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# Wave equations in the Kerr–de Sitter spacetime: The full subextremal range

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**Abstract.** We prove that solutions to linear wave equations in a subextremal Kerr–de Sitter spacetime have asymptotic expansions in quasinormal modes up to a decay order given by the normally hyperbolic trapping, extending the result of the second named author (2013). The main novelties are a different way of obtaining a Fredholm setup that defines the quasinormal modes and a new analysis of the trapping of lightlike geodesics in the Kerr–de Sitter spacetime, both of which apply in the full subextremal range. In particular, this reduces the question of decay for solutions to wave equations to the question of mode stability.

**Keywords:** subextremal Kerr–de Sitter spacetime, resonances, quasinormal modes, radial points, normally hyperbolic trapping.

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## 1. Introduction

Kerr–de Sitter metrics are Lorentzian metrics (which we take of signature  $(-, +, +, +)$ ) on  $\mathbb{R}_{t_*} \times (0, \infty)_r \times S^2$  solving Einstein’s equation  $\text{Ric}(g) = \Lambda g$  with a cosmological constant  $\Lambda > 0$ . They depend on three parameters: apart from the cosmological constant  $\Lambda$ , these are the mass  $m > 0$  and the angular momentum  $a \in \mathbb{R}$  of the black hole the metric corresponds to. Here the ‘black hole’ nature corresponds to the presence of certain null-hypersurfaces, called horizons, which for the Kerr–de Sitter metrics lie at certain values of  $r$ . The metric is given in terms of a quartic polynomial  $\mu$ , and the three parameters then have to satisfy an additional condition, namely that  $\mu$  has four distinct real roots; the significance of this condition is due to the horizons lying at the roots of  $\mu$ . Kerr–de Sitter metrics corresponding to these parameters are called *subextremal*.

In [21], the wave equation on Kerr–de Sitter spacetimes was analyzed for a large but not complete range of the parameters by showing that it fits into the Fredholm framework

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based on microlocal analysis developed there; the result was an expansion of solutions of the wave equation into a finite number of terms, corresponding to *quasinormal modes* described below in Section 1.2, modulo an exponentially decaying tail. That paper then formed the basis of the approach to non-linear wave equations by Hintz and Vasy [12, 13] which culminated in the proof of the stability of slowly rotating Kerr–de Sitter black holes in [14]; the first black hole stability result without symmetry assumptions. The purpose of this paper is to complete [21] by extending its results to the full subextremal range. In particular, we show, in the full subextremal range, the expansion of solutions of the wave equation into a finite number of terms, given by quasinormal modes, modulo an exponentially decaying tail, see Theorem 1.6. While for linear equations the question whether any of these quasinormal mode terms is non-decreasing is irrelevant, it becomes highly relevant for non-linear wave equations, where the absence (perhaps apart from certain well-understood ones) of quasinormal modes corresponding to non-decaying terms is called *mode stability*, see Section 1.2.

It turns out that the additional limitations of [21] had two separate origins. The basic approach of [21] was to Fourier transform the wave operator along the Killing vector field  $-\partial_{t_*}$  to obtain a family of operators,  $P_\sigma$ , depending on the Fourier dual variable  $\sigma$ . These operators are non-elliptic, hence their analysis involves the Hamilton flow in the characteristic set of the principal symbol. One of the additional assumptions in [21], denoted by (6.13) there, was to ensure that for each  $\sigma \in \mathbb{C}$ ,  $P_\sigma$  satisfies a non-trapping condition; it was shown there that indeed in the limiting case of (6.13), trapping appears. It was shown that under this *classical non-trapping* condition the family  $P_\sigma$  is Fredholm for each  $\sigma$  with the analytic Fredholm theorem applicable. The other additional assumption of [21] concerned the behavior of the family  $P_\sigma$  for  $\sigma$  with large real part and bounded imaginary part, namely in this large parameter (or as rescaled there, semiclassical) sense, which is stronger than the fixed  $\sigma$  consideration, the only trapping is normally hyperbolic trapping; this corresponds to the photon sphere when  $a = 0$ . This *semiclassical normally hyperbolic trapping* assumption was shown under the additional condition (6.27) in [21], namely that  $|a| < \frac{\sqrt{3}}{2}m$ . We recall that, with a basic microlocal analysis background, the analysis of [21] was self-contained apart from using the normally hyperbolic trapping analytic results (estimates for a microlocalized at the trapping version of  $P_\sigma^{-1}$ ) of Wunsch and Zworski [23]. Normally hyperbolic trapping has since been investigated in numerous papers [6, 7, 11, 18], giving a more precise version of [23], but for the present purposes [23] still suffices.

In this paper, we remove these limitations in two steps. First, we change the Killing vector field with respect to which we perform the Fourier transform. It turns out that for an appropriate choice of the Killing vector field, the Fourier transformed operator  $P_\sigma$  is classically non-trapping in the full subextremal range. Second, we show that in fact the semiclassical normally hyperbolic trapping holds in the full subextremal range as well. Since these were the only two additional limitations in [21] in the subextremal range, this immediately implies that all of the results of [21] in fact hold in the full subextremal range.

While in this paper we focus on the scalar wave equation, the changes when turning to tensorial wave equations in general are minor, as already shown earlier in [14], mainly affecting certain threshold quantities. We will discuss this elsewhere.

As a comparison, we mention that it has been known for some time that the full subextremal range for the vanishing cosmological constant ( $\Lambda = 0$ ) Kerr spacetime,  $|a| < m$ , behaves in the same way as for small  $|a|$ , see [2, 3, 5, 8, 20] and references therein, and even mode stability is known there, see [1, 20, 22] and references therein.

### 1.1. Kerr–de Sitter spacetimes

We now describe the subextremal condition in more detail. The polynomial  $\mu$  is given by

$$\mu(r) := (r^2 + a^2)\left(1 - \frac{\Lambda r^2}{3}\right) - 2mr,$$

and the subextremality condition is that it has four distinct real roots

$$r_- < r_C < r_e < r_c,$$

which is equivalent to the discriminant condition

$$-\left(1 + \frac{\Lambda a^2}{3}\right)^4 \left(\frac{a}{m}\right)^2 + 12\left(1 - \frac{\Lambda a^2}{3}\right)\Lambda a^2 + \left(1 - \frac{\Lambda a^2}{3}\right)^3 - 9\Lambda m^2 > 0. \quad (1.1)$$

It follows that there is a unique  $r_0 \in (r_e, r_c)$ , such that

$$\mu'(r_0) = 0.$$

As we will see,  $r_0$  will play a crucial role in choosing the Killing vector field for the Fourier transform. The domain of outer communication  $M$  in the subextremal Kerr–de Sitter spacetime is given (in Boyer–Lindquist coordinates) by the real analytic spacetime

$$M := \mathbb{R}_t \times (r_e, r_c)_r \times S_{\phi, \theta}^2$$

with real analytic metric

$$\begin{aligned} g &= (r^2 + a^2 \cos^2(\theta)) \left( \frac{dr^2}{\mu(r)} + \frac{d\theta^2}{c(\theta)} \right) \\ &\quad + \frac{c(\theta) \sin^2(\theta)}{b^2(r^2 + a^2 \cos^2(\theta))} (adt - (r^2 + a^2)d\phi)^2 \\ &\quad - \frac{\mu(r)}{b^2(r^2 + a^2 \cos^2(\theta))} (dt - a \sin^2(\theta)d\phi)^2, \end{aligned}$$

where

$$b := 1 + \frac{\Lambda a^2}{3}, \quad c(\theta) := 1 + \frac{\Lambda a^2}{3} \cos^2(\theta).$$

One easily verifies that this metric extends real analytically to the north and south poles  $\theta = 0, \pi$ .

The Boyer–Lindquist coordinates are singular at the roots of  $\mu$ . We therefore extend this metric real analytically over the future event horizon and the future cosmological horizon, corresponding to the roots  $r = r_e$  and  $r = r_c$ , respectively. One way to do this is by the following coordinate change:

$$t_* := t - \Phi(r), \quad \phi_* := \phi - \Psi(r),$$

where  $\Phi$  and  $\Psi$  satisfy

$$\Phi'(r) = b \frac{r^2 + a^2}{\mu(r)} f(r), \quad \Psi'(r) = b \frac{a}{\mu(r)} f(r),$$

where

$$f: (r_e - \delta, r_c + \delta) \rightarrow \mathbb{R}$$

is a real analytic function such that

$$f(r_e) = -1, \quad f(r_c) = 1. \tag{1.2}$$

The new form of the metric is

$$\begin{aligned} g_* &= (r^2 + a^2 \cos^2(\theta)) \frac{1 - f(r)^2}{\mu(r)} dr^2 \\ &\quad - \frac{2}{b} f(r) (dt_* - a \sin^2(\theta) d\phi_*) dr \\ &\quad - \frac{\mu(r)}{b^2 (r^2 + a^2 \cos^2(\theta))} (dt_* - a \sin^2(\theta) d\phi_*)^2 \\ &\quad + \frac{c(\theta) \sin^2(\theta)}{b^2 (r^2 + a^2 \cos^2(\theta))} (adt_* - (r^2 + a^2) d\phi_*)^2 \\ &\quad + (r^2 + a^2 \cos^2(\theta)) \frac{d\theta^2}{c(\theta)}, \end{aligned} \tag{1.3}$$

which extends real analytically to

$$\mathbb{R}_{t_*} \times (r_e - \delta, r_c + \delta)_r \times S_{\phi_*, \theta}^2.$$

The two real analytic lightlike hypersurfaces

$$\mathcal{H}_e^+ := \mathbb{R}_{t_*} \times \{r_e\} \times S_{\phi_*, \theta}^2, \quad \mathcal{H}_c^+ := \mathbb{R}_{t_*} \times \{r_c\} \times S_{\phi_*, \theta}^2$$

are called the *future event horizon* and *future cosmological horizon*, respectively. Note that the real analytic Killing vector fields  $\partial_t$  and  $\partial_\phi$ , in Boyer–Lindquist coordinates, extend to real analytic Killing vector fields  $\partial_{t_*}$  and  $\partial_{\phi_*}$  over the horizons.

