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Quantum integrable systems and concentration of plasmon resonance

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Abstract. We are concerned with the quantitative mathematical understanding of surface plasmon resonance (SPR) when $d \geq 3$. SPR is the resonant oscillation of conducting electrons at the interface between negative and positive permittivity materials and forms the basis of many cutting-edge applications of metamaterials. It has recently been found that the SPR concentrates due to a curvature effect. In this paper, we derive sharper and more explicit characterizations of the SPR concentration at high-curvature places in both static and quasi-static regimes. The study boils down to analyzing the geometry of the so-called Neumann–Poincaré (NP) operators, which are certain pseudodifferential operators acting on the interfacial boundary. We propose to study the joint Hamiltonian flow of an integrable system given by the moment map defined by the NP operator. Via considering the Heisenberg picture and lifting the joint flow to a joint wave propagator, we establish a more general version of quantum ergodicity on each leaf of the foliation of this integrable system, which can then be used to establish the desired SPR concentration results. The mathematical framework developed in this paper leverages the Heisenberg picture of quantization and extends some results on quantum integrable systems via generalizing the concept of quantum ergodicity, which can be of independent interest in spectral theory and potential theory.

Keywords: surface plasmon resonance, localization, quantum integrable system, quantum ergodicity, high curvature, Neumann–Poincaré operator, quantization.

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1. Introduction

1.1. Physical background and motivation

In this paper, we are concerned with the quantitative mathematical understanding of surface plasmon resonance (SPR) when $d \geq 3$. SPR is the resonant oscillation of conducting electrons at the interface between negative and positive permittivity materials and forms the basis of many cutting-edge applications of metamaterials. To motivate the study, we briefly discuss the mathematical setup of SPR.

Let D be a bounded C^∞ domain in $\mathbb{R}^d$, $d \geq 2$, with connected complement $\mathbb{R}^d \setminus \bar{D}$. Let γ_c and γ_m be two real constants with $\gamma_m \in \mathbb{R}_+$ given and fixed. Let

$$\gamma_D = \gamma_c \chi(D) + \gamma_m \chi(\mathbb{R}^d \setminus \bar{D}); \tag{1.1}$$

here and in what follows, χ stands for the characteristic function of a domain. Consider the following homogeneous problem for a potential field $u \in H^1_{\text{loc}}(\mathbb{R}^d)$,

$$\mathcal{L}_{\gamma_D} u = 0 \quad \text{in } \mathbb{R}^d, \qquad u(x) = \mathcal{O}(|x|^{1-d}) \quad \text{as } |x| \to \infty, \tag{1.2}$$

where $\mathcal{L}_{\gamma_D} u := \nabla(\gamma_D \nabla u)$. It is clear that $u \equiv 0$ is a trivial solution to (1.2). If there exists a nontrivial solution u to (1.2), then γ_c is called a *plasmonic eigenvalue* and u is the associated *plasmonic eigenfunction*. It is apparent that a plasmonic eigenvalue must be negative, since otherwise by the ellipticity of the partial differential operator (PDO) $\mathcal{L}_{\gamma_D}$, (1.2) admits only a trivial solution. That is, the negativity of γ_c may enable that $\text{Ker}(\mathcal{L}_{\gamma_D}) \neq \emptyset$ which consists of the nontrivial solutions to (1.2). In the physical scenario, the nontrivial kernel can induce a resonant field in a standard way. In fact, let us consider the following electrostatic problem for $u \in H^1_{\text{loc}}(\mathbb{R}^d)$:

$$\begin{cases} \nabla \cdot (\gamma_D \nabla u) = 0 \quad \text{in } \mathbb{R}^d, \\ (u - u_0)(x) = \mathcal{O}(|x|^{1-d}) \quad \text{as } |x| \to \infty, \end{cases} \tag{1.3}$$

where u_0 is a harmonic function in $\mathbb{R}^d$ that signifies an incident field, and u is the incurred electric potential field. In the physical setting, γ_c and γ_m respectively specify the dielectric constants of the inclusion D and the matrix space $\mathbb{R}^d \setminus \bar{D}$. If γ_c is a plasmonic eigenvalue and moreover if u_0 is properly chosen so that $\mathcal{L}_{\gamma_D} u_0$ sits in the space spanned by $\mathcal{L}_{\gamma_D}$ acting on the plasmonic eigenfunctions, it is clear that a resonant field can be induced which is a linear superposition of the fields in $\text{Ker}(\mathcal{L}_{\gamma_D})$. It is not surprising that the resonant field exhibits a highly oscillatory pattern. However, it is highly intriguing that the high oscillation mainly propagates along the material interface, namely ∂D. This phenomenon is referred to as the *surface plasmon resonance* (SPR). The SPR forms the basis for an array of industrial and engineering applications including highly sensitive biological detectors to invisibility cloaks [12, 22, 24, 34, 42, 44, 52, 57, 59, 75]. Its theoretical understanding also arouses growing interest in the mathematical literature [3, 6–8, 15, 18, 26, 33, 40, 41, 43, 45, 46], especially its intriguing and delicate connection to the spectral theory of the Neumann–Poincaré (NP) operator described in what follows.

The *NP operator* is a classical weakly singular boundary integral operator in potential theory [5, 31] and is defined by

$$\mathcal{K}_{\partial D}^*[\phi](x) := \frac{1}{\varpi_d} \int_{\partial D} \frac{\langle x-y, \nu(x)\rangle}{|x-y|^d} \phi(y)\, d\sigma(y), \quad x \in \partial D, \tag{1.4}$$

where ϖ_d is the surface area of the unit sphere in $\mathbb{R}^d$ and $\nu(x)$ is the unit outward normal at $x \in \partial D$. In studying the plasmonic eigenvalue problem (1.2), we shall also need the following single-layer potential:

$$\mathcal{S}_{\partial D}[\phi](x) := \int_{\partial D} \Gamma(x-y)\phi(y)\, d\sigma(y), \quad x \in \mathbb{R}^d, \tag{1.5}$$

where Γ is the fundamental solution to $-\Delta$ in $\mathbb{R}^d$:

$$\Gamma(x-y) = \begin{cases} -\frac{1}{2\pi} \log|x-y| & \text{if } d = 2, \\ \frac{1}{(2-d)\varpi_d} |x-y|^{2-d} & \text{if } d > 2. \end{cases} \tag{1.6}$$

The following jump relation holds across ∂D for $\phi \in H^{-1/2}(\partial D)$:

$$\frac{\partial}{\partial \nu} (\mathcal{S}_{\partial D}[\phi])^{\pm}(x) = \left(\pm\tfrac{1}{2}\mathrm{Id} + \mathcal{K}_{\partial D}^*\right)[\phi](x), \quad x \in \partial D, \tag{1.7}$$

where $\pm$ signify the traces taken from the inside and outside of D respectively, and Id is the identity operator. Using (1.4)–(1.7), it can be directly verified that the plasmonic eigenvalue problem (1.2) is equivalent to the spectral problem of determining $\lambda(\gamma_c, \gamma_m) := (\gamma_c + \gamma_m)/[2(\gamma_c - \gamma_m)]$ and a nontrivial surface density distribution $\phi \in H^{-1/2}(\partial D, d\sigma)$ such that

$$u(x) = \mathcal{S}_{\partial D}[\phi](x), \quad x \in \mathbb{R}^d; \qquad \mathcal{K}_{\partial D}^*[\phi](x) = \lambda(\gamma_c, \gamma_m)\phi(x), \quad x \in \partial D. \tag{1.8}$$

That is, in order to determine the plasmonic eigenvalue γ_c of (1.2), it is sufficient to determine the eigenvalues of the NP operator $\mathcal{K}_{\partial D}^*$. On the other hand, in order to understand the peculiar behaviour of the plasmonic resonant field, one needs to study the quantitative properties of the NP eigenfunctions in (1.8) as well as the associated single-layer potentials in (1.5).

The NP operator $\mathcal{K}_{\partial D}^*$ is compact and hence its eigenvalues are discrete, infinitely many and accumulating at zero. A classical result is that $\lambda(\mathcal{K}_{\partial D}^*) \subset (-1/2, 1/2]$, which ensures the negativity of a plasmonic eigenvalue in (1.8). Due to their connection to the SPR discussed above, the quantitative properties of the NP eigenvalues have been extensively studied in recent years; see e.g. [4, 14, 30, 37, 45, 46] and the references cited therein. As mentioned earlier, the SPR mainly oscillates around the material interface ∂D, which is rigorously justified in [10]. It is found mainly through numerics in [14] that the SPR tends to concentrate at high-curvature places on ∂D. In [2], this peculiar curvature effect of the SPR is theoretically explained when D is convex. In [20, 21], a specific (possibly curved) nanorod geometry is considered, and it is shown that the SPR concentrates at the two ends of the nanorod, where both the mean and Gaussian curvatures are high.

In this paper, by developing and exploring new technical tools, we shall derive sharper and more explicit characterizations of the SPR concentration phenomenon driven by the extrinsic curvature when $d \geq 3$. Note that according to the discussion in [2], in order to study the SPR concentration, it is sufficient to consider the concentration of the NP eigenfunctions in (1.8) driven by the extrinsic curvature. Moreover, in addition to the static problem (1.2), we shall also study the quasi-static regime, which will be described in Section 6. Finally, we mention related work on polariton resonance associated with elastic metamaterials [9, 17, 19, 38, 39, 47], and the mathematical framework developed in this paper can be extended to study the geometric properties of polariton resonance.

1.2. Discussion of the technical novelty

In order to provide a global view of the technical contributions of this article, we briefly discuss the mathematical strategies and the new technical tools that are proposed and developed for tackling the concentration of the NP eigenfunctions and hence the SPR.

The layer-potential operators are pseudodifferential operators whose principal symbols encode the geometric character of ∂D when $d \geq 3$. We point out that they are of Cauchy type when $d = 2$. Our main idea in this work is to analyze the quantum ergodicity properties of these operators when $d \geq 3$ and under an addtional assumption, which we refer to as Assumption (A) in Section 4. To that end, we study the joint Hamiltonian flow of commuting Hamiltonians, one of which is the principal symbol of the NP operator. Using this, we derive a new version of the generalized Weyl law that the asymptotic average of the magnitude of the joint eigenfunctions of the joint spectrum sitting inside a given polytope in a neighbourhood of each point is directly proportional to a weighted volume of the pre-image of the polytope by the moment map at the respective point. In particular, this generalizes the related results in [56, 67, 68] for quantum integrable systems, as well as [60] where a pointwise generalized Weyl law of the Laplacian is proved.

Then, we lift the joint Hamiltonian flow to a joint wave propagator via the Heisenberg picture. We obtain a quantum ergodicity result on each leaf of the foliation of the underlying integrable system. This extends the classical results [16, 27, 36, 58, 65, 66, 71–74]. By using our quantum ergodicity result, we further obtain a subsequence (of density 1) of eigenfunctions such that their pointwise absolute value weakly converges to a weighted average of ergodic measures over each leaf, where this weighted average at different points again relates to the volume of the pre-image of the polytope by the moment map at the respective point. We provide explicit upper and lower bounds of the aforementioned volume as functions only depending on the principal curvatures. When the joint flow is ergodic with respect to the Liouville measures on each leaf, we obtain a more explicit description of the weighted average. With that, we provide a more explicit and sharper characterization of the localization of the plasmon resonance driven by the associated extrinsic curvature at a specific boundary point when $d \geq 3$. In fact, we provide an explicit and motivating example of a manifold with rotational symmetry, where the joint flow and the Lagrangian foliation can be explicitly worked out, and the bounds via the principal curvatures can also be calculated explicitly. From our result, we get a quantitative understanding of the plasmon resonances from the dynamical properties of the Hamiltonian flows. To the best of our knowledge, the first time when a quantum integrable system is considered to show eigenfunction concentration on Lagrangian submanifolds is in [65], where the Laplacian eigenfunctions are discussed.

Finally, we remark that at first glance, the use of the term "quantum ergodicity" is a bit paradoxical when investigating a quantum integrable system, since as is conventionally known the descriptions of a (complete) integrable system and that of ergodicity are almost on the opposite sides of the spectrum of a dynamical system. However, our discussion concerns the ergodicity on the leaves of the foliation given by the integrable system, say e.g. the Lagrangian tori if we have a complete integrable system, and therefore no paradox emerges.

The paper is organized as follows. Sections 2 and 3 are devoted to preliminaries, included for the sake of completeness and self-containedness of the paper. In Section 2, we briefly recall the principal symbols of the layer-potential operators following the discussions in [1,2] as well as [51,64]. In Section 3, we provide a general brief introduction to quantum integrable systems. In Section 4, we establish a generalized Weyl law over quantum integrable systems, and generalize the argument of quantum ergodicity over each leaf of the foliation to obtain a variance-like estimate. Sections 5 and 6 are respectively devoted to the quantitative results on concentration of plasmon resonances in the static and quasi-static regimes when $d \geq 3$.

2. Potential operators as pseudodifferential operators

2.1. *h-pseudodifferential operators*

Let us consider the manifold $M = \mathbb{R}^{2d}$ or $M = T^*X$, with the symplectic form $\omega = \sum_{i=1}^{d} dx_i \wedge d\xi_i$, where X is a d-dimensional closed manifold. The h-pseudodifferential operators acting on the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^d)$ (or $L^2(X)$) give semiclassical operators. To start with, we let $S^m(\mathbb{R}^{2d})$ be the Hörmander class (symbol class) of order m whose elements are functions $f \in C^\infty(\mathbb{R}^{2d})$ such that, for $m \in \mathbb{R}$,

$$|\partial^\alpha_{(x,\xi)} f| \leq C_\alpha \langle (x,\xi)\rangle^m, \quad (x,\xi) \in \mathbb{R}^{2d}, \tag{2.1}$$

for every $\alpha \in \mathbb{N}^{2d}$. Here, $\langle z \rangle := (1 + |z|^2)^{1/2}$.

Definition 2.1. Let $f \in S^m(\mathbb{R}^{2d})$. The *h-pseudodifferential operators* of *symbol* f are given on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ by the following expressions:

$$\text{(Left)} \quad (\mathrm{Op}^L_{f,h}(u))(x) := \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \exp\left(\frac{\mathrm{i}}{h}(x-y)\cdot\xi\right) f(y,\xi) u(y)\, dy\, d\xi;$$

$$\text{(Weyl)} \quad (\mathrm{Op}^W_{f,h}(u))(x) := \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \exp\left(\frac{\mathrm{i}}{h}(x-y)\cdot\xi\right) f\left(\frac{x+y}{2},\xi\right) u(y)\, dy\, d\xi;$$

(Right) $$(\mathrm{Op}^R_{f,h}(u))(x) := \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \exp\left(\frac{\mathrm{i}}{h}(x-y)\cdot\xi\right) f(x,\xi) u(y)\, dy\, d\xi.$$

h-pseudodifferential operators also give rise to semiclassical operators on $M = T^*X$ (X is a closed d-dimensional manifold). Let X be covered by a collection $\{U_1, \dots, U_\ell\}$ of smooth charts such that each U_i, $1 \le i \le \ell$, is a convex bounded domain of $\mathbb{R}^d$. There exists a partition of unity $\chi_1^2, \dots, \chi_\ell^2$ subordinate to the cover $\{U_1, \dots, U_\ell\}$. Let $S^m(T^*X)$, for $m \in \mathbb{R}$, be the space of functions $f \in C^\infty(T^*X)$ such that

$$|\partial_x^\alpha \partial_\xi^\beta f(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|} \tag{2.2}$$

for all $\alpha, \beta \in \mathbb{N}^n$. Define the operator on X to be

$$\mathrm{Op}^{L/W/R}_{f,h}(u) := \sum_{j=1}^{\ell} \chi_j \cdot (\mathrm{Op}^{L/W/R})^j_{f,h}(\chi_j u), \quad u \in C^\infty(X), \tag{2.3}$$

where $(\mathrm{Op}^{L/W/R})^j_{h,f}$ are the pseudodifferential operators on U_j with principal symbol $f\chi_j^2$. Following [54], the operators $\mathrm{Op}^{L/W/R}_{h,f}$ are all pseudodifferential operators on X with principal symbol f.

Proposition 2.2. *Let $S^m(T^*X)$ be a Hörmander class, $\mathbf{I} = (0,1]$, $h \in \mathbf{I}$, and $\mathcal{H}_h = L^2(X)$ (independent of h). Then all of the above quantizations $\mathrm{Op}^{L/R}_{h,f}$ and $\mathrm{Op}^W_{h,f}$ defined in (2.3) for $f \in S^m(T^*X)$ form a space of semiclassical operators.*

Proof. We refer the readers to [54] for a proof of this theorem for $\mathrm{Op}^W_{f,h}$. Note that after applying the operator $\exp(\pm i\frac{h}{2}\partial_x\partial_\xi)$, the Weyl quantization $\mathrm{Op}^W_{f,h}$ and left/right quantizations $\mathrm{Op}^{L/R}_{f,h}$ differ only in the higher-order term. Hence, Beals' criterion applies to $\mathrm{Op}^W_{f,h}$ if and only if it applies to $\mathrm{Op}^{L/R}_{f,h}$, which readily completes the proof. ■

From now on, whenever we do not specify whether it is left, right or Weyl, we presume $\mathrm{Op}_{f,h} := \mathrm{Op}^R_{f,h}$ is the right quantization. We notice that Weyl quanitization is symmetric in the L^2 metric by definition. In fact, if we do not specify the cover $\{U_i\}_{1\le i\le l}$, an operator so defined (via any of the quantizations $\mathrm{Op}^{L/W/R}_{f,h}$) is unique up to $h\Phi\mathrm{SO}^{m-1}_h$ if $f \in S^m(T^*X)$.

