1-regular functions on $\Omega_\pm \cup \Gamma_\pm$ that solve

$$\Delta u_\pm = 0 \quad \text{in } \Omega_+ \cup \Omega_-, \tag{A.1}$$
$$u_+ = 0,\ |\nabla u_+| = 1 \quad \text{on } \Gamma_+ \setminus \Gamma_-, \tag{A.2}$$
$$u_- = 0,\ |\nabla u_-| = 1 \quad \text{on } \Gamma_- \setminus \Gamma_+, \tag{A.3}$$
$$|\nabla u_+| = |\nabla u_-| \ge 1 \quad \text{on } \Gamma_+ \cap \Gamma_- \tag{A.4}$$

and are such that

$$1 - \varepsilon \le \partial_{x_d} u_\pm \le 1 + \varepsilon \quad \text{on } \Omega_\pm.$$

Then, the function $u = u_+ - u_-$ is the unique minimizer of the two-phase functional in $\Omega := B_1' \times (-1, 1)$, with $u = u_+ - u_-$ as boundary datum on $\partial\Omega$.

Proof. We first notice that

$$x_d - 2\varepsilon \leq u(x', x_d) \leq x_d + 2\varepsilon \quad \text{for } (x', x_d) \in B_1' \times (-2, 2).$$

Let $v = v_+ - v_-$ be a minimum of the two-phase functional in $\Omega = B_1' \times (-1, 1)$, with boundary datum u on $\partial\Omega$. Then,

$$x_d - 2\varepsilon \leq v(x', x_d) \leq x_d + 2\varepsilon \quad \text{for } (x', x_d) \in \Omega.$$

Now, consider the family of functions (which are all defined on Ω when $|t| < 1$)

$$u_t(x', x_d) := u(x', x_d - t).$$

Then u_t are solutions to (A.1)–(A.4) in $B_1' \times (-1, 1)$ and are monotone, that is,

$$u_t \leq u_s \quad \text{whenever } t \leq s.$$

Now, for t small enough, we have that $u_t \leq x_d - 2\varepsilon \leq v(x)$. Let $t \leq 0$ be the largest parameter for which $u_t \leq v$. In particular,

$$\{u_t > 0\} \subset \{v > 0\} \quad \text{and} \quad \{u_t < 0\} \supset \{v < 0\}.$$

Suppose that $t < 0$. By the monotonicity of $u_\pm$, we have that $u_t < u = v$ on $\partial\Omega$. Thus, u_t touches v from below at a point $(x', x_d) \in \Omega$ and we have the following three possibilities:

(1) $u_t(x', x_d) = v(x', x_d) > 0$;

(2) $u_t(x', x_d) = v(x', x_d) = 0$ and $(x', x_d) \in \partial\{u_t > 0\} \cap \partial\{v > 0\}$;

(3) $u_t(x', x_d) = v(x', x_d) = 0$ and $(x', x_d) \in \partial\{u_t < 0\} \cap \partial\{v < 0\}$.

Now, (1) cannot happen by the strict maximum principle. Suppose that (2) holds. Then, both $\partial\{v > 0\}$ and v_+ are C^1-regular in a neighborhood of (x', x_d). Since u_t touches v from below, we have that

$$|\nabla u_t^+|(x', x_d) \leq |\nabla v_+|(x', x_d).$$

Now, if both gradients are strictly greater than one, then both u_t and v are harmonic in a neighborhood of (x', x_d), so by the strong maximum principle and the unique continuation property they should coincide. Then, since at least one of the gradients should be smaller than 1, so necessarily $|\nabla u_t^+|(x', x_d) = 1$. In order to rule out this possibility, we consider two further cases. Suppose first that $|\nabla v_+|(x', x_d) > 1$. Then,

$$(x', x_d) \in \partial\{v > 0\} \cap \partial\{v < 0\}.$$

This means that $(x', x_d) \in \partial\{u_t > 0\} \cap \partial\{u_t < 0\}$ and $|\nabla u_t^-|(x', x_d) = 1$. But this is impossible since $-u_t^-$ should remain smaller than $-v_-$. Finally, the last possibility is that $|\nabla v_+|(x', x_d) = |\nabla u_t^+|(x', x_d) = 1$. But this is impossible since it violates the Hopf maximum principle. Thus, we have shown that (2) cannot happen. By the same argument, (3) cannot happen either. Then, the only possibility is that $t = 0$, so $v \geq u$ in Ω. Analogously, $v \leq u$ in Ω, so we have that $u = v$ in Ω. ∎

Acknowledgments. We thank the anonymous referees for the careful reading of the manuscript and the useful suggestions and comments.

Funding. G. D. P. was partially supported by the NSF grant DMS 2055686 and by the Simons Foundation. L. S. was partially supported by the NSF grant DMS 1810645 and by the NSF Career Grant DMS 2044954. B. V. was supported by the European Research Council (ERC), under the European Union's Horizon 2020 research and innovation program, through the project ERC VAREG – *Variational approach to the regularity of the free boundaries* (grant agreement No. 853404). B. V. acknowledges also the support of the projects MIUR-PRIN Grant 2022R537CS "NO^3", granted by the European Union – Next Generation EU; PRA_2022_14 GeoDom financed by the University of Pisa and the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Pisa, CUP I57G22000700001.

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