**Remark 1.1.** We claim that there is a real analytic choice of  $f$ , such that the constant  $t_*$  hypersurfaces are everywhere spacelike. It is straightforward to check that this is equivalent to

$$1 - \frac{a^2 \mu(r)}{(r^2 + a^2)^2} > f(r)^2. \tag{1.4}$$

The left-hand side in (1.4) is indeed positive, since  $\mu(r) < r^2 + a^2$  for  $r > 0$ . Let

$$f_0: (r_e - \delta, r_c + \delta) \rightarrow \mathbb{R}$$

be any real analytic function satisfying (1.2) and write

$$f(r) = f_0(r) + \mu(r)f_1(r).$$

The function  $f$  satisfies (1.2) for any choice of real analytic function

$$f_1: (r_e - \delta, r_c + \delta) \rightarrow \mathbb{R}.$$

Define

$$f_{\pm}(r) := -\frac{f_0(r)}{\mu(r)} \pm \frac{1}{\mu(r)} \sqrt{1 - \frac{a^2 \mu(r)}{(r^2 + a^2)^2}}$$

for any  $r \notin \{r_e, r_c\}$ . Note that  $f_-$  has a continuous extension on  $(r_e - \delta, r_c)$  and  $f_+$  has a continuous extension on  $(r_e, r_c + \delta)$ , and

$$\lim_{r \downarrow r_e} f_+(r) = \infty = -\lim_{r \uparrow r_c} f_-(r).$$

Condition (1.4) becomes

$$f_1(r) \in \begin{cases} (f_-(r), \infty), & r \in (r_e - \delta, r_e], \\ (f_-(r), f_+(r)), & r \in (r_e, r_c), \\ (-\infty, f_+(r)), & r \in [r_c, r_c + \delta). \end{cases}$$

There are many real analytic functions  $f_1$  satisfying this condition, making sure that the constant  $t_*$  hypersurfaces are everywhere spacelike.

### 1.2. Main result

#### Assumption 1.2.

- Let  $(M_*, g_*)$  be a subextremal Kerr–de Sitter spacetime, extended over the future event horizon and the future cosmological horizon, where

$$M_* := \mathbb{R}_{t_*} \times (r_e - \delta, r_c + \delta)_r \times S_{\phi_*, \theta}^2,$$

with  $\delta > 0$  small enough so that the boundary hypersurfaces

$$\mathbb{R}_{t_*} \times \{r_e - \delta\} \times S_{\phi_*, \theta}^2, \quad \mathbb{R}_{t_*} \times \{r_c + \delta\} \times S_{\phi_*, \theta}^2$$

are *spacelike* and with  $f$  chosen as in Remark 1.1 so that the hypersurfaces

$$\{t_* = c\} \times (r_e - \delta, r_c + \delta)_r \times S_{\phi_*, \theta}^2$$

are *spacelike* for all  $c \in \mathbb{R}$ .

- Let  $A$  be a smooth complex valued function on  $M_*$  such that

$$\partial_{t_*} A = \partial_{\phi_*} A = 0.$$

We let  $P$  be the linear wave operator given by

$$P = \square + A.$$

For any subset  $\mathcal{U} \subset M_*$ , we use the notation  $C^\infty(\mathcal{U})$  and  $C^\omega(\mathcal{U})$  for the smooth and real analytic complex functions on  $\mathcal{U}$ , respectively.

*1.2.1. Quasinormal modes.* One of the main novelties in this paper is a new definition of quasinormal modes. More precisely, we define the quasinormal modes with respect to a different Killing vector field than in previous literature. As mentioned above, there is a unique  $r_0 \in (r_e, r_c)$  such that

$$\mu'(r_0) = 0.$$

**Definition 1.3** (Quasinormal mode). A complex function

$$u \in C^\infty(M_*)$$

is called a *quasinormal mode*, with *quasinormal mode frequency*  $\sigma \in \mathbb{C}$ , if

$$\left(\partial_{t_*} + \frac{a}{r_0^2 + a^2} \partial_{\phi_*}\right)u = -i\sigma u \quad \text{and} \quad Pu = 0.$$

Quasinormal modes and mode frequencies are also called resonant states and resonances.

This certain choice of  $r_0$  is important in order to get a Fredholm theory for the induced mode equation, which applies in the full subextremal range.

**Remark 1.4.** We can write any quasinormal mode as

$$u = e^{-i\sigma t_*} v_\sigma,$$

where

$$\left(\partial_{t_*} + \frac{a}{r_0^2 + a^2} \partial_{\phi_*}\right)v_\sigma = 0.$$

Our first main result is the following.

**Theorem 1.5** (Discrete set of quasinormal modes). *Let  $(M_*, g_*)$  and  $P$  be as in Assumption 1.2. Then there is a discrete set of quasinormal mode frequencies. More precisely, there is a discrete set  $\mathcal{A} \subset \mathbb{C}$  such that*

$$\sigma \in \mathcal{A}$$

*if and only if there is a quasinormal mode*

$$u \in C^\infty(M_*)$$

*with frequency  $\sigma$ . Moreover, for each  $\sigma \in \mathcal{A}$ , the space of quasinormal modes is finite-dimensional. If the coefficients of  $P$  are real analytic, then the quasinormal modes are real analytic.*

The discrete set of quasinormal modes on the Kerr–de Sitter spacetime is analogous to eigenfunctions of an elliptic operator on, e.g., a compact manifold without boundary. The quasinormal mode frequencies are analogous to the corresponding eigenvalues. Just as eigenvalues and eigenfunctions depend on the operator, i.e., generally change when the coefficients change, the quasinormal modes and frequencies of course also depend on the operator.

This theorem is proved in Section 2 by showing that  $\sigma \in \mathcal{A}$  is equivalent to the lack of invertibility of a certain Fredholm operator  $P_\sigma$ . Since  $P_\sigma$  depends analytically on  $\sigma$  in the appropriate sense, the analytic Fredholm theorem guarantees the discreteness of  $\mathcal{A}$ .

A difference between the eigenfunctions and eigenvalues of elliptic self-adjoint operators and the present case is that the meromorphic family  $P_\sigma^{-1}$  can have higher-order poles, though they are finite rank. These Laurent coefficients then give rise to *generalized quasinormal modes* which play a role in the asymptotic expansion of solutions of wave equations below.

While the Killing vector field in the definition of quasinormal modes may seem curious, and thus the condition on  $v_\sigma$  in Remark 1.4 odd, one way to think about this, and indeed this is how the proof proceeds in Section 2, is to change coordinates, namely replace  $\phi_*$  by

$$\psi_* = \phi_* - \frac{a}{r_0^2 + a^2} t_*,$$

while keeping  $t_*$  unchanged (but call it  $\tau_*$  =  $t_*$  for clarity). In the new coordinates, the quasinormal modes are functions of the new ‘spatial’ variables,  $r$ ,  $\theta$ ,  $\psi_*$ , i.e., are annihilated by  $\partial_{\tau_*}$ . One can think of this step as a refinement of the earlier coordinate changes (namely, the introduction of  $t_*$ ,  $\phi_*$  in place of  $t$ ,  $\phi$ ) that were necessitated by the horizons. While the metric is already well behaved after those coordinate changes, thus the wave operator has smooth coefficients and is non-degenerate, the solvability theory for non-elliptic operators is much more intricate than for elliptic operators, and this further refinement plays a key role in the analysis.

*1.2.2. Asymptotic expansion.* Our second main result concerns the behavior of solutions to the wave equation. For the statement of this, we use the standard Sobolev spaces based on the cylindrical geometry of  $M_*$ , i.e., use the vector fields  $\partial_{t_*}$ ,  $\partial_r$  and vector fields on the sphere. For instance, considering

$$(r_e - \delta, r_c + \delta)_r \times S_{\theta, \phi_*}^2 \subset \mathbb{R}_y^3$$

as spherical coordinates, for non-negative integers  $s$ , we simply have

$$\|u\|_{\bar{H}^s}^2 = \sum_{j+|\beta| \leq s} \|\partial_{t_*}^j \partial_y^\beta u\|_{L^2(M_*, dg_*)}^2.$$

Here the bar over  $H$  corresponds to Hörmander’s notation for extendible distributions, see [15]. We remark that if one compactifies  $M_*$  via replacing  $t_*$  by  $T_* = e^{-t_*}$ , adding  $T_* = 0$  as an ideal boundary, then these spaces are Melrose’s b-Sobolev spaces, see [17],

which is how the result was phrased in [21]. Changing to the variables  $(\tau_*, \psi_*)$  leaves this definition unchanged, up to equivalence of norms, i.e., we could equally well write, with  $(r_e - \delta, r_c + \delta)_r \times S_{\theta, \psi_*}^2 \subset \mathbb{R}_z^3$  being spherical coordinates,

$$\|u\|_{\bar{H}^s}^2 = \sum_{j+|\beta|\leq s} \|\partial_{\tau_*}^j \partial_z^\beta u\|_{L^2(M_*, dg_*)}^2.$$

We then have the following.

**Theorem 1.6** (The asymptotic expansion of waves). *Let  $(M_*, g_*)$  and  $P$  be as in Assumption 1.2 and let  $t_0 \in \mathbb{R}$ . There are  $C, \delta > 0$  such that for  $0 < \varepsilon < C$  and*

$$s > \frac{1}{2} + \beta\varepsilon,$$

where  $\beta$  is defined in (2.3), any solution to

$$Pu = f$$

with  $f \in e^{-\varepsilon t_*} \bar{H}^{s-1+\delta}(M_*)$  and with  $\text{supp}(u) \cup \text{supp}(f) \subset \{t_* > t_0\}$  has an asymptotic expansion

$$u - \sum_{j=1}^N \sum_{k=0}^{k_j} t_*^k e^{-i\sigma_j t_*} v_{jk} \in e^{-\varepsilon t_*} \bar{H}^s(M_*),$$

where  $\sigma_1, \dots, \sigma_N$  are the (finitely many) quasinormal mode frequencies with

$$\text{Im } \sigma_j > -\varepsilon$$

and  $k_j$  is their multiplicity, and where  $e^{-i\sigma_j t_*} v_{jk}$  are the  $C^\infty$  (generalized) quasinormal modes with frequency  $\sigma_j$  which are real analytic if the coefficients of  $P$  are such.

Theorem 1.6 implies that we always have decay apart from finitely many terms. Moreover, decay for all solutions  $u$  is equivalent to proving that

$$\text{Im}(\sigma) < 0$$

for all  $\sigma \in \mathcal{A}$ . On the other hand, decay for all solutions to the zero resonance amounts to proving that for all  $\sigma \in \mathcal{A} \setminus \{0\}$ , one has  $\text{Im}(\sigma) < 0$ , and in addition  $\sigma = 0$  is simple (with thus no generalized quasinormal modes). This is what is commonly known as *mode stability* for the operator  $P$ , which in general will depend strongly on the lower-order terms. For this, only the relatively simple case of small  $|a|$  is known, see Dyatlov’s paper [4]: for the wave operator, if  $a = 0$ , this (in the second sense, corresponding to decay to constants) can be shown explicitly, and if  $|a|$  is small, it follows by perturbation stability of the Fredholm setup; for the Klein–Gordon operator of small positive mass parameter the first (full decay) statement holds for small  $|a|$ . We remark that recently mode stability was extended to a larger range of parameters by Casals and Teixeira da Costa [1].