2.2. *Geometric description of ∂D*

For the subsequent need, we briefly introduce the geometric description of $D \subset \mathbb{R}^d$. Let $\mathbb{X} : \mathbb{R}^{d-1} \supset U \ni \mathbf{u} = (u_1, \dots, u_{d-1}) \mapsto \mathbb{X}(u) \in \partial D \subset \mathbb{R}^d$ be a regular parametrization of the surface ∂D and let $\mathbb{X}_j := \frac{\partial \mathbb{X}}{\partial u_j}$, $j = 1, \dots, d-1$. We denote $\times_{j=1}^{d-1} \mathbb{X}_j = \mathbb{X}_1 \times \cdots \times \mathbb{X}_{d-1}$. Since $\mathbb{X}$ is regular, we know $\times_{j=1}^{d-1}\mathbb{X}_j$ is nonzero, and the normal vector $\nu := \times_{j=1}^{d-1}\mathbb{X}_j / |\times_{j=1}^{d-1}\mathbb{X}_j|$ is well-defined. Let $\bar{\nabla}$ be the standard covariant derivative on the ambient space $\mathbb{R}^d$, and $\mathbf{II}$ be the second fundamental form given by

$$\mathbf{II}(\mathbf{v}, \mathbf{w}) = -\langle \bar{\nabla}_{\mathbf{v}} \nu, \mathbf{w} \rangle \nu = \langle \nu, \bar{\nabla}_{\mathbf{v}} \mathbf{w} \rangle \nu, \quad (\mathbf{v}, \mathbf{w}) \in T(\partial D) \times T(\partial D).$$

Define

$$\mathcal{A}(x) := (\mathcal{A}_{ij}(x)) = \langle \mathbf{II}_x(\mathbb{X}_i, \mathbb{X}_j), \nu_x \rangle, \quad x \in \partial D.$$

Let $g = (g_{ij})$ be the induced metric tensor on ∂D and $(g^{ij}) = g^{-1}$. Finally, we write $\mathcal{H}(x), x \in \partial D$, for the mean curvature satisfying

$$\mathrm{tr}_{g(x)}(\mathcal{A}(x)) := \sum_{i,j=1}^{d-1} g^{ij}(x)\mathcal{A}_{ij}(x) =: (d-1)\mathcal{H}(x).$$

Throughout, we assume $\mathcal{A}(x) \neq 0$ for all $x \in \partial D$.

2.3. *Principal symbols of layer potential operators*

Throughout, with a slight abuse of notation, we shall also denote by $\mathcal{S}_{\partial D}$ the single-layer potential operator which is given in (1.5) but with $x \in \partial D$. This should be clear from the context. We let $\mathcal{K}_{\partial D}$ signify the $L^2(\partial D, d\sigma)$-adjoint of the NP operator $\mathcal{K}^*_{\partial D}$. Then $\mathcal{K}^*_{\partial D}$ is symmetrizable on $H^{-1/2}(\partial D, d\sigma)$ (see, e.g., [32]) due to the following Kelley symmetrization identity:

$$\mathcal{S}_{\partial D}\mathcal{K}^*_{\partial D} = \mathcal{K}_{\partial D}\mathcal{S}_{\partial D}. \tag{2.4}$$

In this section, we treat the layer potential operators as pseudodifferential operators when $d \geq 3$ and derive several important properties, especially their principal symbols. In fact, the special three-dimensional case was treated in [45, 46], whereas the general case was considered in [1,2] as well as in [64, Chapter 12, Section C, Proposition C1] and [51, Proposition 2.2]. Since this result forms the starting point for our subsequent analysis, we discuss the main ingredients. First, we introduce a slightly more relaxed symbol class $\widetilde{S}^m(T^*(\partial D))$ (compared to $S^m(T^*(\partial D))$):

$$\bigcup_i U_i = \partial D, \quad F_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^{d-1}, \quad \sum_i \psi_i^2 = 1, \quad \mathrm{supp}(\psi_i) \subset U_i;$$

$$\widetilde{S}^m(U_i \times \mathbb{R}^{d-1} \setminus \{0\}) := \Big\{ a : U_i \times (\mathbb{R}^{d-1} \setminus \{0\}) \to \mathbb{C};$$
$$a \in C^\infty(U_i \times (\mathbb{R}^{d-1} \setminus \{0\})),\ |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C_{\alpha,\beta}(|\xi|)^{m-|\alpha|} \Big\};$$

$$\widetilde{S}^m(T^*(\partial D)) := \Big\{ a : T^*(\partial D) \setminus \partial D \times \{0\} \to \mathbb{C};$$
$$a = \sum_i \psi_i F_i^*([F_i^{-1}]^*(\psi_i)a_i),\ a_i \in \widetilde{S}^m(U_i \times \mathbb{R}^{d-1} \setminus \{0\}) \Big\},$$

where $\pi : T^*(\partial D) \to \partial D$ is the bundle projection. Similar to our discussion in Section 2.1, for a symbol $a \in \widetilde{S}^m(T^*(\partial D))$, we can define the h-pseudodifferential operator $\mathrm{Op}_{a,h}$. In the sequel, we let $\widetilde{\Phi\mathrm{SO}}_h^m$ denote the class of pseudodifferential operators of order m associated with $\widetilde{S}^m(T^*(\partial D))$. We also let $\widetilde{\Phi\mathrm{SO}}^m := \widetilde{\Phi\mathrm{SO}}_1^m$ for $h = 1$.

Theorem 2.3. *Assume that $\partial D \in C^\infty$. When $d \geq 3$, the operators $\mathcal{K}_{\partial D}^*$ and $\mathcal{S}_{\partial D}$ are pseudodifferential operators of order -1 with symbols given in geodesic normal coordinates around each point x by*

$$\begin{aligned} p_{\mathcal{K}_{\partial D}^*}(x,\xi) &= p_{\mathcal{K}_{\partial D}^*,-1}(x,\xi) + \mathcal{O}(|\xi|^{-2}) \\ &= (d-1)\mathcal{H}(x)|\xi|^{-1} - \langle \mathcal{A}(x)\xi, \xi\rangle |\xi|^{-3} + \mathcal{O}(|\xi|^{-2}), \end{aligned} \tag{2.5}$$

and

$$p_{\mathcal{S}_{\partial D}}(x,\xi) = p_{\mathcal{S}_{\partial D},-1}(x,\xi) + \mathcal{O}(|\xi|_{g(x)}^{-2}) = \tfrac{1}{2}|\xi|_{g(x)}^{-1} + \mathcal{O}(|\xi|_{g(x)}^{-2}), \tag{2.6}$$

where the $\mathcal{O}$-terms depend on $\|\mathbb{X}\|_{\mathcal{C}^2}$. The result in (2.5) also holds for $\mathcal{K}_{\partial D}$ if only the leading-order term is concerned.

Using the symmetrization identity (2.4) and the self-adjointness of $\mathcal{S}_{\partial D}$, we have

$$\begin{aligned} \mathcal{K}_{\partial D}^* &= |\mathcal{D}|^{-1}\Bigg\{(d-1)\mathcal{H}(x)\Delta_{\partial D} \\ &\quad - \sum_{i,j,k,l=1}^{d-1} \frac{1}{\sqrt{|g(x)|}}\partial_i g^{ij}(x)\sqrt{|g(x)|}\mathcal{A}_{jk}(x) g^{kl}(x)\partial_l\Bigg\}|\mathcal{D}|^{-2} \bmod \widetilde{\Phi\mathrm{SO}}^{-2}, \\ \mathcal{S}_{\partial D} &= \frac{1}{2}|\mathcal{D}|^{-1} \bmod \widetilde{\Phi\mathrm{SO}}^{-2}, \end{aligned} \tag{2.7}$$

where $\Delta_{\partial D}$ is the surface Laplacian of ∂D, and $|\mathcal{D}|^{-1} := \mathrm{Op}_{|\xi|_{g(x)}^{-1}}$. Moreover, we have $\mathcal{K}_{h,\partial D}^* := \frac{1}{h}|\mathcal{D}|^{-1/2}\mathcal{K}_{\partial D}^*|\mathcal{D}|^{1/2}$, which is self-adjoint modulo $h\widetilde{\Phi\mathrm{SO}}_h^{-2}$. s

Finally, we note that $((\widetilde{\lambda}^i)^2, \phi^i)$ is an eigenpair of $\mathcal{K}_{\partial D}^*$ if and only if $(\widetilde{\lambda}^i/h, |D|^{-1/2}\phi^i)$ is an eigenpair of $\mathcal{K}_{h,\partial D}^*$. Hencefore, we write

$$((\widetilde{\lambda}^i)^2(h), \phi^i(h)) := ((\widetilde{\lambda}^i)^2/h, |D|^{-1/2}\phi^i). \tag{2.8}$$

3. Classical and quantum integrable systems

In this section, we give a brief review of classical and quantum integrable systems, which will be needed in our subsequent analysis.

3.1. Classical integrable systems

Let M be a $2d$-dimensional symplectic manifold with a nondegenerate 2-form ω.

Definition 3.1. A *completely integrable Hamiltonian system* (M, ω, F) on a $2d$-dimensional symplectic manifold (M, ω) is given by a set of d smooth functions $H_1, \dots, H_d \in C^\infty(M)$ that are functionally independent and Poisson-commuting, i.e.,

$$\{H_i, H_j\} := -\omega(X_{H_i}, X_{H_j}) = 0, \quad i, j \in \{1, \dots, d\},$$

where we recall that X_{H_i} is the symplectic gradient vector field given by

$$\iota_{X_{H_i}}\omega = dH_i.$$

The map $F = (H_1, \dots, H_d) : M \to \mathbb{R}^d$ is called the *moment map*.

The level sets of the moment map in a completely integrable system form a Lagrangian foliation.

Definition 3.2. Let $F = (H_1, \dots, H_d)$ be the moment map of a completely integrable system on $\mathbb{R}^{2d}$. A point $\mathfrak{m} \in \mathbb{R}^{2d}$ is said to be *regular* if

$$\operatorname{rank}\{X_{H_1}(\mathfrak{m}), \dots, X_{H_d}(\mathfrak{m})\} = d.$$

If

$$\operatorname{rank}\{X_{H_1}(\mathfrak{m}), \dots, X_{H_d}(\mathfrak{m})\} = r, \quad 0 \le r < d,$$

then $\mathfrak{m}$ is said to be a *singular point of rank* r. The value $F(\mathfrak{m}) \in \mathbb{R}^d$ is called a *regular value* if $\mathfrak{m}$ is a regular point, and a *singular value* if $\mathfrak{m}$ is a singular point.

Suppose that $\mathfrak{m} \in \mathbb{R}^{2d}$ is a singular point of rank r for a completely integrable system $F = (H_1, \dots, H_d)$ on $\mathbb{R}^{2d}$. After replacing the H_i's with invertible linear combinations of H_j's if necessary, we may assume that

$$X_{H_1}(\mathfrak{m}) = \cdots = X_{H_{d-r}}(\mathfrak{m}) = 0,$$

and the X_{H_i}'s are linearly independent for $d - r < i \le d$. The quadratic parts of $H_1, \dots, H_{d-r}$ form an abelian subalgebra $\mathfrak{s}_{\mathfrak{m}}$ of the Lie algebra of quadratic forms, with the Poisson bracket as the Lie bracket.

Definition 3.3. A singular point $\mathfrak{m}$ or rank r is said to be *nondegenerate* if the subalgebra $\mathfrak{s}_{\mathfrak{m}}$ is a Cartan subalgebra of the Lie algebra $\mathfrak{sp}(2d - 2r, \mathbb{R})$ of the symplectic group $\mathrm{Sp}(2d - 2r, \mathbb{R})$.

Remark 3.4. In an obvious way, Definitions 3.2 and 3.3 can be carried over to a completely integrable system (M, ω, F) on a general $2d$-dimensional symplectic manifold.

In 1936, Williamson [70] classified the Cartan subalgebras of the Lie algebra of the symplectic group.

Theorem 3.5 (Williamson). *Let $\mathfrak{s} \subset \mathfrak{sp}(2l, \mathbb{R})$ be a Cartan subalgebra. Then there exist canonical coordinates $(q_1, \dots, q_l, p_1, \dots, p_l)$ for $\mathbb{R}^{2l}$, a triple $(k_{\mathrm{el}}, k_{\mathrm{hy}}, k_{\mathrm{ff}}) \in \mathbb{Z}^3_{\ge 0}$ satisfying the condition $k_{\mathrm{el}} + k_{\mathrm{hy}} + 2k_{\mathrm{ff}} = l$, and a basis $f_1, \dots, f_l$ of $\mathfrak{s}$ such that*

$$f_i = \frac{q_i^2 + p_i^2}{2}, \qquad i = 1, \dots, k_{\mathrm{el}},$$

$$f_j = q_j p_j, \qquad j = k_{\mathrm{el}} + 1, \dots, k_{\mathrm{el}} + k_{\mathrm{hy}},$$

$$f_k = \begin{cases} q_k p_k + q_{k+1} p_{k+1}, & k = k_{\mathrm{el}} + k_{\mathrm{hy}} + 1, k_{\mathrm{el}} + k_{\mathrm{hy}} + 3, \dots, l - 1, \\ q_k p_{k+1} - q_{k+1} p_k, & k = k_{\mathrm{el}} + k_{\mathrm{hy}} + 2, k_{\mathrm{el}} + k_{\mathrm{hy}} + 4, \dots, l. \end{cases}$$

Additionally, two Cartan subalgebras $\mathfrak{s}, \mathfrak{s}' \subset \mathfrak{sp}(2l, \mathbb{R})$ *are conjugate if and only if their corresponding triples are equal.*

The elements of the basis of $\mathfrak{s}$ *are called* elliptic blocks, hyperbolic blocks *or* focus-focus blocks *according to whether they are of the form* $\frac{q_i^2+p_i^2}{2}$, $q_j p_j$ *or a pair* $q_k p_k + q_{k+1} p_{k+1}, q_k p_{k+1} - q_{k+1} p_k$, *respectively.*

Given a completely integrable system $(M, \omega, F = (H_1, \ldots, H_d))$, suppose $\mathfrak{m} \in M$ is a nondegenerate singularity of rank r. Then with the help of Williamson's Theorem, locally one can write the Hamiltonian H_i as f_i for $i = 1, \ldots, k_{\mathrm{el}} + k_{\mathrm{hy}} + 2k_{\mathrm{ff}}$, and $H_i = p_i$ for $i = k_{\mathrm{el}} + k_{\mathrm{hy}} + 2k_{\mathrm{ff}} + 1, \ldots, k_{\mathrm{el}} + k_{\mathrm{hy}} + 2k_{\mathrm{ff}} + r = n$.

3.2. *Quantum integrable systems*

Next, we provide a tool for the discussion of a lift of the classical Hamiltonian system to its operator counterpart. For this purpose, we define quantum integrable systems.

Let M be a $2d$-dimensional symplectic manifold with a nondegenerate 2-form ω. Let $\mathbf{I} \subset (0, 1]$ be any set that accumulates at 0. If $\mathcal{H}$ is a complex Hilbert space, we denote by $\mathcal{L}(\mathcal{H})$ the set of linear (possibly unbounded) self-adjoint operators on $\mathcal{H}$ with a dense domain.

Definition 3.6. A space Ψ of *semiclassical operators* is a subspace of $\prod_{h\in\mathbf{I}} \mathcal{L}(\mathcal{H}_h)$, containing the identity, and equipped with a weak principal symbol map, which is an $\mathbb{R}$-linear map

$$\sigma : \Psi \to C^\infty(M; \mathbb{R}) \tag{3.1}$$

with the following properties:

(1) $\sigma(\mathrm{Id}) = 1$ (normalization);

(2) if $P, Q \in \Psi$ and if $P \circ Q$ is well-defined and is in Ψ, then $\sigma(P \circ Q) = \sigma(P)\sigma(Q)$ (product formula);

(3) if $\sigma(P) \geq 0$, then there exists a function $h \mapsto \varepsilon(h)$, tending to zero as $h \to 0$, such that $P \geq -\varepsilon(h)$ for all $h \in \mathbf{I}$ (wear positivity);

If $P = (P_h)_{h\in\mathbf{I}}$, then $\sigma(P)$ is called the *principal symbol* of P.

Such a family of Hilbert spaces can be obtained e.g. by Weyl quantization (which we will specify later) or geometric quantization with complex polarizations.

Definition 3.7. A *quantum integrable system* on M consists of d semiclassical operators

$$P_1 = (P_{1,h}), \ldots, P_d = (P_{d,h})$$

acting on $\mathcal{H}_h$ which commute, i.e., $[P_{i,h}, P_{j,h}] = 0$ for all $i, j \in \{1, \ldots, d\}$ and all h and whose principal symbols $f_1 := \sigma(P_1), \ldots, f_d := \sigma(P_d)$ form a completely integrable system on M.

Definition 3.8. Suppose P and Q are commuting semiclassical operators on $\mathcal{H}_h$. Then the *joint spectrum* of (P_h, Q_h), denoted as $\Sigma(P_h, Q_h)$, is the support of their joint spectral measure. If $\mathcal{H}_h$ is finite-dimensional (or, more generally, when the joint spectrum is discrete), then

$$\Sigma(P_h, Q_h) = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : \exists v \neq 0,\ P_h v = \lambda_1 v,\ Q_h v = \lambda_2 v\}. \tag{3.2}$$

The joint spectrum of P, Q, denoted by $\Sigma(P, Q)$, is the collection of all joint spectra of (P_h, Q_h), $h \in \mathbf{I}$.