*1.2.3. Quasilinear wave equations.* In fact, our results immediately make the methods of [13] on solving quasilinear equations applicable in the extended range of parameters. That paper uses the compactification  $\overline{M}_*$  of  $M_*$  by adding  $e^{-t_*}$  as a boundary-defining function, mentioned above, to define the b-Sobolev spaces (which we here write as  $\overline{H}$ , following the discussion above), and b-differential operators  $\text{Diff}_b$ ; here we merely state a simplified result as an illustration. Using  $(t_*, y)$  as above,  $du = (\partial_{t_*}u, d_yu)$ , for each  $p = (t_*, y)$  we have an inner product  $g_p(u(p), du(p))$  on  $T_pM$ , where  $g_p: \mathbb{R} \oplus T_p^*M \rightarrow S^2T_p^*M$  depends smoothly, up to the boundaries  $r = r_e - \delta$ ,  $r = r_c + \delta$ , on  $p$  via  $y$  (or indeed via  $(z, e^{-t_*})$ ), and we consider the quasilinear wave equation

$$\square_{g(u, du)}u = f + q(u, du)$$

for real-valued  $u$ , with  $q$  being a polynomial in  $u$ ,  $\partial_{t_*}u$ ,  $\partial_yu$  with vanishing zeroth and first-order terms, and with real coefficients that are smooth functions of  $y$  (or again  $(z, e^{-t_*})$ ). The weighted version of the Sobolev spaces is simply  $\overline{H}^{s, \alpha}$  given by  $u \in \overline{H}^{s, \alpha}$  if  $e^{\alpha t_*}u \in \overline{H}^s$ . For instance, Theorem 4 of [13], in a simplified form (corresponding to Theorems 1 and 2 there) but with an extended range of parameters, becomes the following.

**Theorem 1.7** (cf. [13, Theorems 1 and 4]). *Let  $(M_*, g_*)$  be a subextremal Kerr–de Sitter spacetime, and consider  $L_{g(u, du)} = \square_{g(u, du)}$  with  $g(0, 0) = g_*$ . Suppose further that  $L_0 = L_{g(0, 0)}$  is such that  $L_0$  has a simple resonance at 0 (i.e., the associated Fredholm operator’s inverse has a simple pole), with resonant states spanned by constants, and no other resonances in  $\text{Im } \sigma \geq 0$ , i.e., that mode stability in the second sense holds. Suppose that  $q(u_0, du_0) = 0$  for a fixed constant  $u_0$ . Let  $s \in \mathbb{R}$ .*

*Then, with  $d = 12$ , for  $\alpha > 0$  sufficiently small: If  $f \in \overline{H}^{\infty, \alpha}$  is real valued with a sufficiently small  $\overline{H}^{2d, \alpha}$ -norm (depending on  $s$ ), then the equation  $L_{g(u, du)}u = f + q(u, du)$  has a unique, smooth in  $M_*$ , real valued, global forward solution of the form  $u = u_0 + \tilde{u}$ , where  $\tilde{u} \in \overline{H}^{s, \alpha}$ ,  $u_0 = c\chi$ ,  $c$  is a constant and  $\chi \in C^\infty(M_*)$  is identically 1 for large  $t_*$ .*

## 2. The Fredholm setup

The goal of this section is to prove Theorem 1.5. This generalizes [21] by removing the assumption

$$\left(1 - \frac{\Lambda a^2}{3}\right)^3 > 9\Lambda m^2, \quad (2.1)$$

which was required in [21]. In fact, (2.1) was stated in [21, (6.13)] as the equivalent condition

$$r_0 \in (r_e, r_c), \quad \mu'(r_0) = 0 \Rightarrow a^2 < \mu(r_0),$$

i.e., that the maximum point of  $\mu$  in the domain of outer communication is larger than  $a^2$ . This will not be necessary in the analysis we present below. Vasy considered in [21] the operator

$$\widehat{P}_\sigma u := e^{i\sigma t_*} P(e^{-i\sigma t_*} u),$$

for  $\sigma \in \mathbb{C}$ , where  $t_*$  is as above (or a slight modification with similar properties), and  $u$  only depends on the remaining coordinates  $(\phi_*, r, \theta)$ . This corresponds to the condition

$$\partial_{t_*} u = 0.$$

One may therefore consider  $\widehat{P}_\sigma$  as a linear second-order differential operator

$$\widehat{P}_\sigma: C^\infty(L_*) \rightarrow C^\infty(L_*),$$

where

$$L_* := \{t_* = 0\} \times (r_e - \delta, r_c + \delta)_r \times S^2_{\psi_*, \theta} \subset M_*.$$

In [21], it was shown that  $\widehat{P}_\sigma$  is a Fredholm operator between appropriate Sobolev spaces, assuming (2.1). If (2.1) is violated, the Fredholm theory of [21] does not apply to  $\widehat{P}_\sigma$ . The main reason for this is that there are trapped bicharacteristics of  $\widehat{P}_\sigma$  in  $T^*(L_* \cap M)$  if  $a$  is too large. This is closely related to the fact that the ergoregions of the event horizon and cosmological horizon intersect for large  $a$ . In order to avoid this, we will construct another operator  $P_\sigma$ , which has similar properties in the full subextremal range as  $\widehat{P}_\sigma$  has for  $a$  satisfying (2.1). For this, we first introduce a new coordinate system  $(\tau_*, r, \psi_*, \theta)$ , where

$$\begin{pmatrix} \tau_* \\ \psi_* \end{pmatrix} := \begin{pmatrix} t_* \\ \phi_* - \frac{a}{r_0^2 + a^2} t_* \end{pmatrix} \tag{2.2}$$

with  $r_0 \in (r_e, r_c)$  being uniquely defined by

$$\mu'(r_0) = 0.$$

Note that

$$\partial_{\tau_*} = \partial_{t_*} + \frac{a}{r_0^2 + a^2} \partial_{\phi_*}, \quad \partial_{\psi_*} = \partial_{\phi_*}$$

are both Killing vector fields since  $a$  and  $r_0$  are constant. A better choice than  $\widehat{P}_\sigma$  is to consider the operator

$$P_\sigma u := e^{i\sigma\tau_*} P(e^{-i\sigma\tau_*} u) = e^{i\sigma t_*} P(e^{-i\sigma t_*} u),$$

where  $u$  only depends on coordinates  $(\psi_*, r, \theta)$ , i.e.,

$$\partial_{\tau_*} u = \left( \partial_{t_*} + \frac{a}{r_0^2 + a^2} \partial_{\phi_*} \right) u = 0.$$

Even though  $\tau_* = t_*$ , we get a quite different induced operator  $P_\sigma$  on the modes, assuming  $\partial_{\tau_*} u = 0$  instead of  $\partial_{t_*} u = 0$ . We conclude that

$$P_\sigma: C^\infty(L_*) \rightarrow C^\infty(L_*)$$

is a linear differential operator of second order, where we note that indeed

$$L_* = \{\tau_* = 0\} \times (r_e - \delta, r_c + \delta)_r \times S^2_{\psi_*, \theta}.$$

The principal difference to  $\widehat{P}_\sigma$  is that there are no trapped bicharacteristics of  $P_\sigma$  in  $T^*(L_* \cap M)$  for any parameters in the subextremal range, see Lemma 2.4 below.

This will be used to prove that  $P_\sigma$  is a Fredholm operator between appropriate spaces, which is the main step in proving Theorem 1.5. In order to formulate the Fredholm statement, define  $\beta_e, \beta_c \in \mathbb{R}$  as

$$\beta_{e/c} := \pm 2 \left( 1 + \frac{\Lambda a^2}{3} \right) \frac{r^2 + a^2}{\mu'} \Big|_{r=r_{e/c}}.$$

Since  $\mu'(r_e) > 0$  and  $\mu'(r_c) < 0$ , we note that  $\beta_{e/c} > 0$ . For each  $s \in \mathbb{R}$ , we use the notation

$$\bar{H}^s := \bar{H}^s(L_*)$$

for the extendible Sobolev distributions on  $L_*$ , in the sense of Hörmander [15]. We will prove the following modification of [21, Theorem 1.1], which holds in the full subextremal range.

**Theorem 2.1.** *Define*

$$\beta := \max(\beta_e, \beta_c) > 0 \tag{2.3}$$

and let  $s \geq \frac{1}{2}$ . The family of operators

$$P_\sigma: \{u \in \bar{H}^s \mid P_\sigma u \in \bar{H}^{s-1}\} \rightarrow \bar{H}^{s-1}$$

is an analytic family of Fredholm operators of index 0 for all  $\sigma \in \mathbb{C}$  such that

$$\text{Im } \sigma > \frac{1 - 2s}{2\beta}.$$

Moreover,  $P_\sigma$  is invertible for  $\text{Im } \sigma \gg 1$ .

The proof of this theorem follows from the Fredholm framework for non-elliptic operators, developed in [21], once we have established the necessary results for the bicharacteristics, described above. We begin by studying the bicharacteristics in the domain of outer communication  $M$  and it will be convenient to do the computations in a modification of the Boyer–Lindquist coordinates  $(t, r, \phi, \theta)$ . We define these new coordinates analogous to (2.2) as

$$\begin{pmatrix} \tau \\ \psi \end{pmatrix} := \begin{pmatrix} t \\ \phi - \frac{a}{r_0^2 + a^2} t \end{pmatrix}.$$

Since

$$\partial_\tau = \partial_t + \frac{a}{r_0^2 + a^2} \partial_\phi = \left( \partial_{t_*} + \frac{a}{r_0^2 + a^2} \partial_{\phi_*} \right) \Big|_M = \partial_{t_*} \Big|_M,$$

we may identify the restriction of  $P_\sigma$  to  $L_* \cap M$  with the operator

$$e^{-i\sigma\tau} P(e^{i\sigma\tau} u),$$

where  $u \in C^\infty(M)$ , such that

$$\partial_\tau u = 0.$$

We may therefore identify  $P_\sigma|_{L_* \cap M}$  with the linear second-order differential operator

$$P_\sigma: C^\infty(L) \rightarrow C^\infty(L),$$

where

$$L := \{\tau = 0\} \times (r_e, r_c)_r \times S_{\psi, \theta}^2.$$

In the new coordinates, the dual metric  $G$  is given by

$$\begin{aligned} (r^2 + a^2 \cos^2(\theta))G &= \mu(r)\partial_r^2 + c(\theta)\partial_\theta^2 \\ &+ \frac{b^2}{c(\theta) \sin^2(\theta)} \left( a \sin^2(\theta)\partial_\tau + \frac{r_0^2 + a^2 \cos^2(\theta)}{r_0^2 + a^2} \partial_\psi \right)^2 \\ &- \frac{b^2}{\mu(r)} \left( (r^2 + a^2)\partial_\tau + a \frac{r_0^2 - r^2}{r_0^2 + a^2} \partial_\psi \right)^2. \end{aligned}$$

The principal symbol  $p_\sigma$  of  $P_\sigma$  is thus given by

$$\begin{aligned} (r^2 + a^2 \cos^2(\theta))p_\sigma(\xi) &= \mu(r)\xi_r^2 + c(\theta)\xi_\theta^2 \\ &+ \frac{b^2}{(r_0^2 + a^2)^2} \left( \frac{(r_0^2 + a^2 \cos^2(\theta))^2}{c(\theta) \sin^2(\theta)} - a^2 \frac{(r_0^2 - r^2)^2}{\mu(r)} \right) \xi_\psi^2. \end{aligned}$$

Since the bicharacteristic flow is invariant under conformal rescaling, it suffices to consider

$$q_\sigma := (r^2 + a^2 \cos^2(\theta))p_\sigma.$$

**Lemma 2.2** (No characteristic set at  $r_0$ ). *We have*

$$\text{Char}(P_\sigma) \subset \{r \neq r_0\}.$$

*Proof.* Assume that there is a point in the characteristic set with  $r = r_0$ . Then

$$0 = q_\sigma(\xi)|_{r=r_0} = \mu(r_0)\xi_r^2 + c(\theta)\xi_\theta^2 + \frac{b^2}{(r_0^2 + a^2)^2} \frac{(r_0^2 + a^2 \cos^2(\theta))^2}{c(\theta) \sin^2(\theta)} \xi_\psi^2,$$

and since  $\mu(r_0) > 0$ , this is a sum of positive terms and hence

$$\xi_r = \xi_\theta = \xi_\psi = 0.$$

This proves the lemma. ■

**Remark 2.3** (Ergoregions). One way of interpreting the above lemma is that  $\partial_\tau$  is *timelike* at  $r_0$ . Hence the ergoregions of the event horizon and cosmological horizon, with respect to  $\partial_\tau$ , are disjoint. It is easy to see that this is not the case for  $\partial_t$  in general; the ergoregions may well intersect in that case.