Suppose that $(P_{1,h}), \ldots, (P_{d,h})$ form a quantum integrable system on M. Then the *joint spectrum* of $((P_{1,h}), \ldots, (P_{d,h}))$ is

$$\Sigma((P_{1,h}), \ldots, (P_{d,h})) := \Big\{(\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d : \bigcap_{i=1}^{d} \ker(P_{i,h} - \lambda_i I) \neq \{0\}\Big\}. \tag{3.3}$$

We remark that in case the operators $P_{i,h}$ are not bounded, their commuting is understood in the strong sense: the spectral measures (obtained via the spectral theorem as projector-valued measures) of $P_{i,h}$ and $P_{j,h}$ commute.

4. Generalized Weyl law

In this section, we recall the concept of generalized Weyl law and quantum ergodicity from the pioneering works of Shnirelman [36, 58], Zelditch [71–74], Colin de Verdière [16] and Helffer–Martinez–Robert [27], as well as [60] where a pointwise generalized Weyl law of the Laplacian is proved, and generalize them to the case of a quantum integrable system [56, 67, 68].

4.1. Hamiltonian flows of principal symbols

Consider the Hamiltonian $H : T^*(\partial D) \to \mathbb{R}$:

$$H(x, \xi) := [p_{\mathcal{K}^*_{\partial D},-1}(x, \xi)]^2 \geq 0. \tag{4.1}$$

Throughout the rest of the paper, we impose the following assumption.

Assumption (A). We have $\langle \mathcal{A}(x) g^{-1}(x)\omega, g^{-1}(x)\omega\rangle \neq (d-1)\mathcal{H}(x)$ for all $x \in \partial D$ and all $\omega \in \{\xi : |\xi|^2_{g(x)} = 1\} \subset T^*_x(\partial D)$.

Assumption (A) holds if and only if $\overline{\{H = 1\}} \cap (\partial D \times \{0\}) = \emptyset$, which is further equivalent to the condition that $H \neq 0$ everywhere, and hence to the ellipticity of $\mathcal{K}^*_{\partial D}$. As explored in [51, Corollary 2.3], it is clear that strict convexity of D implies Assumption (A). Meanwhile, as discussed in [1], (at least) when $d = 3$, an application of the Gauss–Bonnet theorem shows that Assumption (A) holds if and only if D is strictly convex. With this, looking at (1.8), it can be directly inferred that $\phi \in C^\infty(\partial D)$. In this article, we always assume the validity of Assumption (A).

Next, we set $\rho(r) = 1 - \exp(-r) : \mathbb{R}_+ \to \mathbb{R}$. It follows that $\rho(r) \geq 0$ and $\rho'(r) > 0$ for all $\mathbb{R}_+$. Moreover, $\rho(1/r^2) \in C^\infty(\mathbb{R})$, with $\partial_r^\ell|_{r=0}[\rho(1/r^2)] = 0$ for all $\ell \in \mathbb{N}$ and

$$|\partial_r^\ell \rho(1/r^2)| \leq C_\ell (1 + |r|^2)^{(-2-\ell)/2}.$$

Define $\widetilde{H}(x, \xi) = \rho(H(x, \xi)) : T^*(\partial D) \to \mathbb{R}$. When $d \geq 3$, it can be directly verified that under Assumption (A) and together with the fact that $H \in \widetilde{S}^{-2}(T^*(\partial D))$, one has $\widetilde{H} \in S^{-2}(T^*(\partial D))$.

Let us now consider k $(\leq d)$ Poisson commuting and functionally independent Hamiltonians $f_1 = \widetilde{H}, f_2, \ldots f_k \in S^m(T^*(\partial D))$ for $m \geq -2$ (if $k = d$, then we will have a completely integrable system on M). Let $F = (f_1, \ldots, f_k)$. We also consider the corresponding h-pseudodifferential operators $\mathrm{Op}_{f_1,h}, \ldots, \mathrm{Op}_{f_k,h}$ acting on $\mathcal{H}_h = L^2(\partial D)$.

Next we consider the following solution under the (joint) Hamiltonian flows:

$$\begin{cases} \frac{\partial}{\partial t_j} a(t_1, \ldots, t_k) = \{f_j, a(t_1, \ldots, t_k)\}, \\ a_0(x, \xi) \in S^m(T^*X), \end{cases} \tag{4.2}$$

which exists since $\{f_i, f_j\} = 0$, where we recall that $\{\cdot, \cdot\}$ is the Poisson bracket given by

$$\{f, g\} := X_f g = -\omega(X_f, X_g).$$

With this notion in hand, we have $\frac{\partial}{\partial t_j} a = X_{f_j} a$, and it is clear that, writing $t = (t_1, \ldots, t_k)$, we have $a(t) = a_0(\gamma(t), p(t))$ where

$$\begin{cases} \frac{\partial}{\partial t_j}(\gamma(t), p(t)) = X_{f_j}(\gamma(t), p(t)), \\ (\gamma(0), p(0)) = (x, \xi) \in M. \end{cases} \tag{4.3}$$

To emphasize the dependence of a on the initial value (x, ξ), we also sometimes write

$$a_{(x,\xi)}(t) = a(t) \quad \text{with} \quad (\gamma(0), p(0)) = (x, \xi).$$

Next we introduce Heisenberg's picture and lift the above flow to the operator level via Egorov's well-known theorem together with the commutativity of $\mathrm{Op}_{f_i,h}$ and $\mathrm{Op}_{f_j,h}$. Since this is a handy extension of Egorov's original theorem (see [1, 23, 28, 29]), we provide a sketch of the proof.

Proposition 4.1. *Under Assumption* (A), *when* $d \geq 3$, *consider the following operator evolution equation for each* $j \in \{1, \ldots, k\}$:

$$\begin{cases} \frac{\partial}{\partial t_j} A_h(t) = \frac{\mathrm{i}}{h}[\mathrm{Op}_{f_j,h}, A_h(t)], \\ A_h(0) = \mathrm{Op}_{a_0,h}. \end{cases} \tag{4.4}$$

For $|t| < C \log(h)$, *it defines a unique Fourier integral operator* (*up to* $h^\infty \Phi\mathrm{SO}_h^{-\infty}$)

$$\begin{aligned} A_h(t_1, \ldots, t_k) &= e^{-\sum_{j=1}^k \frac{\mathrm{i}t_j}{h} \mathrm{Op}_{f_j,h}} A_h(0) e^{\sum_{j=1}^k \frac{\mathrm{i}t_j}{h} \mathrm{Op}_{f_j,h}} + \mathcal{O}(h\Phi\mathrm{SO}_h^{m-1}) \\ &= \mathrm{Op}_{a(t),h} + \mathcal{O}(h\Phi\mathrm{SO}_h^{m-1}). \end{aligned}$$

Proof. First, by noting that $[\mathrm{Op}_a, \mathrm{Op}_b] = \mathrm{Op}_{\{a,b\}} + \mathcal{O}(h\Phi\mathrm{SO}_h^{m+n-2})$ if $a \in S^m(T^*(\partial D))$ and $b \in S^n(T^*(\partial D))$ and using the given condition that $[\mathrm{Op}_{f_i,h}, \mathrm{Op}_{f_j,h}] = 0$ whenever $i \neq j$, one can construct the symbol at the principal level. Then one can construct the full symbol in an inductive manner, and bound the error operator via repeated use of the Calderón–Vaillancourt theorem. By Beals' theorem, the operator is guaranteed to be a Fourier integral operator. The explicit expression of $A_h(t)$ comes from checking the principal symbols, and bounding the error operator via the Zygmund trick. ■

When $d \geq 3$, we proceed to consider $f_1(x,\xi) = \widetilde{H}(x,\xi) = \rho([p_{\mathcal{K}^*_{\partial D}}(x,\xi)]^2)$, and we can immediately see that

$$\mathrm{Op}_{f_1,h} = \mathrm{Op}_{\widetilde{H},h} = \rho([\mathcal{K}^*_{h,\partial D}]^2) \bmod h\Phi\mathrm{SO}_h^{-3}.$$

Let $\{L_{j,h}\}_{j=2}^k$ be a family of pseudodifferential operators such that

$$\mathrm{Op}_{f_j,h} = L_{j,h}(h\Phi\mathrm{SO}_h^m), \quad j = 2, \dots, n.$$

Then we immediately obtain the following corollary.

Corollary 4.2. *Under Assumption* (A)*, when* $d \geq 3$*,*

$$\begin{aligned} A_h(t) &= e^{-\frac{it_1}{h}\rho([\mathcal{K}^*_{h,\partial D}]^2) - \sum_{j=2}^k \frac{it_j}{h} L_{j,h}} A_h(0) e^{\frac{it_1}{h}\rho([\mathcal{K}^*_{h,\partial D}]^2) - \sum_{j=2}^k \frac{it_j}{h} L_{j,h}} \\ &\quad + \mathcal{O}(h\Phi\mathrm{SO}_h^{m-1}) \\ &= \mathrm{Op}_{a(t),h} + \mathcal{O}(h\Phi\mathrm{SO}_h^{m-1}). \end{aligned}$$

4.2. *Trace formula and generalized Weyl law*

We first state the Schwartz functional calculus without proof.

Lemma 4.3 ([28, 29]). *Recall that $\mathcal{S}(\mathbb{R})$ is the space of Schwartz functions on $\mathbb{R}$. Then for $f \in \mathcal{S}(\mathbb{R})$ and $a \in S^m(T^*(\partial D))$, we have $f(\mathrm{Op}_{a,h}) \in \Phi\mathrm{SO}_h^{-\infty}$ and*

$$f(\mathrm{Op}_{a,h}) = \mathrm{Op}_{f(a)} + \mathcal{O}(h\Phi\mathrm{SO}_h^{-\infty}). \tag{4.5}$$

The above lemma leads us to the following trace theorem:

Proposition 4.4 ([16, 28, 29, 58, 62]). *Given $a \in S^m(T^*(\partial D))$, if $\mathrm{Op}_{a,h}$ is in the trace class and $f \in \mathcal{S}(\mathbb{R})$, then*

$$(2\pi h)^{d-1}\mathrm{tr}(f(\mathrm{Op}_{a,h})) = \int_{T^*(\partial D)} f(a)\, d\sigma \otimes d\sigma^{-1} + \mathcal{O}(h),$$

where $d\sigma \otimes d\sigma^{-1}$ is the Liouville measure given by the top form $\omega^{d-1}/(d-1)!$.

Let $(\lambda_1^i(h), \dots, \lambda_k^i(h))$ be elements of the joint spectrum $\Sigma(\mathrm{Op}_h f_1, \dots, \mathrm{Op}_h f_k)$ (for simplicity of notation, let us call it Σ), with the joint eigenstates $\phi^i(h)$, that is, $\mathrm{Op}_h f_j \phi^i(h) = \lambda_j^i(h)\phi^i(h)$, $j \in \{1, \dots, k\}$. We remark that, with this notation, $(\widetilde{\lambda}^i(h), \phi^i(h))$ is an eigenpair of $\mathcal{K}_{h,\partial D}$ if and only if $(\lambda_1^i(h) = \rho(\lambda^i(h))^2, \phi^i(h))$ is an eigenpair of $\rho(\mathcal{K}^2_{h,\partial D})$. Then we can prove the following result.

Proposition 4.5. *Under Assumption* (A)*, when $d \geq 3$, let $\mathscr{C} \subset \mathbb{R}^k$ be a compact convex polytope. Then for any $a_j \in S^m(T^*(\partial D))$, $j \in \{1, \dots, a_k\}$, we have, as $h \to +0$,*

$$(2\pi h)^{d-k} \sum_{(\lambda_1^i(h), \dots, \lambda_k^i(h)) \in \mathscr{C}} c_i \langle \mathrm{Op}_{a,h} \phi^i(h), \phi^i(h) \rangle_{L^2(\partial D, d\sigma)} = \int_{\{F=(f_1,\dots,f_k)\in\mathscr{C}\}} a \, d\sigma \otimes d\sigma^{-1} + o_{\mathscr{C}}(1), \tag{4.6}$$

where $c_i := \|\phi^i\|^{-2}_{H^{-1/2}(\partial D, d\sigma)}$ and the o-term depends on $\mathscr{C}$.

Proof. Let us first take $\mathscr{C} := \prod_{j=1}^k [r_j, s_j]$, a k-dimensional rectangle. Take

$$\chi_{1,\varepsilon}(\rho[\mathcal{K}^*_{h,\partial D}]^2), \chi_{2,\varepsilon}(L_{2,h}), \dots, \chi_{k,\varepsilon}(L_{k,h}),$$

where $\chi_\varepsilon(x) := \prod_{j=1}^k \chi_{j,\varepsilon}(x_j) \in \mathcal{S}(\mathbb{R}^k)$ approximate $\chi_{\prod_{j=1}^k [r_j, s_j]}$. Then $\chi_{1,\varepsilon}(\rho[\mathcal{K}^*_{h,\partial D}]^2) \in \Phi\mathrm{SO}_h^{-\infty}$ and $\chi_{j,\varepsilon}(L_{j,h}) \in \Phi\mathrm{SO}_h^{-\infty}$ for each $j \in \{2, \dots, k\}$ by the functional calculus with the trace formula

$$(2\pi h)^{d-k} \mathrm{tr}\Big(\chi_{1,\varepsilon}(\rho[\mathcal{K}^*_{h,\partial D}]^2) \prod_{j=2}^k \chi_{j,\varepsilon}(L_{j,h}) \mathrm{Op}_{a,h} \chi_{1,\varepsilon}(\rho[\mathcal{K}^*_{h,\partial D}]^2) \prod_{j=2}^k \chi_{j,\varepsilon}(L_{j,h})\Big) = \int_{T^*(\partial D)} a \chi_{1,\varepsilon}(\rho(H))^2 \prod_{j=2}^k \chi_{j,\varepsilon}(f_j)^2 \, d\sigma \otimes d\sigma^{-1} + \mathcal{O}_{\mathscr{C},\varepsilon}(h), \tag{4.7}$$

where $\mathcal{O}$ depends on $\mathscr{C}$ and ε. As $\varepsilon \to 0$ in (4.7), $\chi_{1,\varepsilon}(\rho[\mathcal{K}^*_{h,\partial D}]^2) \prod_{j=2}^k \chi_{j,\varepsilon}(L_{j,h})$ converges to the spectral projection operator when the joint spectrum $(\lambda_1^i(h), \dots, \lambda_k^i(h)) \in \mathscr{C} = \prod_{j=1}^k [r_j, s_j]$, which readily gives (4.6) when $\mathscr{C} = \prod_{j=1}^k [r_j, s_j]$. Next we observe that a general compact convex polytope $\mathscr{C}$ is rectifiable, and hence can be approximated by an arbitrary refinement of a cover by a finite disjoint union of k-dimensional rectangles, and a standard approximation argument leads to (4.6) for a general $\mathscr{C}$. Finally, we notice that $\|\phi^i(h)\|_{L^2(\partial D, d\sigma)} = \|\phi^i\|_{H^{-1/2}(\partial D, d\sigma)}$.

The proof is complete. ■

Taking $a = 1$ in (4.6), one readily has the classical Weyl law [2, 16, 28, 29, 58, 62]. An algebraic proof of this result can also be found in [53, 55].

Corollary 4.6. *Let $\mathscr{C} \subset \mathbb{R}^k$ be a compact convex polytope. Then*

$$\sum_{(\lambda_1^i(h), \dots, \lambda_k^i(h)) \in \mathscr{C}} 1 = (2\pi h)^{k-d} \int_{\{F=(f_1,\dots,f_k)\in\mathscr{C}\}} d\sigma \otimes d\sigma^{-1} + o_{\mathscr{C}}(h^{-n}). \tag{4.8}$$

4.3. Ergodic decomposition theorem and quantum ergodicity on the leaves of the foliation by the integrable system

From now on, denoting

$$\mathcal{T} : T^*(\partial D) \times [0, \infty)^k \to T^*(\partial D), \quad \mathcal{T}((x, \xi), t) = (\gamma(t), p(t)), \tag{4.9}$$

where $(\gamma(\cdot), p(\cdot))$ is as in (4.3), we may adopt the notion in [61] in our case when $[0,\infty)^k$ forms a semigroup (we may in fact extend it to $\mathbb{R}^k$ by extending the Hamiltonian flow, but it is not necessary) and recall the definition of ergodicity for our purpose.

Definition 4.7. Consider the family of maps $\{\mathcal{T}_t(\cdot) := \mathcal{T}(\cdot, t)\}_{t\in[0,\infty)^k}$. Consider an invariant subspace $M \subset T^*(\partial D)$, i.e. $\mathcal{T}_t(M) = M$ for all $t \in [0,\infty)^k$. We call a Radon measure μ over $M \subset T^*(\partial D)$ an *invariant measure with respect to the joint Hamiltonian flow* of the vector fields X_{f_j}, $j \in \{1,\dots,k\}$, on M if

$$[\mathcal{T}_t]_{\#}\mu = \mu \quad \text{for all } t \in [0,\infty)^k,$$

i.e. the push-foward of the measure coincides with the measure itself. We denote the set of such invariant measures by $M_{X_F}(M)$ (which is a convex set). An invariant measure is *ergodic* with respect to the joint Hamiltonian flow of X_{f_j}, $j \in \{1,\dots,k\}$, on M if for any measurable set $A \subset M$,

$$\mu(A \triangle \mathcal{T}_t(A)) = 0 \text{ for all } t \in [0,\infty)^k \implies \mu(A) = 0 \text{ or } 1.$$

We denote the set of such ergodic measures by $M_{X_F,\mathrm{erg}}(M)$ (which can be directly checked to be the set of extremal points of $M_{X_F}(M)$.)