**Lemma 2.4** (No trapping). *For each  $\varepsilon > 0$ , all bicharacteristics of  $P_\sigma$  in  $L$  leave the region*

$$(r_e + \varepsilon, r_c - \varepsilon)_r \times S_{\psi, \theta}^2$$

*both to the future and the past.*

*Proof.* The Hamiltonian vector field is given by

$$H_{q_\sigma} = \sum_{j=1}^3 (\partial_{\xi_j} q_\sigma) \partial_j - (\partial_j q_\sigma) \partial_{\xi_j}.$$

We claim that

$$H_{q_\sigma} r = 0 \Rightarrow H_{q_\sigma}^2 r \begin{cases} < 0 & \text{if } r_e < r < r_0, \\ > 0 & \text{if } r_0 < r < r_c \end{cases} \tag{2.4}$$

in the characteristic set. Assume therefore that

$$0 = H_{q_\sigma} r = \partial_{\xi_r} q_\sigma = 2\xi_r \mu(r).$$

Since  $\mu(r) > 0$  for all  $r \in (r_e, r_c)$ , we conclude that  $\xi_r = 0$ . At such points, the second derivative is given by

$$\begin{aligned} H_{q_\sigma}^2 r|_{\xi_r=0} &= 2(H_{q_\sigma} \xi_r|_{\xi_r=0})\mu(r) = -2(\partial_r q_\sigma|_{\xi_r=0})\mu(r) \\ &= 2 \frac{b^2}{(r_0^2 + a^2)^2} a^2 \partial_r \frac{(r_0^2 - r^2)^2}{\mu(r)} \xi_\psi^2 \\ &= 2 \frac{a^2 b^2}{(r_0^2 + a^2)^2} \frac{\mu(r) \partial_r (r_0^2 - r^2)^2 - \mu'(r) (r_0^2 - r^2)^2}{\mu(r)^2} \xi_\psi^2. \end{aligned}$$

Now, with our choice of  $r_0$ , we note that

$$\mu'(r) \begin{cases} > 0, & r_e < r < r_0, \\ < 0, & r_0 < r < r_c \end{cases} \quad \text{and} \quad \partial_r (r_0^2 - r^2)^2 \begin{cases} < 0, & r_e < r < r_0, \\ > 0, & r_0 < r < r_c. \end{cases}$$

Finally,  $(r_0^2 - r^2)^2 > 0$  for  $r \in (r_e, r_c) \setminus \{r_0\}$  and  $\mu(r) > 0$  for all  $r \in (r_e, r_c)$ , proving (2.4). We may use this to define an escape function

$$\mathcal{E} := e^{C(r-r_0)^2} H_{q_\sigma} r$$

for any  $C > 0$  and note that

$$H_{q_\sigma} \mathcal{E} = e^{C(r-r_0)^2} (2C(r-r_0)(H_{q_\sigma} r)^2 + H_{q_\sigma}^2 r).$$

Since the characteristic set is disjoint from  $\{r = r_0\}$  and by (2.4), we can choose  $C$  large enough so that  $H_{q_\sigma} \mathcal{E}$  is everywhere non-vanishing and has the same sign as  $r - r_0$ . Hence  $\mathcal{E}$  gives an escape function for the bicharacteristics of  $q_\sigma$  in  $L$ , which finishes the proof. ■

We now improve on Lemma 2.4 by including the precise behavior at and beyond the horizons. By Assumption 1.2, the hypersurface  $L_*$  is spacelike, which means that  $dt_*$  is a future pointing timelike one-form everywhere along  $L_*$ . All elements of the characteristic set of  $P$  are lightlike, so we can use this fact to divide the characteristic set of  $P$  into two disjoint sets. Moreover, since there is a natural embedding

$$\text{Char}(P_\sigma) \subseteq \text{Char}(P),$$

we may divide the characteristic set of  $P_\sigma$  as

$$\text{Char}(P_\sigma) = \Sigma_+ \cup \Sigma_-,$$

where

$$\Sigma_\pm = \{\xi \in \text{Char}(P_\sigma) \mid \pm G_*(dt_*, \xi) > 0\},$$

where  $G_*$  is the dual metric induced by  $g_*$ , defined in (1.3). Note that indeed

$$\Sigma_+ \cap \Sigma_- = \emptyset,$$

hence  $\Sigma_+$  and  $\Sigma_-$  are invariant under the bicharacteristic flow.

**Lemma 2.5.** *The conormal bundles  $N^*\{r = r_e\}$  and  $N^*\{r = r_c\}$  are contained in the characteristic set of  $P_\sigma$  and the bicharacteristic flow at these is radial in the generalized sense as in [21]. All other bicharacteristics of  $P_\sigma$  in  $\Sigma_+$  either starts at fiber infinity of*

$$N^*\{r = r_e\} \cap \{\xi_r > 0\}$$

*and ends at  $r = r_e - \delta$  or starts at the fiber infinity of*

$$N^*\{r = r_c\} \cap \{\xi_r < 0\}$$

*and ends at  $r = r_c + \delta$ . The reverse behavior holds for  $\Sigma_-$ . Moreover, the fiber infinity of*

$$N^*\{r = r_e\} \cap \{\pm \xi_r > 0\} \quad \text{and} \quad N^*\{r = r_c\} \cap \{\mp \xi_r > 0\}$$

*are generalized normal source/sink manifolds of the bicharacteristic flow, respectively, in the sense of [21].*

*Proof.* This actually follows from the semiclassical considerations in [21]; here we consider a large parameter framework; the relationship between these is discussed in Section 2.1 therein. Indeed, in the large parameter sense, our new choice of  $\psi_*$  amounts to a new choice of the ‘classical hyperplane’  $\sigma = 0$  relative to the choice in [21] (the classical hyperplane in [21] corresponds to the choice  $\sigma + \frac{a^2}{r_0^2 + a^2} \xi_{\psi_*} = 0$  in the notation of this paper), but at the normal source/sink manifolds,  $\xi_{\psi_*} = 0$ , so these lie at the same location, and the normal source/sink manifold being such even in the large parameter sense for the old choice (which is equivalent to the semiclassical considerations there) implies the same property for our new choice. However, for the convenience of the reader we give a direct argument.

In order to study the characteristic set of  $P_\sigma$  near the horizons, we need to work in the coordinate system  $(\tau_*, r, \psi_*, \theta)$ , defined in (2.2). The dual metric in these coordinates is given by

$$\begin{aligned} (r^2 + a^2 \cos^2(\theta))G_* &= \mu(r)\partial_r^2 - 2bf(r)\left((r^2 + a^2)\partial_{\tau_*} + a\frac{r_0^2 - r^2}{r_0^2 + a^2}\partial_{\psi_*}\right)\partial_r \\ &\quad + c(\theta)\partial_\theta^2 + \frac{b^2}{c(\theta)\sin^2(\theta)}\left(a\sin^2(\theta)\partial_{\tau_*} + \frac{r_0^2 + a^2\cos^2(\theta)}{r_0^2 + a^2}\partial_{\psi_*}\right)^2 \\ &\quad - b^2\frac{1 - f(r)^2}{\mu(r)}\left((r^2 + a^2)\partial_{\tau_*} + a\frac{r_0^2 - r^2}{r_0^2 + a^2}\partial_{\psi_*}\right)^2. \end{aligned}$$

The principal symbol  $p_\sigma$  of  $P_\sigma$  is thus given by

$$\begin{aligned} & (r^2 + a^2 \cos^2(\theta))p_\sigma(\xi) \\ &= \mu(r)\xi_r^2 - 2abf(r)\frac{r_0^2 - r^2}{r_0^2 + a^2}\xi_{\psi_*}\xi_r + c(\theta)\xi_\theta^2 \\ &+ \left(\frac{b^2}{c(\theta)\sin^2(\theta)}\left(\frac{r_0^2 + a^2 \cos^2(\theta)}{r_0^2 + a^2}\right)^2 - b^2\frac{1 - f(r)^2}{\mu(r)}\left(a\frac{r_0^2 - r^2}{r_0^2 + a^2}\right)^2\right)\xi_{\psi_*}^2. \end{aligned} \tag{2.5}$$

As before, since the bicharacteristic flow is invariant under conformal rescaling, it suffices to consider

$$q_\sigma := (r^2 + a^2 \cos^2(\theta))p_\sigma.$$

By Lemma 2.2, there are no bicharacteristics crossing  $r = r_0$ . In this proof, we therefore only study the bicharacteristics where  $r < r_0$ , as the case  $r > r_0$  is similar. Since

$$G_*(dr, dr)|_{r=r_e} = 0 \quad \text{and} \quad dt_* = d\tau_*,$$

we have

$$G_*(dt_*, dr)|_{r=r_e} = G_*(d\tau_*, dr)|_{r=r_e} = b(r_e^2 + a^2) > 0.$$

This means in particular that  $-dr|_{r=r_e}$  is a future pointing lightlike one-form, so it can be used to characterize  $\Sigma_\pm \cap \{r = r_e\}$ . For any  $\xi \in \text{Char}(P_\sigma) \cap \{r = r_e\}$ , it follows that  $\xi \in \Sigma_\pm \cap \{r = r_e\}$  if and only if

$$\pm G_*(dt_*, \xi)|_{r=r_e} > 0,$$

which is equivalent to

$$\pm G_*(-dr, \xi)|_{r=r_e} > 0$$

or

$$\xi|_{r=r_e} = \xi_r dr|_{r=r_e},$$

with  $\pm \xi_r|_{r=r_e} > 0$ . We now compute the Hamiltonian vector field applied to the function  $r$  at the event horizon

$$\begin{aligned} H_{q_\sigma} r|_{r=r_e} &= \sum_{j=1}^3 ((\partial_{\xi_j} q_\sigma) \partial_j - (\partial_j q_\sigma) \partial_{\xi_j}) r \Big|_{r=r_e} \\ &= 2ab \frac{r_0^2 - r_e^2}{r_0^2 + a^2} \xi_{\psi_*} \Big|_{r=r_e} = -G(-dr, \xi)|_{r=r_e}. \end{aligned} \tag{2.6}$$

We conclude that all bicharacteristics in  $\Sigma_+$  crossing the event horizon enter the black hole region, where  $r < r_e$ , and all bicharacteristics in  $\Sigma_-$  crossing the event horizon enter the domain of outer communication, where  $r \in (r_e, r_c)$ . In particular, no bicharacteristic can pass the event horizon twice. On the other hand, Lemma 2.4 implies that each bicharacteristic in  $\Sigma_\pm$  in the domain of outer communication turns in the past/future and approaches the event horizon  $r = r_e$ .

It remains to show that the fiber infinity of

$$N^*\{r = r_e\} \cap \{\pm \xi_r > 0\}$$

is a generalized normal source/sink of the bicharacteristic flow, in the sense of [21]. Let us first check that  $N^*\{r = r_e\}$  is the largest subset of

$$\text{Char}(P_\sigma) \cap \{r = r_e\},$$

which is invariant under the bicharacteristic flow. Indeed, equation (2.6) shows that the Hamiltonian vector field is transversal unless  $\xi_{\psi_*} = 0$ . But if  $\xi_{\psi_*} = 0$ , then equation (2.5) implies that  $\xi_\theta = 0$  and hence

$$\xi \in N^*\{r = r_e\}.$$

Restricting the Hamiltonian vector field to the conormal bundle, we get

$$\xi_r^{-1} H_{q_\sigma}|_{N^*\{r=r_e\}} = 2ab \frac{r_0^2 - r_e^2}{r_0^2 + a^2} \partial_{\psi_*} - \mu'(r_e) \xi_r \partial_{\xi_r}.$$

Since  $\mu'(r_e) > 0$ , this shows that the fiber infinity of

$$N^*\{r = r_e\} \cap \{\pm \xi_r > 0\}$$

is a generalized source/sink of the bicharacteristic flow, in the sense of [21]. We finally need to check that it indeed is a *normal* source/sink in the generalized sense of [21]. This is to say that for a suitable (local) quadratic defining function  $\rho_0$  for  $N^*\{r = r_e\}$  in the characteristic set of  $P_\sigma$ , modulo cubically vanishing terms at  $N^*\{r = r_e\}$ , there is a function  $\beta_1 > 0$  such that

$$\xi_r^{-1} H_{q_\sigma} \rho_0 - \beta_1 \rho_0 \geq 0$$

near  $N^*\{r = r_e\}$ . We can, for example, take  $\rho_0$  to be a multiple of the (modified) Carter constant, i.e.,

$$\rho_0 := \xi_r^{-2} \left( c(\theta) \xi_\theta^2 + \frac{b^2}{c(\theta) \sin^2(\theta)} \left( \frac{r_0^2 + a^2 \cos^2(\theta)}{r_0^2 + a^2} \right)^2 \xi_{\psi_*}^2 \right).$$

Note that

$$\begin{aligned} H_{q_\sigma} \xi_r^2 \rho_0 &= \{q_\sigma, \xi_r^2 \rho_0\} \\ &= \left\{ \mu(r) \xi_r^2 - 2abf(r) \frac{r_0^2 - r^2}{r_0^2 + a^2} \xi_{\psi_*} \xi_r - b^2 \frac{1 - f(r)^2}{\mu(r)} \left( a \frac{r_0^2 - r^2}{r_0^2 + a^2} \right)^2 \xi_{\psi_*}^2, \right. \\ &\quad \left. c(\theta) \xi_\theta^2 + \frac{b^2}{c(\theta) \sin^2(\theta)} \left( \frac{r_0^2 + a^2 \cos^2(\theta)}{r_0^2 + a^2} \right)^2 \xi_{\psi_*}^2 \right\} \\ &\quad + \{ \xi_r^2 \rho_0, \xi_r^2 \rho_0 \} = 0 \end{aligned}$$

since the second Poisson bracket vanishes trivially and the first Poisson bracket vanishes because the first factor only depends on  $(r, \xi_r, \xi_{\psi_*})$  and the second factor only depends on  $(\theta, \xi_\theta, \xi_{\psi_*})$ .

We deduce that

$$\xi_r^{-1} H_{q_\sigma} \rho_0 = \xi_r \rho_0 H_{q_\sigma} \xi_r^{-2} = -2\xi_r^{-2} (H_{q_\sigma} \xi_r) \rho_0,$$

so  $\beta_1 = -2\xi_r^{-2} (H_{q_\sigma} \xi_r)$ . At  $N^*\{r = r_e\}$ , we have

$$\beta_1 = 2\mu'(r_e),$$

which is positive as desired. We have thus shown that the fiber infinity in

$$N^*\{r = r_e\} \cap \{\pm\xi_r > 0\}$$

is a normal source/sink in the generalized sense of [21]. This concludes the proof, since the behavior near  $r = r_c$  is studied analogously. ■

*Proof of Theorem 2.1.* By Lemma 2.5, the dynamics of the bicharacteristics at and beyond the horizons  $\mathcal{H}_e^+ \cap L_*$  and  $\mathcal{H}_c^+ \cap L_*$  are precisely the same as in [21, Section 6.1], i.e., the analysis for  $P_\sigma$  in our setup is similar to the analysis for  $\hat{P}_\sigma$  there. The proof of Theorem 2.1 therefore follows the same lines as the proof of [21, Theorem 1.4]. ■

*Proof of Theorem 1.5.* Since  $P_\sigma$  is invertible for  $\text{Im } \sigma \gg 1$ , analytic Fredholm theory implies that  $P_\sigma$  has a meromorphic extension to the open set

$$\Omega_s := \left\{ \text{Im } \sigma > \frac{1 - 2s}{2\beta} \right\}.$$

In particular,  $P_\sigma$  is invertible everywhere in  $\Omega_s$  apart from a discrete set. Moreover, since  $P_\sigma$  has index zero,  $P_\sigma$  is invertible if and only if the kernel of  $P_\sigma$  is trivial. Since

$$\mathbb{C} = \bigcup_{s \in \mathbb{R}} \Omega_s,$$

we conclude that  $\ker(P_\sigma)$  is non-trivial precisely on a discrete set  $\mathcal{A} \subset \mathbb{C}$ . Following the arguments in the proof of [19, Theorem 1.2] line by line, using Theorem 2.1 in place of [21, Theorem 1.1], it follows that smooth elements in  $\ker(P_\sigma)$  are real analytic if the coefficients of  $P$  are real analytic. ■

**Remark 2.6.** We recall that [19] proves the real analyticity of the quasinormal modes by using yet another Killing vector field, or rather two, one each for the two horizons, with respect to which the Fourier transformed wave operator is of Keldysh type, so has a similar structure to that of the Schwarzschild–de Sitter spacetime relative to the standard  $\partial_{t_*}$ . (The Killing vector fields we use are lightlike at the horizon under study.) Hence, after this reduction, the real analyticity result of Galkowski and Zworski [9] can be used in a local manner at each horizon. Then the mode analyticity with respect to any other Killing vector field which gives rise to a global Fredholm theory, such as ours presently, is deduced by decomposing the quasinormal modes into eigenmodes relative to the vector field  $\partial_{\psi_*}$ , each of which is a quasinormal mode relative to the horizon Killing vector fields as well.

### 3. Normally hyperbolic trapping

The goal of this section is to prove Theorem 1.6. This generalizes [21] by removing the assumption

$$|a| \leq \frac{\sqrt{3}}{2}m, \tag{3.1}$$

which was required in [21, (6.27)]. In the previous section, we studied the mode operator and showed in particular that the bicharacteristics of that operator are non-trapped. As is well known, there are trapped bicharacteristics for the full wave operator, i.e., trapped lightlike geodesics.

In order to apply the *semiclassical* or *high energy* estimates from [21], we need to prove certain properties of the trapping in the domain of outer communication. More precisely, we need to show that the trapping is *normally hyperbolic*. This was done in [21] assuming (3.1). In this section, we prove that the analogous results hold for the full subextremal range. Since we only work in the domain of outer communication, it is convenient to work in Boyer–Lindquist coordinates  $(t, r, \phi, \theta)$  with dual variables  $(\xi_t, \xi_r, \xi_\phi, \xi_\theta)$ . We let  $p$  denote the principal symbol of the wave operator  $P$ .

**Remark 3.1.** Since  $\partial_t$  and  $\partial_\phi$  are Killing vector fields, it follows that  $\xi_t$  and  $\xi_\phi$  are constant along the Hamiltonian flow with respect to  $p$ .

**Theorem 3.2** (Trapping in the subextremal Kerr–de Sitter spacetimes). *For any  $(\xi_t, \xi_\phi) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ , define the function*

$$F(r) := \frac{1}{\mu}((r^2 + a^2)\xi_t + a\xi_\phi)^2.$$

- (a) *Either*
  - *F vanishes at  $r = r_e$  or  $r_c$  and F has no critical point in  $(r_e, r_c)$ , or*
  - *F has precisely one critical point  $r_{\xi_t, \xi_\phi} \in (r_e, r_c)$  and  $F''(r_{\xi_t, \xi_\phi}) > 0$ .*
- (b) *F is positive on the characteristic set in M.*
- (c) *The trapped set in M is*

$$\Gamma = \bigcup_{(\xi_t, \xi_\phi) \in \mathbb{R}^2 \setminus \{(0, 0)\}} \Gamma_{\xi_t, \xi_\phi},$$

where

$$\Gamma_{\xi_t, \xi_\phi} := \{\xi_r = 0, r = r_{\xi_t, \xi_\phi}\} \cap \text{Char}(P).$$

- (d)  $\Gamma$  *is a smooth connected 5-dimensional submanifold of  $T^*M$ , with defining functions  $\xi_r, r - r_{\xi_t, \xi_\phi}$  and  $p$ .*
- (e) *The linearization of the bicharacteristic flow at  $\Gamma$  is given by*

$$\begin{aligned} H_p \begin{pmatrix} r - r_{\xi_t, \xi_\phi} \\ \xi_r \end{pmatrix} &= \frac{1}{r_{\xi_t, \xi_\phi}^2 + a^2 \cos^2(\theta)} \begin{pmatrix} 0 & 2\mu(r_{\xi_t, \xi_\phi}) \\ b^2 F''(r_{\xi_t, \xi_\phi}) & 0 \end{pmatrix} \begin{pmatrix} r - r_{\xi_t, \xi_\phi} \\ \xi_r \end{pmatrix} \\ &+ \mathcal{O}((r - r_{\xi_t, \xi_\phi})^2 + \xi_r^2). \end{aligned}$$

In particular, the trapping in the domain of outer communication in any subextremal Kerr–de Sitter spacetime is normally hyperbolic trapping in the sense of [23]. The stable  $(s, -)$  and unstable  $(u, +)$  manifolds are the smooth manifolds given by

$$\Gamma^{u/s} = \left\{ \xi_r = \pm \operatorname{sgn}(r - r_{\xi_t, \xi_\phi}) b \sqrt{\frac{F(r) - F(r_{\xi_t, \xi_\phi})}{\mu}} \right\} \cap \operatorname{Char}(P).$$

The main computation in the proof is the following.