We remark that the standard notion of ergodicity, i.e., for any measurable set $A \subset M$,

$$\mathcal{T}_t^{-1}(A) \subset A \text{ for all } t \in [0,\infty)^k \implies \mu(A) = 0 \text{ or } 1,$$

can be readily shown via standard and elementary arguments. We also remark that this definition of ergodicity is not that of the joint ergodicity of the family of commuting one-parameter subgroups generated by each X_{f_i} as introduced in e.g. [11,61] (which is instead a generalization of the mixing properties).

Now, similar to our study in [2], let us consider the set

$$\{F = (f_1,\dots,f_k) = (\rho(1), e_2,\dots,e_k)\},$$

denoted by $F_{(1,e_2,\dots,e_k)}$, for the functions $f_j \in S^m(T^*(\partial D))$ defined as above, with $f_1 = \widetilde{H}$. Let us denote by $\sigma_{F_{(1,e_2,\dots,e_k)}}$ the Liouville measure on $F_{(1,e_2,\dots,e_k)} \subset T^*(\partial D)$, and by $\sigma_{F_{(E,e_2,\dots,e_k)}}$ that on $F_{(E,e_2,\dots,e_k)} := \{F = (f_1,\dots,f_k) = (\rho(E), e_2,\dots,e_k)\}$ when $E \neq 1$. For notational sake, from now on, we write

$$\mathcal{F} = \{e := (e_2,\dots,e_k) \in \mathbb{R}^{k-1} : F_{(1,e_2,\dots,e_k)} \neq \emptyset\}.$$

For all such $e \in \mathcal{F}$, since $X_{f_j} f_i = 0$ and $\mathcal{L}_{X_{f_j}} \omega^{d-1} = 0$ for all $i, j \in \{1,\dots,k\}$, we see

$$d\sigma_{F_{(1,e)}} := \lim_{\varepsilon\to 0} (2\varepsilon)^{-k} \chi_{\bigcup_{v\in[-\varepsilon,\varepsilon]^k} F_{(1,e)+v}}\, d\sigma \otimes d\sigma^{-1}$$

is an invariant measure for the (joint) flow on $F_{(1,e)}$.

With the previous notion of ergodicity in hand, we next consider $M_{X_F}(F_{(1,e)})$, which is the set of invariant measures on $F_{(1,e)}$, and $M_{X_F,\mathrm{erg}}(F_{(1,e)})$, the set of ergodic measures on $F_{(1,e)}$. Since $F_{(1,e)}$ has a countable base, the weak-* topology of $M_{X_F}(F_{(1,e)})$ is metrizable, and hence Choquet's theorem can be applied to obtain the following generalized version of the ergodic decomposition theorem of [69].

Lemma 4.8. *Given a probability measure $\eta \in M_{X_F}(M)$, there exists a probability measure $\nu_e \in M(M_{X_F,\mathrm{erg}}(F_{(1,e)}))$ such that*

$$\eta = \int_{M_{X_F,\mathrm{erg}}(F_{(1,e)})} \mu_e \, d\nu_e(\mu_e).$$

Applying Lemma 4.8 to $\sigma_{F_{1,e}}/\sigma_{F_{(1,e)}}(F_{(1,e)})$, we have a probability measure $\nu_e \in M(M_{X_F,\mathrm{erg}}(F_{(1,e)}))$ such that

$$\sigma_{F_{(1,e)}} = \sigma_{F_{(1,e)}}(F_{(1,e)}) \int_{M_{X_F,\mathrm{erg}}(F_{(1,e)})} \mu_e \, d\nu(\mu_e).$$

Note that by rescaling $F_{(E,e)} = E^{-1/2} F_{(1,e)}$, we have $\sigma_{F_{(E,e)}} = E^{(1+k-d)/2} \sigma_{F_{(1,e)}}$. Then, from the smoothness of F and the nondegeneracy of DF (up to a codimension 1 subset), we can see that the decomposition

$$d\sigma \otimes d\sigma^{-1} = \int_{\mathcal{F}} (E^{(1+k-d)/2} dE \otimes d\sigma_{F_{(1,e)}}) \varphi(e) \, de$$

holds for some density $\varphi \in L^1(\mathcal{F}, de)$ (with $\{\varphi = 0\}$ of measure 0 with respect to de) via a change of variable formula.

For any $\mu_e \in M_{X_F,\mathrm{erg}}(F_{(1,e)})$, we let $\mu_{(E,e)} := [m_{E^{-1/2}}]_\# \mu_e \in M_{X_f,\mathrm{erg}}(F_{(E,e)})$ be the push-forward measure given by

$$m_{E^{-1/2}} : T^*(\partial D) \to T^*(\partial D), \quad (x, \xi) \mapsto (x, E^{-1/2} \xi).$$

Then

$$\sigma \otimes \sigma^{-1} = \int_{\mathcal{F}} \int_{(0,\infty) \times M_{X_F,\mathrm{erg}}(F_{(1,e)})} \mu_{(E,e)} \sigma_{F_{(1,e)}}(F_{(1,e)}) h(e) E^{(1+k-d)/2} (dE \otimes d\nu_e)(E, \mu_e) \, de.$$

Next, we shall derive a general version of the quantum ergodicity on the leaves of the foliation by a quantum integrable system. Since $[0, \infty)^k$ is a countable amenable semigroup, ergodicity of a measure μ on $T^*(\partial D)$ is equivalent to the fact that for all $f \in L^2(T^*(\partial D), d\mu)$ (see [61]), with $\prod := \prod_{j=1}^k [0, T_j]$ and $|\prod| := \prod_{j=1}^k |T_j|$,

$$\lim_{T \to 0} \frac{1}{|\prod|} \int_{\prod} f \circ \mathcal{T}_t \, dt = \int_{T^*(\partial D)} f \, d\mu$$

in the $L^2(T^*(\partial D), d\mu)$ metric, which is in fact von Neumann's ergodic theorem [50] in this scenario. Following [1], we have the following application of Birkhoff's [13] and von Neumann's [50] ergodic theorems.

Lemma 4.9. *Under Assumption* (A)*, when* $d \geq 3$*, for any* $r_j \leq s_j$*,* $j \in \{1, \dots, k\}$*, and all* $a_0 \in \widetilde{S}^m(T^*X)$*, we have, as* $T \to \infty$*,*

$$\frac{1}{|\prod|} \int_{\prod} a_{(x,\xi)}(t)\, dt \to \bar{a}(x,\xi) \qquad \text{a.e.-}d\sigma \otimes d\sigma^{-1} \text{ and in } L^2\Big(\bigcap_{j=1}^{k} \{r_j \leq f_j \leq s_j\}, d\sigma \otimes d\sigma^{-1}\Big)$$

for some $\bar{a} \in L^2(\bigcap_{j=1}^k \{r_j \leq f_j \leq s_j\}, d\sigma \otimes d\sigma^{-1})$ *and a.e.-*$(dE \otimes d\nu_e)(E, \mu_e) de$*, with*

$$\bar{a}(x,\xi) = \int_{F_{(E,e)}} a_0\, d\mu_{(E,e)} \qquad \text{a.e.-}d\mu_{(E,e)}.$$

Proof. By Birkhoff's and von Neumann's ergodic theorems [13, 50] on $\chi_{\bigcap_{j=1}^k \{r_j \leq f_j \leq s_j\}} \times d\sigma \otimes d\sigma^{-1}$, we have, as $T \to \infty$,

$$\frac{1}{|\prod|} \int_{\prod} a_{(x,\xi)}(t)\, dt \to \bar{a}(x,\xi) \qquad \text{a.e.-}d\sigma \otimes d\sigma^{-1} \text{ and in } L^2\Big(\bigcap_{j=1}^{k} \{r_j \leq f_j \leq s_j\}, d\sigma \otimes d\sigma^{-1}\Big)$$

for some $\bar{a} \in L^2(\bigcap_{j=1}^k \{r_j \leq f_j \leq s_j\}, d\sigma \otimes d\sigma^{-1})$ invariant under the joint Hamiltonian flow. Set

$$\mathcal{E} := \Big\{(x,\xi) \in \bigcap_{j=1}^{k} \{r_j \leq f_j \leq s_j\} : \limsup_{T} \Big| \frac{1}{|\prod|} \int_{\prod} a_{(x,\xi)}(t)\, dt - \bar{a}(x,\xi) \Big| > 0 \Big\}.$$

It is clearly seen that $\sigma \otimes \sigma^{-1}(\mathcal{E}) = 0$. Next, we can show by Lemma 4.8 that

$$\int_{\mathcal{F} \cap \prod_{j=2}^k [r_j, s_j]} \int_{[r_1, s_1] \times M_{X_{F,\mathrm{erg}}}(F_{(1,e)})} \mu_{(E,e)}(\mathcal{E}) \sigma_{F_{(1,e)}}(F_{(1,e)}) \varphi(e) \times E^{(1+k-d)/2} (dE \otimes d\nu_e)(E, \mu_e)\, de = \sigma \otimes \sigma^{-1}(\mathcal{E}) = 0.$$

Since $\{\varphi = 0\}$ is of measure zero with respect to de, for a.e.-$(dE \otimes d\nu_e)(E, \mu_e) de$ we have $\mu_{(E,e)}(\mathcal{E}) = 0$. Meanwhile, by using the Birkhoff and von Neumann's ergodic theorems [13, 50] again, on each leaf we have

$$\frac{1}{|\prod|} \int_{\prod} a_{(x,\xi)}(t)\, dt \to \int_{F_{(E,e)}} a_0\, d\mu_{(E,e)} \qquad \text{a.e.-}\mu_{(E,e)} \text{ and in } L^2(F_{(E,e)}, d\mu_{(E,e)}) \text{ as } T \to \infty.$$

Finally, setting

$$\mathcal{E}_{\mu_{(E,e)}} := \Big\{(x,\xi) \in \bigcap_{j=1}^{k} \{r_j \leq f_j \leq s_j\} : \limsup_{T} \Big| \frac{1}{|\prod|} \int_{\prod} a_{(x,\xi)}(t)\, dt - \int_{F_{(E,e)}} a_0\, d\mu_{(E,e)} \Big| > 0 \Big\},$$

we can show that $\mu_{(E,e)}(\mathcal{E}_{\mu_{(E,e)}}) = 0$. Therefore, a.e.-$(dE \otimes d\nu_e)(E, \mu_e)de$, $\mu_{(E,e)}(\mathcal{E} \cup \mathcal{E}_{\mu_{(E,e)}}) = 0$. By the uniqueness of the limit, the proof can be readily concluded. ■

With the above preparations, we can establish the following theorem that will play an important role in our subsequent analysis.

Theorem 4.10. *Let $\mathcal{C} \subset \mathbb{R}^k$ be a compact convex polytope and $c_i := \|\phi^i\|^{-2}_{H^{-1/2}(\partial D, d\sigma)}$. Then under Assumption* (A)*, when $d \geq 3$, the following* (*variance-like*) *estimate holds as $h \to +0$:*

$$\frac{1}{\sum_{(\lambda_1^i(h),\ldots,\lambda_k^i(h))\in\mathcal{C}} 1} \sum_{(\lambda_1^i(h),\ldots,\lambda_k^i(h))\in\mathcal{C}} c_i^2 \big| \langle A_h \phi^i(h), \phi^i(h)\rangle_{L^2(\partial D, d\sigma)} - \langle \mathrm{Op}_{\bar{a},h}\phi^i(h), \phi^i(h)\rangle_{L^2(\partial D, d\sigma)} \big|^2 \to 0. \tag{4.10}$$

Proof. Via considering the Hamiltonian flow of the principal symbol, we can lift the Birkhoff and von Neumann theorems to the operator level. Set $A_h(0) = A_h$. From the definition of $\phi^i(h)$, we have, for each i,

$$\begin{aligned} &\langle A_h(t)\phi^i(h), \phi^i(h)\rangle_{L^2(\partial D, d\sigma)} \\ &= \langle A_h(0) e^{\frac{it_1}{h}\rho([\mathcal{K}^*_{h,\partial D}]^2) - \sum_{j=2}^k \frac{it_j}{h} L_{j,h}} \phi^i(h), e^{\frac{it_1}{h}\rho([\mathcal{K}^*_{h,\partial D}]^2) - \sum_{j=2}^k \frac{it_j}{h} L_{j,h}} \phi^i(h)\rangle_{L^2(\partial D, d\sigma)} \\ &\quad + \mathcal{O}_t(h) \\ &= \langle A_h \phi^i(h), \phi^i(h)\rangle_{L^2(X, d\sigma)} + \mathcal{O}_t(h), \end{aligned} \tag{4.11}$$

where we make use of Proposition 4.1 as well as the definition of the NP eigenfunctions in (2.8). Averaging both sides of (4.11) with respect to T, we arrive at

$$\langle \Upsilon \phi^i(h), \phi^i(h)\rangle_{L^2(\partial D, d\sigma)} = \langle A_h \phi^i(h), \phi^i(h)\rangle_{L^2(X, d\sigma)} + \mathcal{O}_T(h),$$

where

$$\Upsilon := \frac{1}{|\prod|} \int_{\prod} A_h(t)\, dt.$$

By using Proposition 4.1 again, one can directly verify that

$$\frac{1}{|\prod|} \int_{\prod} A_h(t)\, dt - \mathrm{Op}_{\bar{a},h} = \mathrm{Op}_{\frac{1}{|\prod|}\int_{\prod} a_{(x,\xi)}(t)\, dt - \bar{a}} + \mathcal{O}_T(h).$$

Next by using the Cauchy–Schwarz inequality, we have

$$\left| \frac{\langle \mathrm{Op}_{\bar{a},h}\phi^i(h), \phi^i(h)\rangle_{L^2(\partial D, d\sigma)}}{\langle \phi^i(h), \phi^i(h)\rangle_{L^2(\partial D, d\sigma)}} - \frac{\langle A_h\phi^i(h), \phi^i(h)\rangle_{L^2(\partial D, d\sigma)}}{\langle \phi^i(h), \phi^i(h)\rangle_{L^2(\partial D, d\sigma)}} \right|^2 \leq \frac{\langle \Xi^* \Xi \phi^i(h), \phi^i(h)\rangle_{L^2(\partial D, d\sigma)}}{\langle \phi^i(h), \phi^i(h)\rangle_{L^2(\partial D, d\sigma)}} + \mathcal{O}_T(h^2) \tag{4.12}$$

with

$$\Xi := \Upsilon - \mathrm{Op}_{\bar{a},h}.$$

Therefore, summing over the joint spectrum of ϕ^i of (4.12) and applying (4.6) and (4.8), we have

$$\frac{1}{\sum_{(\lambda_1^i(h),\dots,\lambda_k^i(h))\in\mathcal{C}} 1} \sum_{(\lambda_1^i(h),\dots,\lambda_k^i(h))\in\mathcal{C}} c_i^2 \big| \langle A_h \phi^i(h), \phi^i(h)\rangle_{L^2(\partial D, d\sigma)} - \langle \mathrm{Op}_{\bar{a},h}\phi^i(h), \phi^i(h)\rangle_{L^2(\partial D, d\sigma)} \big|^2$$
$$\leq \frac{\int_{\{F=(f_1,\dots,f_k)\in\mathcal{C}\}} \big| \frac{1}{|\prod|}\int_{\prod} a_{(x,\xi)}(t)\,dt - \bar{a}\big|^2 \, d\sigma\otimes d\sigma^{-1}}{\int_{\{F=(f_1,\dots,f_k)\in\mathcal{C}\}} d\sigma\otimes d\sigma^{-1}} + o_{\mathcal{C},T}(1). \tag{4.13}$$

Finally, by noting that the first term on the right-hand side of (4.13) goes to zero as $T = (T_1,\dots,T_k)$ goes to infinity, one can readily have (4.10), which completes the proof. ∎

With Theorem 4.10, together with Chebyshev's trick and a diagonal argument, we obtain the following quantum ergodicity result, which generalizes the relevant results in [16, 25, 36, 58, 62, 63, 65, 66, 71–74].

Corollary 4.11. *Let $\mathcal{C}\subset\mathbb{R}^k$ be a compact convex polytope. Under Assumption* (A)*, when $d\geq 3$, there exists $S(h)\subset J(h) := \{i\in\mathbb{N} : (\lambda_1^i(h),\dots,\lambda_k^i(h))\in\mathcal{C}\}$ such that for all $a_0\in S^m(T^*X)$ we have, as $h\to +0$,*

$$\max_{i\in S(h)} c_i |\langle (A_h - \mathrm{Op}_{\bar{a},h})\phi^i(h), \phi^i(h)\rangle_{L^2(X,d\sigma)}| = o_{\mathcal{C}}(1) \text{ and } \frac{\sum_{i\in S(h)} 1}{\sum_{i\in J(h)} 1} = 1 + o_{\mathcal{C}}(1). \tag{4.14}$$

Note that the choice of $S(h)$ is independent of a_0.

To our best knowledge, the first time when a quantum integrable system is considered to show eigenfunction concentration on Lagrangian submanifolds is in [65], where the Laplacian eigenfunctions are discussed instead.

Remark 4.12. It is indeed a bit paradoxical to refer to Theorem 4.10 as stating quantum ergodicity when we now have a quantum integrable system: as is customarily understood, (complete) integrable systems and ergodicity are almost on the opposite sides in the description of a dynamical system. However, our discussion concerns ergodicity on the leaves of the foliation given by the integrable system (e.g. the Lagrangian tori if we have a complete integrable system), and therefore no paradox emerges.

5. Localization/concentration of plasmon resonances in electrostatics

We are now in a position to present one of our main results on the localization/concentration of plasmon resonances in electrostatics.

5.1. *Consequences of the generalized Weyl law and quantum ergodicity*

In the following, we let $\sigma_{x,F_{(1,e)}}$ signify the Liouville measure on $F_{(1,e)}(x) := \{F(x,\cdot) = (\rho(1), e)\} \subset T_x^*(\partial D)$. By the generalized Weyl law of Section 4, we can obtain the following result, which characterizes the local behaviour of the NP eigenfunctions and their relative magnitude.