**Proposition 3.3.** *We have*

$$h(r) := 2\mu \partial_r(r\mu' - 4\mu) - \mu'(r\mu' - 4\mu) < 0$$

for all  $r \in (r_e, r_c)$ .

We postpone the proof of Proposition 3.3 as it will take up a significant amount of this section.

*Proof of Theorem 3.2, assuming Proposition 3.3.* First we prove claim (a). If  $\xi_t = 0$ , then the claim is clearly true with  $r_{\xi_t, \xi_\phi} = r_0$ , we may therefore assume that  $\xi_t \neq 0$ . We now consider critical points of  $F$  in  $(r_e, r_c)$ . Defining

$$f(r) := ((r^2 + a^2)\xi_t + a\xi_\phi)\mu' - 4r\xi_t\mu,$$

we note that

$$F'(r) = -\frac{(r^2 + a^2)\xi_t + a\xi_\phi}{\mu^2} f(r),$$

which vanishes if either

$$(r^2 + a^2)\xi_t + a\xi_\phi = 0 \tag{3.2}$$

or

$$f(r) = 0 \tag{3.3}$$

for some  $r \in (r_e, r_c)$ . We claim that all critical points of  $F$  in  $(r_e, r_c)$  are local strict minima. Note first that since

$$-4r\xi_t\mu(r) \neq 0$$

for all  $r \in (r_e, r_c)$ , there is no critical point of  $F$  satisfying both (3.2) and (3.3) simultaneously. Any critical point  $r \in (r_e, r_c)$  of  $F$ , satisfying (3.2), satisfies

$$F''(r) = -\frac{2r\xi_t}{\mu^2} (-4r\xi_t\mu) = 8\frac{r^2\xi_t^2}{\mu} > 0,$$

and is therefore a local strict minima. It thus remains to study case (3.3), which is equivalent to

$$(r^2 + a^2)\xi_t + a\xi_\phi = \frac{4r\mu\xi_t}{\mu'},$$

since  $\mu' \neq 0$  at the roots of  $f$  as  $-4\xi_t\mu \neq 0$ . At such points, we compute that

$$\begin{aligned} f'(r) &= \mu''(r)((r^2 + a^2)\xi_t + a\xi_\phi) + 2r\mu'(r)\xi_t - 4\mu(r)\xi_t - 4r\mu'(r)\xi_t \\ &= \xi_t \left( \mu''(r) \frac{4r\mu(r)}{\mu'(r)} - 4\mu(r) - 2r\mu'(r) \right) \\ &= \frac{2\xi_t}{\mu'(r)} (2r\mu(r)\mu''(r) - 2\mu(r)\mu'(r) - r\mu'(r)^2) \\ &= \frac{2\xi_t}{\mu'(r)} (2\mu(r)\partial_r(r\mu'(r) - 4\mu(r)) - \mu'(r)(r\mu'(r) - 4\mu(r))) \\ &= \frac{2\xi_t}{\mu'(r)} h(r), \end{aligned}$$

where  $h$  was defined in Proposition 3.3. It follows that all critical points of  $F$ , satisfying (3.3), satisfy

$$F''(r) = -\frac{(r^2 + a^2)\xi_t + a\xi_\phi}{\mu^2} f'(r) = -8r \frac{\xi_t^2}{\mu\mu'^2} h(r) > 0$$

by Proposition 3.3. We conclude that all critical points  $r \in (r_e, r_c)$  of  $F$  are local strict minima. Now, if  $(r^2 + a^2)\xi_t + a\xi_\phi$  vanishes precisely at  $r = r_e$  or  $r_c$ , then  $F \rightarrow \infty$  at the other end point of  $[r_e, r_c]$  and it follows that  $F$  cannot have critical points for they would all be local strict minima. In the remaining case,  $F \rightarrow \infty$  as  $r \rightarrow r_e$  and  $r_c$ , and we conclude that there is a unique strict minimum  $r_{\xi_t, \xi_\phi} \in (r_e, r_c)$ . This proves claim (a).

We continue by proving claim (b), i.e., that

$$(r^2 + a^2)\xi_t + a\xi_\phi \neq 0$$

in the characteristic set. The principal symbol  $p$  of  $P$  is given by

$$\begin{aligned} (r^2 + a^2 \cos^2(\theta))p(\xi) &= \mu(r)\xi_r^2 + c(\theta)\xi_\theta^2 + \frac{b^2}{c(\theta) \sin^2(\theta)} (a \sin^2(\theta)\xi_t + \xi_\phi)^2 \\ &\quad - \frac{b^2}{\mu(r)} ((r^2 + a^2)\xi_t + a\xi_\phi)^2. \end{aligned}$$

Assume there is a point in the characteristic set satisfying  $(r^2 + a^2)\xi_t + a\xi_\phi = 0$ , then

$$a \sin^2(\theta)\xi_t + \xi_\phi = 0$$

and we get a solution to the linear equation

$$\begin{pmatrix} r^2 + a^2 & a \\ a \sin^2(\theta) & 1 \end{pmatrix} \begin{pmatrix} \xi_t \\ \xi_\phi \end{pmatrix} = 0,$$

and the determinant of the matrix is

$$r^2 + a^2 \cos^2(\theta) > 0.$$

This implies that  $\xi_t = \xi_\phi = 0$ , which in turn implies that  $\xi_r = \xi_\theta = 0$ .

Let us now prove claim (c). For this, recall first that  $\text{Char}(P)$  is invariant under  $H_p$ . Since the bicharacteristic flow is invariant under conformal changes, let us for simplicity study

$$\begin{aligned} q(\xi) &= (r^2 + a^2 \cos^2(\theta))p(\xi) \\ &= \mu(r)\xi_r^2 + c(\theta)\xi_\theta^2 + \frac{b^2}{c(\theta)\sin^2(\theta)}(a \sin^2(\theta)\xi_t + \xi_\phi)^2 \\ &\quad - \frac{b^2}{\mu(r)}((r^2 + a^2)\xi_t + a\xi_\phi)^2. \end{aligned}$$

Since  $H_q \xi_t = H_q \xi_\phi = 0$ , it follows that  $H_q r_{\xi_t, \xi_\phi} = 0$ , and we use this to compute

$$H_q(r - r_{\xi_t, \xi_\phi}) = 2\mu(r)\xi_r, \tag{3.4}$$

$$H_q \xi_r = -\partial_r \left( \mu \xi_r^2 - \frac{b^2}{\mu(r)} ((r^2 + a^2)\xi_t + a\xi_\phi)^2 \right) = -\mu' \xi_r^2 + b^2 F'(r). \tag{3.5}$$

Evaluating these at  $\Gamma$ , both expressions vanish and it follows that  $\Gamma$  is invariant under  $H_q$ . We claim that

$$x \notin \Gamma, \quad H_q(r - r_{\xi_t, \xi_\phi})^2|_x = 0 \Rightarrow (H_q)^2(r - r_{\xi_t, \xi_\phi})^2|_x > 0. \tag{3.6}$$

To prove this, assume that

$$0 = H_q(r - r_{\xi_t, \xi_\phi})^2|_x = 2(r - r_{\xi_t, \xi_\phi})H_q r|_x,$$

which implies that either  $r = r_{\xi_t, \xi_\phi}$  or  $H_q r = 0$  at  $x$ . We compute

$$(H_q)^2(r - r_{\xi_t, \xi_\phi})^2|_x = 2(H_q r)^2|_x + 2(r - r_{\xi_t, \xi_\phi})(H_q)^2 r|_x.$$

Now, if  $r|_x = r_{\xi_t, \xi_\phi}$ , we see that

$$(H_q)^2(r - r_{\xi_t, \xi_\phi})^2|_x = 2(H_q r)^2|_x \geq 0$$

and it vanishes if and only if  $H_q r|_x = 0$ , in which case  $\xi_r|_x = 0$  and hence  $x \in \Gamma$ . Similarly, if instead  $H_q r|_x = 0$ , then  $\xi_r|_x = 0$  and we conclude that

$$(H_q)^2(r - r_{\xi_t, \xi_\phi})^2|_x = 2(r - r_{\xi_t, \xi_\phi})(H_q)^2 r|_x = 4(r - r_{\xi_t, \xi_\phi})\mu(r)b^2 F'(r)|_x.$$

Recalling that  $r - r_{\xi_t, \xi_\phi}$  and  $F'(r)$  have the same sign, we conclude that this is positive unless  $r = r_{\xi_t, \xi_\phi}$ , in which case  $x \in \Gamma$ . This proves claim (3.6). For any  $C > 0$ , we define the function

$$\mathcal{E} := e^{C(r - r_{\xi_t, \xi_\phi})^2} H_q(r - r_{\xi_t, \xi_\phi})^2.$$

We get

$$H_q \mathcal{E} = e^{C(r - r_{\xi_t, \xi_\phi})^2} (C(H_q(r - r_{\xi_t, \xi_\phi}))^2 + (H_q)^2(r - r_{\xi_t, \xi_\phi})^2),$$

which by (3.6) is positive precisely away from  $\Gamma$  if  $C$  is large enough. Thus  $\mathcal{E}$  provides a globally defined escape function which grows along each bicharacteristic not in  $\Gamma$

and vanishes identically at  $\Gamma$ . Note that  $\mathcal{E}$  is also an escape function in the case when  $(r^2 + a^2)\xi_t + a\xi_\phi$  vanishes at  $r_e$  or  $r_c$ , by substituting  $r_{\xi_t, \xi_\phi}$  in the expression for  $\mathcal{E}$  with that point. This proves in particular that  $\Gamma$  is precisely the trapped set in  $M$ , which is claim (c).

For claim (d), we need to prove that

$$d(r - r_{\xi_t, \xi_\phi}), \quad d\xi_r \quad \text{and} \quad dp$$

are linearly independent at  $\Gamma$ . This follows by noting that

$$dp\left(\frac{\partial}{\partial r}\right)\Big|_\Gamma = \frac{\partial p}{\partial r}\Big|_\Gamma = \frac{1}{r^2 + a^2 \cos^2(\theta)}(\mu'\xi_r^2 - b^2 F')\Big|_\Gamma = 0.$$

Finally, the linearization claim in (e) follows immediately from (3.4) and (3.5) by Taylor expanding  $F'$  around  $r = r_{\xi_t, \xi_\phi}$ .