Theorem 5.1. *Given any $x \in \partial D$, let $\{\chi_{x,\delta}\}_{\delta>0}$ be a family of smooth nonnegative bump functions compactly supported in $B_\delta(x)$ with $\int_{\partial D} \chi_{p,\delta}\, d\sigma = 1$. Under Assumption* (A), *when $d \geq 3$, fixing a compact convex polytope $\mathcal{C} \subset \mathcal{F} \subset \mathbb{R}^{k-1}$, $[r,s] \subset \mathbb{R}$, $\alpha \in \mathbb{R}$ and $p, q \in \partial D$, there exists $\delta(h)$ depending on $\mathcal{C}$, p, q, r, s and α such that, as $h \to +0$, we have $\delta(h) \to 0$ and*

$$\frac{\sum_{(\lambda_1^i(h),\ldots,\lambda_k^i(h))\in[r,s]\times\mathcal{C}} c_i \int_{\partial D} \chi_{p,\delta(h)}(x)\left||D|^\alpha \phi^i(x)\right|^2 d\sigma(x)}{\sum_{(\lambda_1^i(h),\ldots,\lambda_k^i(h))\in[r,s]\times\mathcal{C}} c_i \int_{\partial D} \chi_{q,\delta(h)}(x)\left||D|^\alpha \phi^i(x)\right|^2 d\sigma(x)}$$
$$= \frac{\int_{\mathcal{F}} \int_{F_{(1,e)}(p)} |\xi|_{g(y)}^{1+2\alpha}\, d\sigma_{p,F_{(1,e)}}\, h(e)\, de}{\int_{\mathcal{F}} \int_{F_{(1,e)}(q)} |\xi|_{g(y)}^{1+2\alpha}\, d\sigma_{q,F_{(1,e)}}\, h(e)\, de} + o_{\mathcal{C},r,s,p,q,\alpha}(1), \qquad (5.1)$$

where $c_i := \|\phi^i\|_{H^{-1/2}(\partial D, d\sigma)}^{-2}$. In particular, if $\alpha = -1/2$, the RHS term of (5.1) *is the ratio between the volumes of $\bigcup_{e\in\mathcal{F}} F_{(1,e)}(\cdot)$ at the respective points.*

Proof. Taking $p \in \partial D$, we consider $a(x,\xi) := \chi_{p,\delta}(x)|\xi|_{g(x)}^{1+2\alpha}$ in (4.6). Since

$$\mathrm{Op}_{a,h} = h^{1+2\alpha}|D|^{1/2+\alpha}\mathrm{Op}_{\chi_{p,\delta}(x),h}|D|^{1/2+\alpha} - h\mathrm{Op}_{\tilde{a}_{p,\delta},h}$$

for some $\tilde{a}_{p,\delta} \in S^{2\alpha}(T^*(\partial D))$, after applying (4.6) once more upon $\tilde{a}_{p,\delta}$ we have

$$\begin{aligned}(2\pi h)^{d+2\alpha} &\sum_{(\lambda_1^i(h),\ldots,\lambda_k^i(h))\in[r,s]\times\mathcal{C}} c_i \int_{\partial D} \chi_{p,\delta}(x)\left||D|^\alpha\phi^i(x)\right|^2 d\sigma(x) \\ &= \int_{\{(f_1,\ldots,f_k)\in[r,s]\times\mathcal{C}\}} \chi_{p,\delta}(x)|\xi|_{g(x)}^{1+2\alpha}\, d\sigma\otimes d\sigma^{-1} \\ &\quad + h\int_{\{(f_1,\ldots,f_k)\in[r,s]\times\mathcal{C}\}} \tilde{a}_{p,\delta}\, d\sigma\otimes d\sigma^{-1} + o_{\mathcal{C},r,s,\alpha}(1).\end{aligned} \qquad (5.2)$$

With (5.2), we find, after choosing another point $q \in \partial D$ and taking a quotient between the two, that

$$\frac{\sum_{(\lambda_1^i(h),\ldots,\lambda_k^i(h))\in[r,s]\times\mathcal{C}} c_i \int_{\partial D} \chi_{p,\delta}(x)\left||D|^\alpha\phi^i(x)\right|^2 d\sigma(x)}{\sum_{(\lambda_1^i(h),\ldots,\lambda_k^i(h))\in[r,s]\times\mathcal{C}} c_i \int_{\partial D} \chi_{q,\delta}(x)\left||D|^\alpha\phi^i(x)\right|^2 d\sigma(x)}$$
$$= \frac{\int_{\{(f_1,\ldots,f_k)\in[r,s]\times\mathcal{C}\}} \chi_{p,\delta}(x)|\xi|_{g(x)}^{1+2\alpha}\, d\sigma\otimes d\sigma^{-1} + h\int_{\{(f_1,\ldots,f_k)\in[r,s]\times\mathcal{C}\}} \tilde{a}_{p,\delta}\, d\sigma\otimes d\sigma^{-1}}{\int_{\{(f_1,\ldots,f_k)\in[r,s]\times\mathcal{C}\}} \chi_{q,\delta}(x)|\xi|_{g(x)}^{1+2\alpha}\, d\sigma\otimes d\sigma^{-1} + h\int_{\{(f_1,\ldots,f_k)\in[r,s]\times\mathcal{C}\}} \tilde{a}_{q,\delta}\, d\sigma\otimes d\sigma^{-1}}$$
$$+ o_{\mathcal{C},r,s,\alpha}(1).$$

Now, for any given h, we can choose $\delta(h)$ depending on $\mathscr{C}, r, s, p, q, \alpha$ such that as $h \to +0$, we have $\delta(h) \to 0$ (much more slowly than h) and

$$\left| h \int_{\{(f_1,\ldots,f_k)\in[r,s]\times\mathscr{C}\}} \widetilde{a}_{p,\delta(h)}\, d\sigma \otimes d\sigma^{-1} \right| + \left| h \int_{\{(f_1,\ldots,f_k)\in[r,s]\times\mathscr{C}\}} \widetilde{a}_{q,\delta(h)}\, d\sigma \otimes d\sigma^{-1} \right| \to 0.$$

We also find that as $h \to +0$, with this choice of $\delta(h)$ we have $\delta(h) \to 0$, and for $y = p, q$,

$$\int_{\{(f_1,\ldots,f_k)\in[r,s]\times\mathscr{C}\}} \chi_{y,\delta(h)}(x) |\xi|_{g(x)}^{1+2\alpha}\, d\sigma \otimes d\sigma^{-1} \to \int_{\{(f_1(y,\cdot),\ldots,f_k(y,\cdot))\in[r,s]\times\mathscr{C}\}} |\xi|_{g(y)}^{1+2\alpha}\, d\sigma^{-1}.$$

Therefore,

$$\frac{\sum_{(\lambda_1^i(h),\ldots,\lambda_k^i(h))\in[r,s]\times\mathscr{C}} c_i \int_{\partial D} \chi_{p,\delta(h)}(x) \left| |D|^\alpha \phi^i(x) \right|^2 d\sigma(x)}{\sum_{(\lambda_1^i(h),\ldots,\lambda_k^i(h))\in[r,s]\times\mathscr{C}} c_i \int_{\partial D} \chi_{q,\delta(h)}(x) \left| |D|^\alpha \phi^i(x) \right|^2 d\sigma(x)} = \frac{\int_{\{(f_1(p,\cdot),\ldots,f_k(p,\cdot))\in[r,s]\times\mathscr{C}\}} |\xi|_{g(p)}^{1+2\alpha}\, d\sigma^{-1}}{\int_{\{(f_1(q,\cdot),\ldots,f_k(q,\cdot))\in[r,s]\times\mathscr{C}\}} |\xi|_{g(q)}^{1+2\alpha}\, d\sigma^{-1}} + o_{\mathscr{C},r,s,p,q,\alpha}(1).$$

To conclude our proof, we observe that for all $y = p, q$,

$$\int_{\{(f_1(y,\cdot),\ldots,f_k(y,\cdot))\in[r,s]\times\mathscr{C}\}} |\xi|_{g(y)}^{1+2\alpha}\, d\sigma^{-1} = \left(\int_r^s E^{-k/2-\alpha-d/2} dE \right) \left(\int_{\mathscr{F}} \int_{F_{(1,e)}(y)} |\xi|_{g(y)}^{1+2\alpha}\, d\sigma_{y,F_{(1,e)}}\, h(e)\, de \right),$$

where we recall $h \in L^1(\mathscr{F}, de)$ and $d - 1 - k$ is the generic dimension of $F_{(1,e)}(y)$.

The proof is complete. ■

Theorem 5.1 states that, given $p, q \in \partial D$, the relative magnitude of a c_i-weighted sum of a weighted average of $\left| |D|^\alpha \phi^i \right|^2$ over a small neighbourhood of p compared to that of q asymptotically depends on the ratio between the weighted volume of $\{(f_1(p,\cdot), \ldots, f_k(p,\cdot)) = (\rho(1), e) : e \in \mathscr{C}\}$ and that of $\{(f_1(q,\cdot), \ldots, f_k(q,\cdot)) = (\rho(1), e) : e \in \mathscr{C}\}$. This is critical for our subsequent analysis since it reduces the study to analyzing the aforementioned weighted volumes.

Theorem 5.2. *Under Assumption* (A)*, when $d \geq 3$, there is a family of distributions*

$$\{\Phi_{\mu,e}\}_{\mu\in M_{X_F,\mathrm{erg}}(F_{(1,e)}),\, e\in\mathscr{F}} \subset \mathscr{D}'(\partial D \times \partial D),$$

the Schwartz kernels of $\mathcal{K}_{\mu_e}$, that form a partition of the identity operator Id*:*

$$\mathrm{Id} = \int_{\mathscr{F}} \int_{M_{X_F,\mathrm{erg}}(F_{(1,e)})} \mathcal{K}_{\mu_e}\, d\nu_e(\mu_e)\, de(e) \tag{5.3}$$

in the weak operator topology, and for any given compact convex polytope $\mathcal{C} \subset (0,\infty) \times \mathcal{F} \subset \mathbb{R}^k$, there exists $S(h) \subset J(h) := \{i \in \mathbb{N} : (\lambda_1^i(h), \dots, \lambda_k^i(h)) \in \mathcal{C}\}$ such that for all $\varphi \in C^\infty(\partial D)$, as $h \to +0$,

$$\max_{i \in S(h)} \left| \int_{\partial D} \varphi(x) \Big(c_i \big| |D|^{-1/2} \phi^i(x) \big|^2 - \int_{\mathcal{F}} \int_{M_{X_F,\mathrm{erg}}(F_{(1,e)})} \mu(x,e) g_i(\mu_e)\, d\nu_e(\mu_e)\, de(e) \Big)\, d\sigma(x) \right| = o_{\mathcal{C}}(1). \tag{5.4}$$

In (5.4),

$$\begin{aligned} &g_i(\mu_e) := c_i \langle \mathcal{K}_{\mu_e} |D|^{-1/2} \phi^i, |D|^{-1/2} \phi^i \rangle_{L^2(\partial D, d\sigma)}, \\ &\int_{\mathcal{F}} \int_{M_{X_F,\mathrm{erg}}(F_{(1,e)})} g_i(\mu_e)\, d\nu_e(\mu_e)\, de(e) = 1, \quad \frac{\sum_{i \in S(h)} 1}{\sum_{i \in J(h)} 1} = 1 + o_{\mathcal{C}}(1), \end{aligned} \tag{5.5}$$

and moreover

$$\begin{aligned} &\mu(p,e) \geq 0, \quad \int_{\partial D} \mu(p,e)\, d\sigma(p) = 1, \\ &\frac{\int_{M_{X_F,\mathrm{erg}}(F_{(1,e)})} \mu(p,e)\, d\nu_e(\mu_e)}{\int_{M_{X_F,\mathrm{erg}}(F_{(1,e)})} \mu(q,e)\, d\nu_e(\mu_e)} = \frac{\int_{F_{(1,e)}(p)} d\sigma_{p,F_{(1,e)}}}{\int_{F_{(1,e)}(q)} d\sigma_{q,F_{(1,e)}}} \quad a.e.\text{-}(d\sigma \otimes d\sigma)(p,q). \end{aligned} \tag{5.6}$$

Proof. Let $f, \varphi \in C^\infty(\partial D)$ be given. Set $a(x,\xi) := \varphi(x)$. Then

$$\int_{F_{(E,e)}} \varphi\, d\mu_{E,e} = \int_{F_{(1,e)}} \varphi\, d\mu_e.$$

Take a partition of unity $\{\chi_i\}$ subordinate to $\{U_i\}$. With an abuse of notation via identification of points with the local trivialization $\{F_i\}$, by Lemmas 4.8 and 4.9 we have

$$\begin{aligned} &[\mathrm{Op}_{\bar{\varphi},h} f](y) \\ &= \int_{\mathcal{F}} \int_{(0,\infty) \times M_{X_F,\mathrm{erg}}(F_{(1,e)})} \sum_l \Big(\int_{F_{(E,e)}} \exp(\langle x-y,\xi\rangle/h) \bar{a}(x,\xi) \chi_l(x) f(x)\, d\mu_{(E,e)}(x) \Big) \\ &\qquad \times \sigma_{F_{(1,e)}}(F_{(1,e)}) h(e) E^{(1+k-d)/2}\, (dE \otimes d\nu_e)(E,\mu_e)\, de(e) \\ &= \int_{\mathcal{F}} \int_{(0,\infty) \times M_{X_F,\mathrm{erg}}(F_{(1,e)})} \\ &\qquad \times \sum_l \Big(\int_{F_{(1,e)}} \varphi\, d\mu_e \Big) \Big(\int_{F_{(1,e)}} \exp(\langle x-y, E^{-1/2}\xi\rangle/h) \chi_l(x) f(x) d\mu_e(x) \Big) \\ &\qquad \times \sigma_{F_{(1,e)}}(F_{(1,e)}) h(e) E^{(1+k-d)/2}\, (dE \otimes d\nu_e)(E,\mu_e)\, de(e). \end{aligned} \tag{5.7}$$

On the other hand, considering $\mathrm{Id} = \mathrm{Op}_{1,h} = \mathrm{Op}_{\bar{1},h}$ (which is independent of h), one can show that

$$
\begin{aligned}
&[\mathrm{Op}_{1,h} f](y)\\
&= \int_{\mathcal{F}} \int_{(0,\infty)\times M_{X_F,\mathrm{erg}}(F_{(1,e)})} \sum_l \Big(\int_{F_{(E,e)}} \exp(\langle x-y,\xi\rangle/h)\chi_l(x) f(x) d\mu_{(E,e)}(x) \Big)\\
&\qquad\qquad \times \sigma_{F_{(1,e)}}(F_{(1,e)}) h(e) E^{(1+k-d)/2}\,(dE\otimes d\nu_e)(E,\mu_e)\,de(e)\\
&= \int_{\mathcal{F}} \int_{(0,\infty)\times M_{X_F,\mathrm{erg}}(F_{(1,e)})} \sum_l \Big(\int_{F_{(1,e)}} \exp(\langle x-y, E^{-1/2}\xi\rangle/h)\chi_l(x) f(x)\,d\mu_e(x) \Big)\\
&\qquad\qquad \times \sigma_{F_{(1,e)}}(F_{(1,e)}) h(e) E^{(1+k-d)/2}\,(dE\otimes d\nu_e)(E,\mu_e)\,de(e).
\end{aligned}
\tag{5.8}
$$

Defining $\mathcal{K}_\mu$ (which is again independent of h) to be such that

$$
\begin{aligned}
[\mathcal{K}_{\mu_e} f](y) := \int_{(0,\infty)} \sum_l \Big(\int_{F_{(1,e)}} \exp(\langle x-y, E^{-1/2}\xi\rangle/h)\chi_l(x) f(x)\,d\mu_e(x) \Big)&\\
\times \sigma_{F_{(1,e)}}(F_{(1,e)}) h(e) E^{(1+k-d)/2}\,dE(E),&
\end{aligned}
$$

by definition in the weak operator topology we have

$$
\mathrm{Id} = \int_{\mathcal{F}} \int_{M_{X_F,\mathrm{erg}}(F_{(1,e)})} \mathcal{K}_{\mu_e}\,d\nu_e(\mu_e)\,de(e).
$$

That is,

$$
\langle f, f\rangle_{L^2(\partial D,d\sigma)} = \int_{\mathcal{F}} \int_{M_{X_F,\mathrm{erg}}(F_{(1,e)})} \langle \mathcal{K}_{\mu_e} f, f\rangle_{L^2(\partial D,d\sigma)}\,d\nu_e(\mu_e)\,de(e),
$$

whereas

$$
\begin{aligned}
&\langle \mathrm{Op}_{\overline{\varphi},h} f, f\rangle_{L^2(\partial D,d\sigma)}\\
&\qquad = \int_{\mathcal{F}} \int_{M_{X_F,\mathrm{erg}}(F_{(1,e)})} \Big(\int_{F_{(1,e)}} \varphi\,d\mu_e \Big) \langle \mathcal{K}_{\mu_e} f, f\rangle_{L^2(\partial D,d\sigma)}\,d\nu_e(\mu_e)\,de.
\end{aligned}
$$

Recall that $d\sigma_{F_{(1,e)}}(x,\xi)/\sigma_F(F_{(1,e)}) = d\mu_e(x,\xi) d\nu_e(\mu_e)$ is a probability measure. We now apply the disintegration theorem to the measure $d\mu_e(x,\xi)d\nu_e(\mu_e)$ and obtain a disintegration $d\mu_{p,e}(x,\xi)d\nu_e(\mu_e)\otimes d\sigma(p)$, where the measure-valued map $(\mu_e,p)\mapsto \mu_{p,e}$ is a $d\nu_e\otimes d\sigma$-measurable function together with $\mu_{p,e}(F_{(1,e)}\setminus(F_{(1,e)}(p)\cap \mathrm{spt}(\mu_e))) = 0$ a.e.-$d\nu_e\otimes d\sigma$. Therefore,

$$
\begin{aligned}
&\langle \mathrm{Op}_{\overline{\varphi},h} f, f\rangle_{L^2(\partial D,d\sigma)}\\
&= \int_{\mathcal{F}} \int_{M_{X_F,\mathrm{erg}}(F_{(1,e)})\times\partial D} \int_{F_{(1,e)}} \varphi \langle \mathcal{K}_{\mu_e} f, f\rangle_{L^2(\partial D,d\sigma)}\,d\mu_{p,e}\,(d\nu_e\otimes d\sigma)(\mu_e,p)\,de(e).
\end{aligned}
$$

Also observe that

$$\int_{F_{(1,e)}} \varphi \, d\mu_{p,e} = \int_{F_{(1,e)}(p)} \varphi \, d\mu_{p,e} = \varphi(p)\mu_{p,e}(F_{(1,e)}).$$

If we denote

$$\mu(p,e) := \mu_{p,e}(F_{(1,e)}) \geq 0,$$

then a.e.-$d\nu_e(\mu_e)$, the function $\mu(\cdot, e)$ is in $L^1(\partial\Omega, d\sigma)$. As a result of disintegration we have, a.e.-$d\nu_e(\mu_e)$,

$$\int_{\partial\Omega} \mu(p,e) \, d\sigma(p) = \mu_e(F_{(1,e)}) = 1.$$

Furthermore,

$$\begin{aligned}
&\langle \mathrm{Op}_{\overline{\varphi},h} f, f \rangle_{L^2(\partial D, d\sigma)} \\
&\quad = \int_{\mathcal{F}} \int_{M_{X_F,\mathrm{erg}}(F_{(1,e)}) \times \partial D} \varphi(x)\mu(x,e) \langle \mathcal{K}_{\mu_e} f, f \rangle_{L^2(\partial D, d\sigma)} \, (d\nu_e \otimes d\sigma)(\mu_e, x) \, de(e).
\end{aligned}$$

Finally, we choose $f = \phi^i(h) = |D|^{-1/2}\phi^i$ and apply (4.14) to obtain the conclusion of our theorem. Note that the choice of $S(h)$ is independent of $\varphi \in C^\infty(\partial D)$. The ratio in the last line of the theorem comes from the fact that a.e.-$d\sigma(p)$ we have by definition

$$\begin{aligned}
\int_{M_{X_F,\mathrm{erg}}(F_{(1,e)})} \mu(p,e) \, d\nu_e(\mu_e) &:= \int_{M_{X_F,\mathrm{erg}}(F_{(1,e)})} \mu_{p,e}(F_{(1,e)}(p)) \, d\nu_e(\mu_e) \\
&= \frac{\int_{F_{(1,e)}(p)} d\sigma_{p,F_{(1,e)}}}{\int_{F_{(1,e)}} d\sigma_{F_{(1,e)}}}. \qquad (5.9)
\end{aligned}$$

The proof is complete. ■

Theorem 5.2 indicates that most of the function $c_i \left| |D|^{-1/2}\phi^i(x) \right|^2$ weakly converges to a $g_i(\mu_e) d\nu_e(\mu_e)$-weighted average of $\mu(x,e)$ on each leaf $F_{(1,e)}$, where the ratio between a $d\nu_e(\mu_2)$-weighted average of $\mu(p,e)$ and that of $\mu(q,e)$ depends solely on the ratio between the volume of $F_{(1,e)}(p)$ and that of $F_{(1,e)}(q)$.

For the sake of completeness, we also give the following corollary, which generalizes a similar result in [2] and can be viewed as a generalization of quantum ergodicity over the leaves of the foliation generated by the integrable system.

Corollary 5.3. *Under Assumption* (A)*, when $d \geq 3$, if the joint Hamiltonian flow given by X_{f_j}'s is ergodic on $F_{(1,e)}$ with respect to the Liouville measure for each $e \in \mathcal{F}$, then there is a family of distributions $\{\Phi_e\}_{e \in \mathcal{F}} \subset \mathcal{D}'(\partial D \times \partial D)$, the Schwartz kernels of $\mathcal{K}_e$, that form a partition of the identity operator* Id*:*

$$\mathrm{Id} = \int_{\mathcal{F}} \mathcal{K}_e \, de(e) \qquad (5.10)$$

in the weak operator topology, and for any given compact convex polytope $\mathscr{C} \subset (0,\infty) \subset \mathbb{R}^k$*, there exists* $S(h) \subset J(h) := \{i \in \mathbb{N} : (\lambda_1^i(h), \dots, \lambda_k^i(h)) \in \mathscr{C}\}$ *such that for all* $\varphi \in C^\infty(\partial D)$*, as* $h \to +0$*,*

$$\max_{i \in S(h)} \left| \int_{\partial D} \varphi(x) \left(c_i \left| |D|^{-1/2} \phi^i(x) \right|^2 - \int_{\mathscr{F}} \frac{\sigma_{x,F_{(1,e)}}(F_{(1,e)}(x))}{\sigma_{F_{(1,e)}}(F_{(1,e)})} g_i(e)\, de(e) \right) d\sigma(x) \right| = o_{\mathscr{C}}(1). \tag{5.11}$$

In (5.11),

$$\begin{aligned} &g_i(e) := c_i \langle \mathcal{K}_e |D|^{-1/2}\phi^i, |D|^{-1/2}\phi^i \rangle_{L^2(\partial D, d\sigma)}, \\ &\int_{\mathscr{F}} g_i(e)\, de(e) = 1, \quad \frac{\sum_{i \in S(h)} 1}{\sum_{i \in J(h)} 1} = 1 + o_{\mathscr{C}}(1). \end{aligned} \tag{5.12}$$

Proof. The conclusion follows by noting that if the joint flow of X_{f_j}'s is ergodic with respect to $\sigma_{F_{(1,e)}}$ for each $e \in \mathscr{F}$, then $\sigma_{F_{(1,e)}} \in M_{X_F,\mathrm{erg}}(F_{(1,e)})$ and we can take $\nu = \delta_{\sigma_{F_{(1,e)}}}$, the Dirac measure of $\sigma_{F_{(1,e)}} \in M_{X_F,\mathrm{erg}}(F_{(1,e)})$. ∎

By Corollary 5.3, we see that if the joint flow of X_{f_j} is ergodic on $F_{(1,e)}$ with respect to the Liouville measure for all $e \in \mathscr{F}$, then most of the function $c_i \left| |D|^{-1/2}\phi^i(x) \right|^2$ weakly converges to a $\frac{g_i(e)}{\sigma_{F_{(1,e)}}(F_{(1,e)})}$-weighted average of the volumes of $F_{(1,e)}(x)$ over $e \in \mathscr{F}$. Therefore the value of the eigenfunction at $x \in \partial D$ goes high as the volume of $F_{(1,e)}(x)$ goes up for each leaf indexed by $e \in \mathscr{F}$.

5.2. *Localization/concentration of plasmon resonance at high-curvature points*

From Theorems 5.1 and 5.2, it is clear that the relative magnitude of the NP eigenfunction ϕ^i at a point x depends on the (weighted) volume of each leaf $F_{(1,e)}(x)$. Therefore, in order to understand the localization of plasmon resonance, it is essential to obtain a better description of this volume. Again, similar to [2], this volume heavily depends on the magnitude of the second fundamental form $\mathcal{A}(x)$ at the point x. As we will see in this subsection, in general, the higher the magnitude of $\mathcal{A}(x)$, the larger the volume of the characteristic variety. In particular, in a relatively simple case when the values of the second fundamental form at two points are constant multiples of each other, we have the following volume comparison.

Lemma 5.4. *Let* $p, q \in \partial D$ *be such that* $\mathcal{A}(p) = \beta \mathcal{A}(q)$ *for some* $\beta > 0$ *and* $g(p) = g(q)$*. Then* $|F_{(1,e)}(p)| = \beta^{d-2} |F_{(1,e)}(q)|$*. Moreover,*

$$\int_{F_{(1,e)}(p)} |\xi|_{g(p)}^{1+2\alpha}\, d\sigma_{p,F_{(1,e)}} = \beta^{d-1+2\alpha} \int_{F_{(1,e)}(q)} |\xi|_{g(q)}^{1+2\alpha}\, d\sigma_{q,F_{(1,e)}}.$$

Proof. From the -2-homogeneity of H, we have $H(p,\xi) = H(q, \xi/\beta)$, and therefore $\{F(x,\xi) = (\rho(1), e_2, \dots, e_k)\} = \beta\{F(q,\xi) = (\rho(1), e_2, \dots, e_k)\}$, which readily yields the conclusion of the theorem. ∎

Similar to [2], localization can be better understood via a more delicate volume comparison of the characteristic variety at different points with the help of Theorems 5.1 and 5.2 and Corollary 5.3. However, it is difficult to give a more explicit comparison of the volumes of $F_{(1,e)}(p)$ and $F_{(1,e)}(q)$ depending on their respective second fundamental forms $\mathcal{A}(p)$ and $\mathcal{A}(q)$. The following lemma provides a detour to control how the (weighted) volume of $F_{(1,e)}(p)$ depends on the principal curvatures $\{\kappa_i(p)\}_{i=1}^{d-1}$.

Lemma 5.5. *Let $G_\alpha^e : \partial D \times \mathbb{R}^{d-1} \to \mathbb{R}$ be given as*

$$G_\alpha^e\big(p, \{\kappa_i\}_{i=1}^{d-1}\big) := \int_{\bigcap_{l=2}^k \{f_l(p,\omega \sum_{i=1}^{d-1} \widetilde{\kappa}_i \omega_i^2)=e_l\}} \Big|\sum_{i=1}^{d-1} \widetilde{\kappa}_i \omega_i^2\Big|^{d-1+2\alpha} \sqrt{\sum_{i=1}^{d-1} \widetilde{\kappa}_i^2 \omega_i^2}\, d\omega_{p,e}, \tag{5.13}$$

where

$$\widetilde{\kappa}_i := \sum_{j=1}^{d-1} \kappa_j - \kappa_i. \tag{5.14}$$

Then

$$G_\alpha^e\big(p, \{\kappa_i(p)\}_{i=1}^{d-1}\big) \le \int_{F_{(1,e)}(p)} |\xi|_{g(p)}^{1+2\alpha}\, d\sigma_{p,F_{(1,e)}} \le 2G_\alpha^e\big(p, \{\kappa_i(p)\}_{i=1}^{d-1}\big). \tag{5.15}$$

Proof. We first fix a point p and choose geodesic normal coordinates with the principal curvatures along the directions ξ_i. In doing so, we can simplify the expression of $H(p,\xi) = 1$. In fact, we then have

$$H(p,\xi) = \Big(\sum_{i=1}^{d-1} \widetilde{\kappa_i(p)} \xi_i^2\Big)^2 \Big/ \Big(\sum_{i=1}^{d-1} \xi_i^2\Big)^3.$$

Due to the -2-homogeneity of $H(p,\xi)$ with respect to ξ, we parametrize the surface $\{H(p,\xi) = 1\}$ by $\omega \in \mathbb{S}^{d-2}$ with $\xi(\omega) := r(\omega)\omega$. Then we have

$$r(\omega) = \sum_{i=1}^{d-1} \widetilde{\kappa_i(p)} \omega_i^2,$$

Now we substitute the above parametrization to obtain

$$F_{(1,e)}(p) = \bigcap_{l=2}^k \Big\{(p, r(\omega)\omega) : \omega \in \mathbb{S}^{d-2},\ f_l\Big(p, \omega \sum_{i=1}^{d-1} \widetilde{\kappa_i(p)} \omega_i^2\Big) = e_l\Big\}.$$

Writing the $(d-1-k)$-Hausdorff measure of the variety $\bigcap_{l=2}^k \{f_l(p, \omega \sum_{i=1}^{d-1} \widetilde{\kappa_i(p)} \omega_i^2) = e_l\}$ on $\mathbb{S}^{d-2}$ as

$$d\omega_{p,e} := \delta_{\bigcap_{l=2}^k \{\omega \in \mathbb{S}^{d-2} : f_l(p, \omega \sum_{i=1}^{d-1} \widetilde{\kappa_i(p)} \omega_i^2) = e_l\}}(d\omega)$$

and following a similar argument to that for [2, Lemma 4.5], we can show that

$$\begin{aligned}
&\Big|\sum_{i=1}^{d-1}\widetilde{\kappa_i(p)}\omega_i^2\Big|^{d-1+2\alpha}\sqrt{\sum_{i=1}^{d-1}\widetilde{\kappa_i(p)}^2\omega_i^2 d\omega_{p,e}}\\
&\qquad\le |\xi|_{g(p)}^{1+2\alpha}d\sigma_{p,F_{(1,e)}}\le 2\Big|\sum_{i=1}^{d-1}\widetilde{\kappa_i(p)}\omega_i^2\Big|^{d-1+2\alpha}\sqrt{\sum_{i=1}^{d-1}\widetilde{\kappa_i(p)}^2\omega_i^2 d\omega_{p,e}}.
\end{aligned}$$

The proof is complete. ■

By Lemma 5.5, we can readily see that in order to compare the ratios of the magnitudes of the NP eigenfunctions, one can actually compare the ratios of the magnitudes of the principal curvatures at the respective point. For instance, if it happens that $\min_i|\widetilde{\kappa_i(p)}|\gg\max_i|\widetilde{\kappa_i(q)}|$, then it is clear that the weighted volume of $F_{(1,e)}(p)$ is much bigger than that at q.

In the next subsection, we discuss a motivating example which shows how the above lemma can be simplified and provide a precise and concrete description.

5.3. A motivating example: Surface with rotational symmetry

In what follows, we discuss a motivating example, which illustrates how the knowledge of another commuting Hamiltonian simplifies the understanding of the Hamiltonian flow of our concern and provides an explicit description of eigenfunction concentration.

Example 5.6. Let $D\subset\mathbb{R}^3$ be convex and suppose $G:=\left\{\left(\begin{smallmatrix}U&0\\0&1\end{smallmatrix}\right):U\in\mathrm{SO}(2)\right\}\subset\mathrm{SO}(3)$ is such that $G(D)=D$, i.e. D (and hence ∂D) is invariant under the rotation group G. Then, writing $(x_1,x_2,x_3,\xi_1,\xi_2,\xi_3)$ for coordinates in $T^*(\mathbb{R}^3)$, we recall the Lie algebra isomorphism

$$\begin{aligned}
j:\mathfrak{so}(3)=\{A\in\mathbb{R}^{3\times3}:A+A^T=0\}&\to\mathbb{R}^3,\\
\begin{pmatrix}0&-a_3&a_2\\a_3&0&-a_1\\-a_2&a_1&0\end{pmatrix}&\mapsto(a_1,a_2,a_3),
\end{aligned}$$

where $j([A,B])=j(A)\times j(B)$ is the three-dimensional cross product, and the moment map

$$\mu:T^*(\mathbb{R}^3)\to\mathfrak{so}(3)^*,\quad \mu(x,\xi)=\xi\times x.$$

Therefore the Hamiltonian that generates the one-parameter subgroup $G\subset\mathrm{SO}(3)$ is given by

$$\langle\mu(x,\xi),\vec{e}_3\rangle=\langle\xi\times x,(0,0,1)\rangle=\xi_1x_2-\xi_2x_1.$$

Now we define $F=(f_1,f_2)$, where $f_1(x,\xi)=\widetilde{H}(x,\xi)$ defined as above and

$$f_2:T^*(\partial D)\to\mathbb{R},\quad f_2(x,\xi)=\langle\mu(\iota(x,\xi)),\vec{e}_3\rangle,$$

where $\iota : T^*(\partial D) \to T^*(\mathbb{R}^3)$ is the canonical embedding. Notice that $\{f_i, f_j\} = \delta_{ij}$ for $i = 1, 2$ since $G(\partial D) = \partial D$ and for all $g \in G$, rotational symmetry gives $g^*\widetilde{H} = \widetilde{H}$, and hence $\{f_2, f_1\} = X_{f_2}(\widetilde{H}) = 0$. Hence (f_1, f_2) forms a completely integrable system.