The full stable and unstable submanifolds then are given as in Dyatlov’s paper [5, Proposition 3.5]. By equation (3.5), we note that

$$H_q(\mu\xi_r^2 - b^2(F - F(r_{\xi_t, \xi_\phi}))) = 0,$$

which shows that the flow of  $H_q$  leaves  $\Gamma^{u/s}$  invariant. Moreover, note that

$$\Gamma = \Gamma^u \cap \Gamma^s.$$

The defining functions for  $\Gamma^{u/s}$  are

$$\xi_r \mp \operatorname{sgn}(r - r_{\xi_t, \xi_\phi})b\sqrt{\frac{F(r) - F(r_{\xi_t, \xi_\phi})}{\mu}} \quad \text{and} \quad p,$$

and since  $dp(\partial_{\xi_r})|_\Gamma = 0$ , it follows that  $\Gamma^{u/s}$  are smooth submanifolds of  $\operatorname{Char}(P)$  near  $\Gamma$ . A simple rewriting taking into account the eigenvectors of the linearization, or indeed the sign of the escape function  $\mathcal{E}$ , namely negative on the stable (so  $\operatorname{sgn} \xi_r = -\operatorname{sgn}(r - r_{\xi_t, \xi_\phi})$  there), positive on the unstable manifold (so  $\operatorname{sgn} \xi_r = \operatorname{sgn}(r - r_{\xi_t, \xi_\phi})$  there), gives the conclusion. ■

In order to prove Proposition 3.3, we need the following two lemmas. The first one gives three useful ways of rewriting  $h$ .

**Lemma 3.4.** *We have*

$$h(r) = -\frac{1}{r}(r\mu' - 4\mu)^2 - 4\mu\left(3m - \frac{4a^2}{r}\right) \tag{3.7}$$

$$= 4\Lambda m r^4 - 4b^2 r^3 + 12m\left(1 - \frac{\Lambda a^2}{3}\right)r^2 - 12m^2 r + 4ma^2 \tag{3.8}$$

$$= \frac{4\Lambda r^4}{3}\left(3m - \frac{4a^2}{r}\right) - 4\left(1 - \frac{\Lambda a^2}{3}\right)^2\left(r - \frac{m}{1 - \frac{\Lambda a^2}{3}}\right)^3 + 4m^3\left(\frac{a^2}{m^2} - \frac{1}{1 - \frac{\Lambda a^2}{3}}\right) \tag{3.9}$$

for all  $r \in \mathbb{R}$ .

**Remark 3.5.** Note that the discriminant condition (1.1) ensures that

$$1 - \frac{\Lambda a^2}{3} > 0,$$

hence the expressions in the lemma make sense.

*Proof.* Note that

$$\begin{aligned} r\mu' - 4\mu &= -2\left(1 - \frac{\Lambda a^2}{3}\right)r^2 + 6mr - 4a^2, \\ \partial_r(r\mu' - 4\mu) &= -4\left(1 - \frac{\Lambda a^2}{3}\right)r + 6m = \frac{2}{r}(r\mu' - 4\mu) - 6m + \frac{8a^2}{r}. \end{aligned}$$

Inserting this gives

$$\begin{aligned} 2\mu\partial_r(r\mu' - 4\mu) - \mu'(r\mu' - 4\mu) &= 2\mu\left(\frac{2}{r}(r\mu' - 4\mu) - 6m + \frac{8a^2}{r}\right) - \mu'(r\mu' - 4\mu) \\ &= (r\mu' - 4\mu)\left(\frac{4\mu}{r} - \mu'\right) - 4\mu\left(3m - \frac{4a^2}{r}\right) \\ &= -\frac{1}{r}(r\mu' - 4\mu)^2 - 4\mu\left(3m - \frac{4a^2}{r}\right), \end{aligned}$$

proving identity (3.7). Identity (3.8) now follows by

$$\begin{aligned} &-\frac{1}{r}(r\mu' - 4\mu)^2 - 4\mu\left(3m - \frac{4a^2}{r}\right) \\ &= -\frac{1}{r}\left(-2\left(1 - \frac{\Lambda a^2}{3}\right)r^2 + 6mr - 4a^2\right)^2 \\ &\quad - 4\left(-\frac{\Lambda r^4}{3} + \left(1 - \frac{\Lambda a^2}{3}\right)r^2 - 2mr + a^2\right)\left(3m - \frac{4a^2}{r}\right) \\ &= -4\left(1 - \frac{\Lambda a^2}{3}\right)^2 r^3 - 36m^2 r - \frac{16a^4}{r} + 24m\left(1 - \frac{\Lambda a^2}{3}\right)r^2 \\ &\quad - 16\left(1 - \frac{\Lambda a^2}{3}\right)a^2 r + 48ma^2 + 4\Lambda m r^4 - 12m\left(1 - \frac{\Lambda a^2}{3}\right)r^2 \\ &\quad + 24m^2 r - 12ma^2 - \frac{16\Lambda a^2}{3}r^3 + 16\left(1 - \frac{\Lambda a^2}{3}\right)a^2 r - 32ma^2 + \frac{16a^4}{r} \\ &= 4\Lambda m r^4 - 4\left(1 + \frac{\Lambda a^2}{3}\right)^2 r^3 + 12m\left(1 - \frac{\Lambda a^2}{3}\right)r^2 - 12m^2 r + 4ma^2, \end{aligned}$$

as claimed. Finally, identity (3.9) follows by

$$\begin{aligned} &4\Lambda m r^4 - 4\left(1 + \frac{\Lambda a^2}{3}\right)^2 r^3 + 12m\left(1 - \frac{\Lambda a^2}{3}\right)r^2 - 12m^2 r + 4ma^2 \\ &= 4\Lambda m r^4 - 4\left(1 + \frac{\Lambda a^2}{3}\right)^2 r^3 + 4\left(1 - \frac{\Lambda a^2}{3}\right)^2 r^3 - 4\left(1 - \frac{\Lambda a^2}{3}\right)^2 r^3 \\ &\quad + 12m\left(1 - \frac{\Lambda a^2}{3}\right)r^2 - 12m^2 r + \frac{4m^3}{1 - \frac{\Lambda a^2}{3}} + 4m^3\left(\frac{a^2}{m^2} - \frac{1}{1 - \frac{\Lambda a^2}{3}}\right) \\ &= \frac{4\Lambda r^4}{3}\left(3m - \frac{4}{r}a^2\right) - 4\left(1 - \frac{\Lambda a^2}{3}\right)^2\left(r - \frac{m}{1 - \frac{\Lambda a^2}{3}}\right)^3 + 4m^3\left(\frac{a^2}{m^2} - \frac{1}{1 - \frac{\Lambda a^2}{3}}\right). \blacksquare \end{aligned}$$

**Lemma 3.6.** *Given  $a \in \mathbb{R}$  and  $m > 0$ , there is a (potentially empty) interval  $(\Lambda_0, \Lambda_1)$  such that  $a$ ,  $m$  and  $\Lambda \geq 0$  satisfy the discriminant condition (1.1) if and only if*

$$\Lambda \in (\Lambda_0, \Lambda_1),$$

in which case we have

$$\frac{a^2}{m^2} < \frac{9}{8} \frac{1}{1 - \frac{\Lambda a^2}{3}}. \tag{3.10}$$

If  $|a| > m$ , then  $\Lambda_0 > 0$ .

*Proof.* Writing  $\gamma := \frac{\Lambda a^2}{3}$ , the discriminant condition (1.1) becomes

$$-(1 + \gamma)^4 \frac{a^2}{m^2} + 36(1 - \gamma)\gamma + (1 - \gamma)^3 - 27\gamma \frac{m^2}{a^2} > 0.$$

We introduce

$$v := \frac{\gamma}{(1 - \gamma)^2}, \quad \alpha := \frac{m^2}{a^2(1 - \gamma)}.$$

Multiplying with  $\frac{m^2}{a^2}$  and using that

$$(1 + \gamma)^4 = ((1 - \gamma)^2 + 4\gamma)^2 = \frac{\gamma^2}{v^2} + 8\frac{\gamma^2}{v} + 16\gamma^2,$$

we note that (1.1) is equivalent to

$$\frac{\gamma^2}{v^2}(-1 - 8v - 16v^2 + 36v\alpha + \alpha - 27v\alpha^2) > 0,$$

which in turn is equivalent to

$$-16v^2 + (-27\alpha^2 + 36\alpha - 8)v - 1 + \alpha > 0.$$

Since this is a quadratic expression in  $v$  and since  $(0, 1) \ni \gamma \mapsto v$  is injective, this proves the first assertion by Remark 3.5. The discriminant of the quadratic expression in  $v$  is  $\alpha(9\alpha - 8)^3$ , which is positive if and only if (3.10) is satisfied. Finally, in case  $|a| > m$ , equation (1.1) implies that  $\Lambda = 0$  is not allowed, proving the last assertion. ■

**Remark 3.7.** The case of Reissner–Nordström–de Sitter spacetimes gives rise to a very similar subextremality condition, which was explicitly analyzed in [10, Proposition 3.2], following [16].

*Proof of Proposition 3.3.*

*Step 1: The case when  $r \in (\frac{4a^2}{3m}, r_c) \cap (r_e, r_c)$ .* This is immediate from (3.7), by noting that

$$3m - \frac{4a^2}{r} > 0, \quad \mu(r) > 0,$$

in the interval.

Step 2: The case when  $\frac{1}{1-\frac{\Lambda a^2}{3}} \geq \frac{a^2}{m^2}$ . Since

$$\mu'(r) = -\frac{4\Lambda r^3}{3} + 2\left(1 - \frac{\Lambda a^2}{3}\right)r - 2m,$$

we know that

$$\mu'(r) < 0 \quad \text{for all } r \in \left[0, \frac{m}{1 - \frac{\Lambda a^2}{3}}\right].$$

Since we know that

$$r_- < 0 < r_C < r_e < r_c \quad \text{and} \quad \mu(0) = a^2 > 0,$$

it follows that

$$r_e > \frac{m}{1 - \frac{\Lambda a^2}{3}}.$$

The proof in this case is completed by combining this with (3.9) and step 1.