Next we look at G^e_α, where $e = e_2 \in \mathcal{F} \subset \mathbb{R}$. Let us first consider the case when $e_2 \neq 0$. With this system, we find that the two principal curvatures satisfy, for $i = 1, 2$,

$$\kappa_i(p) = \kappa_i(p_z),$$

where $\partial D \ni p \mapsto (p_x, p_y, p_z) \in \mathbb{R}^3$ is the canonical embedding (which can be represented via the parametrization $\mathbb{X}$), and hence in (5.14),

$$\widetilde{\kappa}_1(p) := \kappa_2(p_z), \quad \widetilde{\kappa}_2(p) := \kappa_1(p_z).$$

Moreover, for $\omega = (\cos(\theta), \sin(\theta)) \in \mathbb{S}^1$, let $\{v_i(p)\}_{i=1,2} \in T_p(\partial D) \cong T^*_p(\partial D)$ be the principal directions. Then $A(p)v_i(p) = \kappa_i(p)v_i(p)$ in geodesic normal coordinates and denoting $\iota(p,\xi) = (p_x, p_y, p_z, (\xi_p)_x, (\xi_p)_y, (\xi_p)_z) \in \mathbb{R}^6$ with $(v_1(p))_z = 0$ (which makes the choice unique), we have $f_2(p, \omega \sum_{i=1}^2 \widetilde{\kappa}_i(p)\omega_i^2) = e_2$ if and only if

$$\begin{aligned} e_2 &= \big(\kappa_2(p_z)^2\cos^2(\theta) + \kappa_1(p_z)^2\sin^2(\theta)\big) \\ &\quad \times \big(\cos(\theta)\big((v_1(p))_x p_y - (v_1(p))_y p_x\big) + \sin(\theta)\big((v_2(p))_x p_y - (v_2(p))_y p_x\big)\big) \\ &= \big(\kappa_2(p_z)^2\cos^2(\theta) + \kappa_1(p_z)^2\sin^2(\theta)\big) \\ &\quad \times \sin(\theta + \widetilde{\theta}(p)) \times \sqrt{\big((v_1(p))_x p_y - (v_2(p))_y p_x\big)^2 + \big((v_2(p))_x p_y - (v_1(p))_y p_x\big)^2}, \end{aligned} \tag{5.16}$$

where
$$\widetilde{\theta}(p) = \tan^{-1}\left(\frac{(v_1(p))_x p_y - (v_1(p))_y p_x}{(v_2(p))_x p_y - (v_2(p))_y p_x}\right).$$

The RHS term in (5.16) is invariant by the rotational action (recall $g^* f_2 = f_2$ for all $g \in G$), and therefore is a function of (z_p, θ) only. θ can now be found via the tangent-half-angle formula and a solution to a sixth order polynomial. In any case, the solution set is either an empty set if e_2 is large, or a set of a finite number of points. Let us also denote by $|e_2|_{\max}$ the extremal value of $|e_2|$ such that the solution set of (5.16) is non-empty, i.e.

$$\mathcal{F} = [-|e_2|_{\max}, |e_2|_{\max}].$$

For a given (p_z, e_2), we denote by $N(p_z, e_2)$ the number of solutions to (5.16). If $0 < |e_2| \leq |e_2|_{\max}$, we can see that $1 \leq N(p_z, e_2) \leq 2$ and the solutions have the same value of $\cos(\theta)$. Denoting them as $\theta_l(p_z, e_2)$, $l = 1, \ldots, N(p_z, e_2)$, we have, in (5.13),

$$\begin{aligned} G_\alpha\big(p, \{\kappa_i(p)\}_{i=1}^2\big) &= 2\big(\kappa_2(p_z)\cos^2(\theta_1(p_z, e_2)) + \kappa_1(p_z)\sin^2(\theta_1(p_z, e_2))\big)^{2+2\alpha} \\ &\quad \times \sqrt{\kappa_2(z_p)^2\cos^2(\theta_1(p_z, e_2)) + \kappa_1(z_p)^2\sin^2(\theta_1(p_z, e_2))}. \end{aligned}$$

It is now clear that $G^e_\alpha(p, \{\kappa_i(p)\}_{i=1}^2)$ increases separately when either of $\kappa_1(p), \kappa_2(p)$ increases. We remark that when $|e_2| = |e_2|_{\max}$, the only solution to (5.16) is $\theta = 0$ if $e_2 > 0$

and $\theta = \pi$ if $e_2 < 0$, i.e. $N(p_z, e_2) = 1$. We next check when $e_2 = 0$. If $(p_x, p_y) \neq (0, 0)$, taking $(v_1(p))_z = 0$, we only have $\theta = -\tan^{-1}(\frac{p_y}{p_x}), \pi - \tan^{-1}(\frac{p_y}{p_x})$ that satisfies (5.16), and hence

$$G^e_\alpha\big(p, \{\kappa_i(p)\}_{i=1}^2\big) = 2\kappa_1(p_z)^{3+2\alpha}.$$

If $p_x = p_y = 0$, we have $\widetilde{\kappa}_1(p) = \widetilde{\kappa}_2(p) = \kappa(p_z)$, and however we choose the directions $v_i(p)$, (5.16) is satisfied for all $\theta \in [0, 2\pi)$, and in that case

$$G^e_\alpha\big(p, \{\kappa_i(p)\}_{i=1}^2\big) = \int_0^{2\pi} \kappa(p_z)^{3+2\alpha}\, d\theta = 2\pi\kappa(p_z)^{3+2\alpha}.$$

Again, we can see that in either situation $G_\alpha(p, \{\kappa_i(p)\}_{i=1}^2)$ increases separately when either of $\kappa_1(p), \kappa_2(p)$ increases.

Next, we look more closely at the flow induced by $X_{\widetilde{\widetilde{H}}}$ on $F_{(1,e_2)}$. We first focus on the case $0 < |e_2| < |e_2|_{\max}$. We denote by $p_{z,\max,e_2}$ and $p_{z,\min,e_2}$ respectively the largest and smallest values of p_z for $(p, \xi) \in F_{(1,e_2)}$. Then we find that

$$\begin{aligned} &F_{(1,e_2)} \\ &= \bigcup_{p_z \in [p_{z,\min,e_2}, p_{z,\max,e_2}]} \Big\{\Big(p, r(p_z, e_2)\big(\cos(\theta_l(p_z, e_2))v_1(p) + \sin(\theta_l(p_z, e_2))v_2(p)\big)\Big) : \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad l = 1, \dots, N(p_z, e_2)\Big\}, \end{aligned}$$

where $r(p_z, e_2) := \kappa_2(z_p)^2 \cos^2(\theta_1(p_z, e_2)) + \kappa_1(z_p)^2 \sin^2(\theta_1(p_z, e_2))$. Note that if $p = p_{z,\min,e_2}$ or $p = p_{z,\max,e_2}$, then $\theta = 0$ or π and p satisfies

$$|e_2| = \kappa_2(p_z)^2 \sqrt{p_x^2 + p_y^2},$$

where the RHS is again a function of p_z only. One can show that on $F_{(1,e_2)}$, we have $X_{\widetilde{H}} = 0$ if and only if $\xi = 0$. Assume that we start with $(x, \xi) \in F_{(1,e_2)}$ and flow accordingly with $a_t = (p(t), \xi(t))$. Then there exists an interval $[s_1, s_2]$ containing 0 with smallest length such that $p_z'(s_1) = p_z'(s_2) = 0$ (which coincides with that when $p_z = p_{z,\max,e_2}$ or $p_z = p_{z,\min,e_2}$). Via reflection symmetry, for all $t \in \mathbb{R}$ and $n \in \mathbb{Z}$ we have

$$a_{t-s_1+2n(s_2-s_1)} = a_{-(t-s_1)+n(s_2-s_1)} = a_{t-s_1}.$$

That is, a_t is $2(s_2 - s_1)$-periodic and symmetric about s_2. Hence the orbit of the flow generated by $X_{\widetilde{H}} = 0$ on $F_{(1,e_2)}$ can be either periodic or quasi-periodic. Note that if $p_{z,\min,e_2} < p_z < p_{z,\max,e_2}$, then $N(p_z, e_2) = 2$; whereas, at $p_z = p_{z,\max,e_2}$ or $p_z = p_{z,\min,e_2}$, we have $\theta_l(p_z, e_2) = 0$ or π, and hence $N(p_z, e_2) = 1$. Therefore, tracing the points, we have

$$F_{(1,e_2)} \cong \mathbb{T}^2,$$

which is a smooth 2-dimensional surface. Moreover, only two cases are possible if $0 < |e_2| < |e_2|_{\max}$:

(1) When a Hamiltonian curve is periodic on $\mathbb{T}^2$, a shift of the curve gives all the periodic orbits of the Hamiltonian flow. Hence all the ergodic measures of $X_{\tilde{H}}$ on $F_{(1,e_2)}$ can be indexed as $\{\mu_{(1,e_2),a}\}_{a\in[0,1)}$ where supp $\mu_{(1,e_2),a}$ contains the point (p,ξ) such that

$$(p_x, p_y, p_z) = \Big(\sqrt{p_x^2+p_y^2}\cos(2a\theta_{\mathrm{per}}), \sqrt{p_x^2+p_y^2}\sin(2a\theta_{\mathrm{per}}), p_{z,\max,e_2}\Big)$$

with
$$\theta_{\mathrm{per}} = \cos^{-1}\left(\frac{p_x(s_2)p_x(s_1)+p_y(s_2)p_y(s_1)}{\sqrt{p_x(s_2)^2+p_y(s_2)^2}\sqrt{p_x(s_1)^2+p_y(s_1)^2}}\right).$$

(2) When a curve is quasi-periodic on $\mathbb{T}^2$, the flow of $X_{\tilde{H}}$ is ergodic on the torus $\mathbb{T}^2$ with respect to the Liouville measure restricted to $\mathbb{T}^2$.

If $|e_2| = |e_2|_{\max}$, we find that $N(p_z, e_2) = 1$, and

$$F_{(1,e_2)} \cong \mathbb{S}^1.$$

If $e_2 = 0$, we can verify that $N(p_z, e_2) = 2$, and the solutions to (5.16) are $\theta = \pi/2, \pi/3$. The flow is periodic with $\theta_{\mathrm{per}} = \pi$. We have

$$F_{(1,e_2)} \cong \mathbb{T}^2,$$

and all the ergodic measures of $X_{\tilde{H}}$ on $F_{(1,e_2)}$ can be indexed as $\{\mu_{(1,e_2),a}\}_{a\in[0,1)}$ where supp $\mu_{(1,e_2),a}$ contains the point (p,ξ) such that

$$(p_x, p_y, p_z) = \Big(\sqrt{p_x^2+p_y^2}\cos(2\pi a), \sqrt{p_x^2+p_y^2}\sin(2\pi a), p_{z,\max,e_2}\Big).$$

Finally, we notice that in all the cases, the joint flow given by X_F is ergodic on $F_{(1,e_2)}$ with respect to the Liouville measure for all possible values of $(1, e_2)$ such that $F_{(1,e_2)}$ is nonempty: this holds trivially when $F_{(1,e_2)} \cong \mathbb{S}^1$ and if the flow of $X_{\tilde{H}}$ is quasi-periodic on $F_{(1,e_2)} \cong \mathbb{T}^2$; and it holds when the flow of $X_{\tilde{H}}$ is periodic on $F_{(1,e_2)} \cong \mathbb{T}^2$ since G induces a transitive action on the index set $\{\mu_{(1,e_2),a}\}_{a\in[0,1)}$. Therefore Corollary 5.3 applies to obtain a density-one subsequence $i \in S(h) \subset J(h)$ of $c_i\big||D|^{-1/2}\phi^i(x)\big|^2$ weakly converging to $\int_{-|e_2|_{\max}}^{|e_2|_{\max}} \frac{\sigma_{x,F_{(1,e)}}(F_{(1,e_2)}(x))}{\sigma_{F_{(1,e_2)}}(F_{(1,e_2)})} g_i(e_2)\, de(e_2)$ where

$$G^e_{-1/2}\big(p, \{\kappa_i(p)\}_{i=1}^2\big) \le \sigma_{x,F_{(1,e)}}(F_{(1,e_2)}(x)) \le 2G^e_{-1/2}\big(p, \{\kappa_i(p)\}_{i=1}^2\big),$$

with $G^e_{-1/2}(p, \{\kappa_i(\cdot)\}_{i=1}^{d-1})$ in Lemma 5.5 given by

$$\begin{aligned} &G^e_{-1/2}\big(p, \{\kappa_i(p)\}_{i=1}^2\big) \\ &\quad = \begin{cases} 2\sqrt{\kappa_2(z_p)^2\cos^2(\theta_1(p_z,e_2)) + \kappa_1(z_p)^2\sin^2(\theta_1(p_z,e_2))} & \text{when } e_2 \neq 0, \\ 2\pi\kappa(p_z) & \text{when } e_2 = 0. \end{cases}\end{aligned}$$

It is now clear that the absolute value of this function is separately increasing when each of the κ_i is increasing.

A direct check of the (singular) fibration $\widetilde{\pi} : \{\widetilde{H} = 1\} \to \mathcal{F} := [-|e_2|_{\max}, |e_2|_{\max}]$ with

$$\widetilde{\pi}^{-1}(e_2) = \begin{cases} \mathbb{T}^2 & \text{when } |e_2| < |e_{2,\max}|, \\ \mathbb{S}^1 & \text{when } |e_2| = |e_{2,\max}|, \end{cases}$$

shows that

$$\{\widetilde{H} = 1\} \cong S(\mathbb{S}^2) \cong \mathrm{SO}(3) \cong \mathbb{RP}^3$$

via diffeomorphism where $S(\mathbb{S}^2)$ is the sphere bundle of $\mathbb{S}^2$. Via the well known Gysin sequence, we obtain

$$\begin{aligned} &H^0(\{\widetilde{H} = 1\}; \mathbb{Z}) = \mathbb{Z}, \qquad && H^1(\{\widetilde{H} = 1\}; \mathbb{Z}) = 0, \\ &H^2(\{\widetilde{H} = 1\}; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}, && H^3(\{\widetilde{H} = 1\}; \mathbb{Z}) = \mathbb{Z}. \end{aligned}$$

Now, one can verify that

$$\bigcup_{e_2 \in \mathcal{F}} \widetilde{\pi}^{-1}(e_2) = \mathbb{S}^1 \cup (\mathbb{T}^2 \times (-|e_2|_{\max}, |e_2|_{\max})) \cup \mathbb{S}^1 \cong S(\mathbb{S}^2).$$

Remark 5.7. Our previous description and analysis may extend to systems that are nearly integrable via a perturbation analysis. Here, the system is given as a Kolmogorov non-degenerate perturbation of a (completely) integrable system (of class at least $C^{2(d-1),\alpha}$) via a classical KAM theory [35, 48, 49]. In such a case, if $k = n = d - 1$, it is known that for an ε-perturbation of the system, the flow will stay quasi-periodic on the surviving invariant Lagrangian tori which foliate/occupy $1 - O(\sqrt{\varepsilon})$ of the space. The invariant measures will then be localized on the surviving invariant Lagrangian tori. The dynamics in the remaining $O(\sqrt{\varepsilon})$-space may on the other hand be complicated, e.g. Arnold diffusion may occur. However, when $d - 1 = 2$, i.e. $d = 3$, a topological obstruction prevents the Arnold diffusion from happening.

In the previous example with rotational symmetry with $d = 3$, we may perturb a rotational symmetric shape to a shape of a thin rod, and our result echoes that in [20].

6. Localization/concentration of plasmon resonances for quasi-static wave scattering

In this section, we extend all of the electrostatic results to the quasi-static case governed by the Helmholtz system. We refer to [2] for the discussion of the physical background; moreover, by following the treatment therein, the concentration result in the quasi-static regime can be obtained by directly modifying the relevant results in the previous section. Hence, in what follows, we shall be brief in our discussion.

Let $\varepsilon_0, \mu_0, \varepsilon_1, \mu_1$ be real constants and assume that ε_0 and μ_0 are positive. Let D be as in Section 1, and set

$$\mu_D = \mu_1 \chi(D) + \mu_0 \chi(\mathbb{R}^d \setminus \bar{D}), \quad \varepsilon_D = \varepsilon_1 \chi(D) + \varepsilon_0 \chi(\mathbb{R}^d \setminus \bar{D}).$$

Let $\omega \in \mathbb{R}_+$ denote the angular frequency of the operating wave. Set $k_0 := \omega\sqrt{\varepsilon_0\mu_0}$ and $k_1 := \omega\sqrt{\varepsilon_1\mu_1}$ with $\Im k_j \geq 0$, $j = 0, 1$. Let u_0 be an entire solution to $(\Delta + k_0^2)u_0 = 0$ in $\mathbb{R}^d$. Consider the Helmholtz scattering problem for $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$:

$$\begin{cases} \nabla \cdot \left(\frac{1}{\mu_D}\nabla u\right) + \omega^2\varepsilon_D u = 0 & \text{in } \mathbb{R}^d, \\ \left(\frac{\partial}{\partial|x|} - \mathrm{i}k_0\right)(u - u_0) = o(|x|^{-(d-1)/2}) & \text{as } |x| \to \infty. \end{cases} \tag{6.1}$$

Henceforth, we assume that $\omega \ll 1$, or equivalently $k_0 \ll 1$, which is known as the *quasi-static regime*.

Let

$$\Gamma_k(x - y) := C_d(k|x - y|)^{-(d-2)/2}H^{(1)}_{(d-2)/2}(k|x - y|) \tag{6.2}$$

be the outgoing fundamental solution to $-(\Delta + k^2)$, where C_d is some dimensional constant and $H^{(1)}_{(d-2)/2}$ is the Hankel function of the first kind and order $(d-2)/2$. We introduce the following single-layer and NP operators associated with a given wavenumber $k \in \mathbb{R}_+$:

$$\mathcal{S}^k_{\partial D}[\phi](x) := \int_{\partial D} \Gamma_k(x - y)\phi(y)\, d\sigma(y), \quad x \in \partial D, \tag{6.3}$$

$$(\mathcal{K}^k_{\partial D})^*[\phi](x) := \int_{\partial D} \partial_{\nu_x}\Gamma_k(x - y)\phi(y)\, d\sigma(y), \quad x \in \partial D. \tag{6.4}$$

Following the discussion in [2], in the spirit of (1.8), we consider the following generalized plasmon resonance problem when $\omega \in \mathbb{R}_+$: find a nontrivial $\phi \in H^{-1/2}(\partial D, d\sigma)$ such that for some $m \in \mathbb{N}$,

$$\left\{\frac{1}{2}\left(\frac{1}{\mu_0}\mathrm{Id} + \frac{1}{\mu_1}(\mathcal{S}^{k_1}_{\partial D})^{-1}\mathcal{S}^{k_0}_{\partial D}\right) + \frac{1}{\mu_0}(\mathcal{K}^{k_0}_{\partial D})^* - \frac{1}{\mu_1}(\mathcal{K}^{k_1}_{\partial D})^*(\mathcal{S}^{k_1}_{\partial D})^{-1}\mathcal{S}^{k_0}_{\partial D}\right\}^m \phi = 0. \tag{6.5}$$

Here, (ε_1, μ_1) is said to be a (pair) *generalized plasmonic eigenvalue*. We emphasize that m must be finite in the equation. When $m = 1$, we refer to this problem as the *plasmon resonance problem* for $\omega \in \mathbb{R}_+$, and (ε_1, μ_1) as a (pair) *generalized plasmonic eigenvalue*.

The following two lemmas in [2] characterize the plasmon resonance when $\omega \ll 1$.

Lemma 6.1. *Under Assumption* (A) *and supposing $\omega \ll 1$, a solution*

$$((\mu_0, \mu_1, \varepsilon_0, \varepsilon_1, \omega), m, \phi_{\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,\omega,m})$$

to (6.5) *with unit L^2-norm has the following property for all $s \in \mathbb{R}$:*

$$\begin{cases} \left\| |D|^s\phi_{\mu_0,\mu_1,\varepsilon_0,\varepsilon_1,\omega,m} - |D|^s\phi^i \right\|_{C^0(\partial D)} = \mathcal{O}_{i,s}(\omega^2), \\ \lambda(\mu_0^{-1}, \mu_1^{-1}) - \widetilde{\lambda}^i = \mathcal{O}_i(\omega^2), \end{cases}$$

for some eigenpair $(\widetilde{\lambda}^i, \phi^i)$ of the Neumann–Poincaré operator $\mathcal{K}^*_{\partial D}$, and $m \leq m_i$, where m and m_i signify the algebraic multiplicities of λ and λ_i, respectively.

Lemma 6.2. *Given any nonzero* $\lambda_i \in \sigma(\mathcal{K}^*_{\partial D})$ *and for any* $(\widetilde{\mu}_0, \widetilde{\mu}_1) \in D_i := \{(\mu_0, \mu_1) \in \mathbb{C}^2 \setminus \{(0,0)\} : \lambda(\widetilde{\mu}_0^{-1}, \widetilde{\mu}_1^{-1}) = \widetilde{\lambda}^i,\ \mu_0 - \mu_1 \neq 0\}$ *(which is nonempty), there exists* $0 < \omega_i \ll 1$ *such that for all* $\omega < \omega_i$*, the set*

$$\{(\mu_0, \mu_1, \varepsilon_0, \varepsilon_1) \in \mathbb{C}^2 \setminus \{\mu_0 - \mu_1 = 0\} \times (\mathbb{C} \setminus \mathbb{R}^+)^2 : \text{there are } m \in \mathbb{N}, \phi \in H^{-1/2}(\partial D, \sigma) \text{ such that } ((\mu_0, \mu_1, \varepsilon_0, \varepsilon_1, \omega), \phi, m) \text{ satisfies } (6.5)\}$$

forms a complex codimension 1 *surface in a neighbourhood of* $(\widetilde{\mu}_0, \widetilde{\mu}_1)$.

By Lemma 6.2, we easily see that there are infinitely many choices of (ε_1, μ_1) such that the (genearalized) plasmon resonance occurs around $\widetilde{\lambda}^i$. Combining this with a similar perturbation argument as in the proof of Lemma 6.2, our conclusion on the plasmon resonance in the electrostatic case transfers to the Helmholtz transmission problem to show concentration of plasmon resonances at high-curvature points.

Theorem 6.3. *Given any* $x \in \partial D$*, let* $\{\chi_{x,\delta}\}_{\delta>0}$ *be a family of smooth nonnegative bump functions compactly supported in* $B_\delta(x)$ *with* $\int_{\partial D} \chi_{p,\delta}\, d\sigma = 1$*. Under Assumption* (A)*, when* $d \geq 3$*, fixing a compact convex polytope* $\mathcal{C} \subset \mathcal{F} \subset \mathbb{R}^{k-1}$*,* $[r,s] \subset \mathbb{R}$*,* $\alpha \in \mathbb{R}$ *and* $p, q \in \partial D$*, we have a choice of* $\delta(h)$ *and* $\omega(h)$*, both depending on* $\mathcal{C}$*,* p*,* q *and* α*, such that for any* $\omega < \omega(h)$*, there exists*

$$\big((\mu_{0,i}, \mu_{1,i}, \varepsilon_{0,i}, \varepsilon_{1,i}, \omega), m_i, \phi_{\mu_{0,i}, \mu_{1,i}, \varepsilon_{0,i}, \varepsilon_{1,i}, \omega, m_i}\big)$$

solving (6.5)*, and as* $h \to +0$*, we have* $\delta(h) \to 0$*,* $\omega(h) \to 0$ *and*

$$\frac{\sum_{(\lambda_1^i(h),\ldots,\lambda_k^i(h)) \in [r,s]\times\mathcal{C}} c_i \int_{\partial D} \chi_{p,\delta(h)}(x) \big||D|^\alpha \phi_{\mu_{0,i},\mu_{1,i},\varepsilon_{0,i},\varepsilon_{1,i},\omega,m_i}(x)\big|^2\, d\sigma(x)}{\sum_{(\lambda_1^i(h),\ldots,\lambda_k^i(h)) \in [r,s]\times\mathcal{C}} c_i \int_{\partial D} \chi_{q,\delta(h)}(x) \big||D|^\alpha \phi_{\mu_{0,i},\mu_{1,i},\varepsilon_{0,i},\varepsilon_{1,i},\omega,m_i}(x)\big|^2\, d\sigma(x)} = \frac{\int_{\mathcal{F}} \int_{F_{(1,e)}(p)} |\xi|_{g(y)}^{1+2\alpha}\, d\sigma_{p,F_{(1,e)}}\, h(e)\, de}{\int_{\mathcal{F}} \int_{F_{(1,e)}(q)} |\xi|_{g(y)}^{1+2\alpha}\, d\sigma_{q,F_{(1,e)}}\, h(e)\, de} + o_{\mathcal{C},r,s,p,q,\alpha}(1),$$

where $c_i := |\phi^i|^{-2}_{H^{-1/2}(\partial D, d\sigma)}$*. Here, the o-term depends on* $\mathcal{C}$*,* r*,* s*,* p*,* q *and* α*.*

Proof. From Theorem 5.1, we have a choice of $\delta(h)$ depending on $\mathcal{C}$, p, q and α such that, for any given $\varepsilon > 0$, there exists h_0 depending on $\mathcal{C}$, p, q, α such that for all $h < h_0$,

$$\left| \frac{\sum_{(\lambda_1^i(h),\ldots,\lambda_k^i(h)) \in [r,s]\times\mathcal{C}} c_i \int_{\partial D} \chi_{p,\delta(h)}(x) \big||D|^\alpha \phi^i(x)\big|^2\, d\sigma}{\sum_{(\lambda_1^i(h),\ldots,\lambda_k^i(h)) \in [r,s]\times\mathcal{C}} c_i \int_{\partial D} \chi_{q,\delta(h)}(x) ||D|^\alpha \phi^i(x)|^2\, d\sigma} - \frac{\int_{\mathcal{F}} \int_{F_{(1,e)}(p)} |\xi|_{g(y)}^{1+2\alpha}\, d\sigma_{p,F_{(1,e)}}\, h(e)\, de}{\int_{\mathcal{F}} \int_{F_{(1,e)}(q)} |\xi|_{g(y)}^{1+2\alpha}\, d\sigma_{q,F_{(1,e)}}\, h(e)\, de} \right| \leq \varepsilon. \quad (6.6)$$

Now for each $h < h_0$, from Lemma 6.2, there exists

$$\widetilde{\omega}(h) := \min_{\{i \in \mathbb{N} : (\lambda_1^i(h),\ldots,\lambda_k^i(h)) \in \mathcal{C}\}} \omega_i$$

such that for all $\omega < \omega(h)$, there exists

$$\big((\mu_{0,i}, \mu_{1,i}, \varepsilon_{0,i}, \varepsilon_{1,i}, \omega), m_i, \phi_{\mu_{0,i},\mu_{1,i},\varepsilon_{0,i},\varepsilon_{1,i},\omega,m_i}\big)$$

solving (6.5). By Lemma 6.1, upon rescaling $\phi_{\mu_{0,i},\mu_{1,i},\varepsilon_{0,i},\varepsilon_{1,i},\omega,m_i}$ (without relabelling it), we have

$$\big\| |D|^\alpha \phi_{\mu_{0,i},\mu_{1,i},\varepsilon_{0,i},\varepsilon_{1,i},\omega,m_i} - |D|^\alpha \phi^i \big\|_{C^0(\partial D)} \le C_{i,\alpha}\omega^2.$$

In particular, we can choose a smaller $\omega(h) < \widetilde{\omega}(h)$ depending on $\mathscr{C}$, r, s, p, q, α such that for all $\omega < \omega(h)$, we have

$$\Big\| \big||D|^\alpha \phi_{\mu_{0,i},\mu_{1,i},\varepsilon_{0,i},\varepsilon_{1,i},\omega,m_i}\big|^2 - \big||D|^\alpha \phi^i\big|^2 \Big\|_{C^0(\partial D)} \le \quad 10^{-2}\varepsilon/\Theta,$$

where

$$\Theta := \sum_{(\lambda_1^i(h),\ldots,\lambda_k^i(h))\in[r,s]\times\mathscr{C}} c_i \Big/ \min\Big\{1, \min_{y=p,q}\Big\{\Big(\sum_{(\lambda_1^i(h),\ldots,\lambda_k^i(h))\in[r,s]\times\mathscr{C}} \vartheta_i\Big)^{-2}\Big\}\Big\}$$

and

$$\vartheta_i := c_i \int_{\partial D} \chi_{y,\delta(h)}(x) \big||D|^\alpha \phi^i(x)\big|^2 \, d\sigma.$$

Therefore, with this choice of $\omega(h)$ we have, for all $\omega < \omega(h)$,

$$\left| \frac{\sum_{(\lambda_1^i(h),\ldots,\lambda_k^i(h))\in[r,s]\times\mathscr{C}} c_i \int_{\partial D} \chi_{p,\delta(h)}(x) \big||D|^\alpha \phi_{\mu_{0,i},\mu_{1,i},\varepsilon_{0,i},\varepsilon_{1,i},\omega,m_i}(x)\big|^2 \, d\sigma(x)}{\sum_{(\lambda_1^i(h),\ldots,\lambda_k^i(h))\in[r,s]\times\mathscr{C}} c_i \int_{\partial D} \chi_{q,\delta(h)}(x) \big||D|^\alpha \phi_{\mu_{0,i},\mu_{1,i},\varepsilon_{0,i},\varepsilon_{1,i},\omega,m_i}(x)\big|^2 \, d\sigma(x)} - \frac{\int_{\mathscr{F}} \int_{F_{(1,e)}(p)} |\xi|_{g(y)}^{1+2\alpha} \, d\sigma_{p,F_{(1,e)}} \, h(e) \, de}{\int_{\mathscr{F}} \int_{F_{(1,e)}(q)} |\xi|_{g(y)}^{1+2\alpha} \, d\sigma_{q,F_{(1,e)}} \, h(e) \, de} \right| \le \varepsilon. \tag{6.7}$$

Combining (6.7) with (6.6) readily yields our conclusion.

The proof is complete. ■

In a similar manner, we obtain the following result.

Theorem 6.4. *Under Assumption* (A), *when* $d \ge 3$, *given a compact convex polytope* $\mathscr{C} \subset \mathbb{R} \times \mathscr{F} \subset \mathbb{R}^k$, *there exists* $S(h) \subset J(h) := \{i \in \mathbb{N} : (\lambda_1^i(h), \ldots, \lambda_k^i(h)) \in \mathscr{C}\}$ *and* $\omega(h)$ *such that, for all* $\varphi \in C^\infty(\partial D)$ *and any* $\omega < \omega(h)$, *there exists*

$$\big((\mu_{0,i}, \mu_{1,i}, \varepsilon_{0,i}, \varepsilon_{1,i}, \omega), m_i, \phi_{\mu_{0,i},\mu_{1,i},\varepsilon_{0,i},\varepsilon_{1,i},\omega,m_i}\big)$$

solving (6.5) *such that as* $h \to +0$, *we have* $\omega(h) \to 0$ *and*

$$\max_{i\in S(h)} \left| \int_{\partial D} \varphi(x) \Big(c_i \big||D|^{-1/2} \phi_{\mu_{0,i},\mu_{1,i},\varepsilon_{0,i},\varepsilon_{1,i},\omega,m_i}(x)\big|^2 - \int_{\mathscr{F}} \int_{M_{X_F,\mathrm{erg}}(F_{(1,e)})} \mu(x,e) g_i(\mu_e) \, d\nu_e(\mu_e) \, de(e) \Big) \, d\sigma(x) \right| = o_{r,s}(1). \tag{6.8}$$

Here, $S(h)$, $\{g_i : \bigcup_{e\in\mathcal{F}} M_{X_F,\mathrm{erg}}(F_{(1,e)}) \to \mathbb{C}\}_{i\in\mathbb{N}}$ and $\mu(p,e)$ are as in Theorem 5.2. In particular,

$$\frac{\int_{M_{X_F,\mathrm{erg}}(F_{(1,e)})} \mu(p,e)\, d\nu_e(\mu_e)}{\int_{M_{X_F,\mathrm{erg}}(F_{(1,e)})} \mu(q,e)\, d\nu_e(\mu_e)} = \frac{\int_{F_{(1,e)}(p)} d\sigma_{p,F_{(1,e)}}}{\int_{F_{(1,e)}(q)} d\sigma_{q,F_{(1,e)}}} \quad a.e.\text{-}(d\sigma\otimes d\sigma)(p,q).$$

If the joint Hamiltonian flow given by X_{f_j}'s is ergodic on $F_{(1,e)}$ with respect to the Liouville measure for each $e\in\mathcal{F}$, then

$$\max_{i\in S(h)} \left| \int_{\partial D} \varphi(x) \Big(c_i \big| |D|^{-1/2} \phi_{\mu_{0,i},\mu_{1,i},\varepsilon_{0,i},\varepsilon_{1,i},\omega,m_i}(x) \big|^2 \right.$$
$$\left. - \int_{\mathcal{F}} \frac{\sigma_{x,F_{(1,e)}}(F_{(1,e)}(x))}{\sigma_{F_{(1,e)}}(F_{(1,e)})} g_i(e)\, de(e) \Big) d\sigma(x) \right| = o_{\mathcal{C}}(1),$$

where $\{g_i : \mathcal{F} \to \mathbb{C}\}_{i\in\mathbb{N}}$ is now defined as in Corollary 5.3.

Proof. Let $\mathcal{C}$ be a compact convex polytope and $\varphi \in C^\infty(\partial D)$. Given $\varepsilon > 0$, by Theorem 5.2, considering h_0 small enough such that for all $h < h_0$, we have

$$\max_{i\in S(h)} \left| \int_{\partial D} \varphi(x) \Big(c_i \big| |D|^{-1/2} \phi^i(x) \big|^2 \right.$$
$$\left. - \int_{\mathcal{F}} \int_{M_{X_F,\mathrm{erg}}(F_{(1,e)})} \mu(x,e) g_i(\mu_e)\, d\nu_e(\mu_e)\, de(e) \Big) d\sigma(x) \right| \le \varepsilon.$$

Now, for each $h < h_0$, from Lemma 6.2, there exists $\tilde{\omega}(h) = \min\{\min_{i\in S(h)} \omega_i, 1\}$ such that for all $\omega < \tilde{\omega}(h)$, there exists

$$\big((\mu_{0,i}, \mu_{1,i}, \varepsilon_{0,i}, \varepsilon_{1,i}, \omega), m_i, \phi_{\mu_{0,i},\mu_{1,i},\varepsilon_{0,i},\varepsilon_{1,i},\omega,m_i}\big)$$

solving (6.5). By Lemma 6.1, again upon rescaling $\phi_{\mu_{0,i},\mu_{1,i},\varepsilon_{0,i},\varepsilon_{1,i},\omega,m_i}$ (without relabelling it), we have

$$\max_{i\in S(h)} c_i \left| \int_{\partial D} \varphi(x) \Big(\big| |D|^{-1/2} \phi^i(x) \big|^2 - \big| |D|^{-1/2} \phi_{\mu_{0,i},\mu_{1,i},\varepsilon_{0,i},\varepsilon_{1,i},\omega,m_i}(x) \big|^2 \Big) d\sigma(x) \right|$$
$$\le C_{S(h)} \|\varphi\|_{C^0(\partial D)} \omega^2. \tag{6.9}$$

We may now choose

$$\omega(h) \le \min\{\varepsilon, \tilde{\omega}(h), \tilde{\omega}(h)/C_{S(h)}\}.$$

Then for all $\omega < \omega(h)$, we finally infer from (6.9) and Corollary 5.3 that

$$\max_{i\in S(h)} \left| \int_{\partial D} \varphi(x) \Big(c_i \big| |D|^{-1/2} \phi_{\mu_{0,i},\mu_{1,i},\varepsilon_{0,i},\varepsilon_{1,i},\omega,m_i}(x) \big|^2 \right.$$
$$\left. - \int_{\mathcal{F}} \int_{M_{X_F,\mathrm{erg}}(F_{(1,e)})} \mu(x,e) g_i(\mu_e)\, d\nu_e(\mu_e)\, de(e) \Big) d\sigma(x) \right| \le (1 + \|\varphi\|_{C^0(\partial D)})\varepsilon.$$

The proof is complete. ∎

We remark that a similar conclusion holds for the explicit motivating example discussed in Section 5.3 in the quasi-static case when $\omega \ll 1$, which we choose not to repeat.

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