Step 3: The remaining case. We are left with proving the statement when

$$\frac{1}{1 - \frac{\Lambda a^2}{3}} < \frac{a^2}{m^2}, \quad r \in \left(r_e, \frac{4a^2}{3m}\right].$$

The assertion in Proposition 3.3 is the negativity of

$$h(r) := 2\mu f' - \mu' f,$$

where

$$f(r) := r\mu' - 4\mu = -2\left(\left(1 - \frac{\Lambda a^2}{3}\right)r^2 - 3mr + 2a^2\right).$$

By Lemma 3.6, we need to check the condition in an interval  $\Lambda \in (\Lambda_0, \Lambda_1)$ , where  $\Lambda_0 > 0$ . We first claim that  $h(r)$  decreases with increasing  $\Lambda$  for all  $r \in (r_e, \frac{4a^2}{3m})$ . Differentiating  $h$  with respect to  $\Lambda$ , using equation (3.8), gives

$$\frac{\partial}{\partial \Lambda} h(r) = 4mr^4 - \frac{8}{3}a^2br^3 - 4ma^2r^2 = 4m^3r^2\left(\left(\frac{r}{m}\right)^2 - \frac{2}{3}\frac{a^2}{m^2}b\frac{r}{m} - \frac{a^2}{m^2}\right),$$

which is negative for all  $r \in (r_1, r_2)$ , where

$$\frac{r_{1/2}}{m} = \frac{a^2b}{3m^2} \mp \sqrt{\frac{a^4b^2}{9m^4} + \frac{a^2}{m^2}}.$$

The claim is proven if we can show that  $(r_1, r_2) \subseteq (r_e, r_c)$ . Since

$$r_1 < 0 < r_e,$$

it remains to show that

$$r_2 \geq \frac{4a^2}{3m},$$

which is equivalent to

$$\sqrt{\frac{a^4}{9m^4} \left(1 + \frac{\Lambda a^2}{3}\right)^2 + \frac{a^2}{m^2}} \geq \frac{4a^2 - a^2b}{3m^2} = \frac{3 - \frac{\Lambda a^2}{3}}{3m^2} a^2,$$

which in turn is equivalent to

$$\frac{a^2}{m^2} \geq \left( \left(3 - \frac{\Lambda a^2}{3}\right)^2 - \left(1 + \frac{\Lambda a^2}{3}\right)^2 \right) \frac{a^4}{9m^4} = 8 \left(1 - \frac{\Lambda a^2}{3}\right) \frac{a^4}{9m^4}.$$

This inequality is equivalent to (3.10), so the proof of the claim is complete.

It therefore suffices to prove the negativity of  $h(r)$  for all  $r \in (r_e, \frac{4a^2}{3m}]$ , in the limit when  $\Lambda = \Lambda_0$ . By definition of  $\Lambda_0$ ,  $\mu$  has at least a double root when  $\Lambda = \Lambda_0$ . Since  $\mu(0) = a^2 > 0$ , independent of  $\Lambda$ , there is still a simple negative root  $r_- < 0$ . Since  $\mu$  decreases with  $\Lambda$ , the case  $r_C < r_e = r_c$  is excluded for  $\Lambda = \Lambda_0$ , for it would contradict that  $\mu$  has four distinct real roots for any  $\Lambda > \Lambda_0$ . Similarly, the case  $r_C = r_e = r_c$  can be excluded, since also  $\mu'$  decreases with  $\Lambda$  and would in that case not have three distinct real roots for any  $\Lambda > \Lambda_0$ . We conclude that  $r_- < r_C = r_e < r_c$  when  $\Lambda = \Lambda_0$ . In this case,  $\mu(r_e) = \mu'(r_e) = 0$  and we note that  $r_e$  satisfies the equation

$$0 = r_e \mu'(r_e) - 4\mu(r_e) = -2 \left( \left(1 - \frac{\Lambda_0 a^2}{3}\right) r_e^2 - 3m r_e + 2a^2 \right). \tag{3.11}$$

Since

$$r\mu' - 4\mu = r^5 \partial_r (r^{-4} \mu)$$

and since  $r^{-4} \mu$  has two positive zeros  $r_C = r_e$  and  $r_c$ , we conclude that  $r_e$  coincides with the smaller root of (3.11):

$$r_e = \frac{3m}{2(1 - \frac{\Lambda_0 a^2}{3})} - \frac{m}{2(1 - \frac{\Lambda_0 a^2}{3})} \sqrt{9 - 8 \frac{a^2}{m^2} \left(1 - \frac{\Lambda_0 a^2}{3}\right)}.$$

The idea is now to estimate the Taylor expansion at  $r_e$  for  $r \geq r_e$ . We write

$$h(r) = \sum_{k=0}^4 h^{(k)}(r_e) \frac{(r - r_e)^k}{k!}.$$

It is immediate from (3.8) that

$$\frac{h^{(4)}(r_e)}{4!} = 4\Lambda_0 m.$$

Using that

$$\mu(r_e) = \mu'(r_e) = f(r_e) = 0,$$

one computes that

$$\begin{aligned} h(r_e) &= h'(r_e) = h''(r_e) = 0, \\ h'''(r_e) &= 3\mu''(r_e) f''(r_e) - \mu'''(r_e) f'(r_e). \end{aligned}$$

We note that

$$f'(r) = -4\left(1 - \frac{\Lambda_0 a^2}{3}\right)r + 6m = r\mu''(r) - 3\mu'(r)$$

and hence

$$f'(r_e) = 2m\sqrt{9 - 8\frac{a^2}{m^2}\left(1 - \frac{\Lambda_0 a^2}{3}\right)} = r_e\mu''(r_e).$$

Moreover,

$$f''(r_e) = r_e\mu'''(r_e) - 2\mu''(r_e).$$

Inserting this, we get

$$\begin{aligned} h'''(r_e) &= -6\mu''(r_e)^2 + 2r_e\mu''(r_e)\mu'''(r_e) \leq 2f'(r_e)\mu'''(r_e) \\ &= -32\Lambda_0 m r_e \sqrt{9 - 8\frac{a^2}{m^2}\left(1 - \frac{\Lambda_0 a^2}{3}\right)}. \end{aligned} \quad (3.12)$$

Using that

$$r_e \geq \frac{m}{1 - \frac{\Lambda_0 a^2}{3}},$$

the Taylor expansion at  $r = r_e$  can be estimated as

$$\frac{h(r)}{(r - r_e)^3} \leq -\frac{16}{3} \frac{\Lambda_0 m^2}{1 - \frac{\Lambda_0 a^2}{3}} \sqrt{9 - 8\frac{a^2}{m^2}\left(1 - \frac{\Lambda_0 a^2}{3}\right)} + 4\Lambda_0 m(r - r_e)$$

for  $r \geq r_e$ . Now, the above bound is increasing with  $r$ , it therefore suffices to check the negativity at

$$\hat{r} := \frac{3m}{2\left(1 - \frac{\Lambda_0 a^2}{3}\right)} \geq \frac{3}{2} \frac{8}{9} \frac{a^2}{m} = \frac{4a^2}{3m}.$$

We get

$$\begin{aligned} \frac{h(\hat{r})}{(\hat{r} - r_e)^3} &\leq -\frac{16}{3} \frac{\Lambda_0 m^2}{1 - \frac{\Lambda_0 a^2}{3}} \sqrt{9 - 8\frac{a^2}{m^2}\left(1 - \frac{\Lambda_0 a^2}{3}\right)} \\ &\quad + 2\frac{\Lambda_0 m^2}{1 - \frac{\Lambda_0 a^2}{3}} \sqrt{9 - 8\frac{a^2}{m^2}\left(1 - \frac{\Lambda_0 a^2}{3}\right)} < 0. \end{aligned}$$

We have in particular shown that  $h(r) < 0$  when  $r \in (r_e, \frac{4a^2}{3m}]$ ,  $\Lambda = \Lambda_0$ . It follows that  $h < 0$  on this interval for all  $\Lambda \in (\Lambda_0, \Lambda_1)$ . The proof is completed by applying step 1. ■

**Remark 3.8.** We also present a somewhat different approach to proving (3.12), which was the key computation above. By (3.8), we have, with  $\Lambda = \Lambda_0$ ,  $r = r_e$ ,

$$h'''(r) = 24(4\Lambda m r - (1 + \gamma)^2),$$

and want to prove that

$$h'''(r_e) \leq -96\Lambda r_e \sqrt{1 - \frac{8}{9\alpha}}, \quad (3.13)$$

where, as in Lemma 3.6,  $\alpha = \frac{m^2}{a^2(1-\gamma)}$ ,  $\gamma = \frac{\Lambda a^2}{3}$ , and  $\alpha \in [\frac{8}{9}, 1]$ . Also,

$$r_e = \frac{3m}{2(1-\gamma)} \left(1 - \sqrt{1 - \frac{8}{9\alpha}}\right).$$

Thus,

$$h'''(r_e) = 24 \left( \frac{6\Lambda m^2}{1-\gamma} \left(1 - \sqrt{1 - \frac{8}{9\alpha}}\right) - (1 + \gamma)^2 \right),$$

and

$$\Lambda m^2 = 3 \frac{\Lambda a^2}{3} \frac{m^2}{a^2(1-\gamma)} (1-\gamma) = 3\alpha\gamma(1-\gamma), \tag{3.14}$$

so

$$\frac{h'''(r_e)}{24} = 18\alpha\gamma \left(1 - \sqrt{1 - \frac{8}{9\alpha}}\right) - (1 + \gamma)^2.$$

On the other hand, the right-hand side of (3.13) divided by 24 is

$$-4 \frac{3\Lambda m^2}{2(1-\gamma)} \left(1 - \sqrt{1 - \frac{8}{9\alpha}}\right) \sqrt{1 - \frac{8}{9\alpha}},$$

which is, using expression (3.14) for  $\Lambda m^2$ ,

$$-18\alpha\gamma \left(1 - \sqrt{1 - \frac{8}{9\alpha}}\right) \sqrt{1 - \frac{8}{9\alpha}}.$$

Subtracting this from  $h'''(r_e)/24$ , one wants to show that it  $\leq 0$ . But this is

$$18\alpha\gamma \left(1 - \sqrt{1 - \frac{8}{9\alpha}}\right) \left(1 + \sqrt{1 - \frac{8}{9\alpha}}\right) - (1 + \gamma)^2,$$

which simplifies to

$$18\gamma\alpha \left(1 - 1 + \frac{8}{9\alpha}\right) - (1 + \gamma)^2 = 16\gamma - (1 + \gamma)^2,$$

which is exactly the negative of the quadratic polynomial one gets in the discriminant when one wants to assure that  $r\mu' - \mu = r^2(r^{-1}\mu)'$  has distinct real roots, as is the case in the subextremal range, namely one each in  $(r_C, r_e)$  and in  $(r_e, r_c)$  corresponding to the critical points of  $r^{-1}\mu$  there, i.e.,  $\gamma$  has to be such that the negative of this is  $\geq 0$ , so it is indeed  $\leq 0$ .

Finally, we combine Theorem 3.2 with the results of [21] in the proof of Theorem 1.6.

*Proof of Theorem 1.6.* We again consider the operator

$$P_\sigma: \{u \in \bar{H}^s \mid P_\sigma u \in \bar{H}^{s-1}\} \rightarrow \bar{H}^{s-1}$$

from Theorem 2.1. The semiclassical trapping is corresponding to the trapping of bicharacteristics of the full wave operator  $P$ . Given that Theorem 3.2 implies that the trapping is normally hyperbolic, the proof of the semiclassical estimates and consequently the proof of Theorem 1.6 proceeds completely analogous to the proof of [21, Theorem 1.4]. ■

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